Chapter 7
Hamiltonian Systems

It is well known fact that Newton's equation of deterministic motion correctly describes the motion of a particle or a system of particles in an inertial frame. In Newtonian set up there is no chance for unpredictable nature of motion. On the other hand, sometimes the particle may be restricted in its motion so that it is forced to follow a specified path or some forces may act on the particles to keep them on the surface. Thus it is out of question to treat such cases using the Newtonian formalism. Besides, if the forces of constraints acting on a system are unknown to us in advance then the Newton's equation of motion remains undefined. To get over such situation, Lagrangian mechanics, introduced by renowned Italian mathematician Joseph Louis Lagrange in 1788, provides a technique of two kinds. In the first kind of Lagrange's formulation, Newton's equation of motion is solved by evaluating the forces of constraints using the constraint relations. But it is a tedious procedure. Moreover, Newton's equation of motion is applicable in an inertial frame only. The forces of constraints operating on a dynamical system regulate some of these coordinates to vary independently this means that all the coordinates which describe the configuration of a dynamical system moving under the forces of constraints may not necessarily be independent. Consequently, the resulting equations of motion are not independent. So, to describe the configuration of the dynamical system and also to obtain a general equation of motion valid in any coordinate system, a set of independent coordinates is required. This gives a general equation of motion which is known as Lagrange's equation of motion. It is valid in any coordinate system, and the knowledge of constraint forces is not necessary for its derivation instead the knowledge of work, energy and principle of virtual work are needed. Thus, Lagrangian's method can provide a much fresher way of solving some physical systems compared to Newtonian mechanics, in particular for the system moving under some constraints. Thus Lagrangian mechanics is a reformulation of classical mechanics in terms of arbitrary coordinates.

In Lagrangian mechanics, the Lagrange's equation of motion is a second order differential equation, where the Lagrangian variables are the generalized coordinates and generalized velocities with time t as parameter. Apart from Lagrangian formulation there is another formulation in terms of Hamiltonian function. The corresponding dynamics is called Hamiltonian dynamics named after the famous Scottish mathematician Sir William Rowan Hamilton (1805–1865). Hamilton

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originated this formulation of classical mechanics in 1833 which is applicable to a holonomic system described by a set of generalized coordinates. Hamiltonian mechanics is founded on the basis of Lagrangian formulation where the basic variables are the generalized coordinates and the generalized momenta. This reformulation provides a deeper understanding of the equations of motion of a dynamical system compared to the Lagrangian formulation and makes possible to write the equation of motion in a very stylish, yet simple way. The main beneficial thing about this formulation is that rather than providing a more convenient way of solving a particular problem, Hamiltonian mechanics gives a deeper understanding of the general structure of classical mechanics. Also, it makes clear its relationship with the quantum mechanics and other related areas of science. In this chapter we shall learn the basics of Lagrangian and Hamiltonian mechanics, and also Hamiltonian flows in the phase space, symplectic transformations and Hamilton–Jacobi equation.

7.1 Generalized Coordinates

The position of a point in space is generally specified by its position vector with respect to a fixed set of coordinate system or by the help of three Cartesian coordinates (x, y, z) of that point. Generally, the positions of N points are determined by N vectors or by $3N$ Cartesian coordinates. But the position of a system can be determined not only by using Cartesian coordinates, but there also exists alternative coordinates systems or alternative parameters by which one can determine the position of a system completely at any time t. These coordinates are called the generalized coordinates. Therefore generalized coordinates are the independent coordinates which completely specify or describe the configuration of a dynamical system at any given time. Now if we consider a set of quantities say, $q_1, q_2, \ldots, q_n$, defining the position of a dynamical system as generalized coordinates of the system then the set of their first order derivatives $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$ are called as generalized velocities. Any set of parameters which gives the representation of the configuration of a dynamical system without any ambiguity can serve the purpose of generalized coordinates. So one can use angles, axes, moments or any set of parameters as the generalized coordinates. But one should be careful while making choice of generalized coordinates as it is totally dependent on skill. Correct choice of generalized coordinates; make the problem look easy while the problem becomes difficult to handle for a wrong choice of the generalized coordinates. Some examples of generalized coordinates are as follows:

(i) For a simple pendulum of length l, the generalized coordinate is the angular displacement θ from the vertical.
(ii) For a spherical pendulum of fixed length l, the generalized coordinates are θ, ϕ; θ, ϕ being the spherical polar coordinates.

(iii) Consider a rod lying on a plane surface. The generalized coordinates are x, y, θ, where (x, y) are the coordinates of one end of the rod and θ is the angle between x-axis and the rod.

(iv) Consider a lamina lying in a plane. For this case, the generalized coordinates are x, y, θ, where (x, y) are the coordinates of the centroid and θ is the angle made by a line fixed in the plane.

So far we have defined the generalized coordinate and the generalized velocities. The generalized momentum is the product of mass and the generalized velocity. If p_j is the generalized momentum of the jth particle whose mass is m_j and generalized velocity is $\dot{q}_j$ then $p_j = m_j\dot{q}_j$.

7.1.1 Configuration and Phase Spaces

The configuration of a dynamical system is described instantaneously by the generalized coordinates. The n-generalized coordinates $q_1, q_2, \ldots, q_n$ correspond to a particular point in the n-dimensional space. The n-dimensional space spanned by these n-generalized coordinates of a dynamical system is called the configuration space of that system. The state of the system changes with time and the system point traces out a curve in moving through the configuration space. The curve traces out by the system point is known as trajectory or the path of the motion of the dynamical system. On the other hand phase space is generally a $2n$-dimensional space spanned by n generalized coordinates and n generalized momenta where the qualitative behavior of a dynamical system is represented geometrically. A $2n$-dimensional space spanned by n generalized coordinates and n generalized momenta of a dynamical system is called the phase space of that dynamical system. At any instant of time a point in a phase space is called the phase point. As the dynamical system evolves with time the phase point moves through the phase space thereby traced a path, known as phase curve. When one additional dimension in terms of time t is added to the phase space then the phase space is a $(2n + 1)$-dimensional space which is called as state space.

For instance, Hamiltonian system which does not depend on time t explicitly is a $2n$-dimensional phase space. The axes of a Hamiltonian system give the values of generalized coordinate q and generalized momentum p. Hamiltonian of such systems are conserved quantities and gives the energy of the system. The trajectories of Hamiltonian system therefore can go only to those regions of phase space where the energy of the system remains same as to the initial point of the trajectory. The trajectories of a Hamiltonian system are thus confined to a $2n - 1$-dimensional constant energy surface.

7.2 Classification of Systems

A dynamical system is said to be holonomic if it is possible to give arbitrary and independent variations to the generalized coordinates without breaching the constraint relations. Otherwise it is called nonholonomic system. In mechanics the constraint is very important and can be found in Sommerfeld [1], Goldstein [2] and Arnold [3].

More specifically, a system is said to be holonomic if it contains only the holonomic constraints. If there is any nonholonomic constraint then the system is called nonholonomic. For instance, if $q_1, q_2, \ldots, q_n$ be the n generalized coordinates of a dynamical system then for a holonomic system it is possible to change q_r to $(q_r + \delta q_r)$ without changing the other coordinates.

Again consider two particles of masses m_1 and m_2 connected by a string of length l moving in space. If $\vec{r}_1$ and $\vec{r}_2$ are the position vectors of masses at time t then clearly we have $|\vec{r}_2 - \vec{r}_1| \leq l$ or $l^2 - (\vec{r}_2 - \vec{r}_1)^2 \geq 0$. In this case the system is a holonomic system with unilateral constraint. If the conditions of the constraints are expressed by means of non-integrable relations of the following form $a_m \delta t + \sum_{j=1}^{n} a_{jm} \delta q_j = 0$ for m = 1,2, …, $k(< n)$ where a's are functions of generalized coordinates then the system is called a non-holonomic dynamical system.

Example 7.1 Examine whether the motion of a vertical wheel on a horizontal plane is holonomic or non holonomic.

Solution Consider the motion of a vertical wheel of radius a rolling on a perfectly rough horizontal plane specified by the coordinate axes O_x, O_y. The contact point P traces out some curve C on the xy-plane. If θ be the angle of rotation of the wheel when the contact point P has travelled a distance s (measured from P_o) along the curve then $s = a\theta$ (assuming that the wheel rolls without sliding). Now, $\delta s = a\,\delta\theta$ (Fig. 7.1).

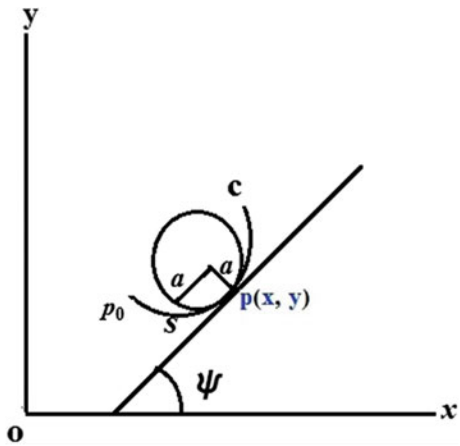


Fig. 7.1 Motion of a vertical wheel on a horizontal plane

If the coordinates of P are (x, y) and if the tangent at P makes an angle ψ with O_x then

$$\delta x = \cos\psi\,\delta s = a\cos\psi\,\delta\theta$$

and

$$\delta y = \sin\psi\,\delta s = a\sin\psi\,\delta\theta$$

The coordinates (x, y, θ, ψ) form a nonholonomic system.

7.2.1 Degrees of Freedom

The degrees of freedom of a system are the minimum number of generalized coordinates needed to describe the configuration of a system or to specify the exact position of an object of that system. In other words, the minimum number of independent parameters necessary to describe the configuration of the dynamical system at any time is called the degrees of freedom.

We shall now give some examples of degrees of freedom which will help in understanding the idea more clearly.

Example 7.2 If a system is made up of N particles, we need $3N$ coordinates to specify the positions of all the particles of the system. If a system of N particles are subjected to C constraints (i.e. if some of the particles are connected by C relations), there will be ($3N$-C) number of independent coordinates only. So, the number of degrees of freedom is ($3N$-C).

Example 7.3 If a point mass is constrained to move in a plane (two dimensions) the number of spatial coordinates necessary to describe its motion is two. So the degrees of freedom in this case are two.

Example 7.4 Consider a particle moving on a surface $x^2 + y^2 + z^2 = a^2$. In this case the degrees of freedom is 2, though the degrees of freedom in 3-dimensional Cartesian coordinate system is 3. Again if the particle is inside the sphere (i.e., $x^2 + y^2 + z^2 - a^2 < 0$), then the degrees of freedom is 3.

Example 7.5 Consider a system of three free objects. The system has 9 degrees of freedom. If by imposing some constraints the free spaces between the objects are fixed, then the number of degrees of freedom of the system will be 9 − 3 = 6. These six degrees of freedom can be chosen in any way. For example, the three coordinates of the centre of mass with the 3 angles of their inclinations to a fixed frame of reference.

The set of coordinates used to describe a system can be selected freely, keeping in mind that, the number of coordinates minus the number of constraints must give the number of degrees of freedom for that system.

7.2.1.1 Some Important Features of the Degrees of Freedom of a System

1. The number of degrees of freedom is independent of the choice of coordinate system.
2. The number of coordinates and number of constraints do not have to be the same for all possible choices.
3. There are freedoms of choices of origin, coordinate system.

7.3 Basic Problem with the Constraints

The most fundamental problem associated with the forces of constraints is that they are unknown beforehand. So, in the absence of knowledge of the total force acting on the system, it is impossible to solve Newton's equation of motion which is a relation between the total force and the acquired acceleration. The total force is the sum of the externally applied force and the force of constraints. Let us try to overcome this situation.

Consider the motion of a particle of mass m under the velocity dependent (nonholonomic) constraint

$$g\left(\vec{r}, \dot{\vec{r}}, t\right) = 0 \tag{7.1}$$

Let $\vec{f}^{(a)}$ and $\vec{f}$ be the externally applied forces and constraint forces respectively acting on the particle. So, the total force acting on the particle is given by $\vec{F} = \vec{f}^{(a)} + \vec{f}$. The Newton's equation of motion therefore becomes

$$m\ddot{\vec{r}} = \vec{F} = \vec{f}^{(a)} + \vec{f} \tag{7.2}$$

The numbers of equations are four whereas the numbers of unknowns are six. Therefore, the problem does not possess unique solution. To obtain a unique solution one needs additional constraint relations. The search for additional relations gives rise to Lagranges's equations of motion of the first kind. We shall now give the derivation of Lagrange's equation of first kind.

7.3.1 Lagrange Equation of Motion of First Kind

Consider a holonomic, bilateral constraint given by

$$g(\vec{r}, t) = 0 \tag{7.3}$$

Differentiating Eq. (7.3) with respect to time t we get,

$$\frac{\partial g}{\partial t} + \frac{\partial g}{\partial \vec{r}} \cdot \dot{\vec{r}} = 0 \tag{7.4}$$

Again differentiating Eq. (7.4) with respect to time 't' we get,

$$\frac{\partial^2 g}{\partial t^2} + \frac{\partial^2 g}{\partial \vec{r} \partial t} \cdot \dot{\vec{r}} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial g}{\partial \vec{r}}\right) \cdot \dot{\vec{r}} + \frac{\partial g}{\partial \vec{r}} \cdot \ddot{\vec{r}} = 0 \tag{7.5}$$

The above constraint relation on the total acceleration ($\ddot{\vec{r}}$) is therefore directly affected by the vector $\frac{\partial g}{\partial \vec{r}}$. Only the component of acceleration (hence the force) parallel to the vector $\frac{\partial g}{\partial \vec{r}}$ enters the above constraint relation due to scalar nature of the product $\frac{\partial g}{\partial \vec{r}} \cdot \ddot{\vec{r}}$. In other words, $\vec{f}$ must be parallel to $\frac{\partial g}{\partial \vec{r}}$, that is

$$\vec{f} = \lambda \frac{\partial g}{\partial \vec{r}} \tag{7.6}$$

where λ is a scalar.

Let us consider a nonholonomic, bilateral constraint of the form $g\left(\vec{r}, \dot{\vec{r}}, t\right) = 0$ and taking time derivative we get, $\frac{\partial g}{\partial t} + \frac{\partial g}{\partial \vec{r}}.\dot{\vec{r}} + \frac{\partial g}{\partial \dot{\vec{r}}}.\ddot{\vec{r}} = 0$.

Arguing as above, one can get,

$$\vec{f} = \lambda \frac{\partial g}{\partial \dot{\vec{r}}}. \tag{7.7}$$

Since, $g(\vec{r}, t) = 0$ or, $g\left(\vec{r}, \dot{\vec{r}}, t\right) = 0$ is given, hence $\vec{f}$ is known except for λ.

Now there are four unknowns and four independent equations which can give simultaneous solution for $\vec{r}$. Hence $\vec{f}$ can be uniquely specified along with λ. Newton's equations of motion now take the following form:

For holonomic one particle system

$$m\ddot{\vec{r}} - \vec{f}^{(a)} - \lambda \frac{\partial g(\vec{r}, t)}{\partial \vec{r}} = 0, \tag{7.8}$$

and for nonholonomic system

$$m\ddot{\vec{r}} - \vec{f}^{(a)} - \lambda \frac{\partial g\left(\vec{r}, \dot{\vec{r}}, t\right)}{\partial \dot{\vec{r}}} = 0. \tag{7.9}$$

The Eq. (7.8) or (7.9) is sometimes called Lagrange's equation of motion of the first kind and λ is called Lagrange's multiplier.

This can easily be generalized for the motion of a system of N particles having k bilateral constraints, viz.

$$g_i(\vec{r}, t) = 0, \; i = 1, 2, \ldots, k \quad \text{(for holonomic system)}$$

and

$$g_i\left(\vec{r}, \dot{\vec{r}}, t\right) = 0, \; i = 1, 2, \ldots, k \quad \text{(for nonholonomic system).}$$

Thus Lagrange's equations of motion of the first kind for the jth particle having mass m_j become

$$m_j\ddot{\vec{r}}_j - \vec{f}_j^{(a)} - \sum_{i=1}^{k} \lambda_i \frac{\partial g_i(\vec{r}_j, t)}{\partial \vec{r}_j} = 0, \; j = 1, 2, \ldots, N \quad \text{(for holonomic system)} \tag{7.10}$$

$$m_j\ddot{\vec{r}}_j - \vec{f}_j^{(a)} - \sum_{i=1}^{k} \lambda_i \frac{\partial g_i\left(\vec{r}_j, \dot{\vec{r}}_j, t\right)}{\partial \dot{\vec{r}}_j} = 0, \; j = 1, 2, \ldots, N \quad \text{(for nonholonomic system)} \tag{7.11}$$

$\vec{f}_j^{(a)}$ being the total externally applied force on the jth particle of the system.

These vector equations for holonomic and nonholonomic systems can be applied to the systems containing scleronomic or rheonomic bilateral constraints forms. The total number of scalar equations is $3N + k$ ($3N$ equations for motion and k number of constraints). The total number of unknowns are $3N + k$ ($3N$ components for $\vec{r}$ and k number of unknown λ). Since these equations are coupled, so obtaining solutions of these equations become rather complicated. So, Lagrange's equations of motion of the first kind are of little help and find a few applications in practice. But if solved then the solution provides the complete description of the dynamical problems of diversified nature.

Let us now show that the order of differentiation is immaterial in Lagrange's equation of motion.

Suppose that the dynamical system be comprised of N particles of masses $m_i (i = 1, 2, \ldots, N)$. Let $\vec{r}_i$ be the position vector of the ith particle having mass m_i. The position of the system at time t is specified by n generalized coordinates denoted by $q_1, q_2, \ldots, q_n$. Then each $\vec{r}_i$ is a function of $q_1, q_2, \ldots, q_n$ and time t, that is $\vec{r}_i = \vec{r}_i\,(q_1, q_2, \ldots, q_n)$.

Time derivative of the generalized coordinate q_i is called the generalized velocity of the ith particle. The velocity of the ith particle is given by

$$\dot{\vec{r}} = \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}$$

Differentiating again this equation with respect to generalized coordinate 'q_j' we have,

$$\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j} \text{for } j = 1, 2, \ldots, n$$

Again,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) = \frac{\partial \dot{\vec{r}}_i}{\partial q_j}$$

This proves that the order of differentiation with respect to 't' and 'q_j' are immaterial.

7.3.2 Lagrange Equation of Motion of Second Kind

Let the system contains N particles of masses $m_i (i = 1, 2, \ldots, N)$. The position of the system at time t is specified by n generalized coordinates $q_1, q_2, \ldots, q_n$. If $\vec{r}_i$ be the position vector of the ith mass then

$$r_i = \vec{r}_i(q_1, q_2, \ldots, q_n)(i = 1, 2, \ldots, N)$$

From the generalized D'Alembert's principle we have

$$\sum_{i=1}^{N} \left(\vec{\mathrm{F}}_\mathrm{i} - m_i \ddot{\vec{r}}_i\right) . \delta \vec{r}_i = 0 \tag{7.12}$$

where $\vec{F}_i$'s being the external forces acting on the system and $\delta \vec{r}_i$'s are the small instantaneous virtual displacements consistent with the constraints.

From Eq. (7.12) we have,

$$\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i . \delta \vec{r}_i = \sum_{i=1}^{N} \vec{\mathrm{F}}_\mathrm{i} . \delta \vec{r}_i \tag{7.13}$$

Since $\vec{r}_i = \vec{r}_i(q_1, q_2, \ldots, q_n)$, $\delta\vec{r}_i = \sum_{e=1}^{n} \frac{\partial \vec{r}_i}{\partial q_e}\delta q_e$.

Then $\delta w = \sum_{i=1}^{N} \vec{\mathrm{F}}_\mathrm{i}.\delta\vec{r}_i = \sum_{i=1}^{N} \vec{\mathrm{F}}_\mathrm{i}.\left(\sum_{e=1}^{n} \frac{\partial \vec{r}_i}{\partial q_e}\right)\delta q_e = \sum_{e=1}^{n}\left(\sum_{i=1}^{N} \vec{\mathrm{F}}_\mathrm{i}.\frac{\partial \vec{r}_i}{\partial q_e}\right)\delta q_e = \sum_{e=1}^{n} Q_e \delta q_e$

where $Q_e = \frac{\partial w}{\partial q_e}$ being the generalized force associated with the generalized coordinates $q_e, (e = 1, 2, \ldots, n)$.

Now,

$$\begin{aligned}
\sum_{i=1}^{N} \mathrm{m_i}\ddot{\vec{r}}_i.\delta\vec{r}_i &= \sum_{i=1}^{N} m_i\ddot{\vec{r}}_i.\left(\sum_{e=1}^{n}\frac{\partial \vec{r}_i}{\partial q_e}\delta q_e\right) = \sum_{e=1}^{n}\left(\sum_{i=1}^{N}\mathrm{m_i}\ddot{\vec{r}}_i.\frac{\partial \vec{r}_i}{\partial q_e}\right)\delta q_e \\
&= \sum_{e=1}^{n}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{N}\mathrm{m_i}\dot{\vec{r}}_i.\frac{\partial \vec{r}_i}{\partial q_e}\right) - \sum_{i=1}^{N}\mathrm{m_i}\dot{\vec{r}}_i.\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \vec{r}_i}{\partial q_e}\right)\right]\delta q_e \\
&= \sum_{e=1}^{n}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{N}\mathrm{m_i}\dot{\vec{r}}_i.\frac{\partial \vec{r}_i}{\partial q_e}\right) - \sum_{i=1}^{N}\mathrm{m_i}\dot{\vec{r}}_i.\frac{\partial}{\partial q_e}\left(\frac{\mathrm{d}\vec{r}_i}{\mathrm{d}t}\right)\right]\delta q_e \\
&= \sum_{e=1}^{n}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{N}\mathrm{m_i}\dot{\vec{r}}_i.\frac{\partial \vec{r}_i}{\partial q_e}\right) - \sum_{i=1}^{N}\mathrm{m_i}\dot{\vec{r}}_i.\frac{\partial \dot{\vec{r}}_i}{\partial q_e}\right]\delta q_e
\end{aligned}$$

T = Kinetic Energy of the system $= \frac{1}{2}\sum_{i=1}^{N} m_\mathrm{i}\dot{\vec{r}}_i^2$

Therefore, $\frac{\partial T}{\partial \dot{q}_e} = \sum_{i=1}^{n} m_i\dot{\vec{r}}_i.\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_e} = \sum_{i=1}^{n} m_i\dot{\vec{r}}_i.\frac{\partial \vec{r}_i}{\partial q_e}$ (since, $\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_e} = \frac{\partial \vec{r}_i}{\partial q_e}$).

$$\frac{\partial T}{\partial q_e} = \sum_{i=1}^{N}\mathrm{m_i}\dot{\vec{r}}_i.\frac{\partial \dot{\vec{r}}_i}{\partial q_e}$$

Thus, $\sum_{i=1}^{N} m_i\ddot{\vec{r}}_i.\delta\vec{r}_i = \sum_{e=1}^{n}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_e}\right) - \frac{\partial T}{\partial q_e}\right]\delta q_e$.

Substituting the above expression into Eq. (7.13) and transferring all the terms in one side we have,

$$\sum_{e=1}^{n}\left[Q_e - \left\{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_e}\right) - \frac{\partial T}{\partial q_e}\right\}\right]\delta q_e = 0 \tag{7.14}$$

Case (i) System with n degrees of freedom

In this case the coordinates are free coordinates and can be varied arbitrarily. So, the coefficients of each δq_e must vanish separately, giving

$$Q_e - \left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_e}\right) - \frac{\partial T}{\partial q_e}\right\} = 0 \quad \text{for} \quad e = 1, 2, \ldots, n$$

or,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_e}\right) - \frac{\partial T}{\partial q_e} = Q_e \quad \text{for} \quad e = 1, 2, \ldots, n \tag{7.15}$$

These equations are second order differential equations and are called as the Lagrange's equations of motion of a dynamical system with n degrees of freedom.

If in addition, the system is conservative then $Q_e = -\frac{\partial V}{\partial q_e}$ where $V = V(q_e, t)$ is the potential function.

Substituting the value for Q_e in Eq. (7.15) we have,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_e}\right) - \frac{\partial T}{\partial q_e} = -\frac{\partial V}{\partial q_e}$$

If we assume that V is independent of the generalized velocity $\dot{q}_e$, then we can write the above equation as $\frac{d}{dt}\left\{\frac{\partial (T-V)}{\partial \dot{q}_e}\right\} - \frac{\partial (T-V)}{\partial q_e} = 0$.

If we set $L = T - V$, known as Lagrangian of the system then Lagrange's equations of motion can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_e}\right) - \frac{\partial L}{\partial q_e} = 0, e = 1, 2, \ldots, n$$

Note that if the system contains some forces derivable from a potential function and some other forces not derivable from a potential function then the Lagrange's equation of motion can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_e}\right) - \frac{\partial L}{\partial q_e} = Q_e^{/}, e = 1, 2, \ldots, n \tag{7.16}$$

where all the potential forces have been included in the Lagrangian L and the non-potential forces are given by $Q_e^{/}$.

Case (ii) Holonomic dynamical system with k bilateral constraints

For a holonomic dynamical system with k bilateral constraints, the generalized coordinates are connected by k independent relations of the following form:

$$f_j(q_1, q_2, \ldots, q_n, t) = 0,\ j = 1, 2, \ldots, k(k < n) \tag{7.17}$$

Let us now consider a virtual change of the system at time t consistent with the constraints in which the coordinates $q_1, q_2, \ldots, q_n$ are changed to

$q_1 + \delta q_1, q_2 + \delta q_2, \ldots, q_n + \delta q_n$. Therefore, from Eq. (7.17) we have, $f_j(q_1 + \delta q_1, q_2 + \delta q_2, \ldots, q_n + \delta q_n, t) = 0$, this can be expanded in a series like

$$f_j(q_1, q_2, \ldots, q_n, t) + \sum_{e=1}^{n} \frac{\partial f_j}{\partial q_e} \delta q_e + O(\delta q_e)^2 = 0$$

Since changes δq_e are small we have

$$\sum_{e=1}^{n} \frac{\partial f_j}{\partial q_e} \delta q_e = 0 \quad \text{for} \quad j = 1, 2, \ldots, k (k < n) \tag{7.18}$$

It is evident from (7.18) that the changes $\delta q_1, \delta q_2, \ldots, \delta q_k$ are not independent. We now introduce k arbitrary parameters $\lambda_1, \lambda_2, \ldots, \lambda_k$. We now multiply the Eq. (7.18) by these k parameters and sum up to obtain $\sum_{j=1}^{k} \lambda_j \sum_{e=1}^{n} \frac{\partial f_j}{\partial q_e} \delta q_e = 0$ or,

$$\sum_{e=1}^{n} \sum_{j=1}^{k} \left(\lambda_j \frac{\partial f_j}{\partial q_e} \right) \delta q_e = 0 \tag{7.19}$$

Adding Eq. (7.19) to the Eq. (7.14), we get

$$\sum_{e=1}^{n} \left[Q_e - \left\{ \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_e} \right) - \frac{\partial T}{\partial q_e} \right\} + \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_e} \right] \delta q_e = 0 \tag{7.20}$$

Now choose the parameters $\lambda_1, \lambda_2, \ldots, \lambda_k$ in such a way that the coefficients of $\delta q_1, \delta q_2, \ldots, \delta q_k$ vanish separately. This gives

$$Q_e - \left\{ \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_e} \right) - \frac{\partial T}{\partial q_e} \right\} + \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_e} = 0 \quad \text{for} \quad e = 1, 2, \ldots, k\,(<n)$$

or,

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_e} \right) - \frac{\partial T}{\partial q_e} = Q_e + \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_e} \text{for } e = 1, 2, \ldots, k(<n) \tag{7.21}$$

Now Eq. (7.19) takes the following form

$$\sum_{e=k+1}^{n} \left[Q_e - \left\{ \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{q}_e} \right) - \frac{\partial T}{\partial q_e} \right\} + \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_e} \right] \delta q_e = 0 \tag{7.22}$$

Since the variations $\delta q_{k+1}, \delta q_{k+2}, \ldots, \delta q_n$ are arbitrary and independent we must have

$$Q_e - \left\{ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_e}\right) - \frac{\partial T}{\partial q_e} \right\} + \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_e} = 0 \quad \text{for} \quad e = k+1, k+2, \ldots, n$$

or,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_e}\right) - \frac{\partial T}{\partial q_e} = Q_e + \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_e} \quad \text{for} \quad e = k+1, k+2, \ldots, n. \tag{7.23}$$

The Eqs. (7.20) and (7.22) together give

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_e}\right) - \frac{\partial T}{\partial q_e} = Q_e + \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_e} \quad \text{for} \quad e = 1, 2, \ldots, k, k+1, k+2, \ldots, n \tag{7.24}$$

These are the Lagrange's equations of motion for a holonomic dynamical system with k bilateral constraints.

These equations have $(n + k)$ unknown quantities $q_1, q_2, \ldots, q_n; \lambda_1, \lambda_2, \ldots, \lambda_k$. In order to solve the n number of equations given by (7.24) we have to supply k equations of constraints.

Case (iii) Nonholonomic dynamical system

In this case the changes $\delta q_1, \delta q_2, \ldots, \delta q_k$ are connected by k-nonintegrable relations of the following form

$$a_j\, \delta t + \sum_{e=1}^{n} a_{je} \delta q_e = 0 \quad \text{for} \quad j = 1, 2, \ldots, k(<n) \tag{7.25}$$

where a_j's are the functions of the coordinates.

For virtual changes at time t,

$$\sum_{e=1}^{n} a_{je} \delta q_e = 0 \quad \text{for}\, j = 1, 2, \ldots, k(<n) \tag{7.26}$$

From Eq. (7.25) it is clear that the changes $\delta q_1, \delta q_2, \ldots, \delta q_k$ are not independent. We now multiply Eq. (7.25) by k arbitrary parameters $\lambda_j (j = 1, 2, \ldots, k)$ and sum up to obtain

$$\sum_{j=1}^{k} \lambda_j \sum_{e=1}^{n} a_{je} \delta q_e = 0 \quad \text{or,} \quad \sum_{e=1}^{n} \left(\sum_{j=1}^{k} \lambda_j a_{je} \right) \delta q_e = 0 \tag{7.27}$$

Adding Eq. (7.27) with the (7.14), we get

$$\sum_{e=1}^{n}\left[Q_e-\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_e}\right)-\frac{\partial T}{\partial q_e}\right\}+\sum_{j=1}^{k}\lambda_j a_{je}\right]\delta q_e=0 \tag{7.28}$$

Let us choose $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that the coefficients $\delta q_1, \delta q_2, \ldots, \delta q_k$ vanish separately.

This gives $Q_e-\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_e}\right)-\frac{\partial T}{\partial q_e}\right\}+\sum_{j=1}^{k}\lambda_j a_{je}=0$ for $e=1,2,\ldots,n$

$$\text{or, } \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_e}\right)-\frac{\partial T}{\partial q_e}=Q_e+\sum_{j=1}^{k}\lambda_j a_{je} \text{ for } e=1,2,\ldots,n \tag{7.29}$$

Equations (7.29) are the Lagrange's equations of motion for nonholonomic dynamical system.

The equations of constraints are added in the modified form:

$$a_j+\sum_{e=1}^{n}a_{je}\dot{q}_e=0 \quad \text{for} \quad j=1,2,\ldots,k(<n) \tag{7.30}$$

If $Q_e=-\frac{\partial V}{\partial q_e}$ and $L=T-V$ then we have,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_e}\right)-\frac{\partial L}{\partial q_e}=\sum_{j=1}^{k}\lambda_j a_{je}$$

7.3.2.1 Physical Significance of λ's

Let us suppose that we remove the constraints of the system and instead of constraints let us apply external forces Q'_e in such a manner so as to keep the motion of the system unchanged. Clearly, the extra applied forces must be equal to the forces of constraints. Then under the influence of these forces Q'_e the equations of motion are $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_e}\right)-\frac{\partial L}{\partial q_e}=Q'_e$ for $e=1,2,\ldots,n$.

But this must be identical with $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_e}\right)-\frac{\partial L}{\partial q_e}=\sum_{j=1}^{k}\lambda_j a_{je}$ for $e=1,2,\ldots,n$.

Hence one can identify $\sum_{j=1}^{k}\lambda_j a_{je}$ with Q'_e, the generalized forces of constraints.

7.3.2.2 Cyclic Coordinates (Ignorable Coordinates)

If a coordinate is explicitly absent in the Lagrangian function L of a dynamical system then the coordinate is called a cyclic or ignorable coordinate. Thus if q_k is a cyclic coordinate, then the form of the Lagrangian is $L \equiv L(\dot{q}_k, t)$ and $\frac{\partial L}{\partial q_k} = 0$. The cyclic co-ordinate is very important in deriving Hamilton's equations of motion.

Example 7.6 Find the Lagrange's equation of motion of a simple pendulum

Solution The generalized coordinate of a simple pendulum of length l is the angle variable θ. The velocity of the ball is $l\dot{\theta}$ where l is the length of the string of the pendulum. A simple pendulum oscillating in a vertical plane constitutes a conservative holonomic dynamical system with one degree of freedom.

Here, Kinetic Energy $= T = \frac{1}{2}ml^2\dot{\theta}^2$, m is the mass of the ball.

Potential Energy $= V = mgh = mgl(1 - \cos\theta)$.

Therefore, Lagrangian of the system $= L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)$.

Lagrange's equation of motion is $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$ or, $ml^2\ddot{\theta} + mgl\sin\theta = 0$

or, $ml^2\ddot{\theta} = -mgl\sin\theta$ or, $\ddot{\theta} = -\frac{g}{l}\sin\theta \simeq -\frac{g}{l}\theta$ (if the amplitude of oscillation is small then θ is small and so $\sin\theta \simeq \theta$).

Time period is given by $2\pi\sqrt{\frac{l}{g}}$, g is the acceleration due to gravity.

Example 7.7 For a dynamical system Lagrangian is given by $L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z) + A\dot{x} + B\dot{y} + C\dot{z}$ where A, B, C are functions of (x, y, z). Show that Lagrange's equations of motion are $\ddot{x} + \dot{y}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) + \dot{z}\left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right) + \frac{\partial V}{\partial x} = 0$ and similar ones.

Solution The generalized coordinates for the given dynamical system are x, y, z.

Now the Lagrange's equation of motion corresponding to x-coordinate is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

or, $\frac{\mathrm{d}}{\mathrm{d}t}(\dot{x} + A) - \left[-\frac{\partial V}{\partial x} + \frac{\partial A}{\partial x}\dot{x} + \frac{\partial B}{\partial x}\dot{y} + \frac{\partial C}{\partial x}\dot{z}\right] = 0$

or, $\ddot{x} + \frac{\mathrm{d}A}{\mathrm{d}t} + \frac{\partial V}{\partial x} - \frac{\partial A}{\partial x}\dot{x} - \frac{\partial B}{\partial x}\dot{y} - \frac{\partial C}{\partial x}\dot{z} = 0$

or, $\ddot{x} + \dot{y}\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) + \dot{z}\left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right) + \frac{\partial V}{\partial x} = 0$ since $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\partial A}{\partial x}\dot{x} + \frac{\partial A}{\partial y}\dot{y} + \frac{\partial A}{\partial z}\dot{z}$.

In an analogous way one can obtain the other two Lagrange's equations of motion corresponding to y and z coordinates.

Example 7.8 Obtain the Lagrangian and also the Lagrange's equation of motion of a harmonic oscillator.

Solution A harmonic oscillator consists of a single particle of mass m moving in a straight line which can be taken as x-axis (see Fig. 7.2). The particle is attracted

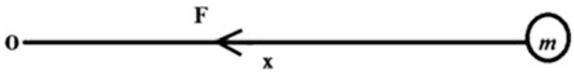


Fig. 7.2 Sketch of a harmonic oscillator

towards the origin by a force which varies proportionally with the distance of the particle from the origin.

Then the Kinetic Energy of the harmonic oscillator is given by $T = \frac{1}{2}m\dot{x}^2$ whereas the Potential Energy is given by $V = \frac{1}{2}kx^2$, k being a constant.

Then the Lagrangian of the motion is $L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$.

Lagrange's equation of motion is $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$ or, $m\ddot{x} + kx = 0$.

Example 7.9 Find the Lagrangian of motion of a particle of unit mass moving in a central force field under a force that varies inversely as the square of the distance from the centre O (Fig. 7.3).

Solution Here r, θ are the generalized coordinates. Let V be the potential.

Then, $-\frac{\mathrm{d}V}{\mathrm{d}r} = F = -\frac{\mu}{r^2}$ which on integration gives $V = -\frac{\mu}{r} + \text{Constant}$ $T =$ Kinetic Energy of the particle = $\frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$.

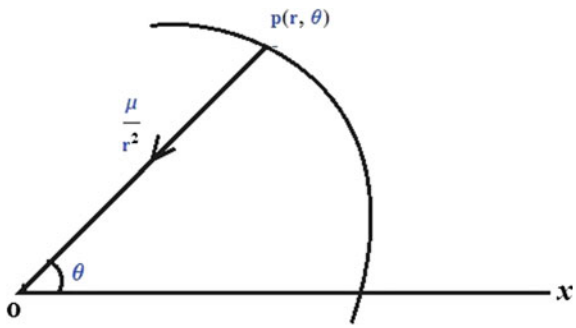
The Lagrangian of the motion is $L = T - V = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{\mu}{r} - \text{Constant}$.

Now, Lagrange's equation of motion is given by $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_e}\right) - \frac{\partial L}{\partial q_e} = 0, e = 1, 2$

Here, $q_1 = r, q_2 = \theta$.

Lagrange's equation of motion corresponding to r-coordinate gives $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$ or, $\frac{\mathrm{d}}{\mathrm{d}t}(\dot{r}) - r\dot{\theta}^2 + \frac{\mu}{r^2} = 0$ or, $\ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2} = 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$ or, $\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) = 0$ which gives $r^2\dot{\theta} = \text{constant}$

Fig. 7.3 Motion under a central force field



Theorem 7.1 *For a scleronomous, conservative dynamical system of n degrees of freedom the quantity* $\left(\sum_{k=1}^{n} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L\right)$ *is a constant of motion where* $q_1, q_2, \ldots, q_n$ *are the n generalized coordinates and L is the Lagrangian of the system.*

Proof Differentiating the quantity $\sum_{k=1}^{n} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L$ with respect to t, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{k=1}^{n} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L\right) = \sum_{k=1}^{n} \ddot{q}_k \frac{\partial L}{\partial \dot{q}_k} + \sum_{k=1}^{n} \dot{q}_k \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\mathrm{d}L}{\mathrm{d}t}$$

Now, $L = L(q_k, \dot{q}_k)$

Therefore, $\frac{\mathrm{d}L}{\mathrm{d}t} = \sum_{k=1}^{n} \frac{\partial L}{\partial q_k} \dot{q}_k + \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k$

$$\text{Thus,} \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{k=1}^{n} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L\right) = \sum_{k=1}^{n} \ddot{q}_k \frac{\partial L}{\partial \dot{q}_k} + \sum_{k=1}^{n} \dot{q}_k \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \sum_{k=1}^{n} \frac{\partial L}{\partial q_k} \dot{q}_k - \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k$$
$$= \sum_{k=1}^{n} \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k}\right] \dot{q}_k = 0,$$

since by Lagrange's equations of motion $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$ for $k = 1, 2, \ldots, n$.

Therefore, $\left(\sum_{k=1}^{n} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L\right) = \text{constant} = E$ (say)

Thus when the Lagrangian is independent of time, Lagrange's equations possess the integral of motion, known as the energy integral of the system.

Theorem 7.2 *For a scleronomous, conservative dynamical system of n degrees of freedom the total energy* $E = T + V$ *is constant*

Proof We know that $L = T - V = T(q_k, \dot{q}_k) - V(q_k)$

Now,

$$E = \sum_{k=1}^{n} \dot{q}_k \frac{\partial (T - V)}{\partial \dot{q}_k} - (T - V) = \sum_{k=1}^{n} \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} - T + V$$
$$= 2T - T + V = T + V$$

since, $\frac{\partial V}{\partial \dot{q}_k} = 0$ in a velocity independent potential field and since in a scleronomous system, Kinetic Energy T is a homogeneous quadratic function of generalized velocities $\dot{q}_k (k = 1, 2, \ldots, n)$ i.e. $2T = \sum_{k=1}^{n} \dot{q}_k \frac{\partial T}{\partial \dot{q}_k}$ (by Euler's formula for homogeneous function). In such a system the total energy is conserved.

Theorem 7.3 (Law of conservation of generalized momentum) *The generalised momentum corresponding to a cyclic coordinate of a system is an integral of motion or constant of motion.*

Proof Lagrange's equation of motion corresponding to the coordinate q_k is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0, L \text{ being the Lagrangian}$$

If q_k is cyclic coordinate then $\frac{\partial L}{\partial q_k} = 0$. Thus from the above equation, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0 \text{ i.e. } \frac{\partial L}{\partial \dot{q}_k} = \text{constant or, } p_k = \text{constant} \tag{7.31}$$

where $p_k = \frac{\partial L}{\partial \dot{q}_k}$ is the generalized momentum corresponding to the generalized coordinate q_k.

Therefore, the generalized momentum conjugate to a cyclic coordinate is an integral of motion. So, when the Lagrangian is time independent there is an energy integral for the system. In case of cyclic co-ordinate there is also an integral for the system, known as a momentum integral.

Theorem 7.4 *Let q_j be a cyclic coordinate such that δq_j corresponds to a rotation of the system of particles around some axis, then the angular momentum of the system is conserved.*

Proof Let the system be conservative. Then the potential energy V depends on position only. It is well known that kinetic energy T depends on translational velocities which are not affected by rotation. As the position coordinate q_j is affected by rotation δq_j, the kinetic energy T does not depend on the coordinate q_j.

So,

$$\frac{\partial V}{\partial \dot{q}_j} = 0 \quad \text{and} \quad \frac{\partial T}{\partial \mathrm{q_j}} = 0 \tag{7.32}$$

Now, Lagrange's equation of motion corresponding to the generalized coordinate q_j can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j} \tag{7.33}$$

Using (7.32) in (7.33) we have, $\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\partial (T-V)}{\partial \dot{q}_j}\right\} = -\frac{\partial V}{\partial q_j}$ or, $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = -\frac{\partial V}{\partial q_j}$ or, $\frac{\mathrm{d}}{\mathrm{d}t}\left(p_j\right) = -\frac{\partial V}{\partial q_j}$ i.e.

$$\dot{p}_j = Q_j \tag{7.34}$$

where $Q_j = -\frac{\partial V}{\partial q_j}$ is the generalized force corresponding to the generalized coordinate q_j.

As q_j is a rotational coordinate so p_j is the component of the total angular momentum along the axis of rotation.

The magnitude of change in the position vector $\vec{r}_i$ due to the change in rotational coordinate q_j is $|\delta\vec{r}_i| = \vec{r}_i \sin\theta\,\delta q_j$ which gives $\left|\frac{\partial\vec{r}_i}{\partial q_j}\right| = \vec{r}_i \sin\theta$

Now, $\frac{\partial\vec{r}_i}{\partial q_j}$ is perpendicular to both $\vec{r}_i$ and $\vec{n}$ where $\vec{n}$ is the unit vector along the axis of rotation. Therefore, $\frac{\partial\vec{r}_i}{\partial q_j} = \vec{n}\times\vec{r}_i$.

Now, $p_j = \frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i\vec{v}_i.\frac{\partial\vec{r}_i}{\partial q_j} = \sum_i m_i\vec{v}_i.\vec{n}\times\vec{r}_i = \vec{n}.\sum_i \vec{L}_i = \vec{n}.\vec{L}$

where $\vec{L} = \sum_i \vec{L}_i = \sum_i m_i\vec{v}_i\times\vec{r}_i$ is the total angular momentum along the axis of rotation.

It is known that if q_j is cyclic then the generalized momentum p_j is constant.

Hence, one can find that if the rotational coordinate is cyclic, the component of total angular momentum along the axis of rotation remains constant.

Corollary 7.1 *If the rotational coordinate is cyclic, the component of the applied torque along the axis of rotation vanishes.*

Proof Generalised force Q_j is given by

$$Q_j = \sum_i \vec{F}_i.\frac{\partial\vec{r}_i}{\partial q_j},$$

$\vec{F}_i$ being the force acting on the ith particle of the system and $\vec{r}_i$ is the position vector of the ith particle.

Again, $Q_j = \sum_i \vec{F}_i.\vec{n}\times\vec{r}_i = \vec{n}.\sum_i \vec{F}_i\times\vec{r}_i = \vec{n}.\sum_i \vec{N}_i = \vec{n}.\vec{N}$, $\vec{N} = \sum_i \vec{N}_i = \sum_i \vec{F}_i\times\vec{r}_i$ being the total torque acting on the system.

From Eq. (7.34) we have $Q_j = 0$ since p_j is constant.

Thus if the rotational coordinate is cyclic, the component of the applied torque along the axis of rotation vanishes.

Example 7.10 The Lagrangian of a particle of mass m moving in a central force field is given by (in polar coordinates)

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r).$$

Discuss its motion.

Solution Clearly, r, θ are the generalized coordinates and since θ is not present in the Lagrangian L, it is the cyclic coordinate.

Therefore,

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = mr^2\dot{\theta} = \text{constant}$$

which indicates that angular momentum of the particle is a constant of motion.

Lagrange's equation of motion corresponding to the r-coordinate is $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$ or, $m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0$ or, $m\ddot{r} - \frac{p_\theta^2}{mr^3} + \frac{\partial V}{\partial r} = 0$.

As p_θ is constant, so we have to deal with only one degree of freedom.

7.3.2.3 Routh's Process for the Ignoration of Coordinates

Consider a dynamical system with n degrees of freedom specified by n generalized coordinates $q_1, q_2, \ldots, q_n$. Let the system has j-cyclic coordinates $q_1, q_2, \ldots, q_j$ ($j < n$). We shall show that the dynamical system has (n-j) degrees of freedom. Clearly the generalized momenta against the cyclic coordinates would be

$$p_k = \frac{\partial L}{\partial \dot{q}_k} = \text{constant} = \beta_\mathrm{k} \quad \text{for} \quad k = 1, 2, \ldots, j \tag{7.35}$$

Let us define a new function R as

$$R = L - \sum_{k=1}^{j} \beta_k \dot{q}_k, L \tag{7.36}$$

being the Lagrangian of the system

With the help of (7.36), R can be expressed as a function of $q_{j+1}, q_{j+2}, \ldots, q_n; \dot{q}_{j+1}, \dot{q}_{j+2}, \ldots, \dot{q}_n; \beta_1, \beta_2, \ldots, \beta_j; t$

that is, $R = R\left(q_{j+1}, q_{j+2}, \ldots, q_n; \dot{q}_{j+1}, \dot{q}_{j+2}, \ldots, \dot{q}_n;\ \beta_1, \beta_2, \ldots, \beta_j; t\right)$.

The function R is called the Routhian function.

Taking a virtual change of R we get from Eq. (7.36),

$$\delta R = \delta L - \sum_{k=1}^{j} \delta\beta_k \dot{q}_k - \sum_{k=1}^{j} \beta_k \delta\dot{q}_k. \tag{7.37}$$

Also, $L = L(q_{j+1}, q_{j+2}, \ldots, q_n; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; t)$.

$$\begin{aligned} \therefore \delta L &= \sum_{k=j+1}^{n} \frac{\partial L}{\partial q_k} \delta q_k + \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \\ &= \sum_{k=j+1}^{n} \frac{\partial L}{\partial q_k} \delta q_k + \sum_{k=1}^{j} \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k + \sum_{k=j+1}^{n} \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k. \end{aligned} \tag{7.38}$$

Substituting (7.38) in the Eq. (7.37) we get

$$
\begin{aligned}
&\sum_{k=j+1}^{n} \frac{\partial R}{\partial q_k}\delta q_k + \sum_{k=j+1}^{n} \frac{\partial R}{\partial \dot{q}_k}\delta \dot{q}_k + \sum_{k=1}^{j} \frac{\partial R}{\partial \beta_k}\delta \beta_k \\
&= \sum_{k=j+1}^{n} \frac{\partial L}{\partial q_k}\delta q_k + \sum_{k=1}^{j} \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k + \sum_{k=j+1}^{n} \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k - \sum_{k=1}^{j} \delta\beta_k\, \dot{q}_k - \sum_{k=1}^{j} \beta_k \delta \dot{q}_k \quad (7.39)\\
&= \sum_{k=j+1}^{n} \frac{\partial L}{\partial q_k}\delta q_k + \sum_{k=j+1}^{n} \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k - \sum_{k=1}^{j} \delta\beta_k\, \dot{q}_k,
\end{aligned}
$$

since by Eq. (7.35), $\frac{\partial L}{\partial \dot{q}_k} = \beta_k$ for $k = 1, 2, \ldots, j$.

Now Eq. (7.39) involves changes δq_k, $\delta \dot{q}_k (k = j+1, j+2, \ldots, n)$ and $\delta \beta_k (k = 1, 2, \ldots, j)$ which are arbitrary and independent.

This leads to the following equations

$$\frac{\partial R}{\partial q_k} = \frac{\partial L}{\partial q_k}, \frac{\partial R}{\partial \dot{q}_k} = \frac{\partial L}{\partial \dot{q}_k} \quad \text{for} \quad k = j+1, j+2, \ldots, n \tag{7.40}$$

And

$$-\frac{\partial R}{\partial \beta_k} = \dot{q}_k \quad \text{for} \quad k = 1, 2, \ldots, j. \tag{7.41}$$

By Lagrange's equation of motion we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \quad \text{for} \quad k = j+1, j+2, \ldots, n.$$

Using (7.40) we have,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial R}{\partial \dot{q}_k}\right) - \frac{\partial R}{\partial q_k} = 0 \quad \text{for} \quad k = j+1, j+2, \ldots, n.$$

From (7.41) we have

$$q_k = -\int \frac{\partial R}{\partial \beta_k} dt + \text{constant} \quad \text{for} \quad k = 1, 2, \ldots, j \tag{7.42}$$

This shows that R behaves as the Lagrangian L of a new dynamical system having (n-j) degrees of freedom.

Example 7.11 Use the method of ignorable coordinates to reduce the degrees of freedom of a spherical pendulum to one.

Solution A spherical pendulum is a simple pendulum of length l. The bob of the pendulum moves on the surface of a sphere of radius equal to the length of the pendulum.

Let (l, θ, ϕ) be the position of the bob at time t in spherical polar coordinate system. Let h be the height of the bob from the horizontal plane.

$$\text{Potential energy} = V = mgh = mg\{l - l\cos(\pi - \theta)\} = mgl(1 + \cos\theta)$$

$$\text{Kinetic energy} = T = \frac{1}{2}m\left\{\left(l\dot{\theta}\right)^2 + \left(l\sin\theta\dot{\phi}\right)^2\right\} = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right).$$

Now, Lagrangian of the system $= L = T - V = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) - mgl$ $(1 + \cos\theta)$.

Clearly, ϕ is a cyclic coordinate.

$\therefore \frac{\partial L}{\partial \dot{\phi}} = p_\phi = \text{constant} = \beta_\phi$ which gives $ml^2\sin^2\theta\dot{\phi} = \beta_\phi$ or, $\dot{\phi} = \frac{\beta_\phi}{ml^2\sin^2\theta}$.

Now, Routhian function is given by

$$R = L - \sum_{k=1}^{j}\beta_k\dot{q}_k = L - \beta_\phi\dot{\phi} = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) - mgl(1 + \cos\theta) - \beta_\phi\dot{\phi}$$

$$= \frac{1}{2}ml^2\left(\dot{\theta}^2 + \frac{\beta_\phi^2}{m^2l^4\sin^2\theta}\right) - mgl(1 + \cos\theta) - \frac{\beta_\phi^2}{ml^2\sin^2\theta}$$

$$= \frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}\frac{\beta_\phi^2}{ml^2\sin^2\theta} - mgl(1 + \cos\theta)$$

$$= R(\theta, \dot{\phi}, \beta_\phi)$$

The equation of motion is $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial R}{\partial \dot{\theta}}\right) - \frac{\partial R}{\partial \theta} = 0$,,

$$\text{or, } ml^2\ddot{\theta} - \frac{\beta_\phi^2\cos\theta}{ml^2\sin^3\theta} - mgl\sin\theta = 0.$$

The above equation can be interpreted as representing a system with single degree of freedom.

Example 7.12 In a dynamical system the kinetic energy and the potential energy are given by $T = \frac{1}{2}\frac{\dot{q}_1^2}{a + bq_2^2} + \frac{1}{2}\dot{q}_2^2, V = c + dq_2^2$ where a, b, c, d are constants.

Determine $q_1(t)$ and $q_2(t)$ by Routh's process for ignorable coordinates.

Solution Here, Lagrangian is given by

$$L = \frac{1}{2}\frac{\dot{q}_1^2}{a + bq_2^2} + \frac{1}{2}\dot{q}_2^2 - c - dq_2^2.$$

Clearly, q_1 is a cyclic coordinate and so, $\frac{\partial L}{\partial q_1} = 0$.

Generalized momentum corresponding to the coordinate q_1 is $p_1 = \frac{\partial L}{\partial \dot{q}_1} =$ constant $= \beta_1$(say) i.e. $\frac{\dot{q}_1}{a+bq_2^2} = \beta_1$. Now the Routhian function is given by

$$\begin{aligned} R = L - \beta_1 \dot{q}_1 &= \frac{1}{2}\frac{\dot{q}_1^2}{a+bq_2^2} + \frac{1}{2}\dot{q}_2^2 - c - dq_2^2 - \beta_1 \dot{q}_1 \\ &= \frac{1}{2}\frac{\beta_1^2 \left(a+bq_2^2\right)^2}{\left(a+bq_2^2\right)} + \frac{1}{2}\dot{q}_2^2 - c - dq_2^2 - \beta_1^2\left(a+bq_2^2\right) \\ &= -\frac{1}{2}\beta_1^2\left(a+bq_2^2\right) + \frac{1}{2}\dot{q}_2^2 - c - dq_2^2. \end{aligned}$$

The equation of motion for the coordinate q_2 is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial R}{\partial \dot{q}_2}\right) - \frac{\partial R}{2} = 0, \text{ or, } \ddot{q}_2 + \beta_1^2 b q_2 + 2dq_2 = 0, \text{ or, } \ddot{q}_2 + A^2 q_2 = 0$$

where $A^2 = \beta_1^2 b + 2d$.

Thus we have, $q_2 = B\,\sin(At + \in)$ where B and $\in$ are arbitrary constants.

Again, we have $\frac{\dot{q}_1}{a+bq_2^2} = \beta_1$ or, $\dot{q}_1 = \beta_1(a+bq_2^2) = \beta_1\left\{a + bB^2\sin^2(At + \in)\right\}$.

Upon integration the above relation, we have

$$\begin{aligned} q_1 &= \beta_1 at + \frac{1}{2}\beta_1 bB^2 \int \{1 - \cos 2(At + \in)\}dt \\ &= \beta_1 at + \frac{1}{2}\beta_1 bB^2 t - \frac{1}{4A}\beta_1 bB^2 \sin 2(At + \in) + C \end{aligned}$$

where C is an arbitrary constant.

7.4 Hamilton Principle

Sir W.R. Hamilton gave his famous principle of least action, also known as Hamilton's principle of least action in 1834 which states that

The variation of the integral $\int_1^2 L dt$ between the actual path and any neighboring virtual path of a dynamical system moving from one configuration to another, coterminous in both space and time with the actual path, is zero. That is,

$$\delta \int_{1}^{2} L(q, \dot{q}, t)dt = 0,$$

L being the Lagrangian of the dynamical system, $q, \dot{q}$ being the generalized coordinate and generalized velocity, respectively and δ denotes the variation of the integral.

7.5 Noether Theorem

Conservation laws of a dynamical system in classical mechanics where time t is one of the independent variable and other variables are spatial variables is the first integral of motion or constant of motion of the system. Conservation laws reduces the degrees of freedom of a system, thus makes the system simple to integrate. But, deducing conservation laws of a system is not an easy task. One can notice that the idea of conservation laws and symmetry with respect to group of transformations are interrelated for instance, the translational symmetry provides the conservation of linear momentum, rotational symmetry provides the conservation of angular momentum etc.

German mathematician Emalie Emmy Noether in 1918 gave a general theory of conservation laws and symmetry transformations recognizing the importance of the relation between the symmetry and conservation laws. Actually, Noether gave two theorems in this regard. Herein we only give the statement of the first theorem; we shall discuss other in the later chapter.

Theorem 7.5 (Noether's First theorem) *Every conservation law gives rise to a one-parameter symmetry group of transformation and vice versa.*

Let us now explain this theorem

Consider that L be the Lagrangian of a system in a coordinate system $(q, \dot{q}, t)$ and L' be the Lagrangian in the coordinate system $(q', \dot{q}', t)$ obtained under the coordinate transformations $q' = q'(q, \dot{q}, t), \dot{q}' = \dot{q}'(q, \dot{q}, t)$. Then this transformation of coordinates is said to be a symmetry transformation of the Lagrangian if

$$L'(q', \dot{q}', t) = L(q, \dot{q}, t).$$

Noether's theorem states that if the coordinates of a Lagrangian of a system has a set of continuous symmetry transformations $\bar{t} = \mathcal{P}(t, \in)$ where $\mathcal{P}(\in = 0) = t$ and ϵ, $\bar{q}_k = Q_k(q_k, \in)$ where $Q_k(\in = 0) = q_k$, ϵ is a continuous parameter then the functional $I = \int_{x_1}^{x_2} L(q_k, \dot{q}_k, t)dt$ with arbitrary end points x_1 and x_2 is an invariant with a set of quantities that remain conserved along the trajectories of the system, given by:

$$\sum_{k=1}^{n} p_k \frac{\mathrm{d}Q_k}{\mathrm{d}\epsilon}\bigg|_{\{\epsilon=0\}} - \left(\sum p_k\dot{q}_k - L(q_k, \dot{q}_k, t)\right)\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}\epsilon}\bigg|_{\{\epsilon=0} \} = \text{constant}$$

where $p_k = \frac{\partial L}{\partial \dot{q}_k}$ is the momentum conjugate to generalized coordinate q_k.

Example 7.13 Show that if the functional $I = \int L(q, \dot{q}, t)dt$ is invariant with respect to the homogeneity in time t then the total energy of a system is conserved.

Solution Homogeneity of time t means that the physical laws governing a system do not change for any arbitrary shift in the origin of time. For such a system the Lagrangian L is independent of time t. For homogeneity in time t, one have the transformations $\bar{t} = t + \in, \bar{q} = q$. Therefore $\mathcal{P} = t + \in$ and $Q = q$. Hence the conserved quantity is $p\dot{q} - L(q, \dot{q})$, which is the total energy of the system.

Example 7.14 Show that if the functional $I = \int L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)dt$ is invariant with respect to the homogeneity of space then the total linear momentum of the system is conserved.

Solution Homogeneity of space means that the space possesses the same property everywhere. For homogeneity in x, one has the transformations $\bar{t} = t, \bar{x} = x + \in$, $\bar{y} = y, \bar{z} = z$. Therefore $\mathcal{P} = t$ and $Q = (x + \in, y, z)$. Hence the conserved quantity is $p_x =$ constant. Similarly, for homogeneity in y gives $p_y =$ constant and for homogeneity in z gives $p_z =$ constant. Accordingly the total linear momentum $p = p_x\hat{i} + p_y\hat{j} + p_z\hat{k}$ is conserved.

Example 7.15 Show that if the functional $I = \int L(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)\mathrm{d}t$ is invariant with respect to the isotropy of space then the total angular momentum of the system is conserved.

Solution Isotropy of space means that an arbitrary rotation of a system about an axis does not change the system. For isotropy of space about z-axis, one has the transformations $\bar{t} = t, \bar{x} = x \cos\theta + y \sin\theta$,

$\bar{y} = y \cos\theta - x \sin\theta$, $\bar{z} = z$. Therefore $\mathcal{P} = t$ and $Q(\theta) = (x \cos\theta + y \sin\theta$, $y \cos\theta - x \sin\theta, z)$.

Now $\frac{\partial Q}{\partial\theta}\big|_{\theta=0} = (y, -x, 0)$. Hence the conserved quantity is $yp_x - xp_y =$ constant. Similarly for isotropy about x-axis, $yp_z - zp_y =$ constant and for isotropy about y-axis, $xp_z - zp_x =$ constant. Thus the components of angular momentum are conserved quantity under rotation of space. Accordingly the total Angular momentum $H = q_k \times p_k$ is conserved.

Remarks Conservation theorem of generalized momentum is a particular case of Noether's theorem.

7.6 Legendre Dual Transformations

Legendre dual transformation as the name suggests is a transformation that transforms functions on a vector space to functions on the dual space. Legendre transformation is a standard technique for generating a new pair of independent variables (x, z) from an initial pair (x, y). The transformation is completely invertible i.e. applying the transformation twice one gets the initial pair of variables (x, y).

If $L = L(q_k, \dot{q}_k, t)$ is the Lagrangian of a dynamical system where q_k is the generalized coordinate, $\dot{q}_k$ is the generalized velocity then the generalized momenta are given by $p_k = \frac{\partial L}{\partial \dot{q}_k}, k = 1, 2, \ldots, n$. If one wants to eliminate $\dot{q}_k$ in terms of p_k then this elimination considers L as a function of q_k, $\frac{\partial L}{\partial \dot{q}_k}$ and time. The transformation which does this (mathematically) is known as Legendre transformation.

Theorem 7.6 *Let a function $F(x_1, x_2, \ldots, x_n)$ depending explicitly on the n independent variables $x_1, x_2, \ldots, x_n$ be transformed to another function $G(y_1, y_2, \ldots, y_n)$, which is expressed explicitly in terms of the new set of n independent variables $y_1, y_2, \ldots, y_n$. These new variables are connected by the old variables by a given set of relations*

$$y_i = \frac{\partial F}{\partial x_i}, i = 1, 2, \ldots, n \tag{7.43}$$

and the form of G is given by

$$G(y_1, y_2, \ldots, y_n) = \sum_{i=1}^{n} x_i y_i - F(x_1, x_2, \ldots, x_n) \tag{7.44}$$

Then the variables $x_1, x_2, \ldots, x_n$ satisfy the dual transformations viz. *the relations:*

$$x_i = \frac{\partial G}{\partial y_i}, i = 1, 2, \ldots, n \tag{7.45}$$

and

$$F(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i y_i - G(y_1, y_2, \ldots, y_n) \tag{7.46}$$

The transformations (7.46) between the two set of variables given by the Eqs. (7.43) and (7.45) are known as Legendre dual transformations.

Proof It is given that,

$$G(y_1, y_2, \ldots, y_n) = \sum_{i=1}^{n} x_i y_i - F(x_1, x_2, \ldots, x_n)$$
$$\therefore \delta G = \sum_{i=1}^{n} \frac{\partial G}{\partial y_i} \delta y_i.$$

Again, $\delta G = \sum_{i=1}^{n} x_i \delta y_i + \sum_{i=1}^{n} y_i \delta x_i - \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \delta x_i$.

Thus we have,

$$\begin{aligned} \sum_{i=1}^{n} \frac{\partial G}{\partial y_i} \delta y_i &= \sum_{i=1}^{n} x_i \delta y_i + \sum_{i=1}^{n} y_i \delta x_i - \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} \delta x_i \\ &= \sum_{i=1}^{n} x_i \delta y_i + \sum_{i=1}^{n} \left(y_i - \frac{\partial F}{\partial x_i} \right) \delta x_i. \\ \therefore x_i &= \frac{\partial G}{\partial y_i}, i = 1, 2, \ldots, n \end{aligned}$$

since it is given that, $y_i = \frac{\partial F}{\partial x_i}, i = 1, 2, \ldots, n$ and δy_i's are arbitrary (as all y_i's are independent).

This proves the duality of the transformations.

Relation (7.46) can simply be obtained by rearranging terms of the relation (7.44). Moreover, starting from the relation (7.45) one can easily prove the relation (7.43), exactly in the same fashion.

7.7 Hamilton Equations of Motion

In Lagrangian mechanics, $q_k, \dot{q}_k$ are treated as the Lagrangian variables with time t as a parameter where q_k are the n-generalized coordinates and $\dot{q}_k$ are the n-generalized velocities (k = 1, 2,…, n). In Hamiltonian mechanics, the basic variables are the generalized coordinates q_k and the generalized momenta p_k(k = 1, 2, …, n) defined by the equations

$$p_k = \frac{\partial L}{\partial \dot{q}_k}, k = 1, 2, \ldots, n \quad (7.47)$$

where L is the Lagrangian of the system. With the aid of the Legendre dual transformation the Lagrangian of a system can be converted into a new function H, known as Hamiltonian function, by the relation

$$H = \sum_{k=1}^{n} \dot{q}_k p_k - L \tag{7.48}$$

where $L = L(q_k, \dot{q}_k, t)$.

With the help of (7.47), $\dot{q}_k$'s can be eliminated from the right hand side of (7.48) which can then be expressed in terms of (q_k, p_k, t) so that $H = H(q_k, p_k, t)$.

Theorem 7.7 *The system of n second order differential equation known as Lagranges's equation of motion are equivalent to the 2n first order differential equation known as Hamilton's equation of motion given by*

$$\dot{q}_k = \frac{\partial H}{\partial p_k},\ \dot{p}_k = -\frac{\partial H}{\partial q_k}$$

where H is the Legender dual transformation of the Lagrangian function L given by

$$H = \sum p_i\dot{q}_i - L(q_i,\dot{q}_i, t)$$

Proof The Legender dual transformation of the Lagrangian $L(q_k, \dot{q}_k, t)$ is given by

$$H = \sum p_i\dot{q}_i - L(q_i, \dot{q}_i, t)$$

Let us consider a virtual change of H at time t. Then,

$$\delta H = \sum_{k=1}^{n} \delta\dot{q}_k p_k + \sum_{k=1}^{n} \dot{q}_k \delta p_k - \delta L(q_k, \dot{q}_k, t) \tag{7.49}$$

Now, $H = H(q_k, p_k, t)$.

$$\therefore \delta H = \sum_{k=1}^{n} \frac{\partial H}{\partial q_k}\delta q_k + \sum_{k=1}^{n} \frac{\partial H}{\partial p_k}\delta p_k + \frac{\partial H}{\partial t}\delta t \tag{7.50}$$

Again,

$$\delta L = \sum_{k=1}^{n} \frac{\partial L}{\partial q_k}\delta q_k + \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k}\delta \dot{q}_k + \frac{\partial L}{\partial t}\delta t \tag{7.51}$$

Substituting (7.50) and (7.51) in (7.49) we have,

$$\begin{aligned}&\sum_{k=1}^{n}\frac{\partial H}{\partial q_k}\delta q_k+\sum_{k=1}^{n}\frac{\partial H}{\partial p_k}\delta p_k+\frac{\partial H}{\partial t}\delta t\\&=\sum_{k=1}^{n}\delta\dot{q}_k p_k+\sum_{k=1}^{n}\dot{q}_k\delta p_k-\sum_{k=1}^{n}\frac{\partial L}{\partial q_k}\delta q_k-\sum_{k=1}^{n}\frac{\partial L}{\partial\dot{q}_k}\delta\dot{q}_k-\frac{\partial L}{\partial t}\delta t\\&=\sum_{k=1}^{n}\dot{q}_k\delta p_k-\sum_{k=1}^{n}\dot{p}_k\delta q_k-\frac{\partial L}{\partial t}\delta t,\end{aligned}\tag{7.52}$$

as $p_k=\frac{\partial L}{\partial\dot{q}_k}, k=1,2,\ldots,n$ so Lagrange's equations of motion give

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{q}_k}\right)-\frac{\partial L}{\partial q_k}=0 \text{ i.e. } \dot{p}_k=\frac{\partial L}{\partial q_k} \text{ for } k=1,2,\ldots,n.$$

Equation (7.52) is true for all arbitrary variations. Hence we get,

$$\frac{\partial H}{\partial q_k}=-\dot{p}_k,\frac{\partial H}{\partial p_k}=\dot{q}_k \quad \text{for} \quad k=1,2,\ldots,n$$

and

$$\frac{\partial H}{\partial t}=-\frac{\partial L}{\partial t}.$$

Thus we have

$$\dot{q}_k=\frac{\partial H}{\partial p_k},\ \dot{p}_k=-\frac{\partial H}{\partial q_k} \quad \text{for}\, k=1,2,\ldots,n\tag{7.53}$$

These equations are called Hamilton's equations of motion of a dynamical system.

Theorem 7.8 *If the Lagrangian does not explicitly depend on time, Hamiltonian is also independent of time and Hamiltonian is a constant of motion or first integral of the system.*

Proof We have Hamiltonian H as defined by $H=\sum_{k=1}^{n}\dot{q}_k p_k-L$, where L is the Lagrangian of the system. Now, if $H=H(q_k,p_k,t)$ then using Hamilton's equations of motion and $\frac{\partial H}{\partial t}=-\frac{\partial L}{\partial t}$, we have

$$\frac{\mathrm{d}H}{\mathrm{d}t}=\sum_{i=1}^{n}\frac{\partial H}{\partial q_i}\dot{q}_i+\sum_{i=1}^{n}\frac{\partial H}{\partial p_i}\dot{p}_i+\frac{\partial H}{\partial t}=-\sum_{i=1}^{n}\dot{p}_i\dot{q}_i+\sum_{i=1}^{n}\dot{q}_i\dot{p}_i-\frac{\partial L}{\partial t},$$

If in addition the Lagrangian does not explicitly depend on time that is, $\frac{\partial L}{\partial t}=0$ then $\frac{\partial H}{\partial t}=0$.

Hence the Hamiltonian is also independent of time t.

Moreover, using the above result we have $\frac{dH}{dt} = 0$, that is $H =$ constant. Thus, Hamiltonian is a constant of motion.

Theorem 7.9 *For a scleronomous dynamical system in a velocity independent potential field the total energy (sum of the Kinetic Energy and Potential Energy) remains conserved.*

Proof If the Hamiltonian H is explicitly independent of time t then $H = H(q_k, p_k)$, $k = 1, 2, \ldots, n$ where p_k's are the generalized momenta corresponding to the generalized coordinates q_k of the system.

Now,

$$\frac{dH}{dt} = \sum_{k=1}^{n} \frac{\partial H}{\partial q_k} \delta q_k + \sum_{k=1}^{n} \frac{\partial H}{\partial p_k} \delta p_k \tag{7.54}$$

By Hamilton's equations of motion we have,

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \; \dot{p}_k = -\frac{\partial H}{\partial q_k} \quad \text{for } k = 1, 2, \ldots, n.$$

Substituting these in (7.54) we get,

$$\frac{dH}{dt} = 0 \,\text{which gives}\, H = \text{constant.} \tag{7.55}$$

Moreover, in a scleronomous system, the kinetic Energy T is a homogeneous quadratic function of generalized velocities. Therefore, by Euler's theorem on homogeneous function we have,

$$2T = \sum_{k=1}^{n} \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} \tag{7.56}$$

We have,

$$H = \sum_{k=1}^{n} \dot{q}_k p_k - L = \sum_{k=1}^{n} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - (T - V) = \sum_{k=1}^{n} \dot{q}_k \frac{\partial (T - V)}{\partial \dot{q}_k} - T + V$$

Since in a velocity independent potential field, $\frac{\partial V}{\partial \dot{q}_k} = 0$,

$\therefore\ H = \sum_{k=1}^{n} \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} - T + V = 2T - T + V = T + V$. [using (7.56)]

Therefore, in a scleronomous dynamical system in a velocity independent potential field the total energy (sum of the Kinetic Energy and Potential Energy) remains conserved.

Theorem 7.10 *If all the coordinates of a dynamical system are cyclic they can be determined by integration. In particular, if the system is scleronomous, the coordinates are all linear functions of time t.*

Proof It is known that a coordinate q_k is said to be cyclic if it is explicitly absent from the Hamiltonian.

Since $H = \sum_{k=1}^{n} \dot{q}_k p_k - L$, so q_k's are cyclic coordinates.

As all the coordinates are cyclic, so $H = H(p_k, t)$ where p_k's are the n-generalized momenta of the system.

Hamilton's equations of motion give $\dot{q}_k = \frac{\partial H}{\partial p_k} = f_k(p_k, t)$, k = 1, 2, …, n.

Since q_k's are the cyclic coordinates, p_k = constant = β_k(say), k = 1, 2, …, n.

Therefore, $\dot{q}_k = f_k(\beta_k, t)$, this on integration gives

$$q_k = \int f_k(\beta_k, t)dt + \text{constant}, k = 1, 2, \ldots, n.$$

So, the coordinates can be determined by integration.

For a scleronomous system, $\dot{q}_k = f_k(p_k)$, which on integration gives

$$q_k = \int f_k(\beta_k)\mathrm{d}t + \text{constant} = f_k(\beta_k)\int \mathrm{d}t + \text{constant} = t f_k(\beta_k) + \text{constant}\ , \ \text{for } k = 1,\ 2,\ \ldots, n.$$

So, the coordinates are linear functions of time t.

Example 7.16 Obtain the equation of motion of a simple pendulum using Hamiltonian function.

Solution Here, θ is the generalized coordinate. The velocity of the ball is $l\dot{\theta}$ where l is the length of the string of the pendulum. A simple pendulum oscillating in a vertical plane constitutes a conservative holonomic dynamical system.

Here, Kinetic Energy = $T = \frac{1}{2}ml^2\dot{\theta}^2$, m is the mass of the ball.

Potential Energy = $V = mgh = mgl(1 - \cos\,\theta)$.

Therefore, Lagrangian of the system = $L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\,\theta)$.

Since the system is scleronomous and conservative,

$$H = T + V = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1 - \cos\,\theta)$$

Generalized momentum $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$ or, $\dot{\theta} = \frac{p_\theta}{ml^2}$.

$$\therefore H = \frac{1}{2}ml^2\left(\frac{p_\theta}{ml^2}\right)^2 + mgl(1 - \cos\,\theta) = \frac{1}{2}\frac{p_\theta^2}{ml^2} + mgl(1 - \cos\,\theta) = H(\theta, p_\theta).$$

Hamilton's equations of motion are given by

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2} \text{ i.e. } p_\theta = ml^2\dot{\theta} \quad \text{and} \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mgl\,\sin\,\theta$$

$$\therefore ml^2\ddot{\theta} = -mgl \sin\theta \text{ or, } \ddot{\theta} = -\frac{g}{l}\sin\theta \simeq -\frac{g}{l}\theta \text{ (when } \theta \text{ is very small).}$$

Time period is given by $2\pi\sqrt{\frac{l}{g}}$, g is the acceleration due to gravity.

Example 7.17 For a dynamical system Hamiltonian H is given by

$$H = q_1p_1 - q_2p_2 - aq_1^2 + bq_2^2$$

Find q_1, q_2, p_1, p_2 in terms of time t. Hence find the corresponding equation of motion of the dynamical system.

Solution Hamilton's equations of motion give

$$\dot{q}_1 = \frac{\partial H}{\partial p_1} = q_1 \text{ which on integration gives } q_1 = Ce^t,\ C = \text{constant}$$

$$\dot{p}_1 = -\frac{\partial H}{\partial q_1} = -p_1 + 2aq_1 \text{ or, } \dot{p}_1 + p_1 = 2aCe^t.$$

Multiplying both sides by the integrating factor e^t we get,

$$\frac{\mathrm{d}}{\mathrm{d}t}(p_1e^t) = 2aCe^{2t}$$

Integrating this we get, $p_1 = aCe^t + De^{-t}, D = \text{constant}$

$$\dot{q}_2 = \frac{\partial H}{\partial p_2} = -q_2 \text{ or, } q_2 = Ae^{-t}, A = \text{constant}$$

$$\dot{p}_2 = -\frac{\partial H}{\partial q_2} = p_2 - 2bq_2 = p_2 - 2Abe^{-t} \text{ or, } \dot{p}_2 - p_2 - 2Abe^{-t} = 0$$

Multiplying both sides by the integrating factor e^{-t} we get, $\frac{\mathrm{d}}{\mathrm{d}t}(p_2e^{-t}) = -2Abe^{-2t}$ which on integration gives $p_2 = Abe^{-t} + Be^t, B = \text{constant}$. Therefore the required equation of motion is

$$\dot{p}_1 + p_1 - 2aCe^t = 0$$
$$\dot{p}_2 - p_2 - 2Abe^{-t} = 0$$

Example 7.18 Hamiltonian of a dynamical system is given by $H = \frac{1}{2}\sum_{i=1}^{3}\left(p_i^2 + \mu^2q_i^2\right)$ where p_i, q_i are the n generalized momenta and generalized coordinates, μ is a constant. Show that $F_1 = q_2p_3 - q_3p_2$ and $F_2 = \mu q_1 \cos \mu t - p_1 \sin \mu t$ are constants of motion.

Solution Now, $H = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2 + \mu^2 q_1^2 + \mu^2 q_2^2 + \mu^2 q_3^2\right)$.

$$F_1 = q_2 p_3 - q_3 p_2 = F_1(q_2, q_3, p_2, p_3)$$

$$\therefore \frac{dF_1}{dt} = \frac{\partial F_1}{\partial q_2}\dot{q}_2 + \frac{\partial F_1}{\partial q_3}\dot{q}_3 + \frac{\partial F_1}{\partial p_2}\dot{p}_2 + \frac{\partial F_1}{\partial p_3}\dot{p}_3$$

$$= \frac{\partial F_1}{\partial q_2}\frac{\partial H}{\partial p_2} + \frac{\partial F_1}{\partial q_3}\frac{\partial H}{\partial p_3} - \frac{\partial F_1}{\partial p_2}\frac{\partial H}{\partial q_2} - \frac{\partial F_1}{\partial p_3}\frac{\partial H}{\partial q_3}$$

(using Hamilton′s equations of motion)

$$= p_3 p_2 - p_2 p_3 + q_3 \mu^2 q_2 - q_2 \mu^2 q_3 = 0$$

So, $F_1 =$ constant.
Again, we have

$$\frac{dF_2}{dt} = \frac{\partial F_2}{\partial q_1}\dot{q}_1 + \frac{\partial F_2}{\partial p_1}\dot{p}_1 + \frac{\partial F_2}{\partial t} = \frac{\partial F_2}{\partial q_1}\frac{\partial H}{\partial p_1} - \frac{\partial F_2}{\partial p_1}\frac{\partial H}{\partial q_1} + \frac{\partial F_2}{\partial t}$$

(using Hamilton′s equations of motion)

$$= p_1 \mu \cos \mu t + q_1 \mu^2 \sin \mu t - q_1 \mu^2 \sin \mu t - p_1 \mu \cos \mu t = 0$$

So, $F_2 =$ constant.

7.7.1 Differences Between Lagrangian and Hamiltonian of a Dynamical System

1. Lagrangian $L(q, \dot{q}, t)$ is described always in the configuration space of appropriate dimension and the configuration space is described only by the total number of generalized coordinates of the system whereas Hamiltonian $H(q, p, t)$ is described in the phase space of requisite dimension set by the equal number of generalized coordinates and generalized momenta.
2. Lagrangian mechanics provides a second order differential equation of motion corresponding to each degree of freedom of the system whereas Hamiltonian mechanics provides us two first order differential equations of motion corresponding to each degree of freedom.
3. Hamilton’s equations of motion can only be derived for holonomic dynamical systems whereas Lagrange’s equations of motion can be derived for holonomic as well as nonholonomic dynamical systems.

So far an introduction of basic concepts of Hamiltonian function has been given in the context of classical mechanics. We shall now give the concept with respect to the flow of a dynamical system.

7.8 Hamiltonian Flows

We have knowledge that solutions of a system of differential equations are geometrically curves which depict a flow in the space $\mathbb{R}^n$. The system of differential equation represents a vector field where the direction of the tangent at a given point of the solution curve is given by the first order differential equation written in a solved form. Herein we shall discuss the dynamical properties of the flow generated by Hamiltonian vector fields in the phase space.

A system of differential equation on $\mathbb{R}^{2n}$ corresponding to n degrees of freedom of a system is said to be Hamiltonian flows or systems if it can be expressed as

$$\dot{q}_i = \frac{\partial H}{\partial p_i},\ \dot{p}_i = -\frac{\partial H}{\partial q_i},\ i = 1, 2, \ldots, n \tag{7.57}$$

where $H(\underset{\sim}{x}) = H(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n)$ is a twice continuously differentiable function. The dimension of the phase space is $2n$ for the n generalized coordinates q_i and n generalized momenta p_i, $i = 1, 2, \ldots, n$. The function H is called Hamiltonian of the system as defined earlier. Furthermore, Hamiltonian functions are commutative. The Hamiltonian vector field is derived from the Hamiltonian function H denoted by $X_H(x)$ and is given by $X_H(x) \equiv \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right)$. From Liouville theorem (see Chap. 1) it has been established that the flow induced by a time independent Hamiltonian is volume preserving. The flow takes place on bounded energy manifolds and all orbits will return after some time to a neighbourhood of the starting point. This is a consequence of the recurrence theorem by Poincarè stated below.

Theorem 7.11 (Poincarè Recurrence Theorem) *Let f be a bijective, continuous, volume preserving mapping of a bounded domain $D \subset \mathbb{R}^n$ into itself. Then each neighbourhood U of each point in D contains a point x which returns to U after repeated applications of the mapping $f^{(n)}x \in U$ for some $n \in \mathbb{N}$.*

It follows that $H(x)$ is a first integral of $\dot{x} = f(x)$ if and only if $\frac{\mathrm{d}H}{\mathrm{d}t} = \nabla H \cdot f(x) = 0 \quad \forall x \in \mathbb{R}^n, H(x)$ is identically constant on any open subset of $\mathbb{R}^n$. Furthermore, if there is a first integral, that is, a Hamiltonian function then the orbits of the system are contained in the one-parameter family of level curves $H(x) = k$.

Given a system $\dot{x} = f(x), x \in \mathbb{R}^n$, a set $S \subseteq \mathbb{R}^n$ which is the union of whole orbits of the system, is called an invariant set for the system. For example, for a Hamiltonian system in $\mathbb{R}^2$ the level sets $H(x_1, x_2) = k$ are invariant sets, since H is constant along any orbit. More generally, a function $H : \mathbb{R}^n \to \mathbb{R}$ of class C^1, is called a first integral of the system $\dot{x} = f(x)$ if H is constant on every orbit, $\frac{\mathrm{d}}{\mathrm{d}t} H(x, t) = \nabla H(x(t)) \cdot f(x(t)) = 0 \quad \forall t$.

In general, the first integral of a dynamical system is defined as follows.

First integral of a system A continuously differentiable function $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^n$ is said to be a first integral of the system $\dot{x} = F(x), x = (x_1, x_2, \ldots, x_n) \in X \subseteq \mathbb{R}^n$ on the region $D \subseteq X$ if

$$\mathcal{D}_t f(x(t)) = 0,\ x \in X \subseteq \mathbb{R}^2$$

where $\mathcal{D}_t f = \frac{\partial f}{\partial x}\dot{x} = \frac{\partial f}{\partial x_1}\dot{x}_1 + \frac{\partial f}{\partial x_2}\dot{x}_2 + \cdots + \frac{\partial f}{\partial x_n}\dot{x}_n,\ x = (x_1, x_2, \ldots, x_n)$ and $\mathcal{D}_t$ is called the orbital derivative.

The first integral $f(x(t))$ is constant for any solution $x(t)$ of the system i.e. $f(x(t)) = C$(constant) and is therefore also called as constant of motion. The first integral of a system when exists is not unique for if $f(x)$ is a first integral of a system then $f(x) + C$ and $Cf(x)$, where $C \in \mathbb{R}$ are also the first integral of the system. The first integral of a system, as the name suggests is obtained by integrating just once the system $\dot{x} = F(x)$.

The level curve of the first integral of a system is denoted by L_c and is defined as $L_c = \{x | f(x) = C\}$. The first integral of motion f is constant on every trajectory of the system. Hence every trajectory of a system is a member of some level curve of f. Each level curve is therefore a union of trajectories of the system. The level set which contains family of trajectories of the system is called an integral manifold.

A system is said to be conservative if it has a first integral of motion on the whole plane i.e. $D = \mathbb{R}^n$.

Theorem 7.12 *If $f \in C^1(E)$, where E is an open, simply connected subset of $\mathbb{R}^2$, then the system $\dot{x} = \mathrm{f}(x)$ is a Hamiltonian system on E if and only if $\nabla \cdot \mathrm{f}(x) = 0 \quad \forall x \in E$.*

Proof Hamiltonian system is given by $\dot{q} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial q}$, therefore

$$\nabla \cdot \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right) = \frac{\partial^2 H}{\partial q \partial p} - \frac{\partial^2 H}{\partial p \partial q} = 0 \quad \text{for}(q, p) \in E$$

since $\mathrm{f}(x) \equiv (f_1(q,p),\ f_2(q,p)) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right)$ is a continuously differentiable function.

Hence, $\nabla \cdot \mathrm{f}(x) = 0 \quad \forall x \in E$ for a Hamiltonian system, where $\mathrm{f}(x) \equiv (f_1(q,p), f_2(q,p)) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right)$.

For the converse part suppose that $\dot{x} = f_1(x,y), \dot{y} = f_2(x,y)$ where $f_1, f_2 \in C^1(E)$ and $\nabla \cdot \mathrm{f}(x) = 0 \quad \forall x \in E$ holds. Therefore

$$\begin{aligned}
&\nabla \cdot (f_1(x,y), f_2(x,y)) = 0 \\
&\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = 0 \\
&\frac{\partial f_1}{\partial x} = -\frac{\partial f_2}{\partial y}
\end{aligned}$$

Clearly, the system is Hamiltonian.

Theorem 7.13 *The flow defined by a Hamiltonian system with one degree of freedom is area preserving.*

Proof The rate of change of area of a system $\dot{x} = f(x), x = (x, y), f = (f_1, f_2)$ is given by

$$\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}$$

Now for a Hamiltonian system $\frac{\partial f_1}{\partial x} = -\frac{\partial f_2}{\partial y}$(since $\nabla \cdot \mathrm{f}(x) = 0$) hence

$$\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}t} = 0 \text{ or, } A = \text{constant}$$

Thus the area of the flow generated by the Hamiltonian system is preserved.

7.8.1 Integrable and Non-Integrable Systems

The most crucial aspect of any system is its integrability. An integrable system is that system whose solution curves and therefore the geometry of the flow in its embedding space can be determined with certainty. At this juncture, question may arise that when the Hamiltonian system will be integrable. The answer to this question is given as follows:

When the degrees of freedom of a Hamiltonian system are equal to the constants of motion of the system then the Hamiltonian system is said to be integrable. For instance, all Hamiltonian system with only one degree of freedom for which the Hamiltonian function H is analytic are integrable. Moreover, all Hamiltonian systems for which Hamilton's equations of motion are linear in generalized coordinates and momenta are integrable. Furthermore, if it is possible to separate the nonlinear Hamilton's equations of motion into decoupled one degree freedom systems, then the system is integrable.

On the other hand, if the degrees of freedom are more than the constants of motion of the system then the system is called non-integrable.

We have seen in our earlier discussions that if a coordinate is explicitly absent in a system then the corresponding momentum is a constant of motion. But for a Hamiltonian system it is not always the momenta which are constants of motion but there are constants of motion which are expressed in terms of the generalized coordinates and generalized momenta.

Generally the trajectory of an integrable system with n-degrees of freedom moves on an n-dimensional surface of a torus which lies in the $2n$-dimensional phase space. Now if the motion of the trajectory is constrained by some constants of motion then the dimension of the surface in which it stays in the phase space would get reduced. For instance, the trajectories of a Hamiltonian system which has k numbers of constants of motion, lie on a $(2n - k)$-dimensional surface in the phase space. Since the motion of the trajectories is always restricted to these surfaces so these surfaces are called as invariant tori. Nevertheless, systems with more than one degree of freedom may not be integrable. If a Hamiltonian system with two degrees of freedom is integrable then there exist exactly two constants of motion. The trajectories of the system therefore would move on the two dimensional surface $(2 \times 2 - 2 = 2)$ of a torus lying in a four dimensional phase space. The trajectories of the system are periodic or quasi-periodic, and do not show any chaotic behavior. Now if the system is slightly nonintegrable due to some perturbation then the constants of motion of the system is no longer constant except the energy of the system. Since the energy of the Hamiltonian system is conserved even now so, the trajectories of the system are constrained to move on a three dimensional surface $(2 \times 2 - 1 = 3)$ in the four dimensional phase space. The trajectories of this three dimensional motion of the system are no longer periodic and displays chaotic motion. When the amount of non-integrability get increased then the trajectories of the system move off the tori and the tori are destroyed. The trajectories of the system therefore can move throughout the phase space without any restriction. Note that for non-integrability, a system must have at least two degrees of freedom which implies a phase space of at least four dimensions. The phase portraits of such system are very difficult to obtain, (see Hilborn [13]).

Theorem 7.14 *For any Hamiltonian system, the Hamiltonian* $\left(x,p\right)$ *is a conserved quantity or first integral of the system.*

Solution Let us consider a Hamiltonian system with x as generalized coordinate and p as generalized momentum. Now, Hamilton's equation of motion is given by

$$\left.\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial x} \end{aligned}\right\} \tag{7.58}$$

We can write $\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial H}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}t} = \dot{x}\frac{\partial H}{\partial x} + \dot{p}\frac{\partial H}{\partial p} = \frac{\partial H}{\partial p}\frac{\partial H}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial H}{\partial p} = 0$

Hence, the Hamiltonian $H\left(x,p\right)$ = constant of motion or integral of motion along the trajectories of the system.

Example 7.19 Show that the system $\dot{x} = -y, \dot{y} = x$ is conservative.

Solution The given system can be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}, y \neq 0$$

The solution of this equation is $x^2 + y^2 = c$, $y \neq 0$, where c is a positive constant. However

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\mathrm{x}) = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} = -2xy + 2xy = 0 \quad \forall (x, y) \in \mathbb{R}^2$$

Hence the given system is a conservative system.

Example 7.20 Find out whether the system $\dot{x} = x$, $\dot{y} = y$ is a conservative system or not.

Solution The given system can be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x},\ x \neq 0$$

The solution of this equation is $y = cx$, $x \neq 0$, where c is a positive constant. However the first integral of the system $f(x, y) = \frac{y}{x}$, $x \neq 0$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\mathrm{x}) = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} = -y/x + y/x = 0 \quad \forall\ (x, y)\backslash(0, y) \in \mathbb{R}^2$$

Hence the given system is not a conservative system.

Example 7.21 Find a first integral of the system

$$\dot{x} = xy,\ \dot{y} = \ln x, x > 1$$

in the region indicated. Hence, sketch the phase portrait.

Solution The given system of equation can be written as a single system

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\ln x}{xy}$$

Separation of variables yields

$$ydy = \frac{\ln x}{x} dx$$

$$\therefore y^2 = (\ln x)^2 + \text{constant}$$

This is the first integral of the system. Since the first integral $f(x, y) = y^2 - (\ln x)^2$ of the given system satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\mathrm{x}) = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} = -2y \ln x + 2y \ln x = 0 \quad \forall (x, y)\backslash(0, y) \in \mathbb{R}^2$$

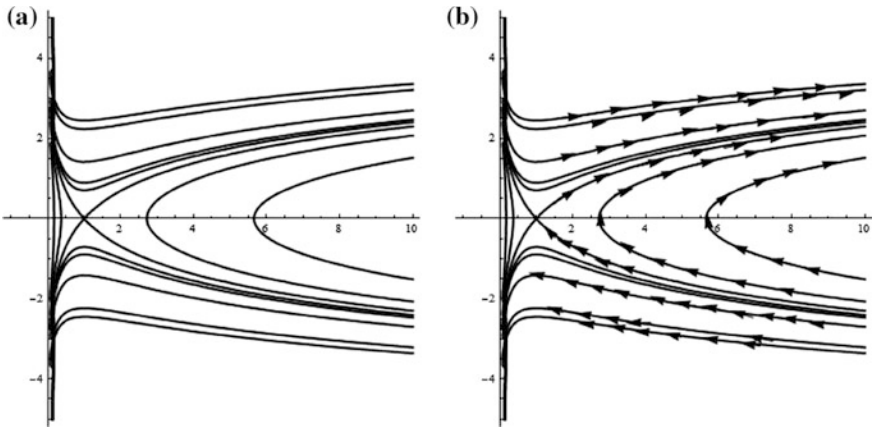


Fig. 7.4 **a** Level curves and **b** Phase portrait of the first integral

hence the system is not conservative. The sketch the phase portrait of the given system in the $x - \dot{x}$ plane is shown in Fig. 7.4. The direction of the trajectories is given by the direction of $\dot{x}$. Now $\dot{x} > 0$ when $y > 0$ and $\dot{x} < 0$ for $y < 0$. Hence the direction of the trajectories are from left to right in the upper half-plane and from right to left in the lower half plane. The level curves give the shape of the trajectories without the directions.

Example 7.22 Find a Hamiltonian H for a moving particle along a straight line, given by the equation of motion

$$\ddot{x} = -x + \beta x^2, \beta > 0, x \,\text{is the displacement}$$

Sketch the Level curve of the Hamiltonian H in the phase plane.

Solution The given nonlinear equation is converted into the following system of equations

$$\dot{x} = y$$
$$\dot{y} = -x + \beta x^2$$

If we consider the variable x as generalized coordinate and y as genaralised momentum then we have

$$\frac{\partial H}{\partial y} = y$$
$$\frac{\partial H}{\partial x} = x - \beta x^2$$
$$\therefore \frac{\mathrm{d}H}{\mathrm{d}t} = \dot{x}\frac{\partial H}{\partial x} + \dot{y}\frac{\partial H}{\partial y} = \dot{x}(x - \beta x^2) + y\dot{y} = 0$$

and the Hamiltonian of the system is given by $H(x, y) = \frac{x^2}{2} - \frac{\beta}{3}x^3 + \frac{y^2}{2}$.

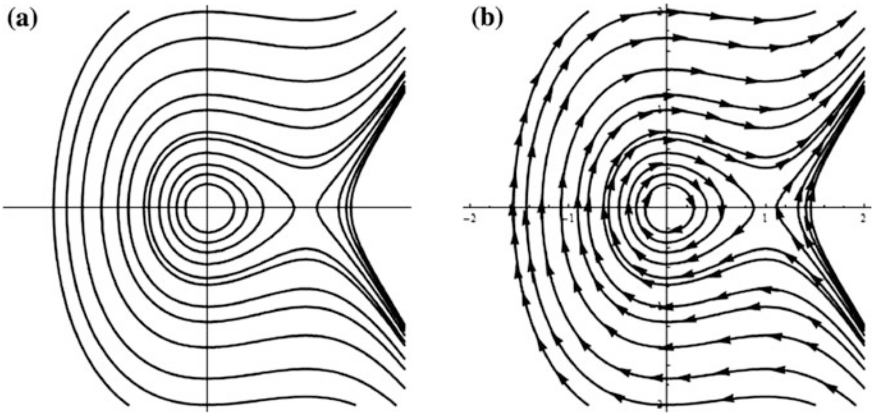


Fig. 7.5 Level curve and phase portrait of $H(x,y) = \frac{x^2}{2} - \frac{\beta}{3}x^3 + \frac{y^2}{2}$ for $\beta = 1$

Now we sketch the phase portrait of the given system in the $x - \dot{x}$ plane shown in Fig. 7.5. The direction of the trajectories is given by the direction of $\dot{x}$. Now $\dot{x} > 0$ when $y > 0$ and $\dot{x} < 0$ for $y < 0$. Hence the direction of the trajectories are from left to right in the upper half-plane and from right to left in the lower half plane.

Example 7.23 Find a Hamiltonian H for the undamped pendulum, given by the equation of motion

$$\ddot{x} + \sin x = 0$$

Sketch the level curves of the Hamiltonian H in the phase plane.

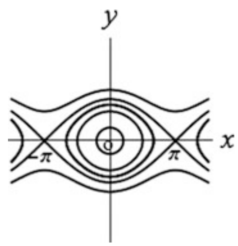
Solution The given equation of pendulum can be written as the following system of equation

$$\dot{x} = y$$
$$\dot{y} = -\sin x$$

If we consider the variable x as the generalized coordinate and y as the generalized velocity then we will have

$$\frac{\partial H}{\partial y} = y$$
$$\frac{\partial H}{\partial x} = \sin x$$

Fig. 7.6 Level curve of undamped pendulum



Hence $\frac{\mathrm{d}H}{\mathrm{d}t} = \dot{x}\frac{\partial H}{\partial x} + \dot{y}\frac{\partial H}{\partial y} = \dot{x}\,\sin\,x + \dot{y}y = 0$

Whence Hamiltonian $H = -\cos\,x + \frac{y^2}{2}$. The level curves are shown in Fig. 7.6.

Example 7.24 Find the Hamiltonian for the system

$$\dot{x} = y$$
$$\dot{y} = -x + x^3$$

Also, sketch the Hamiltonian of the system in the phase plane.

Solution If x is the generalized coordinate of the system and y the generalized velocity, then clearly,

$$\frac{\partial H}{\partial x} = x - x^3 \quad \text{and} \quad \frac{\partial H}{\partial y} = y$$

Hence $\frac{\mathrm{d}H}{\mathrm{d}t} = \dot{x}\frac{\partial H}{\partial x} + \dot{y}\frac{\partial H}{\partial y} = \dot{x}(x - x^3) + \dot{y}y = 0$. Therefore the Hamiltonian of the system is

$$H = \frac{x^2}{2} - \frac{x^4}{4} + \frac{y^2}{2}$$

Now we sketch the phase portrait of the given system in the $x - \dot{x}$ plane shown in Fig. 7.7. The direction of the trajectories is given by the direction of $\dot{x}$. Now $\dot{x} > 0$ when $y > 0$ and $\dot{x} < 0$ for $y < 0$. Hence the direction of the trajectories are from left to right in the upper half-plane and from right to left in the lower half plane.

Example 7.25 Find a conserved quantity for the system $\ddot{x} = a - e^x$ and also sketch the phase portrait for $a < 0, = 0,$ and > 0.

Solution The given system can be written as the following system of equation

$$\dot{x} = p$$
$$\dot{p} = a - e^x$$

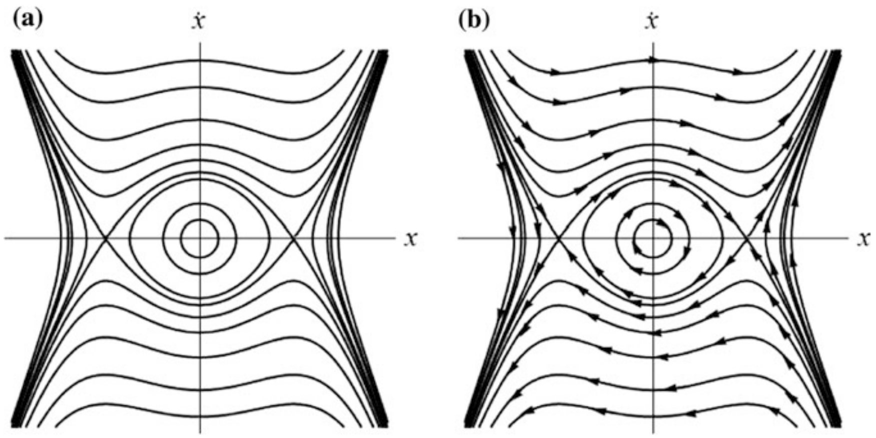


Fig. 7.7 **a** Level curves of $H(x,y) = \frac{x^2}{2} - \frac{x^4}{4} + \frac{y^2}{2}$. **b** Phase portrait of the system $\dot{x} = y,\ \dot{y} = -x + x^3$

For this system $\frac{\partial H}{\partial p} = p$ and $\frac{\partial H}{\partial x} = e^x - a$. Therefore, the Hamiltonian H is given by

$$dH = (e^x - a)dx + pdp$$
$$H = e^x - ax + \frac{p^2}{2}$$

The given system of equation is a Hamiltonian system and the Hamiltonian H is the conserved quantity of the system. When $a = 0$ then the given system has no fixed points and for $a = -1$ the system has complex fixed points $((2n+1)i\pi, 0)$ while for $a = 1$ the only real fixed point is $(0, 0)$ which is a center and the trajectories in the phase plane are closed curves about the fixed point origin. The sketch the phase portrait of the given system in the $x - \dot{x}$ plane shown in Fig. 7.8.

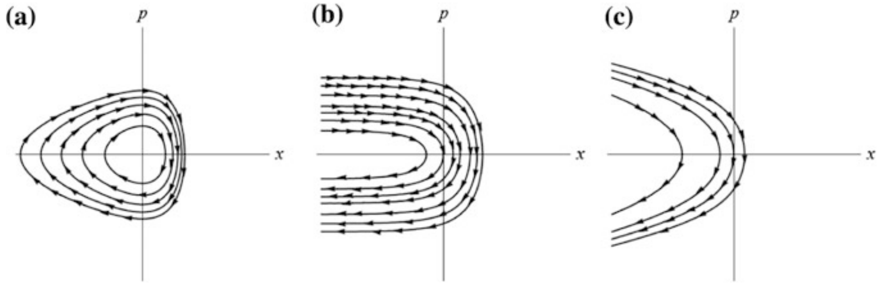


Fig. 7.8 Phase portrait of the Hamiltonian $H = e^x - ax + \frac{p^2}{2}$ for $a > 0, a = 0$ and $a < 0$, **a** $a = 1$, **b** $a = 0$, **c** $a = -1$

The direction of the trajectories is given by the direction of $\dot{x}$. Now $\dot{x} > 0$ when $p > 0$ and $\dot{x} < 0$ for $p < 0$. Hence the direction of the trajectories are from left to right in the upper half-plane and from right to left in the lower half plane.

Example 7.26 Show that the Hamiltonian H is $\frac{p^2}{2m} + \frac{\mu x^2}{2}$ for a simple harmonic oscillator of mass m and spring constant μ. Also, show that H is the total energy.

Solution The equation of motion of a simple linear harmonic oscillator with spring constant μ is given by

$$m\ddot{x} + \mu x = 0$$

This equation can be written as a system of equation given below

$$\dot{x} = \frac{p}{m}$$
$$\dot{p} = -\mu x$$

Now, $\frac{\partial H}{\partial p} = \frac{p}{m}$ and $\frac{\partial H}{\partial x} = \mu x$. Hence the Hamiltonian $H(x, p)$ is given by

$$dH = \frac{\partial H}{\partial x} dx + \frac{\partial H}{\partial p} dp$$
$$H = \mu x dx + \frac{p}{m} dp = \frac{\mu x^2}{2} + \frac{p^2}{2m}$$

Since the given system is a Hamiltonian system, the conserved quantity is the Hamiltonian function H. The only fixed point of the system is the origin $(0, 0)$ obtained by solving $\frac{\partial H}{\partial p} = \frac{p}{m} = 0$ and $\frac{\partial H}{\partial x} = \mu x = 0$. The trajectories of the system are ellipse. The sketch of the phase portrait is given in the Fig. 7.9. In the upper half plane $x > 0$, $\dot{x} > 0$ hence the direction of the trajectories is from left to right and in

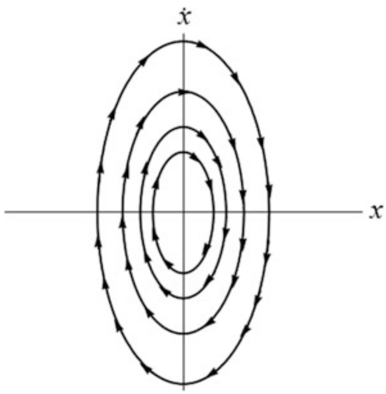


Fig. 7.9 Phase portrait of simple harmonic oscillator

the lower half plane ($x<0$), $\dot{x}<0$ hence the direction of the trajectories is from right to left.

Now the total energy of the system = Kinetic energy of the system + potential energy of the system.

Kinetic energy $=\frac{1}{2}m\dot{x}^2$ and potential energy $=-\int -\mu x dx=\frac{\mu x^2}{2}$ (since the system is conservative and the force acting on the system is $-\mu x$). Therefore the total energy of the system is $\frac{m\dot{x}^2}{2}+\frac{\mu x^2}{2}=\frac{p^2}{2m}+\frac{\mu x^2}{2}$ (Since $\dot{x}=\frac{p}{m}$). Hence, the Hamiltonian H is equal to the total energy of the system.

7.8.2 Critical Points of Hamiltonian Systems

We have learnt the qualitative analysis of a nonlinear dynamical system in Chap. 3 by evaluating the fixed points of the system and various behaviors in its neighbourhood. The fixed points of a conservative Hamiltonian system $\dot{x}=H_y, \dot{y}=-H_x$ are given by

$$\frac{\partial H}{\partial x}=0, \frac{\partial H}{\partial y}=0.$$

The Hamiltonian H is a conserved quantity along any phase path of the system, and gives the shapes of the trajectories of the flow generated by the Hamiltonian vector field X_H in the phase plane of the system. And the fixed points of the system give the local dynamics in its neighbourhood. Moreover, if the Hamiltonian H of a system is known than one can directly yield the fixed points of the system.

Lemma 7.1 *If the origin is a focus of the Hamiltonian system*

$$\dot{x}=\frac{\partial H}{\partial y}, \dot{y}=-\frac{\partial H}{\partial x}$$

Then the origin is not a strict local maximum or minimum of the Hamiltonian function $H(x,y)$.

Theorem 7.15 *Any nondegenerate critical point of an analytic Hamiltonian system*

$$\dot{x}=\frac{\partial H}{\partial y}, \dot{y}=-\frac{\partial H}{\partial x} \tag{7.59}$$

is either a saddle or a center; Again (x_0, y_0) *is a saddle for* (7.59) *iff it is a saddle of the Hamiltonian function* $H(x,y)$ *and a strict local maximum or minimum of the function* $H(x,y)$ *is a center for* (7.59).

Proof Suppose that critical point of the Hamiltonian system

$$\dot{x} = H_y(x, y), \dot{y} = -H_x(x, y)$$

is at origin. Therefore $H_x(0,0) = 0$, $H_y(0,0) = 0$. The linearized system of the Hamiltonian system at the origin is given by

$$\dot{x} = Ax \tag{7.60}$$

where $A = \begin{pmatrix} H_{yx}(0,0) & H_{yy}(0,0) \\ -H_{xx}(0,0) & -H_{xy}(0,0) \end{pmatrix}$

Trace of $A = 0$ and the det $A = H_{xx}(0,0)H_{yy}(0,0) - H_{xy}^2(0,0)$. The critical point at the origin is a saddle of the function $H(x, y)$ if and only if it is saddle of the Hamiltonian system. Now origin is a saddle of the Hamiltonian system if and only if it is a saddle of the linearized system (7.60) i.e. $\det A < 0$. Also if $\mathrm{tr} A = 0$ and det $A > 0$ then the origin is a center for the system (7.60) and then the origin is either a center or focus for the Hamiltonian system. Now if the nondegenerate critical point (0, 0) is a strict local maximum or minimum of the Hamiltonian $H(x, y)$ then det $A > 0$ and then according to the above lemma the origin is not a focus for the Hamiltonian system (7.59) i.e. the origin is a center for the Hamiltonian system (7.59).

Example 7.27 Hamiltonian H for the undamped pendulum, is given by $H = 1 - \cos x + \frac{y^2}{2}$. Calculate its fixed points and sketch the phase portrait of the Hamiltonian H.

Solution The fixed points of the system is given by

$$\frac{\partial H}{\partial x} = \sin x = 0,$$
$$\frac{\partial H}{\partial y} = y = 0$$

Hence the fixed points are $(n\pi, 0)$, $n \in \mathbb{Z}$.

We shall now analyze the trajectories in the neighbourhood of $(0,0)$. In this case the linearized system is

$$\dot{x} = y$$
$$\dot{y} = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots$$

Neglecting the higher order terms, the linearized system is obtained as

$$\dot{x} = y$$
$$\dot{y} = -x$$

In matrix form it can be written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The eigen values of the system are complex with real part zero. Hence the origin $(0, 0)$ is the center of the system. About the fixed point $(-\pi, 0)$ the Hamiltonian equation of motion can be linearized by introducing $(x, y) \to (x - \pi, y)$. We obtain

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\sin(x - \pi) = \sin(\pi - x) \\ &= \sin x = x - \frac{x^3}{3!} + \cdots \end{aligned}$$

Neglecting higher order terms we get the linearized system as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x \end{aligned}$$

In matrix form the above system can be written as $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

In this case the system has real distinct eigen values of opposite signs. Hence the fixed point $(-\pi, 0)$ is a saddle of the system.

About the fixed point $(\pi, 0)$ the Hamiltonian equation of motion can be linearized by introducing $(x, y) \to (x + \pi, y)$. We obtain

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\sin(\pi + x) = \sin x \\ &= x - \frac{x^3}{3!} + \cdots \end{aligned}$$

In this case also the linearized system is (Fig. 7.10)

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x \end{aligned}$$

Hence the fixed point $(\pi, 0)$ is a saddle of the system.

Now we will see the nature of the fixed points $(2\pi, 0)$. Here introducing $(x, y) \to (x + 2\pi, y)$ we have the

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\sin(2\pi + x) = -\sin x \\ &= -x + \frac{x^3}{3!} + \cdots \end{aligned}$$

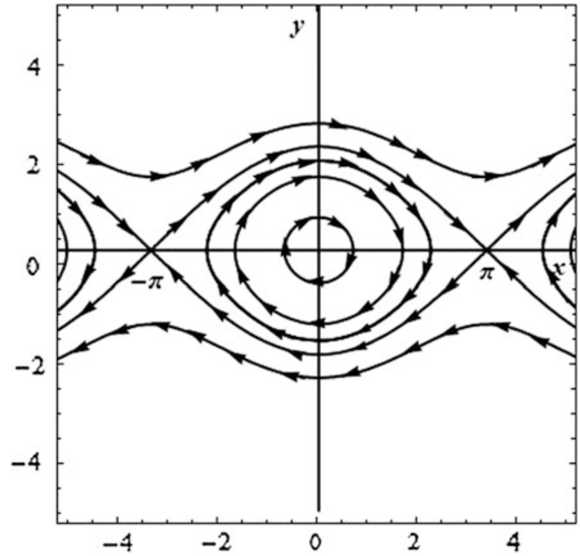


Fig. 7.10 Phase portrait of undamped pendulum

Neglecting the higher order terms we have the linearized system

$$\dot{x} = y$$
$$\dot{y} = -x$$

In matrix form the above system can be written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

which gives complex eigen values with real part zero, hence $(2\pi, 0)$ is a center similarly the fixed point $(-2\pi, 0)$ is also a center. Hence, we can conclude that the fixed points $(2n\pi, 0), n \in \mathbb{Z}$ are centers and the fixed points $((2n+1)\pi, 0), n \in \mathbb{Z}$ are saddles. The phase portrait of the given system is given in Fig. 7.10. The given system is a Hamiltonian system hence the system is a conservative system with no dissipation and the total energy of the system is given by the Hamiltonian H, which is the conserved quantity. The separatrices of the system divide the phase space into two types of qualitatively different behaviors of the trajectories. Each trajectory corresponds to a particular value of the energy H. The trajectories inside the sepratrices around the fixed points $(2n\pi, 0), n \in \mathbb{Z}$ have small values of the energy and are nearly circles which describes the usual to and fro (Oscillatory) motion of the pendulum about the equilibrium points. The sepratrices which connects the saddles at $(\pm\pi, 0)$ corresponds to the motion with total energy $H = 2$ and the pendulum tends to the unstable vertical position as $t \to \pm\infty$. The trajectories outside the sepratrix loops are hyperbolas with total energy $H > 2$ and the pendulum swing over the top. The motion corresponds to clockwise motion for negative angular velocity and counterclockwise motion for positive angular velocity.

7.8.3 Hamiltonian and Gradient Systems

We have already defined the gradient systems in Chap. 4. Herein we will discuss the relationship of the gradient system with the Hamiltonian system. For convenience we are giving the definition of gradient system once again as follows

Definition 7.6 A system given by $\dot{x} = -\text{grad}\, F(x)$, $\text{grad}\, F = \left(\frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}\right)^T$ and the function $F \in C^2(E)$, where E is an open subset of $\mathbb{R}^n$ is called a gradient system on E.

The critical points or fixed points of a gradient system is given by the function $F(x)$ where $\text{grad}\, F(x) = 0$. The points for which $\text{grad}\, F(x) \neq 0$ are called regular points of the function $F(x)$. At regular points of $F(x)$ the vector $\text{grad}\, F(x)$ is perpendicular to the level surface $F(x) =$ constant through the regular point.

We know that any system orthogonal to a two dimensional system $\dot{x} = f(x, y), \dot{y} = g(x, y)$ is given as $\dot{x} = g(x, y), \dot{y} = -f(x, y)$. The critical points of these two systems are same. Moreover, the centre of a planar system corresponds to the nodes of its orthogonal system. Also the saddle and foci of the planar system are the saddles and foci of its orthogonal system. At regular points the trajectories of the planar system and its orthogonal system are orthogonal to each other. Again if the planar system is a Hamiltonian system then the system orthogonal to this system is a gradient system and conversely. It follows that there is an interesting relationship between the gradient system and the Hamiltonian system. The following theorem gives the relationship between the Hamiltonian and gradient system.

Theorem 7.16 *The planar system given by $\dot{x} = f(x, y), \dot{y} = g(x, y)$ is a Hamiltonian system if and only if the system orthogonal to it, given by $\dot{x} = g(x, y), \dot{y} = -f(x, y)$ is a gradient system.*

Proof Suppose that the system

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned} \tag{7.61}$$

is a Hamiltonian system, therefore $\nabla \cdot (f, g) = 0$

i.e.$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0$. So there exist a function say H for which we can write $f = \frac{\partial H}{\partial y}$ and $g = -\frac{\partial H}{\partial x}$. Now the system orthogonal to this system is

$$\begin{aligned} \dot{x} &= g(x, y) \\ \dot{y} &= -f(x, y) \end{aligned} \tag{7.62}$$

which can be written as

$$\dot{x} = -\frac{\partial H}{\partial x}, \dot{y} = -\frac{\partial H}{\partial y}$$

or $\dot{x} = -\text{grad}\, H(x)$, $x = (x, y)$ where $\text{grad}\, H = \left(\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}\right)$. This is by definition a gradient system. Conversely suppose that the system (7.61) orthogonal to (7.60) is a gradient system therefore there exists a function say H such that we can write $g = -\frac{\partial H}{\partial x}$ and $f = \frac{\partial H}{\partial y}$. For this the system (7.60) can be written as

$$\dot{x} = \frac{\partial H}{\partial y}, \dot{y} = -\frac{\partial H}{\partial x}$$

For which it can be easily checked that $\nabla \cdot \left(\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x}\right) = 0$. Hence the system (7.61) is a Hamiltonian system when the system (7.62) is a gradient system.

Note that the trajectories of the gradient system (7.62) cross the surface $H(x, y) = \text{constant}$ orthogonally.

For the Hamiltonian system in higher dimensional spaces say for n degrees of freedom is given by

$$\left.\begin{aligned} \dot{x} &= \frac{\partial H}{\partial y} \\ \dot{y} &= -\frac{\partial H}{\partial x} \end{aligned}\right\} \tag{7.63}$$

Then the system orthogonal to (7.63)

$$\left.\begin{aligned} \dot{x} &= -\frac{\partial H}{\partial x} \\ \dot{y} &= -\frac{\partial H}{\partial y} \end{aligned}\right\} \tag{7.64}$$

is a gradient system in $\mathbb{R}^{2n}$.

Example 7.28 For the Hamiltonian function $H(x, y) = y \sin x$, sketch the phase portraits of the Hamiltonian system and its gradient system.

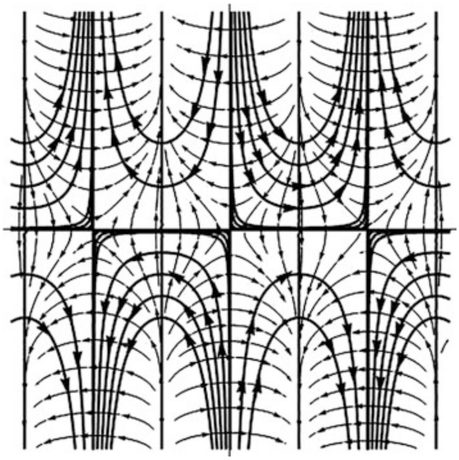


Fig. 7.11 Phase portrait of the Hamiltonian and its gradient system

Solution For $H(x, y) = y \sin x$, the Hamiltonian system is given by

$$\begin{aligned} \dot{x} &= \sin x \\ \dot{y} &= -y \cos x \end{aligned} \tag{7.65}$$

Therefore its gradient system is given by

$$\begin{aligned} \dot{x} &= -y \cos x \\ \dot{y} &= -\sin x \end{aligned}$$

The critical points of the Hamiltonian system and the gradient system are at $(n\pi, 0), n \in \mathbb{Z}$. The phase portrait of the Hamiltonian and Gradient system is shown in Fig. 7.11.

7.9 Symplectic Transformations

The symplectic or canonical transformation is an important coordinate transformation from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n}$ as it preserves the flow generated by the Hamiltonian vector field X_H of the Hamiltonian system. Under symplectic transformation, Hamilton equation of motion is form invariant. Also, the Hamilton H in terms of new coordinate, obtained under this transformation is such that the new system becomes comparatively easier as it reveals all cyclic coordinates and conserved quantities. We will first define the symplectic form.

7.9.1 Symplectic Forms

Let $\underset{\sim}{x} = (q, p)$, then the Hamiltonian system can be written as $\dot{\underset{\sim}{x}} = JDH(x)$, where $J = \begin{pmatrix} O & I \\ -I & O \end{pmatrix}$, O, I denotes the $n \times n$ null and unit matrices respectively and $DH(\dot{\underset{\sim}{x}}) = \left(\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}\right)$.

A bilinear form Ω is said to be symplectic on the phase space $\mathbb{R}^{2n}$ if it is a skew-symmetric and nondegenerate that is, the matrix representation of the bilinear form is non-singular and skew-symmetric. A vector space is said to be a symplectic vector space if it is furnished with a symplectic form. A symplectic form for the vector space, particularly for the phase space $\mathbb{R}^{2n}$ is given by

$$\Omega(u, \upsilon) = \langle u, J\upsilon \rangle, u, \upsilon \in \mathbb{R}^{2n}$$

where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product on $\mathbb{R}^{2n}$ and J is the nonsingular, skew-symmetric matrix defined above. This particular symplectic structure on $\mathbb{R}^{2n}$ is called the Canonical symplectic form.

7.9.2 Symplectic Transformation

An $r(\geq 1)$ times continuously differentiable diffeomorphism $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is said to be a Symplectic or Canonical transformation if, $\Omega(u, \upsilon) = \Omega(D\phi(x)u, D\phi(x)\upsilon)$ $\forall x, u, \upsilon \in \mathbb{R}^{2n}$.

7.9.3 Derivation of Hamilton's Equations from Symplectic Form

Hamilton's equation of motion in phase space can be derived from the symplectic form. For this consider the phase space $\mathbb{R}^{2n}$ for the Hamiltonian vector field $X_H(\underset{\sim}{x}) \equiv \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right)$ obtained from the Hamiltonian function H. The symplectic structure for $X_H(x)$ is expressed by

$$\Omega(X_H(x), \upsilon) = \langle DH(x), \upsilon \rangle, x \in \mathcal{U} \subset \mathbb{R}^{2n}, \upsilon \in \mathbb{R}^{2n} \tag{7.66}$$

Now if $X_H(x) = (\dot{q}, \dot{p})$ is an arbitrary vector field on some subset $\mathcal{U} \subset \mathbb{R}^{2n}$ with $DH = \left(\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}\right)$ then the above equation becomes

$$\Omega((\dot{q},\dot{p}),\upsilon) = \langle(\dot{q},\dot{p}),J\upsilon\rangle = \left\langle\left(\frac{\partial H}{\partial q},\frac{\partial H}{\partial p}\right),\upsilon\right\rangle \tag{7.67}$$

Now, $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = -J^T$, therefore we get

$$\langle(\dot{q},\dot{p}),J\upsilon\rangle = \langle -J(\dot{q},\dot{p}),\upsilon\rangle = \langle(-\dot{p},\dot{q}),\upsilon\rangle \tag{7.68}$$

Substituting (7.68) into (7.67), we get

$$\langle(-\dot{p},\dot{q}),\upsilon\rangle = \left\langle\left(\frac{\partial H}{\partial q},\frac{\partial H}{\partial p}\right),\upsilon\right\rangle$$

For the symplectic form, the inner product must be nondegenerate. Using linearity principle of symplectic form and for fixed $\upsilon \in \mathbb{R}^{2n}$, the above form can be written as

$$\left\langle(-\dot{p},\dot{q}) - \left(\frac{\partial H}{\partial q},\frac{\partial H}{\partial p}\right),\upsilon\right\rangle = 0$$

This relation holds for all υ. By non degeneracy of the symplectic form we have

$$\begin{aligned}(-\dot{p},\dot{q}) - \left(\frac{\partial H}{\partial q},\frac{\partial H}{\partial p}\right) &= 0 \\ \Rightarrow \dot{p} = \frac{\partial H}{\partial q}, \dot{q} &= -\frac{\partial H}{\partial p}.\end{aligned}$$

Hence, the Hamiltonian canonical equations are established.

Example 7.29 Show that a transformation $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is symplectic with respect to the canonical symplectic form if $D\phi(x)^T JD\phi(x) = J \quad \forall x, u, \upsilon \in \mathbb{R}^{2n}$.

Solution We have for symplectic transformation $\Omega(u,\upsilon) = \Omega(D\phi(x)u, D\phi(x)\upsilon)$ $\forall x, u, \upsilon \in \mathbb{R}^{2n}$. Now for a canonical symplectic form this relation can be written as

$$\langle u, J\upsilon\rangle = \langle D\phi(x)u, JD\phi(x)\upsilon\rangle = \langle u, (D\phi(x))^T JD\phi(x)\upsilon\rangle$$

Since this holds for all $u, \upsilon \in \mathbb{R}^{2n}$, therefore $(D\phi(x))^T JD\phi(x) = J$.

Theorem 7.17. *The flow generated by the Hamiltonian vector field X_H defined on some open convex set $\mathcal{U} \in \mathbb{R}^{2n}$ is a one parameter family of symplectic (canonical) transformation and conversely if the flow generated by a vector field comprise of symplectic transformation for each t, then the vector field is a Hamiltonian vector field.*

For proof see the book of Stephen Wiggins [11].

7.10 Poisson Brackets

Poisson bracket is a connection between a pair of dynamical variables for any holonomic system which remains invariant under any symplectic transformation. This relation is helpful in testing whether a phase space transformation is symplectic or not. Also, using Poisson bracket new integrals of motion can be constructed from those already known. Poisson bracket of the variables of the Hamiltonian system in the phase space $\mathbb{R}^{2n}$ is given as follows:

The Poisson bracket of any two $C^r, r \geq 2$ (r times continuously differentiable functions) functions F, G of $x \in \mathbb{R}^{2n}$ such that $F, G : \mathcal{U} \subset \mathbb{R}^{2n} \to \mathbb{R}$ is a function defined by the following notation:

$$\{F, G\} = \Omega(X_F, X_G) = \langle X_F, JX_G \rangle \tag{7.69}$$

From (7.69), we can write

$$\begin{aligned}\{F, G\} &= \Omega(X_F, X_G) = \langle X_F, JX_G \rangle = \left\langle J^T X_F, X_G \right\rangle = \langle -JX_F, X_G \rangle = -\langle JX_F, X_G \rangle \\ &= -\langle X_G, JX_F \rangle = -\{G, F\}\end{aligned}$$

This implies that the Poisson bracket is anti-symmetric.

If $F \equiv H$, the Hamiltonian vector fields $X_H(x)$ obtained from the Hamilton function H is given by

$$X_H(x) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right), \text{ and similarly } X_G(x) = \left(\frac{\partial G}{\partial p}, -\frac{\partial G}{\partial q}\right),$$

The Poisson bracket of H and G is therefore given as

$$\{H, G\} = \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial G}{\partial q_i},$$

this implies that

$$\{F, F\} = 0. \tag{7.70}$$

The Hamilton's equation of motion $\dot{q}_i = \frac{\partial H}{\partial p_i}$ and $\dot{p}_i = -\frac{\partial H}{\partial q_i}$ can be written in terms of Poisson bracket. For this, the rate of change of any function F along the trajectories generated by the Hamiltonian vector field X_H is given by

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \sum_{i=1}^{n}\left(\frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial p_i}\dot{p}_i\right) = \sum_{i=1}^{n}\left(\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}\right) = \{F, H\} \tag{7.71}$$

So the above relation is an alternative way to write Hamilton's equation of motion for all function $F : \mathcal{U} \to \mathbb{R}$. Again the Hamilton's equation of motion

$\dot{q} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial q}$ can be easily established from the relation $\frac{\mathrm{d}F}{\mathrm{d}t} = \{F, H\}$. For that consider the time derivative of the function F, given as $\frac{\mathrm{d}F}{\mathrm{d}t} = \sum_{i=1}^{n} \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial p_i}\dot{p}_i$, then subtracting the latter relation from the former, we have

$$\{F, H\} - \sum_{i=1}^{n} \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial p_i}\dot{p}_i = 0$$

Now since the Poisson bracket of F, H is $\sum_{i=1}^{n} \frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}$, we have

$$\sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} - \dot{q}_i\right)\frac{\partial F}{\partial q_i} - \left(\frac{\partial H}{\partial q_i} + \dot{p}_i\right)\frac{\partial F}{\partial p_i} = 0$$

which gives $\dot{q}_i = \frac{\partial H}{\partial p_i}$ and $\dot{p}_i = -\frac{\partial H}{\partial q_i}$.

The relation (7.70) and (7.71) yields the following proposition.

Proposition 7.1 *The Hamiltonian $H(q, p)$ is constant along trajectories of the Hamiltonian vector field X_H.*

In other words, any function F which satisfies $\dot{F} = \{F, H\} = 0$ is an integral or constant of motion with respect to the flow generated by the Hamiltonian vector field X_H. We have already established this result without using Poisson bracket that has a fine geometric structure and can be obtained as follows:

We have $\{F, H\} = \langle X_F, JX_H\rangle = -\langle JX_F, X_H\rangle = 0$

Now, $JX_F = J\begin{pmatrix} \frac{\partial F}{\partial P} \\ -\frac{\partial F}{\partial q} \end{pmatrix} = \begin{pmatrix} O & I \\ -I & O \end{pmatrix}\begin{pmatrix} \frac{\partial F}{\partial P} \\ -\frac{\partial F}{\partial q} \end{pmatrix} = -\begin{pmatrix} \frac{\partial F}{\partial q} \\ \frac{\partial F}{\partial p} \end{pmatrix}$. Thus the vector $-\left(\frac{\partial F}{\partial q}, \frac{\partial F}{\partial p}\right)$ is the vector perpendicular to the surface due to the level set of the integral of motion F, at each point at which it is evaluated. Thus geometrically the vector field is tangent to the surface given by the level set of the integral (see Wiggins [11]).

The Poisson bracket of any two functions F and G satisfies the following properties

(i) $\{F, G\}_{p,q} = -\{G, F\}_{p,q} = \{G, F\}_{q,p}$.
(ii) $\{F, C\} = 0$, if C is a constant.
(iii) $\{F_1 + F_2, G\} = \{F_1, G\} + \{F_2, G\}$.
(iv) $\{F_1F_2, G\} = F_1\{F_2, G\} + \{F_1, G\}F_2$.
(v) For any three functions F, G, H, Poisson bracket satisfies the Jacobi identity

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$$

Furthermore, the Poisson bracket of any two functions F and G remains invariant under a symplectic transformation that is, $\{F, G\}_{p,q} = \{F, G\}_{P,Q}$, where P, Q are obtained from p, q under symplectic transformation. Again the Poisson bracket of the variables P, Q obtained from p, q, given by $\{P_i, Q_j\} = \delta_{ij}$, $\{P_i, P_j\} = 0 = \{Q_i, Q_j\}$ are useful for testing the canonicality of any phase space transformation.

7.11 Hamilton–Jacobi Equation

Here we have another formulation of motion of a system that is, Hamilton–Jacobi equation. At this point one may ask why to formulate any other method when we already have Hamilton's canonical equations of motions. The answer is that Hamilton's equations of motions are system of first order differential equation which though looks simple is usually harder to solve. Thus Hamilton's formulation though provides us a simpler formulation over the Lagrangian by giving liberty in choosing the generalized coordinate and momenta, but the difficulty in solving the problem remains same. Hamilton–Jacobi equation on the other hand is a single partial differential equation. Even though, it is also not easy to solve this equation, but can be solved when variables are separable. Hamilton–Jacobi equation was formulated by Carl Gustov Jacob Jacobi (1804–1851) which is useful in particular for solving conservative periodic systems. This theory is regarded as the most complex, yet significant and strong approach for solving problems in classical mechanics. By using Hamilton–Jacobi equation all hidden constants of motion in spite of having complicated form can be found out. We shall now give the formulation of Hamilton–Jacobi equation as follows:

Suppose that $\bar{q}(t)$ is the extremal of the action integral $\int_{t_0}^{t} L(q, \dot{q}, t)\mathrm{d}t$ with $\bar{q}(t_0) = q_0$ and $\bar{q}(t) = q$, where q_0 and t_0 are fixed. Mathematically this can be written as

$$A(q,t) = \int_{t_0}^{t} L(\bar{q}, \dot{\bar{q}}, t)dt \tag{7.72}$$

For an infinitesimal change h, its differential is given by

$$\begin{aligned} dA(q,t)h &= \int_{t_0}^{t} (L(q_i + h, \dot{q}_i + \dot{h}, t) - L(q_i, \dot{q}_i, t))dt + o(h^2) \\ &= \int_{t_0}^{t} \sum_{i=1}^{N} \left(\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_i} \right) h_i dt + \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{q}_i} h_i \Big|_{t_0}^{t} \quad \text{(using integration by parts)} \end{aligned}$$

this gives $dA(q,t)h = \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{q}_i} h_i = \sum_{i=1}^{N} p_i h_i$ (by using $p_i = \frac{\partial L}{\partial \dot{q}_i}$), the first term vanishes since $\bar{q}(t)$ is an extremal of the action integral and $\bar{q}(t_0) = q_0$ is fixed for each extremal. Since the variation is taken only for q in computing the differential of A one obtains the relation $p_i = \frac{\partial A}{\partial q_i}$. Again from the action integral (7.72) we have $\frac{dA}{dt} = L$.

Therefore

$$\frac{dA}{dt} = \sum_{i=1}^{N} \frac{\partial A}{\partial q_i}\dot{q}_i + \frac{\partial A}{\partial t} = L \Rightarrow \sum_{i=1}^{N} p_i\dot{q}_i + \frac{\partial A}{\partial t} = L \Rightarrow L - \sum_{i=1}^{N} p_i\dot{q}_i = \frac{\partial A}{\partial t}$$

which can be rewritten as

$$H(q_i, p_i, t) = \frac{\partial A}{\partial t} \text{ or}, \frac{\partial A}{\partial t} - H\left(q_i, \frac{\partial A}{\partial q_i}, t\right) = 0 \tag{7.73}$$

The above equation is a first order partial differential equation for the function $A(q,t)$ containing $(n+1)$ partial derivatives, known as Hamilton–Jacobi equation. The trajectories of the Hamilton's canonical equation can be obtained from the solutions of the Hamilton–Jacobi equation which follows from the following Jacobi theorem, proved by Jacobi in the year 1845.

Theorem 7.18 *If the Hamilton Jacobi equation given by* (7.73) *admits a complete integral* $A = f(q_1, q_2, \ldots, q_N, t; c_1, c_2, \ldots, c_N) + \alpha$ *then the equations* $\frac{\partial f}{\partial \alpha_i} = \beta_i, \frac{\partial f}{\partial q_i} = p_i$ *with the 2N arbitrary constant* a_i, c_i, α_i, β_i, *respectively gives the 2N parameter family of solutions of Hamilton's equations* $\dot{q}_i = \frac{\partial H}{\partial p_i}, \dot{p}_i = -\frac{\partial H}{\partial q_i}$ $i = 1, \ldots, N$.

Proof For a complete integral of the Hamilton Jacobi equation we must have $\det\left(\frac{\partial^2 f}{\partial q \partial \alpha}\right) \neq 0$ and therefore one can get the solution of the N equations $\frac{\partial f}{\partial \alpha_i} = \beta_i$ for q_i as a function of t and the $2N$ constants α_i, β_i. Substituting these functions in $\frac{\partial f}{\partial q_i} = p_i$ one will obtain p_i as a function of t and the $2N$ constants α_i, β_i. Now in order to obtain the proof of the theorem differentiate $\frac{\partial f}{\partial \alpha_i} = \beta_i$ with respect to t, which gives

$$\frac{\partial^2 f}{\partial t \partial \alpha_i} + \sum_{j=1}^{N} \frac{\partial^2 f}{\partial q_j \partial \alpha_i}\frac{dq_j}{dt} = 0 \tag{7.74}$$

Now differentiate $\frac{\partial f}{\partial t} + H(q, \frac{\partial f}{\partial q}, t) = 0$ with respect to α_i which gives

$$\frac{\partial^2 f}{\partial t \partial \alpha_i} + \sum_{j=1}^{N} \frac{\partial H}{\partial p_j}\frac{\partial^2 f}{\partial q_j \partial \alpha_i} = 0 \tag{7.75}$$

Now using $\det\left(\frac{\partial^2 f}{\partial q \partial \alpha}\right) \neq 0$ and subtracting (7.75) from (7.74), one will obtain

$$\sum_{j=1}^{N} \left(\frac{dq_j}{dt} - \frac{\partial H}{\partial p_j}\right)\frac{\partial^2 f}{\partial q_j \partial \alpha_i} = 0 \Rightarrow \dot{q}_j = \frac{\partial H}{\partial p_j} \quad j = 1, 2, \ldots$$

Similarly, differentiating $\frac{\partial f}{\partial q_i} = p_i$, with respect to t, we have

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = \frac{\partial^2 f}{\partial t \partial q_i} + \sum_{j=1}^{N} \frac{\partial^2 f}{\partial q_j \partial q_i} \frac{\mathrm{d}q_j}{\mathrm{d}t} \tag{7.76}$$

Again differentiating $\frac{\partial f}{\partial t} + H(q, \frac{\partial f}{\partial q}, t) = 0$ with respect to q_i, we will have

$$\frac{\partial^2 f}{\partial t \partial q_i} + \sum_{j=1}^{N} \frac{\partial H}{\partial p_j} \frac{\partial^2 f}{\partial q_j \partial q_i} + \frac{\partial H}{\partial q_i} = 0 \tag{7.77}$$

Now substituting $\dot{q}_j = \frac{\partial H}{\partial p_j}$ in (7.76) and then using (7.76) in (7.77), we will have

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad i = 1, 2, \ldots, N$$

Let us elaborate this with the following examples.

Example 7.30 Find the trajectory of the motion of a free particle using Hamilton–Jacobi equations.

Solution The Hamiltonian of a free particle is $H = \frac{1}{2} m (\dot{q})^2$ and the momentum $p = m\dot{q}$, therefore $H = \frac{p^2}{2m}$. Again $p = \frac{\partial A}{\partial q}$. Therefore the Hamilton–Jacobi equation becomes $\frac{1}{2m} \left(\frac{\partial A}{\partial q} \right)^2 + \frac{\partial A}{\partial t} = 0$.

Solving this equation one can get, $A(q, E, t) = \sqrt{2mE}q - Et$ where E is the non-additive constant.

This is a complete integral.

Now, using the solution of the Hamilton–Jacobi equation, the solution of the Hamilton's equation is obtained as follows

$$\begin{aligned} \frac{\partial A}{\partial \alpha} &= \beta \Rightarrow \beta = \sqrt{\frac{m}{2E}} q - t \Rightarrow q = \sqrt{\frac{2E}{m}} (t + \beta) \\ \text{and} p &= \frac{\partial A}{\partial q} = \sqrt{2mE} \end{aligned}$$

The constant β can be obtained when the initial condition for the trajectories are prescribed.

Example 7.31 Find the trajectories of the simple harmonic oscillator using Hamilton–Jacobi equation.

Solution For harmonic oscillator, the Hamiltonian function $H = \frac{1}{2m} (p^2 + m^2 \omega^2 q^2)$.

Now, $p = \frac{\partial A}{\partial q}$ gives the Hamilton–Jacobi equation $\frac{1}{2m}\left[\left(\frac{\partial A}{\partial q}\right)^2 + m^2\omega^2 q^2\right] + \frac{\partial A}{\partial t} = 0$

Let us try to solve this using method of separation of variables. Suppose $A(q;\alpha;t) = W(q;\alpha) - \alpha t$, α is a non-additive constant.

We have from Hamilton-Jacobi equation, $\frac{1}{2m}\left[\left(\frac{\partial W}{\partial q}\right)^2 + m^2\omega^2 q^2\right] = \alpha \Rightarrow W = \sqrt{2m\alpha}\int\sqrt{1 - \frac{m^2\omega^2 q^2}{2\alpha}}dq.$

Thus the solution of the Hamilton–Jacobi equation is obtained as

$$A(q;\alpha;t) = \sqrt{2m\alpha}\int\sqrt{1 - \frac{m^2\omega^2 q^2}{2\alpha}}dq - \alpha t$$

Now using the above solution of Hamilton–Jacobi equation, the solution of Hamilton's canonical equation can be obtained as follows $\beta = \frac{\partial A}{\partial \alpha}$ gives $\beta = \sqrt{\frac{m}{2\alpha}}\int\frac{\mathrm{d}q}{\sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}}} - t$ which on integration gives $t + \beta = \frac{1}{\omega}\sin^{-1} q\sqrt{\frac{m\omega^2}{2\alpha}}$.

Thus we have,

$$\begin{aligned} q &= \sqrt{\frac{2\alpha}{m\omega^2}}\sin\,\omega(t+\beta), p = \frac{\partial S}{\partial q} = \frac{\partial W}{\partial q} = \sqrt{2m\alpha - m^2\omega^2 q^2} \\ &= \sqrt{2m\alpha}\cos\,\omega(t+\beta) \end{aligned}$$

Remarks Consider the time independent Hamilton–Jacobi equation given by

$$H\left(q, \frac{\partial G_2}{\partial q}\right) + \frac{\partial G_2}{\partial t} = 0 \tag{7.78}$$

To solve this let us use the method of separation of variables and take

$$G_2(q,t) = W(q) + T(t) \tag{7.79}$$

Thus we have,

$$H\left(q, \frac{\partial W}{\partial q}\right) = E, \frac{\partial T}{\partial t} = -E \tag{7.80}$$

The first Eq. (7.80) is known as time independent Hamilton–Jacobi equation and E represents the total constant energy. We then write

$$G_2 = W(q_1, q_2, \ldots, q_n; \alpha_1, \alpha_2, \ldots, \alpha_n) - E(\alpha_1, \alpha_2, \ldots, \alpha_n)t$$

with the canonical transformation

$$p = \frac{\partial W(q;\alpha)}{\partial q}, \beta = \frac{\partial W(q;\alpha)}{\partial \alpha} - \frac{\partial E(\alpha)}{\partial \alpha} t. \tag{7.81}$$

This may be interpreted as a transformation from $(q, p) \to (Q, P)$ where $Q = \beta + \frac{\partial E}{\partial \alpha} t, P = \alpha$. The transformed Hamiltonian is $E(P)$ since $E = E(\alpha_1, \alpha_2, \ldots, \alpha_n) = E(\alpha) = E(P)$.

7.12 Exercises

1. Prove that for a scleronomous system the kinetic energy T is a homogeneous quadratic function of $\dot{q}_k, k = 1, 2, ..., n$ and hence $2T = \sum_{k=1}^{n} \dot{q}_k \frac{\partial T}{\partial \dot{q}_k}$
2. Find the Lagrangian and Lagrange's equation of motion for a simple harmonic oscillator of mass m and spring constant k freely moving in a plane. Show that this system gives two independent integrals.
3. Find the energy integral for the simple pendulum and hence show that it is constant on trajectories.
4. Formulate Lagrange's equations of motion for the double pendulum.
5. Find Hamiltonian and Hamilton's equations of motion for a particle of mass m moving in the xy-plane under the influence of a central force depending on the distance from the origin.
6. Find the Hamiltonian function $H(x, y)$ for the system $\dot{x} = y,\ \dot{y} = -\mu(e^{-x} - e^{-2x})$.
7. Prove that node and focus types fixed points cannot exists for Hamiltonian systems.
8. Prove that $\frac{dH}{dt} = \frac{\partial H}{\partial t}$ where H is the Hamiltonian.
9. State and prove the theorem for conservation of linear momentum.
10. Prove the theorem for conservation of angular momentum.
11. State and prove the theorem for conservation of energy.
12. A particle of mass m moves in a plane. Find Hamilton's equations of motion.
13. Find the Hamilton's equations of motion for a compound pendulum oscillating in a vertical plane about a fixed horizontal axis.
14. A bead is sliding on a uniformly rotating wire in a force-free space. Obtain the equations of motion in terms of Hamiltonian.
15. Construct the Hamiltonian and find the Hamilton's equations of motion of a coplanar double pendulum placed in a uniform gravitational field.
16. A particle of mass m is attracted to a fixed point O by an inverse square law of force $F_r = -\frac{\mu}{r^2}$ where $\mu (> 0)$ is a constant. Using Hamiltonian, obtain the equations of motion of the particle.

17. Use Hamiltonian to obtain the Hamilton's equations of motion of a projectile in space.

18. The Hamiltonian of a dynamical system is given by $H = qp^2 - qp + bp$ where b is a constant. Obtain the equations of motion.

19. Using cylindrical coordinates obtain the Hamilton's equations of motion for a particle of mass m moving inside the frictionless cone $x^2 + y^2 = z^2 \tan^2 \alpha$.

20. If the kinetic energy $T = \frac{1}{2} m r^2$ and the potential energy $V = \frac{1}{r}\left(1 + \frac{r^2}{c^2}\right)$, find the Hamiltonian.

    Determine whether (i) $H = T + V$ and (ii) $\frac{dH}{dt} = 0$.

21. For the Hamiltonian $H = q_1 p_1 - q_2 p_2 - a q_1^2 - b q_2^2$, solve the Hamilton's equations of motion and prove that $\frac{p_2 - b q_2}{q_1} =$ constant and $q_1 q_2 =$ constant, a,b are constants and q_1, q_2, p_1, p_2 are generalized coordinates.

22. The Lagrangian of a system of one degree of freedom can be written as $L = \frac{m}{2}\left(\dot{q}^2 \sin \omega t + q\dot{q}\omega \sin 2\omega t + q^2 \omega^2\right)$. Determine the corresponding Hamiltonian. Is it conserved?
23. Define first integral of a system. When a system is conservative? What can you say about the dynamics of conservative system?
24. Find the first integrals of the following system and also check whether the systems are conservative or not
    (i) $\dot{x} = y,\ \dot{y} = x^2 + 1$ (ii) $\dot{x} = x(y+1),\ \dot{y} = -y(x+1)$ (iii) $\dot{x} = -(1-y)x,\ \dot{y} = -(1-x)y$
    (iv) $\dot{x} = -x^3,\ \dot{y} = -x^2 y$ (v) $\dot{x} = -x + 2xy^2,\ \dot{y} = -x^2 y^3$
25. Find the Hamiltonian H of the following system
    (i) $\dot{x} = 4p,\ \dot{p} = -2x$ (ii) $\dot{x} = -2p,\ \dot{p} = -2x$ (iii) $\dot{x} = \sin x,\ \dot{p} = -p\cos x$ (iv) $\dot{x} = -2p + 4,\ \dot{p} = 2 - 2x$
    (v) $\dot{x} = -2x - 2y - 2,\ \dot{p} = -2x + 2y - 2$
    Also sketch the phase portraits and level curves.
26. Show that the system

$$\dot{x} = a_{11}x + a_{12}y + Ax^2 - 2Bxy + Cy^2$$
$$\dot{y} = a_{21}x + a_{11}y + Dx^2 - 2Axy + By^2$$

    is a Hamiltonian system with one degree of freedom.
27. For each of the following Hamiltonian functions sketch the phase portraits for the Hamiltonian system and the gradient system. Also draw both the phase portraits on the same phase plane.
    (i) $H(x,y) = x^2 + 2y^2$ (ii) $H(x,y) = x^2 - y^2$ (iii) $H(x,y) = \frac{\lambda x^2}{2} + \frac{y^2}{2}$ (iv) $H(x,y) = \frac{x^2}{2} - \frac{x^4}{4} + \frac{y^2}{2}$
    (iv) $H(x,y) = \frac{x^2}{2} - \frac{\beta}{3}x^3 + \frac{y^2}{2}$
28. Prove that the transformation $U : \mathbb{R}^2 \to \mathbb{R}^2$ is symplectic iff it is both area and orientation preserving.
29. Prove that the matrix S satisfies the properties (i) $s^T = -s = s^{-1}$ (ii) $\det(s) = 1$.
30. For a symplectic differentiation $h : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, prove that
    (i) the mapping h preserves volumes in $\mathbb{R}^{2n}$,
    (ii) h^{-1} is symplectic.
    (iii) $\left[Dh(x)\right]^{-1} = S^T \left[Dh(x)\right]^T S$
31. Prove that the composition of two symplectic transformation is a symplectic transformation.
32. Show that two dimensional volume preserving vector fields are Hamiltonian.
33. Prove that the Poisson bracket is anti-symmetric.
34. Show that Poisson bracket satisfies the Jacobi identity $\{F,\{G,H\}\} + \{G,\{H,F\}\} + \{H,\{F,G\}\} = 0$.
35. Using Poisson bracket show that if time t is explicitly absent in the Hamiltonian of a system, then the energy of the system is conserved.

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