Chapter 18
An Improved Adomian Decomposition Method for Nonlinear ODEs

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Abstract This work deals with getting approximate solution of boundary value problem consists of nonlinear ordinary differential equations in a series of exponential instead of power of independent variable in traditional Adomian decomposition method (TADM). As a consequence: (i) in contrast to TADM the vanishing boundary condition for localized solution can be implemented in a straightforward way, (ii) the convergence of the series obtained through the modification proposed here found to be faster than the same obtained by employing TADM, and (iii) for most of the problems, the sum of the series converges to the exact analytic solution to the equation involved. The efficiency of the modification of TADM has been illustrated for physical problems with varied nonlinearities.

Keywords Nonlinear ordinary differential equation · Boundary value problem · Improved Adomian decomposition method

18.1 Introduction

In many branches of applied mathematics, physical, biological, and engineering sciences, evolution of physical processes are found to be described by nonlinear ordinary or partial differential equations (ODEs/PDEs). The solution of such equations helps one to understand the nature of evolution of the process. But in most of the cases, it is not possible to find the exact solution to the equation used as the mathematical model for the description of the physical process of interest. A few analytical methods such as symmetry method based on Lie theory [1, 2], Prelle-Singer method [3], method based on Jacobi last multiplier [4], etc., analytical methods for approximate solution such as tanh method [5, 6], homotopy analysis method (HAM) [7, 8], Adomian decomposition method (ADM) [9–21], etc., numerical methods, viz., finite

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S. Sarkar et al. (eds.), *Applied Mathematics*, Springer Proceedings
in Mathematics & Statistics 146, DOI 10.1007/978-81-322-2547-8_18

difference/element methods are used to find the solution of this problems. Among the approximation methods mentioned above, ADM is found to be the simplest one. Using ADM, Adomian and his collaborators [9–14], Wazwaz [15–21] as well as other researchers obtained the approximate solutions as the sum of finite number of terms with the leading term as the polynomial in independent variable involved in the problem. But in their approach, the boundary condition in case of infinite domain cannot be implemented in a straight forward way. Instead, it is desirable to express the successive terms in their approximate solution as a rational function with the help of Padé approximant to accommodate boundary conditions. Naturally, question arises whether straightforward method can be developed which is able to provide a rapidly convergent series approximation of the solution to the differential equation involving the physical processes that incorporate boundary conditions at $\pm\infty$ in a straightforward way in both cases of finite as well as infinite domain.

In this paper, we have addressed this problem and developed an recursive scheme for solving two-point nonlinear boundary value problems through a modification of the conventional ADM. Here we have introduced an operator associated with the linear part of the differential equation and derived a straightforward formula involving such operator for correction terms associated to the nonlinear part of the equation. We designate this method as the improved Adomian decomposition method (IADM), provides the solution in a series of exponentials instead of power of independent variable, appears in case of conventional ADM. Expansion in series of exponential perhaps is the source of accelerated convergence of the method proposed here.

The organization of this paper is as follows. The improved Adomian decomposition method (IADM) within finite domain has been discussed in Sect. 18.2. Its extension to infinite domain has been presented in Sect. 18.3. Our findings on utility of the proposed IADM developed in previous two sections have been illustrated in Sect. 18.4.

18.2 IADM in Finite Domain $[a, b]$

We consider here a two-point boundary value problem of the form

$$y''(x) - \lambda^2 y(x) = \mathscr{N}[y](x) + g(x), \; a \le x \le b \tag{18.1}$$

within finite domain $[a, b]$ subject to the Dirichlet boundary condition

$$y(a) = \alpha, \; y(b) = \beta \tag{18.2}$$

where $N[y]$ is an nonlinear term in y, and $g(x)$ is the inhomogeneous or source term, continuous over $[a, b]$. Instead of shifting the linear term $\lambda^2 y(x)$ of (18.1) into R.H.S in conventional ADM, we incorporate it into the operator $\hat{\mathscr{O}}[\cdot] \equiv \frac{d^2}{dx^2} - \lambda^2$, so that (18.1) can now be recast into the form

$$\hat{\mathscr{O}}[y](x) = \mathscr{N}[y](x) + g(x), \;\; a \le x \le b. \tag{18.3}$$

It is important to mention here that the linear operator $\hat{\mathscr{O}}[\cdot]$ can be written in the form

$$\hat{\mathscr{O}}[\cdot](x) = \mathrm{e}^{\lambda x}\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-2\lambda x}\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\lambda x}[\cdot]\right) \tag{18.4}$$

which plays the fundamental role in expressing the solution in terms of rapidly convergent series of exponentials. One may reinterpret the inverse operator $\hat{\mathscr{O}}^{-1}$ as a twofold integral operator given by

$$\hat{\mathscr{O}}^{-1}[\cdot](x) = \mathrm{e}^{-\lambda x}\int_a^x \mathrm{e}^{2\lambda x'}\int_a^{x'} \mathrm{e}^{-\lambda x''}[\cdot](x'')\mathrm{d}x''\mathrm{d}x'. \tag{18.5}$$

Note that representing inverse operator by integrals for a linear operator with variable coefficient is also possible whenever it is factorizable. Application of $\hat{\mathscr{O}}^{-1}$ given in (18.5) to $y''(x) - \lambda^2 y(x)$, one gets

$$\begin{aligned}
\hat{\mathscr{O}}^{-1}\left[y''(x) - \lambda^2 y(x)\right] &= \mathrm{e}^{-\lambda x}\int_a^x \mathrm{e}^{2\lambda x'}\int_a^{x'} \mathrm{e}^{-\lambda x''}\left(y''(x'') - \lambda^2 y(x'')\right)\mathrm{d}x''\mathrm{d}x' \\
&= \mathrm{e}^{-\lambda x}\int_a^x \mathrm{e}^{2\lambda x'} \\
&\quad \left(\mathrm{e}^{\lambda x'}y'(x') - \mathrm{e}^{-\lambda a}y'(a) + \lambda \mathrm{e}^{-\lambda x'}y(x') - \lambda \mathrm{e}^{\lambda a}y(a)\right)\mathrm{d}x' \\
&= y(x) - y(a)\mathrm{e}^{-\lambda(a-x)} - \mathrm{e}^{-\lambda a}\left(y'(a) + \lambda y(a)\right)\left(\frac{\mathrm{e}^{\lambda x} - \mathrm{e}^{-\lambda x}}{2\lambda}\right).
\end{aligned} \tag{18.6}$$

Operating $\hat{\mathscr{O}}^{-1}$ on both sides of (18.3) followed by using (18.6) one gets

$$\begin{aligned}
y(x) = y(a)\mathrm{e}^{-\lambda(a-x)} + \mathrm{e}^{-\lambda a}\left(y'(a) + \lambda y(a)\right)\left(\frac{\mathrm{e}^{\lambda x} - \mathrm{e}^{-\lambda x}}{2\lambda}\right) \\
+ \hat{\mathscr{O}}^{-1}[\mathscr{N}[y]](x) + \hat{\mathscr{O}}^{-1}[g](x),
\end{aligned} \tag{18.7}$$

which involve an unknown term $y'(a)$. To eliminate $y'(a)$, we substitute $x = b$ in Eq. (18.7) and solve for $e^{-\lambda a}\left(y'(a) + \lambda y(a)\right)$ to get

$$\mathrm{e}^{\lambda a}\left(y'(a) + \lambda y(a)\right) = \frac{2\lambda\left(y(b) - y(a)\mathrm{e}^{-\lambda(a-b)} - \hat{\mathscr{O}}^{-1}[\mathscr{N}[y]](b) - \hat{\mathscr{O}}^{-1}[g](b)\right)}{\mathrm{e}^{\lambda b} - \mathrm{e}^{-\lambda b}}. \tag{18.8}$$

Eliminating $\mathrm{e}^{-\lambda a}\left(y'(a) + \lambda y(a)\right)$ from (18.7) with the help of (18.8) gives the expression for $y(x)$ involving inverse operator

$$y(x) = y_0(x) - \hat{\mathscr{O}}^{-1}[\mathscr{N}[y]](b)\frac{\mathrm{e}^{\lambda x} - \mathrm{e}^{-\lambda x}}{\mathrm{e}^{\lambda b} - \mathrm{e}^{-\lambda b}} + \hat{\mathscr{O}}^{-1}[\mathscr{N}[y]](x). \tag{18.9}$$

One can now apply the relevant steps of ADM for evaluating terms involving nonlinear operator $\mathscr{N}[y](x)$ where leading term $y_0(x)$ is given by

$$y_0(x) = y(a)\mathrm{e}^{-\lambda(a-x)} + \frac{y(b) - y(a)\mathrm{e}^{-\lambda(a-b)} - \hat{\mathscr{O}}^{-1}[g](b)}{\mathrm{e}^{\lambda b} - \mathrm{e}^{-\lambda b}}\left\{\mathrm{e}^{\lambda x} - \mathrm{e}^{-\lambda x}\right\} + \hat{\mathscr{O}}^{-1}[g](x). \tag{18.10}$$

The successive corrections can be obtained recursively using the formula

$$y_{n+1}(x) = \hat{\mathscr{O}}^{-1}[\mathscr{A}_n](x) - \frac{\hat{\mathscr{O}}^{-1}[\mathscr{A}_n](b)}{\left(\mathrm{e}^{\lambda b} - \mathrm{e}^{-\lambda b}\right)}\left(\mathrm{e}^{\lambda x} - \mathrm{e}^{-\lambda x}\right), \ n \leq 0, \tag{18.11}$$

where $A_n(x), \ n \geq 0$ are Adomain polynomial for nonlinear term given by the formula

$$\mathscr{A}_m(x) = \frac{1}{m!}\left[\frac{d^m}{d\varepsilon^m}\mathscr{N}\left(\sum_{k=0}^{\infty} y_k \varepsilon^k\right)\right]_{\varepsilon=0}, \ m \geq 0. \tag{18.12}$$

18.3 IADM in Infinite Domain

Whenever the domain of independent variable become infinite, we write the inverse operator $\hat{\mathscr{O}}^{-1}$ as a twofold integral operator without limit given by

$$\hat{\mathscr{O}}^{-1}[\cdot](x) = \mathrm{e}^{-\lambda x}\int \mathrm{e}^{2\lambda x}\int e^{-\lambda x}[\cdot](x)\,\mathrm{d}x\,\mathrm{d}x. \tag{18.13}$$

In this case, operation of $\hat{\mathscr{O}}^{-1}$ on $y''(x) - \lambda^2 y(x)$ gives

$$\begin{aligned}\hat{\mathscr{O}}^{-1}\left(y''(x) - \lambda^2(x)\right) &= \mathrm{e}^{-\lambda x}\int \mathrm{e}^{2\lambda x}\int \mathrm{e}^{-\lambda x}\left(y''(x) - \lambda^2 y(x)\right)\mathrm{d}x\,\mathrm{d}x \\ &= \mathrm{e}^{-\lambda x}\int \mathrm{e}^{2\lambda x}\left(\mathrm{e}^{-\lambda x}y'(x) + \lambda\mathrm{e}^{-\lambda x}y(x) + c\right)\mathrm{d}x \\ &= y(x) + \frac{c}{2\lambda}\mathrm{e}^{\lambda x} - d\mathrm{e}^{-\lambda x}. \end{aligned} \tag{18.14}$$

involving two arbitrary constants c and d. Operating $\hat{\mathscr{O}}^{-1}$ on both sides of $\hat{\mathscr{O}}^{-1}[y](x) = N[y](x) + g(x)$ and use of (18.14), leads to

$$y(x) = -\frac{c}{2\lambda}\mathrm{e}^{\lambda x} + d\mathrm{e}^{-\lambda x} + \hat{\mathscr{O}}^{-1}[\mathscr{N}[y]](x) + \hat{\mathscr{O}}^{-1}[g](x). \tag{18.15}$$

Assuming $\lambda > 0$ and using the vanishing boundary condition $y(\infty) = 0$ for localized solution of (18.1) within $[0, \infty)$, we can obtain $c = 0$. Thus

$$y(x) = d\mathrm{e}^{-\lambda x} + \hat{\mathscr{O}}^{-1}[\mathscr{N}[y]](x) + \hat{\mathscr{O}}^{-1}[g](x),\ x \in [0, \infty). \tag{18.16}$$

The correction to the leading order due to presence of nonlinearities are obtained by executing steps followed in conventional ADM with

$$y_{n+1}(x) = \hat{\mathscr{O}}^{-1}\left[\mathscr{A}_n\right](x),\ n \geq 0 \tag{18.17}$$

with

$$y_0(x) = d\mathrm{e}^{-\lambda x} + \hat{\mathscr{O}}^{-1}[g](x), \tag{18.18}$$

where $\mathscr{A}_n(x),\ n \geq 0$ are Adomain polynomial for nonlinear term can be obtained using the formula (18.12). It is important to note that whenever the domain becomes $(-\infty, 0]$, instead of using vanishing boundary condition $y(\infty) = 0$, for localized solution (18.15) we use $y(-\infty) = 0$ and get

$$y(x) = -\frac{c}{2\lambda}\mathrm{e}^{\lambda x} + \hat{\mathscr{O}}^{-1}[\mathscr{N}[y]](x) + \hat{\mathscr{O}}^{-1}[g](x),\ x \in (-\infty, 0] \tag{18.19}$$

so that higher order corrections over leading order approximation

$$y_0(x) = -\frac{c}{2\lambda}\mathrm{e}^{\lambda x} + \hat{\mathscr{O}}^{-1}[g](x) \tag{18.20}$$

can obtained recursively form

$$y_{n+1}(x) = \hat{\mathscr{O}}^{-1}[\mathscr{A}]_n(x),\ n \geq 0. \tag{18.21}$$

In case of $\lambda < 0$, one has to proceed in the same way by retaining the term involving $\mathrm{e}^{\lambda x}$.

18.4 Illustrative Example

Our findings on getting approximate solution for nonlinear ODEs within finite and infinite domain by using IADM proposed here have been summarized in Tables 18.1 and 18.2, respectively.

Table 18.1 Solution of nonlinear ODE in finite domain by IADM

Problem	Linear term	Nonlinear term	Leading order approximation	Series solution	Exact solution
$y''(x) - 2y(x) = 2y(x)^3$ with boundary condition $y(0) = 0,\ y\left(\frac{\pi}{4}\right) = -1$	$y''(x) - 2y(x)$	$2y(x)^3$	$-\frac{\sinh\left(\sqrt{2}x\right)}{\sinh\left(\sqrt{2}\frac{\pi}{4}\right)}$	$-\frac{\sinh\left(\sqrt{2}x\right)}{\sinh\left(\sqrt{2}\frac{\pi}{4}\right)} + 0.0347465 \sin\left(\sqrt{2}x\right)$ $-\frac{1}{32}\cosh\left(\frac{\pi}{2\sqrt{2}}\right)^3 \left(-12\sqrt{2}x\,\text{csch}\left(\sqrt{2}x\right)\right)$ $+9\sinh\left(\sqrt{2}x\right) + \sinh\left(3\sqrt{2}x\right) + \cdots$	$y(x) = -\tan(x)$

Table 18.2 Solution of nonlinear ODEs in infinite domain by IADM

Equations	Korteweg-de Varies equation $u_t + 6uu_x + u_{xxx} = 0$	Zakharov equation $iE_t + E_{xx} = \eta E$ $\eta_{tt} - \eta_{xx} = \left(\lvert E\rvert^2\right)_{xx}$	Camassa-Holm equation $u_t + 2ku_x - u_{xxt} + 3uu_x$ $-2u_x u_{xx} - uu_{xxx} = 0$
Similarity variables	$\xi = c(x - vt)$ $u(x,t) = U(\xi)$	$E(x,t) = e^{iv}u(\xi)$ $\eta = \eta(\xi)$ $v = cx + dt$ $\xi = x - 2ct$	$\xi = (x - ct)$, $u(x,t) = v(\xi) - k$
Reduced ODE	$U''(\xi) - \frac{v}{c^2}U(\xi) = -\frac{3}{c^2}U(\xi)^2$	$u''(\xi) - \left(c^2 + d\right)u(\xi) = \frac{u(\xi)^3}{\left(4c^2 - 1\right)}$	$v''(\xi) - v(\xi) =$ $\frac{1}{2(k+c)}\left(v'(\xi)^2 + 2v(\xi)v''(\xi) - 3v(\xi)^2\right)$
Leading order approximation	$de^{-\frac{\sqrt{v}}{c}\xi}$	$\gamma e^{-\sqrt{(c^2+d)}\xi}$	$de^{-\xi}$ when $\xi \succ 0$ $de^{+\xi}$ when $\xi \prec 0$
Sum of series	$u(x,t) = \frac{v}{2}\operatorname{sech}^2\left(\frac{\sqrt{v}}{2c}(x - vt) + m\right)$ where $m = -\frac{1}{2}\log\left(\frac{d}{2v}\right)$	$E(x,t) = \sqrt{2\left(1 - 4c^2\right)\left(c^2 + d\right)}\, e^{i(cx+dt)}$ $\operatorname{sech}\left(\sqrt{c^2 + d}\,(x - 2ct) + m\right)$ $\eta(x,t) =$ $2\left(c^2 + d\right)\operatorname{sech}^2\left(\sqrt{c^2 + d}\,(x - 2ct) + m\right)$ where $m - \log\left(\frac{\gamma}{\sqrt{8}(1-4c^2)(c^2+d)}\right)$	$u(x,t) =$ $\begin{cases} de^{-(x-ct)} - k \text{ for } \xi \succ 0 \\ de^{+(x-ct)} - k \text{ for } \xi \prec 0 \end{cases}$

18.5 Conclusions

In this work, an improvement over conventional ADM has been proposed. The consequence is to get an approximate solution of nonlinear ODE in the series of exponential. As a result, the approximate solution become rapidly convergent and found to converges to the exact analytic solution for both kind of problems defined over bounded and unbounded domains. From this study, it also appears that conventional ADM can further be improved for problem consists of variable coefficient in their linear part in order to get rapidly convergent approximate solution of nonlinear ODEs used as mathematical models for physical processes.

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