Determinantal Approach to Hermite-Sheffer Polynomials

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Abstract In this article, the determinantal definition for the Hermite-Sheffer polynomials is established using linear algebra tools. Further, the Hermite-Sheffer matrix polynomials are introduced by means of their generating function.

Keywords Hermite polynomials · Sheffer polynomials · Determinantal definition

1 Introduction

The Hermite polynomials are frequently used in many branches of pure and applied mathematics, engineering, and physics. The 2-variable Hermite Kampé de Feriet polynomials (2VHKdFP) $H_n(x, y)$ are defined as [2]:

$$H_n(x,y) = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{y^k x^{n-2k}}{k!(n-2k)!} \tag{1}$$

and specified by the generating function

$$\exp(xt + yt^2) = \sum_{n=0}^{\infty} H_n(x,y) \frac{t^n}{n!}. \tag{2}$$

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These polynomials are the solutions of the heat equation

$$\frac{\partial}{\partial y} H_n(x,y) = \frac{\partial^2}{\partial x^2} H_n(x,y), \tag{3}$$

$$H_n(x,0) = x^n. \tag{4}$$

A polynomial sequence $\{s_n(x)\}_{n=0}^{\infty}(s_n(x)$ being a polynomial of degree $n)$ is called Sheffer A-type [10, p. 17], if $S_n(x)$ possesses the exponential generating function of the form:

$$A(t)\exp(xH(t)) = \sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!}, \tag{5}$$

where $A(t)$ and $H(t)$ have (at least the formal) expansions:

$$A(t) = \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{n!}, \alpha_0 \neq 0 \tag{6}$$

and

$$H(t) = \sum_{n=1}^{\infty} b_n \frac{t^n}{n!}, b_1 \neq 0, \tag{7}$$

respectively.

The Sheffer sequence for $(1, H(t))$ is called the associated Sheffer sequence for $H(t)$ and the Sheffer sequence for $(A(t), t)$ becomes the Appell sequence [1] for $A(t)$.

Thus, the associated Sheffer sequence $p_n(x)$ is defined by the generating function

$$\exp(xH(t)) = \sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!} \tag{8}$$

and the Appell sequence $A_n(x)$ is defined by the generating function

$$A(t)\exp(xt) = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!}. \tag{9}$$

The Appell and Sheffer sequences arise in numerous problems of applied mathematics, theoretical physics, approximation theory and several other mathematical branches, which appear in even the most basic problems of quantum mechanics.

In the past few decades, there has been a renewed interest in Sheffer polynomials. Properties of Sheffer polynomials are naturally handled within the framework of modern classical umbral calculus by Roman [10]. Another aspect of such study is to find the differential equation and recursive formula for Sheffer polynomial sequences. The mathematical concept of a Sheffer polynomial set has been used as a tool for determining how a signal changes when its complex cepstrum is scaled by a constant, see for example [11].

The Hermite-based Sheffer polynomials (HSP) ${}_Hs_n(x,y)$ are introduced in [8] by combining the 2VHKdFP $H_n(x,y)$ and Sheffer polynomials $s_n(x)$. The HSP are defined by the generating function of the form:

$$A(t)\exp(xH(t)+y(H(t))^2)=\sum_{n=0}^{\infty}{}_Hs_n(x,y)\frac{t^n}{n!}. \tag{10}$$

Since, for $A(t)=1$, the Sheffer polynomials $s_n(x)$ reduce to the associated Sheffer polynomials $p_n(x)$. Therefore, by taking $A(t)=1$ on the l.h.s. of Eq. (10), we obtain the generating function of the Hermite-associated Sheffer polynomials (HASP) ${}_Hp_n(x,y)$ as:

$$\exp(xH(t)+y(H(t))^2)=\sum_{n=0}^{\infty}{}_Hp_n(x,y)\frac{t^n}{n!}. \tag{11}$$

The determinantal forms of the Appell and Sheffer sequences are studied by Costabile and Longo in [4, 5] respectively. The determinantal approach considered in [4, 5] provides motivation to consider the determinantal forms of the mixed special polynomials.

In this article, the determinantal definition of the HSP ${}_Hs_n(x,y)$ is established using the generating functions of the HSP ${}_Hs_n(x,y)$ and HASP ${}_Hp_n(x,y)$. Further, the Hermite-Sheffer matrix polynomials are introduced by means of the generating function.

2 Determinantal Approach

The determinantal definition of the HSP ${}_Hs_n(x,y)$ is established using certain linear algebraic tools. In order to derive the determinantal definition of the HSP ${}_Hs_n(x,y)$, we prove the following result:

Theorem 1 *The HSP ${}_Hs_n(x,y)$ of degree n are defined by*

$$ {}_Hs_0(x,y) = \frac{1}{\beta_0}, \quad (12)$$

$$ {}_Hs_n(x,y) = \frac{(-1)^n}{(\beta_0)^{n+1}} \times \begin{vmatrix} 1 & {}_Hp_1(x,y) & {}_Hp_2(x,y) & \cdots & {}_Hp_{n-1}(x,y) & {}_Hp_n(x,y) \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}, \quad n = 1, 2, \ldots, \quad (13)$$

where

$$\begin{aligned} \beta_0 &= \tfrac{1}{\alpha_0}, \\ \beta_n &= -\tfrac{1}{\alpha_0}\left(\sum_{k=1}^{n}\binom{n}{k}\alpha_k\,\beta_{n-k}\right), \quad n = 1, 2, \ldots. \end{aligned}$$

Proof Let ${}_Hs_n(x,y)$ be a sequence of polynomials with generating function given in Eq. (10) and α_n, β_n, be two numerical sequences such that

$$A(t) = \alpha_0 + \frac{t}{1!}\alpha_1 + \frac{t^2}{2!}\alpha_2 + \cdots + \frac{t^n}{n!}\alpha_n + \cdots, \; n = 0, 1, \ldots; \; \alpha_0 \neq 0, \quad (14)$$

$$\widehat{A}(t) = \beta_0 + \frac{t}{1!}\beta_1 + \frac{t^2}{2!}\beta_2 + \cdots + \frac{t^n}{n!}\beta_n + \cdots, \; n = 0, 1, \ldots; \; \beta_0 \neq 0, \quad (15)$$

satisfying

$$A(t)\widehat{A}(t) = 1. \quad (16)$$

Then, according to the Cauchy-product rules, we find

$$A(t)\widehat{A}(t) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\alpha_k\,\beta_{n-k}\frac{t^n}{n!},$$

by which, we have

$$\sum_{k=0}^{n}\binom{n}{k}\alpha_k\,\beta_{n-k} = \begin{cases} 1 & \text{for}\, n=0, \\ 0 & \text{for}\, n>0. \end{cases} \tag{17}$$

Hence,

$$\begin{cases} \beta_0 = \frac{1}{\alpha_0}, \\ \beta_n = -\frac{1}{\alpha_0}\left(\sum_{k=1}^{n}\binom{n}{k}\alpha_k\,\beta_{n-k}\right), \quad n=1,2,\ldots. \end{cases} \tag{18}$$

Let us multiply both sides of Eq. (10) by $\widehat{A}(t)$, so that we have

$$A(t)\widehat{A}(t)\exp(xH(t)+y(H(t))^2) = \widehat{A}(t)\sum_{n=0}^{\infty}{}_Hs_n(x,y)\frac{t^n}{n!}. \tag{19}$$

In view of Eqs. (11), (15) and (16), we find

$$\sum_{n=0}^{\infty}{}_Hp_n(x,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty}{}_Hs_n(x,y)\frac{t^n}{n!}\sum_{k=0}^{\infty}\beta_k\frac{t^k}{k!}. \tag{20}$$

By multiplying the series on the r.h.s. of Eq. (20) according to Cauchy-product rules, the previous equality leads to the following system of infinite equations in the unknown ${}_Hs_n(x,y)$, $n=0,1,\ldots$,

$$\begin{cases} {}_Hs_0(x,y)\beta_0 = 1, \\ {}_Hs_0(x,y)\beta_1 + {}_Hs_1(x,y)\beta_0 = {}_Hp_1(x,y), \\ {}_Hs_0(x,y)\beta_2 + \binom{2}{1}{}_Hs_1(x,y)\beta_1 +_H s_2(x,y)\beta_0 = {}_Hp_2(x,y), \\ \vdots {}_Hs_0(x,y)\beta_n + \binom{n}{1}{}_Hs_1(x,y)\beta_{n-1} + \cdots + {}_Hs_n(x,y)\beta_0 = {}_Hp_n(x,y). \end{cases} \tag{21}$$

From the first equation of system (21), we get assertion (12). Also, the special form of system (21) (lower triangular) allows us to work out the unknown ${}_Hs_n(x,y)$. Operating with the first $n+1$ equations simply by applying the Cramer's rule, we find

$$
{}_Hs_n(x,y) = \frac{\begin{vmatrix} \beta_0 & 0 & 0 & \cdots & 0 & 1 \\ \beta_1 & \beta_0 & 0 & \cdots & 0 & {}_Hp_1(x,y) \\ \beta_2 & \binom{2}{1}\beta_1 & \beta_0 & \cdots & 0 & {}_Hp_2(x,y) \\ . & . & . & \cdots & . & . \\ \beta_{n-1} & \binom{n-1}{1}\beta_{n-2} & \binom{n-1}{2}\beta_{n-3} & \cdots & \beta_0 & {}_Hp_{n-1}(x,y) \\ \beta_n & \binom{n}{1}\beta_{n-1} & \binom{n}{2}\beta_{n-2} & \cdots & \binom{n}{n-1}\beta_1 & {}_Hp_n(x,y) \end{vmatrix}}{\begin{vmatrix} \beta_0 & 0 & 0 & \cdots & 0 & 0 \\ \beta_1 & \beta_0 & 0 & \cdots & 0 & 0 \\ \beta_2 & \binom{2}{1}\beta_1 & \beta_0 & \cdots & 0 & 0 \\ . & . & . & \cdots & . & . \\ \beta_{n-1} & \binom{n-1}{1}\beta_{n-2} & \binom{n-1}{2}\beta_{n-3} & \cdots & \beta_0 & 0 \\ \beta_n & \binom{n}{1}\beta_{n-1} & \binom{n}{2}\beta_{n-2} & \cdots & \binom{n}{n-1}\beta_1 & \beta_0 \end{vmatrix}}, \quad n = 1, 2, \ldots. \tag{22}
$$

Now, expanding the determinant in the denominator of Eq. (22) and transposing the determinant in the numerator, we find

$$
{}_{H}s_n(x,y) = \frac{1}{(\beta_0)^{n+1}} \begin{vmatrix} \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \cdots & \binom{n-1}{2}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \cdots & \binom{n-1}{2}\beta_{n-3} & \binom{n}{1}\beta_{n-2} \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \beta_0 & \binom{n}{n-1}\beta_1 \\ 1 & {}_{H}p_1(x,y) & {}_{H}p_2(x,y) & \cdots & {}_{H}p_{n-1}(x,y) & {}_{H}p_n(x,y) \end{vmatrix}, \tag{23}
$$

where $n = 1, 2, \ldots.$ which after n circular row exchanges, that is, after moving the ith row to the $(i + 1)$th position for $i = 1, 2, \ldots, n-1$, yields assertion (13).

Several important polynomials belong to the Sheffer family. By considering the suitable values of the coefficients in the determinantal definition of the Hermite-Sheffer polynomials, the determinantal definitions of the corresponding members can be obtained.

3 Concluding Remarks

Special matrix functions appear in statistics, Lie group theory and number theory [3]. Hermite matrix polynomials have been introduced and studied in [7] for matrices in $\mathbb{C}^{n\times n}$ whose eigenvalues are all situated in the right open half-plane.

We recall that the Hermite matrix polynomials $H_n(x;A)$ are defined by [7]:

$$
H_n(x;A) = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k}{(n-2k)!k!} (x\sqrt{2A})^{n-2k}, n \geq 0 \tag{24}
$$

and are specified by the generating function of the form [7]:

$$
\exp\left(xt\sqrt{2A} - t^2 I\right) = \sum_{n=0}^{\infty} H_n(x;A) \frac{t^n}{n!}, \tag{25}
$$

where A is a matrix in $\mathbb{C}^{n\times n}$ such that

$$
Re(\mu) \not\leq 0, \quad \mu \in \sigma(A). \tag{26}
$$

Recently, Khan and Raza [9] introduced the 2-variable Hermite matrix polynomials (2VHMaP) $H_n(x,y;A)$. The 2VHMaP $H_n(x,y;A)$ are defined by the generating function of the form:

$$\exp\left(xt\sqrt{\frac{A}{2}}+yt^2I\right)=\sum_{n=0}^{\infty}H_n(x,y;A)\frac{t^n}{n!} \tag{27}$$

It is also shown in [9] that the 2VHMaP $H_n(x,y;A)$ are quasi-monomial with respect to the following multiplicative and derivative operators:

$$\widehat{M}_H=\left(x\sqrt{\frac{A}{2}}+2y\left(\sqrt{\frac{A}{2}}\right)^{-1}\frac{\partial}{\partial x}\right) \tag{28}$$

and

$$\widehat{P}_H=\left(\sqrt{\frac{A}{2}}\right)^{-1}\frac{\partial}{\partial x}, \tag{29}$$

respectively.

Here, we combine the 2-variable Hermite matrix polynomials (2VHMaP) $H_n(x,y;A)$ with Sheffer polynomials $s_n(x)$ to introduce a mixed family.

In order to introduce the Hermite-Sheffer matrix polynomials (HSMaP) denoted by ${}_Hs_n(x,y;A)$ by means of generating function, we prove the following result:

Theorem 2 *The HSMaP* ${}_Hs_n(x,y;A)$ *are defined by the following generating equation*:

$$\exp\left(x\sqrt{\frac{A}{2}}H(t)+y(H(t))^2I\right)=\sum_{n=0}^{\infty}{}_Hs_n(x,y;A)\frac{t^n}{n!}, \tag{30}$$

where A is a matrix in $\mathbb{C}^{n\times n}$ and satisfies condition (26).

Proof Replacing x by the multiplicative operator $\widehat{M}_H$ of 2VHMaP $H_n(x,y;A)$ in the generating function (5) of the Sheffer polynomials $s_n(x)$, we find

$$A(t)\exp(\widehat{M}_H H(t))=\sum_{n=0}^{\infty}s_n(\widehat{M}_H)\frac{t^n}{n!}. \tag{31}$$

Using the expression of $\hat{M}_H$ given in Eq. (28) in the above equation, we have

$$A(t)\exp\left(\left(x\sqrt{\frac{A}{2}}+2y\left(\sqrt{\frac{A}{2}}\right)^{-1}\frac{\partial}{\partial x}\right)H(t)\right) = \sum_{n=0}^{\infty} s_n\left(x\sqrt{\frac{A}{2}}+2y\left(\sqrt{\frac{A}{2}}\right)^{-1}\frac{\partial}{\partial x}\right)\frac{t^n}{n!}. \tag{32}$$

Using the notations $X := x\sqrt{\frac{A}{2}}$, so that $\frac{\partial}{\partial X} := \left(\sqrt{\frac{A}{2}}\right)^{-1}\frac{\partial}{\partial x}$ and then decoupling the exponential operator on the l.h.s. of Eq. (32) using the Crofton-type identity [6]:

$$f\left(X+2\lambda\frac{\partial}{\partial X}\right)\{1\} = \exp\left(\lambda\frac{\partial}{\partial X}\right)\{f(X)\} \tag{33}$$

and denoting the resultant Hermite-Sheffer matrix polynomials (HSMaP) on the r.h.s. by ${}_Hs_n(x,y;A)$, that is,

$${}_Hs_n(x,y;A) = s_n(\widehat{M}_H) = s_n\left(x\sqrt{\frac{A}{2}}+2y\left(\sqrt{\frac{A}{2}}\right)^{-1}\frac{\partial}{\partial x}\right), \tag{34}$$

we get assertion (30) after simplification.

Certain properties such as the series definition, operational representation, recurrence relations, differential equations, determinantal definition, etc., for the HSMaP ${}_Hs_n(x,y;A)$ are yet to be explored. This aspect will be taken up in the next investigation.

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References

1. Appell, P.: Sur Une Classe de Polynômes. Ann. Sci. École. Norm. Sup. **9**(2), 119–144 (1880)
2. Appell, P., Kampé de Fériet, J.: Fonctions Hypergéométriques et Hypersphériques: Polynômes d' Hermite. Gauthier-Villars, Paris (1926)
3. Constantine, A.G., Muirhead, R.J.: Partial differential equations for hypergeometic functions of two argument matrix. J. Multivariate Anal. **3**, 332–338 (1972)
4. Costabile, F.A., Longo, E.: A determinantal approach to Appell polynomials. J. Comput. Appl. Math. **234**(5), 1528–1542(2010)
5. Costabile, F.A., Longo, E.: An algebraic approach to Sheffer polynomial sequences. Integral Transforms Spec. Funct. **25**(4), 295–311 (2013)

6. Dattoli, G., Ottaviani, P.L., Torre, A., Vázquez, L.: Evolution operator equations: integration with algebraic and finite difference methods, applications to physical problems in classical and quantum mechanics and quantum field theory. Riv. Nuovo Cimento Soc. Ital. Fis. (4) **20**(2), 1–133 (1997)
7. Jódar, L., Company, R.: Hermite matrix polynomials and second order matrix differential equations. Approx. Theory Appl. (N.S.) **12**(2), 20–30 (1996)
8. Khan, S., Al-Saad, M.W.M., Yasmin, G.: Some properties of Hermite-based Sheffer polynomials. Appl. Math. Comput. **217**(5), 2169–2183 (2010)
9. Khan, S., Raza, N.: 2-variable generalized Hermite matrix polynomials and Lie Algebra representation. Rep. Math. Phys. 66(2), 159–174 (2010)
10. Roman, S.: The Umbral Calculus. Academic Press, New York (1984)
11. Schmidlin, D.J.: The bond between causal complex Cepstra and Sheffer polynomials. IEEE Trans. Circuits Syst. II: Express Briefs **52**(2), pp. 99–103 (2005)