Chapter 2
Linear Systems

This chapter deals with linear systems of ordinary differential equations (ODEs), both homogeneous and nonhomogeneous equations. Linear systems are extremely useful for analyzing nonlinear systems. The main emphasis is given for finding solutions of linear systems with constant coefficients so that the solution methods could be extended to higher dimensional systems easily. The well-known methods such as eigenvalue–eigenvector method and the fundamental matrix method have been described in detail. The properties of fundamental matrix, the fundamental theorem, and important properties of exponential matrix function are given in this chapter. It is important to note that the set of all solutions of a linear system forms a vector space. The eigenvectors constitute the solution space of the linear system. The general solution procedure for linear systems using fundamental matrix, the concept of generalized eigenvector, solutions of multiple eigenvalues, both real and complex, are discussed.

2.1 Linear Systems

Consider a linear system of ordinary differential equations as follows:

$$\left.\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= \dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_1 \\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= \dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_2 \\
&\vdots \\
\frac{\mathrm{d}x_n}{\mathrm{d}t} &= \dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_n
\end{aligned}\right\} \tag{2.1}$$

where $a_{ij}, b_j (i, j = 1, 2, \ldots, n)$ are all given constants. The system (2.1) can be written in matrix notation as

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$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x} + \underset{\sim}{b} \tag{2.2}$$

where $\underset{\sim}{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^t$, $\underset{\sim}{b} = (b_1, b_2, \ldots, b_n)^t$ are the column vectors and $A = [a_{ij}]_{n\times n}$ is the square matrix of order n, known as the coefficient matrix of the system. The system (2.2) is said to be homogeneous if $\underset{\sim}{b} = \underset{\sim}{0}$, that is, if all b_i's are identically zero. On the other hand, if $\underset{\sim}{b} \neq \underset{\sim}{0}$, that is, if at least one b_i is nonzero, then the system is called nonhomogeneous. We consider first linear homogeneous system as

$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x} \tag{2.3}$$

A differentiable function $\underset{\sim}{x}(t)$ is said to be a solution of (2.3) if it satisfies the equation $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$. Let $\underset{\sim}{x}_1(t)$ and $\underset{\sim}{x}_2(t)$ be two solutions of (2.3). Then any linear combination $\underset{\sim}{x}(t) = c_1\underset{\sim}{x}_1(t) + c_2\underset{\sim}{x}_2(t)$ of $\underset{\sim}{x}_1(t)$ and $\underset{\sim}{x}_2(t)$ is also a solution of (2.3). This can be shown very easily as below.

$$\dot{\underset{\sim}{x}} = c_1\dot{\underset{\sim}{x}}_1 + c_2\dot{\underset{\sim}{x}}_2$$

and so

$$A\underset{\sim}{x} = A(c_1\underset{\sim}{x}_1 + c_2\underset{\sim}{x}_2) = c_1A\underset{\sim}{x}_1 + c_2A\underset{\sim}{x}_2 = c_1\dot{\underset{\sim}{x}}_1 + c_2\dot{\underset{\sim}{x}}_2 = \dot{\underset{\sim}{x}}.$$

The solution $\underset{\sim}{x} = c_1\underset{\sim}{x}_1 + c_2\underset{\sim}{x}_2$ is known as **general solution** of the system (2.3). Thus the general solution of a system is the linear combination of the set of all solutions of that system (superposition principle). Since the system is linear, we may consider a nontrivial solution of (2.3) as

$$\underset{\sim}{x}(t) = \underset{\sim}{\alpha}\, e^{\lambda t} \tag{2.4}$$

where $\underset{\sim}{\alpha}$ is a column vector with components $\underset{\sim}{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)^t$ and λ is a number. Substituting (2.4) into (2.3) we obtain

$$\lambda\underset{\sim}{\alpha}\, e^{\lambda t} = A\underset{\sim}{\alpha}\, e^{\lambda t}$$

$$\text{or, } (A - \lambda I)\underset{\sim}{\alpha} = \underset{\sim}{0} \tag{2.5}$$

where I is the identity matrix of order n. Equation (2.5) gives a nontrivial solution if and only if

$$\det(A - \lambda I) = 0 \tag{2.6}$$

On expansion, Eq. (2.6) gives a polynomial equation of degree n in λ, known as the *characteristic equation* of matrix A. The roots of the characteristic equation (2.6) are called the **characteristic roots** or **eigenvalues** or **latent roots** of A. The vector $\underset{\sim}{\alpha}$, which is a nontrivial solution of (2.5), is known as an **eigenvector** of A corresponding to the eigenvalue λ. If $\underset{\sim}{\alpha}$ is an eigenvector of a matrix A corresponding to an eigenvalue λ, then $\underset{\sim}{x}(t) = e^{\lambda t}\underset{\sim}{\alpha}$ is a solution of the system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$. The set of linearly independent eigenvectors constitutes a solution space of the linear homogeneous ordinary differential equations which is a vector space. All properties of vector space hold good for the solution space. We now discuss the general solution of a linear system below.

2.2 Eigenvalue–Eigenvector Method

As we know, the solution of a linear system constitutes a linear space and the solution is formed by the eigenvectors of the matrix. There may have four possibilities according to the eigenvalues and corresponding eigenvectors of matrix A. We proceed now case-wise as follows.

Case I: Eigenvalues of A are real and distinct

If the coefficient matrix A has real distinct eigenvalues, then it has linearly independent (L.I.) eigenvectors. Let $\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_n$ be the eigenvectors corresponding to the eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$ of matrix A. Then each $\underset{\sim}{x}_j(t) = \underset{\sim}{\alpha}_j e^{\lambda_j t}, j = 1, 2, \ldots, n$ is a solution of $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$. The general solution is a linear combination of the solutions $\underset{\sim}{x}_j(t)$ and is given by

$$\underset{\sim}{x}(t) = \sum_{j=1}^{n} c_j \underset{\sim}{x}_j(t)$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants. In $\mathbb{R}^2$, the solution can be written as

$$\underset{\sim}{x}(t) = \sum_{j=1}^{2} c_j \underset{\sim}{\alpha}_j e^{\lambda_j t} = c_1 \underset{\sim}{\alpha}_1 e^{\lambda_1 t} + c_2 \underset{\sim}{\alpha}_2 e^{\lambda_2 t}.$$

Case II: Eigenvalues of A are real but repeated

In this case matrix A may have either n linearly independent eigenvectors or only one or many ($<n$) linearly independent eigenvectors corresponding to the repeated

eigenvalues. The generalized eigenvectors have been used for linearly independent eigenvectors. We discuss this case in the following two sub-cases.

Sub-case 1: Matrix A has linearly independent eigenvectors
Let $\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_n$ be n linearly independent eigenvectors corresponding to the repeated real eigenvalue λ of matrix A. In this case the general solution of the linear system is given by

$$\underset{\sim}{x}(t) = \sum_{i=1}^{n} c_i \underset{\sim}{\alpha}_i e^{\lambda t}.$$

Sub-case 2. Matrix A has only one or many ($<n$) linearly independent eigenvectors
First, we give the definition of generalized eigenvector of A. Let λ be an eigenvalue of the $n \times n$ matrix A of multiplicity $m \le n$. Then for $k = 1, 2, \ldots, m$, any nonzero solution of the equation $(A - \lambda I)^k \underset{\sim}{v} = \underset{\sim}{0}$ is called a **generalized eigenvector** of A. For simplicity consider a two dimensional system. Let the eigenvalues be repeated but only one eigenvector, say $\underset{\sim}{\alpha}_1$ be linearly independent. Let $\underset{\sim}{\alpha}_2$ be a generalized eigenvector of the 2×2 matrix A. Then $\underset{\sim}{\alpha}_2$ can be obtained from the relation $(A - \lambda I)\underset{\sim}{\alpha}_2 = \underset{\sim}{\alpha}_1 \Rightarrow A\underset{\sim}{\alpha}_2 = \lambda \underset{\sim}{\alpha}_2 + \underset{\sim}{\alpha}_1$. So the general solution of the system is given by

$$\underset{\sim}{x}(t) = c_1 \underset{\sim}{\alpha}_1 e^{\lambda t} + c_2 (t \underset{\sim}{\alpha}_1 e^{\lambda t} + \underset{\sim}{\alpha}_2 e^{\lambda t}).$$

Similarly, for an $n \times n$ matrix A, the general solution may be written as $\underset{\sim}{x}(t) = \sum_{i=1}^{n} c_i \underset{\sim}{x}_i(t)$, where

$$\begin{aligned}
\underset{\sim}{x}_1(t) &= \underset{\sim}{\alpha}_1 e^{\lambda t}, \\
\underset{\sim}{x}_2(t) &= t\underset{\sim}{\alpha}_1 e^{\lambda t} + \underset{\sim}{\alpha}_2 e^{\lambda t}, \\
\underset{\sim}{x}_3(t) &= \tfrac{t^2}{2!}\underset{\sim}{\alpha}_1 e^{\lambda t} + t\underset{\sim}{\alpha}_2 e^{\lambda t} + \underset{\sim}{\alpha}_3 e^{\lambda t}, \\
&\vdots \\
\underset{\sim}{x}_n(t) &= \tfrac{t^{n-1}}{(n-1)!}\underset{\sim}{\alpha}_1 e^{\lambda t} + \cdots + \tfrac{t^2}{2!}\underset{\sim}{\alpha}_{n-2} e^{\lambda t} + t\underset{\sim}{\alpha}_{n-1} e^{\lambda t} + \underset{\sim}{\alpha}_n e^{\lambda t}.
\end{aligned}$$

Case III: Matrix A has non-repeated complex eigenvalues
Suppose the real $n \times n$ matrix A has m-pairs of complex eigenvalues $a_j \pm i b_j, j = 1, 2, \ldots, m$. Let $\underset{\sim}{\alpha}_j \pm i \underset{\sim}{\beta}_j, j = 1, 2, \ldots, m$ denote the corresponding eigenvectors. Then the solution of the system $\dot{\underset{\sim}{x}}(t) = A\underset{\sim}{x}(t)$ for these complex eigenvalues is given by

$$\underset{\sim}{x}(t) = \sum_{j=1}^{m} c_j \underset{\sim}{u}_j + d_j \underset{\sim}{v}_j$$

where $\underset{\sim}{u}_j = \exp(a_j t)\{\underset{\sim}{\alpha}_j \cos(b_j t) - \underset{\sim}{\beta}_j \sin(b_j t)\}$, $\underset{\sim}{v}_j = \exp(a_j t)\{\underset{\sim}{\alpha}_j \sin(b_j t) + \underset{\sim}{\beta}_j \cos(b_j t)\}$ and $c_j, d_j (j = 1, 2, \ldots, m)$ are arbitrary constants. We discuss each of the above cases through specific examples below.

Example 2.1 Find the general solution of the following linear homogeneous system using eigenvalue-eigenvector method:

$$\dot{x} = 5x + 4y$$
$$\dot{y} = x + 2y.$$

Solution In matrix notation, the system can be written as $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, where $\underset{\sim}{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$. The eigenvalues of A satisfy the equation

$$\begin{aligned} &\det(A - \lambda I) = 0 \\ &\Rightarrow \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0 \\ &\Rightarrow (5-\lambda)(2-\lambda) - 4 = 0 \\ &\Rightarrow \lambda^2 - 7\lambda + 6 = 0. \end{aligned}$$

The roots of the characteristic equation $\lambda^2 - 7\lambda + 6 = 0$ are $\lambda = 1, 6$. So the eigenvalues of A are real and distinct. We shall now find the eigenvectors corresponding to these eigenvalues.

Let $\underset{\sim}{e} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ be the eigenvector corresponding to the eigenvalue $\lambda_1 = 1$. Then

$$\begin{aligned} &(A - I)\underset{\sim}{e} = \underset{\sim}{0} \\ &\Rightarrow \begin{pmatrix} 5-1 & 4 \\ 1 & 2-1 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &\Rightarrow \begin{pmatrix} 4e_1 + 4e_2 \\ e_1 + e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &\Rightarrow 4e_1 + 4e_2 = 0,\ e_1 + e_2 = 0. \end{aligned}$$

We can choose $e_1 = 1$, $e_2 = -1$. So, the eigenvector corresponding to the eigenvalue $\lambda_1 = 1$ is $\underset{\sim}{e} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Again, let $\underset{\sim}{e}' = \begin{pmatrix} e_1' \\ e_2' \end{pmatrix}$ be the eigenvector corresponding to the eigenvalue $\lambda_2 = 6$. Then

$$\begin{aligned} & (A - 6I)\underset{\sim}{e}' = \underset{\sim}{0} \\ \Rightarrow \quad & \begin{pmatrix} 5-6 & 4 \\ 1 & 2-6 \end{pmatrix} \begin{pmatrix} e_1' \\ e_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \Rightarrow \quad & \begin{pmatrix} -e_1' + 4e_2' \\ e_1' - 4e_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \Rightarrow \quad & -e_1' + 4e_2' = 0\, e_1' - 4e_2' = 0. \end{aligned}$$

We can choose $e_1' = 4, e_2' = 1$. So, the eigenvector corresponding to theeigen-value $\lambda_2 = 6$ is $\underset{\sim}{e}' = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$. The eigenvectors $\underset{\sim}{e}$, $\underset{\sim}{e}'$ are linearly independent. Hence the general solution of the system is given as

$$\underset{\sim}{x}(t) = c_1 \underset{\sim}{e}\; e^t + c_2 \underset{\sim}{e}' e^{6t} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{6t}$$

or, $\left. \begin{aligned} x(t) &= c_1 e^t + 4c_2 e^{6t} \\ y(t) &= -c_1 e^t + c_2 e^{6t} \end{aligned} \right\}$, where c_1, c_2 are arbitrary constants.

Example 2.2 Find the general solution of the linear system

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Solution The characteristic equation of matrix A is

$$\begin{aligned} & \det(A - \lambda I) = 0 \\ & \text{or, } \begin{vmatrix} 3-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = 0 \\ & \text{or, } (3-\lambda)^2 = 0 \\ & \text{or, } \lambda = 3, 3. \end{aligned}$$

So, the eigenvalues of A are 3, 3, which are real and repeated. Clearly, $\underset{\sim}{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\underset{\sim}{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are two linearly independent eigenvectors corresponding to the repeated eigenvalue $\lambda = 3$. Thus, the general solution of the system is

$$\underset{\sim}{x}(t) = c_1 \underset{\sim}{e}_1 e^{\lambda t} + c_2 \underset{\sim}{e}_2 e^{\lambda t}$$

$$\Rightarrow \quad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{3t} = \begin{pmatrix} c_{1e^{3t}} \\ c_{2e^{3t}} \end{pmatrix}.$$

$$\Rightarrow \quad x(t) = c_1 e^{3t},\ y(t) = c_2 e^{3t}, \text{where } c_1,\ c_2 \text{ are arbitrary constants.}$$

Example 2.3 Find the general solution of the system

$$\dot{x} = 3x - 4y$$
$$\dot{y} = x - y$$

using eigenvalue-eigenvector method.

Solution The characteristic equation of matrix A is

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \quad \begin{vmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^2 - 2\lambda + 1 = 0$$

$$\Rightarrow \quad \lambda = 1, 1$$

So matrix A has repeated real eigenvalues $\lambda = 1,\ 1$.

Let $\underset{\sim}{e} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ be the eigenvector corresponding to the eigenvalue $\lambda = 1$. Then

$$(A - I)\underset{\sim}{e} = \underset{\sim}{0}$$

$$\Rightarrow \quad \begin{pmatrix} 3-1 & -4 \\ 1 & -1-1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} 2e_1 - 4e_2 \\ e_1 - 2e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad 2e_1 - 4e_2 = 0, e_1 - 2e_2 = 0$$

We can choose $e_1 = 2, e_2 = 1$. Therefore, $\underset{\sim}{e} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Let $\underset{\sim}{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ be the generalized eigenvector corresponding to the eigenvalue $\lambda = 1$. Then

$$
\begin{aligned}
&(A-I)\underset{\sim}{g} = \underset{\sim}{e}\\
\Rightarrow\quad &\begin{pmatrix} 3-1 & -4 \\ 1 & -1-1 \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}\\
\Rightarrow\quad &\begin{pmatrix} 2g_1 - 4g_2 \\ g_1 - 2g_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}\\
\Rightarrow\quad &2g_1 - 4g_2 = 2, g_1 - 2g_2 = 1
\end{aligned}
$$

We can choose $g_2 = 1, g_1 = 3$. Therefore $\underset{\sim}{g} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

Therefore the general solution of the system is

$$
\underset{\sim}{x}(t) = c_1 \underset{\sim}{e}\, e^t + c_2\left(\underset{\sim}{e}\; t e^t + \underset{\sim}{g}\; e^t\right)
$$

$$
\text{or,}\quad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} te^t + c_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^t
$$

or, $\left.\begin{aligned} x(t) &= \{2c_1 + (2t+3)c_2\}e^t \\ y(t) &= \{c_1 + (t+1)c_2\}e^t \end{aligned}\right\}$, where c_1 and c_2 are arbitrary constants.

Example 2.4 Find the general solution of the linear system

$$
\begin{aligned}
\dot{x} &= 10x - y\\
\dot{y} &= 25x + 2y
\end{aligned}
$$

Solution Given system can be written as

$$
\dot{\underset{\sim}{x}} = A\underset{\sim}{x}, \text{ where } A = \begin{pmatrix} 10 & -1 \\ 25 & 2 \end{pmatrix} \text{ and } \underset{\sim}{x} = \begin{pmatrix} x \\ y \end{pmatrix}.
$$

The characteristic equation of matrix A is

$$
\begin{aligned}
&\det(A - \lambda I) = 0\\
\Rightarrow\quad &\begin{vmatrix} 10-\lambda & -1 \\ 25 & 2-\lambda \end{vmatrix} = 0\\
\Rightarrow\quad &\lambda^2 - 12\lambda + 45 = 0\\
\Rightarrow\quad &\lambda = 6 \pm 3i.
\end{aligned}
$$

Therefore, matrix A has a pair of complex conjugate eigenvalues $6 \pm 3i$.

Let $\underset{\sim}{e} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ be the eigenvector corresponding to the eigenvalue $\lambda = 6 + 3i$. Then

$$\begin{aligned}
&(A-(6+3i)I)\underset{\sim}{e} = \underset{\sim}{0} \\
\Rightarrow\quad &\begin{pmatrix} 10-6-3i & -1 \\ 25 & 2-6-3i \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\Rightarrow\quad &\begin{pmatrix} (4-3i)e_1 - e_2 \\ 25e_1-(4+3i)e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\Rightarrow\quad &(4-3i)e_1 - e_2 = 0, 25e_1-(4+3i)e_2 = 0.
\end{aligned}$$

A nontrivial solution of this system is

$$e_1 = 1,\ e_2 = 4-3i.$$

Therefore $\underset{\sim}{e} = \begin{pmatrix} 1 \\ 4-3i \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + i\begin{pmatrix} 0 \\ -3 \end{pmatrix} = \underset{\sim}{\alpha}_1 + i\underset{\sim}{\alpha}_2$, where $\underset{\sim}{\alpha}_1 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ and $\underset{\sim}{\alpha}_2 = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$.

Similarly, the eigenvector corresponding to the eigenvalue $\lambda = 6-3i$ is $\underset{\sim}{e}' = \begin{pmatrix} 1 \\ 4+3i \end{pmatrix} = \underset{\sim}{\alpha}_1 - i\underset{\sim}{\alpha}_2$. Therefore,

$$\underset{\sim}{u}_1 = e^{at}\left(\underset{\sim}{\alpha}_1 \cos bt - \underset{\sim}{\alpha}_2 \sin bt\right) = e^{6t}\left\{\begin{pmatrix} 1 \\ 4 \end{pmatrix}\cos 3t - \begin{pmatrix} 0 \\ -3 \end{pmatrix}\sin 3t\right\}$$

and

$$\underset{\sim}{v}_1 = e^{at}\left(\underset{\sim}{\alpha}_1 \sin bt + \underset{\sim}{\alpha}_2 \cos bt\right) = e^{6t}\left\{\begin{pmatrix} 1 \\ 4 \end{pmatrix}\sin 3t + \begin{pmatrix} 0 \\ -3 \end{pmatrix}\cos 3t\right\}.$$

Therefore, the general solution is

$$\begin{aligned}
\underset{\sim}{x}(t) &= c_1\underset{\sim}{u}_1 + d_1\underset{\sim}{v}_1 \\
&= e^{6t}\left\{\begin{pmatrix} 1 \\ 4 \end{pmatrix} c_1 \cos 3t - \begin{pmatrix} 0 \\ -3 \end{pmatrix} c_1 \sin 3t\right\} \\
&\quad + e^{6t}\left\{\begin{pmatrix} 1 \\ 4 \end{pmatrix} d_1 \sin 3t + \begin{pmatrix} 0 \\ -3 \end{pmatrix} d_1 \cos 3t\right\} \\
&= e^{6t}\begin{pmatrix} c_1 \cos 3t + d_1 \sin 3t \\ (4c_1-3d_1)\cos 3t + (3c_1+4d_1)\sin 3t \end{pmatrix} \\
\Rightarrow\quad x(t) &= e^{6t}(c_1\cos 3t + d_1 \sin 3t), \\
y(t) &= e^{6t}[(4c_1-3d_1)\cos 3t + (3c_1+4d_1)\sin 3t]
\end{aligned}$$

where c_1 and d_1 are arbitrary constants.

Example 2.5 Find the solution of the system

$$\dot{x} = x - 5y, \dot{y} = x - 3y$$

satisfying the initial condition $x(0) = 1$, $y(0) = 1$. Describe the behavior of the solution as $t \to \infty$.

Solution The characteristic equation of matrix A is

$$\begin{aligned} & \det(A - \lambda I) = 0 \\ \Rightarrow \quad & \begin{vmatrix} 1-\lambda & -5 \\ 1 & -3-\lambda \end{vmatrix} = 0 \\ \Rightarrow \quad & \lambda^2 + 2\lambda + 2 = 0 \\ \Rightarrow \quad & \lambda = -1 \pm i. \end{aligned}$$

So, matrix A has a pair of complex conjugate eigenvalues $(-1 \pm i)$.

Let $\underset{\sim}{e} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ be the eigenvector corresponding to the eigenvalue $\lambda = -1 + i$. Then

$$\begin{aligned} & (A - (-1+i)I)\underset{\sim}{e} = \underset{\sim}{0} \\ \Rightarrow \quad & \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \Rightarrow \quad & \begin{pmatrix} (2-i)e_1 - 5e_2 \\ e_1 - (2+i)e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \Rightarrow \quad & (2-i)e_1 - 5e_2 = 0, e_1 - (2+i)e_2 = 0. \end{aligned}$$

A nontrivial solution of this system is

$$e_1 = 2 + i, e_2 = 1.$$

Therefore $\underset{\sim}{e} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \underset{\sim}{\alpha}_1 + i\underset{\sim}{\alpha}_2$, where $\underset{\sim}{\alpha}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\underset{\sim}{\alpha}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Similarly, the eigenvector corresponding to the eigenvalue $\lambda = -1 - i$ is $\underset{\sim}{e}' = \begin{pmatrix} 2-i \\ 1 \end{pmatrix} = \underset{\sim}{\alpha}_1 - i\underset{\sim}{\alpha}_2$.

$$\therefore \underset{\sim}{u}_1 = e^{at}\left(\underset{\sim}{\alpha}_1 \cos bt - \underset{\sim}{\alpha}_2 \sin bt\right) = e^{-t}\left\{\begin{pmatrix}2\\1\end{pmatrix}\cos t - \begin{pmatrix}1\\0\end{pmatrix}\sin t\right\} \text{ and}$$

$$\underset{\sim}{v}_1 = e^{at}\left(\underset{\sim}{\alpha}_1 \sin bt + \underset{\sim}{\alpha}_2 \cos bt\right) = e^{-t}\left\{\begin{pmatrix}2\\1\end{pmatrix}\sin t + \begin{pmatrix}1\\0\end{pmatrix}\cos t\right\}$$

Therefore, the solution of the system is

$$\begin{aligned}\underset{\sim}{x}(t) &= x(0)\underset{\sim}{u}_1 + y(0)\underset{\sim}{v}_1\\ &= e^{-t}\left\{\begin{pmatrix}2\\1\end{pmatrix}\cos t - \begin{pmatrix}1\\0\end{pmatrix}\sin t\right\} + e^{-t}\left\{\begin{pmatrix}2\\1\end{pmatrix}\sin t + \begin{pmatrix}1\\0\end{pmatrix}\cos t\right\}\\ &= e^{-t}\left\{\begin{pmatrix}2\\1\end{pmatrix}(\cos t + \sin t) + \begin{pmatrix}1\\0\end{pmatrix}(\cos t - \sin t)\right\}.\end{aligned}$$

When $t \to \infty$, $e^{-t} \to 0$. So, in this case $\underset{\sim}{x}(t) \to \underset{\sim}{0}$, that is, the solution of the system is stable in the usual sense.

Example 2.6 Find the solution of the system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, where

$$A = \begin{pmatrix}-1 & 2 & 3\\ 0 & -2 & 1\\ 0 & 3 & 0\end{pmatrix}.$$

Solution The characteristic equation of A is

$$\begin{aligned}&\det(A - \lambda I) = 0\\ \Rightarrow\quad &\begin{vmatrix}-1-\lambda & 2 & 3\\ 0 & -2-\lambda & 1\\ 0 & 3 & -\lambda\end{vmatrix} = 0\\ \Rightarrow\quad &(\lambda+1)(\lambda-1)(\lambda+3) = 0\\ \Rightarrow\quad &\lambda = -1, 1, -3\end{aligned}$$

Therefore the eigenvalues of matrix A are $\lambda = -1, 1, -3$.

We shall now find the eigenvector corresponding to each of the eigenvalues.

Let $\underset{\sim}{e} = \begin{pmatrix}e_1\\e_2\\e_3\end{pmatrix}$ be the eigenvector corresponding to the eigenvalue $\lambda = -1$.

Then

$$
\begin{aligned}
&(A+1)\underset{\sim}{e} = \underset{\sim}{0} \\
&\Rightarrow \quad \begin{pmatrix} -1+1 & 2 & 3 \\ 0 & -2+1 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
&\Rightarrow \quad 2e_2 + 3e_3 = 0,\ -e_2 + e_3 = 0,\ 3e_2 + e_3 = 0 \\
&\Rightarrow e_2 = e_3 = 0 \text{ and } e_1 \text{ is arbitrary.}
\end{aligned}
$$

We choose $e_1 = 1$. Therefore, the eigenvector corresponding to the eigenvalue $\lambda = -1$ is $\underset{\sim}{e} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Similarly, the eigenvectors corresponding to $\lambda = 1$ and $\lambda = -3$ are, respectively, $\underset{\sim}{g} = \begin{pmatrix} 11/2 \\ 1 \\ 3 \end{pmatrix}$ and $\underset{\sim}{\alpha} = \begin{pmatrix} 1/2 \\ 1 \\ -1 \end{pmatrix}$. Therefore the general solution is

$$
\begin{aligned}
\underset{\sim}{x}(t) &= c_1 \underset{\sim}{e}\, e^{-t} + c_2 \underset{\sim}{g}\, e^{t} + c_3 \underset{\sim}{\alpha}\, e^{-3t} \\
&= c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 11/2 \\ 1 \\ 3 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 1/2 \\ 1 \\ -1 \end{pmatrix} e^{-3t}
\end{aligned}
$$

where c_1,c_2 and c_3 are arbitrary constants.

Example 2.7 Solve the system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, where

$$
A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}.
$$

Solution The characteristic equation of matrix A is

$$
\begin{aligned}
&\det(A - \lambda I) = 0 \\
&\Rightarrow \quad \begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = 0 \\
&\Rightarrow \quad \lambda = 4, -2, -2.
\end{aligned}
$$

So (-2) is a repeated eigenvalue of A. The eigenvector for the eigenvalue $\lambda_1 = 4$ is given as $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. The eigenvector corresponding to the repeated eigenvalue $\lambda_2 = \lambda_3 = -2$ is $\begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix}^T$ such that

$$\begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which is equivalent to

$$3e_1 - 3e_2 + 3e_3 = 0, \quad 3e_1 - 3e_2 + 3e_3 = 0, \quad 6e_1 - 6e_2 + 6e_3 = 0,$$

that is, $e_1 - e_2 + e_3 = 0$.

We can choose $e_1 = 1$, $e_2 = 1$ and $e_3 = 0$, and so we can take one eigenvector as $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Again, we can choose $e_1 = 0$, $e_2 = 1$ and $e_3 = 1$. Then we obtain another eigenvector $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Clearly, these two eigenvectors are linearly independent. Thus, we have two linearly independent eigenvectors corresponding to the repeated eigenvalue -2. Hence, the general solution of the system is given by

$$\underset{\sim}{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-2t} + c_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{-2t}$$

where c_1, c_2 and c_3 are arbitrary constants.

Example 2.8 Solve the system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ where

$$A = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Solution Here matrix A has two pair of complex conjugate eigenvalues $\lambda_1 = -1 \pm i$ and $\lambda_2 = 1 \pm i$. The corresponding pair of eigenvectors is

$$\underset{\sim}{w}_1 = \underset{\sim}{\alpha}_1 \pm i\underset{\sim}{\beta}_1 = \begin{pmatrix} \pm i \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \pm i \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and}$$

$$\underset{\sim}{w}_2 = \underset{\sim}{\alpha}_2 \pm i\underset{\sim}{\beta}_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \pm i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Therefore, the general solution of the system is expressed as

$$\begin{aligned}
\underset{\sim}{x}(t) &= \sum_{j=1}^{2} c_j \underset{\sim}{u}_j + d_j \underset{\sim}{v}_j \\
&= c_1 e^{-t} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cos t - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \sin t \right\} + c_2 e^{t} \left\{ \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \sin t \right\} \\
&\quad + d_1 e^{-t} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \sin t + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cos t \right\} \\
&\quad + d_2 e^{t} \left\{ \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \sin t + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \cos t \right\} \\
&= \begin{pmatrix} e^{-t}(d_1 \cos t - c_1 \sin t) \\ e^{-t}(c_1 \cos t + d_1 \sin t) \\ e^{t}\{(d_2 - c_2)\cos t - (d_2 + c_2)\sin t\} \\ e^{t}(c_2 \cos t + d_2 \sin t) \end{pmatrix}
\end{aligned}$$

where $c_j, d_j (j = 1, 2)$ are arbitrary constants.

2.3 Fundamental Matrix

A set $\{\underset{\sim}{x}_1(t), \underset{\sim}{x}_2(t), \ldots, \underset{\sim}{x}_n(t)\}$ of solutions of a linear homogeneous system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ is said to be a **fundamental set** of solutions of that system if it satisfies the following two conditions:

(i) The set $\{\underset{\sim}{x}_1(t), \underset{\sim}{x}_2(t), \ldots, \underset{\sim}{x}_n(t)\}$ is linearly independent, that is, for $c_1, c_2, \ldots, c_n \in \mathbb{R}$, $c_1\underset{\sim}{x}_1 + c_2\underset{\sim}{x}_2 + \cdots + c_n\underset{\sim}{x}_n = \underset{\sim}{0} \Rightarrow c_1 = c_2 = \cdots = c_n = 0$.

(ii) For any solution $\underset{\sim}{x}(t)$ of the system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, there exist $c_1, c_2, \ldots, c_n \in \mathbb{R}$ such that $\underset{\sim}{x}(t) = c_1\underset{\sim}{x}_1(t) + c_2\underset{\sim}{x}_2(t) + \cdots + c_n\underset{\sim}{x}_n(t), \forall t \in \mathbb{R}$.

The solution, expressed as a linear combination of a fundamental set of solutions of a system, is called a **general solution** of the system.

Let $\{\underset{\sim}{x}_1(t), \underset{\sim}{x}_2(t), \ldots, \underset{\sim}{x}_n(t)\}$ be a fundamental set of solutions of the system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ for $t \in I = [a, b];\ a, b \in \mathbb{R}$. Then the matrix

$$\Phi(t) = \left(\underset{\sim}{x}_1(t), \underset{\sim}{x}_2(t), \ldots, \underset{\sim}{x}_n(t)\right)$$

is called a **fundamental matrix** of the system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, $\underset{\sim}{x} \in \mathbb{R}^n$. Since the set $\{\underset{\sim}{x}_1(t), \underset{\sim}{x}_2(t), \ldots, \underset{\sim}{x}_n(t)\}$ is linearly independent, the fundamental matrix $\Phi(t)$ is nonsingular. Now the general solution of the system is

$$\begin{aligned}\underset{\sim}{x}(t) &= c_1\underset{\sim}{x}_1(t) + c_2\underset{\sim}{x}_2(t) + \cdots + c_n\underset{\sim}{x}_n(t)\\ &= \left(\underset{\sim}{x}_1(t), \underset{\sim}{x}_2(t), \ldots, \underset{\sim}{x}_n(t)\right)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}\\ &= \Phi(t)\,\underset{\sim}{c}\end{aligned}$$

where $\underset{\sim}{c} = (c_1, c_2, \ldots c_n)^t$ is a constant column vector. If the initial condition is $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$, then

$$\begin{aligned}&\Phi(0)\,\underset{\sim}{c} = \underset{\sim}{x}_0\\ &\Rightarrow \underset{\sim}{c} = \Phi^{-1}(0)\,\underset{\sim}{x}_0 [\text{Since } \Phi(t) \text{ is nonsingular for all } t].\end{aligned}$$

Thus the solution of the initial value problem $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ with the initial conditions $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$ can be expressed in terms of the fundamental matrix $\Phi(t)$ as

$$\underset{\sim}{x}(t) = \Phi(t)\Phi^{-1}(0)\underset{\sim}{x}_0 \tag{2.7}$$

Note that two different homogeneous systems cannot have the same fundamental matrix. Again, if $\Phi(t)$ is a fundamental matrix of $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, then for any constant C, $C\Phi(t)$ is also a fundamental matrix of the system.

Example 2.9 Find the fundamental matrix of the system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, where $A = \begin{pmatrix} 1 & -2 \\ -3 & 2 \end{pmatrix}$. Hence find its solution.

Solution The characteristic equation of matrix A is

$$\begin{aligned}
&|A - \lambda I| = 0 \\
\Rightarrow \quad & \begin{vmatrix} 1-\lambda & -2 \\ -3 & 2-\lambda \end{vmatrix} = 0 \\
\Rightarrow \quad & (1-\lambda)(2-\lambda) - 6 = 0 \\
\Rightarrow \quad & \lambda^2 - 3\lambda - 4 = 0 \\
\Rightarrow \quad & \lambda = -1, 4.
\end{aligned}$$

So, the eigenvalues of matrix A are -1, 4, which are real and distinct.

Let $\underset{\sim}{e} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ be the eigenvector corresponding to the eigenvalue $\lambda_1 = -1$. Then

$$\begin{aligned}
&(A+I)\underset{\sim}{e} = \underset{\sim}{0} \\
\Rightarrow \quad & \begin{pmatrix} 1+1 & -2 \\ -3 & 2+1 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\Rightarrow \quad & 2e_1 - 2e_2 = 0, -3e_1 + 3e_2 = 0.
\end{aligned}$$

A nontrivial solution of this system is $e_1 = 1, e_2 = 1$.

$$\therefore \underset{\sim}{e} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Again, let $\underset{\sim}{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ be the eigenvector corresponding to the eigenvalue $\lambda_2 = 4$. Then

$$(A - 4I)\underset{\sim}{g} = \underset{\sim}{0}$$
$$\Rightarrow \begin{pmatrix} 1-4 & -2 \\ -3 & 2-4 \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\Rightarrow 3g_1 + 2g_2 = 0$$

Choose $g_1 = 2, g_2 = -3$. Therefore, $\underset{\sim}{g} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

Therefore the eigenvectors corresponding to the eigenvalues $\lambda = -1,\ 4$ are respectively $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$, which are linearly independent. So two fundamental solutions of the system are

$$\underset{\sim}{x}_1(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}, \underset{\sim}{x}_2(t) = \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{4t}$$

and a fundamental matrix of the system is

$$\Phi(t) = \begin{pmatrix} \underset{\sim}{x}_1(t) & \underset{\sim}{x}_2(t) \end{pmatrix} = \begin{pmatrix} e^{-t} & 2e^{4t} \\ e^{-t} & -3e^{4t} \end{pmatrix}.$$

Now $\Phi(0) = \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}$ and so $\Phi^{-1}(0) = \frac{1}{5}\begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}$.

Therefore the general solution of the system is given by

$$\underset{\sim}{x}(t) = \Phi(t)\Phi^{-1}(0)\underset{\sim}{x}_0 = \frac{1}{5}\begin{pmatrix} e^{-t} & 2e^{4t} \\ e^{-t} & -3e^{4t} \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}\underset{\sim}{x}_0$$
$$= \frac{1}{5}\begin{pmatrix} 3e^{-t} + 2e^{4t} & 2e^{-t} - 2e^{4t} \\ 3e^{-t} - 3e^{4t} & 2e^{-t} + 3e^{4t} \end{pmatrix}\underset{\sim}{x}_0.$$

2.3.1 General Solution of Linear Systems

Consider a simple linear equation

$$\dot{x} = ax \tag{2.8}$$

with initial condition $x(0) = x_0$, where a and x_0 are certain constants. The solution of this initial value problem (IVP) is given as $x(t) = x_0 e^{at}$. Then we may expect that the solution of the initial value problem for $n \times n$ system

$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x} \text{ with } \underset{\sim}{x}(0) = \underset{\sim}{x}_0 \tag{2.9}$$

can be expressed in term of exponential matrix function as

$$\underset{\sim}{x}(t) = e^{At}\underset{\sim}{x}_0 \tag{2.10}$$

where A is an $n \times n$ matrix. Comparing (2.10) with the solution obtained by the fundamental matrix, we have the relation

$$e^{At} = \Phi(t)\Phi^{-1}(0) \tag{2.11}$$

Thus we see that if $\Phi(t)$ is a fundamental matrix of the system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, then $\Phi(0)$ is invertible and $e^{At} = \Phi(t)\Phi^{-1}(0)$. Note that if $\Phi(0) = I$, then $\Phi^{-1}(0) = I$ and so, $e^{At} = \Phi(t)I = \Phi(t)$.

Example 2.10 Does $\Phi(t) = \begin{pmatrix} 2e^t & -e^{-3t} \\ -4e^t & 2e^{-3t} \end{pmatrix}$ a fundamental matrix for a system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$?

Solution We know that if $\Phi(t)$ is a fundamental matrix, then $\Phi(0)$ is invertible.

Here $\Phi(t) = \begin{pmatrix} 2e^t & -e^{-3t} \\ -4e^t & 2e^{-3t} \end{pmatrix}$. So, $\Phi(0) = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$.

Since $\det(\Phi(0)) = 4 - 4 = 0$, $\Phi(0)$ is not invertible and hence the given matrix is not a fundamental matrix for the system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$.

Example 2.11 Find e^{At} for the system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, where $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$.

Solution The characteristic equation of A is

$$\begin{aligned} & |A - \lambda I| = 0 \\ \Rightarrow \quad & \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = 0 \\ \Rightarrow \quad & (\lambda - 1)^2 - 4 = 0 \\ \Rightarrow \quad & \lambda = 3, -1. \end{aligned}$$

So, the eigenvalue of A are $\lambda = 3, -1$. The eigenvector corresponding to the eigenvalues $\lambda = 3, -1$ are, respectively, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, which are linearly independent. So, two fundamental solutions of the system are $\underset{\sim}{x}_1(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}, \underset{\sim}{x}_2(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$. Therefore a fundamental matrix of the system is

$$\Phi(t) = \begin{pmatrix} \underset{\sim}{x}_1(t) & \underset{\sim}{x}_2(t) \end{pmatrix} = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}.$$

Now, $\Phi(0) = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$ and $\Phi^{-1}(0) = -\frac{1}{4}\begin{pmatrix} -2 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{pmatrix}$.

Therefore,

$$\begin{aligned} e^{At} &= \Phi(t)\Phi^{-1}(0) \\ &= \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(e^{3t}+e^{-t}) & \frac{1}{4}(e^{3t}-e^{-t}) \\ (e^{3t}-e^{-t}) & \frac{1}{2}(e^{3t}+e^{-t}) \end{pmatrix}. \end{aligned}$$

2.3.2 Fundamental Matrix Method

The fundamental matrix can be used to obtain the general solution of a linear system. The fundamental theorem gives the existence and uniqueness of solution of a linear system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, $\underset{\sim}{x} \in \mathbb{R}^n$ subject to the initial conditions $\underset{\sim}{x}_0 \in \mathbb{R}^n$. We now present the fundamental theorem.

Theorem 2.1 (Fundamental theorem) *Let A* be an $n \times n$ *matrix. Then for given any initial condition* $\underset{\sim}{x}_0 \in \mathbb{R}^n$, *the initial value problem* $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ *with* $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$ *has the unique solution* $\underset{\sim}{x}(t) = e^{At}\underset{\sim}{x}_0$.

Proof The initial value problem is

$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x}, \quad \underset{\sim}{x}(0) = \underset{\sim}{x}_0 \tag{2.12}$$

We have

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots \tag{2.13}$$

Differentiating (2.13) w.r.to t,

$$\begin{aligned} \frac{d}{dt}(e^{At}) &= \frac{d}{dt}\left(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots\right) \\ &= \frac{d}{dt}(I) + \frac{d}{dt}(At) + \frac{d}{dt}\left(\frac{A^2t^2}{2!}\right) + \frac{d}{dt}\left(\frac{A^3t^3}{3!}\right) + \cdots \end{aligned}$$

The term by term differentiation is valid because the series of e^{At} is convergent for all t under the operator.

$$\text{or, } \frac{d}{dt}\left(e^{At}\right) = \varphi + A + A^2 t + \frac{A^3 t^2}{2\,!} + \frac{A^4 t^3}{3!} + \cdots$$
$$= A\left(I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \cdots\right)$$
$$= Ae^{At}.$$

Therefore,

$$\frac{d}{dt}\left(e^{At}\right) = Ae^{At} \tag{2.14}$$

This shows that the matrix $\underset{\sim}{x} = e^{At}$ is a solution of the matrix differential equation $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$. The matrix e^{At} is known as the fundamental matrix of the system (2.12). Now using (2.14)

$$\frac{d}{dt}\left(e^{At}\underset{\sim}{x}_0\right) = \frac{d}{dt}\left(e^{At}\right)\underset{\sim}{x}_0 = Ae^{At}\underset{\sim}{x}_0$$
$$\Rightarrow \quad \dot{\underset{\sim}{x}} = \frac{d}{dt}(\underset{\sim}{x}) = A\underset{\sim}{x},$$

where $\underset{\sim}{x} = e^{At}\underset{\sim}{x}_0$.

Also, $\underset{\sim}{x}(0) = \left[e^{At}\underset{\sim}{x}_0\right]_{t=0} = [e^{At}]_{t=0}\underset{\sim}{x}_0 = I\underset{\sim}{x}_0 = \underset{\sim}{x}_0$. Thus $\underset{\sim}{x}(t) = e^{At}\underset{\sim}{x}_0$ is a solution of (2.12). We prove the uniqueness of solution as follows. Let $\underset{\sim}{x}(t)$ be a solution of (2.12) and $\underset{\sim}{y}(t) = e^{-At}\underset{\sim}{x}(t)$ be its another solution. Then

$$\dot{\underset{\sim}{y}}(t) = -Ae^{-At}\underset{\sim}{x}(t) + e^{-At}\dot{\underset{\sim}{x}}(t)$$
$$= -Ae^{-At}\underset{\sim}{x}(t) + Ae^{-At}\underset{\sim}{x}(t) = 0.$$

This implies $\underset{\sim}{y}(t)$ is constant. At $t = 0$, for $t \in \mathbb{R}$, it shows that $\underset{\sim}{y}(t) = \underset{\sim}{x}_0$. Therefore any solution of the IVP (2.12) is given as $\underset{\sim}{x}(t) = e^{At}\underset{\sim}{y}(t) = e^{At}\underset{\sim}{x}_0$. This completes the proof.

2.3.3 Matrix Exponential Function

From the fundamental theorem, the general solution of a linear system can be obtained using the exponential matrix function. The exponential matrix function has some interesting properties in which the general solution can be obtained easily. For an $n \times n$ matrix A, the **matrix exponential function** e^A of A is defined as

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{A^2}{2!} + \cdots \tag{2.15}$$

Note that the infinite series (2.15) converges for all $n \times n$ matrix A. If $A = [a]$, a 1×1 matrix, then $e^A = [e^a]$ (see the book by L. Perko [1]). We now discuss some of the important properties of matrix exponential function e^A.

Property 1 *If $A = \varphi$, the null matrix, then $e^{At} = I$.*

Proof By definition

$$\begin{aligned} e^{At} &= I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots \\ &= I + \varphi t + \frac{\varphi^2t^2}{2!} + \frac{\varphi^3t^3}{3!} + \cdots \\ &= I. \end{aligned}$$

So, $e^{At} = I$ for $A = \varphi$.

Property 2 *Let $A = I$, the identity matrix. Then*

$$e^{At} = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} = Ie^t.$$

Proof We know that $e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$. Therefore

$$\begin{aligned} e^{It} &= I + It + \frac{I^2t^2}{2\,!} + \frac{I^3t^3}{3\,!} + \cdots \\ &= I + It + \frac{It^2}{2\,!} + \frac{It^3}{3\,!} + \cdots \\ &= I\left(1 + t + \frac{t^2}{2\,!} + \frac{t^3}{3\,!} + \cdots\right) \\ &= Ie^t = e^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix}. \end{aligned}$$

Note If $A = \alpha I$, α being a scalar, then

$$e^{At} = e^{\alpha It} = Ie^{\alpha t} = \begin{bmatrix} e^{\alpha t} & 0 \\ 0 & e^{\alpha t} \end{bmatrix}.$$

Property 3 *Suppose* $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$*, a diagonal matrix. Then*

$$e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

Proof By definition

$$\begin{aligned} e^{Dt} &= I + Dt + \frac{D^2t^2}{2!} + \frac{D^3t^3}{3!} + \cdots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} t + \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^2 \frac{t^2}{2!} + \cdots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} t + \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix} \frac{t^2}{2!} + \cdots \\ &= \begin{bmatrix} 1 + \lambda_1 t + \frac{\lambda_1^2 t^2}{2!} + \cdots & 0 \\ 0 & 1 + \lambda_2 t + \frac{\lambda_2^2 t^2}{2!} + \cdots \end{bmatrix} \\ &= \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}. \end{aligned}$$

Property 4 *Let* $P^{-1}AP = D$*,* D *being a diagonal matrix. Then*

$$e^{At} = Pe^{Dt}P^{-1} = P\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}P^{-1}, \text{ where } D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Proof We have

$$\begin{aligned} e^{At} &= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{A^k t^k}{k!} \\ &= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{\left(PDP^{-1}\right)^k t^k}{k!} \left[\because D = P^{-1}AP, \text{ so } A = PDP^{-1}\right] \\ &= \lim_{n\to\infty} \sum_{k=0}^{n} \frac{\left(PD^kP^{-1}\right) t^k}{k!} \begin{bmatrix} \left(PDP^{-1}\right)^k = \left(PDP^{-1}\right)\left(PDP^{-1}\right)\cdots\left(PDP^{-1}\right) \\ = PD(P^{-1}P)D(P^{-1}P)\cdots(P^{-1}P)DP^{-1} \\ = PD^kP^{-1} \end{bmatrix} \\ &= P\left(\lim_{n\to\infty} \sum_{k=0}^{n} \frac{D^k t^k}{k!}\right)P^{-1} \\ &= Pe^{Dt}P^{-1} \\ &= P\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}P^{-1} \end{aligned}$$

Property 5 *Let N be a nilpotent matrix of order k. Then e^{Nt} is a series containing finite terms only.*

Proof A matrix N is said to be a nilpotent matrix of order or index k if k is the least positive integer such that $N^k = \varphi$ but $N^{k-1} \neq \varphi$, φ being the null matrix.

Since N is a nilpotent matrix of order k, $N^{k-1} \neq \varphi$ but $N^k = \varphi$.

Therefore

$$\begin{aligned} e^{Nt} &= I + Nt + \frac{N^2t^2}{2\,!} + \frac{N^3t^3}{3\,!} + \cdots + \frac{N^{k-1}t^{k-1}}{(k-1)\,!} + \frac{N^kt^k}{k\,!} + \cdots \\ &= I + Nt + \frac{N^2t^2}{2\,!} + \frac{N^3t^3}{3\,!} + \cdots + \frac{N^{k-1}t^{k-1}}{(k-1)\,!} \end{aligned}$$

which is a series of finite terms only.

Property 6 *If* $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, *then* $e^{At} = e^{aIt}[I\cos(bt) + J\sin(bt)]$, *where* $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *and* $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Proof We have

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = aI + bJ, \text{ where } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Therefore

$$\begin{aligned} e^{At} &= e^{aIt + bJt} \\ &= e^{aIt} \cdot e^{bJt} = e^{aIt}\left[I + bJt + \frac{(bJt)^2}{2!} + \frac{(bJt)^3}{3!} + \cdots\right] \\ &= e^{aIt}\left[I\left(1 - \frac{b^2t^2}{2!} + \frac{b^4t^4}{4!} + \cdots\right) + J\left(bt - \frac{b^3t^3}{3!} + \cdots\right)\right] \\ &= e^{aIt}[I\cos(bt) + J\sin(bt)] \left[\begin{array}{l} \because J^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I \\ J^3 = J^2J = (-I)J = -J \\ J^4 = J^3J = (-J)J = -J^2 = I \\ \text{etc.}\ldots \end{array}\right] \end{aligned}$$

Property 7 $e^{A+B} = e^A e^B$, *provided AB = BA.*

Proof Suppose $AB = BA$. Then by Binomial theorem,

$$(A+B)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} A^{n-k} B^k = n! \sum_{j+k=n} \frac{A^j B^k}{j!k!}.$$

Therefore

$$\begin{aligned} e^{A+B} &= \sum_{n=0}^{\infty} \frac{(A+B)^n}{n!} = \sum_{n=0}^{\infty} \sum_{j+k=n} \frac{A^j B^k}{j!k!} \\ &= \sum_{j=0}^{\infty} \frac{A^j}{j!} \sum_{k=0}^{\infty} \frac{B^k}{k!} \\ &= e^A e^B. \end{aligned}$$

It is true that $e^{A+B} = e^A e^B$ if $AB = BA$. But in general $e^{A+B} \neq e^A e^B$.

Property 8 *For any* $n \times n$ *matrix* A, $\frac{d}{dt}(e^{At}) = Ae^{At}$.

Proof By definition

$$\begin{aligned} e^{At} &= I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots \\ \therefore \frac{d}{dt}\left(e^{At}\right) &= \frac{d}{dt}\left(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots\right) \\ &= \frac{d}{dt}(I) + \frac{d}{dt}(At) + \frac{d}{dt}\left(\frac{A^2t^2}{2!}\right) + \frac{d}{dt}\left(\frac{A^3t^3}{3!}\right) + \cdots \end{aligned}$$

The term by term differentiation is valid because the series of e^{At} is convergent for all t under the operator.

$$\begin{aligned} \text{or, } \frac{d}{dt}\left(e^{At}\right) &= \varphi + A + A^2t + \frac{A^3t^2}{2!} + \frac{A^4t^3}{3!} + \cdots \\ &= A\left(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots\right) \\ &= Ae^{At}. \end{aligned}$$

Therefore, $\frac{d}{dt}(e^{At}) = Ae^{At}$.

We now establish the important result below.

Result Multiplying both sides of $\frac{d}{\mathrm{d}t}\left(e^{At}\right) = Ae^{At}$ by $\Phi(0)$ in right, we have

$$\begin{aligned}
&\frac{d}{\mathrm{d}t}\left(e^{At}\right)\Phi(0) = Ae^{At}\Phi(0)\\
&\Rightarrow \frac{d}{\mathrm{d}t}\left(e^{At}\Phi(0)\right) = Ae^{At}\Phi(0)\\
&\Rightarrow \frac{d}{\mathrm{d}t}\left(\Phi(t)\Phi^{-1}(0)\Phi(0)\right) = A\Phi(t)\Phi^{-1}(0)\Phi(0) \text{ [since } e^{At} = \Phi(t)\Phi^{-1}(0)]\\
&\Rightarrow \frac{d}{\mathrm{d}t}(\Phi(t)) = \dot{\Phi}(t) = A\Phi(t).
\end{aligned}$$

This shows that the fundamental matrix $\Phi(t)$ must satisfy the system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$. This is true for all t. So, it is true for $t = 0$. Putting $t = 0$ in $\dot{\Phi}(t) = A\Phi(t)$, we get

$$\dot{\Phi}(0) = A\Phi(0) \Rightarrow A = \dot{\Phi}(0)\Phi^{-1}(0).$$

This gives that the coefficient matrix A can be expressed in terms of the fundamental matrix $\Phi(t)$.

Example 2.12 Does $\Phi(t) = \begin{pmatrix} e^t & e^{-2t} \\ 2e^t & 3e^{-2t} \end{pmatrix}$ a fundamental matrix for the system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$? If so, then find the matrix A.

Solution We know that if $\Phi(t)$ is a fundamental matrix, then $\Phi(0)$ is invertible.

Here $\Phi(t) = \begin{pmatrix} e^t & e^{-2t} \\ 2e^t & 3e^{-2t} \end{pmatrix}$. So, $\Phi(0) = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$.

Since $\det(\Phi(0)) = 3 - 2 = 1 \neq 0$, $\Phi(0)$ is invertible. Hence the given matrix is a fundamental matrix for the system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$. We shall now find the coefficient matrix A.

We have $\Phi(0) = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$. So $\Phi^{-1}(0) = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$.

Also $\dot{\Phi}(t) = \begin{pmatrix} e^t & -2e^{-2t} \\ 2e^t & -6e^{-2t} \end{pmatrix}$, and $\dot{\Phi}(0) = \begin{pmatrix} 1 & -2 \\ 2 & -6 \end{pmatrix}$.

Therefore the matrix A is

$$A = \dot{\Phi}(0)\Phi^{-1}(0) = \begin{pmatrix} 1 & -2 \\ 2 & -6 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 7 & -3 \\ 18 & -8 \end{pmatrix}.$$

Example 2.13 Find e^{At} for the matrix $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$. Hence find the solution of the system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$.

Solution We see that the eigenvectors corresponding to the eigenvalues $\lambda = 2, 4$ of A are respectively $\underset{\sim}{e} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\underset{\sim}{g} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, which are linearly independent. Therefore, two fundamental solutions of the system are $\underset{\sim}{x}_1(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$ and $\underset{\sim}{x}_2(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$. So a fundamental matrix of the system is

$$\Phi(t) = \begin{pmatrix} \underset{\sim}{x}_1(t) & \underset{\sim}{x}_2(t) \end{pmatrix} = \begin{pmatrix} e^{2t} & e^{4t} \\ -e^{2t} & e^{4t} \end{pmatrix}.$$

We find $\Phi(0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and $\Phi^{-1}(0) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Therefore

$$e^{At} = \Phi(t)\Phi^{-1}(0) = \frac{1}{2}\begin{pmatrix} e^{2t} & e^{4t} \\ -e^{2t} & e^{4t} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{2t} + e^{4t} & e^{4t} - e^{2t} \\ e^{4t} - e^{2t} & e^{2t} + e^{4t} \end{pmatrix}.$$

By fundamental theorem, the solution of the system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ is

$$\underset{\sim}{x}(t) = e^{At}\underset{\sim}{x}_0 = \frac{1}{2}\begin{pmatrix} e^{2t} + e^{4t} & e^{4t} - e^{2t} \\ e^{4t} - e^{2t} & e^{2t} + e^{4t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

where $\underset{\sim}{x}_0 = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ is an arbitrary constant column vector.

2.4 Solution Procedure of Linear Systems

The general solution of a linear homogeneous system can be easily deduced from the fundamental theorem. According to this theorem the solution of $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ with $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$ is given as $\underset{\sim}{x}(t) = e^{At}\underset{\sim}{x}_0$ and this solution is unique.

For a simple change of coordinates $\underset{\sim}{x} = P\underset{\sim}{y}$ where P is an invertible matrix, the equation $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ is transformed as

$$\begin{aligned} \dot{\underset{\sim}{x}} &= A\underset{\sim}{x} \\ \Rightarrow P\dot{\underset{\sim}{y}} &= AP\underset{\sim}{y} \\ \Rightarrow \dot{\underset{\sim}{y}} &= P^{-1}AP\,\underset{\sim}{y} \\ \Rightarrow \dot{\underset{\sim}{y}} &= C\underset{\sim}{y}\,, \text{ where } C = P^{-1}AP. \end{aligned}$$

The initial conditions $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$ become $\underset{\sim}{y}(0) = P^{-1}\underset{\sim}{x}(0) = P^{-1}\underset{\sim}{x}_0 = \underset{\sim}{y}_0$. So, the new system is $\dot{\underset{\sim}{y}} = C\underset{\sim}{y}$ with $\underset{\sim}{y}(0) = \underset{\sim}{y}_0$, where $C = P^{-1}AP$.

It has the solution

$$\underset{\sim}{y}(t) = e^{Ct}\underset{\sim}{y}_0.$$

Hence the solution of the original system is

$$\underset{\sim}{x}(t) = P\underset{\sim}{y}(t) = Pe^{Ct}\underset{\sim}{y}_0 = Pe^{Ct}P^{-1}\underset{\sim}{x}_0.$$

We see that $e^{At} = Pe^{Ct}P^{-1}$. The matrix P is chosen in such a way that matrix C takes a simple form. We now discuss three cases.

(i) **Matrix *A* has distinct real eigenvalues**

Let $P = \left(\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_n\right)$ so that, P^{-1} exists. The matrix C is obtained as $C = P^{-1}AP$ which is a diagonal matrix. Hence the exponential function of C becomes

$$e^{Ct} = \mathrm{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t}).$$

Therefore we can write the solution of $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ with $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$ as $\underset{\sim}{x}(t) = e^{At}\underset{\sim}{x}_0 = Pe^{Ct}P^{-1}\underset{\sim}{x}_0$. So

$$\underset{\sim}{x}(t) = P\,\mathrm{diag}(e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t})P^{-1}\underset{\sim}{x}_0$$

where $\underset{\sim}{x}_0 = (c_1, c_2, \ldots, c_n)^t$ is an arbitrary constant.

(ii) **Matrix *A* has real repeated eigenvalues**

In this case the following theorems are relevant (proofs are available in the book Hirsch and Smale [2]) for finding general solution of a linear system when matrix A has repeated eigenvalues.

Theorem 2.2 *Let the* $n \times n$ matrix A *have real eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *repeated according to their multiplicity. Then there exists a basis of generalized eigenvectors* $\{\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_n\}$ *such that the matrix* $P = (\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\alpha}_n)$ *is invertible and* $A = S + N$, *where* $P^{-1}SP = \mathrm{diag}(\lambda_1, \lambda_1, \ldots, \lambda_n)$ *and* $N(=A - S)$ *is nilpotent of order* $k \le n$, *and* S *and* N *commute.*

Using the theorem the linear system subject to the initial conditions $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$ has the solution

$$\underset{\sim}{x}(t) = P\text{diag}(e^{\lambda_j t})P^{-1}\left[I + Nt + \cdots + \frac{N^{k-1}t^{k-1}}{(k-1)!}\right]\underset{\sim}{x}_0.$$

(iii) Matrix *A* has complex eigenvalues

Theorem 2.3 *Let A be a* $2n \times 2n$ *matrix with complex eigenvalues* $a_j \pm ib_j, j = 1, 2, \ldots, n$*. Then there exists generalized complex eigenvectors* $(\underset{\sim}{\alpha}_j \pm i\underset{\sim}{\beta}_j), j = 1, 2 \ldots, n$ *such that the matrix* $P = (\underset{\sim}{\beta}_1, \underset{\sim}{\alpha}_1, \underset{\sim}{\beta}_2, \underset{\sim}{\alpha}_2, \ldots, \underset{\sim}{\beta}_n, \underset{\sim}{\alpha}_n)$ *is invertible and* $A = S + N$, *where* $P^{-1}SP = \text{diag}\begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}$, *and* $N(=A - S)$ *is a nilpotent matrix of order* $k \leq 2n$, *and S and N commute.*

Using the theorem the linear system of equations subject to the initial conditions $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$ has the solution

$$\underset{\sim}{x}(t) = P\text{diag}(e^{a_j t})\begin{bmatrix} \cos(b_j t) & -\sin(b_j t) \\ \sin(b_j t) & \cos(b_j t) \end{bmatrix}P^{-1}\left[I + Nt + \cdots + \frac{N^k t^k}{k!}\right]\underset{\sim}{x}_0.$$

For a 2×2 matrix A with complex eigenvalues $(\alpha \pm i\beta)$ the solution is given by

$$\underset{\sim}{x}(t) = Pe^{\alpha t}\begin{pmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{pmatrix}P^{-1}\underset{\sim}{x}_0.$$

Example 2.14 Solve the initial value problem

$$\dot{x} = x + y, \dot{y} = 4x - 2y$$

with initial condition $\underset{\sim}{x}(0) = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

Solution The characteristic equation of matrix A is

$$\begin{aligned} & |A - \lambda I| = 0 \\ \Rightarrow & \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = 0 \\ \Rightarrow & (\lambda - 1)(\lambda + 2) - 4 = 0 \\ \Rightarrow & \lambda^2 + \lambda - 6 = 0 \\ \Rightarrow & \lambda = 2, -3 \end{aligned}$$

So the eigenvalues of matrix A are 2, −3, which are real and distinct.

Let $\underset{\sim}{e} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ be the eigenvector corresponding to the eigenvalue $\lambda_1 = 2$. Then

$$\begin{aligned} &(A - 2I)\underset{\sim}{e} = 0 \\ \Rightarrow \quad &\begin{pmatrix} 1-2 & 1 \\ 4 & -2-2 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \Rightarrow \quad &-e_1 + e_2 = 0, 4e_1 - 4e_2 = 0 \end{aligned}$$

A nontrivial solution of this system is $e_1 = 1, e_2 = 1$.

$$\therefore \underset{\sim}{e} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Again let $\underset{\sim}{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ be the eigenvector corresponding to the eigenvalue $\lambda_2 = -3$. Then

$$\begin{aligned} &(A + 3I)\underset{\sim}{g} = 0 \\ \Rightarrow \quad &\begin{pmatrix} 1+3 & 1 \\ 4 & -2+3 \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \Rightarrow \quad &4g_1 + g_2 = 0, 4g_1 + g_2 = 0 \end{aligned}$$

A nontrivial solution of this system is $g_1 = 1, g_2 = -4$.

$$\therefore \underset{\sim}{g} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

Let $P = \begin{pmatrix} \underset{\sim}{e}, \underset{\sim}{g} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}$. Then $P^{-1} = -\frac{1}{5}\begin{pmatrix} -4 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}$

$$\begin{aligned} \therefore C = P^{-1}AP &= \frac{1}{5}\begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 8 & 2 \\ -3 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 10 & 0 \\ 0 & -15 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \end{aligned}$$

$$\therefore e^{Ct} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{pmatrix}$$

Therefore by the fundamental theorem, the solution of the system is

$$\underset{\sim}{x}(t) = e^{At}\underset{\sim}{x}_0 = Pe^{Ct}P^{-1}\underset{\sim}{x}_0$$
$$= \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{pmatrix}\frac{1}{5}\begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}\underset{\sim}{x}_0$$
$$= \frac{1}{5}\begin{pmatrix} 4e^{2t} + e^{-3t} & e^{2t} - e^{-3t} \\ 4e^{2t} - 4e^{-3t} & e^{2t} + 4e^{-3t} \end{pmatrix}\underset{\sim}{x}_0$$
$$\Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \frac{4}{5}e^{2t} + \frac{1}{5}e^{-3t} & \frac{1}{5}e^{2t} - \frac{1}{5}e^{-3t} \\ \frac{4}{5}e^{2t} - \frac{4}{5}e^{-3t} & \frac{1}{5}e^{2t} + \frac{4}{5}e^{-3t} \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} e^{2t} + e^{-3t} \\ e^{2t} - 4e^{-3t} \end{pmatrix}$$
$$\Rightarrow x(t) = e^{2t} + e^{-3t}, y(t) = e^{2t} - 4e^{-3t}.$$

Example 2.15 Solve the system

$$\dot{x}_1 = -x_1 - 3x_2, \dot{x}_2 = 2x_2.$$

Also sketch the phase portrait.

Solution The characteristic equation of matrix A is

$$|A - \lambda I| = 0$$
$$\Rightarrow \begin{vmatrix} -1-\lambda & -3 \\ 0 & 2-\lambda \end{vmatrix} = 0$$
$$\Rightarrow (\lambda - +)(\lambda - 2) = 0$$
$$\Rightarrow \lambda = -1, 2$$

The eigenvalues of matrix A are -1, 2, which are real and distinct.

Let $\underset{\sim}{e} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ be the eigenvector corresponding to the eigenvalue $\lambda_1 = -1$. Then

$$(A + I)\underset{\sim}{e} = 0$$
$$\Rightarrow \begin{pmatrix} -1+1 & -3 \\ 0 & 2+1 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\Rightarrow -3e_2 = 0, 3e_2 = 0$$
$$\Rightarrow e_2 = 0 \text{ and } e_1 \text{ is arbitrary.}$$

Choose $e_1 = 1$ so that $\underset{\sim}{e} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Again, let $\underset{\sim}{g} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ be the eigenvector corresponding to the eigenvalue $\lambda_2 = 2$. Then

$$
\begin{aligned}
&(A-2I)\underset{\sim}{g} = \underset{\sim}{0} \\
\Rightarrow \quad & \begin{pmatrix} -1-2 & -3 \\ 0 & 2-2 \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\Rightarrow \quad & g_1 + g_2 = 0
\end{aligned}
$$

Choose $g_1 = 1, g_2 = -1$. Then $\underset{\sim}{g} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Let $P = \left(\underset{\sim}{e}, \underset{\sim}{g}\right) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$. Then $P^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

Therefore

$$
\begin{aligned}
C = P^{-1}AP &= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & -3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} -1 & -1 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}
\end{aligned}
$$

and so $e^{Ct} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix}$.

Therefore by fundamental theorem, the solution of the system is

$$
\begin{aligned}
\underset{\sim}{x}(t) &= e^{At}\underset{\sim}{x}_0 = Pe^{Ct}P^{-1}\underset{\sim}{x}_0 \\
&= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\underset{\sim}{x}_0 \\
&= \begin{pmatrix} e^{-t} & e^{-t} - e^{2t} \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\
\Rightarrow \quad & \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} e^{-t} & e^{-t} - e^{2t} \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} (c_1 + c_2)e^{-t} - c_2e^{2t} \\ c_2e^{2t} \end{pmatrix} \\
\Rightarrow \quad & x_1(t) = c_1e^{-t} + c_2\left(e^{-t} - e^{2t}\right), x_2(t) = c_2e^{2t}
\end{aligned}
$$

where c_1, c_2 are arbitrary constants. The phase diagram is presented in Fig. 2.1.

Example 2.16 Solve the following system using the fundamental theorem.

$$
\begin{aligned}
\dot{x} &= 5x + 4y \\
\dot{y} &= -x + y
\end{aligned}
$$

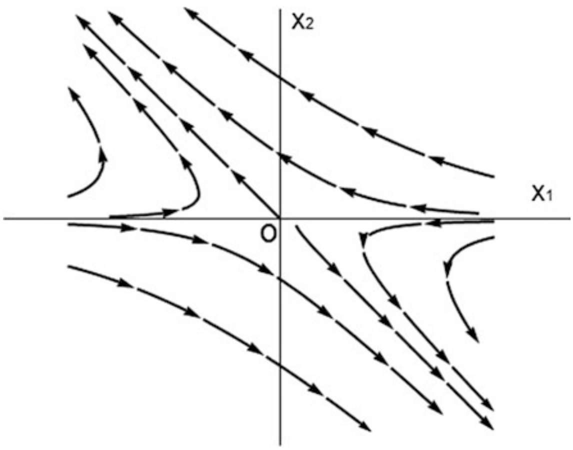

Fig. 2.1 A typical phase portrait of the system

Solution The characteristic equation of matrix A is

$$
\begin{aligned}
& |A - \lambda I| = 0 \\
\Rightarrow \quad & \begin{vmatrix} 5-\lambda & 4 \\ -1 & 1-\lambda \end{vmatrix} = 0 \\
\Rightarrow \quad & (\lambda - 1)(\lambda - 5) + 4 = 0 \\
\Rightarrow \quad & \lambda^2 - 6\lambda + 9 = 0 \\
\Rightarrow \quad & \lambda = 3, 3.
\end{aligned}
$$

This shows that matrix A has an eigenvalue $\lambda = 3$ of multiplicity 2. Then $S = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and $N = A - S = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$. Clearly, matrix N is a nilpotent matrix of order 2. So, the general solution of the system is given by

$$
\begin{aligned}
\underset{\sim}{x}(t) = e^{At}\underset{\sim}{x}_0 = e^{(S+N)t}\underset{\sim}{x}_0 = e^{St}e^{Nt}\underset{\sim}{x}_0 &= \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{bmatrix}[I + Nt]\underset{\sim}{x}_0 \\
&= \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{bmatrix}\begin{bmatrix} 1+2t & 4t \\ -t & 1-2t \end{bmatrix}\underset{\sim}{x}_0.
\end{aligned}
$$

Example 2.17 Find the general solution of the system of linear equations

$$
\begin{aligned}
\dot{x} &= 4x - 2y \\
\dot{y} &= 5x + 2y
\end{aligned}
$$

Solution The characteristic equation of matrix A is

$$\begin{aligned}&|A-\lambda I|=0\\ \Rightarrow\quad &\begin{vmatrix}4-\lambda & -2\\ 5 & 2-\lambda\end{vmatrix}=0\\ \Rightarrow\quad &(\lambda-4)(\lambda-2)+10=0\\ \Rightarrow\quad &\lambda^2-6\lambda+18=0\\ \Rightarrow\quad &\lambda=\frac{6\pm\sqrt{36-72}}{2}=3\pm 3i.\end{aligned}$$

So matrix A has a pair of complex conjugate eigenvalues $3 \pm 3i$

Let $\underset{\sim}{e}=\begin{pmatrix}e_1\\ e_2\end{pmatrix}$ be the eigenvector corresponding to the eigenvalue $\lambda_1=3+3i$. Then

$$\begin{aligned}&(A-(3+3i)I)\underset{\sim}{e}=\underset{\sim}{0}\\ \Rightarrow\quad &\begin{pmatrix}4-(3+3i) & -2\\ 5 & 2-(3+3i)\end{pmatrix}\begin{pmatrix}e_1\\ e_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\\ \Rightarrow\quad &\begin{pmatrix}1-3i & -2\\ 5 & -1-3i\end{pmatrix}\begin{pmatrix}e_1\\ e_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\\ \Rightarrow\quad &(1-3i)e_1-2e_2=0, 5e_1+(1+3i)e_2=0\end{aligned}$$

A nontrivial solution of this system is $e_1=2,\ e_2=1-3i$.

$$\therefore \underset{\sim}{e}=\begin{pmatrix}2\\ 1-3i\end{pmatrix}.$$

Similarly, the eigenvector corresponding to the eigenvalue $\lambda_2=3-3i$ is $\underset{\sim}{g}=\begin{pmatrix}2\\ 1+3i\end{pmatrix}$.

Let $P=\begin{pmatrix}0 & 2\\ -3 & 1\end{pmatrix}$. Then $P^{-1}=\frac{1}{6}\begin{pmatrix}1 & -2\\ 3 & 0\end{pmatrix}$.

Let $C=P^{-1}AP$. Then $C=P^{-1}AP=\frac{1}{6}\begin{pmatrix}1 & -2\\ 3 & 0\end{pmatrix}\begin{pmatrix}4 & -2\\ 5 & 2\end{pmatrix}\begin{pmatrix}0 & 2\\ -3 & 1\end{pmatrix}=\begin{pmatrix}3 & -3\\ 3 & 3\end{pmatrix}$.

So,

$$e^{Ct}=e^{3t}\begin{pmatrix}\cos 3t & -\sin 3t\\ \sin 3t & \cos 3t\end{pmatrix}.$$

Therefore, the solution of the system is

$$\underset{\sim}{x}(t) = e^{At}\underset{\sim}{x}_0 = Pe^{Ct}P^{-1}\underset{\sim}{x}_0$$
$$= \frac{1}{6}e^{3t}\begin{pmatrix} 0 & 2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} \cos 3t & -\sin 3t \\ \sin 3t & \cos 3t \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix}\underset{\sim}{x}_0.$$

Example 2.18 Solve the initial value problem $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, with $\underset{\sim}{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, where $A = \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}$, $\underset{\sim}{x} = \begin{pmatrix} x \\ y \end{pmatrix}$. Also sketch the solution curve in the phase plane $\mathbb{R}^2$.

Solution The characteristic equation of matrix A is

$$|A - \lambda I| = 0$$
$$\Rightarrow \begin{vmatrix} -2-\lambda & -1 \\ 1 & -2-\lambda \end{vmatrix} = 0$$
$$\Rightarrow (\lambda+2)^2 + 1 = 0$$
$$\Rightarrow \lambda^2 + 4\lambda + 5 = 0$$
$$\Rightarrow \lambda = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i.$$

So matrix A has a pair of complex conjugate eigenvalues $-2 \pm i$

Let $\underset{\sim}{e} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ be the eigenvector corresponding to the eigenvalue $\lambda_1 = -2 + i$. Then

$$(A - (-2+i)I)\underset{\sim}{e} = 0$$
$$\Rightarrow \begin{pmatrix} -2-(-2+i) & -1 \\ 1 & -2-(-2+i) \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\Rightarrow -ie_1 - e_2 = 0, e_1 - ie_2 = 0$$

A nontrivial solution of this system is $e_1 = 1, e_2 = -i$.

$\therefore \underset{\sim}{e} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$. Similarly, the eigenvector corresponding to the eigenvalue $\lambda_2 = -2 - i$ is $\underset{\sim}{g} = \begin{pmatrix} 1 \\ i \end{pmatrix}$. Let $P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then $P^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and

$$C = P^{-1}AP = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}.$$

So,

$$e^{Ct} = e^{-2t}\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Hence the solution of the system is

$$\begin{aligned}
\underset{\sim}{x}(t) &= e^{At}\underset{\sim}{x}_0 = Pe^{Ct}P^{-1}\underset{\sim}{x}_0 \\
&= e^{-2t}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\underset{\sim}{x}_0. \\
&= e^{-2t}\begin{pmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\underset{\sim}{x}_0 \\
&= e^{-2t}\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= e^{-2t}\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \\
\therefore x(t) &= e^{-2t}\cos t, y(t) = e^{-2t}\sin t.
\end{aligned}$$

Phase Portrait The phase portrait of the solution curve is shown in Fig. 2.2.

Example 2.19 Solve the system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ with $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$, where $A = \begin{pmatrix} 2 & 1 & 3 & -1 \\ 0 & 2 & 2 & -1 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

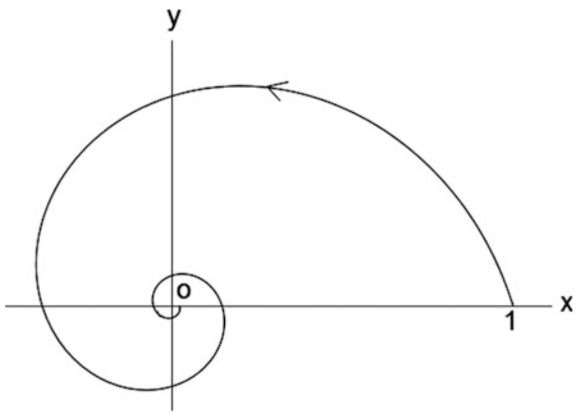

Fig. 2.2 Phase portrait of the solution curve

Solution Clearly, matrix A has the eigenvalue $\lambda = 2$ with multiplicity 4. Therefore,

$$S = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \text{ and } N = A - S = \begin{pmatrix} 0 & 1 & 3 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check that the matrix N is nilpotent of order 4. Therefore, the solution of the system is

$$\underset{\sim}{x}(t) = e^{St}\left(I + Nt + \frac{N^2t^2}{2!} + \frac{N^3t^3}{3!}\right)\underset{\sim}{x}_0.$$

2.5 Nonhomogeneous Linear Systems

The most general form of a nonhomogeneous linear system is given as

$$\dot{\underset{\sim}{x}}(t) = A(t)\underset{\sim}{x}(t) + \underset{\sim}{b}(t) \tag{2.16}$$

where $A(t)$ is an $n \times n$ matrix, usually depends on time and $\underset{\sim}{b}(t)$ is a time dependent column vector. Here we consider matrix $A(t)$ to be time independent, that is, $A(t) \equiv A$. Then (2.16) becomes

$$\dot{\underset{\sim}{x}}(t) = A\underset{\sim}{x}(t) + \underset{\sim}{b}(t) \tag{2.17}$$

The corresponding homogeneous system is given as

$$\dot{\underset{\sim}{x}}(t) = A\underset{\sim}{x}(t) \tag{2.18}$$

We have described solution techniques for homogeneous system (2.18). We now find the solution of the nonhomogeneous system (2.17), subject to initial conditions $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$.

As discussed earlier if $\Phi(t)$ be the fundamental matrix of (2.18) with $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$, then the solution of (2.18) is given by

$$\underset{\sim}{x}(t) = \Phi(t)\Phi^{-1}(0)\underset{\sim}{x}_0$$

We assume that

$$\underset{\sim}{x}(t) = \Phi(t)\Phi^{-1}(0)\underset{\sim}{x}_0 + \Phi(t)\Phi^{-1}(0)\underset{\sim}{u}(t) \tag{2.19}$$

be the solution of the nonhomogeneous linear system (2.17). Then the initial conditions are obtained as $\underset{\sim}{u}(0) = 0$. Differentiating (2.19) with respect to t, we get

$$\dot{\underset{\sim}{x}}(t) = \dot{\Phi}(t)\Phi^{-1}(0)\underset{\sim}{x}_0 + \dot{\Phi}(t)\Phi^{-1}(0)\underset{\sim}{u}(t) + \Phi(t)\Phi^{-1}(0)\dot{\underset{\sim}{u}}(t) \tag{2.20}$$

Substituting (2.20) and (2.19) into (2.17),

$$\begin{aligned}\dot{\Phi}(t)\Phi^{-1}(0)\underset{\sim}{x}_0 + \dot{\Phi}(t)\Phi^{-1}(0)\underset{\sim}{u}(t) + \Phi(t)\Phi^{-1}(0)\dot{\underset{\sim}{u}}(t)\\ = A\Phi(t)\Phi^{-1}(0)\underset{\sim}{x}_0 + A\Phi(t)\Phi^{-1}(0)\underset{\sim}{u}(t) + \underset{\sim}{b}(t)\end{aligned} \tag{2.21}$$

Since $\Phi(t)$ is a fundamental matrix solution of (2.18),

$$\dot{\Phi}(t) = A\Phi(t).$$

Using this in (2.21), we get

$$\begin{aligned}\Phi(t)\Phi^{-1}(0)\dot{\underset{\sim}{u}}(t) &= \underset{\sim}{b}(t)\\ \Rightarrow \dot{\underset{\sim}{u}}(t) &= \Phi(0)\dot{\Phi}^{-1}(t)\underset{\sim}{b}(t)\end{aligned}$$

Integrating w.r.to t and using $\underset{\sim}{u}(0) = 0$, we get

$$\underset{\sim}{u}(t) = \int_0^t \Phi(0)\Phi^{-1}(t)\underset{\sim}{b}(t)\mathrm{d}t.$$

Hence the general solution of the nonhomogeneous system (2.17) subject to $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$ is given by

$$\underset{\sim}{x}(t) = \Phi(t)\Phi^{-1}(0)\underset{\sim}{x}_0 + \Phi(t)\int_0^t \Phi^{-1}(\alpha)\underset{\sim}{b}(\alpha)d\alpha \tag{2.22}$$

Example 2.20 Find the solution of the nonhomogeneous system $\dot{x} = x + y + t,\ \dot{y} = -y + 1$ with the initial conditions $x(0) = 1$, $y(0) = 0$.

Solution In matrix notation, the system takes the form $\dot{\underset{\sim}{x}}(t) = A\underset{\sim}{x}(t) + \underset{\sim}{b}(t)$, where $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ and $\underset{\sim}{b}(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}$.

The initial conditions become $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$, where $\underset{\sim}{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Matrix A has eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1$ with corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Therefore

$$\Phi(t) = \begin{pmatrix} e^t & e^{-t} \\ 0 & -2e^{-t} \end{pmatrix}.$$

This gives

$$\Phi^{-1}(t) = \frac{1}{2}\begin{pmatrix} 2e^{-t} & e^{-t} \\ 0 & -e^t \end{pmatrix},\ \Phi(0) = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \text{and } \Phi^{-1}(0) = \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}.$$

Therefore the required solution is

$$\begin{aligned}
\underset{\sim}{x}(t) &= \Phi(t)\Phi^{-1}(0)\underset{\sim}{x}_0 + \Phi(t)\int_0^t \Phi^{-1}(\alpha)\underset{\sim}{b}(\alpha)\mathrm{d}\alpha \\
&= \frac{1}{2}\Phi(t)\left\{\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} 2e^{-\alpha} & e^{-\alpha} \\ 0 & -e^{\alpha} \end{pmatrix}\begin{pmatrix} \alpha \\ 1 \end{pmatrix}\mathrm{d}\alpha\right\} \\
&= \frac{1}{2}\Phi(t)\left\{\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 3-(2t+3)e^{-t} \\ 1-e^t \end{pmatrix}\right\} \\
&= \frac{1}{2}\begin{pmatrix} e^t & e^{-t} \\ 0 & -2e^{-t} \end{pmatrix}\begin{pmatrix} 5-(2t+3)e^{-t} \\ 1-e^t \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 5e^t - 2t - 4 + e^{-t} \\ 2 - 2e^{-2t} \end{pmatrix}.
\end{aligned}$$

Example 2.21 Prove that the flow evolution operator $\phi_t(\underset{\sim}{x}) = e^{At}\underset{\sim}{x}$ satisfies the following properties:

(i) $\phi_0(\underset{\sim}{x}) = \underset{\sim}{x}$,

(ii) $\phi_{-t} \circ \phi_t(\underset{\sim}{x}) = \underset{\sim}{x}$,

(iii) $\phi_t \circ \phi_s(\underset{\sim}{x}) = \phi_{t+s}(\underset{\sim}{x})$

for all s, $t \in \mathbb{R}$ and $\underset{\sim}{x} \in \mathbb{R}^n$. Is $\phi_t \circ \phi_s = \phi_s \circ \phi_t$?

Solution We have

(i) $\phi_0(\underset{\sim}{x}) = e^{A\cdot 0}\underset{\sim}{x} = \underset{\sim}{x}$.

(ii) $\phi_{-t} \circ \phi_t(\underset{\sim}{x}) = \phi_{-t}(\underset{\sim}{y}) = e^{-At}\underset{\sim}{y} = e^{-At}e^{At}\underset{\sim}{x} = \underset{\sim}{x}$, where $\underset{\sim}{y} = e^{At}\underset{\sim}{x}$.

(iii) $\phi_t \circ \phi_s(\underset{\sim}{x}) = \phi_t(\underset{\sim}{y}) = e^{At}\underset{\sim}{y} = e^{At}e^{As}\underset{\sim}{x} = e^{A(t+s)}\underset{\sim}{x} = \phi_{t+s}(\underset{\sim}{x})$.

Now,

$$\phi_t \circ \phi_s(\underset{\sim}{x}) = \phi_t(\underset{\sim}{y}) = e^{At}\underset{\sim}{y} = e^{At}e^{As}\underset{\sim}{x} = e^{As}e^{At}\underset{\sim}{x} = \phi_s(\underset{\sim}{z}) = \phi_s \circ \phi_t(\underset{\sim}{x})$$

for all $\underset{\sim}{x} \in \mathbb{R}^n$, where $\underset{\sim}{z} = e^{As}\underset{\sim}{x}$.

Hence $\phi_t \circ \phi_s = \phi_s \circ \phi_t$. This indicates that the given flow evolution operator is commutative.

2.6 Exercises

1. Prove that for a square matrix A of order n, the set of solutions of the linear homogeneous system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ in $\mathbb{R}^n$ forms an n-dimensional vector space.

2. Find the eigenvalues and the corresponding eigenvectors of the following matrices:

(i) $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$ (iii) $\begin{pmatrix} 2 & 7 \\ 5 & -10 \end{pmatrix}$ (iv) $\begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$ (v) $\begin{bmatrix} 1 & 3 \\ \sqrt{2} & 3\sqrt{2} \end{bmatrix}$

(vi) $\begin{pmatrix} 1 & -2 & 5 \\ 0 & 6 & -1 \\ 3 & -2 & 1 \end{pmatrix}$

3. (a) Consider the matrix $A = \begin{pmatrix} p & 0 \\ 1 & 1 \end{pmatrix}$. Find the value(s) of p for which the matrix A has repeated eigenvalues.

(b) Find the 2×2 matrix A whose eigenvalues are 1, 4 and the corresponding eigenvectors are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

(c) Find all 2×2 matrices A whose eigenvalues are 0 and 1.

4. Consider the linear homogeneous system

$$\dot{x} = -4x + y, \dot{y} = -2x - y.$$

(a) Write the system as $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$.

(b) Show that the characteristic polynomial is $\lambda^2 + 5\lambda + 6$

(c) Find the eigenvalues and the corresponding eigenvectors of the matrix A.

(d) Find the general solution of the system.

(e) Solve the system subject to the initial condition $\underset{\sim}{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

5. Find the general solution to each of the following system of homogeneous linear equations:

(i) $\dot{x} = x + 3y, \dot{y} = x - y$

(ii) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, i = \sqrt{-1}$

(iii) $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ where $A = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}$

(iv) $\dot{\underset{\sim}{x}}(t) = A\underset{\sim}{x}(t)$ where $A = \begin{pmatrix} 5 & 4 \\ -1 & 0 \end{pmatrix}$

(v) $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, where $A = \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}$

(vi) $\dot{x} = -5x, \dot{y} = -5y$

(vii) $\dot{\underset{\sim}{x}} = \begin{pmatrix} a & b \\ c & a \end{pmatrix} \underset{\sim}{x}$, where $bc > 0$.

(viii) $\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

(ix) $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ where $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

(x) $\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$

(xi) $\dot{x} = y, \dot{y} = z, \dot{z} = x + y - z$

(xii) $\dot{x} = x + 2y - z, \dot{y} = y + z, \dot{z} = -y + z$

(xiii) $\dot{x} = x,\ \dot{y} = 2y - 3z,\ \dot{z} = x + 3y + 2z$

(xiv) $\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$

(xv) $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, where $A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

6. Solve the following initial value problems:

(i) $\dot{x} = 9x + 5y, \dot{y} = -6x - 2y\,; x(0) = 1, y(0) = 0$.

(ii) $\dot{\underset{\sim}{x}} = \begin{pmatrix} 3 & -1 \\ 1 & 5 \end{pmatrix} \underset{\sim}{x},\ \underset{\sim}{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(iii) $\dot{\underset{\sim}{x}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \underset{\sim}{x},\ \underset{\sim}{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(iv) $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}, \underset{\sim}{x}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, where $A = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}$.

(v) $\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},\ \underset{\sim}{x}(\pi/2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(vi) $\dot{\underset{\sim}{x}}(t) = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix} \underset{\sim}{x}(t),\ \underset{\sim}{x}(0) = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

7. Find the solution of the IVP $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ subject to the initial condition $\underset{\sim}{x}(0) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, where $A = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}$ and $\underset{\sim}{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. Also draw the diagram for the solution set.

8. Convert the second order differential equation $\ddot{x} + a\dot{x} + bx = 0$ to a system of two first order differential equations. Find all values of a and b for which the system has real, distinct eigenvalues. Also find the general solution of the system. Find the solution of the system that satisfies the initial condition $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Draw the diagram for the solution set.

9. Find the general solution of the system $\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, where $a + d \neq 0$ and $ad - bc = 0$. Also sketch the diagram.

10. Consider the system $\dot{\underset{\sim}{x}} = \begin{pmatrix} 0 & 1 \\ -k & -b \end{pmatrix} \underset{\sim}{x}$, where $b \geq 0, k > 0$.

a) For what values of k and b does the system has
   (i) Complex conjugate eigenvalues?
   (ii) Repeated real eigenvalues?
   (iii) Real and distinct eigenvalues?

b) Find the general solution of the system in each case.

11. Solve the following second order differential equation after reducing them into a system of two first order differential equations:

(i) $\ddot{x}+x=0$ with $x(0)=1, \dot{x}(0)=0$

(ii) $\ddot{x}+3\dot{x}+5x=0$ with $x(0)=1, \dot{x}(0)=-1$.

12. Find the general solution of the system below and determine the possible values of α, β so that the initial value problem has a solution that tends to zero as $t \to \infty$

$$\dot{\underset{\sim}{x}}=\begin{pmatrix} 5 & -1 \\ 7 & 3 \end{pmatrix}\underset{\sim}{x},\ \underset{\sim}{x}(0)=\begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

13. (a) What do you mean by a fundamental matrix of a homogeneous system of linear equations?

(b) Show that two different homogeneous systems cannot have the same fundamental matrix.

(c) Let $\Phi(t)$ be a fundamental matrix of the system $\dot{\underset{\sim}{x}}=A\underset{\sim}{x}$. Show that for any non-zero constant k, $k\Phi(t)$ is also a fundamental matrix of the system.

14. Find the fundamental matrix of the following systems and hence find the solution of each system:

(i) $\dot{\underset{\sim}{x}}=\begin{pmatrix} 1 & -2 \\ -3 & 2 \end{pmatrix}\underset{\sim}{x}$

(ii) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}=\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(iii) $\dot{\underset{\sim}{x}}=\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\underset{\sim}{x}$

(iv) $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 9 & 5 \\ -6 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

(v) $\dot{\underset{\sim}{x}}=A\underset{\sim}{x}$, where $A=\begin{pmatrix} 3 & -1 \\ 1 & 5 \end{pmatrix}$

(vi) $\dot{x}=x+y, \dot{y}=-5x-3y$

(vii) $\dot{\underset{\sim}{x}}=\begin{pmatrix} 5 & 4 \\ -1 & 0 \end{pmatrix}\underset{\sim}{x}$

15. Find the fundamental matrix of the system

$$\dot{\underset{\sim}{x}}=\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}\underset{\sim}{x}$$

and use it to find the solution of the system satisfying the initial condition $\underset{\sim}{x}(0)=\begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

16. Find a fundamental matrix of the system

$$\dot{x} = 2x - y, \dot{y} = 3x - 2y.$$

Also, find the fundamental matrix $\Phi(t)$ satisfying $\Phi(0) = I$. Find the solution of the system satisfying the initial condition $x(0) = -1, y(0) = 1$.

17. Does $\Phi(t) = \begin{pmatrix} 2e^{3t} & 2e^{-2t} \\ 3e^{3t} & 5e^{-2t} \end{pmatrix}$ a fundamental matrix of the system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$? If yes, then find the coefficient matrix A.

18. Does $\Phi(t) = \begin{pmatrix} 2e^{4t} & -2 \\ -e^{4t} & 1 \end{pmatrix}$ a fundamental solution of a system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$?

19. Find e^{At} and then solve the linear system $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ for

(i) $A = \begin{pmatrix} 9 & -5 \\ 4 & 5 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ (iii) $A = \begin{pmatrix} -4 & 12 \\ -3 & 8 \end{pmatrix}$ (iv) $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

20. Compute the exponentials of the following matrices:

(i) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (ii) $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$, $a, b \in \mathbb{R}$ (iii) $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $a, b \in \mathbb{R}$ (iv) $\begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix}$

(v) $\begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}$

21. If $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ then prove that $e^A = e^a \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix}$

22. If $AB = BA$, then show that

(i) $e^A e^B = e^B e^A$ (ii) $Ae^B = Be^A$ (iii) $e^{(A+B)t} = e^{At} e^{Bt}$.

23. If $\underset{\sim}{\alpha}$ is an eigenvector of the matrix A corresponding to the eigenvalue λ, then show that $\underset{\sim}{\alpha}$ is also an eigenvector of the matrix e^A corresponding to the eigenvalue e^λ.

24. Consider the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$.

(i) Compute e^A directly from the expression.

(ii) Compute e^A by diagonalizing the matrix A.

25. Find the solution of the following systems using fundamental theorem:

(i) $\dot{\underset{\sim}{x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \underset{\sim}{x}$

(ii) $\dot{x} = A\underset{\sim}{x}$, where $A = \begin{pmatrix} 5 & -3 \\ 3 & -1 \end{pmatrix}$

(iii) $\dot{\underset{\sim}{x}} = \begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix} \underset{\sim}{x}$

(iv) $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, where $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$

(v) $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, where $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}$

(vi) $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, where $A = \begin{bmatrix} 0 & -2 & -1 & -1 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(vii) $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$, where $A = \begin{bmatrix} 0 & -2 & -1 & -1 \\ 1 & -2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

26. Solve the following system and sketch its phase portrait

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

27. Solve the initial value problem $\dot{\underset{\sim}{x}} = \begin{pmatrix} -2 & -1 \\ 1 & 2 \end{pmatrix} \underset{\sim}{x}$, $\underset{\sim}{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and sketch the solution curve in the phase plane $\mathbb{R}^2$.

28. Find the solution of the problem

$$\ddot{x} + \alpha\dot{x} + \beta x = f(t), \quad x(0) = 1, \dot{x}(0) = 0$$

where $\alpha, \beta > 0$ are constants and $f(t)$ is a function of t.

29. Find the solution curve of the system $\dot{x} = x + y + 1, \dot{y} = x + y$ subject to the initial condition $x(0) = a,\ y(0) = b$, where a, b are some constants.

30. Consider the non-homogeneous linear system

$$\dot{\underset{\sim}{x}}(t) = A\underset{\sim}{x}(t) + \underset{\sim}{b}(t)$$

Now apply the co-ordinate transformation $\underset{\sim}{x} = P\underset{\sim}{y}$, $\underset{\sim}{x}, \underset{\sim}{y} \in \mathbb{R}^n$, where P is a $n \times n$ non-singular matrix. Find the transformed system. Hence show that every non-homogeneous system in $\mathbb{R}^2$ can be transformed into a non-homogeneous system with a Jordan matrix.

31. Does the translation property always hold for non-autonomous system of equations? Justify your answer.

32. Show that if the coefficient matrix A of a non-homogeneous system $\dot{\underset{\sim}{x}}(t) = A\underset{\sim}{x}(t) + \underset{\sim}{b}(t)$ in $\mathbb{R}^2$ has two real distinct eigenvalues, then the system can be decomposed.

33. Show that $\underset{\sim}{x}(t) = \underset{\sim}{x}_0 e^t$ is a trajectory passing through the point $\underset{\sim}{x}_0$ of a linear vector field $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ where A is a constant matrix.

34. Show that $\underset{\sim}{x}(t+\tau) = \underset{\sim}{x}_0 e^{A(t+\tau)}, t, \tau \in \mathbb{R}$ is also a solution of $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$ subject to the initial condition $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$. Does it violate the uniqueness of solution? Justify.

35. Define a flow in $\mathbb{R}^2$. Write the properties of flow $\phi(t, \underset{\sim}{x})$. Show that $\phi(t, \underset{\sim}{x}) = e^{At}\underset{\sim}{x}$ satisfies all properties of flow.

36. Find the flow evolution operator $\phi(t, \underset{\sim}{x})$ for the following systems:

(i) $\dot{x} = -x, \dot{y} = -2y$,

(ii) $\dot{x} = xy, \dot{y} = y^2$,

(iii) $\dot{r} = r(1-r), \dot{\theta} = 1$,

(iv) $\ddot{x} + \dot{x} + x = 0$,

(v) $\dot{x} = y, \dot{y} = (x - y)$.

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