Chapter 4
Stability Theory

Stability of solutions is an important qualitative property in linear as well as nonlinear systems. The objective of this chapter is to introduce various methods for analyzing stability of a system. In fact, stability of a system plays a crucial role in the dynamics of the system. In the context of differential equations rigorous mathematical definitions are often too restrictive in analyzing the stability of solutions. Different kinds of methods on stability were developed in the theory of differential equations. We begin with the stability analysis of linear systems. Stability theory originates from the classical mechanics, the laws of statics and dynamics. The ideas in mechanics had been enriched by many mathematicians and physicists like *Evangelista Torricelli* (1608–1647), *Christiaan Huygens* (1629–1695), *Joseph-Louis Lagrange* (1736–1813), *Henri Poincaré* (1854–1912), and others. In the beginning of the twentieth century the principles of stability in mechanics were generalized by the Russian mathematician *A.M. Lyapunov* (1857–1918). There are many stability theories in the literature but we will discuss a few of them in this chapter which are practically the most useful.

4.1 Stability of Linear Systems

This section describes the stability analysis of a linear system of homogeneous first-order differential equations. The systems with constant coefficients can be written as

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_j; \quad i = 1, 2, \ldots, n \tag{4.1}$$

where $a_{ij} (i, j = 1, 2, \ldots, n)$ are constants. In matrix notation, (4.1) can be written as

$$\dot{\underset{\sim}{x}} = A \underset{\sim}{x} \tag{4.2}$$

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where A is an $n \times n$ matrix and $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)^t$ is a column vector. The characteristic equation of (4.2) is $\det(A - \lambda I) = 0$. Depending upon the roots of the characteristic equation the following cases may arise for stability of solutions of (4.2):

(i) If all the roots of the characteristic equation of (4.2) have negative real part, then all solutions of (4.2) are asymptotically stable. Moreover, the solutions tend to the equilibrium point origin as $t \to \infty$;
(ii) If at least one root of the characteristic equation has a positive real part, then all solutions are unstable;
(iii) If the characteristic equation has simple roots, purely imaginary or zero and the other roots exist and have a negative real part, then all solutions of the system are stable, but not asymptotically.

In case of nonhomogeneous linear systems we prove the following theorem for stability.

Theorem 4.1 *The solutions of the nonhomogeneous linear system* $\dot{x}_i = \sum_{j=1}^{n} a_{ij}(t)x_j + b_i(t);\ i = 1, 2, \ldots, n$ *are all simultaneously either stable or unstable.*

Proof Let $\underset{\sim}{f}(t) = (f_1(t), f_2(t), \ldots, f_n(t))$ be any particular solution of the nonhomogeneous linear system

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij}(t)x_j + b_i(t),\ i = 1, 2, \ldots, n \tag{4.3}$$

Consider the transformation $y_i(t) = x_i(t) - f_i(t), i = 1, 2, \ldots, n,$ which transforms the particular solution $\underset{\sim}{f}(t)$ *of* (4.3) into a trivial solution. Applying this transformation to (4.3), we get the homogeneous linear system

$$\dot{y}_i = \sum_{j=1}^{n} a_{ij}(t)y_j(t),\ i = 1, 2, \ldots, n \tag{4.4}$$

Thus any particular solution of (4.3) has the same stability behavior as that of the trivial solution of (4.4). Suppose that the trivial solution of (4.4) is stable. Then by definition of stability, for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that for every other solution $y_i, i = 1, 2, \ldots, n$ of (4.4),

$$|y_i(t_0) - 0| < \delta \Rightarrow |y_i(t) - 0| < \varepsilon\ \forall t \geq t_0.$$

Substituting $y_i(t) = x_i(t) - f_i(t), i = 1, 2, \ldots, n,$ we see that for every solution $x_i(t), i = 1, 2, \ldots, n$ of (4.3),

$$|x_i(t_0) - f_i(t_0)| < \delta \Rightarrow |x_i(t) - f_i(t)| < \varepsilon \ \forall t \geq t_0.$$

This implies that the particular solution $f_i(t), i = 1, 2, \ldots, n$ of (4.3) is stable. One can prove the instability of the particular solution similarly. This completes the proof.

4.2 Methods for Stability Analysis

There does not exist a single method which will suffice for stability analysis of a system. We begin with the Lyapunov stability analysis.

(I) **Lyapunov method**

First, we shall explain Lyapunov method with respect to equilibrium points of a system. Let $\underset{\sim}{x}^*$ be the equilibrium point of a nonlinear system $\underset{\sim}{x}^* = \underset{\sim}{f}(\underset{\sim}{x}),\ \underset{\sim}{x} \in \mathbb{R}^n$. If any orbit that passes close to the equilibrium point stays close to it for all time, then we say that the equilibrium point $\underset{\sim}{x}^*$ is Lyapunov stable. Mathematically, it is defined as follows:

An equilibrium point $\underset{\sim}{x}^*$ of a system $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x}), \underset{\sim}{x} \in \mathbb{R}^n$ is said to be *Lyapunov stable* if and only if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that the orbit $\underset{\sim}{\phi}\left(t, \underset{\sim}{x}\right)$ of the system satisfies the following relation:

$$\|\underset{\sim}{x} - \underset{\sim}{x}^*\| < \delta \Rightarrow \|\underset{\sim}{\phi}(t, \underset{\sim}{x}) - \underset{\sim}{x}^*\| < \varepsilon,\ \forall t \geq 0.$$

(Starts near $\underset{\sim}{x}^*$) (Stayed nearby orbit)

The equilibrium point $\underset{\sim}{x}^*$ is said to be *asymptotically stable* if

(i) it is stable, and
(ii) the orbit $\underset{\sim}{\phi}(t, \underset{\sim}{x})$ approaches to $\underset{\sim}{x}^*$ as $t \to \infty$.

Thus, for asymptotically stable equilibrium point we can find a $\delta > 0$ such that

$$\|\underset{\sim}{x} - \underset{\sim}{x}^*\| < \delta \Rightarrow \|\underset{\sim}{\phi}(t, \underset{\sim}{x}) - \underset{\sim}{x}^*\| \to 0 \text{ as } t \to \infty.$$

For an asymptotic stable equilibrium point $\underset{\sim}{x}^*$, the set $D(\underset{\sim}{x}^*) = \{\underset{\sim}{x} \in \mathbb{R}^n | \lim_{t\to\infty} \|\underset{\sim}{\phi}(t, \underset{\sim}{x}) - \underset{\sim}{x}^*\| = 0\}$ is called the domain of asymptotic stability of $\underset{\sim}{x}^*$. If $D = \mathbb{R}^n$, then $\underset{\sim}{x}^*$ is globally stable (asymptotically).

An equilibrium point $\underset{\sim}{x}^*$ which satisfies only the condition (ii) of the definition of asymptotic stability is called *quasi-asymptotically stable*. An equilibrium point which is not stable is said to be *unstable*. The diagrammatic representations of Lyapunov, asymptotic, and quasi-asymptotic stabilities about the equilibrium point are shown in Fig. 4.1a–c.

The solution $u(t)$ of a system is said to be **uniformly** stable if there exists a $\delta(\varepsilon) > 0$ for all $\varepsilon > 0$ such that for any other solution $v(t)$, the inequality $|u(t_0) - v(t_0)| < \delta$ implies $|u(t) - v(t)| < \varepsilon$ for all $t \geq t_0$. The solution $u(t)$ is said to be unstable when no such δ exists. Again, a stable solution $u(t)$ is said to be asymptotically stable if $|u(t) - v(t)| \to 0$ as $t \to \infty$. From this stability criterion we see that the Lyapunov stability condition is quite restrictive. The two neighboring solutions remain close to each other at the same time. We now discuss few less restrictive stability methods below.

(II) Poincaré method

This stability criterion is related with different time scales, say t' and t. Let Γ' and Γ be two orbits represented by $\underset{\sim}{x}(t)$ and $\underset{\sim}{y}(t)$, respectively, for all t. The orbit Γ is orbitally stable if for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if $\|\underset{\sim}{x}(0) - \underset{\sim}{y}(\tau)\| < \delta$ for some time τ, then there exists $t'(t)$ such that $\|\underset{\sim}{x}(t) - \underset{\sim}{y}(t')\| < \varepsilon,\ \forall t > 0$. The orbit is said to be asymptotically stable if the orbit Γ' tends toward Γ as $t \to \infty$. This is the most significant test for stability analysis but it is very difficult to establish mathematically.

(III) Lagrange method

This is a simple criterion for stability analysis. The solutions of the system $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x}, t)$ are said to be bounded stable if $\|\underset{\sim}{x}(t)\| \leq M < \infty,\ \forall t$. This is also known as bounded stability.

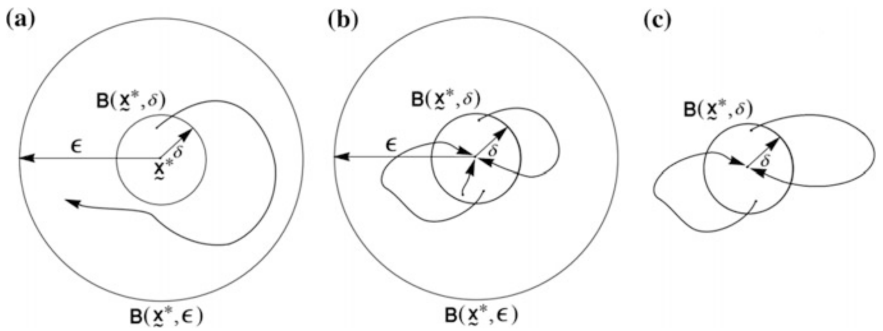

Fig. 4.1 **a** Lyapunov stability, **b** asymptotic stability, and **c** quasi-asymptotic stability of an equilibrium point $\underset{\sim}{x}^*$

(IV) Lyapunov's direct method

The Russian mathematician A.M. Lyapunov generalized the stability conditions which are used in analyzing the stability of a system, in particular the stability in the neighborhood of equilibrium points of a system. This is known as Lyapunov's second method or direct method for stability. He nicely introduced a scalar function, $L(\underset{\sim}{x})$ later called it the Lyapunov function, such that $L(\underset{\sim}{x}^*) = 0$ and $L(\underset{\sim}{x}) > 0$ when $\underset{\sim}{x} \neq \underset{\sim}{x}^*$ in the neighborhood of $\underset{\sim}{x}^*$, the equilibrium point of the system $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x})$. The function $L \equiv L(x_1, x_2, \ldots, x_n)$ is said to be positively (resp. negatively) definite in a domain $D \subset \mathbb{R}^n$ if $L(\underset{\sim}{x}) > 0$ (resp. <0) for all $\underset{\sim}{x} \in D, \underset{\sim}{x} \neq \underset{\sim}{0}$. Similarly, L is called positively (resp. negatively) semi-definite in D if $L(\underset{\sim}{x}) \geq 0$ (resp. ≤ 0) for all $\underset{\sim}{x} \in D$. When the function $L(t, \underset{\sim}{x})$ depends explicitly on time t, these definitions can be redefined as follows:

The function $L(t, \underset{\sim}{x})$ is said to be positively (resp. negatively) definite in D if there exists a function $G(\underset{\sim}{x})$ in D such that $G(\underset{\sim}{x})$ is continuous in D, $G(\underset{\sim}{0}) = 0$ and $0 < G(\underset{\sim}{x}) \leq L(t, \underset{\sim}{x})$ (resp. $L(t, \underset{\sim}{x}) \leq G(\underset{\sim}{x}) < 0$) for all $\underset{\sim}{x} \in D \backslash \{0\}$, $t \geq t_0$. Similarly, the semi-definite functions can be defined. The total derivative or orbital derivative of L in the direction of the vector field $\underset{\sim}{f}(\underset{\sim}{x})$ is defined as

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \underset{\sim}{f} \cdot \nabla L = \underset{\sim}{f}(\underset{\sim}{x}) \frac{\partial L}{\partial \underset{\sim}{x}}.$$

Let $D \subseteq \mathbb{R}^n$ be an open neighborhood of the equilibrium point $\underset{\sim}{x}^*$. Then the function $L : D \to \mathbb{R}$, satisfying the following properties:

(i) L is continuously differentiable,

(ii) $L > 0$ for all $\underset{\sim}{x} \in D \backslash \{\underset{\sim}{x}^*\}$ and $L\left(\underset{\sim}{x}^*\right) = 0$,

is called a Lyapunov function. Moreover, if $\frac{\mathrm{d}L}{\mathrm{d}t} \leq 0$ in D, then $\underset{\sim}{x}^*$ is stable. This condition implies that the point $\underset{\sim}{x}(t)$ moves along a path where $L(\underset{\sim}{x})$ does not increase. Hence, $\underset{\sim}{x}(t)$ will remain close to the point $\underset{\sim}{x}^*$ and come to $\underset{\sim}{x}^*$ if $\frac{\mathrm{d}L}{\mathrm{d}t} = 0$. There is no systematic procedure to deduct the Lyapunov function $L(\underset{\sim}{x})$. However, in case of conservative system it L is the energy of the system. In fact, Lyapunov constructed this function on the basis of the principle of energy in mechanics.

Theorem 4.2 (*Lyapunov theorem*) *Suppose that the origin is an equilibrium point of* $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x}), \underset{\sim}{x} \in \mathbb{R}^n$ and let $L = L(x_1, x_2, \ldots, x_n)$ *be a Lyapunov function in a neighborhood D of the origin. If*

(i) the orbital derivative $\dot{L} \leq 0$ *in D, that is, if* $\dot{L}$ *is negative semi-definite in D, the origin is stable,*

(ii) $\dot{L} < 0$ *in* $D \backslash \{0\}$, *that is, if* $\dot{L}$ *negative definite in D, then the origin is asymptotically stable,*

(iii) $\dot{L} > 0$, *that is, positive definite in D, the origin is unstable.*

For proof see Hartmann [1].

For an application of the theorem we illustrate the stability of pendulum problem.

Simple undamped pendulum: Consider the simple pendulum problem governed by the equation

$$ml\ddot{\theta} = -mg\sin\theta,$$
$$\text{that is, } \ddot{\theta} = -\left(\frac{g}{l}\right)\sin\theta \tag{4.5}$$

in which a bob of mass m is suspended from a light string of length l, where θ represents the angle between the string and the vertical axis at some instant t, and g is the acceleration due to gravity. With $x = \theta$ and $y = \dot{\theta}$, we can rewrite the Eq. (4.5) as a system of equations

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\left(\frac{g}{l}\right)\sin x \end{aligned}\right\} \tag{4.6}$$

Consider the function $L(x,y) = \frac{1}{2}ml^2y^2 + mgl(1-\cos x),\ (x,y) \in \mathbb{R}^2$, which is simply the total energy of the system. Let $G = \left\{(x,y) \in \mathbb{R}^2 : -2\pi < x < 2\pi\right\}$. We see that $L(0,0) = 0$ and $L > 0$ in $G \backslash \{(0,0)\}$. Therefore, L is positive definite in G. We now calculate the derivate $\dot{L}$ of L along the trajectory of (4.6) as

$$\dot{L} = \frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\partial L}{\partial x}\dot{x} + \frac{\partial L}{\partial y}\dot{y} = [mgl\sin x]y + ml^2y\left[-\left(\frac{g}{l}\right)\sin x\right] = 0$$

Thus, conditions of Theorem 4.2 are satisfied. Hence, the fixed point origin is stable. Note that the origin is not asymptotically stable, since $\dot{L} \equiv 0$.

Damped pendulum: Consider the damped pendulum governed by the equation

$$ml\ddot{\theta} = -mg\sin\theta - \mu l\dot{\theta} \tag{4.7}$$

which is simply obtained by taking into account the effect of damping force (frictional force) $\mu l\dot{\theta}$, $\mu > 0$ being the coefficient of friction. As previous, with $x = \theta$ and $y = \dot{\theta}$, we can rewrite Eq. (4.7) as

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\left(\tfrac{g}{l}\right)\sin x - \left(\tfrac{\mu}{m}\right)y \end{aligned}\right\} \tag{4.8}$$

The origin $O(0,0)$ is a fixed point of the system. We now determine its stability. As earlier, consider the function

$$L(x, y) = \frac{1}{2}ml^2y^2 + mgl(1 - \cos x),\ (x, y) \in \mathbb{R}^2.$$

Then L is positive definite in G, as defined earlier. Now calculate $\dot{L}$ along the trajectory of (4.8) as follows:

$$\dot{L} = \frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\partial L}{\partial x}\dot{x} + \frac{\partial L}{\partial y}\dot{y} = [mgl \sin x]y + ml^2y\left[-\left(\frac{g}{l}\right)\sin x - \left(\frac{\mu}{m}\right)y\right] = -\mu l^2y^2.$$

Now, in G we can find points, in particular $(x, y) = (\pi/2, 0)$, such that $\dot{L} = 0$. So, $\dot{L} \le 0$ in G. Therefore, by Theorem 4.2 the fixed point origin is stable. However, the phase portrait near the origin gives some other picture: the origin is asymptotically stable (see Fig. 3.14). So, we discard this particular choice of L and consider a more general form of L as

$$L(x, y) = \frac{1}{2}ml^2\left[ax^2 + 2bxy + cy^2\right] + mgl(1 - \cos x)$$

We shall now determine the values of a, b, and c for which the origin are asymptotically stable, that is, L is positive definite and $\dot{L}$ is negative definite in some neighborhood of the origin. It can be shown that the first right-hand member in the expression of L is positive definite if and only if $a > 0,\ c > 0,\ ac - b^2 > 0$. The orbital derivative $\dot{L}$ of the Lyapunov function L is given by

$$\begin{aligned}\dot{L} = \frac{\partial L}{\partial x}\dot{x} + \frac{\partial L}{\partial y}\dot{y} &= ml^2\left[ax + by + \left(\frac{g}{l}\right)\sin x\right]y + ml^2(bx + cy)\left[-\left(\frac{g}{l}\right)\sin x - \left(\frac{\mu}{m}\right)y\right] \\ &= ml^2\left[\left\{a - b\left(\frac{\mu}{m}\right)\right\}xy + \left(\frac{g}{l}\right)(1 - c)y \sin x - \left(\frac{g}{l}\right)bx \sin x + \left\{b - c\left(\frac{\mu}{m}\right)\right\}y^2\right]\end{aligned}$$

The right-hand side of the above expression contains two sign indefinite terms, xy and $y \sin x$. We need to discard them in our problem and it leads to the following relations:

$$a - b\left(\frac{\mu}{m}\right) = 0,\ 1 - c = 0 \Rightarrow b = a\left(\frac{m}{\mu}\right),\ c = 1.$$

With this choice, $\dot{L}$ takes the form

$$\dot{L} = -ml^2\left[\left(\frac{g}{l}\right)\left(\frac{m}{\mu}\right)ax \sin x + \left\{\left(\frac{\mu}{m}\right) - \left(\frac{m}{\mu}\right)a\right\}y^2\right].$$

To make $\dot{L}$ negative definite, we must have

$$\left(\frac{\mu}{m}\right) - \left(\frac{m}{\mu}\right)a > 0 \Rightarrow 0 < a < \left(\frac{\mu}{m}\right)^2.$$

Then $0 < b < \left(\frac{\mu}{m}\right)$. Now, the term $x \sin x > 0$ if all $x : -\pi < x < \pi$ with $x \neq 0$. Let $N = \{(x, y) \in \mathbb{R}^2 : -\pi < x < \pi\}$. Then L is positive definite and $\dot{L}$ is negative definite in N. Therefore, by Theorem 4.2 the fixed point origin is asymptotically stable, as required.

Theorem 4.3 *Consider a nonautonomous system* $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(t, \underset{\sim}{x})$ *with* $\underset{\sim}{f}(t, \underset{\sim}{0}) = \underset{\sim}{0}$, $\underset{\sim}{x} \in D \subseteq \mathbb{R}^n$, *and* $t \geq t_0$. *The Lyapunov function* $L(t, \underset{\sim}{x})$ *is defined in a neighborhood of the origin and positively definite for* $t \geq t_0$. *Then*

(i) *if the orbital derivative is negatively semi-definite, the solution is stable;*
(ii) *if the orbital derivative is negative definite, the solution is asymptotically stable; and*
(iii) *if the orbital derivative in positive definite, the solution is unstable.*

Example 4.1 Show that the solution of the autonomous system $\dot{x} = y$, $\dot{y} = -x$ with $x(0) = 0$, $y(0) = 0$ is stable in the sense of Lyapunov.

Solution The solution of the system with $x(0) = x_0$, $y(0) = y_0$ is given as

$$x(t) = x_0 \cos t + y_0 \sin t, \quad y(t) = -x_0 \sin t + y_0 \cos t$$

and the solution subject to the given initial condition is $x(t) = 0, \quad y(t) = 0$. Choose an arbitrary real $\varepsilon > 0$. We have to find a $\delta(\varepsilon) > 0$ such that for $|x_0 - 0| < \delta$ and $|y_0 - 0| < \delta$,

$$|x(t) - 0| = |x_0 \cos t + y_0 \sin t| < \varepsilon, \text{ and } |y(t) - 0| = |-x_0 \sin t + y_0 \cos t| < \varepsilon$$

hold for all $t \geq 0$. We see that

$$|x_0 \cos t + y_0 \sin t| \leq |x_0 \cos t| + |y_0 \sin t| \leq |x_0| + |y_0|.$$

Similarly, $|-x_0 \sin t + y_0 \cos t| \leq |x_0| + |y_0|$. Take $\delta = \varepsilon/2$. This gives

$$\begin{aligned} &\text{for } |x_0| < \delta \text{ and } |y_0| < \delta \\ &\Rightarrow |x_0 \cos t + y_0 \sin t| < \varepsilon/2 + \varepsilon/2 = \varepsilon, \ \forall t \geq 0. \end{aligned}$$

Hence, the solution $x(t) = 0, \quad y(t) = 0$ is stable in the sense of Lyapunov but the stability is not asymptotic.

Example 4.2 Prove that each solution of the equation $\dot{x} + x = 0$ is asymptotically stable.

Solution The general solution is given by $x(t) = Ae^{-t}$, where A is an arbitrary constant. The solutions $x_1(t)$ and $x_2(t)$ of the equation that satisfy the initial conditions $x_1(t_0) = x_1^0$ and $x_2(t_0) = x_2^0$ are $x_1(t) = x_1^0 e^{-(t-t_0)}$ and $x_2(t) = x_2^0 e^{-(t-t_0)}$, respectively. We see that

$$|x_2(t) - x_1(t)| = \left|x_2^0 - x_1^0\right| e^{-(t-t_0)} \to 0 \text{ as } t \to \infty.$$

This implies that every solution of the equation is asymptotically stable.

Example 4.3 Prove that all solutions of the system $\dot{x} = \sin^2 x$ are bounded on $(-\infty, +\infty)$ but the solution $x(t) = 0$ is unstable as $t \to \infty$.

Solution Clearly, $x = n\pi$; $n = 0, \pm 1, \pm 2, \ldots$ are the obvious solutions of the given equation. Other solutions are obtained as

$$\begin{aligned} &\operatorname{cosec}^2 x \, \mathrm{d}x = \mathrm{d}t \Rightarrow \cot x = \cot x_0 - t \quad [\text{assuming } x(0) = x_0] \\ &\Rightarrow x = \cot^{-1}(\cot x_0 - t), \; x_0 \neq n\pi. \end{aligned}$$

All above solutions are bounded on $(-\infty, +\infty)$. The solution $x(t) = 0$ is unstable as $t \to \infty$, because for any $x_0 \in (0, \pi)$ we have $\lim_{t \to \infty} x(t) = \pi$. So, boundedness of solution does not imply that it is stable. Similarly, stability of a solution does not ensure that it is bounded. Thus, bounded and stability of solutions are independent properties of a system.

Example 4.4 Using suitable Lyapunov functions, examine the stabilities for the following systems: (i) $\ddot{x} + x = 0$, (ii) $\dot{x} = x, \quad \dot{y} = -y$ at the origin.

Solution (i) The given system can be written as $\dot{x} = y$, $\dot{y} = -x$. The origin is the equilibrium point of the system. We take Lyapunov function as $L(x, y) = x^2 + y^2$, which is positive definite in the neighborhood of the origin and $L(0, 0) = 0$. The orbital derivative of L is given by

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\partial L}{\partial x}\dot{x} + \frac{\partial L}{\partial y}\dot{y} = 2xy - 2xy = 0.$$

Hence, $\frac{\mathrm{d}L}{\mathrm{d}t}$ is semi-negative definite. So, the system is stable at (0, 0). The phase paths of the system are obtained as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y} \Rightarrow x^2 + y^2 = k^2, \quad k \neq 0.$$

which represent concentric circles with center at the origin. Hence, the system is not asymptotically stable at the origin.

(ii) We take Lyapunov function $L(x, y) = x^2 - y^2$, in the neighborhood of origin, which is positive definite in arbitrarily close to (0, 0) ($L > 0$ along the straight line $y = 0$) and $L(0, 0) = 0$. The orbital derivative of L is

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\partial L}{\partial x}\dot{x} + \frac{\partial L}{\partial y}\dot{y} = 2(x^2 + y^2) > 0.$$

So the equilibrium point origin is unstable. In this case, the origin is a saddle point. The path of the system is $xy = k$, k being an arbitrary constant, which is a rectangular hyperbola.

Example 4.5 Examine different stability criteria satisfied by the linear harmonic oscillator $\ddot{x} + x = 0$.

Solution The harmonic oscillator can be written as a system of differential equations as

$$\dot{x} = y, \dot{y} = -x.$$

The solution of the system is given by

$$x(t) = A\cos t + B\sin t, y(t) = -A\sin t + B\cos t,$$

where A and B are constants. Let $\varepsilon > 0$ be given. Assume $u(t) = A_1\cos t + B_1\sin t$ and $v(t) = A_2\cos t + B_2\sin t$ are two solutions of the equation, where $A_1^2 + A_2^2 \neq 0$ and $B_1^2 + B_2^2 \neq 0$. Then we get

$$\begin{aligned} |u(t) - v(t)| &= |(A_1 - A_2)\cos t + (B_1 - B_2)\sin t| \\ &\leq |A_1 - A_2||\cos t| + |B_1 - B_2||\sin t| \\ &\leq |A_1 - A_2| + |B_1 - B_2| < \varepsilon \end{aligned}$$

if $|A_1 - A_2| < \varepsilon/2$ and $|B_1 - B_2| < \varepsilon/2$. Take $\delta = \varepsilon/2$. Then, $\delta = \delta(\varepsilon) > 0$ and $|u(0) - v(0)| \leq |A_1 - A_2| < \delta$. Thus, the solution is uniformly stable.

Now, take $L(x, y) = x^2 + y^2$ as Lyapunov function, which is the energy of the harmonic oscillator. Then

$$\frac{\mathrm{d}L}{\mathrm{d}t} = 2x\dot{x} + 3y\dot{y} = 2xy - 2xy = 0.$$

Hence, the origin of the harmonic oscillator is stable in the sense of Lyapunov but it is not asymptotically stable. This can be shown easily that the system is orbitally stable in the sense of Poincaré but not asymptotically stable.

The solutions of the system $\dot{x} = y, \dot{y} = -x$ are given by

$$x(t) = A\cos t + B\sin t,\ y(t) = -A\sin t + B\cos t,$$

where $A, B \in \mathbb{R}$ are arbitrary constants. Then

$$\|\underset{\sim}{x}(t)\| = \sqrt{A^2 + B^2} < \infty \text{ for all } t.$$

Hence, the solutions of the harmonic oscillator have bounded stability in the sense of Lagrange.

Example 4.6 Show that the system $\dot{x} = -y(x^2 + y^2)^{1/2}, \dot{y} = x(x^2 + y^2)^{1/2}$ is orbitally stable but not Lyapunov stable.

Solution Convert the system into the polar coordinates (r, θ) using $x = r\cos\theta, y = r\sin\theta$. In (r, θ) coordinates, the system becomes

$$\dot{r} = 0, \dot{\theta} = r,$$

which has the solution

$$r = r_0, \theta = r_0 t + \theta_0,$$

where $r_0 = r(0)$ and $\theta_0 = \theta(0)$, the initial condition of the system. Therefore, the solution of the original system is given by

$$x(t) = r_0\cos(r_0 t + \theta_0), y(t) = r_0\sin(r_0 t + \theta_0).$$

This shows that the amplitude and frequency of the solutions depend upon r_0. Hence, the system is orbitally stable. The solutions represent concentric circles with center at the origin. Consider two neighboring points $(r_0, 0)$ and $(r_0 + \varepsilon, 0)$ on the concentric circles as two initial solutions, where ε is very small. After some time t, these two points move to $(r_0, r_0 t)$ and $(r_0, (r_0 + \varepsilon)t)$. This yields the angle difference between the solutions as

$$\Delta\theta = (r_0 + \varepsilon)t - r_0 t = \varepsilon t.$$

Hence, when $t = (2n + 1)\pi/\varepsilon, \Delta\theta = (2n + 1)\pi$, that is, the solutions are diametrically opposite to one another and in this case, the distance between them is $2r_0 + \varepsilon$. So, there always exists a time at which the solutions move further away from each other. Hence, the solution is not stable in the sense of Lyapunov.

Example 4.7 Investigate the stability of the system

$$\frac{dx}{dt} = -(x - 2y)\left(1 - x^2 - 3y^2\right)$$
$$\frac{dy}{dt} = -(y + x)\left(1 - x^2 - 3y^2\right)$$

at the fixed point origin.

Solution Take the Lyapunov function $L(x, y) = x^2 + 2y^2$. Then L is positive definite in the neighborhood of (0, 0) and $L(0, 0) = 0$. The orbital derivative is calculated as

$$\begin{aligned}\frac{\mathrm{d}L}{\mathrm{d}t} &= \frac{\partial L}{\partial x}\dot{x} + \frac{\partial L}{\partial y}\dot{y} = 2x\left[-(x-2y)(1-x^2-3y^2)\right] + 4y\left[-(y+x)(1-x^2-3y^2)\right] \\ &= -2(x^2+2y^2)(1-x^2-3y^2) < 0, \text{in the neighbourhood of} (0,0)\end{aligned}$$

and is equal to zero only when $x = y = 0$. So, $\dot{L}$ is negative definite, and hence the fixed point origin is asymptotically stable.

Example 4.8 Using a suitable Lyapunov function shows that the origin is an asymptotically stable equilibrium point of the system

$$\begin{aligned}\dot{x} &= -2y + yz - x^3 \\ \dot{y} &= x - xz - y^3 \\ \dot{z} &= xy - z^3\end{aligned}$$

Solution Obviously, (0, 0, 0) is the equilibrium point of the system. We take $L(x, y, z) = x^2 + 2y^2 + z^2$ as a Lyapunov function for which we can test the stability of the equilibrium point origin. The orbital derivative of L is given by

$$\begin{aligned}\dot{L} &= 2x\dot{x} + 4y\dot{y} + 2z\dot{z} \\ &= 2x\left(-2y + yz - x^3\right) + 4y\left(x - xz - y^3\right) + 2z\left(xy - z^3\right) \\ &= -4xy + 2xyz - 2x^4 + 4xy - 4xyz - 4y^4 + 2xyz - 2z^4 \\ &= -\left(2x^4 + 4y^4 + 2z^4\right) < 0, \text{ and } \dot{L} = 0 \text{ only at } (0,0,0).\end{aligned}$$

This implies that $\dot{L}$ is negative definite for $(x, y, z) \neq (0, 0, 0)$. Hence, by Lyapunov theorem on stability, the origin is an asymptotically stable.

4.3 Stability of Linearized Systems

Let us consider a nonlinear system represented as

$$\dot{x} = \underset{\sim}{f}(\underset{\sim}{x}); \underset{\sim}{x} \in \mathbb{R}^n. \tag{4.9}$$

Without loss of generality we assume that $\underset{\sim}{x} = \underset{\sim}{0}$ is an equilibrium point of the system. So when $\| \underset{\sim}{x} \| \ll 1$, we can expand $\underset{\sim}{f}(\underset{\sim}{x})$ in the form of a Taylor series in a neighborhood of $\underset{\sim}{x} = \underset{\sim}{0}$. Neglecting second- and higher order terms. we get a linear system as

$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x} \tag{4.10}$$

where $A = J(0)$, the Jacobian of the system evaluated at the origin. The linear system (4.10) is known as the linearization of the nonlinear system (4.9). An equilibrium point of a system is hyperbolic if the corresponding Jacobian matrix evaluated at the point has eigenvalues with nonzero real part. If not, then it is said to be non-hyperbolic. The flows in the neighborhood of hyperbolic fixed point retain the character under sufficiently small perturbation. On the other hand, non-hyperbolic fixed points and corresponding flows are easily changed under small perturbation. The non-hyperbolic fixed points are weak, whereas hyperbolic fixed points are robust in the context of flows. The phase portrait near a hyperbolic fixed point of a nonlinear system is topologically equivalent to the phase portrait of the corresponding linear system. This means that there is a homeomorphism which maps the local phase portrait onto the other preserving directions of trajectories. A homeomorphism is a continuous map with a continuous inverse. The flow near a hyperbolic fixed point is structurally stable. A phase portrait is said to be structurally stable if its topology does not change under an arbitrarily small perturbation to the vector field of the system. For example, the phase portrait of a saddle point (hyperbolic type) is structurally stable, whereas the center (non-hyperbolic type) is not structurally stable. By adding a small amount of damping force to the undamped pendulum equation makes, the center becomes a spiral. For hyperbolic fixed points and their flows we discuss some important theorems.

Theorem 4.4 (Hartman–Grobman) Let $\underset{\sim}{x} = \underset{\sim}{0}$ *be a hyperbolic equilibrium point of the nonlinear system* (4.9) *with* $\underset{\sim}{f} \in C^1$ (*continuously differentiable of order one*). *Then the stability type of the equilibrium point origin for the nonlinear system is same as that of the linear system* $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, *which is the linearization of* (4.9) *in the neighborhood of* $\|\underset{\sim}{x}\| \ll 1$. *Also, there exists a homeomorphism* $H(\underset{\sim}{x})$ *which maps the orbits of the nonlinear system* (4.9) *onto the orbits of the corresponding linear system in the neighborhood of the origin.*

The Hartman–Grobman theorem gives a very important result in the local qualitative theory of a dynamical system. This theorem shows that near a hyperbolic-type equilibrium point, the nonlinear system has the same qualitative behavior (locally) as the corresponding linearized system. Also, one can find the local solution of the nonlinear system through homeomorphism.

Example 4.9 Using Hartman–Grobman theorem discuss the local stability of the equilibrium point for the system $\dot{x} = x - y^2, \dot{y} = -y$. Also, find the homeomorphic mapping.

Solution Clearly, the origin is the only equilibrium point of the system. At the origin, the Jacobian matrix of the nonlinear system is given by

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The matrix has nonzero real eigenvalues 1, −1. The origin is of hyperbolic type and it is a saddle. According to Hartman–Grobman theorem, it is a saddle-type equilibrium point of the given nonlinear system. We shall now find the homeomorphic mapping H. The solutions of the nonlinear system and the corresponding linearized system with the initial conditions $x(0) = x_0, y(0) = y_0$ are given by

$$x(t) = x_0 e^t + \frac{y_0^2}{3}(e^{-2t} - e^t), y(t) = y_0 e^{-t}$$
$$\text{and} \quad x(t) = x_0 e^t, y(t) = y_0 e^{-t},$$

respectively. Therefore, the flow of the nonlinear system is

$$\varphi_t(x, y) = \begin{pmatrix} xe^t + \frac{y^2}{3}(e^{-2t} - e^t) \\ ye^{-t} \end{pmatrix}.$$

and the flow of the linear system is

$$e^{At} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

Now consider the map

$$H(x, y) = \begin{pmatrix} x - \frac{y^2}{3} \\ y \end{pmatrix}.$$

Clearly, H is continuous, and $H^{-1}(x, y) = \begin{pmatrix} x + \frac{y^2}{3} \\ y \end{pmatrix}$ exists and also continuous, that is, the mapping H is a homeomorphism. Now, for all $(x, y) \in \mathbb{R}^2$ and for all $t \geq 0$, we see that

$$
\begin{aligned}
H(\varphi_t(x,y)) &= H\begin{pmatrix} xe^t + \dfrac{y^2}{3}(e^{-2t} - e^t) \\ ye^{-t} \end{pmatrix} \\
&= \begin{pmatrix} xe^t + \dfrac{y^2}{3}(e^{-2t} - e^t) - \dfrac{y^2}{3}e^{-2t} \\ ye^{-t} \end{pmatrix} \\
&= \begin{pmatrix} xe^t - \dfrac{y^2}{3}e^t \\ ye^{-t} \end{pmatrix} \\
&= \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} x - \dfrac{y^2}{3} \\ y \end{pmatrix} \\
&= e^{At}H(x,y).
\end{aligned}
$$

Therefore, $H \circ \varphi_t = e^{At} \circ H \; \forall t \geq 0$. This relation shows that the two flows are connected by the mapping H.

4.4 Topological Equivalence and Conjugacy

Two autonomous systems are said to be topologically equivalent in a neighborhood of the origin if there exists a homeomorphism $H : U \rightarrow V$, where U and V are two open sets containing the origin, such that the trajectories of nonlinear system (4.9) in U are mapped onto the trajectories of the corresponding linear system (4.10) in V and preserve their orientation by time in the sense that if a trajectory is directed from $\underset{\sim}{x}_1$ to $\underset{\sim}{x}_2$ in U, then its image is directed from $H(\underset{\sim}{x}_1)$ to $H(\underset{\sim}{x}_2)$ in V. If the homeomorphism H preserves the parameterization by time, then the systems (4.9) and (4.10) are said to be topologically conjugate in a neighborhood of the origin.

The following theorem is very useful for topologically equivalent of two linear systems.

Theorem 4.5 *Two linear systems $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ and $\dot{\underset{\sim}{y}} = B\underset{\sim}{y}$, whose all eigenvalues have nonzero real parts, are topologically equivalent if and only if the number of eigenvalues with positive (and corresponding negative) real parts are the same for both the systems (see* Arnold [2]).

Example 4.10 Show that the systems $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ and $\dot{\underset{\sim}{y}} = B\underset{\sim}{y}$ where $A = \begin{pmatrix} -2 & -5 \\ -5 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 0 \\ 0 & -7 \end{pmatrix}$ are topologically conjugate.

Solution Consider the map $H(\underset{\sim}{x}) = C\underset{\sim}{x}$, where

$$C = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Clearly, the matrix C is invertible with the inverse

$$C^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Also, it is easy to verify that $B = CAC^{-1}$, that is, A and B are similar matrices. Then

$$e^{Bt} = Ce^{At}C^{-1}.$$

Let $\underset{\sim}{y} = H(\underset{\sim}{x}) = C\underset{\sim}{x}$. Then $\underset{\sim}{x} = C^{-1}\underset{\sim}{y}$ and

$$\dot{\underset{\sim}{y}} = C\dot{\underset{\sim}{x}} = CA\underset{\sim}{x} = CAC^{-1}\underset{\sim}{y} = B\underset{\sim}{y}.$$

Let $\underset{\sim}{x}(t) = e^{At}\underset{\sim 0}{x}$ be the solution of the system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ with the initial condition $\underset{\sim}{x}(0) = \underset{\sim 0}{x}$. Then $\underset{\sim}{y}(t) = C\underset{\sim}{x}(t) = Ce^{At}\underset{\sim 0}{x} = e^{Bt}C\underset{\sim 0}{x}$. This shows that if $\underset{\sim}{x}(t) = e^{At}\underset{\sim 0}{x}$ is a solution of the first system through $\underset{\sim 0}{x}$, then $\underset{\sim}{y}(t) = e^{Bt}C\underset{\sim 0}{x}$ is a solution of the second system through $C\underset{\sim 0}{x}$. Thus the mapping H maps the trajectories of the first system onto the trajectories of the second and since $Ce^{At} = e^{Bt}C$, and H also preserves the parameterization. The map H is a homeomorphism. Therefore, the given two systems are topologically conjugate. Note that the map $H(\underset{\sim}{x}) = C\underset{\sim}{x}$ is simply a rotation through 45° as shown in Fig. 4.2.

4.5 Linear Subspaces

The dynamics of a system may be restricted to manifolds which are embedded in the phase space. We give very formal definition of manifold below.

Manifold: The concept of manifold is very important in dynamical system, especially in stability theory, bifurcation, etc. A manifold in the n-dimensional Euclidean space $\mathbb{R}^n$ is defined as an $m(m \leq n)$-dimensional continuous region embedded in $\mathbb{R}^n$ and is represented by equations, say $f_j(\underset{\sim}{x}) = 0, j = 1, 2, \ldots, n - m$ in $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. In other words, an n-dimensional topological

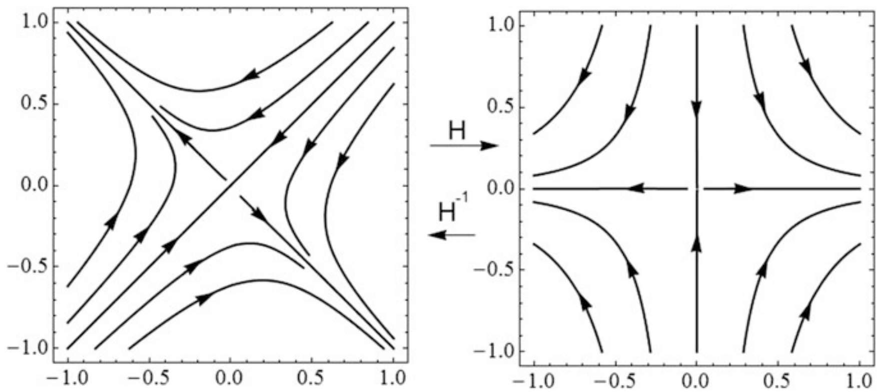

Fig. 4.2 Topologically equivalent flows

manifold is a Housdroff space (topological space such that any two distinct points possess distinct neighborhoods) such that every point has an open neighborhood N_i which is homeomorphic to an open set of E^n. If the functions $f_j(\underset{\sim}{x})$ are differentiable, then the manifold is called a differentiable manifold. More specifically, let M be a differentiable manifold. We may consider simply an open set of an Euclidean space, or a sphere or a torous as examples. A function on a differential manifold M is a diffeomorphism iff it is smooth, invertible, and its inverse is also smooth. On the other hand, an endomorphism of M is a smooth function from M to itself. A curve is an example of a one-dimensional manifold, and a surface is a two-dimensional manifold (see the book Tu [3] for details on manifolds). Our next target is to find the manifolds for some dynamical systems. First, consider the simple linear harmonic oscillator represented by the equation $m\ddot{x} = -kx$. With $\dot{x} = y$ we have the system

$$\dot{x} = y, \dot{y} = -\left(\frac{k}{m}\right)x.$$

This is a conservative system and its phase space is the two-dimensional Euclidean plane $\mathbb{R}^2$. It is easy to show that the Hamiltonian of the harmonic oscillator is constant and is given by $H(x,y) = \frac{1}{2}my^2 + \frac{1}{2}kx^2 = \text{constant}$. The Hamiltonian represents a one-dimensional differential manifold in $\mathbb{R}^2$ and all solutions of the system lie on this manifold. The manifold is a system of ellipses in the phase plane. All these ellipses are topologically equivalent to the unit circle $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ under the homeomorphism $h(x,y) = \left(\frac{x}{\sqrt{2H/k}}, \frac{y}{\sqrt{2H/m}}\right)$. Since $\frac{dH}{dt} = 0$ for all $(x,y) \in \mathbb{R}^2$, the Hamiltonian H is an integral of motion (this notion will discuss in later chapter) and the manifold H is also known as an *integral manifold*. All these manifolds for different values of constants are topologically equivalent to the unit circle S.

Linear Subspaces: Consider the linear system represented by (4.2), where A is a matrix of order $n \times n$ with real entries. The subspaces spanned by the eigenvectors of the corresponding eigenvalues of A can be categorized into three different subspaces, namely, stable, unstable, and center subspaces. These subspaces are defined as follows:

Let $\lambda_j = a_j \pm ib_j; i = \sqrt{-1}$ be the eigenvalues and $w_j = u_j \pm iv_j; j = 1, 2, \ldots, k$ be the corresponding eigenvectors of the matrix A. Depending upon the sign of a_j, the real part of λ_j, the three subspaces of the system (4.2) are defined as follows:

Stable subspace: The stable subspace E^s is generated by the eigenvectors of λ_j for which $a_j < 0$. That is, $E^s = \text{span}\{u_j, v_j | a_j < 0\}$.

Unstable subspace: The unstable subspace E^u is spanned by the eigenvectors of λ_j with $a_j > 0$. That is,

$$E^u = \text{span}\{u_j, v_j | a_j > 0\}.$$

Center subspace The center subspace occurs when the eigenvalues are purely imaginary. It is defined as $E^c = \text{span}\{u_j, v_j | a_j = 0\}$.

Example 4.11 Find the linear subspaces for the system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ with $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$, where

$$A = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 1 \end{pmatrix}.$$

Solution The characteristic equation of A has the roots $\lambda = -3,\ 2 \pm i$. So, the fixed points are hyperbolic type. The eigenvector corresponding to $\lambda_1 = -3$ is $(1, 0, 0)^t$, and that for $\lambda_2 = 2 + i$ is

$$w_2 = \begin{pmatrix} 0 \\ 1+i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = u_2 + iv_2, \text{where } u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Therefore, the stable and unstable subspaces are given by

$$\begin{aligned} E^s &= \text{span}\{u_j, v_j | a_j < 0\} \\ &= \text{span}\{(1, 0, 0)^t\} = x\text{-axis in the phase space} \end{aligned}$$

and

$$E^u = \text{span}\{u_j, v_j | a_j > 0\} = \text{span}\left[(0,1,0)^t, (0,1,1)^t\right] = yz\text{-plane}.$$

There is no center subspace E^C, since no eigenvalue is purely imaginary.

Example 4.12 Find all linear subspaces of the system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, where
$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$.

Solution Clearly, the origin is the unique equilibrium point of the system. The eigenvalues of the matrix A are $1, \pm i$. It can be easily obtained that the eigenvectors corresponding to $\lambda_1 = 1$ is $w_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and that for $\lambda_2 = i$ is $w_2 = \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = u_2 + iv_2$, where $u_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Since the system has positive and purely imaginary eigenvalues, it has unstable and center subspaces, given by

$$E^u = \text{span}\{u_j, v_j | a_j > 0\} = \text{span}\left[(0,1,0)^t\right] = y\text{-axis in the phase space}$$

and

$$\begin{aligned} E^c &= \text{span}\{u_j, v_j | a_j = 0\} = \text{span}\left[(1,0,\ 0)^t, (0,0,1)^t\right] \\ &= xz\text{-plane in the phase space} \end{aligned}$$

These two subspaces are presented in Fig. 4.3. Note that the system has no stable subspace, since it has no negative eigenvalue.

Theorem 4.6 *Consider a system* $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, *where A is a n* $\times$ *n matrix with real entries. Then phase space* $\mathbb{R}^n$ *can be decomposed as*

$$\mathbb{R}^n = E^u \oplus E^s \oplus E^c$$

where E^u, E^s, *and* E^c *are the unstable, stable, and of the system, respectively. Furthermore, these subspaces are invariant with respect to the flow.*

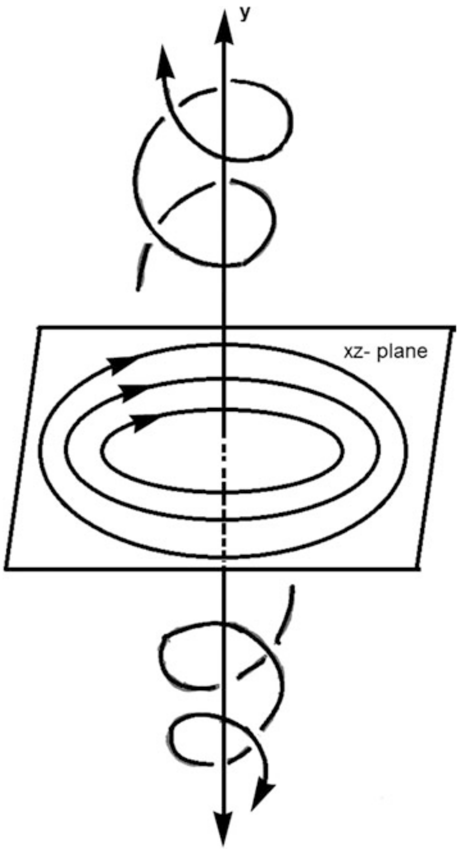

Fig. 4.3 Unstable and center subspaces of the given system

4.6 Hyperbolicity and Its Persistence

The flow in the neighborhood of hyperbolic fixed point has some special characteristic features. The special flow characteristic around the hyperbolic fixed point is called hyperbolicity. There are two important theorems, namely (i) Hartman–Grobman theorem and (ii) Stable manifold theorem for hyperbolic fixed points. The first theorem proves that there exists a continuous invertible map in some neighborhood of the hyperbolic fixed point which maps the nonlinear flow to the linear flow preserving the sense of time and the second theorem implies that the local structure of hyperbolic fixed points of nonlinear flows is the same as the linear flows in terms of the existence and transversality of local stable and unstable manifolds. We now define the local stable and unstable manifolds as follows:

Let U be some neighborhood of a hyperbolic fixed point $\underset{\sim}{x}^*$. The local stable manifold, denoted by $W^s_{loc}(\underset{\sim}{x}^*)$, is defined as

$$W^s_{loc}(\underset{\sim}{x}^*) = \left\{ \underset{\sim}{x} \in U | \phi_t(\underset{\sim}{x}) \to \underset{\sim}{x}^* \text{ as } t \to \infty, \phi_t(\underset{\sim}{x}) \in U \; \forall t \geq 0 \right\}.$$

Similarly, the local unstable manifold is defined as

$$W^u_{loc}(\underset{\sim}{x}^*) = \left\{ \underset{\sim}{x} \in U | \phi_t(\underset{\sim}{x}) \to \underset{\sim}{x}^* \text{ as } t \to -\infty, \phi_t(\underset{\sim}{x}) \in U \; \forall t \leq 0 \right\}.$$

The stable manifold theorem states that these manifolds exist and have the same dimension as the stable and unstable manifolds of the corresponding linear system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, if $\underset{\sim}{x}^*$ is a hyperbolic equilibrium point, and that they are tangential to the manifolds of linear system at $\underset{\sim}{x}^*$. This notion is known as hyperbolicity of a system.

Hyperbolic flow: If all the eigenvalues of the $n \times n$ matrix A are nonzero, then the flow $e^{At} : \mathbb{R}^n \to \mathbb{R}^n$ is called a hyperbolic flow, and the linear system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ is then called a hyperbolic linear system.

Invariant manifold: An invariant set $D \subset \mathbb{R}^n$ is said to be a $C^r(r \geq 1)$ invariant manifold if the set D has a structure of a C^r differentiable manifold. Similarly, the positively and negatively invariant manifolds are defined. In other words, a subspace $D \subseteq \mathbb{R}^n$ is said to be invariant if any flow starting in this subspace will remain within it for all future time.

The linear subspaces E^s, E^u, and E^c are all invariant subspaces of the linear system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ with respect to the flow e^{At}.

Theorem 4.7 *(Stable manifold theorem) Let* $\underset{\sim}{x}^* = 0$ *be a hyperbolic equilibrium point of the system* $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x}), \underset{\sim}{x} \in C^1$, *and* E^s *and* E^u *be the stable an unstable manifolds of the corresponding linear system* $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$. *Then there exists local stable and unstable manifolds* $W^s_{loc}(0)$ *and* $W^u_{loc}(0)$ *of the nonlinear system with the same dimension as that of* E^s *and* E^u, *respectively. These manifolds are tangential to* E^s *and* E^u, *respectively, at the origin and are smooth as the function* $\underset{\sim}{f}$.

Let $\underset{\sim 0}{x}$ be a hyperbolic fixed point of the nonlinear system. Then $\underset{\sim 0}{x}$ is called a sink if all the eigenvalues of the linear system have strictly negative real parts, and a source if all the eigenvalues have strictly positive real parts. Otherwise, $\underset{\sim 0}{x}$ is a saddle. A sketch of stable and unstable manifolds is given in Fig. 4.4.

Example 4.13 Find the local stable and unstable manifolds of the system $\dot{x} = x - y^2, \dot{y} = -y$.

Solution The system has the unique equilibrium point at the origin, (0, 0). Also, the origin is a saddle equilibrium point of the corresponding linearized system $\dot{x} = x, \dot{y} = -y$ with the invariant linear stable and unstable subspaces as

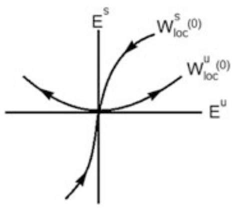

Fig. 4.4 Stable and unstable manifolds at the origin

$$E^s(0,0) = \{(x,y) : x = 0\} \text{ and } E^u(0,0) = \{(x,y) : y = 0\},$$

respectively. Therefore by stable manifold theorem, the system has local stable and unstable manifolds

$$W^s_{loc}(0,0) = \left\{(x,y) : x = S(y), \frac{\partial S}{\partial y}(0) = 0\right\} \text{ and}$$
$$W^u_{loc}(0,0) = \left\{(x,y) : y = U(x), \frac{\partial U}{\partial x}(0) = 0\right\},$$

respectively. We now find these manifolds.

Stable manifold: For the local stable manifold, we expand $S(y)$ as a power series in the neighborhood of the origin as follows:

$$S(y) = \sum_{i \geq 0} s_i y^i = s_0 + s_1 y + s_2 y^2 + s_3 y^3 + \cdots.$$

Since at $y = 0, S = 0$ and $\frac{\partial S}{\partial x} = 0$, we have $s_0 = s_1 = 0$. Therefore,

$$x = S(y) = \sum_{i \geq 2} s_i y^i = s_2 y^2 + s_3 y^3 + s_4 y^4 + s_5 y^5 + \cdots\cdots.$$

Now,

$$\begin{aligned} \dot{x} = x - y^2 &= \left(s_2 y^2 + s_3 y^3 + s_4 y^4 + s_5 y^5 + \cdots\right) - y^2 \\ &= (s_2 - 1) y^2 + s_3 y^3 + s_4 y^4 + s_5 y^5 + \cdots \end{aligned}$$

Again,

$$\begin{aligned} x = S(y) \Rightarrow \dot{x} = \frac{\partial S}{\partial y}\dot{y} &= (2s_2 y + 3s_3 y^2 + 4s_4 y^3 + 5s_5 y^4 + \cdots)(-y) \\ &= -(2s_2 y^2 + 3s_3 y^3 + 4s_4 y^4 + 5s_5 y^5 + \cdots) \end{aligned}$$

Therefore, we have

$$(s_2 - 1)y^2 + s_3y^3 + s_4y^4 + s_5y^5 + \cdots = -(2s_2y^2 + 3s_3y^3 + 4s_4y^4 + 5s_5y^5 + \cdots).$$

Equating the coefficients of like powers of y from both sides of the above relation, we get

$$s_2 = 1/3,\ s_3 = s_4 = \cdots = 0.$$

Therefore, $x = \frac{y^2}{3}$, and hence, the local stable manifold of the nonlinear system in the neighborhood of the equilibrium point origin is

$$W^s_{loc}(0,0) = \left\{(x, y) : x = \frac{y^2}{3}\right\}.$$

Unstable manifold: For the local unstable manifold we expand $U(x)$ as

$$U(x) = \sum_{i \geq 0} u_i x^i = u_0 + u_1 x + u_2 x^2 + u_3 x^3 + \cdots.$$

As previous, $u_0 = u_1 = 0$. Therefore,

$$y = U(x) = \sum_{i \geq 2} u_i x^i = u_2x^2 + u_3x^3 + u_4x^4 + u_5x^5 + \cdots.$$

Now,

$$\dot{y} = -y = -(u_2x^2 + u_3x^3 + u_4x^4 + u_5x^5 + \cdots).$$

But

$$\begin{aligned}\dot{y} = \frac{\partial U}{\partial x}\dot{x} &= (2u_2x + 3u_3x^2 + 4u_4x^3 + 5u_5x^4 + \cdots)(x - y^2)\\ &= (2u_2x + 3u_3x^2 + 4u_4x^3 + 5u_5x^4 + \cdots)\left\{x - \left(u_2x^2 + u_3x^3 + u_4x^4 + u_5x^5 + \cdots\right)^2\right\}\end{aligned}$$

Therefore, we must have

$$\begin{aligned}(u_2x^2 + u_3x^3 + u_4x^4 + u_5x^5 + \cdots) =& (2u_2x + 3u_3x^2 + 4u_4x^3 + 5u_5x^4 + \cdots)\\ &\left\{\left(u_2x^2 + u_3x^3 + u_4x^4 + u_5x^5 + \cdots\right)^2 - x\right\}\end{aligned}$$

Equating the coefficients of like powers of x from both sides of the above relation, we get

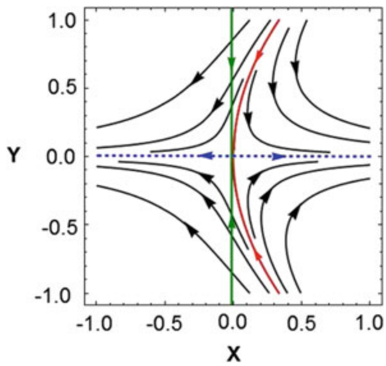

Fig. 4.5 Local stable and unstable manifolds at the equilibrium point origin

$$u_2 = u_3 = u_4 = \cdots = 0.$$

Therefore, $y = U(x) = 0$. Hence, the local unstable manifold of the nonlinear system in the neighborhood of the origin is (Fig. 4.5)

$$W^u_{loc}(0,0) = \{(x,y) : y = 0\} = E^u(0,0).$$

4.6.1 Persistence of Hyperbolic Fixed Points

In the previous section, we have seen important features of hyperbolic fixed points that near a hyperbolic fixed point the nonlinear and its corresponding linear systems have the same qualitative features locally. In this section, we study another important feature that hyperbolic equilibrium points persist their character under sufficiently small perturbation. Let the origin be a hyperbolic fixed point of the linear system $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x}); x \in \mathbb{R}^n$. Consider the perturbed system

$$\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x}) + \varepsilon \underset{\sim}{g}(\underset{\sim}{x}) \tag{4.11}$$

where $\underset{\sim}{g}$ is a smooth vector field defined in $\mathbb{R}^n$ and ε is a sufficiently small perturbation quantity. The fixed points of (4.11) are given by

$$\underset{\sim}{f}(\underset{\sim}{x}) + \varepsilon \underset{\sim}{g}(\underset{\sim}{x}) = 0.$$

Expanding in Taylor series about $\underset{\sim}{x} = \underset{\sim}{0}$ and using $\underset{\sim}{f}(\underset{\sim}{0}) = \underset{\sim}{0}$, we get

$$Df(\underset{\sim}{0})\underset{\sim}{x} + \varepsilon\left[\underset{\sim}{g}(\underset{\sim}{0}) + D\underset{\sim}{g}(\underset{\sim}{0})\underset{\sim}{x}\right] + O\left(|\underset{\sim}{x}|^2\right) = 0$$

$$\Rightarrow \left[D\underset{\sim}{f}(\underset{\sim}{0}) + \varepsilon D\underset{\sim}{g}(\underset{\sim}{0})\right]\underset{\sim}{x} + \varepsilon\underset{\sim}{g}(\underset{\sim}{0}) + O\left(|\underset{\sim}{x}|^2\right) = 0$$

Since the origin is hyperbolic, the eigenvalues of $D\underset{\sim}{f}(\underset{\sim}{0})$ are nonzero and so the eigenvalues of $\left[D\underset{\sim}{f}(\underset{\sim}{0}) + \varepsilon D\underset{\sim}{g}(\underset{\sim}{0})\right]$ are nonzero for sufficiently small ε. Hence, $\det\left[D\underset{\sim}{f}(\underset{\sim}{0}) + \varepsilon D\underset{\sim}{g}(\underset{\sim}{0})\right] \neq 0$, that is, $\left[D\underset{\sim}{f}(\underset{\sim}{0}) + \varepsilon D\underset{\sim}{g}(\underset{\sim}{0})\right]^{-1}$ exists. Therefore the fixed points of (4.11) are given by

$$\underset{\sim}{x}^* = \varepsilon\left[D\underset{\sim}{f}(\underset{\sim}{0}) + \varepsilon D\underset{\sim}{g}(\underset{\sim}{0})\right]^{-1}\underset{\sim}{g}(\underset{\sim}{0}) + O\left(|\underset{\sim}{x}|^2\right).$$

We now determine whether the point is hyperbolic or not. Since ε is small, we can find a neighborhood of $\varepsilon = 0$ in which the eigenvalues of $\left[D\underset{\sim}{f}(\underset{\sim}{x}) + \varepsilon D\underset{\sim}{g}(\underset{\sim}{x})\right]$ have nonzero real part for sufficiently small $\underset{\sim}{x}$. So, for sufficiently small ε, the eigenvalues of the perturbed equation do not change. So, the equilibrium points retain their character, that is, they are of hyperbolic type. This proves that the character of hyperbolic fixed point remains unchanged when the system undergoes small perturbation.

Theorem 4.8 (*Center manifold theorem*) *Consider a nonlinear system* $\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x})$ *where* $\underset{\sim}{f} \in C^r(E), r \geq 1$, *E being an open subset of* $\mathbb{R}^n$ *containing a non-hyperbolic fixed point, say* $\underset{\sim}{x}^* = 0$ *of the system. Suppose that the Jacobian matrix,* $J = D\underset{\sim}{f}(0)$, *of the system at the origin has j eigenvalues with positive real parts, k eigenvalues with negative real parts, and* $m(= n - j - k)$ *eigenvalues with zero real parts. Then there exists a j-dimensional* C^r*-class unstable manifold* $W^u(0)$, *a k-dimensional* C^r*-class stable manifold* $W^s(0)$, *and an m-dimensional* C^r*-class center manifold* $W^c(0)$ *tangent to subspaces* E^u, E^s, E^c *of the corresponding linear system* $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ *at the origin, respectively. Furthermore, these manifolds are invariant under the flow* ϕ_t *of the nonlinear system. The manifolds* $W^s(0)$ *and* $W^u(0)$ *are unique but the local center manifold* $W^c(0)$ *is not unique.*

Example 4.14 Find the manifolds of the system $\dot{x} = x, \dot{y} = y^2$.

Solution The system has a non-hyperbolic fixed point at the origin. The unstable subspace $E^u(0,0)$ of the linearized system at the origin is the x-axis and the center subspace is the y-axis. No stable subspace occurs for this system. Using the technique of power series expansion, discussed in Example 4.13, we see that the unstable manifold at the origin is the x-axis and its center manifold is the y-axis, that is, the line $x = 0$. However, there are other center manifolds of the system. From the equations, we have

$$\frac{dy}{dx} = \frac{y^2}{x},$$

which have the solution $x = ke^{-1/y}$ for $y \neq 0$. Thus, the center manifold of the origin is

$$W^c_{loc}(0,0) = \left\{(x,y) \in \mathbb{R}^2 : x = ke^{-1/y} \text{ for } y > 0, x = 0 \text{ for } y \leq 0\right\}.$$

It represents a one-parameter (k) family of center manifolds of the origin. Note that if we use the technique of power series expansion for the center manifold, we only get $x = 0$ as the center manifold. This example also shows that the center manifold is not unique.

4.7 Basin of Attraction and Basin Boundary

Let $\underset{\sim}{x}^*$ be an attracting fixed point of the linear system (4.2). We define the basin of attraction in some neighborhood of $\underset{\sim}{x}^*$ subject to some initial condition $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$ to be the set of points such that $\underset{\sim}{x}(t) \rightarrow \underset{\sim}{x}^*$ as $t \rightarrow \infty$. The boundary of this attracting set is called the basin boundary, also known as separatrix, separating the stable and unstable regions.

We discuss the basin of attraction and basin boundary with the help of the model for two interacting species. The well-known Lotka–Volterra model is considered which exhibits the basin of attraction and basin boundary for some situations. Consider the Lotka–Volterra model represented by the system of equations as

$$\dot{x} = x(3 - x - 2y), \dot{y} = y(3 - 2x - y)$$

where $x(t)$ and $y(t)$ are the populations the two interacting species, say rabbits and sheep, respectively, and $x, y \geq 0$. We shall first find the fixed points of the system, which can be obtained by solving the equations

$$x(3 - x - 2y) = 0 \text{ and } y(3 - 2x - y) = 0.$$

Solving we get four fixed points, $(0,0), (0,3), (3,0), (1,1)$. The Jacobian matrix of the system is given by

$$J(x,y) = \begin{pmatrix} 3-2x-2y & -2x \\ -2y & 3-2x-2y \end{pmatrix}.$$

At the fixed point (0, 0), $J(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$. The eigenvalues of $J(0,0)$ are 3, 3, which are positive. So, the fixed point origin is an unstable node. All trajectories leave the origin parallel to the eigenvector $(0,1)^T$ for $\lambda = 3$ which spans the y-axis. The phase portrait near the origin is shown in Fig. 4.6.

At the fixed point (0, 3), $J(0,3) = \begin{pmatrix} -3 & 0 \\ -6 & -3 \end{pmatrix}$, which has the eigenvalues −3, −3. So, the fixed point (0, 3) is a stable node. Trajectories approach along the eigen direction with the eigenvalue $\lambda = -3$ spanning the eigenvector $(0,1)^T$. The phase portrait near the fixed point (0, 3) which is a stable node looks like as presented in Fig. 4.7.

At (3, 0), we have $J(3,0) = \begin{pmatrix} -3 & -6 \\ 0 & -3 \end{pmatrix}$. The eigenvalues of $J(3,0)$ are −3, −3. So, as previous the fixed point (3, 0) is also a stable node. Trajectories approach along the eigen direction with the eigenvalue $\lambda = -3$ spanning the eigenvector $(1,0)^T$. The phase portrait near the fixed point (3, 0) is depicted in Fig. 4.8.

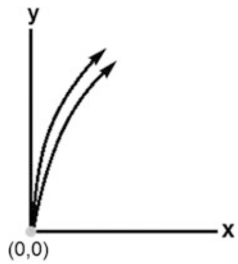

Fig. 4.6 Local phase portrait near the fixed point origin

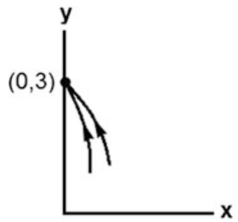

Fig. 4.7 Local phase portrait near the fixed point (0, 3)

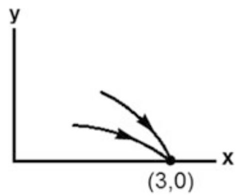

Fig. 4.8 Local phase portrait near the fixed point (3, 0)

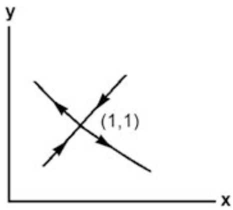

Fig. 4.9 Local phase portrait near the fixed point (1, 1)

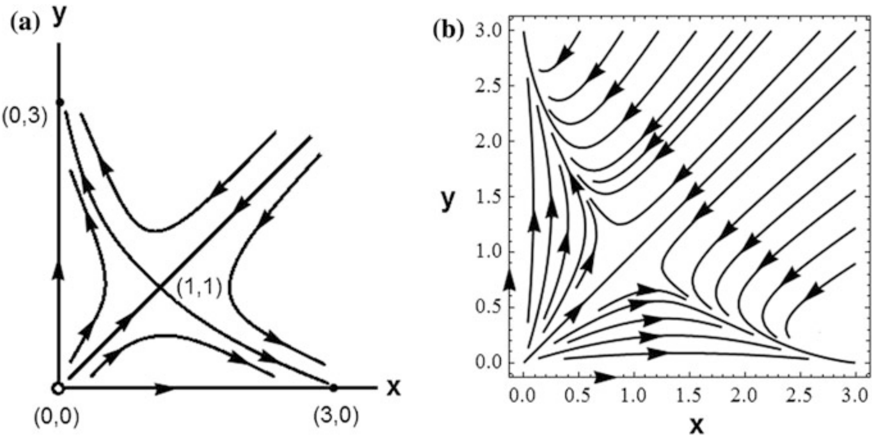

Fig. 4.10 Phase trajectories of the given system

At (1, 1), we calculate $J(1,1) = \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}$, which gives two distinct eigenvalues, 1, −3, with opposite signs. Therefore, the fixed point (1, 1) is a saddle. The phase portrait near (1, 1) is shown in Fig. 4.9.

The x and y axes represent the straight line trajectories because $\dot{x} = 0$ when $x = 0$ and $\dot{y} = 0$ when $y = 0$. All trajectories of the system are presented in Fig. 4.10. This figure also clearly depicts the attracting points and the basin boundary of the model. The attracting points of the system are (3, 0) and (0, 3). The basin boundary of the two attracting points is the straight line $y = x$, which is also the separatrix of the system.

4.8 Exercises

1. Examine Lyapunov, Poincaré, and Lagrange stability criteria for the following equations:

   (i) $\dot{x} = 0$,
   (ii) $\dot{x} + x = 0$,
   (iii) $\dot{x} = y, \dot{y} = 0$.

2. Find the general solution of the nonlinear oscillator $\dot{x} = -y(x^2+y^2)^{\frac{1}{2}}$, $\dot{y} = x(x^2+y^2)^{\frac{1}{2}}$. Also, examine whether it is Lyapunov or orbitally stable.
3. Define Lyapunov function and Lyanupov stability. Examine the stability in the Lyapunov sense for the following equations:

   (i) $\dot{x}+x = 2,\ x(0) = 1$,
   (ii) $\dot{x}-x = 2,\ x(0) = -1$,
   (iii) $\dot{x} = 5,\ x(0) = 0$

4. Using a suitable Lyapunov function, prove that the system $\dot{x} = -x+4y,\ \ \dot{y} = -x-y^3$ has no closed orbits.
5. Examine asymptotic stability through the construction of suitable Lyapunov function L for the system $\dot{x} = 2y(z-1), \dot{y} = -x(z-1), \dot{z} = xy$.
6. Using suitable Lyapunov functions examine the stability at the equilibrium point origin for the following systems:

   (i) $\dot{x} = y+x^3,\ \dot{y} = x-y^3$
   (ii) $\dot{x} = y - x\,g(x,\,y),\ \dot{y} = -x - y\,g(x,\,y)$ where the function $g(x,\,y)$ can be expanded in a convergent power series with $g(0,\,0) = 0$,
   (iii) $\dot{x} = 2xy+x^3,\ \dot{y} = x^2-y^5$,
   (iv) $\dot{x} = y-x^3,\ \ \dot{y} = -x-y^3$

7. Investigate the stability of the system $\dot{x} = -5y-2x^3,\ \ \dot{y} = 5x-3y^3$ at (0, 0) using Lyapunov direct method.
8. State Hartman–Grobmann theorem and discuss its significance. Using theorem describe the local stability behavior near equilibrium points of the following nonlinear systems (i) $\dot{x} = y^2-x+2, \dot{y} = x^2-y^2$. (ii) $\dot{x} = -y,\ \ \dot{y} = x-x^5$. Also, draw the phase portrait.
9. Find the stable, unstable, and center subspaces for the linear system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ when the matrix A is given by

   (i) $A = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$
   (ii) $A = \begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix}$,
   (iii) $A = \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}$,
   (iv) $A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$,
   (v) $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$,
   (vi) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,

(vii) $A = \begin{pmatrix} 10 & -1 & 0 \\ 25 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

(viii) $A = \begin{pmatrix} 0 & -2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

10. Obtain the local stable and unstable manifolds for the system $\dot{x} = -x$, $\dot{y} = y + x^2$ and give a rough sketch of the manifolds.
11. Obtain the stable and unstable manifolds for the system
$\dot{x} = -x + \sigma + \frac{x^2}{y}, \dot{y} = -y + x^2$,where σ is a parameter.
12. Find the fixed point and investigate their stability for the system
$\dot{x} = x(3 - 2x - y)$
$\dot{y} = y(2 - x - y)$
Also, draw the basin of attraction and basin boundary.
13. Find the basin of attraction and basin boundary for the following systems:

(i) $\dot{x} = x(1 - x - 2y), \dot{y} = y(1 - 2x - y)$
(ii) $\dot{x} = x(1 - x - 2y), \dot{y} = y(1 - 3x - y)$
(iii) $\dot{x} = x(1 - x - 3y), \dot{y} = y(1 - 2x - y)$
(iv) $\dot{x} = x(1 - x - 5y), \dot{y} = 2y(1 - 3x - y)$
(v) $\dot{x} = x(3 - x - 2y), \dot{y} = y(2 - x - y)$.

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