Chapter 4
Covering Spaces

This chapter continues the study of the fundamental groups and is designed to utilize the power of the fundamental groups through a study of covering spaces. The fundamental groups are deeply connected with covering spaces. Algebraic features of the fundamental groups are expressed by the geometric language of covering spaces. Main interest in the study of this chapter is to establish an exact correspondence between the various connected covering spaces of a given base space B and subgroups of its fundamental group $\pi_1(B)$, like Galois theory, with its correspondence between field extensions and subgroups of Galois groups, which is an amazing result. Historically, the systemic study of covering spaces appeared during the late 19th century and early 20th century through the theory of Riemann surfaces. But its origin was found before the invention of the fundamental groups by H. Poincaré in 1895. Poincaré introduced the concept of universal covering spaces in 1883 to prove a theorem on analytic functions.

The theory of covering spaces is of great importance not only in topology but also in other branches of mathematics such as complex analysis, geometry, Lie groups and also in some areas beyond mathematics. A covering space is a locally trivial map with discrete fibers. The objects of this nature can be classified by algebraic objects related to fundamental groups. The exponential map $p : \mathbf{R} \to S^1$ defined by $p(x) = e^{2\pi i x}$, $x \in \mathbf{R}$ is a powerful covering projection and $(\mathbf{R}, p)$ is the universal covering space of S^1. Chapter 3 has utilized this map as a tool for computing $\pi_1(S^1)$. Covering spaces likewise provide useful general tools for computation of fundamental groups. The fundamental group is instrumental for classifying the topological spaces which can be covering spaces of a given base space B. For a large class of spaces, the possible covering spaces of B are determined by the subgroups of $\pi_1(B)$. Moreover, the theory of covering spaces facilitates to determine the fundamental groups of several spaces.

More precisely, this chapter considers a class of mappings $p : X \to B$, called the 'covering projections' from a space X, called a covering space, to a space B, called base space, to which the properties of the exponential map p are extended. Moreover,

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this chapter introduces the concepts of fibrations and cofibrations born in geometry and topology and proves some classical results such as Borsuk–Ulum theorem and Hurewicz theorem for a fibration.

For this chapter the books Croom (1978), Hatcher (2002), Rotman (1988), Spanier (1966), Steenrod (1951), and some others are referred in the Bibliography.

4.1 Covering Spaces: Introductory Concepts and Examples

This section introduces the concept of covering spaces. Covering spaces displays the first example of the power of the fundamental groups in classifying topological spaces. Algebraic features of the fundamental groups $\pi_1(B)$ of the base space B are expressed in the geometric language of covering spaces of B.

4.1.1 Introductory Concepts

This subsection introduces the concept of a covering space with illustrative examples. Recall that a topological space X is path-connected if each pair of points in X can be joined by a path in X. A space that satisfies this property locally is called 'locally path-connected.' If X is a disconnected space, a maximal path-connected subset of the space X is called a path component and is not a proper subset of any path-connected subset of X. The path components of a subset B of X are the path components of B in its subspace topology. For example, each interval and each ray in the real line are both path-connected and locally path-connected. On the other hand, the subspace $[-1, 0) \cup (0, 1]$ of $\mathbf{R}$ is not path-connected but it is locally path-connected. The deleted comb space is path-connected but not locally path-connected. The space of rationals $\mathbf{Q}$ is neither connected nor locally connected.

Definition 4.1.1 Let X and B be topological spaces and let $p : X \to B$ be a continuous surjective map. An open set U of B is said to be evenly covered by p if $p^{-1}(U)$ is a union of disjoint open sets S_i, called sheets such that $p|_{S_i} : S_i \to U$ is a homeomorphism for each i and U is called an admissible open set in B.

Example 4.1.2 Consider the exponential map $p : \mathbf{R} \to S^1$ defined by

$$p(x) = e^{2\pi i x} = \cos 2\pi x + i \sin 2\pi x, x \in \mathbf{R}.$$

Then the open set $U = S^1 - \{1\}$ is evenly covered by p, since $p^{-1}(U) = \bigcup_{n\in\mathbf{Z}}(n - \frac{1}{2}, n + \frac{1}{2})$. Clearly, the sheets are open intervals.

Definition 4.1.3 Let B be a topological space. The pair (X, p) is called a covering space of B if

(i) X is a path-connected topological space;
(ii) the map $p : X \to B$ is continuous;
(iii) each $b \in B$ has an open neighborhood which is evenly covered by p.

The map p is called the covering projection and an open set in B which is evenly covered by p is called p-admissible or simply admissible.

Remark 4.1.4 Some authors do not assume X to be path-connected but assume p to be surjective while defining a covering space.

Example 4.1.5 The exponential map $p : \mathbf{R} \to S^1$ defined in Example 4.1.2 is a covering projection and hence $(\mathbf{R}, p)$ is a covering space of S^1. Because the open sets $U_1 = S^1 - \{-1\}$ and $U_2 = S^1 - \{1\}$ are evenly covered by p. Thus each point of S^1 has an admissible open neighborhood in S^1. In fact, any proper connected arc of S^1 is evenly covered by p. The same argument shows that the map $p : \mathbf{R} \to S^1$ defined by $p(t) = e^{i\alpha t}$, where $\alpha \in \mathbf{R}$ is a fixed nonzero real number, is also a covering projection.

Example 4.1.6 For any positive integer n, let $p_n : S^1 \to S^1$ be the map defined by $p_n(z) = z^n, z \in S^1$. Then (S^1, p_n) is a covering space of S^1. Because, in polar coordinates, p_n is given by $p_n(1, \theta) = (1, n\theta)$. The map p_n wraps the circle around itself n times. Let U be an open arc on S^1 subtended by an angle θ, $0 \leq \theta \leq 2\pi$, and containing a point x. Then $p^{-1}(U)$ consists of n open arcs each determining an angle θ/n and each containing one nth root of x. Each of these n open arcs is mapped homeomorphically onto U. Thus any proper arc in S^1 is an admissible neighborhood. Consequently, (S^1, p_n) is a covering space of S^1.

Example 4.1.7 Consider the map $f : \mathbf{R}^2 \to S^1 \times S^1$ from the plane to the torus defined by $f(t_1, t_2) = (e^{2\pi i t_1}, e^{2\pi i t_2})$, $(t_1, t_2) \in \mathbf{R}^2$. Then $(\mathbf{R}^2, f)$ is a covering space of $S^1 \times S^1$.

For any point $(z_1, z_2) \in S^1 \times S^1$, let U be a small rectangle formed by the product of two open arcs in S^1 containing z_1 and z_2, respectively. Then U is an admissible neighborhood whose inverse image consists of a countably infinite family of open rectangles in the plane $\mathbf{R}^2$. This example is essentially a generalizaton of the covering projection $p : \mathbf{R} \to S^1$.

Theorem 4.1.8 *Let (X_1, p_1) be a covering space of B_1, (X_2, p_2) be a covering space of B_2, then $(X_1 \times X_2, p_1 \times p_2)$ is a covering space of $B_1 \times B_2$, where $p_1 \times p_2 : X_1 \times X_2 \to B_1 \times B_2$ is defined by $(p_1 \times p_2)(x, y) = (p_1(x), p_2(y))$.*

Proof Let $(b_1, b_2) \in B_1 \times B_2$ and U_1 be an open neighborhood of b_1 and U_2 be an open neighborhood of b_2 which are evenly covered by p_1 and p_2, respectively. Then $U_1 \times U_2$ is a neighborhood of (b_1, b_2) in $B_1 \times B_2$ which is evenly covered by $p_1 \times p_2$. ❑

Example 4.1.9 Consider the exponential map $p : \mathbf{R} \to S^1$ defined by

$$p(x) = e^{2\pi i x} = \cos 2\pi x + i \sin 2\pi x, x \in \mathbf{R}.$$

Then the map $(p, p) : \mathbf{R} \times \mathbf{R} \to S^1 \times S^1$ is a covering projection. In fact for every positive integer n, the product map $p \circ p \circ \cdots \circ p = p^n : \mathbf{R}^n \to T^n$ is a covering projection, where $T^n = \prod_1^n S^1$ is the n-dimensional torus.

Theorem 4.1.10 *Let $p : (X, x_0) \to (B, b_0)$ be a covering projection. If X is path-connected, then there is a surjection $\psi : \pi_1(B, b_0) \to p^{-1}(b_0)$. If X is simply connected, then ψ is a bijection.*

Proof Following the technique for computation of $\pi_1(S^1, 1)$ (see Theorem 3.3.9 of Chap. 3) the theorem can be proved. ❑

Remark 4.1.11 Everytopological space is not necessarily a covering space. The following is an example of a topological space X which is not a covering space of Y.

Example 4.1.12 Let X be a rectangle which is mapped by the projection p onto the first coordinate to an interval Y. Let U be an interval in Y. Then $p^{-1}(U)$ is a strip in X consisting of all points above U (as shown in Fig. 4.1).

This strip cannot be mapped by p homeomorphically onto U. Hence U is not evenly covered by p. Consequently, (X, p) is not a covering space of Y.

Example 4.1.13 (*Infinite and finite spirals*) Let X be an infinite spiral, and $p : X \to S^1$ be the projection described in Fig. 4.2.

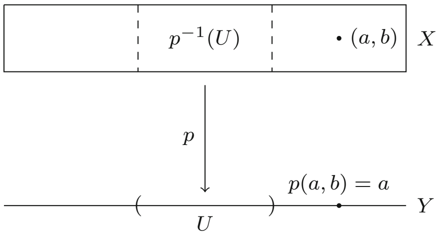


Fig. 4.1 Example of a neighborhood which is not evenly covered

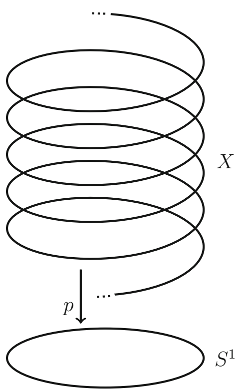


Fig. 4.2 Infinite spiral with projection p

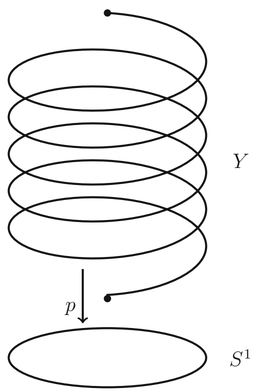


Fig. 4.3 Finite spiral with projection p

Each point of X is projected by p to the point on the circle directly below it. Then (X, p) is a covering space of S^1. On the other hand, if $p : Y \to S^1$ is a finite spiral projection as shown in Fig. 4.2, then (Y, p) is not covering space of S^1, because if x_0 and x_1 are the end points of the spiral Y, then the points $p(x_0)$ and $p(x_1)$ as shown in Fig. 4.3 have no admissible neighborhoods.

4.1.2 Some Interesting Properties of Covering Spaces

This subsection presents some properties of covering spaces.

Proposition 4.1.14 *Let $p : (X, x_0) \to (B, b_0)$ be a covering space. Then the induced homomorphism $p_* : \pi_1(X, x_0) \to \pi_1(B, b_0)$ is a monomorphism and the subgroup $p_*(\pi_1(X, x_0))$ in $\pi_1(B, b_0)$ consists of homotopy class of loops in B based at b_0 which lifts to X starting at x_0 are loops.*

Proof Let an element $\alpha \in \ker p_*$ be represented by a loop $\tilde{f}_0 : I \to X$ with a homotopy $F_t : I \to B$ of $f_0 = p \circ \tilde{f}_0$ to the trivial loop f_1 (Fig. 4.4).

Hence there exists a lifted homotopy of loop $\widetilde{F}_t : I \to X$ started at $\tilde{f}_0$ and ending with a constant loop (because the lifted homotopy $\widetilde{F}_t$ is a homotopy of paths fixing the end points, since t varies each point of $\widetilde{F}_t$ gives a path lifting a constant path, which is therefore constant). Hence $[\tilde{f}_0] = 0$ in $\pi_1(X, x_0)$ shows that p_* is injective. ❑

We now state the following two other properties of covering spaces whose proofs are given in Sect. 4.5.2.

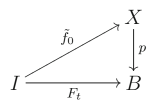


Fig. 4.4 Homotopy diagram corresponding to a lifting of f_0 to $\tilde{f}_0$

Proposition 4.1.15 *Let $p : (X, x_0) \to (B, b_0)$ be a covering space and $f : (Y, y_0) \to (B, b_0)$ be a map, where Y is path-connected and locally path-connected. Then a lift $\tilde{f} : (Y, y_0) \to (X, x_0)$ of f exists iff $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(X, x_0))$.*

Proposition 4.1.16 *Given a covering space $p : X \to B$ and a map $f : Y \to X$ with two lifting $\tilde{f_1}, \tilde{f_2} : Y \to X$ that agree at some point of Y, then if Y is connected, $\tilde{f_1} = \tilde{f_2}$, i.e., $\tilde{f_1}(y) = \tilde{f_2}(y)$, $\forall\, y \in Y$.*

4.1.3 Covering Spaces of $\mathbf{R}P^n$

This subsection studies covering spaces of real projective spaces $\mathbf{R}P^n$ and computes fundamental group of $\mathbf{R}P^2$.

Definition 4.1.17 (*Real projective plane*) Let $\mathbf{R}P^2$ be the real projective plane defined as a quotient space of the 2-sphere S^2 obtained by identifying each point x of S^2 with its antipodal point $-x$ and $p : S^2 \to \mathbf{R}P^2$ be the natural map which identifies each pair of antipodal points i.e., p maps each x to its equivalence class. We topologize $\mathbf{R}P^2$ by defining V to be open in $\mathbf{R}P^2$ if and only if $p^{-1}(V)$ is open in S^2. With this topology $\mathbf{R}P^2$ becomes a topological space.

Theorem 4.1.18 *The projective space $\mathbf{R}P^2$ is a surface and (S^2, p) is a covering space of $\mathbf{R}P^2$.*

Proof First we show that $p : S^2 \to \mathbf{R}P^2$ is a covering map. Given $y \in \mathbf{R}P^2$, we choose $x \in p^{-1}(y)$. We then choose an ϵ-neighborhood U of x in S^2 for some $\epsilon < 1$, using the Euclidean metric d of $\mathbf{R}^3$. If $A : S^2 \to S^2$ is the antipodal map sending z to its antipodal point $-z$, then U contains no pair $\{z, A(z)\}$ of antipodal points of S^2, since $d(z, a(z)) = 2$. Consequently, the map $p : U \to p(U)$ is bijective. The antipodal map $A : S^2 \to S^2$, given by $A(z) = -z$ is a homeomorphism of S^2 and hence $A(U)$ is open in S^2. Since $p^{-1}(p(U)) = U \cup A(U)$, this set is also open in S^2. Consequently, $p(U)$ is open in $\mathbf{R}P^2$ and hence p is an open map. Thus the bijective map $p : U \to p(U)$ is continuous and open. Hence it is a homeomorphism. Similarly, $p : A(U) \to p(A(U)) = p(U)$ is a homeomorphism. The set $p^{-1}(p(U))$ is thus the union of two open sets U and $A(U)$, each of which is mapped homeomorphically by p onto $p(U)$. Hence $p(U)$ is a neighborhood of $p(x) = y$, which is evenly covered by p. Consequently, (S^2, p) is a covering space of $\mathbf{R}P^2$. For the first part, let $\{U_n\}$ be countable basis of S^2. Then $\{p(U)\}$ is a countable basis of $\mathbf{R}P^2$. Clearly, $\mathbf{R}P^2$ is a Hausdorff space. Let y_1 and y_2 be two points of $\mathbf{R}P^2$. The set $p^{-1}(y_1) \cup p^{-1}(y_2)$ consists of four points. Let 2ϵ be the minimum distance between them. Let U_1 be the ϵ-neighborhood of one of the points $p^{-1}(y_1)$ and U_2 be the ϵ-neighborhood of one of the points $p^{-1}(y_2)$. Then the sets $U_1 \cup A(U_1)$ and $U_2 \cup A(U_2)$ are disjoint. Consequently, $p(U_1)$ and $p(U_2)$ are disjoint neighborhoods of y_1 and y_2, respectively, in $\mathbf{R}P^2$. Since S^2 is a surface and every point of $\mathbf{R}P^2$ has a neighborhood homeomorphic to an open subset of S^2, the space $\mathbf{R}P^2$ is also a surface. ❑

A generalization of Theorem 4.1.18 for $n > 1$ is now given.

Theorem 4.1.19 *(S^n, p) is a covering space of $\mathbf{R}P^n$, where p is the map identifying antipodal points of S^n for $n > 1$.*

Proof The sets $E_i^+ = \{(x_1, x_2, \ldots, x_{n+1}) \in S^n : x_i > 0\}$ and $E_i^- = \{(x_1, x_2, \ldots, x_{n+1}) \in S^n : x_i < 0\}$ are open sets and cover S^n. The map $p|_{E_i^+}$ is 1-1, continuous and open. Hence if $U_i = p(E_i^+) = p(E_i^-)$, then $p^{-1}(U_i) = E_i^+ \cup E_i^-$. The sets E_i^+ and E_i^- are disjoint open sets, and homeomorphic to U_i. This shows that $p : S^n \to \mathbf{R}P^n$ is a covering space. This asserts that (S^n, p) is a covering space of $\mathbf{R}P^n$. ❑

Definition 4.1.20 The multiplicity of a covering space (X, p) of B is the cardinal number of a fiber. If the multiplicity is n, we say that (X, p) is an n-sheeted covering space of B or that (X, p) is an n-fold cover of B.

Example 4.1.21 **(i)** (S^2, p) is a double covering of $\mathbf{R}P^2$.

(ii) The number of sheets of $(\mathbf{R}, p)$ of S^1 is countably infinite.
Because, p identifies pairs of antipodal points, the number of sheets of this covering in (i) is 2. On the other hand, for the (ii) covering projection $p : \mathbf{R} \to S^1$ (see Example 4.1.2) maps each integer and only the integers to $1 \in S^1$. Thus $p^{-1}(1) = \mathbf{Z}$ and hence the number of sheets of this covering is countably infinite.

Theorem 4.1.22 $\pi_1(\mathbf{R}P^2, y) \cong \mathbf{Z}_2$.

Proof The projection $p : S^2 \to \mathbf{R}P^2$ is covering map by Theorem 4.1.18. Since S^2 is simply connected, we apply Theorem 4.1.10, which gives a bijective correspondence between $\pi_1(\mathbf{R}P^2, y)$ and the set $p^{-1}(y)$. Since $p^{-1}(y)$ is a two-element set, $\pi_1(\mathbf{R}P^2, y)$ is a group of order 2. Since any group of order 2 is isomorphic to $\mathbf{Z}_2$, it follows that $\pi_1(\mathbf{R}P^2, y) \cong \mathbf{Z}_2$. ❑

Remark 4.1.23 For computing $\pi_1(\mathbf{R}P^n, y)$ by using the universal covering space (S^n, q) of $\mathbf{R}P^n$, where q identifies the antipodal points of S^n, (see Sect. 4.6.2), use topological group action see Corollary 4.10.4.

4.2 Computing Fundamental Groups of Figure-Eight and Double Torus

We now consider some topological spaces whose fundamental groups are nonabelian. This section constructs covering spaces for computation of fundamental groups of some spaces such as figure-eight and double torus whose fundamental groups are not abelian. For computing the fundamental group of figure-eight by graph-theoretic method see Sect. 4.10.6.

Example 4.2.1 (*figure-eight*) The figure-eight F is the union of two circles A and B with a point x_0 in common. We now describe a certain covering space X for F.

Let X be the subspace of the plane consisting of the x-axis and the y-axis, along with the small circles tangent to these axes, one circle tangent to the x-axis at each nonzero integer point and one circle tangent to the y-axis at each nonzero integer point as shown in Fig. 4.5.

The projection map p wraps the x-axis around the circle A and wraps the y-axis around the other circle B; in each case the integer points are mapped by p into the base point x_0 of F. Then each circle tangent to an integer point on the x-axis is mapped homeomorphically by p onto B; on the other hand, each circle tangent to an integer point on the y-axis is mapped homeomorphically onto A; in each case the point of tangency is mapped onto the point x_0. Then p is a covering map.

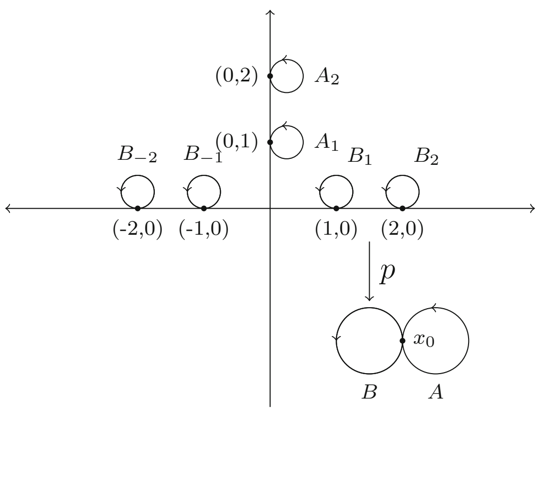
Theorem 4.2.2 *The fundamental group of the figure-eight is not abelian.*

Proof Let $\tilde{f} : I \to X$ be the path $\tilde{f}(t) = (t, 0)$, going along the x-axis from the origin $(0, 0)$ to the point $(1, 0)$. Let $\tilde{g} : I \to X$ be the path $\tilde{g}(t) = (0, t)$, going along the y-axis from the origin $(0, 0)$ to the point $(0, 1)$. Let $f = p \circ \tilde{f}$ and $g = p \circ \tilde{g}$. Then f and g are loops, in the figure-eight F based at x_0, going around the circles A and B, respectively. We claim that $f * g$ and $g * f$ are not path homotopic. We lift each of these paths to a path in X beginning at the origin. The path $f * g$ lifts to a path that goes along the x-axis from the origin to $(1, 0)$, and then goes once around the circle tangent to the x-axis at $(1, 0)$. But the path $g * f$ lifts to a path in X that goes along the y-axis from the origin to $(0, 1)$, and then goes once around the circle tangent to the y-axis at $(0, 1)$. Since the lifted paths do not end at the same point, $f * g$ and $g * f$ cannot be path homotopic. Therefore we conclude that $[f * g] \neq [g * f]$ and hence $[f] \cdot [g] \neq [g] \cdot [f]$ proves that the fundamental group of the figure eight is not abelian. ❑

Remark 4.2.3 For computing the fundamental group of figure-eight by graph-theoretic method see Sect. 4.10.6.

Corollary 4.2.4 *The fundamental group of the double torus T_2 is not abelian.*

Fig. 4.5 Figure-eight



Proof Figure-eight F is a retract of $T_2 \Rightarrow$ the inclusion map $i : (F, x_0) \hookrightarrow (T_2, x_0)$ induces a monomorphism $i_* : \pi_1(F, x_0) \to \pi_1(T_2, x_0) \Rightarrow \pi_1(T_2, x_0)$ is not abelian, since $\pi_1(F, x_0)$ is not abelian. ❑

Remark 4.2.5 For computing fundamental groups of some orbit spaces see Sect. 4.10.2.

4.3 Path Lifting and Homotopy Lifting Properties

This section continues the study of covering spaces and displays basic properties of covering spaces such as path lifting and homotopy lifting properties (PLP and HLP). We begin with characterization of locally path-connected spaces.

Recall the following definitions.

Definition 4.3.1 A topological space X is said to be locally path-connected if for each point $x \in X$ and every neighborhood U_x of x, there is an open set V with $x \in V \subset U_x$ such that any two points in V can be joined by a path in U_x.

Definition 4.3.2 A topological space X is said to be semilocally path-connected if for every point $x \in X$, there is an open neighborhood U_x of x such that every closed path in U_x at x is nullhomotopic in X.

Proposition 4.3.3 *A topological space X is locally path-connected if and only if each path component of each open subset of X is open.*

Proof Left as an exercise. ❑

Theorem 4.3.4 *Every covering projection $p : X \to B$ is an open mapping for any locally path-connected space X.*

Proof Let X be a locally path-connected space such that $p : X \to B$ be a covering projection and V be an open set in X. We claim that $p(V)$ is open in B. Let $b \in p(V)$ and $x \in p^{-1}(b)$ and U be an admissible neighborhood for b. Then x is a point of V such that $p(x) = b$. Let W be the component of $p^{-1}(U)$ which contains x. Since X is locally path connected, W is open in X by Proposition 4.3.3. Since p maps W homeomorphically onto U, p maps the open set $W \cap V$ to the open subset $p(W \cap V)$ in B. Then $b \in p(W \cap V) \subseteq p(V)$. Since b is an arbitrary point of $p(V)$, it follows that $p(V)$ is a union of open sets and hence $p(V)$ is an open set. Consequently p is an open mapping. ❑

Theorem 4.3.5 *Let (X, p) be a covering space of B and Y be a space. If f and g are continuous maps from Y to X for which $p \circ f = p \circ g$, as shown in Fig. 4.6, then the set $A = \{y \in Y : f(y) = g(y)\}$ (i.e., the set of points of Y at which f and g agree) is both open and closed in Y. (Y is not assumed to be path-connected or locally path-connected).*

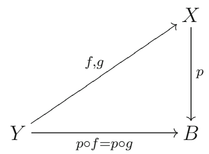


Fig. 4.6 Triangular diagram involving f, g and p

Proof To prove that A is open, let $y \in A$ and U be an admissible neighborhood of $(p \circ f)(y)$. Then the path component V of $p^{-1}(U)$ to which $f(y)$ belongs is an open set in X and hence $f^{-1}(V)$ and $g^{-1}(V)$ are open in Y. Since $f(y) \in V$ and $f(y) = g(y)$, then $y \in f^{-1}(V) \cap g^{-1}(V)$. We claim that $f^{-1}(V) \cap g^{-1}(V)$ is a subset of A and conclude that A is open, since it contains a neighborhood of each of its points. Let $t \in f^{-1}(V) \cap g^{-1}(U)$. Then $f(t), g(t) \in V$ and $(p \circ f)(t) = (p \circ g)(t)$. Since p maps V homeomorphically onto U, it follows that $f(t) = g(t)$ and hence $t \in A$. Thus it follows that A is an open set.

Next we prove that A is closed. Suppose A is not closed and let t be a limit point of A not in A. Then $f(t) \neq g(t)$. The point $(p \circ f)(t) = (p \circ g)(t)$ has an elementary neighborhood U such that the points $f(t)$ and $g(t)$ must be in distinct path components V_1 and V_2 of $p^{-1}(U)$. Since $t \in f^{-1}(V_1) \cap g^{-1}(V_2)$ which is an open set in Y, $f^{-1}(V_1) \cap g^{-1}(V_2)$ must contain a point $y \in A$. But this implies a contradiction, since $V_1 \cap V_2 = \emptyset$ and $f(y) = g(y) \in V_1 \cap V_2$. Hence all limit points of A must lie in A and therefore A is closed. ❑

Corollary 4.3.6 *Let (X, p) be a covering space of B, and let f, g be continuous maps from a connected space Y into X such that $p \circ f = p \circ g$. If f and g agree at a point of Y, then $f = g$.*

Proof Let Y be a connected space. Then the only sets that are both open and closed in Y are Y and $\emptyset$. Hence by Theorem 4.3.5 it follows that either $A = Y$ or $A = \emptyset$. This implies that either $f(y) = g(y)$ at every $y \in Y$ or $f(y) \neq g(y)$ at every $y \in Y$. By hypothesis $f(y) = g(y)$ at some $y \in Y$. Thus $A \neq \emptyset$ and hence $A = Y$. Consequently, $f(y) = g(y)$, $\forall\, y \in Y$ shows that $f = g$. ❑

Remark 4.3.7 The Corollary 4.3.6 gives the uniqueness of the lifting of a map and generalizes Proposition 3.3.2 of Chap. 3.

We now consider lifting problems. What is lifting problem?

Let $p : X \to B$ be a continuous surjective map (not necessarily a covering projection). Given a subspace A of X and a continuous map $f : A \to B$, does there exist a continuous map $\tilde{f} : A \to X$ such that $p \circ \tilde{f} = f$?

In other words, can we find a continuous map $\tilde{f} : A \to X$ making the diagram in Fig. 4.7 commutative? If such $\tilde{f}$ exists, then $\tilde{f}$ is called a lift of f. The satisfactory answer is available if p is covering projection.

Fig. 4.7 Lifting of a map f

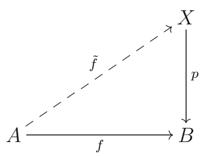
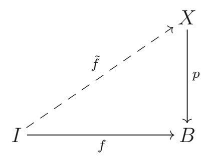


Fig. 4.8 Lifting of a path f



Definition 4.3.8 Let (X, p) be a covering space of B and let $f : I \to B$ be a path in B. A path $\tilde{f} : I \to X$ in X such that $p \circ \tilde{f} = f$, is called a lifting or covering path of f,i.e., if it makes the diagram as shown in Fig. 4.8 commutative.

If $F : I \times I \to B$ be a homotopy, then a homotopy $\tilde{F} : I \times I \to X$ for which $p \circ \tilde{F} = F$, is called a lifting or covering homotopy of F.

We now generalize Theorem 3.3.3 and its Corollary 3.3.4. of Chap. 3.

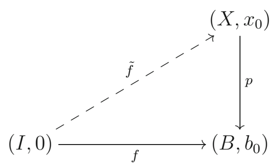
Theorem 4.3.9 (The Path Lifting Property) *Let (X, p) be a covering space of B and $f : I \to B$ be a path in B beginning at a point $b_0 \in B$. If $x_0 \in p^{-1}(b_0)$, then there is a unique covering path $\tilde{f} : I \to X$ as shown in Fig. 4.9 of f beginning at x_0 such that $p \circ \tilde{f} = f$.*

Proof **Existence of $\tilde{f}$**: Suppose $[a, b] \subset I$ is such that $f([a, b]) \subset U$, where U is an admissible neighborhood of $y = f(a)$ in B. Let $x \in f^{-1}(y)$. Then x lies in a unique sheet S (say). Define

$$\tilde{g} : ([a, b], a) \to (X, x), \text{ by } \tilde{g} = (p|_S)^{-1} \circ (f|_{[a,b]})$$

such that $p \circ \tilde{g} = f|_{[a,b]}$. Let U_t be an admissible neighborhood of $f(t)$ for each $t \in I$. Then $\{f^{-1}(U_t), t \in I\}$, being an open cover of the compact metric space I has a Lebesgue number λ. This shows that if $0 < \delta < \lambda$ and Y is a subset of I of diameter less than δ, then $Y \subset f^{-1}(U_t)$ for some $t \in I$. Thus $f(Y) \subset U_t$ partitions I with points $t_1 = 0, t_2, \ldots, t_k = 1$, where $t_{i+1} - t_i < \delta$ for $1 \leq i \leq k - 1$. Then there is a continuous map $\tilde{g}_1 : [0, t_2] \to X$ satisfying $p \circ \tilde{g}_1 = f|_{[0,t_2]}$ and $\tilde{g}_1(o) = x_0$.

Fig. 4.9 Path lifting property (PLP)



Similarly, there is a continuous map $\tilde{g}_2 : [t_2, t_3] \to X$ satisfying $p \circ \tilde{g}_2 = f|_{[t_2,t_3]}$ and $\tilde{g}_2(t_2) = \tilde{g}_1(t_2)$. In this way, for $1 \le i \le k-2$, there is a continuous map

$$\tilde{g}_{i+1} : [t_{i+1}, t_{i+2}] \to X$$

satisfying $p \circ \tilde{g}_{i+1} = f|_{[t_{i+1},t_{i+2}]}$ and $\tilde{g}_{i+1}(t_{i+1}) = \tilde{g}_i(t_{i+1})$. Using gluing lemma, and assembling the functions g_i, we obtain a continuous function $\tilde{f} : I \to X$, where $\tilde{f}(t) = \tilde{g}_i(t)$ if $t \in [t_i, t_{i+1}]$.
The uniqueness of $\tilde{f}$: It follows from Corollary 4.3.6, because I is connected, and by assumption any two lifts of f agree at the point $0 \in I$. ❑

Corollary 4.3.10 (Homotopy Lifting Property) *Let (X, p) be a covering space of B and $F : I \times I \to B$ be a homotopy such that $F(0,0) = b_0$. If $x_0 \in p^{-1}(b_0)$, then there exists a unique homotopy $\tilde{F} : I \times I \to X$ such that $\tilde{F}(0,0) = x_0$.*

Proof Proceed as in Theorem 4.3.9 by subdividing $I \times I$ into rectangles (in place of I). ❑

We can prove in a similar way the general form of the Homotopy Lifting Property.

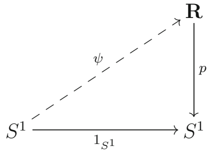
Theorem 4.3.11 (The Generalized Homotopy Lifting Property) *Let (X, p) be a covering space of B and A be a compact space. If $f : A \to X$ is continuous and $F : A \times I \to B$ is a homotopy starting from $p \circ f$, then there is a homotopy $\tilde{F} : A \times I \to X$ starting from f and lifts F. Furthermore, if F is a homotopy relative to a subset A' of A, then $\tilde{F}$ is also so.*

4.4 Lifting Problems of Arbitrary Continuous Maps

This section gives a necessary and sufficient condition for lifting of an arbitrary continuous map $f : A \to X$ by applying the tools of fundamental groups. More precisely, given a covering space (X, p) of B and a continuous map $f : A \to X$, can we find a continuous map $\tilde{f} : A \to X$ such that $p \circ \tilde{f} = f$? The answer is positive if f is a path or a homotopy between paths by the Path Lifting Property (Theorem 4.3.9), and the Homotopy Lifting Property (Corollary 4.3.10), respectively. To the contrary the answer is negative for an arbitrary continuous map f. For more results see Chap. 16.

Example 4.4.1 The exponential map $p : \mathbf{R} \to S^1$ defined by $p(t) = e^{2\pi i t}$ is a covering projection. The identity map $1_{S^1} : S^1 \to S^1$ cannot be lifted to a continuous map $\psi : S^1 \to \mathbf{R}$ making the triangle in Fig. 4.10 commutative. Otherwise, $p \circ \psi = 1_{S^1} \Rightarrow \psi$ is injective $\Rightarrow \psi$ is an embedding of S^1 into $\mathbf{R}$, since S^1 is compact $\Rightarrow \psi(S^1)$ is a closed interval homeomorphic to S^1, since any compact connected subset of $\mathbf{R}$ must be a closed interval. This is impossible, since a closed interval cannot be homeomorphic to S^1.

Fig. 4.10 Covering projection for exponential map p



Remark 4.4.2 We now give a necessary and sufficient condition under which an arbitrary continuous map $f : A \to X$ can be lifted. The methods of algebraic topology are now applied to solve such problems.

Theorem 4.4.3 (Lifting Theorem) *Let (X, p) be a covering space of B. Given a connected and locally path-connected space A, let $f : A \to B$ be any continuous map. Then given any three points $a_0 \in A$, $b_0 \in B$ and $x_0 \in X$ such that $f(a_0) = b_0$ and $p(x_0) = b_0$, there exists a unique continuous map $\tilde{f} : A \to X$ satisfying $\tilde{f}(a_0) = x_0$ such that $p \circ \tilde{f} = f$ if and only if $f_*(\pi_1(A, a_0)) \subset p_*(\pi_1(X, x_0))$.*

Proof Suppose that $\exists$ a continuous map $\tilde{f} : A \to X$ satisfying the given conditions. Then the diagram in Fig. 4.11 is commutative. Hence the diagram in Fig. 4.12 is also commutative (by the functorial property of π_1). Consequently, $f_*(\pi_1(A, a_0)) = p_*(\tilde{f}_*(\pi_1(A, a_0))) \subseteq p_*(\pi_1(X, x_0))$. Conversely, let the algebraic condition $f_*(\pi_1(A, a_0)) \subset p_*(\pi_1(X, x_0))$ holds.

Since A is connected, A has only one component. Again since A is locally path-connected, this component is a path component. Hence A is path-connected.

Let $a \in A$. We take a path $u : I \to A$ such that $u(0) = a_0$ and $u(1) = a$. Then $f \circ u : I \to B$ is a path such that $(f \circ u)(0) = f(u(0)) = f(a_0) = b_0$. By path Lifting Property, Theorem 4.3.9, $\exists$ a unique path $\tilde{u} : I \to X$ that lifts $f \circ u$ in X with $\tilde{u}(0) = x_0$ as shown in Fig. 4.13. Define a map

$$\tilde{f} : A \to X, a \mapsto \tilde{u}(1).$$

Fig. 4.11 Lifting of f to $\tilde{f}$

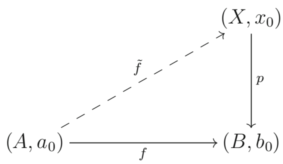
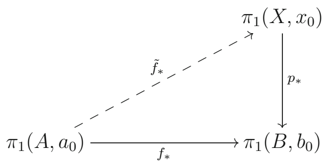


Fig. 4.12 Induced homomorphisms of f and $\tilde{f}$



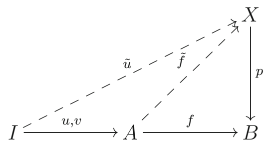


Fig. 4.13 Diagram for lifting theorem

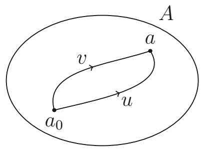


Fig. 4.14 Two paths u and v in A

To show that $\tilde{f}$ is well defined, choose another path v from a_0 to a as shown in Fig. 4.14. Let $\tilde{v}$ be the unique path in X lifting $f \circ v$ for which $\tilde{v}(0) = x_0$, i.e., $p \circ \tilde{v} = f \circ v$ and $\tilde{v}(0) = x_0$.

Now $u * v^{-1}$ is a closed path in A at a_0. Then $f \circ (u * v^{-1}) = (f \circ u) * (f \circ v^{-1})$ is a closed path in B at b_0. Again since

$$[(f \circ u) * (f \circ v^{-1})] = f_*[u * v^{-1}] \in f_*\pi_1(A, a_0) \subseteq p_*\pi_1(X, x_0) \text{(by hypothesis)},$$

there exists a closed path α in X at x_0 such that

$$(f \circ u) * (f \circ v^{-1}) \simeq p \circ \alpha \ \text{rel}\ \dot{I}.$$

Hence

$$(f \circ u) * (f \circ v^{-1}) * (p \circ \tilde{v}) \simeq (p \circ \alpha) * (p \circ \tilde{v}) \ \text{rel}\ \dot{I};$$

$$f \circ u \simeq p \circ (\alpha * \tilde{v}) \ \text{rel}\ \dot{I}, \ \text{since}\ p \circ \tilde{v} = f \circ v.$$

Again by homotopy lifting property (see Corollary 4.3.10)

$$\tilde{u} \simeq \alpha * \tilde{v} \ \text{rel}\ \dot{I} \ \text{with}\ \tilde{u}(1) = (\alpha * \tilde{v})(1) = \tilde{v}(1).$$

This shows that $\tilde{f}$ is well defined.

$\tilde{f}$ **is continuous**: Let $a \in A$ and U be an open neighborhood of $\tilde{f}(a)$. To show the continuity of $\tilde{f}$, we have to find an open neighborhood V_a of a with $\tilde{f}(V_a) \subset U$. We take an open admissible neighborhood V of $p\tilde{f}(a) = f(a)$ such that $V \subset p(U)$. Let W be the path component of $p^{-1}(V)$ which contains the point $\tilde{f}(a)$, and let V' be an open admissible neighborhood of $f(a)$ such that $V' \subseteq p(U \cap W)$. Then the path component of $p^{-1}(V')$ containing $\tilde{f}(a)$ must be contained in U. Since f is continuous and path-connected A is locally connected, $\exists$ a path-connected neighborhood V_a of a such that $f(V_a) \subset V$. Then $\tilde{f}(V_a) \subset U$. ❑

Corollary 4.4.4 *Let A be simply connected and locally path-connected and $f : (A, a_0) \to (B, b_0)$ be continuous. If (X, p) is a covering space of B and if $x_0 \in p^{-1}(b_0)$, then $\exists$ a unique lifting $\tilde{f} : (A, a_0) \to (X, x_0)$ of f.*

Proof A is simply connected $\Rightarrow \pi_1(A, a_0) = 0 \Rightarrow p_*\pi_1(A, a_0) = \{0\} \subset p_*\pi_1(X, x_0)$. Then $\exists$ a unique lifting $\tilde{f} : (A, a_0) \to (X, x_0)$ of f. ❑

Corollary 4.4.5 *Let B be a connected and locally path-connected space, and (X, p) and (Y, q) be covering spaces of B. Let $b_0 \in B$ and $x_0 \in X$, $y_0 \in Y$ be base points with $p(x_0) = b_0 = q(y_0)$. If $p_*\pi_1(X, x_0) = q_*\pi_1(Y, y_0)$, then there exists a unique continuous map $f : (Y, y_0) \to (X, x_0)$ such that $p \circ f = q$.*

Example 4.4.6 (S^n, p) is a covering space of $\mathbf{R}P^n$ of multiplicity 2. Since S^n is simply connected for $n \geq 2$, it follows that if $x_0 \in p^{-1}(b_0), b_0 \in \mathbf{R}P^n$, then for any continuous map $f : (S^n, s_0) \to (\mathbf{R}P^n, b_0)$, there exists a unique lifting $\tilde{f} : (S^n, s_0) \to (S^n, x_0)$.

4.5 Covering Homomorphisms: Their Classifications and Galois Correspondence

This section defines covering homomorphisms between covering spaces of the base space B and classify the covering spaces with the help of conjugacy classes of the fundamental group $\pi_1(B)$. This classification establishes an exact correspondence between the various connected covering spaces of a given space B and subgroups of its fundamental group $\pi_1(B)$, like Galois theory, with its correspondence between field extensions and subgroups of Galois groups. There is a natural question: given a space B, how many distinct covering spaces of B, we can find? Before answering this question, we explain what is meant by distinct covering spaces of B.

4.5.1 Covering Homomorphisms and Deck Transformations

This subsection introduces the concepts of covering homomorphisms and deck transformations.

Definition 4.5.1 Let (X, p) and (Y, q) be covering spaces of the same space B. A covering homomorphism h from (X, p) to (Y, q) is a continuous map $h : X \to Y$ such that the diagram in Fig. 4.15 is commutative. If in addition, h is a homeomorphism, then h is called an isomorphism. If there is an isomorphism from (X, p) to (Y, q), then they are called isomorphic or equivalent covering spaces, otherwise, they are said to be distinct covering spaces. An isomorphism of a covering space onto itself is called an automorphism or a deck transformation.

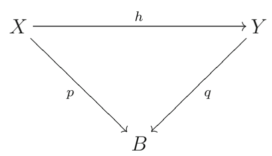


Fig. 4.15 Covering homomorphism

Remark 4.5.2 A homomorphism of covering spaces is a covering projection i.e.,if $h : X \to Y$ is a homomorphism of covering spaces, then (X, h) is a covering space of Y.

Proposition 4.5.3 *Covering spaces of a space B and their homomorphisms form a category.*

Proof We take covering spaces of B as the class of objects and their homomorphisms as the class of morphisms. Let (X, p) be a covering space of B. Then $1_X : X \to X$ is a covering homomorphism. If (X, p), (Y, q) and (Z, r) are covering spaces of B and $h : X \to Y, g : Y \to Z$ are covering homomorphisms, then $g \circ h : X \to Z$ is also a covering homomorphism from (X, p) to (Z, r). ❑

Isomorphisms in this category are just the isomorphisms of covering spaces as defined above.

Let $\mathcal{A}ut(X/B)$ be the set of all automorphisms of covering space (X, p) of B.

Proposition 4.5.4 $(\mathcal{A}ut(X/B), \circ)$ *is a group under usual composition of maps.*

Proof The identity map $1_X : X \to X$ is itself an automorphism and the inverse of an automorphism is again an automorphism. Consequently, $\mathcal{A}ut(X/B)$ is a group under usual composition of maps. ❑

Definition 4.5.5 $\mathcal{A}ut(X/B)$ is called the automorphism group of the covering space (X, p) of B. These automorphisms are also known as the covering transformations or deck transformations of the covering space (X, p) of B.

Let $p : X \to B, q : Y \to B$ be covering projections. Then (X, p) and (Y, q) are covering spaces of B. Suppose $g, h : X \to Y$ are two covering homomorphisms. We now consider each of g and h as liftings of the map $p : X \to B$ with respect to the covering projection $q : Y \to B$ (Fig. 4.16).

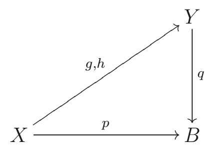


Fig. 4.16 Uniqueness of lifting

Consequently, if X is connected and g and h both agree at a single point of X, then $g = h$. This proves the following proposition.

Proposition 4.5.6 *Let $g, h : X \to Y$ be two covering homomorphisms from the covering space (X, p) to the covering space (Y, q) of B. If X is connected and $g(x_0) = h(x_0)$ for some $x_0 \in X$, then $g = h$.*

4.5.2 Classification of Covering Spaces by Using Group Theory

This subsection characterizes and classifies covering spaces of a space B with the help of conjugacy classes of the group $\pi_1(B)$. The following two results of algebra are used in this subsection.

(i) If H and K are subgroups of a group G, then they are conjugate subgroups iff $H = g^{-1}Kg$ for some $g \in G$.
(ii) If H and K are subgroups of a group G, then the G-sets G/H and G/K are G-isomorphic iff H and K are conjugate subgroups in G.

Theorem 4.5.7 *Let (X, p) be a covering space of B, where X and B are path-connected. If $b_0 \in B$, then the groups $p_*\pi_1(X, y)$, as y runs over $Y = p^{-1}(b_0)$, form a conjugacy class of subgroups of $\pi_1(B, b_0)$.*

Proof To prove the theorem we have to prove:

(a) for any $y_0, y_1 \in Y$, the subgroups $p_*\pi_1(X, y_0)$ and $p_*\pi_1(X, y_1)$ are conjugate;
(b) any subgroup of $\pi_1(B, b_0)$ conjugate to $p_*\pi_1(X, y_0)$ is equal to $p_*\pi_1(X, y)$ for some $y \in Y$.

(a) Let $u : I \to X$ be a path from y_0 to y_1. Then the function $\beta_u : \pi_1(X, y_0) \to \pi_1(X, y_1)$ defined by $\beta_u([f]) = [\bar{u} * f * u]$, $\forall [f] \in \pi_1(X, y_0)$, is an isomorphism (by Theorem 3.1.18). In particular, $\beta_u\pi_1(X, y_0) = \pi_1(X, y_1) \Rightarrow (p_* \circ \beta_u)\pi_1(X, y_0) = p_*\pi_1(X, y_1)$. It follows from the definition of β_u that $(p_* \circ \beta_u)\pi_1(X, y_0) = [p \circ u]^{-1} p_*\pi_1(X, y_0)[p \circ u] \Rightarrow p_*\pi_1(X, y_1)$ and $p_*\pi_1(X, y_0)$ are conjugate subgroups of $\pi_1(B, b_0)$.
(b) Let H be a subgroup of $\pi_1(B, b_0)$ such that H is conjugate to $p_*\pi_1(X, y_0)$ for some $[g] \in \pi_1(B, b_0)$. Then $H = [g]^{-1} p_*\pi_1(X, y_0)[g]$. Let $\tilde{g}$ be the unique lifting of g in X starting at y_0. Then $\tilde{g}(1) = y$(say)$\in Y$. Now proceeding as in (a), we have

$$\begin{aligned} p_*\pi_1(X, y) &= [p \circ \tilde{g}]^{-1} p_*\pi_1(X, y_0)[p \circ \tilde{g}] \\ &= [g]^{-1} p_*\pi_1(X, y_0)[g] = H \\ \Rightarrow p_*\pi_1(X, y) &= H. \end{aligned}$$

We conclude that the set $\{p_*\pi_1(X, y) : y \in Y\}$ forms a complete conjugate class of subgroups of the group $\pi_1(B, b_0)$.

□

Definition 4.5.8 The conjugacy class of subgroups $\{p_*\pi_1(X, y) : y \in Y = p^{-1}(b_0)\}$ described above is called the conjugate class determined by the covering space (X, p) of B.

We now characterize covering spaces of a base space B with the help of conjugacy classes of subgroups of $\pi_1(B)$.

Theorem 4.5.9 *Let B be path-connected and locally path-connected. Let (X, p) and (Y, q) be path-connected covering spaces of B; let $p(x_0) = q(y_0) = b_0$. Then the covering spaces (X, p) and (Y, q) are isomorphic if and only if $p_*\pi_1(X, x_0)$ and $q_*\pi_1(Y, y_0)$ are conjugate subgroups of $\pi_1(B, b_0)$ (i.e., iff they determine the same conjugacy class of subgroups of $\pi_1(B, b_0)$).*

Proof Suppose that the covering spaces (X, p) and (Y, q) are isomorphic. Then there exists a homeomorphism $h : Y \to X$ such that $p \circ h = q$ i.e., making the diagram in Fig. 4.17 commutative.

Let $h(y_0) = x_1$. Then h induces an isomorphism $h_* : \pi_1(Y, y_0) \to \pi_1(X, x_1) \Rightarrow h_*(\pi_1(Y, y_0)) = \pi_1(X, x_1) \Rightarrow (p_* \circ h_*)(\pi_1(Y, y_0)) = p_*(\pi_1(X, x_1))$. Hence $q_*(\pi_1(Y, y_0)) = p_*\pi_1(X, x_1)$. By Theorem 4.5.7, $p_*\pi_1(X, x_1)$ is a subgroup of $\pi_1(B, b_0)$ and conjugate to the subgroup $p_*\pi_1(X.x_0)$. Consequently, $p_*\pi_1(X, x_0)$ and $q_*\pi_1(Y, y_0)$ are conjugate subgroups of $\pi_1(B, b_0)$. For the converse, let the two subgroups of $\pi_1(B, b_0)$ be conjugate. By Theorem 4.5.7 we can choose a different base point y_0 in Y such that the two groups are equal. We now consider the diagram in Fig. 4.18 where q is a covering map. The space X is path-connected; it is also locally path-connected, being locally homeomorphic to B. Moreover, $p_*\pi_1(X, x_0) \subseteq q_*\pi_1(Y, y_0)$. In fact, these two groups are equal. By Theorem 4.5.7, we can lift the map p to $\tilde{p} : X \to Y$ such that $\tilde{p}(x_0) = y_0$. Then $q \circ \tilde{p} = p$.

Reversing the role of X and Y in this discussion, we see that $q : Y \to B$ can also be lifting to $\tilde{q} : Y \to X$ such that $\tilde{q}(y_0) = x_0$ as shown in Fig. 4.19.

We claim that $\tilde{p}$ and $\tilde{q}$ as shown in Fig. 4.20 are inverses of each other. Consider the diagram in Fig. 4.21.

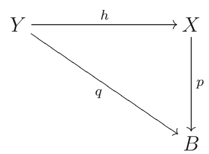


Fig. 4.17 Isomorphisms of covering spaces

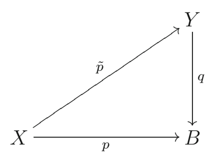


Fig. 4.18 Lifting of p

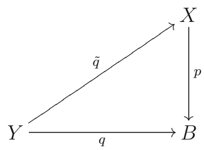


Fig. 4.19 Lifting of q

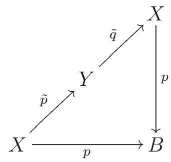


Fig. 4.20 Liftings of p and q

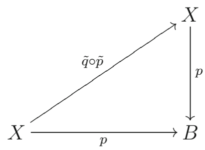


Fig. 4.21 Lifting of p involving $\tilde{p}$ and $\tilde{q}$

Now $\tilde{q} \circ \tilde{p} : X \to X$ is a lifting of the map $p : X \to B$ satisfying the condition $(\tilde{q} \circ \tilde{p})(x_0) = x_0$. The identity map $1_X : X \to X$ is another such lifting of p. Hence by uniqueness of lifting it follows that $\tilde{q} \circ \tilde{p} = 1_X$. Similarly, $\tilde{p} \circ \tilde{q} = 1_Y$. Consequently, $\tilde{p} : X \to Y$ is a homeomorphism and hence the covering spaces (X, p) and (Y, q) are isomorphic. ❑

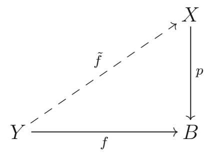
Remark 4.5.10 For any covering space (X, p) of B, the subgroups $\{p_*(\pi_1(X, x)) : x \in p^{-1}(b)\}$ form a conjugacy class of subgroups of $\pi_1(B, b)$. The above Theorem 4.5.9 shows that a conjugacy class of a subgroup of $\pi_1(B, b)$ determines completely the covering spaces upto isomorphisms.

Recall that

Definition 4.5.11 A topological space X is said to be simply connected if it is path-connected and $\pi_1(X, x_0) = 0$ for some $x_0 \in X$ (hence for every $x_0 \in X$).

Example 4.5.12 Consider the covering spaces of S^1. $\pi_1(S^1, 1)$ is abelian $\Rightarrow$ two subgroups of $\pi_1(B, b_0)$ are conjugate if and only if they are equal. Consequently, two covering spaces of S^1 are isomorphic if and only if they correspond to the same subgroup of $\pi_1(S^1) \cong \mathbf{Z}$. The subgroups of $\mathbf{Z}$ are given by $< n >$, consisting of all multiples of n, for $n = 0, 1, 2, \ldots$. The covering space $(\mathbf{R}, p)$ of S^1 corresponds to the trivial subgroup of $\mathbf{Z}$, because $\mathbf{R}$ is simply connected. On the other hand, the covering space (S^1, p) of S^1 defined by $p(z) = z^n$ corresponds to the subgroup $< n >$ of $\mathbf{Z}$. We conclude that every path-connected covering space of S^1 is isomorphic to one of these coverings i.e., any covering space of S^1 must be isomorphic either to $(\mathbf{R}, p)$ or to one of the coverings (S^1, q_n), where $q_n(z) = z^n$, $z \in S^1$ wraps S^1 around itself n times.

Fig. 4.22 Lifting of f to X



Example 4.5.13 Consider the double covering (S^1, p) over $\mathbf{R}P^2$. Since S^2 is simply connected, $\pi_1(S^2, s) = 0$ and hence the conjugacy class contains only the trivial subgroup.

Example 4.5.14 The plane $\mathbf{R}^2$ is simply connected. Consequently, the conjugacy class of $(\mathbf{R}^2, r)$ over the torus also contains only the trivial subgroup.

Example 4.5.15 Let X denote an infinite spiral and let $q : X \to S^1$ denote the projection map projecting each point on X to the point on the circle directly beneath it. Then (X, q) is a covering space of S^1. Since X is contractible, it has trivial fundamental group. Consequently, (X, q) determines the conjugacy class of $\pi_1(S^1)$ consisting of only the trivial subgroup. $(\mathbf{R}, p)$ also determines the conjugacy class of $\pi_1(S^1)$ consisting of only the trivial subgroup. We conclude that (X, q) and $(\mathbf{R}, p)$ are isomorphic covering spaces of S^1 by Theorem 4.5.9.

We recall the following proposition (see Sect. 4.1.2).

Proposition 4.5.16 *Let $p : (X, x_0) \to (B, b_0)$ be a covering space. Then the induced homomorphism $p_* : \pi_1(X, x_0) \to \pi_1(B, b_0)$ is a monomorphism and the subgroup $p_*(\pi_1(X, x_0))$ in $\pi_1(B, b_0)$ consists of homotopy classes of loops in B based at b_0 which lift to X starting at x_0 are loops.*

Remark 4.5.17 If $p : X \to B$ is a covering map, then p is also onto. But its induced homomorphism

$$p_* : \pi_1(X, x_0) \to \pi_1(B, b_0)$$

need not be an epimorphism. However, p_* is a monomorphism.

Proposition 4.5.18 *Let $p : (X, x_0) \to (B, b_0)$ be a covering space and $f : (Y, y_0) \to (B, b_0)$ be a map, where Y is path-connected and locally path-connected. Then a lift $\tilde{f} : (Y, y_0) \to (X, x_0)$ of f (as shown in Fig. 4.22) exists iff $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(X, x_0))$.*

Proof Since $f_*([\alpha]) = (p * \circ \tilde{f}_*)[\alpha] \in p_*(\pi_1(X, x_0))$, $\forall\, [\alpha] \in \pi_1(Y, y_0)$, it follows that $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(X, x_0))$.

Conversely, let $y \in Y$ and β be a path in Y from y_0 to y. Then the path $f \circ \beta$ in B starting at b_0 has a unique lifting $(\widetilde{f \circ \beta})$ starting at x_0. Define

$$\tilde{f} : (Y, y_0) \to (X, x_0),\ y \mapsto (\widetilde{f \circ \beta})(1).$$

Clearly, $\tilde{f}$ is well defined and continuous. ❑

Proposition 4.5.19 *Given a covering space $p : X \to B$ and a map $f : Y \to X$ with two liftings $\tilde{f}_1, \tilde{f}_2 : Y \to X$ that agree at some point of Y, if Y is connected, then $\tilde{f}_1 = \tilde{f}_2$, i.e., $\tilde{f}_1(y) = \tilde{f}_2(y)$, $\forall\, y \in Y$.*

Proof Let $y \in Y$ and U be an open neighborhood of $f(y)$ in B such that $p^{-1}(U)$ is a disjoint union of open sets $\widetilde{U}_i$ each of which is mapped homeomorphically onto U by p. Suppose $\widetilde{U}_1$ and $\widetilde{U}_2$ are the $\widetilde{U}_i$'s containing $\tilde{f}_1(y)$ and $\tilde{f}_2(y)$, respectively. By continuity of $\tilde{f}_1$ and $\tilde{f}_2$ there is neighborhood N_y of y mapped into $\widetilde{U}_1$ by $\tilde{f}_1$ and $\widetilde{U}_2$ by $\tilde{f}_2$. If $\tilde{f}_1(y) \neq \tilde{f}_2(y)$, then $\widetilde{U}_1 \neq \widetilde{U}_2$. Hence $\widetilde{U}_1$ and $\widetilde{U}_2$ are disjoint open sets and $\tilde{f}_1 \neq \tilde{f}_2$ throughout the neighborhood N_y. Again if $\tilde{f}_1(y) = \tilde{f}_2(y)$, then $\widetilde{U}_1 = \widetilde{U}_2$ and hence $\tilde{f}_1 = \tilde{f}_2$ on N_y, because $p \circ \tilde{f}_1 = p \circ \tilde{f}_2$ and p is injective on $\widetilde{U}_1 = \widetilde{U}_2$. This shows that the set of points where $\tilde{f}_1$ and $\tilde{f}_2$ agree is a both open and closed set in Y. ❑

4.5.3 Classification of Covering Spaces and Galois Correspondence

This subsection considers the problem of classifying all different covering spaces of a fixed base space B. The main thrust of this classification is given in the Galois correspondence between connected covering spaces of B and subgroups of $\pi_1(B)$. The Galois correspondence ψ arises from the function that assigns to each covering space $p : (X, x_0) \to (B, b_0)$ the subgroup $p_*(\pi_1(X, x_0))$ of $\pi_1(B, b_0)$. By Proposition 4.5.16, this correspondence ψ is injective. To show that ψ is surjective, we have to show that corresponding to each subgroup G of $\pi_1(B, b_0)$, there is a covering space $p : (X, x_0) \to (B, b_0)$ such that $p_*\pi_1(X, x_0) = G$.

Definition 4.5.20 A topological space X is said to be semilocally simply connected if each point $x \in X$ has a neighborhood U_x such that the map induced by inclusion $i : U_x \hookrightarrow X$ is trivial, i.e., $i_* : \pi_1(U_x, x) \to \pi_1(X, x)$ is trivial (equivalently, every closed path in U_x at x is nullhomotopic in X).

Definition 4.5.21 A topological space X is said to be semilocally path-connected if for every point $x \in X$, there is an open neighborhood U_x of x such that every closed path in U_x at x is nullhomotopic in X.

Theorem 4.5.22 *Let B be a path-connected, locally path-connected and semilocally path space. Then for each subgroup G of $\pi_1(B, b_0)$ there is a covering space $p : X_G \to B$ such that $p_*(\pi_1(X_G, x_0)) = G$ for some suitable chosen base point $x_0 \in X_G$.*

Proof Let $b \in B$. Since B is semilocally path-connected, there is an open neighborhood W_b of b such that every closed path in W_b at b is nullhomotopic in B. Again since X is locally path-connected, $\exists$ an open connected neighborhood U_b of b such that $b \in U_b \subset W_b$. Clearly, every closed path in U_b at b is null homotopic in B and U_b is evenly covered by p.

Construction of X_G: Let $P(B, b_0)$ be the family of all paths f in B with $f(0) = b_0$, topologized by the compact open topology. Define a binary relation $f_1 \sim f_2$ mod G iff $f_1(1) = f_2(1)$ and $[f_1 * f_2^{-1}] \in G$. Then '$\sim$' is an equivalence relation. The equivalence class of $f \in P(B, b_0)$ is denoted by $[f]$. Let X_G denote the set of all such equivalence classes, topologized by the quotient topology. If c_0 is the constant path at b_0, define $x_0 = \langle c_0 \rangle_G \in X_G$ and $p : X_G \to B, [f]_G \mapsto f(1)$. Then $p(x_0) = b_0$. Since any two paths in the basic neighborhoods $U_{[f_1]_G}$ and $U_{[f_2]_G}$ are identified in X_G, the whole neighborhoods are identified. Consequently, the natural projection $p : X_G \to B$ is a covering space with $p(x_0) = b_0$. Then the image of $p_* : \pi_1(X_G, x_0) \to \pi_1(B, b_0)$ is precisely G. Because, for any loop β in B based at b_0, its lifting to X_G starting at $x_0 = \langle c_0 \rangle_G$ ends at $[\beta]_G$ and hence the image of this lifted path in X_G is a loop iff $[\beta]_G \sim [c_0]_G$ (equivalently, $[\beta] \in G$). ❑

Remark 4.5.23 Every group G can be realized as the fundamental group of the topological space X_G.

Corollary 4.5.24 *Let B be a connected, locally path-connected, semilocally simply connected space. Then every covering space $q : Y \to B$ is isomorphic (equivalent) to a covering spaces of the form $p : X_G \to B$.*

Proof Let $b_0 \in B$ be a base point of B and $y_0 \in Y$ lie in the fiber over b_0. If $G = q_*\pi_1(Y, y_0)$, then $p_*\pi_1(X_G, x_0) = G$. Hence Theorem 4.5.9 shows that the covering spaces $p : X_G \to B$ and $q : Y \to B$ are isomorphic. ❑

Corollary 4.5.25 *Let B be a connected, locally path-connected, semilocally simply connected space. If $p : X \to B$ is a covering space of B, then every open contractible set V in B is evenly covered by p.*

Proof Since if V is an open path-connected set in B for which every closed path in V is nullhomotopic in B, then V is evenly covered by p. In particular, if $b \in V$, then $p^{-1}(V) = \bigcup_{x \in p^{-1}(b)} (V, x)$ and contractible open sets are evenly covered in every covering space of the form $p : X_G \to B$. Then the corollary follows from Corollary 4.5.24. ❑

Corollary 4.5.26 *Let B be a connected and locally path-connected space. Then B has a universal covering space X (i.e., X is simply connected) iff X is semilocally simply connected.*

Proof Theorem 4.5.22 proves sufficiency of the condition. Definition 4.5.20 gives the necessity of the condition. ❑

Theorem 4.5.27 (Classification theorem) *Let B be a path-connected and locally path-connected space. Then the two coverings $p : X \to B$ and $q : Y \to B$ are isomorphic via a homeomorphism $f : X \to Y$ taking a base point $x_0 \in p^{-1}(b_0)$ to a base point $y_0 \in q^{-1}(b_0)$ iff $p_*(\pi_1(X, x_0)) = q_*(\pi_1(Y, y_0))$.*

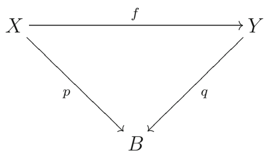


Fig. 4.23 Diagram for two isomorphic coverings of B

Proof Suppose there is a homeomorphism $f : (X, x_0) \to (Y, y_0)$ as shown in Fig. 4.23. Then the two relations $p = q \circ f$ and $q = p \circ f^{-1}$ show that $p_*(\pi_1(X, x_0)) = q_*(\pi_1(Y, y_0))$.

For the converse, let $p_*(\pi_1(X, x_0)) = q_*(\pi_1(Y, y_0))$. Then by the lifting criterion, we may lift p to $\tilde{p} : (X, x_0) \to (Y, y_0)$ with $q \circ \tilde{p} = p$. Similarly, we obtain $\tilde{q} : (Y, y_0) \to (X, x_0)$ with $p \circ \tilde{q} = q$. Then by unique lifting property, it follows that $\tilde{p} \circ \tilde{q} = 1_d$ and $\tilde{q} \circ \tilde{p} = 1_d$, since these composed lifts fix the base points. Consequently, p_* and q_* are inverse isomorphisms. ❑

Remark 4.5.28 We now present a generalization of the above classification theorem in the following form.

Theorem 4.5.29 (Classification theorem in general form) *Let B be a path-connected, locally path-connected and semilocally simply connected space. Then there exists a bijection between the set of base point preserving isomorphism classes of path-connected covering spaces $p : (X, x_0) \to (B, b_0)$ and the set of subgroups of $\pi_1(B, b_0)$, obtained by assigning the subgroups $p_*(\pi_1(X, x_0))$ to the covering spaces (X, x_0). If the base points are ignored, this correspondence gives a bijection between isomorphism classes of path-connected covering spaces $p : X \to B$ and conjugacy classes of subgroups of $\pi_1(B, b_0)$.*

Proof The first part follows from Theorem 4.5.27. For the proof of the second part, we claim that covering space $p : X \to B$, changing the base point x_0 within $\pi^{-1}(b_0)$ corresponds exactly to changing $p_*(\pi_1(X, x_0))$ to a conjugate subgroup of $\pi_1(B, b_0)$. Suppose x_1 is another base point $p^{-1}(b_0)$. Let $\tilde{\alpha}$ is a path from x_0 to x_1. Then $\tilde{\alpha}$ projects to a loop α in B, which represents some element $g \in \pi_1(B, b_0)$. Define G_i by $G_i = p_*(\pi_1(X, x_i))$ for $i = 0, 1$. Then we have an inclusion $g^{-1}G_0g \subset G_1$, since for $\tilde{f}$ a loop at x_0, $\widetilde{\gamma}^{-1} * f * \widetilde{\gamma}^{-1}$ is a loop at x_1. Similarly, $gG_1g^{-1} \subset G_0$. Using conjugation the latter relation by g^{-1} we have $G_1 \subset g^{-1}G_0g$ and hence $g^{-1}G_0g = G_1$. Consequently, changing the base point from x_0 to x_1 changes G_0 to the conjugate subgroup $G_1 = g^{-1}G_0g$. Conversely, to change G_0 to a conjugate subgroup $G_1 = g^{-1}G_0g$, choose a loop β represents g, that lifts to a path $\tilde{\beta}$ starting at x_0 and let $x_1 = \tilde{\beta}(1)$. The earlier argument proves that $G_1 = g^{-1}G_0g$. ❑

Theorem 4.5.30 (Galois correspondence) *Let B be path-connected and locally path-connected space. The Galois correspondence ψ arising from the function that assigns to each covering space $p : (X, x_0) \to (B, b_0)$ the subgroup $p_*(\pi_1(X, x_0))$ of $\pi_1(B, b_0)$ is a bijection.*

Proof ψ **is injective**: it follows from Proposition 4.5.16.

ψ **is surjective**: it follows from classification Theorem 4.5.27, since to each subgroup G of $\pi_1(B, b_0)$, there is a covering space $p : (X, x_0) \to (B, b_0)$ such that $p_*\pi_1(X, x_0) = G$.

Hence this correspondence ψ is a bijection. ❑

Definition 4.5.31 ψ defined in Theorem 4.5.30 is called a Galois correspondence.

4.6 Universal Covering Spaces and Computing $\pi_1(\mathbf{R}P^n)$

This section introduces the concept of a special class of covering spaces, called universal covering spaces and studies them with the help of fundamental groups of their base spaces and computes $\pi_1(\mathbf{R}P^n)$.

4.6.1 Universal Covering Spaces

This subsection opens with the concept of universal covering spaces. For a topological space B, $(B, 1_B)$ is a covering space over B. This covering space does not create in general much interest because it corresponds to the conjugacy class of the entire fundamental group $\pi_1(B, b)$. On the other hand, the covering space corresponding to the conjugacy class of the trivial subgroup $\{0\}$ of $\pi_1(B, b)$ is interesting. This covering space, if it exists for some B, is called the 'universal covering space'.

We now examine the relation between a base space B and its universal covering space.

Definition 4.6.1 Let B be a topological space. A covering space (X, p) of B for which X is simply connected (i.e., X is path-connected and $\pi_1(X, x_0) = 0$ for every $x_0 \in X$) is called the universal covering space of B.

Remark 4.6.2 We now explain the name of the term "universal covering space".

Theorem 4.6.3 **(i)** *Any two universal covering spaces of the same base space B are isomorphic.*

(ii) *If (X, p) is the universal covering space of B and (Y, q) is a covering space of B, then there is a continuous map*

$$\tilde{p} : X \to Y$$

such that $(X, \tilde{p})$ is a covering space of Y.

Proof **(i)** Any universal covering space of B determines the conjugacy class of the trivial subgroup $\Rightarrow$ any two universal covering spaces of B are isomorphic by Theorem 4.5.9.

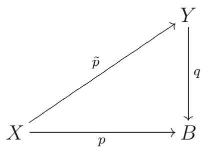


Fig. 4.24 Lifting of p to $\tilde{p}$

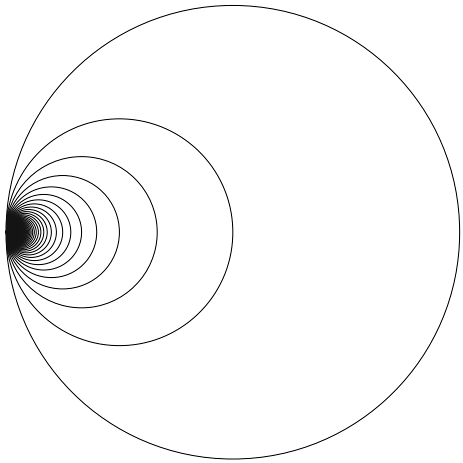


Fig. 4.25 Infinite earring

(ii) We consider the commutative diagram in Fig. 4.24 and choose base points $x_0 \in X$, $b_0 \in B$ and $y_0 \in Y$ such that $p(x_0) = q(y_0) = b_0$. Since $\pi_1(X, x_0) = 0$, $p_*\pi_1(X, x_0) \subset q_*\pi_1(Y, y_0)$. Hence Lifting Theorem 4.4.3 shows the existence of a continuous map $\tilde{p} : (X, x_0) \to (Y, y_0)$ such that $q \circ \tilde{p} = p$ and therefore $\tilde{p}$ is a covering projection. In other words, $(X, \tilde{p})$ is a covering space of Y.

❑

Example 4.6.4 **(i)** $(\mathbf{R}, p)$ is the universal covering space of S^1, where $p(t) = e^{2\pi it}$, since the space of real numbers $\mathbf{R}$ is simply connected.

(ii) $(\mathbf{R}^2, r)$ (in Example 4.5.14) is a universal covering space over the torus, since $\mathbf{R}^2$ is simply connected.

(iii) (S^2, p) is the universal covering space of $\mathbf{R}P^2$.

(iv) (S^n, p_n) is a universal covering space of $\mathbf{R}P^n$, where $p_n : S^n \to S^n$ is the map identifying antipodal points of S^n for $n > 1$ (see Theorem 4.1.19).

Remark 4.6.5 A space may not have a universal covering. We now present an example of a space which has no universal covering.

Example 4.6.6 (Infinite earring or shrinking wedge of circles) Let C_n be the circle of radius $1/n$ in $\mathbf{R}^2$ with center at $(1/n, 0)$, for each $n \geq 1$. Let X be the subspace of $\mathbf{R}^2$ that is the union of these circles as shown in Fig. 4.25.

Then X is the union of a countably infinite collection of circles. The space X is called the 'infinite earring' or 'shrinking wedge of circles' in the plane $\mathbf{R}^2$. Let b_0 the origin. We claim that if U is a neighborhood of b_0 in X, then the homomorphism of

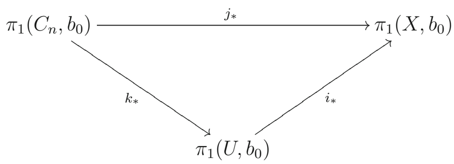


Fig. 4.26 Homomorphisms induced by inclusion maps

fundamental groups induced by the inclusion $i : U \hookrightarrow X$ is not trivial. To show this, let n be a given integer, there is a retraction $r : X \to C_n$ defined by letting r maps each circle C_i for $i \neq n$ to the point b_0. We can choose n sufficiently large such that inclusion $j : U \hookrightarrow X$ and inclusion $k : U \hookrightarrow U$ and thus for sufficiently large n, C_n lies in U. Then in the commutative diagram of groups and homomorphisms induced by inclusions k_* and j_* as shown in Fig. 4.26, j_* is injective.

Hence j_* can not be trivial. This asserts that X has no universal covering.

4.6.2 Computing $\pi_1(\mathbf{R}P^n)$

We now present an interesting result of the universal covering space and utilize this result to compute $\pi_1(\mathbf{R}P^n)$. For an alternative method see Corollary 4.10.4.

Theorem 4.6.7 *Let* (X, p) *be the universal covering space of* B *and* $\mathcal{A}ut(X/B)$ *be the group of all automorphisms of (X,B). Then the automorphism group* $\mathcal{A}ut(X/B)$ *is isomorphic to the fundamental group* $\pi_1(B)$ *of* B*. Moreover, if* $|\pi_1(B)|$ *is the order of the group* $\pi_1(B)$*, then* $|\pi_1(B)|$*=number of sheets of the universal covering space.*

Proof To prove the first part, let $x_0 \in X$ and $p(x_0) = b_0$. We define a map $\psi : \mathcal{A}ut(X/B) \to \pi_1(B, b_0)$ as follows:
$f \in \mathcal{A}ut(X/B) \Rightarrow f$ permutes the points of the fiber $p^{-1}(b_0)$. The point $f(x_0) \in p^{-1}(b_0)$, since $(p \circ f)(x_0) = b_0$. Let u be the path in X joining x_0 and $f(x_0)$. Then $p \circ u$ is a loop in B based at b_0. We define a mapping $\psi : \mathcal{A}ut(X/B) \to \pi_1(B)$ given by $\psi(f) = [p \circ u]$.

ψ **is well defined**: Let v be any other path joining x_0 and $f(x_0)$. Since X is simply connected, u is equivalent to v and hence $[p \circ u] = [p \circ u] \Rightarrow \psi$ is well defined.

ψ **is a homomorphism**: Let $f, g \in \mathcal{A}ut(X/B)$ and u, v be two paths in X joining x_0 to $f(x_0)$ and to $g(x_0)$, respectively. Then $\psi(f) = [p \circ u]$ and $\psi(g) = [p \circ v]$. Clearly, $f \circ v$ is a path joining $f(x_0)$ to $f(g(x_0))$ and hence $u * (f \circ v)$ is a path in X joining x_0 to $f(g(x_0))$. Again $\psi(fg) = [p \circ (u * (f \circ v))] = [(p \circ u) * (p \circ f \circ v)] = [p \circ u][p \circ f \circ v]$. Since $p \circ f = p$, we have $\psi(fg) = [p \circ u * p \circ v] = [p \circ u][p \circ v] = \psi(f)\psi(g)$.

ψ **is a monomorphism**: Let $\psi(f) = \psi(g)$. Then $[p \circ u] = [p \circ v]$, where u, v are paths in X starting at x_0 and ending at $f(x_0)$ and $g(x_0)$, respectively. Consequently,

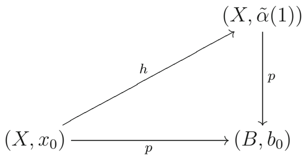


Fig. 4.27 Commutativity of the triangle for lifting of p to h

$p_*[u] = p_*[v] \Rightarrow u$ and v must have the same terminal point by Monodromy Theorem 4.9.3 i.e., $f(x_0) = g(x_0)$ and hence $f = g$ by Proposition 4.5.6, since X is connected.

ψ **is an epimorphism**: Let $\alpha \in \pi_1(B, b_0)$ and $\tilde{\alpha}$ be the unique lifting of the path α in X such that $\tilde{\alpha}(0) = x_0 \in X$. Consider the commutative diagram in Fig. 4.27 obtained by applying Lifting Theorem 4.4.3 to define a continuous lifting h of p such that $h(x_0) = \tilde{\alpha}(1)$.

Since X is a simply connected covering space of B, there exists a homeomorphism $h : X \to X$ such that $h(x_0) = \tilde{\alpha}(1)$. By the same argument, there is also a homeomorphism $k : X \to X$ such that $k(\tilde{\alpha}(1)) = x_0$. Since the homeomorphism $k \circ h : X \to X$ maps x_0 to itself and hence by Proposition 4.5.6 it follows that $h \circ k = 1_X$. This implies that $h \in \mathcal{A}ut(X/B)$ and by definition, $\psi(h) = [p \circ \tilde{\alpha}] = [\alpha]$. This shows that ψ is an isomorphism.

Proof of the last part: Since ψ is one-to-one, it establishes a one-to-one correspondence between $p^{-1}(b_0)$ and a subset of $\pi_1(B, b_0)$. While proving ψ is onto, we showed that every homotopy class $[\alpha]$ in $\pi_1(B, b_0)$ corresponds to a point $\tilde{\alpha}(1)$ in $p^{-1}(b_0)$. Hence it follows that $|p^{-1}(b_0)|$ = number of sheets of (X, p), is the order of $\pi_1(B, b_0)$. ❑

Remark 4.6.8 Last part of the Theorem 4.6.7 also follows from Theorem 4.10.1(iii), since $\pi_1(X) = 0$.

Theorem 4.6.9 $\pi_1(\mathbf{R}P^n) \simeq \mathbf{Z}_2$ *for* $n \geq 2$.

Proof Consider the universal covering space (S^n, q) of $\mathbf{R}P^n$ where q identifies the antipodal points of S^n. Then $|\pi_1(\mathbf{R}P^n)| = 2 \Rightarrow \pi_1(\mathbf{R}P^n) \cong \mathbf{Z}_2$. ❑

Theorem 4.6.10 *The automorphism group* $G = \mathcal{A}ut(X/B)$ *of a universal covering space* (X, p) *of* B *acts on* X *freely.*

Proof It is sufficient to prove that if $g \in G$ and $g(x) = x$ for some $x \in X$, then $g = 1_X$. The group *homeo*(X) of all homeomorphisms of a space X acts on the set X by the action defined by $g \cdot x = g(x)$, where $g \in$ Homeo (X) and $x \in X$. Since the group $\mathcal{A}(X/B)$ is a subgroup of Homeo (X), $\mathcal{A}ut(X/B)$ also acts on the space X by the above action. Thus $g, 1_X : X \to X$ are two covering homomorphisms of corresponding covering projections such that $g(x) = x = 1_X(x)$ for some $x \in X$. This shows that $g = 1_X$ by Proposition 4.5.6, since every path-connected space is connected. ❑

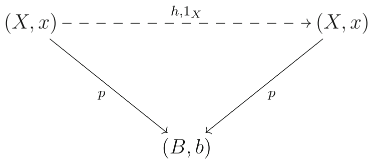


Fig. 4.28 Commutativity of the triangle for the covering space (X, p)

Corollary 4.6.11 *Let (X, p) be covering space of B.*

(i) *If $h \in \text{Cov}\,(X/B) = \mathcal{A}ut(X/B)$ and $h \neq 1_X$, then h has no fixed point.*
(ii) *If $h, g \in \mathcal{A}ut(X/B)$ and $\exists\, x \in X$ with $h(x) = g(x)$, then $h = g$.*

Proof **(i)** Let $\exists\ x \in X$ with $h(x) = x$; let $b = p(x)$. Consider the commutative diagram in Fig. 4.28.
Since both h and 1_X complete the diagram in Fig. 4.28, it follows that $h = 1_X$, a contradiction.
(ii) The map $h^{-1}g \in \mathcal{A}ut(X/B)$ has a fixed point, namely x and so by (i) $h^{-1}g = 1_X \Rightarrow h = g$. ❑

4.7 Fibrations and Cofibrations

This section gives a systematic approach to the lifting and extension problems through representation of maps as fibrations or cofibrations which are dual concepts of each other in some sense and form two important classes of maps in algebraic topology. They are central concepts in homotopy theory. Every continuous map is equivalently expressed up to homotopy as a fibration and also as a cofibration. The concept of fibration first appeared in 1937 implicitly in the work of K. Borsuk (1905–1982). This concept born in geometry and topology provides important strong mathematical tools to invade many other branches of mathematics. More precisely, this section introduces the concepts of fibrations and cofibrations and establishes a connection between a fibration and a covering projection.

The concept of homotopy lifting property (HLP) is very important in algebraic topology, specially in homotopy theory. It is the dual concept of the homotopy extension property (HEP). The concept of HLP leads to the concept of fibration. There is a dual theory to fibration leading to the concept of cofibration. This is a very nice duality principle in homotopy theory.

4.7.1 Homotopy Lifting Problems

This subsection discusses homotopy lifting problems of a map. It is an important problem of algebraic topology and dual to the extension problem. Let $p : X \to B$

Fig. 4.29 Lifting of f

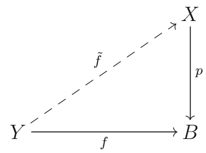


Fig. 4.30 Homotopy Lifting of H

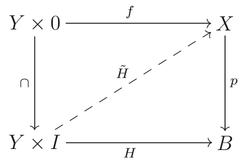
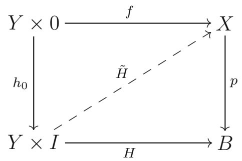


Fig. 4.31 Homotopy Lifting Problem



be a map and Y be a space. If $f : Y \to B$ is a map, then the lifting problem for f is to determine whether there is a continuous map $\tilde{f} : Y \to X$ such that the diagram in Fig. 4.29 is commutative, i.e., $f = p \circ \tilde{f}$. If there exists such a map $\tilde{f} : Y \to X$, we say that f can be lifted to X, and $\tilde{f}$ is called a lifting or a lifting of f. To show that the lifting problem is a problem in the homotopy category, we need the concept of homotopy lifting property(HLP) which is similar to the concept of HEP.

Definition 4.7.1 A continuous map $p : X \to B$ is said to have the HLP with respect to a space Y, if given maps $f : Y \to X$ and $H : Y \times I \to B$ such that $H(y, 0) = pf(y)$ for all $y \in Y$, there is a continuous map $\tilde{H} : Y \times I \to X$ such that $\tilde{H}(y, 0) = f(y)$ for all $y \in Y$ and $H = p \circ \tilde{H}$. If f is regarded as a map of $Y \times 0$ to X, the existence of $\tilde{H}$ is equivalent to the existence of a map represented by the dotted arrow that makes the diagram in Fig. 4.30 commutative.

Let $p : X \to B$ be a map and Y be a space. A homotopy lifting problem is sometimes symbolized by the commutative diagram in Fig. 4.31 where $h_0(y) = (y, 0)$ for all $y \in Y$ and the maps $f : Y \to X$, $H : Y \times I \to B$ are said to constitute the data for the problem in question. The map H is a homotopy of $p \circ f$ and a solution to the problem is a homotopy $\tilde{H} : Y \times I \to X$ of f such that $p \circ \tilde{H} = H$. Thus $\tilde{H}$ lifts the homotopy of H of $p \circ f$ to a homotopy of f.

Proposition 4.7.2 *Let $p : X \to B$ has the HLP with respect to a space Y. If $f \simeq g : Y \to B$, then f can be lifted to X iff g can be lifted to X.*

Proof Similar to the proof of Corollary 3.3.4 of Chap. 3. ❑

Remark 4.7.3 Let $p : X \to B$ and $f : Y \to B$ be two continuous maps. Then f can or cannot be lifted to X is a property of the homotopy class. This implies that the lifting problem for maps $f : Y \to B$ to X is a problem of homotopy category.

4.7.2 Fibration: Introductory Concepts

This subsection introduces the concept of a fibration first implicitly appeared in the work of K. Borsuk in 1937 but explicitly in the work of Whiteney during 1935–1940, first on sphere bundles. Fibrations form an important class of maps in algebraic topology. Covering map is a fibration. The homotopy lifting property leads to the concept of fibration (or Hurewicz fiber space) (Hurewicz 1955). More precisely, a continuous map $p : X \to B$ has the HLP with respect to a space Y if and only if every problem symbolized by the commutative diagram in Fig. 4.31 has a solution.

Definition 4.7.4 A pointed continuous map $p : X \to B$ is called a fibration (or fiber map or Hurewicz fiber space) if p has the HLP with respect to every space. X is called the total space and B is called the base space of the fibration. For $b \in B$, $p^{-1}(b) = F$ is called the fiber over b. A Serre fibration is map $X \to B$ satisfying HLP with respect to disk D^n, $\forall n$. It is sometimes called a weak fibration.

We use the notation "$F \hookrightarrow X \xrightarrow{p} B$ is a fibration" to mean that $p : X \to B$ is a fibration, F is the fiber space over some specific point of B, and $i : F \hookrightarrow X$ is the inclusion map.

Example 4.7.5 The projection

$$p : B \times F \to B, (b, f) \mapsto b$$

is a fibration.

Definition 4.7.6 A fibration $p : X \to B$ is called principal fibration if there is also a space C and a map $g : B \to C$ and a homotopy equivalence (over B, i.e., commuting through B) of X with mapping path space of g defined by

$$P_g = \{(b, \sigma) \in B \times C^I : \sigma_g(0) = *, \sigma_g(1) = g(b)\},$$

$$p_1 : P_g \to B, (b, \sigma) \mapsto b.$$

as shown in Fig. 4.32; C is called the classifying space and g is called the classifying map for the principal fibration.

Theorem 4.7.7 *Given a principal fibration $p : X \to B$, a lifting $\tilde{f}$ of f exists if and only if $g \circ f$ is homotopic to a constant map, where $g : B \to C$ is the classifying map.*

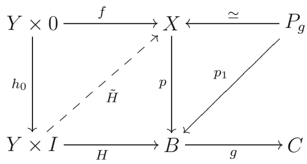


Fig. 4.32 Diagram of the classifying space and classifying map

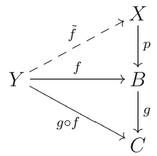


Fig. 4.33 Existence of lifting for a principal fibration

Proof Since p is a principal fibration, there is a homotopy equivalence (over B) $X \simeq P_g$ and hence there exist maps

$$h : X \to P_g \text{ and } k : P_g \to X$$

such that

$$k \circ h \simeq 1_X \text{ and } h \circ k \simeq 1_{P_g} \text{ and } p_1 \circ h = p,\ p \circ k = p_1,$$

where $p_1 : P_g \to B, (b, \sigma) \mapsto b$.

Given $\tilde{f} : Y \to X$, we obtain a homotopy $g \circ f \simeq c$, where $c : Y \to C$ is the constant map $y \mapsto * \in C$ as the composite

$$H : Y \times I \to P_g \to C, (y, t) \mapsto (h \circ \tilde{f})(y) = (f(y), \sigma_y) \mapsto \sigma_y(t) = H_t(y)$$

as shown in Fig. 4.33.

Conversely, let $G : g \circ f \simeq c$. Define

$$G_y : I \to C, t \mapsto G(y, t);$$

$$\tilde{f} : Y \to P_g \to X, y \mapsto (f(y), G_y) \mapsto k(f(y), G_y).$$

Hence

$$(p \circ \tilde{f})(y) = (pok)(f(y), G_y) = p_1(f(y), G_y) = f(y),\ \forall\, y \in Y \Rightarrow p \circ \tilde{f} = f$$

❑

Theorem 4.7.8 *A lifting $\tilde{f}$ of a principal fibration $p : X \to B$ exists iff there exists a map $\tilde{g} : C_f \to C$ extending the classifying map g in the diagram as shown in Fig. 4.34.*

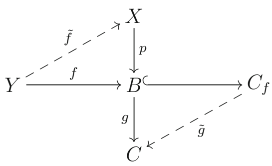


Fig. 4.34 Existence of a map extending the classifying map g

Proof In the category $\mathcal{T}op_*$ of pointed topological spaces the mapping cone C_f is obtained from the mapping cylinder M_f by identifying $Y \times \{0\} \cup \{*\} \times I$ with $*$ in B.

Suppose there is a homotopy

$$H : c \simeq g \circ f : Y \to C,$$

where $c : Y \to * \in C$ is the given constant map. Define

$$\tilde{g} : C_f \to C, \begin{cases} (y, t) \mapsto H(y, t), \\ b \mapsto g(b) \text{ for } b \notin f(Y). \end{cases}$$

Then

$$\tilde{g}(y, 0) = * \text{ and } \tilde{g}(y, 1) = gf(y), \ \forall\, y \in Y.$$

This shows that $\tilde{g}$ is the required extension of g.

Conversely, let $\tilde{g}$ be an extension of g. Then there is a homotopy

$$G : Y \times I \to C, (y, t) \mapsto \tilde{g}(y, t.)$$

Consequently,

$$G(y, 0) = \tilde{g}(y, 0) = *$$

and

$$G(y, 1) = \tilde{g}(y, 1) = (g \circ f)(y), \ \forall\, y \in Y.$$

Hence $g \circ f \simeq c$. ❑

Proposition 4.7.9 *Let $p : X \to B$ be a fibration and α be any path in B such that $\alpha(0) \in p(X)$. Then α can be lifted to a path $\tilde{\alpha}$ in X.*

Proof α can be regarded as a homotopy $\alpha : \{p_0\} \times I \to B$, where $\{p_0\}$ is a one-point space. Let x_0 be a point in X such that $p(x_0) = \alpha(0)$. Then there exists a map $f : \{p_0\} \to X$ such that $pf(p_0) = \alpha(p_0, 0)$. Hence it follows from the HLP of p that there exists a path $\tilde{\alpha}$ in X such that $\tilde{\alpha}(0) = x_0$ and $p \circ \tilde{\alpha} = \alpha$. This shows that $\tilde{\alpha}$ is a lifting of α. ❑

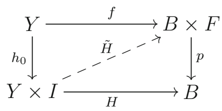


Fig. 4.35 Trivial fibration

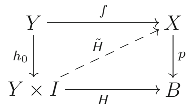


Fig. 4.36 Homotopy lifting property

Example 4.7.10 Let F be any space and $p : B \times F \to B$ be the projection to the first factor. Then p is a trivial fibration and for any $b \in B$, the fiber $p^{-1}(b)$ over b is homeomorphic to F. Because, if the diagram in Fig. 4.35 symbolizes homotopy lifting problem, then the map $\tilde{H} : Y \times I \to B \times F$ defined by $\tilde{H}(y, t) = (H(y, t), pf(y))$ is a solution of the lifting problem.

The projection $p : B \times I \to B$ is said to be a trivial fibration.

Example 4.7.11 For any space X, let $P(X) = M(I, X)$ be the space of all paths in X. Then the map $p : P(X) \to X \times X$, defined by $p(\alpha) = (\alpha(0), \alpha(1))$ is a fibration. Again

$$p_i : P(X) \to X, \alpha \mapsto \alpha(0), \alpha(1)$$

for $i = 1, 2$, respectively, are also fibrations.

Example 4.7.12 Let $p : X \to Y$ be a fibration and $q : Y \to B$ be also a fibration, then their composite $q \circ p : X \to B$ is also a fibration.

Theorem 4.7.13 *Every covering projection is a fibration.*

Proof Let $p : X \to B$ be a covering projection and the diagram in Fig. 4.36 symbolizes a homotopy lifting problem. Then for each $y \in Y$, there exists a unique path $\alpha_y : I \to X$ such that $\alpha_y(0) = f(y)$ and $p\alpha_y(t) = H(y, t)$. Then the map $\tilde{H} : Y \times I \to X, (y, t) \mapsto \alpha_y(t)$ is a continuous map and p is a fibration. ❑

Remark 4.7.14 For a covering projection the lifting is unique but it is not true for an arbitrary fibration.

4.7.3 Cofibration: Introductory Concepts

This subsection conveys the concept of cofibration and studies it in the category $\mathcal{T}op_*$ of pointed topological spaces and pointed maps. Cofibrations form an important class of maps in topology. Geometrically, the concept of cofibrations is less complicated than that of fibrations. There is a very nice duality principle in homotopy theory.

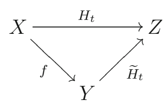


Fig. 4.37 Commutative triangle for cofibration

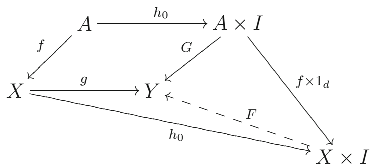


Fig. 4.38 Cofibration of f

For example, if in the definition of a fibration as a map satisfying homotopy lifting property we reverse the directions of all rows, we obtain the dual notion, called a cofibration. This is a continuous map $f : X \to Y$ satisfying the property: given $\tilde{g} : Y \to Z$ in $\mathcal{T}op_*$ and a homotopy $\widetilde{H}_t : Y \to Z$ such that there is a continuous map $H_t : X \to Z$ with the property $\widetilde{H}_t \circ f = H_t$, i.e., making the triangle in Fig. 4.37 commutative.

In this subsection, we work in $\mathcal{T}op_*$ unless specified otherwise.

Definition 4.7.15 A continuous map $f : A \to X$ is said to be a cofibration if for every topological space Y and given a continuous map $g : X \to Y$ and a homotopy $G : A \times I \to Y$ starting with $g \circ f$, there exists a homotopy $F : X \times I \to Y$ starting with g, such that $G = F \circ (f \times 1_d)$ as shown in Fig. 4.38.

Remark 4.7.16 It follows from Definition 4.7.15 that if A is a subspace of X, the inclusion map $i : A \hookrightarrow X$ is a cofibration if the pair (X, A) has the homotopy extension property (HEP) with respect to the given space Y.

Example 4.7.17 For any space A in $\mathcal{T}op_*$, let $CA = A \times I/A \times \{1\} \cup \{*\} \times I$ be the cone of A and $i : A \to CA, a \mapsto [a, 0]$ be the inclusion. Then i is a cofibration.

Proposition 4.7.18 *Given a map $f : (X, x_0) \to (Y, y_0)$ in $\mathcal{T}op_*$, the inclusion $i : Y \hookrightarrow Y \bigcup_f CX$ is a cofibration.*

Proof Let $r : I \times I \to I \times \{0\} \cup \dot{I} \times I$ be a retraction. Then given maps $f : Y \bigcup_g CX \to Z$ in $\mathcal{T}op_*$ and $G : Y \times I \to Z$ with $G(y, 0) = f(y)$, $\forall\, y \in Y$, define $H : CX \times \{0\} \cup X \times I \to Z$ by the rule $H|_{CX\times\{0\}} = f|_{CX}$ and $H|_{X\times I} = G \circ (g \times 1_d)$. Again define $F : (Y \bigcup_g CX) \times I$ by the rule $F|_{Y\times I} = G$ and $F([s, x], t) = H([p_1 \circ r(s, t), x], p_2 \circ r(s, t))$, $\forall\, [s, t] \in CX, t \in I$, where p_1, p_2 are the restrictions to $I \times \{0\} \cup \dot{I} \times I$ of the projections $p_1, p_2 : I \times I \to X$. Then F is well defined and is a continuous map such that $F(y, 0) = f(y)$. ❑

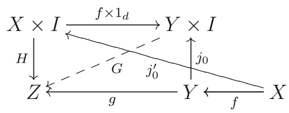


Fig. 4.39 Diagram for cofibration

Proposition 4.7.19 *If $f : X \to Y$ is a cofibration, then f is injective, and in fact it is a homeomorphism onto its image.*

Proof Consider the mapping cylinder M_f of f, the quotient space of $X \times I \cup Y$ in which $(x, 1)$ is identified with $f(x)$. Let $H_t : X \to M_f$ be the homotopy, mapping $x \in X$ to the image $(x, 1-t) \in X \times I$ in M_f, and let $\widetilde{H}_t : Y \hookrightarrow M_f$ be the inclusion. Then the cofibration property of f shows that $\widetilde{H}_t : Y \to M_f$ is such that $\widetilde{H}_t \circ f = H_t$. Restriction to a fixed $t > 0$, shows that f is injective, since H_t is so. Moreover, as H_t is a homeomorphism onto its image $X \times \{1-t\}$, the relation $\widetilde{H}_t \circ f = H_t$ holds. ❑

There is an equivalent definition of cofibration in $\mathcal{T}op_*$.

Definition 4.7.20 A continuous map $f : X \to Y$ in $\mathcal{T}op_*$ is said to be a cofibration if given a topological space Z, a continuous map $g : Y \to Z$ and a homotopy $H : X \times I \to Z$ starting from $g \circ f$, there exists a homotopy $G : Y \times I \to Z$, starting from g such that $H = G \circ (f \times 1_d)$ i.e., making the three triangles as shown in Fig. 4.39 commutative, where $j_0(y) = (y, 0)$, $\forall\, y \in Y$ and $j_0'(x) = (x, 0)$, $\forall\, \in X$.

Remark 4.7.21 Let A be a subspace of a topological space X. Then the inclusion map $i : A \hookrightarrow X$ is a cofibration if the pair (X, A) has the absolute homotopy extension property (see Chap. 2). The converse is not true in general. Because the definition of a cofibration refers to $\mathcal{T}op_*$ but the absolute homotopy extension property refers to maps and homotopies that are not necessarily based.

Theorem 4.7.22 *Every continuous map $f : X \to Y$ in $\mathcal{T}op_*$ is the composite of a cofibration and a homotopy equivalence.*

Proof Let M_f be the mapping cylinder in $\mathcal{T}op_*$ obtained from Y and $(X \times I)/x_0 \times I$ by identifying, for each $x \in X$, the points $(x, 1)$ and $f(x)$. Suppose $g : X \to M_f, x \mapsto [(x, 0)]$ is the inclusion map. Let $h : M_f \to Y$ be the map induced by the identity map 1_Y of Y and the map from $X \times I$ to Y that sends each $[(x, t)]$ to $f(x)$. Then $f = h \circ g$. We claim that h is a homotopy equivalence and g is a cofibration. We first show that g is a cofibration. Given a map $k : M_f \to Z$ in $\mathcal{T}op_*$, and a homotopy $H : X \times I \to Z$ starting from $k \circ g$, define maps

$$G_Y : Y \times I \to Z, (y, s) \mapsto k(y), \ \forall\, s \in I$$

and $G_X : (X \times I) \times I \to Z,\ (x, t, s) \mapsto \begin{cases} k(x, (2t-s)/(2-s)), & 0 \le s \le 2t \\ H(x, s-2t), & 2t \le s \le 1. \end{cases}$

Clearly, G_X is continuous and $G_X(x, 1, s) = k(x, 1) = (k \circ f)(x) = G_Y(f(x), s)$. Hence G_X and G_Y give together a homotopy $G : M_f \times I \to Z$ such that G starts from k and $G \circ (g \times 1_d)(x, s) = G(x, 0, s) = H(x, s)$. Hence $G \circ (g \times 1_d) = H$ shows that g is a cofibration.

Finally we show that h is a homotopy equivalence. Define $j : Y \to M_f$ to be (the restriction) of the identification map onto M_f. Then $h \circ j = 1_Y$ and

$$j \circ h : M_f \to M_f, y \mapsto y \text{ and } (x, t) \mapsto f(x).$$

Define a homotopy

$$H : M_f \times I \to M_f, (y, s) \mapsto y \text{ and } (x, t, s) \mapsto (x, t + s(1 - t)).$$

Clearly, H is continuous and $H : 1_d \simeq j \circ h$. This shows that h is a homotopy equivalence. ❑

Remark 4.7.23 The dual of the Theorem 4.7.22 is true in the sense that every continuous map $f : X \to Y$ in $\mathcal{T}op_*$ is also the composite of a homotopy equivalence and a fiber map.

Theorem 4.7.24 *Let A be a closed subset of a topological space X. Then the inclusion $i : A \hookrightarrow X$ is a cofibration iff $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.*

Proof If $i : A \hookrightarrow X$ is a cofibration, then the given continuous maps $f : X \to X \times \{0\} \cup A \times I, x \mapsto (x, 0)$ and $G : A \times I \to X \times \{0\} \cup A \times I, (a, t) \mapsto (a, t)$ determine a map $H : X \times I \to X \times \{0\} \cup A \times I$ which is a retraction. Conversely, suppose there is a retraction $r : X \times I \to X \times \{0\} \cup A \times I$. Then given a space Y, a map $f : X \to Y$, and a homotopy $G : A \times I \to Y$ with the property that $H(a, 0) = f(i(a)), \forall a \in A$, define a map

$$H : X \times I \to Y, (x, t) \mapsto \begin{cases} (f \circ p_X \circ r)(x, t), & \text{if } (x, t) \in r^{-1}(X \times \{0\}) \\ (G \circ r)(x, t), & \text{if } (x, t) \in r^{-1}(A \times I). \end{cases}$$

Since $X \times \{0\}$ and $A \times I$ are closed in $X \times I$, it follows that H is continuous. Consequently, i is a cofibration. ❑

4.8 Hurewicz Theorem for Fibration and Characterization of Fibrations

This section continues the study of fibrations, characterizes path liftings of fibrations with the help of their fibers and studies Hurewicz theorem. This theorem is due to W. Hurewicz (1904–1956). It gives a sufficient condition for a map $p : X \to B$ to be a fibration (Hurewicz 1955).

Theorem 4.8.1 (Hurewicz) *Let $p : X \to B$ be a continuous map. Suppose B is paracompact and there is an open covering $\{V_i\}$ of B such that, for each V_i, $p|p^{-1}(V_i) : p^{-1}(V_i) \to V_i$ is a fibration. Then p is a fibration.*

Proof The proof is long and complicated. (Dugundji 1966) is referred. ❑

An immediate important consequence of Hurewicz Theorem 4.8.1 gives a sufficient condition for a projection map $p : X \to B$ of a fiber bundle (Chap. 5) to be a fibration.

Corollary 4.8.2 *Let $p : X \to B$ be the projection of a fiber bundle, and suppose B is paracompact, then p is a fibration.*

Theorem 4.8.3 *Let $p : X \to B$ be a covering projection and let $f, g : Y \to X$ be liftings of the same map (i.e., $p \circ f = p \circ g$). If Y is connected and $f(y_0) = g(y_0)$ for some point y_0 of Y, then $f = g$.*

Proof Let $A = \{y \in Y : f(y) = g(y)\}$. Then $A \neq \emptyset$ and A is an open set in Y. To show this, let $y \in A$ and U be an open neighborhood of $pf(y)$ evenly covered by p and let $\tilde{U}$ be an open subset of X containing $f(y)$ such that p maps $\tilde{U}$ homeomorphically onto U. Then $f^{-1}(\tilde{U}) \cap g^{-1}(\tilde{U})$ is an open subset of Y containing y and contained in A. Again let $B = \{y \in Y : f(y) \neq g(y)\}$. If X is assumed to be Hausdorff, then B is open in Y. Otherwise, let $y \in B$ and U be an open neighborhood of $pf(y)$ evenly covered by p. Since $f(y) \neq g(y)$, there are disjoint open sets $\tilde{V}_1$ and $\tilde{V}_2$ of X such that $f(y) \in \tilde{V}_1$ and $g(y) \in \tilde{V}_2$ and p maps each of the sets $\tilde{V}_1$ and $\tilde{V}_2$ homeomorphically onto U. Then $f^{-1}(\tilde{V}_1) \cap g^{-1}(\tilde{V}_2)$ is an open subset of Y containing y and contained in B. Finally, $Y = A \cup B$ and A and B are disjoint open sets imply from the connectedness of Y that either $A = \emptyset$ or $A = Y$. By hypothesis $A \neq \emptyset$ and hence $A = Y$ shows that $f = g$. ❑

Definition 4.8.4 A continuous map $p : X \to B$ is said to have a unique path lifting if, given paths α and β in X such that $p \circ \alpha = p \circ \beta$ and $\alpha(0) = \beta(0)$, then $\alpha = \beta$.

Proposition 4.8.5 *If a continuous map $p : X \to B$ has unique path lifting property, then it has path lifting property for path-connected spaces.*

Proof Let $p : X \to B$ has unique path lifting property and Y be path connected space. If $f, g : Y \to X$ and maps are such that $p \circ f = p \circ g$ and $f(y_0) = g(y_0)$ for some $y_0 \in Y$, we claim that $f = g$. Let $y \in Y$ and α be a path in Y from y_0 to y. Then $f \circ \alpha$ and $g \circ \alpha$ are paths in X that are liftings of some path in B and have the same initial point. Since p has unique path lifting, $f \circ \alpha = g \circ \alpha$ and hence $f(y) = (f \circ \alpha)(1) = (g \circ \alpha)(1) = g(y)$ implies $f = g$, since $\alpha(1) = y$. ❑

We now characterize path liftings of fibrations with the help of their fibers.

Theorem 4.8.6 *Let $p : X \to B$ be a fibration. Then the fibration has unique path lifting iff every fiber has no nonconstant paths.*

Proof Let $p : X \to B$ be a fibration with unique path lifting. Let α be a path in the fiber $p^{-1}(b)$ and β be the constant path in $p^{-1}(b)$ such that $\beta(0) = \alpha(0)$. Then $p \circ \alpha = p \circ \beta \Rightarrow \alpha = \beta \Rightarrow \alpha$ is a constant path. Conversely, let $p : X \to B$ be a fibration such that every fiber has no nontrivial path. If α and β are paths in X such that $p \circ \alpha = p \circ \beta$ and $\alpha(0) = \beta(0)$, then for $t \in I$ define a path γ_t in X by

$$\gamma_t(t') = \begin{cases} \alpha((1-2t')t), \ 0 \leq t' \leq 1/2 \\ \beta((2t'-1)t), \ 1/2 \leq t' \leq 1. \end{cases}$$

Then for each $t \in I$, $\gamma_t : I \to X$ is a path in X from $\alpha(t)$ to $\beta(t)$ and $p \circ \gamma_t$ is a closed path in B, which is homotopic rel $\dot{I}$ to the constant path at $p\alpha(t)$. By HLP of p, there is a map $H : I \times I \to X$ such that $H(t', 0) = \gamma_t(t')$ and H maps $0 \times I \cup I \times 1 \cup 1 \times I$ to the fiber $p^{-1}(p\alpha(t))$. Every Since $p^{-1}(p\alpha(t))$ has no nonconstant paths, F maps $0 \times I$, $I \times 1$ and $1 \times I$ to a single point. Hence it follows that $F(0, 0) = F(1, 0)$. Consequently, $\gamma_t(0) = \gamma_t(1)$ and $\alpha(t) = \beta(t)$. ❑

Proposition 4.8.7 *Let X be pointed topological space with base point x_0 and $P(X)$ be the space of paths in X starting at x_0, then the map*

$$p : P(X) \to X, \alpha \mapsto \alpha(1)$$

is fibration with fiber $\Omega(X)$.

Proof Let Y be an arbitrary space. Given maps $f : Y \to P(X)$ and $G : Y \times I \to X$ with $G_0 = p \circ f : Y \to X$, define a function

$$H : Y \times I \times I \to X,$$

$$(y, t, s) \mapsto \begin{cases} (f(y))(s(t+1)), & 0 \leq s \leq \frac{1}{t+1} \\ G(y, s(t+1) - 1), & \frac{1}{t+1} \leq s \leq 1 \end{cases}$$

Then H is continuous and defines a map $F : Y \times I \to X^I$ such that

$$F(y, t)(0) = f(y)(0) = x_0, \ \forall\, y \in Y, t \in I.$$

Hence $F \in P(X)$ and $F(y, 0)(s) = f(y)s$, $\forall\, y \in Y$, $\forall\, s \in I$. Consequently, $F_0 = f$ and $p \circ F = G$. This implies that F is the required lifting of G. Moreover, $p^{-1}(x_0) = \{\alpha \in P(x) : \alpha(1) = x_0\} = \Omega(X)$. ❑

4.9 Homotopy Liftings and Monodromy Theorem

This section continues the study of covering spaces by presenting some interesting applications of the path lifting property and homotopy lifting property of covering projections and proves Monodromy Theorem which provides a necessary and sufficient condition for two liftings of a covering projection to be equivalent.

4.9.1 Path Liftings and Homotopy Liftings

This subsection discusses path lifting property of a covering projection by using the homotopy lifting property.

Theorem 4.9.1 *Let (X, p) be a covering space of B and $b_0 \in B$. If $x_0 \in p^{-1}(b_0)$, then for any path $f : I \to B$, with $f(0) = b_0$, there exists a unique path $\tilde{f} : I \to X$ such that $\tilde{f}(0) = x_0$ and $p \circ \tilde{f} = f$.*

Proof Let $A = \{a\}$ be a singleton space. We consider the map $f : A \to B$ defined by $f(a) = b_0$. The path f defines a homotopy $F : A \times I \to B$ on A given by $F(a, t) = f(t)$. Then by the Homotopy Lifting Property, $\exists$ a map $\tilde{F} : A \times I \to X$ such that $\tilde{F}(a, 0) = x_0$ and $p \circ \tilde{F} = F$. Consequently, $\tilde{f} : I \to X$ defined by $\tilde{f}(t) = \tilde{F}(a, t), t \in I$, is a path in X starting from x_0 and having the property:

$$(p \circ \tilde{f})(t) = (p \circ \tilde{F}(a, t)) = F(a, t) = f(t),\ \forall t \in I \text{ i.e., } p \circ \tilde{f} = f.$$

Clearly the path $\tilde{f} : I \to X$ is unique. ❑

Remark 4.9.2 If $p : X \to B$ is a covering map, then p is also onto. But its induced homomorphism

$$p_* : \pi_1(X, x_0) \to \pi_1(B, b_0)$$

need not be a epimorphism. However, p_* is a monomorphism.

4.9.2 Monodromy Theorem

This subsection gives a criterion for two path liftings in X to be equivalent through a result known as ‘Monodromy theorem’.

Theorem 4.9.3 (The Monodromy Theorem) *Let (X, p) be a covering space of B and $\tilde{f}, \tilde{g}$ are paths in X with same initial point x_0. Then $\tilde{f}$ and $\tilde{g}$ are equivalent (i.e., $\tilde{f} \simeq \tilde{g}$ rel $\dot{I}$) if and only if $p \circ \tilde{f}$ and $p \circ \tilde{g}$ are equivalent paths in B.*

Proof Let $\tilde{f}, \tilde{g}$ be equivalent paths in X. Then $\exists$ a homotopy $F : \tilde{f} \simeq \tilde{g}$ rel $\dot{I} \Rightarrow p \circ F : I \times I \to B$ is a continuous map such that $p \circ F : p \circ \tilde{f} \simeq p \circ \tilde{g}$ rel $\dot{I}$ and hence $p \circ \tilde{f}$ and $p \circ \tilde{g}$ are equivalent paths in B. Conversely, let $p \circ \tilde{f}$ and $p \circ \tilde{g}$ be equivalent paths in B. Then $\exists$ a continuous map $G : I \times I \to B$ such that $G : p \circ \tilde{f} \simeq p \circ \tilde{g}$ rel $\dot{I}$. By Homotopy Lifting Property, $\exists$ a unique homotopy $\tilde{G} : I \times I \to X$ such that $\tilde{G}(0, 0) = x_0$ and $p \circ \tilde{G} = G$. Restricting $\tilde{G}$ on $(t, 0), t \in I$, we have a path $t \mapsto \tilde{G}(t, 0)$, starting from x_0 and lifting $p \circ \tilde{f}$. Then $t \mapsto \tilde{f}(t)$ is also a path in X starting from x_0 and lifting $p \circ \tilde{f}$. Hence the uniqueness property of the covering paths, $\tilde{G}(t, 0) = \tilde{f}(t)$, $\forall t \in I$. Similarly, $\tilde{G}(t, 1) = \tilde{g}(t)$. Again by restricting $\tilde{G}$ on $(0, s)$, $s \in I$, we have a path $s \mapsto \tilde{G}(0, s)$, which projects under p to the constant path at $b_0 = p(x_0)$. A constant path $s \mapsto x_0$ in X also projects under p to the constant path $s \mapsto x_0$ in X. Hence by uniqueness theorem $s \mapsto \tilde{G}(0, s)$ is a constant path based at x_0. Similarly, the path $s \mapsto \tilde{G}(s, t)$ is a constant path based at some point $x_1 \in p^{-1}(b_0)$. This shows that $\tilde{G}$ is a homotopy between $\tilde{f}$ and $\tilde{g}$ rel $\dot{I}$. Consequently $\tilde{f}$ and $\tilde{g}$ are equivalent paths in X. ❑

Corollary 4.9.4 *Let (X, p) be a covering space of B and $b_0 \in B$, $x_0 \in p^{-1}(b_0)$. Then the induced homomorphism $p_* : \pi_1(X, x_0) \to \pi_1(B, b_0)$ is a monomorphism.*

Proof Let $[\tilde{f}], [\tilde{g}] \in \pi_1(X, x_0)$ and $[\tilde{f}] \neq [\tilde{g}]$. Then $p_*([\tilde{f}]) = [p \circ \tilde{f}]$ and $p_*([\tilde{g}]) = [p \circ \tilde{g}]$. Now $p \circ \tilde{f} \simeq p \circ \tilde{g}$ rel $\dot{I} \Leftrightarrow \tilde{f} \simeq \tilde{g}$ rel $\dot{I}$. But $\tilde{f} \not\simeq \tilde{g}$ rel $\dot{I} \Leftrightarrow p \circ \tilde{f} \not\simeq p \circ \tilde{g}$ rel I, otherwise we arrive at a contradiction by Theorem 4.9.3. This shows that p_* is well defined and injective; hence p_* is a monomorphism. ❑

4.10 Applications and Computations

This section presents applications of covering spaces and computes fundamental groups of some interesting spaces. Finally it presents an application of Galois correspondence arising from the function that assigns to each covering space $p : (X, x_0) \to (B, b_0)$ the subgroup $p_*(\pi_1(X, x_0))$ of $\pi_1(B, b_0)$.

4.10.1 Actions of Fundamental Groups

This subsection considers action of the fundamental group of the base space of a covering space on a fiber. This action plays an important role in the study of the covering space.

Let (X, p) be a covering space of B and $b_0 \in B$. We now consider the action of the fundamental group $\pi_1(B, b_0)$ on the fiber $p^{-1}(b_0) = Y$.

Theorem 4.10.1 *Let (X, p) be a covering space of B and $b_0 \in B$. Let $Y = p^{-1}(b_0)$ be the fiber over b_0. Then*

(i) *$\pi_1(B, b_0)$ acts transitively on Y;*
(ii) *If $x_0 \in Y$, then the isotropy group $G_{x_0} = p_*\pi_1(X, x_0)$; and*
(iii) $|Y| = [\pi_1(B, b_0) : p_*\pi_1(X, x_0)]$.

Proof First we show that Y is a (right) $\pi_1(B, b_0)$-set. We define $\sigma : Y \times \pi_1(B, b_0) \to Y$ by the rule $\sigma(x, [f]) = x \cdot [f] = \tilde{f}(1)$, where $\tilde{f}$ is the unique lifting of $f : (I, 0) \to (B, b_0)$ such that $\tilde{f}(0) = x$. This definition does not depend on the choice of the representative of the class $[f]$ by the Monodromy Theorem 4.9.3. If f is a constant path at b_0, then $\tilde{f}$ is also a constant path at $x \in Y$. Hence $x \cdot [f] = \tilde{f}(1) = x$. Next suppose $[f], [g] \in \pi_1(B, b_0)$. Let $\tilde{f}$ be the lifting of f with $\tilde{f}(0) = x$ and $\tilde{g}$ be the lifting of g with $\tilde{g}(0) = \tilde{f}(1)$. Then $\tilde{f} * \tilde{g}$ is a lifting of $f * g$ that begins at x and ends at $\tilde{g}(1)$. Consequently, $x \cdot [f * g] = (x \cdot [f])[g]$. As a result σ is an action of $\pi_1(B, b_0)$ on Y.

(i) σ **is transitive**: Let $x_0 \in Y$ and x be any point in Y. Since X is path-connected, $\exists$ a path $\tilde{\lambda}$ in X from x_0 to x. Then $p \circ \tilde{\lambda}$ is a closed path in B at b_0 whose lifting with initial point x_0 is $\tilde{\lambda}$. Thus $[p \circ \tilde{\lambda}] \in \pi_1(B, b_))$ and $x_0 \cdot [p \circ \tilde{\lambda}] = \tilde{\lambda}(1) = x$. Hence it follows that $\pi_1(B, b_0)$ acts transitively on Y.

(ii) Let f be a closed path in B at b_0 and $\tilde{f}$ be the lifting of f with $\tilde{f}(0) = x_0$. Let $G = \pi_1(B, b_0)$. Then $G_{x_0} = \{[f] \in \pi_1(B, b_0) : x_0 \cdot [f] = x_0\}$. Hence $[f] \in G_{x_0} \Rightarrow x_0 \cdot [f] = x_0 \Rightarrow \tilde{f}(1) = x_0 = \tilde{f}(0) \Rightarrow \tilde{f} \in \pi_1(X, x_0)$ and $[f] = [p \circ \tilde{f}] \in p_*\pi_1(X, x_0) \Rightarrow \pi_1(B, b_0) = G \subseteq p_*\pi_1(X, x_0)$. For the reverse inclusion, assume $[f] = [p \circ \tilde{g}]$ for some $[\tilde{g}] \in \pi_1(X, x_0)$. Then $\tilde{f} = \tilde{g}$, since both are liftings of f and both have initial point $x_0 \Rightarrow \tilde{f}(1) = \tilde{g}(1) \Rightarrow x_0 \cdot [f] = \tilde{f}(1) = x_0 \Rightarrow [f] \in G_{x_0} \Rightarrow p_*\pi_1(X, x_0) \subseteq G_{x_0}$. Consequently, $G_{x_0} = p_*\pi_1(X, x_0)$.

(iii) Recall that if a group G acts on a set Y and $x_0 \in Y$, then $|\text{orbit of } x_0| = [G : G_{x_0}]$. In particular, if G acts transitively, $|Y| = [G : G_{x_0}]$. Hence in this case, $G = \pi_1(B, b_0)$ and $G_{x_0} = p_*\pi_1(X, x_0)$ by (ii). Consequently, $|Y| = [\pi_1(B, b_0) : p_*\pi_1(X, x_0)]$. ❑

Corollary 4.10.2 *Let (X,p) be the universal covering space of B, then* $|Y| = |\pi_1(B, b_0)|$.

Proof The corollary follows from Theorem 4.10.1(iii), since $\pi_1(X, x_0) = 0$. ❑

Corollary 4.10.3 *If* $n \geq 2$, *then* $\pi_1(\mathbf{R}P^n) \cong \mathbf{Z}_2$.

Proof Since (S^n, p) is a covering space of $\mathbf{R}P^n$ of multiplicity 2, it follows that $[\pi_1(\mathbf{R}P^n, x_0) : p_*\pi_1(S^n, y_0)] = 2$. Again, S^n is simply connected for $n \geq 2 \Rightarrow p_*\pi_1(S^n, y_0) = 0 \Rightarrow |\pi_1(\mathbf{R}P^n, x_0)| = 2 \Rightarrow \pi_1(\mathbf{R}P^n, x_0) \cong \mathbf{Z}_2$. ❑

Corollary 4.10.4 *Let* (X, p) *be a covering space of* $b_0 \in B$, $x_0 \in p^{-1}(b_0)$. *If* $p_* : \pi_1(X, x_0) \to \pi_1(B, b_0)$ *is onto, then the map* $p : X \to B$ *induces an isomorphism*

$$p_* : \pi_1(X, x_0) \to \pi_1(B, b_0).$$

Proof p_* is a monomorphism by Corollary 4.9.4 and hence the corollary follows from the given condition. ❑

4.10.2 Fundamental Groups of Orbit Spaces

This subsection computes the fundamental groups of some important spaces which are obtained as orbit spaces. For example, projective spaces, lens spaces, figure-eight and Klein's bottles are interesting spaces. We represent them as orbit spaces and compute their fundamental groups. A topological group G with identity e acting on a topological space X is said to satisfy the condition **(A)**: if for each $x \in X$, $\exists$ a neighborhood U_x such that, $\Phi_g(U_x) \cap U_x \neq \emptyset \Rightarrow g = e$, where

$$\Phi_g : X \to X, x \mapsto gx$$

is a homeomorphism. This special kind of group action of the group G of homeomorphisms of X is said to act on X properly discontinuously. For example, any action of a finite group on a Hausdorff space is properly discontinuous.

Example 4.10.5 The automorphism group $\mathcal{A}ut(X/B)$ of (X, p) of B satisfies the condition **(A)**.

Definition 4.10.6 A covering space (X, p) of B is said to be regular if $p_*\pi_1(B, b_0)$ is a normal subgroup of $\pi_1(B, b_0)$.

Example 4.10.7 Let B be a connected, locally path-connected space and G satisfies the condition **(A)** on X then (X, p) is a regular covering space of

$$X \bmod G, \text{ where } p : X \to X \bmod G, x \mapsto Gx$$

is the natural projection.

Theorem 4.10.8 *If an action of a topological group G on a topological space X satisfies the condition* **(A)**, *then*

- **(i)** *if X is path-connected, then G is the group of deck transformations of the covering space*
$$p : X \to X \bmod G, \ x \mapsto Gx$$
- **(ii)** *if X is path-connected and locally path-connected, then G is isomorphic to the quotient group $\pi_1(X \bmod G)/p_*\pi_1(X)$.*
- **(iii)** *for any simply connected space X, the groups $\pi_1(X \bmod G)$ and G are isomorphic.*

Proof **(i)** Let X be path-connected. The deck transformation group contains G as a subgroup and equals this group, since if f is any deck transformation, then given any point $x \in X$, x and $f(x)$ are in the same orbit and hence there is some $g \in G$ such that $g(x) = f(x)$. Consequently, $f = g$, since deck transformation of a connected covering space are uniquely determined under this situation.

(ii) It follows from Ex. 28 of Sect. 4.11.

(iii) Let $y \in X \bmod G$. Since X is simply connected, $\pi_1(X, x_0) = \{e\}$, $\forall x_0 \in p^{-1}(y)$ and hence $p_*\pi_1(X, x_0) = \{e\}$. Consequently, Theorem 4.10.8(iii) follows from Theorem 4.10.8(ii). ❑

Remark 4.10.9 We first make geometrical constructions of some orbit spaces and then compute their fundamental groups.

4.10.3 Fundamental Group of the Real Projective Space $\mathbf{R}P^n$

This subsection computes the fundamental group of $\mathbf{R}P^n$ by using group action. We have computed $\pi_1(\mathbf{R}^n) \cong \mathbf{Z}_2$ for $n \geq 2$ in Theorem 4.6.9. Here we give an alternative approach.

Definition 4.10.10 Let $S^n = \{x \in \mathbf{R}^{n+1} : ||x|| = 1\}$ be the n-sphere and $\mathbf{R}P^n$ be the n-dimensional real projective n-space. The antipodal map $A : S^n \to S^n, x \mapsto -x$, generates an action of the two element group $G = \{+1, -1\}$ given by the relation $(+1)x = x$ and $(-1)x = -x$. Then its orbit space S^n mod G is $\mathbf{R}P^n$, the real projective n-space.

Theorem 4.10.11 $\pi_1(\mathbf{R}^n) \cong \mathbf{Z}_2$ *for* $n \geq 2$.

Proof As S^n is simply connected for $n \geq 2$, so from the covering space $p : S^n \to \mathbf{R}P^n$ it follows by the Theorem 4.10.8 that the fundamental group of orbit space is G. Thus $\pi_1(S^n \text{ mod } G) = G \Rightarrow \pi(\mathbf{R}^n) = G \cong \mathbf{Z}_2$ for $n \geq 2$. ❑

Remark 4.10.12 The above action is free in the sense that $gx = x \Rightarrow g = e$. Does there exist any other finite group G acting freely on S^n and defining covering space $S^n \to S^n$ mod G? The answer is Z_2 is the only non-trivial group that can act freely on S^n if n is even (see Chap. 14).

Remark 4.10.13 A generator for $\pi_1(\mathbf{R}P^n)$ is any loop obtained by projecting a path in S^n connecting two antipodal points.

4.10.4 The Fundamental Group of Klein's Bottle

This subsection computes the fundamental group of Klein's bottle. Let G be the group of transformations of the plane generated by a and b. Consider the action of G on $\mathbf{R}^2$ by $a(x, y) = (x + 1, y)$ and $b(x, y) = (1 - x, y + 1)$, $\forall (x, y) \in \mathbf{R}^2$. Then $a^{-1}(x, y) = (x - 1, y)$ and $b^{-1}(x, y) = (1 - x, y - 1)$. Hence $\mathbf{R}^2$ is simply connected and the action satisfies condition **(A)**, then by Theorem 4.10.8, $\pi_1(\mathbf{R}^2 \text{ mod } G) \simeq G$. Now

$$\begin{aligned} b^{-1}ab(x, y) &= b^{-1}a(1 - x, y + 1) \\ &= b^{-1}(2 - x, y + 1) = (1 - 2 + x, y) \\ &= (x - 1, y) = a^{-1}(x, y), \ \forall (x, y) \in \mathbf{R}^2 \Rightarrow b^{-1}ab = a^{-1}. \end{aligned}$$

Therefore $\mathbf{R}^2$ mod G is the Klein's bottle. This gives a representation of Klein's bottle as an orbit space whose fundamental group is generated by a and b.

4.10.5 The Fundamental Groups of Lens Spaces

This subsection computes the fundamental group of lens spaces defined by H. Tietze (1888–1971) in 1908, which are are 3-manifolds. Such spaces constitute an important class of objects in the study of algebraic topology.

Let $m > 1$ be an integer space and p be an integer relatively prime to m and $S^3 = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1|^2 + |z_2|^2 = 1\} \subset \mathbf{C}^2$. Let $\rho = e^{\frac{2\pi i}{m}}$ be a primitive m-th root of unity.

Define a map

$$h : S^3 \to S^3, (z_1, z_2) \mapsto (\rho z_1, \rho^p z_2) = (e^{\frac{2\pi i}{m}} z_1, e^{\frac{2\pi i p}{m}} z_2).$$

Then h is a homeomorphism of S^3 onto itself of period m, i.e., $h^m = 1_d$. Thus h induces an action of $\mathbf{Z}_m$ on S^n by the rule $\mathbf{Z}_m \times S^3 \to S^3, k(z_1, z_2) = h^k(z_1, z_2)$, where k denotes the residue class of the integer k modulo m, i.e., the action is generated by the rotation $z \mapsto e^{\frac{2\pi i}{m}} z$ of the unit sphere $S^3 \subset \mathbf{C}^2 = \mathbf{R}^4$. This action has no fixed point, because the equation $z = e^{\frac{2\pi i r}{m}} z$, where r is an integer such that $0 < r < m$ has a solution $z = 0$ but $z = 0$ is not a point of S^3.

The orbit spaces S^3 mod $\mathbf{Z}_m$ is called a lens space and is denoted by $L(m, p)$. Then the lens space is the quotient space $S^3/\sim$ given by an equivalence relation $\sim$ on S^3, defined by $(z_1, z_2) \sim (z_1', z_2')$ if there exists an integer k such that $k(z_1, z_2) = (z_1', z_2')$, i.e., $(z_1', z_2') = h^k(z_1, z_2)$.

As S^3 is Hausdorff, $\mathbf{Z}_m$ is finite and $\mathbf{Z}_m$ acts on S^3 without fixed point. The above action of $\mathbf{Z}_m$ on S^3 satisfies condition **(A)**. Hence $\mathbf{Z}_m \simeq \pi_1(S^3 \text{ mod } \mathbf{Z}_m) = \pi_1(L(m, p))$. We now extend the method of construction to construct generalized lens spaces. Let $m > 1$ be an integer and $p_1, p_2, \ldots, p_{n-1}$ be integers relatively prime to m and $S^{2n-1} = \{(z_1, z_2, \ldots, z_n) \in \mathbf{C}^n : |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 = 1\} \subset \mathbf{C}^n$. Let $\rho = e^{\frac{2\pi i}{m}}$ be a primitive mth root of unity. Define

$$h : S^{2n-1} \to S^{2n-1}, (z_1, z_2, \ldots, z_n) \mapsto (\rho z_1, \rho^{p_1} z_2, \ldots, \rho^{p_{n-1}} z_n) = (e^{\frac{2\pi i}{m}} z_1, e^{\frac{2\pi i p_1}{m}} z_2, \ldots, e^{\frac{2\pi i p_{n-1}}{m}} z_n)$$

Then h is a homeomorphism of S^3 with period m, i.e., $h^m = 1_d$. Thus as before h induces an action of $\mathbf{Z}_m$ on S^{2n-1} without fixed point by the rule $\mathbf{Z}_m \times S^{2n-1} \to S^{2n-1}, k(z_1, z_2, \ldots, z_n) \mapsto h^k(z_1, z_2, \ldots, z_n), \forall h \in \mathbf{Z}_m$.

The orbit spaces S^{2n-1} mod Z_m is called a generalized lens space and is denoted by $L(m, p_1, \ldots, p_{n-1})$. As before $\pi_1(L(m, p_1, \ldots, p_{n-1})) \cong \mathbf{Z}_m$. As a particular case, for $\mathbf{Z}_2 = \{1_d, a\}$, S^2 mod $\mathbf{Z}_2 = L(2, 1)$, where $1_d : S^2 \to S^2$ is the identity map and $a : S^2 \to S^2$ is the antipodal map and hence $a^2 = 1_d$.

As the action of $\mathbf{Z}_2$ on S^2 yields S^2 mod $\mathbf{Z}_2 = \mathbf{R}P^2$, and its fundamental group $\pi_1(S^2 \text{ mod } \mathbf{Z}_2) \cong \mathbf{Z}_2$.

4.10.6 Computing Fundamental Group of Figure-Eight by Graph-theoretic Method

This subsection computes the fundamental group of figure-eight by graph-theoretic method. We have shown in Theorem 4.2.2 that fundamental group of figure-eight is not abelian.

For an alternative proof let G be a free group on two letters a and b. Define a graph $X = \text{Graph}\,(G, a, b)$ as follows:

The vertices of X are the elements of G. Hence the vertices are the reduced words a and b. The edges of X are of the two types: (g, ga) and (g, gb), $g \in G$. Again (g, ga), (g, gb), (ga^{-1}, g) and are the only four edges corresponding to the vertex g. Now define a map $G \times X \to X$, given by $h \cdot g = hg$, for every $g, h \in G$. Then $h \cdot (g, ga) = (hg, hga)$ and $h \cdot (g, gb) = (hg, hgb)$, for edges (g, ga) and (g, gb).

Let 1_G be the identity element in G. Then $1_G \cdot g = g$ and $1_G \cdot (g, ga) = (g, ga)$ and $1_G \cdot (g, gb) = (g, gb)$. Again $(h_1h_2) \cdot g = h_1h_2 \cdot g$ and $h_1 \cdot (h_2 \cdot g) = h_1 \cdot (h_2 \cdot g) = h_1h_2 \cdot g$, $\forall\, h_1, h_2 \in G$ (Since h_2g be a vertex in X).

$$\begin{aligned}\text{Also }\ h_1 \cdot (h_2 \cdot (g, ga)) &= h_1 \cdot (h_2g, h_2ga) = h_1(h_2g, h_1(h_2ga)) = (h_1(h_2)) \cdot (g, ga)\\ &= h_1(h_2 \cdot (g, gb)) = h_1(h_2g, h_2gb)\\ &= h_1(h_2g, h_1(h_2gb)) = h_1(h_2) \cdot (g, gb).\end{aligned}$$

Clearly, G acts on X. The orbit space X mod G is the Figure-Eight space whose two loops are the images of the edges (g, ga) and (g, gb). As X is simply connected, by using Theorem 4.10.8 it follows that $\pi_1(X \bmod G, *) \cong G$, which is the free group on two generators.

4.10.7 Application of Galois Correspondence

This subsection presents an interesting application of Galois correspondence.

Theorem 4.10.14 *Let (X, p) be a covering space of B. If B is connected, locally path-connected, and semilocally simply connected, then*

(a) *The components of X are in one-to-one correspondence with orbits of the action of $\pi_1(B, b_0)$ on the fiber $p^{-1}(b_0)$;*
(b) *Under the Galois correspondence between connected covering spaces of B and subgroups of $\pi_1(X, x_0)$, the subgroup corresponding to the component of X containing a given lift $\tilde{b}_0 = x_0$ of b_0 is the stabilizer group G_{x_0} of $x_0 \in X$, whose action on the fiber leaves x_0 fixed.*

Proof **(a)** Let $x_0, x_1 \in p^{-1}(b_0)$. If they are in different components of X, $\pi_1(X, x_0)$ cannot map one to the other, since there exists no path-connecting them. Claim that $\pi_1(X, x_0)$ acts transitively on each of the components of X to obtain a bijection. By hypothesis B is locally path-connected, hence X is locally path connected. Clearly, the notions of connected components and path-connected components are the same. If x_0 and x_1 lie in the same component, there exists a path $\alpha : I \to X$ such that $\alpha(0) = x_0$, $\alpha(1) = x_1$. Then $[p \circ \alpha]$ is an element of $\pi_1(B, b_0)$ whose action on $p^{-1}(b_0)$ maps x_0 to x_1. Hence this action is transitive. Then the set of elements of $p^{-1}(b_0)$ in a given component constitutes an orbit, and this produces a bijection.

(b) Choose a given lift x_0 of b_0 in some component X' of X. Under the Galois correspondence, the subgroup of $\pi_1(B, b_0)$ corresponding to X' is the image of $G = \pi_1(X', x_0)$ in the inclusion $p_* : G \to \pi_1(B, b_0)$. Any loop $\alpha \in p_*(G)$ lifts back to a loop in X' by the unique lifting property. Hence α sends x_0 to itself and is an element of the stabilizer group G_{x_0} of x_0.

Conversely, if $\beta \in \pi_1(B, b_0)$ is in the stabilizer group of x_0, then the lift $\tilde{\beta}$ of β is a loop from x_0 to itself and hence $\tilde{\beta} \in G$, which implies $\beta \in p_*(G)$. This shows that $p_*(G)$ is the stabilizer group of x_0. ❑

4.11 Exercises

1. Assume that $f : S^n \to \mathbf{R}^n$ is a continuous map such that $f(-x) = -f(x)$ for any $x \in S^n$. Show that there exists a point $x \in S^n$ such that $f(x) = 0$.
2. Assume that $f : S^n \to \mathbf{R}^n$ is a continuous map. Show that there exists a point $x \in S^n$ such that $f(x) = f(-x)$.
3. Prove that no subspace of $\mathbf{R}^n$ is homeomorphic to S^n.
4. Show that there is no continuous antipode-preserving map $f : S^2 \to S^1$. Use this result to prove Borsuk–Ulum theorem for dimension 2.
5. Let X and B be path-connected spaces and (X, p) be a covering space of B. Let $b_0 \in B$ and $Y = p^{-1}(b_0)$ be the fiber over b_0. Prove the following:

   (i) If $x_0, x_1 \in Y$, then $p_*\pi_1(X, x_0)$ and $p_*\pi_1(X, x_1)$ are conjugate subgroups of $\pi_1(B, b_0)$;

   (ii) If H is a subgroup of $\pi_1(X, x_0)$ which is conjugate to $p_*\pi_1(X, x_0)$ for some $x_0 \in Y$, then there exists a point $x_1 \in Y$ such that $H = p_*\pi_1(X, x_1)$. [Hint: Use Theorem 4.5.7.]

   (iii) A covering space (X, p) of B is said to be regular if $p_*\pi_1(X, x_0)$ is a normal subgroup of $\pi_1(B, b_0)$ for every $b_0 \in B$. If (X, p) is regular covering space of B, show that $p_*\pi_1(X, x_0) = p_*\pi_1(X, x_1)$ for every pair of point x_0, x_1 in the same fiber.

   (iv) If X is simply connected, prove that every covering space (X, p) of B is regular.

   (v) If $\pi_1(B, b_0)$ is abelian, then every covering space of B is regular.

6. Let B be a connected and locally path-connected space and let $b_0 \in B$. Then show that a covering space (X, p) of B is regular if and only if the group $\mathcal{A}ut(X/B)$ acts transitively on the fiber over b_0.

7. Let B be locally path-connected and $b_0 \in B$. Show that two covering spaces (X, p) and (Y, q) of B are isomorphic if and only if the fibers $p^{-1}(b_0)$ and $q^{-1}(b_0)$ are isomorphic $G = \pi_1(B, b_0)$-sets.

8. Let a group G act transitively as a set Y, and let $x, y \in Y$. Prove that $G_x = G_y$ if and only if there exists $f \in \mathcal{A}ut(Y)$ with $f(x) = y$.

9. Show that the graph X described in Sect. 4.10.6 has no cycles.

10. Let (X, p) be a covering space of B, where X is locally path-connected. Let $b_0 \in B$. Given $x_0, x_1 \in Y = p^{-1}(b_0)$, show that there exists an $h \in \text{Cov}\,(X/B)$ with $h(x_0) = x_1$ if and only if there exists $f \in \mathcal{A}ut(Y)$ with $f(x_0) = x_1$.

11. Let (X, p) be a covering space of B, where B is locally path-connected. Let $b_0 \in B$ and let the fiber $p^{-1}(b_0) = y$ be viewed as a $G = \pi_1(B, b_0)$-set. Then show that $\psi : \text{Cov}\,(X/B) \to \mathcal{A}ut(Y)$ defined by $\psi(h) = h|Y$ is isomorphism.

12. Let G be a group acting transitively on a set Y and let $y_0 \in Y$. Let $N_G(G_0)$ denote the normalizer of the isotropy group G_0 of y_0. Show that $\mathcal{A}ut(Y) \cong N_G(G_0)/G_0$.

13. Let (X, p) be a covering space of B, where B is locally path-connected. Show that for $b_0 \in B$ and $x_0 \in p^{-1(b_0)}$, $\mathcal{A}ut(X/B) \cong N_G(p_*\pi(X, x_0))/p_*\pi_1(X, x_0)$. Hence show that $\pi_1(S^1, 1) \cong \mathbf{Z}$.

14. Let (X, p) be a regular covering space of B, where B is locally path-connected. For $b_0 \in B$ and $x_0 \in p^{-1}(b_0)$, show that $\mathcal{A}ut(X/B) \cong \pi_1(B, b_0)/p_*\pi_1(X, x_0)$ by the monodromy group of the regular covering space.

15. Let (X, p) be a universal covering space of B, where B is locally path connected. Show that for any $b_0 \in B$, $\mathcal{A}ut(X/B) \cong \pi_1(B, b_0)$.

16. If B is an H-space, prove that every covering space of B is regular.
[Hint: $\pi_1(B, b_0)$ is abelian for $b_0 \in B \Rightarrow$ every covering space of B is regular.]

17. Let G be a path-connected topological group and H be a discrete normal subgroup of G. If $p : G \to G/H$ is the natural homomorphism, show that (G, p) is covering space of G/H.

18. Let (X, p) be a covering space of B and $b_0, b_1 \in B$. If F_0 and F_1 are the fibers over b_0 and b_1, respectively, show that $|F_0| = |F_1|$ and any two fibers of (X, p) a [Hint: Use Theorem 4.10.1(iii). Since each fiber is discrete and any two fibers have the same cardinal numbers, it follows that any two fibers are homeomorphic.]

19. Show that the map $p : S^1 \to S^1, z \mapsto z^2$ is a covering map. Generalize to the map $p : S^1 \to S^2, z \mapsto z^n$.

20. If $S^1 \to S^1$ is continuous and antipode preserving, show that f is not nullhomotopic.

21. Let B be a path-connected and locally path-connected space. Suppose (X, p) and (Y, q) are covering spaces of B. Let $b_0 \in B$, $x_0 \in X$ and $y_0 \in Y$ be base points with $p(x_0) = b_0 = q(y_0)$. If $q_*\pi_1(Y, y_0) \subset p_*\pi_1(X, x_0)$, show that

- **(i)** there exists a unique continuous map $f : (Y, y_0) \to (X, x_0)$ such that $p \circ f = q$;
- **(ii)** (Y, f) is a covering space of X and so X is a quotient space of Y.

22. Let X, B, Y be path-connected and locally path-connected spaces such that (X, p) is a covering space of B. If $x_0 \in X$, $y_0 \in Y$ and $b_0 \in B$ with $p(x_0) = b_0$, show that for every continuous map $f : (Y, y_0) \to (B, b_0)$ with $f_*\pi_1(Y, y_0) \subset$

$p_*\pi_1(B, b_0)$, there exists a continuous map $\tilde{f} : (Y, y_0) \to (X, x_0)$ such that $p \circ \tilde{f} = f$.

23. Let (X, p) be a covering space of B and $x_0 \in X$, $b_0 \in b$ such that $p(x_0) = b_0$. If X is simply connected, show that b_0 has a neighborhood U such that the inclusion map $i : U \hookrightarrow B$ induces the trivial homomorphism

$$i_* : \pi_1(U, b_0) \to \pi_1(B, b_0).$$

24. Let X be a normal space. Show that the inclusion $i : A \hookrightarrow X$ is a cofibration iff the inclusion $j : A \hookrightarrow U$ is a cofibration for some open neighborhood U of A in X.

25. Let $p : X \to B$ be a fibration and $f : A \to B$ be a continuous map. Show that there exists a bijection between the homotopy sets $C = [g : A \to X : p \circ g = f]$ and $D = [\tilde{g} : A \to X : p \circ \tilde{g} \simeq f]$.

26. Is the map $p : (0, 3) \to S^1,\ x \mapsto e^{2\pi i x}$ a covering map? Justify your answer.

27. Find nontrivial coverings of Möbius strip by itself.

28. Let B be path-connected, locally path-connected and $p : (X, x_0) \to (B, b_0)$ be a covering space. If H is the subgroup $p_*(\pi_1(X, x_0))$ of $\pi_1(B, b_0)$, show that

- **(i)** The automorphism group $\mathcal{A}ut(X/B)$ is isomorphic to the quotient group $N(H)/H$, where $N(H) = \{g \in \pi_1(B, b_0) : gHg^{-1} = H\}$ is the normalizer of H in $\pi_1(B, b_0)$.
- **(ii)** The group $\mathcal{A}ut(X/B)$ is isomorphic to the group $\pi_1(B, b_0)/H$ if X is a regular covering.
- **(iii)** If $p : (X, x_0) \to (B, b_0)$ is universal covering, then $\mathcal{A}ut(X/B) \cong \pi_1(B, b_0)$.

29. Let (X, p) be a universal covering space of a connected topological space B. If $b_0 \in B$ and $x_0 \in X$ are base points such that $x_0 \in p^{-1}(b_0)$, show that the induced homomorphism $p_* : \pi_1(X, x_0) \to \pi_n(B, b_0)$ is an isomorphism for $n \geq 2$. Hence show that $\pi_n(\mathbf{R}P^m) \cong \pi_n(S^m)$ for $n \geq 2$.

30. Let B be a path-connected space and X be a connected covering space of B. Let $p : X \to B$ be a covering projection. Let $b_0 \in B$ and $x_0 \in p^{-1}(b_0)$. Show that for every $n \geq 2$, $p_* : \pi_1(X, x_0) \to \pi_1(B, b_0)$ is an isomorphism. Hence show that for every $n \geq 2$, $\pi_n(S^1, 1) = 0$.

31. Let $f : A \to X$ be a continuous map and $i : A \to M_f$ be the inclusion $i(a) = [a, 0]$. Show that the inclusion $i : A \to M_f$ is cofibration.

[Hint. Use Steenrod theorem, Chap. 2.]

32. Let $p : X \to B$ be a fibration with fiber $F = p^{-1}(b_0)$ and B be path-connected. Let Y be any space. Show that the sequence of sets

$$[Y, F] \xrightarrow{i_*} [Y, X] \xrightarrow{p_*} [Y, B]$$

is exact.

33. Let $i : A \hookrightarrow X$ be a cofibration, with cofiber X/A and $q : X \to X/A$ denote the quotient map. If Y is any path-connected space, then show that the sequence of sets

$$[X/A, Y] \xrightarrow{q_*} [X, Y] \xrightarrow{i_*} [A, Y]$$

is exact.

4.12 Additional Reading

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