Chapter 5
Mathematical Analysis of RDRA Amplitude Coefficients

Abstract Mathematical analysis of amplitude coefficients in rectangular DRA (RDRA) have been evaluated. Rigorous theoretical analysis has been developed for different resonant modes inside RDRA. The resonance phenomenon and its potential use as radiator have been described. The dielectric polarization *P* is equal to the total dipole moment induced in the volume of the material by the electric field. The discontinuity of the relative permittivity at the resonator surface allows a standing electromagnetic wave to be supported in its interior at a particular resonant frequency, thereby leading to maximum confinement of energy within the resonator. Certain field distributions or modes will satisfy Maxwell's equations and boundary condition. Mathematical solution to get amplitude coefficients C_{mnp} along with its phase coefficients has been obtained. These are also known as eigenvector.

Keywords Amplitude coefficients · Resonant modes · Radiation lobes · Fourier transform · Discrete solution · PMC (perfect magnetic conducting) · PEC (perfect electrical conducting) · Dominant mode · Higher-order modes

5.1 Introduction

Rigorous theoretical analysis has been developed for resonant modes in rectangular DRA (RDRA). RDRA resonance phenomenon and its potential, as a radiator have been long back described. Accordingly, external electric fields bring the charges of the molecules of the dielectric into a certain ordered arrangement in space. The dielectric polarization P is equal to the total dipole moment induced in the volume of the material by the electric field. The discontinuity of the relative permittivity at the resonator surface allows a standing electromagnetic wave to be supported in its interior at a particular resonant frequency, thereby leading to maximum confinement of energy within the resonator. Certain field distributions or modes will satisfy Maxwell's equations and boundary conditions. Resonant modes are field structures that can exist inside the RDRA. The RDRA prototype is shown in Fig. 5.1.

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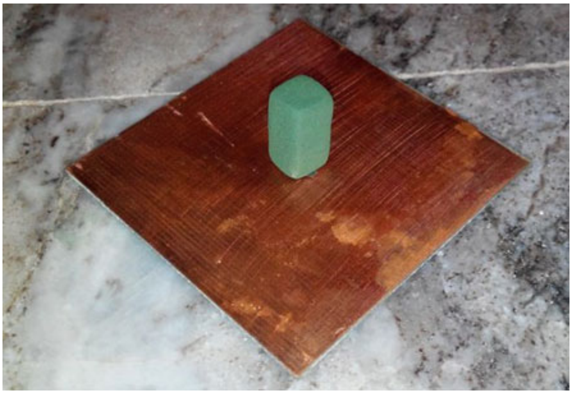


Fig. 5.1 Homogenous dielectric RDRA on ground plane

5.2 Amplitude Coefficients C_{mnp}

Time domain fields can be written as follows: $E_z(x,y,z,t) = \sum_{mnp} \mathrm{Re}(C_{mnp} e^{j\omega(mnp)t} u_{mnp}(x,y,z))$, using orthonormality.

In discrete form,

$$\sum_{m,n,p} |C_{mnp}| u_{mnp}(x,y,z) \cos\left(\omega\left(\omega_{mnp}\right)t + \Psi(mnp)\right)$$

The probe current can be expressed as:

$$E_z\ (x,y,\delta,t) = \int G(x,y) \frac{j\omega\mu I \mathrm{d}l(x^2+y^2)}{4\pi\left(x^2+y^2+\delta^2\right)^{3/2}} e^{(j\omega t - \frac{\omega}{c}\sqrt{x^2+y^2+\delta^2})} I(\omega) e^{j\omega t}\ \mathrm{d}\omega$$

where $G(x,y)$ are the constant terms associated with current.

$$\text{Resonator current} = \sum_p \left|C_{mnp}\right| \sqrt{\frac{2}{d}} \sin\left(\frac{p\pi\delta}{d}\right) \cos(\omega(mnp)t + \Psi(mnp)) u_{m,n}(x,y);$$

$$\text{Probe current } = \int G(x,y) \frac{j\omega\mu I \mathrm{d}l(x^2+y^2)}{4\pi\left(x^2+y^2+\delta^2\right)^{3/2}} I(\omega) e^{jkt} \mathrm{d}\omega$$

$$\left(e^{\left(j\omega t - \frac{\omega}{c}\sqrt{x^2+y^2+\delta^2} + \psi_{mnp}\right)}\right) (u_{m,n}(x,y) \mathrm{d}x\ \mathrm{d}y\ ;$$

The probe current must be equal to the resonator current due to principle of orthonormality.

$$\underline{E}(x,y,z,t) = \sum_{mnp=1}^{\infty} \mathrm{Re}\Big\{C(mnp)e^{j\omega(mnp)t}\,\underline{\psi}^{E}_{mnp}(x,y,z)\Big\} + \sum_{mnp=1}^{\infty} \mathrm{Re}\Big\{D(mnp)e^{j\omega(mnp)t}\,\underline{\bar{\phi}}^{E}_{mnp}(x,y,z)\Big\}$$

$$\underline{H}(x,y,z,t) = \sum_{mnp=1}^{\infty} \mathrm{Re}\Big\{C(mnp)e^{j\omega(mnp)t}\,\underline{\psi}^{H}_{mnp}(x,y,z)\Big\} + \sum_{mnp=1}^{\infty} \mathrm{Re}\Big\{D(mnp)e^{j\omega(mnp)t}\,\underline{\bar{\phi}}^{H}_{mnp}(x,y,z)\Big\}$$

$$\underline{E}_{\perp} = -\frac{\gamma}{h^2}\underline{\nabla}_{\perp}E_z - \frac{j\omega\mu}{h^2}\underline{\nabla}_{\perp}\,H_z x\,\hat{z}$$

From duality

$$\underline{H}_{\perp} = -\frac{\gamma}{h^2}\underline{\nabla}_{\perp}H_z - \frac{j\omega\epsilon}{h^2}\underline{\nabla}_{\perp}\,E_z x\,\hat{z}$$

From above two equations, we obtain E_x and E_y fields as given below:

$$E_x = \frac{m\pi x}{a}\alpha'(mnp)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) - \frac{n\pi}{b}\beta'(mnp)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right);$$
$$E_y = \frac{n\pi y}{b}\alpha'(mnp)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) - \frac{m\pi}{a}\beta'(mnp)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right);$$

and

$$E_z = \sum_{m,n,p}\mathrm{Re}[C(mnp)e^{j\omega(mnp)t}\sqrt{\frac{2}{d}}\sin\left(\frac{p\pi\delta}{d}\right)\,u_{m,n}(x,y)$$

$$E_z = \int \frac{j\omega\mu I\mathrm{d}l(x^2+y^2)}{4\pi\left(x^2+y^2+\delta^2\right)^{3/2}}e^{-j\frac{\omega}{c}\sqrt{x^2+y^2+\delta^2}}.\,I(\omega)e^{jkt}\mathrm{d}\omega$$

Here, $I\ (\omega)$ is the Fourier transform of source current, i.e., $I(t)$ is the probe current

$$I(\omega) = \frac{1}{2}\sum_{mnp}|I(mnp)|\left[\delta(\omega-\omega(mnp))e^{j\varnothing}(mnp) + e^{j\varnothing}(mnp)\delta(\omega+\omega(mnp))\right]$$

$$E_z(x,y,z,t) = \frac{\mu dl(x^2+y^2)}{4\pi\left(x^2+y^2+\delta^2\right)^{3/2}} \;\; \omega(mnp)|I(mnp)|\sin\left(\omega(mnp)\left(t - \frac{\sqrt{x^2+y^2+\delta^2}}{c} + \phi(mnp)\right)\right)$$
$$= \left|C_{mnp}\right|u_{mn}(x,y)\cos(\omega(mnp)t + \psi(mnp))\sqrt{\frac{2}{d}}\sin\left(\frac{p\pi\delta}{d}\right)$$

Hence, $C_{mnp} = \sqrt{\frac{\beta(mnp)^2+\alpha(mnp)^2}{\left[\frac{\sqrt{2}}{\sqrt{d}}\sin\left(\frac{p\pi\delta}{d}\right)\right]^2}}$; amplitude coefficient

$$\psi(mnp) = \tan^{-1}\left[\frac{\alpha_{mnp}\cos(\phi(mnp)) + \beta_{mnp}\sin(\phi(mnp))}{\alpha_{mnp}\sin(\phi(mnp)) - \beta_{mnp}\cos(\phi(mnp))}\right]; \text{Phase}$$

This completely solves the problem of RDRA resonant modes' coefficients in homogeneous medium.

5.3 RDRA Maxwell's Equation-Based Solution

Maxwell's equations with J electric and M magnetic sources:

$$\nabla \times \underline{E} = -j\omega\mu\underline{H} - \underline{M}; \qquad \text{(a)}$$

$$\nabla \times \underline{H} = \underline{J} + j\omega\epsilon\underline{E}; \quad \text{(b)}$$

$$\nabla \times E = \frac{\rho_v}{\varepsilon}; \quad \nabla \times H = \frac{\rho_m}{\mu};$$

where ρ_v is the electric charge density, and ρ_m is the magnetic charge density.

$$\text{For consistency}, -j\omega\varepsilon\nabla \times H - \nabla \times M = 0;$$
$$\nabla \times J + j\omega\varepsilon\nabla \times E = 0;$$
$$\text{i.e.} \quad \nabla \times M + j\omega\rho_m = 0, \nabla \times J + j\omega\rho = 0;$$

namely conservation of electric and magnetic charge:

$$\nabla\times(\nabla \times E) = -j\omega\mu\nabla \times H - \nabla \times M;$$

taking curl on both sides

$$\text{or} \quad \nabla(\nabla \times E) - \nabla^2 E = -j\omega\mu\,(J + j\omega\epsilon E) - \nabla \times M;$$

$$\text{or} \quad \left(\nabla^2 + k^2\right)\underline{E} = \frac{\nabla\rho}{\varepsilon} + j\omega\mu J + \underline{\nabla} \times \underline{H} = \underline{s}\,(\text{electric source}); \quad \text{(c)}$$

i.e., $\underline{E}$ satisfies the Helmholtz equation with source.

$$\text{Likewise,} \quad \nabla \times (\nabla \times \underline{H}) = \nabla \times J + j\omega\varepsilon\nabla \times E$$
$$\text{or} \quad \nabla(\nabla \times H) - \nabla^2 H = \nabla \times J + j\omega\varepsilon(-j\omega\mu H - M)$$
$$\text{or} \quad \left(\nabla^2 + k^2\right)\underline{H} = \frac{\underline{\nabla}\rho_m}{\mu} + j\omega\varepsilon M - \underline{\nabla} \times \underline{J} = \underline{f}$$
$$\text{(magnetic source due to probe);} \quad \text{(d)}$$

Hence, $\underline{H}$ also satisfies Helmholtz equation with source. Rectangular cavity resonator sidewalls are the perfect magnetic conductors (PMC) and top and bottom surfaces are the perfect electric conductors (PEC). Applying these boundary conditions, we get the following equation:

$$H_z = 0; \quad \text{where } x = 0, a \text{ or } y = 0, b$$

So,

$$H_z(x, y, z) = \sum_{m,n \geq 1} \phi_{mn}(z) u_{mn}(x, y|a, b) \tag{5.1}$$

where

$$u_{mn}(x, y|a, b) = \frac{2}{\sqrt{ab}} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \tag{5.2}$$

as we know,

$$H_z^{(0)}(x, y, z) = C_{mn} \frac{2}{\sqrt{ab}} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{p\pi z}{d}\right)$$

Let

$$f_z(x, y, z) = \sum_{mn \geq 1} f_{zmn}(z) u_{mn}[x, y|a, b] \tag{5.3}$$

$$\begin{aligned} \left(\nabla^2 + k^2\right)H_z &= \sum_{mn} \phi''_{mn}(z) + (k^2 - h^2[m, n|a, b]\phi_{mn}(z))u_{mn}[x, y|a, b] \\ &= f_z \Rightarrow \phi''_{mn}(z) + \left(k^2 - h^2[m, n|a, b]\phi_{mn}(z)\right) \\ &= f_{zmn}(z) \end{aligned} \tag{5.4}$$

where $k^2 = \omega^2\mu\epsilon$, and $h^2[x, y|a, b] = \pi^2\left(\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right)$;

$$H_z = 0; \quad \text{for} \quad z = 0, d; \quad \text{completely determines}\, ø_{mn}(z)$$

from (1),

Taking Laplace Transform of (5.4);

$$S^2\phi_{mn}(s) - S\phi_{mn}(0) - \phi'_{mn}(0) + \gamma_z^2[m,n]\widehat{ø}_{mn}(s) + \hat{f}_{zmn}(s)$$

So,

$$\widehat{ø}_{mn}(s) = \frac{\hat{f}_{zmn}(s)}{s^2 + \gamma_z^2[m,n]} + \frac{Sø_{mn}(0) - ø'_{mn}}{S^2 + \gamma_z^2[m,n]} \tag{5.5}$$

where $\gamma_z^2[m,n] = k^2 - h^2[m,n|a,b]$.

Thus,

$$\widehat{ø}_{mn}(z) = \frac{1}{\gamma_z[m,n]} \int_0^z \sin\big(\gamma_z[m,n](z-\xi)\big)\, f_{zmn}(\xi)\mathrm{d}\xi$$
$$+ C_1 \sin\big(\gamma_z[m,n]z\big) + C_2 \cos(\gamma_z[m,n]z)$$

$$ø_{mn}(0) = ø_{mn}(d) = 0 \quad \Rightarrow \quad C_2 = 0,$$

$$C_1 = \frac{-1}{\gamma_z[m,n]\sin(\gamma_z[m,n]d)} \int_0^d \sin(\gamma_z[m,n](d-\xi)f_{zmn}(\xi)\mathrm{d}\xi)$$

So,

$$ø_{mn}(z) = \frac{-1}{\gamma_z[m,n] \times \sin\big(\gamma_z[m,n]d\big)} \sin\big(\gamma_z[m,n]d\big) \int_0^z \sin\big(\gamma_z[m,n](z-\xi)\big) f_{zmn}(\xi)\mathrm{d}\xi$$
$$- \sin\big(\gamma_z[m,n]z\big) \int_0^d \sin\big(\gamma_z[m,n](d-\xi)\big) f_{zmn}(\xi)\mathrm{d}\xi$$

In the limit $k^2 \to \pi^2\left(\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2\right)$, we have, $\gamma_z^2[m,n] \to \left(\frac{\pi p}{d}\right)^2$ and we get,

$$ø_{mn}(z) \to \frac{d}{\pi p}\left\{ \int_0^z \sin\left(\frac{\pi p}{d}(z-\xi)\right) f_{zmn}(\xi)\mathrm{d}\xi - \sin\left(\frac{\pi p z}{d}\right) \lim_{\lambda \to \frac{\pi p}{d}} \frac{\int_0^d \sin\left(\frac{\pi p}{d}(d-\xi)\right) f_{zmn}(\xi)\mathrm{d}\xi}{\sin(\lambda d)} \right\} \tag{5.6}$$

The limit in ∞ showing resonance, when

$$k^2 = \pi^2\left(\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2\right)$$

Let

$$f_{mn}(z) = \sum_{r=1}^{\infty} f_{zmnr}\sqrt{\frac{2}{d}}\sin\left(\sin\frac{r\pi t}{d}\right) \tag{5.7}$$

Then

$$\begin{aligned}&\int_0^d \sin\left(\frac{\pi p}{d}(d-\xi)\right)f_{zmn}(\xi)\mathrm{d}\xi \\ &= \sqrt{\frac{2}{d}}\sum_r f_{zmnr}\int_0^d \sin\left(\frac{\pi p}{d}(d-\xi)\right)\sin\left(\frac{r\pi\xi}{d}\right)\mathrm{d}\xi\end{aligned} \tag{5.8}$$

$$\begin{aligned}&= (-1)^{p+1}\sqrt{\frac{2}{d}}\sum_r f_{zmnr}\int_0^d \sin(\lambda\xi)\sin\left(\frac{r\pi\xi}{d}\right)\mathrm{d}\xi \\ &= (-1)^{p+1}\sqrt{\frac{2}{d}}\sum_r f_{zmnr}\times\frac{1}{2}\int_0^d \cos\left(\left(\lambda-\frac{r\pi}{d}\right)\xi\right) - \cos\left(\left(\lambda+\frac{r\pi}{d}\right)\xi\right)\mathrm{d}\xi \\ &= \frac{1}{2}(-1)^{p+1}\sqrt{\frac{2}{d}}\sum_r f_{zmnr}\left[\frac{\sin\left(\left(\lambda-\frac{r\pi}{d}\right)d\right)}{\left(\lambda-\frac{r\pi}{d}\right)} - \frac{\sin\left(\left(\lambda+\frac{r\pi}{d}\right)d\right)}{\left(\lambda+\frac{r\pi}{d}\right)}\right]\end{aligned} \tag{5.9}$$

Here λ propagation parameter $= kz \approx \frac{\pi p}{d}$

Thus,

$$\begin{aligned}&\frac{1}{\sin(\lambda d)}\int_0^d \sin(\lambda(d-\xi))\ f_{zmn}(\xi)\mathrm{d}\xi \\ &= \frac{1}{2}(-1)^{p+1}\sqrt{\frac{2}{d}}\sum_r f_{zmnr}\ \frac{1}{\sin(\lambda d)}\left[\frac{\sin(\lambda d)(-1)^r}{\left(\lambda-\frac{r\pi}{d}\right)} - \frac{\sin(\lambda d)(-1)^r}{\left(\lambda+\frac{r\pi}{d}\right)}\right] \\ &= \frac{1}{\sqrt{2d}}\ (-1)^{p+1}\sum_r (-1)^r f_{zmnr}\left[\frac{1}{\left(\lambda-\frac{r\pi}{d}\right)} - \frac{1}{\left(\lambda+\frac{r\pi}{d}\right)}\right] \\ &= \frac{(-1)^{p+1}}{\sqrt{2d}}\ \sum_r (-1)^r f_{zmnr}\left[\frac{\frac{2r\pi}{d}}{\left(\lambda^2-\left(\frac{r\pi}{d}\right)^2\right)}\right]\end{aligned} \tag{5.10}$$

Writing $\lambda = \frac{p\pi}{d} + \delta\ (\delta \to 0),$ we get,

$$\begin{aligned} &\frac{1}{\sin(\lambda d)} \int_0^d \sin(\lambda(d-\xi)) \quad f_{zmn}(\xi) \mathrm{d}\xi \\ &\approx \frac{(-1)^{p+1}}{\sqrt{2\mathrm{d}}} (-1)^p \frac{f_{zmnp}}{\delta} \\ &= -\frac{1}{\delta\sqrt{2\mathrm{d}}} f_{zmnp} \qquad \text{(Dominant term)} \end{aligned} \tag{5.11}$$

Hence,

$$\varnothing_{mn}(z) \approx \frac{d}{\pi p} \left\{ \int_0^z \sin(\lambda(z-\xi)) \sum_{r\geq 1} f_{zmnr} \sqrt{\frac{2}{d}} \ \sin\left(\frac{r\pi\xi}{d}\right) \mathrm{d}\xi \ + \sin\left(\frac{p\pi z}{d}\right) \frac{1}{\delta\sqrt{2d}} f_{zmnp} \right\} \tag{5.12}$$

Now

$$\begin{aligned} &\int_0^z \sin(\lambda(z\ \xi)) \sin\left(\frac{r\pi\xi}{d}\right) \mathrm{d}\xi = \frac{1}{2} \int_0^z \left[\cos\left(\lambda z - \left(\lambda + \frac{r\pi}{d}\right)\xi\right) - \cos\left(\lambda z - \left(\lambda - \frac{r\pi}{d}\right)\xi\right)\right] \mathrm{d}\xi \\ &= \frac{1}{2}\left[\frac{\left(\sin\left(\frac{r\pi z}{d}\right) + \sin(\lambda z)\right)}{\left(\lambda + \frac{r\pi}{d}\right)} - \frac{\left(\sin\left(\frac{r\pi z}{d}\right) - \sin(\lambda z)\right)}{\left(\lambda - \frac{r\pi}{d}\right)}\right] \\ &= \frac{1}{2}\left[\frac{\left(\sin\left(\frac{r\pi z}{d}\right) + \sin\left(\frac{p\pi z}{d} + \delta z\right)\right)}{\frac{\pi(p+r)}{d} + \delta} - \frac{\left(\sin\left(\frac{r\pi z}{d}\right) - \sin\left(\frac{p\pi z}{d} + \delta z\right)\right)}{\left(\frac{\pi(p-r)}{d} + \delta\right)}\right] \end{aligned} \tag{5.13}$$

There is no dominant term here, i.e., if $\geq \mathrm{O}(\frac{1}{\delta})$, where O-order.

Hence, for $k^2 = \pi^2\left(\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right) + \left(\frac{\pi p}{d} + \delta\right)^2$

$$\begin{aligned} \varnothing_{mn}(z) &\approx \frac{d}{\pi p}\left(\frac{1}{\delta\sqrt{2d}} f_{zmnp} \sin\left(\frac{\pi p z}{d}\right)\right) \\ &= \frac{1}{\pi p \delta}\sqrt{\frac{d}{2}} f_{zmnp} \sin\left(\frac{p\pi z}{d}\right) \end{aligned} \tag{5.14}$$

Likewise, propagation in *x* direction can be taken as:

$$\left(\nabla^2 + (k)^2\right) H_x = f_x$$
$$H_x(x, y, z) = \sum_{m,n} ø_{xmn} \tilde{u}_{mn}(y, z|b, d)$$
$$\text{Let, } f_x(\vec{x}, y, z) = \sum_{m,n} f_{xmn}(\mathrm{x}) \tilde{u}_{mn}(y, z|b, d)$$

where $\tilde{u}_{mn}(y, z|b, d) = \frac{2}{\sqrt{bd}} \sin\left(\frac{m\pi y}{b}\right) \cos\left(\frac{n\pi y}{b}\right)$; orthogonal 2D half wave Fourier basis function.

Then,

$$ø''_{xmn}(x) + \left(k^2 - h^2[m, n|b, d]\right) ø_{xmn}(x) = f_{xmn}(x)$$

Hence, general solution can be given as follows:

$$ø_{xmn}(x) = \frac{1}{\gamma_x[m, n]} \int_0^x \sin(\gamma_x[m, n](x - \xi)) f_{xmn}(\xi) + C_1 \cos(\gamma_x[m, n]x) C_2 \sin(\gamma_x[m, n]x) \quad (5.15)$$

Likewise,

$$H_y(x, y, z) = \sum_{m,n} ø_{ymn}(y) u_{mn}(x, z|a, d)$$
$$f_y(x, y, z) = \sum_{m,n} f_{ymn}(y) u_{mn}(x, z|a, d)$$

with

$$ø''_{ymn}(y) + \left(k^2 - h^2[m, n|a, d]\right) \varphi_{ymn}(y) = f_{ymn}(y)$$

and with the boundary conditions:

$$E_x = 0 \ \text{ where } \ x = 0, a \ \text{ or } \ z = 0, d;$$
$$E_y = 0 \ \text{ where } \ y = 0, b \ \text{ or } \ z = 0, d;$$

The general solution for $ø_{ymn}(y)$ is given as follows:

$$ø_{ymn}(y) = \frac{1}{\gamma_y[m, n]} \int_0^y \sin\left(\gamma_y[m, n](z - \xi)\right) f_{ymn}(\xi) \mathrm{d}\xi + D_1 \cos(\gamma_y[m, n]y) + D_2 \sin(\gamma_y[m, n]y) \quad (5.16)$$

Here, $\gamma_x[m,n] = (k^2 - h^2[m,n|b,d])^{1/2}$

$$\gamma_y[m,n] = (k^2 - h^2[m,n|a,d])^{1/2}$$

The equation

$$\nabla \times H = J + j\omega\epsilon E$$

gives

$$j\omega\epsilon E_x = H_{z,y} - H_{y,z} - J$$
$$j\omega\epsilon E_y = H_{x,z} - H_{z,x} - J$$

We assume that J on the walls is zero. Then, the boundary conditions yields

$$H_{z,y} - H_{y,z} = 0; \quad \text{where} \quad x = 0, a;$$
$$H_{x,z} - H_{z,x} = 0; \quad \text{where} \quad y = 0, b$$

Recall that H_z has been completely determined.

5.4 RDRA Inhomogeneous Permittivity and Permeability

$$\varepsilon = \varepsilon_0\big(1 + \delta_p X_e(x,y)\big) \tag{5.17}$$

$$\mu = \mu_0\big(1 + \delta_p X_m(x,y)\big) \tag{5.18}$$

At some known frequency ω and δ_p as perturbation parameter, the solution has been worked out using perturbation techniques to determine shift in the frequency.

As per Maxwell's equation,

$$\nabla \times \underline{E} = -j\omega\mu\underline{H}$$
$$\nabla \times \underline{H} = j\omega\varepsilon\underline{E}$$

where boundary conditions are given as follows:

$$0 \le x \le a = W$$
$$0 \le y \le b = L$$
$$0 \le z \le d = h$$

Due to duality $E \rightarrow H, H \rightarrow -E$, and $\mu \leftrightarrow \varepsilon$.

Sidewalls have been taken as PMC (magnetic conductor walls) and top and bottom as PEC (perfect electrical conductor).

$$\begin{aligned} &H_{\tan} = 0; \ \ \text{on side walls} \\ &H_x, H_y = 0; \quad \text{when} \ \ y = 0, b \\ &H_y, H_x = 0; \quad \text{when} \ \ x = 0, d \\ &E_x, E_y = 0; \quad \text{when} \ \ z = 0, d \end{aligned}$$

$$\underline{E} = \underline{E}(x, y)e^{-\gamma z} \tag{5.19}$$

$$\underline{H} = \underline{H}(x, y)e^{-\gamma z} \tag{5.20}$$

Propagation constant is given as:
$h_0^2 = \gamma^2 + \omega^2\mu_0\epsilon_0, = \gamma^2 + k^2$; when $k^2 = \omega^2\mu_0\epsilon_0$

$$E_X = \frac{\gamma}{-h^2}E_{z'x} - \frac{j\omega\mu}{h^2}H_{z'y} \tag{5.21}$$

$$E_Y = \frac{\gamma}{-h^2}E_{z'y} - \frac{j\omega\mu}{h^2}H_{z'x} \tag{5.22}$$

$$H_X = \frac{\gamma}{-h^2}H_{z'x} - \frac{j\omega\epsilon}{h^2}E_{z'y} \tag{5.23}$$

$$H_Y = \frac{\gamma}{-h^2}H_{z'y} - \frac{j\omega\epsilon}{h^2}E_{z'x} \tag{5.24}$$

Top and bottom walls are perfect electric conditions so that

$$E_x, E_y = 0; \quad \text{when} \quad z = 0, d$$

$$\underline{E} \sim \underline{E}(x, y)\exp(-\gamma z), \quad \sim \underline{H} - \underline{H}(x, y)\exp(-\gamma z)$$

$$\underline{E} = \underline{E}_\perp + E_z\hat{z}, \quad \underline{H} = \underline{H}_\perp + H_z\hat{z}.$$

$$\underline{\nabla} = \underline{\nabla}_\perp + \hat{z}\frac{\partial}{\partial z} = \underline{\nabla}_\perp - \gamma\hat{z}$$

$$\underline{\nabla}_\perp E_z \times \hat{z} - \gamma\hat{z} \times E_\perp = -j\omega\mu(\underline{H}_\perp) \tag{5.25}$$

$$\underline{E} = \underline{E}_\perp + E_z\hat{z}, \quad \underline{H} = \underline{H}_\perp + H_z\hat{z}. \tag{5.26}$$

$$\underline{\nabla}_\perp H_z \times \hat{z} - \gamma\hat{z} \times \underline{H}_\perp = j\omega\epsilon\underline{E}_\perp, \tag{5.27}$$

$$\underline{\nabla}_{\perp} \times \underline{H}_{\perp} = j\,w\,\epsilon\,E_z\hat{z} \tag{5.28}$$

Taking $\hat{z} \times$ of (5.25) gives

$$\underline{\nabla}_{\perp}E_z + \gamma\underline{\mathrm{E}}_{\perp} = -j\omega\mu\hat{z} \times \underline{\mathrm{H}}_{\perp} \tag{5.29}$$

Equations (5.11) and (5.13) can be changed as:

$$\begin{bmatrix} j\omega\,\epsilon & \gamma \\ -\gamma & j\omega\mu \end{bmatrix}\begin{bmatrix} E_{\perp} \\ \hat{Z} \times \underline{H}_{\perp} \end{bmatrix} = \begin{bmatrix} \underline{\nabla}_{\perp}H_z \times \hat{z} \\ \underline{\nabla}_{\perp} \quad E_z \end{bmatrix} \tag{5.30}$$

$$\begin{bmatrix} E_{\perp} \\ \hat{z} \times \underline{H}_{\perp} \end{bmatrix} = \frac{\begin{bmatrix} -j\omega\mu & -\gamma \\ \gamma & j\omega\epsilon \end{bmatrix}\begin{bmatrix} \underline{\nabla}_{\perp}H_z \times \hat{Z} \\ \underline{\nabla}_{\perp}E_z \end{bmatrix}}{\omega^2\mu\epsilon + \gamma^2}$$

$$\underline{E}_{\perp} = \frac{-\gamma}{h^2}\underline{\nabla}_{\perp}E_z - \frac{j\omega\mu}{h^2}\underline{\nabla}_{\perp}H_z \times \hat{z} \tag{5.31}$$

$$\hat{z} \times \underline{H}_{\perp} = \frac{\gamma}{h^2}\underline{\nabla}_{\perp}H_z \times \hat{z} + \frac{j\omega\mu}{h^2}\underline{\nabla}_{\perp}E_z \tag{5.32}$$

$$h^2 = h^2(x,y) = \gamma^2 + \omega^2\mu(x,y)\varepsilon(x,y) = h_0^2 + k_0^2\delta\chi(x,y),$$

where $h_0^2 = \gamma^2 + \omega^2\mu_0\varepsilon_0, k^2 = \omega^2\mu_0\varepsilon_0 = \gamma^2 + k^2$

$$\chi(x,y) = \chi_e(x,y) + \chi_m(x,y) + \delta \times \chi_e(x,y)\chi_m(x,y)$$

Taking $\hat{z}$ of (5.32) gives

$$-\left(\frac{\gamma}{h^2}\right)\nabla_{\perp}H_z + \frac{j\omega\varepsilon}{h^2}\hat{z} \times \nabla_{\perp}E_z = \underline{H}_{\perp} \tag{5.33}$$

from Eqs. (5.31) and (5.33),

$$E_X = \frac{\gamma}{-h^2}E_{z,x} - \frac{j\omega\mu}{h^2}H_{z,y}$$
$$E_Y = \frac{\gamma}{-h^2}E_{z,y} + \frac{j\omega\mu}{h^2}H_{z,x}$$
$$H_X = \frac{\gamma}{-h^2}H_{z,x} + \frac{j\omega\epsilon}{h^2}E_{z,y}$$
$$H_y = \frac{\gamma}{-h^2}H_{z,y} - \frac{j\omega\epsilon}{h^2}E_{z,x}$$

From Eqs. (5.25) and (5.26),

$$\nabla_{\perp} \times \left(\frac{\gamma}{h^2}\nabla_{\perp}E_z + \frac{j\omega\mu}{h^2}\nabla_{\perp}H_z \times \hat{Z}\right) - j\omega\mu H_z\hat{z} = 0 \tag{5.34}$$

$$\text{or} \quad \left(\hat{z} \times \underline{\nabla}_{\perp}\left(\frac{\gamma}{h^2}\right), \underline{\nabla}_{\perp}E_z\right) - \left(\underline{\nabla}_{\perp}, \frac{j\omega\mu}{h^2}\underline{\nabla}_{\perp}H_z\right) - j\omega\mu H_z = 0$$

or

$$\underline{\nabla}_{\perp}^2 H_z + h^2 H_z + \left(\underline{\nabla}_{\perp}\left(\frac{j\omega\mu}{h^2}\right), \underline{\nabla}_{\perp}H_z\right) \times \frac{h^2}{j\omega\mu} - \frac{h^2}{j\omega\mu}\left(\hat{Z} \times \underline{\nabla}_{\perp}\left(\frac{\gamma}{h^2}\right), \underline{\nabla}_{\perp}E_z\right) = 0$$

or

$$\left(\underline{\nabla}_{\perp}^2 + h_0^2\right)H_z + \delta\left\{k^2 X H_z \delta^{-1}\log\left(\frac{\mu}{h^2}\right) + (\underline{\nabla}_{\perp}, \underline{\nabla}_{\perp}H_z) + \frac{\gamma k^2}{j\omega\mu h^2}(\underline{\nabla}_{\perp}X_1\underline{\nabla}_{\perp}E_z)\right\} = 0 \tag{5.35}$$

Now, we retain only $0\,(\delta)$ terms.

$$\chi \sim \chi_e + \chi_m$$
$$\frac{\chi_m}{h^2} \sim \frac{\chi_m}{h_0^2}$$
$$\frac{\gamma k^2}{j\omega\mu h^2} \sim \frac{\gamma k^2}{j\omega\mu h_0^2}$$

and (5.35) becomes

$$\begin{aligned}\left(\underline{\nabla}_{\perp}^2 + h_0^2\right)H_z + \delta\big\{k^2\,\chi\,H_z + (\underline{\nabla}_{\perp}\chi_m - \frac{\epsilon^2\underline{\nabla}_{\perp}\chi}{h_0^2}, \underline{\nabla}_{\perp}H_z) \\ + \frac{\gamma k^2}{j\omega\mu_0 h_0^2}(\underline{\nabla}_{\perp}, \chi\underline{\nabla}_{\perp}E_z)\big\} = 0\end{aligned} \tag{5.36}$$

By duality

$$E \to H, \quad H \to -E, \quad \chi_e \leftrightarrow \chi_m$$

$$\epsilon_0 \leftrightarrow \mu_0, \quad \chi \leftrightarrow \chi$$

we get from (5.36)

$$\left(\underline{\nabla}_{\perp}^{2}+h_{0}^{2}\right)E_{z}+\delta\left\{k^{2}\chi E_{z}+\left(\underline{\nabla}_{\perp}\chi_{e}-\frac{k^{2}}{h_{0}^{2}}\underline{\nabla}_{\perp}\chi,E_{z}\right)+\left(\underline{\nabla}_{\perp}\chi_{e}-\frac{k^{2}}{h_{0}^{2}}\underline{\nabla}_{\perp}\chi,E_{z}\right)-\frac{\gamma k^{2}}{j\omega\epsilon_{0}h_{0}^{2}}\left(\underline{\nabla}_{\perp}\chi,\underline{\nabla}_{\perp}H_{z}\right)\right\}=0 \tag{5.37}$$

Boundary conditions are given as:

$$H_z = 0,\ \ x = 0, a\ \text{ and }\ Y = 0, b \quad H_z = 0,\ \ Z = 0, d$$

$$H_X = 0,\ \ Y = 0, b\ \ H_Y = 0,\ \ x = 0, a\ \ E_X = 0,\ x = 0, a$$

$$E_X = E_Y = 0,\ \ Z = 0, d\ \ E_Y = 0\ \ y = 0, b$$

Equations (5.28) and (5.29) are the own fundamental equations, let $h_0^2 = \lambda$.

Let

$$\lambda = \lambda_{m,n}^{(0)} + \delta \times \lambda^{(1)} + 0\left(\delta^2\right)$$

$$E_z = E_z^{(0)} + \delta E_z^{(1)} + 0\left(\delta^2\right) \tag{5.38}$$

$$H_z = H_z^{(0)} + \delta H_z^{(1)} + 0\left(\delta^2\right) \tag{5.39}$$

if there is non-homogeneity $\rightarrow$

$$\lambda_{n,m}^{(0)} = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

$$\left(\underline{\nabla}_{\perp}^{2} + \lambda_{n,m}^{(0)}\right)E_z^{(0)} = 0$$

$$\left(\underline{\nabla}_{\perp}^{2} + \lambda_{n,m}^{(0)}\right)H_z^{(0)} = 0 \tag{5.40}$$

By Eqs. (5.36) and (5.37) $\left(0\left(\delta^0\right)\right)$

$$H_x^{(0)} = \frac{-\gamma}{\lambda_{m,n}^{0}}\ H_{z,x}^{(0)} + \frac{j\omega\epsilon 0}{\lambda_{m,n}^{0}}\quad E_{z,y}^{(0)}$$

Since

$$H_Z^{(0)} = 0,\ \ \text{when}\ \ Y = 0, b \quad H_{z,x}^{(0)} = 0,\ \ \text{when}\ \ y = 0, b$$

Then

$$H_x^{(0)} = 0, \quad \text{when } y = 0, b \quad \rightarrow \quad E_{z,Y}^{(0)} = 0, \quad \text{when } Y = 0, b$$

Likewise

$$E_{z,X}^{(0)} = 0, \quad \text{when } X = 0, a$$

Thus,

$$H_z^{(0)} = C_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \times \frac{2}{\sqrt{ab}} \tag{5.41}$$

$$E_z^{(0)} = D_{mn} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \times \frac{2}{\sqrt{ab}} \tag{5.42}$$

If z-dependent is taken into account, then $H_z^{(0)}, E_z^{(0)}$ must be multiplied by $\exp(\pm\gamma z)$ according to Eq. (5.34),

$$E_x^{(0)} = \mp \frac{-\gamma}{\lambda_{m,n}^0} - \frac{j\omega\mu_0}{\lambda_{m,n}^0} \quad H_{z,Y}^{(0)}$$

and $E_x^{(0)} = 0$, when $z = 0, d$, and $H_z^{(0)} = 0$, when $z = 0, d$.

We get $E_{z,X}^{(0)} = 0$, when $z = 0, d$ then,

$$E_z^{(0)}(x, y, z) = D_{mn} \frac{2}{\sqrt{ab}} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \sin\left(\frac{p\pi\delta}{d}\right) \tag{5.43}$$

$$\gamma = \frac{jp\pi}{d}, \quad p = 1, 2, 3.$$

Since $H_z^{(0)} = 0, \quad \text{when} \;\; z = 0, d,$

$$H_z^{(0)}(x, y, z) = \mathcal{C}_{mn} \frac{2}{\sqrt{ab}} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{p\pi z}{d}\right) \tag{5.44}$$

Frequency of oscillations:

$$\begin{aligned} &\omega = \omega_{mnp} \\ &\gamma^2 + \omega^2 \mu_0 \epsilon_0 = \lambda_{mn}^{(0)} \\ &\text{or} \;\; \omega = \frac{\pi}{\sqrt{\mu_0 \epsilon_0}} \left(\left(\frac{m}{a}\right)^2 + \left(\frac{n}{a}\right)^2 \left(\frac{p}{d}\right)^2 \right)^{1/2}. \end{aligned}$$

5.5 RDRA with Probe Current Excitation

The rectangular cavity has dimensions *a*, *b*, and *d* as shown in Fig. 5.2. Sidewalls are taken as magnetic conductors (PMC), and top and bottom surfaces are as PEC; theoretical fields (modes) solution has been worked under boundary conditions with a square-type feed probe for excitation.

$E_X, E_y = 0$, top and bottom plane being electric walls.

$E_X, E_y = 0$, sidewalls being magnetic walls.

$$H_z(x,y,z,t) = \sum_{mnp} C(m,n,p)\frac{2}{\sqrt{ab}}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi z}{d}\right)\{\cos(\omega(m,n,p)t + \phi(m,n,p))$$

where *m*, *n*, and *p* are the integers (half wave variations in particular direction, i.e., *x*, *y*, *z* directions, respectively); *a*, *b*, and *d* are the dimensions (width, length, and height) of the RDRA, $C(m,n,p)$ and $\phi(m,n,p)$ are the magnitude and phase coefficients of H_z and $D(m,n,p)$ and $\psi(m,n,p)$ for E_z.

Let, orthogonal 2D half wave Fourier basis function $= \frac{2}{\sqrt{ab}}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) = u_{mn}(x,y)$ for convenience.

$$E_z(x,y,z,t) = \sum_{mnp} d(m,n,p)\frac{2}{\sqrt{ab}}\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{d}\right)\{\cos\omega(m,n,p)t + \psi(m,n,p)\}$$

Let, $\frac{2}{\sqrt{ab}}\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) = v_{mn}(x,y)$ for convenience = orthogonal 2D half wave Fourier basis function.

From Lorentz Gauge conditions, $E_z = -j\omega A_z - \frac{\partial\bar{\phi}}{\partial z}$

Therefore, the magnetic vector potential can be given as below in discrete form after taking Fourier transform of A_z.

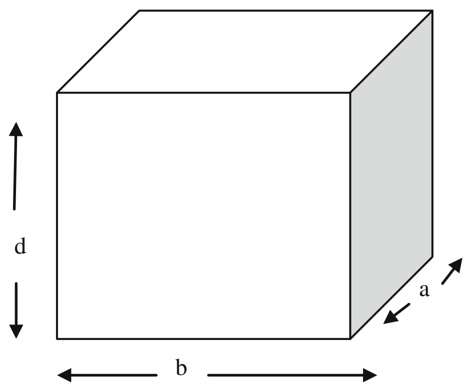


Fig. 5.2 RDRA with square feed probe inserted in $a \times b$ plane

$\hat{A}_z(x, y, z, \omega) = \frac{\mu}{4\pi}\hat{I}(\omega)\frac{\delta l e^{-j\frac{\omega r}{c}}}{r}$; where $\delta l =$ probe length

Div $\underline{\hat{A}} = \frac{\partial \hat{A}_Z}{\partial z}$; need to be computed.

Now, if we insert this probe at the location defined below into the cavity to find the fields pattern, we get:

$$\frac{l_2}{2} < |x - \frac{a}{2}| < \frac{a}{2}, \quad \frac{l_2}{2} < |y - \frac{b}{2}| < \frac{b}{2}$$

Then, the magnetic vector potential will be

$$\frac{\partial \hat{A}_z}{\partial z} = \frac{\mu \hat{I} \delta l}{4 \times \pi} \left[\frac{\partial}{\partial z}\frac{e^{-jkr}}{r}\right) = \frac{\mu \hat{i} \delta l}{4 \times \pi} \left(\frac{\cos\theta}{r^2} - \frac{jk\cos\theta}{r}\right)e^{-jkr} = -\frac{j\omega}{c^2}\underline{\hat{\phi}}$$

and scalar potential will be

$$\underline{\hat{\phi}} = \frac{\mu \hat{I} \delta l j c^2}{4 \times \pi\omega}\cos\theta\left(\frac{1}{r^2} - \frac{jk}{r}\right)e^{-jkr} = \frac{\mu \hat{i} \delta l c^2}{4 \times \pi\omega}\left(\frac{z}{r^3} - \frac{jkz}{r^2}\right)e^{-jkr}$$

Differentiating $\underline{\hat{\phi}}$ w.r.t. z

$$\frac{\partial \underline{\hat{\phi}}}{\partial z} = \frac{\mu \hat{i} \delta l c^2}{4\pi\omega}\left(\frac{1}{r^3} + \frac{3z^2}{r^5} - \frac{jk}{r^2} - \frac{2jkz^2}{r^4} + \left(\frac{z}{r^3} - \frac{jkz}{r^2}\right)\left(\frac{-jkz}{r}\right)\right)e^{-jkr} \tag{5.45}$$

when, $\hat{E}_z = -j\omega\hat{A}_z - \frac{\partial\hat{\phi}}{\partial z}$, substituting $\frac{\partial\hat{\phi}}{\partial z}$ in $\hat{E}_z$,

$$\hat{E}_z = \left[\frac{-j\omega\mu\hat{I}\delta l}{4\pi r} - \frac{\mu\hat{I}\delta l c^2}{4\pi\omega}\left(\frac{1}{r^3} + \frac{3z^2}{r^5} - \frac{jk}{r^2} - \frac{2jkz^2}{r^4} - \frac{jkz^2}{r^4} - \frac{k^2z^2}{r^3}\right)\right]e^{-jkr} \tag{5.46}$$

If we take $\theta = \frac{\pi}{2}, \quad z = 0$

If we take $\theta = \frac{\pi}{2}, \quad z = 0$

$$\hat{E}_z = \frac{\omega c\mu\hat{I}\delta l}{4\pi k}\left[\frac{-jk^2}{r} - \left(\frac{1}{r^3} + \frac{3\ \delta^2}{r^5} - \frac{jk}{r^2} - \frac{2jk\delta^2}{r^4} - \frac{jk\delta^2}{r^4} - \frac{k^2\delta^2}{r^3}\right)\right]$$

Also,

$$r = \sqrt{x^2 + y^2 + \delta^2}$$

$\frac{k^2r^2}{r} = \frac{1}{r}$, for $r = \lambda = 2\pi/k$.

Coulomb component of electric field is dominant in this inductive zone $r^2 \approx \delta^2$ given that $\delta \ll r$.

Minimum of r $\approx l_2$ and Maximum of $r = (a, b)$;

$$kr \ll 1$$

Hence,

$$\hat{E}_z \approx \frac{\mu c \hat{I} \delta l}{4\pi k r^3} \quad \text{and} \quad \approx \frac{\mu c^2 \delta l}{4\pi r^3 j\omega} \hat{I}(\omega);$$

$$E_z(t, x, y\delta) \approx \frac{\delta l}{4\pi\epsilon\left(\sqrt{x^2+y^2}\right)^3} \int_0^t I(\tau)\mathrm{d}\tau \approx \frac{Q(t)\delta l}{4\pi\epsilon r^3} \tag{5.47}$$

Charge flowing through the resonator is $Q(t) = \int_0^t I(\tau)\mathrm{d}\tau$ or equivalently $= \frac{\hat{I}(\omega)}{j\omega} = \hat{Q}(\omega)$,

Here,

$$\frac{Q(t)\delta l}{4\pi\epsilon(x^2+y^2)^{\frac{3}{2}}} \approx \sum_{mnp} D(m,n,p) v_{mn}(x,y) \sin\left(\frac{\pi p \delta}{d}\right) \cos(\omega(m,n,p)t + \psi(m,n,p))$$

and

$$\frac{Q(t)}{4\pi\epsilon(x^2+y^2)^{\frac{3}{2}}} = \sum_{mnp} C(m,n,p) v_{mn}(x,y) \frac{\pi p}{d} \cos(\omega(m,n,p)t + \gamma(m,n,p))$$

For complete solution, we need to compute $D(m,n,p)$ and $\psi(m,n,p)$ coefficients for H_z fields and $C(m,n,p)$ and $\gamma(m,n,p)$ for E_z fields. The $D(m,n,p)$ and $C(m,n,p)$ are the desired resonant modes. For region,

$$\frac{l_2}{2} < |x - \frac{a}{2}| < \frac{a}{2}$$

$$\frac{l_2}{2} < |y - \frac{b}{2}| < \frac{b}{2}$$

$$\frac{Q(t)}{4\pi\epsilon} \int\int \frac{v_{mn}(x,y)}{(x^2+y^2)^{\frac{3}{2}}} \mathrm{d}x\mathrm{d}y = \sum_p \frac{D(m,n,p)\pi p}{d} \cos(\omega(m,n,p)t + \psi(m,n,p)) \tag{5.48}$$

$$\left\{\frac{a+l_2}{2} < x < a;\ 0 < x < \frac{a-l_2}{2}\right\} \cap \left\{\frac{b+l_2}{2} < y < b \cup 0 < y \frac{b-l_2}{2}\right\}$$

$$\langle \cos(\omega t)\cos(\omega t)\rangle = \frac{1}{2}$$
$$\langle \sin(\omega t)\cos(\omega t)\rangle = 0$$

$$\frac{D(m,n,p)\pi p}{2d}\cos(\psi(m,n,p)) = \frac{1}{4\pi\epsilon}\langle Q(t)\cos(\omega(m,n,p)t)\rangle \int \frac{v_{mn}(x,y)}{(x^2+y^2)^{\frac{3}{2}}}\mathrm{d}x\mathrm{d}y$$
$$-\frac{D(m,n,p)\pi p}{2d}\sin(\psi(m,n,p)) = \frac{1}{4\pi\epsilon}\langle Q(t)\sin(\omega(m,n,p)t)\rangle \int \frac{v_{mn}(x,y)\mathrm{d}x\mathrm{d}y}{(x^2+y^2)^{\frac{3}{2}}}$$

Hence,

$$D(m,n,p) = 2\mathrm{d}/\pi p\sin(\psi(m,n,p))\left(\frac{1}{4\pi\epsilon}\langle Q(t)\sin(\omega(m,n,p)t)\rangle \int \frac{v_{mn}(x,y)\mathrm{d}x\mathrm{d}y}{(x^2+y^2)^{\frac{3}{2}}}\right). \tag{5.49}$$

5.6 RDRA Resonant Modes Coefficients in Homogeneous Medium

The basic Maxwell's theory can be applied with boundary conditions to express RDRA resonant fields as superposition of these characteristics frequencies. RDRA is shown in Fig. 5.3. u_{mn} depends on input excitation = orthogonal Fourier basis function, h_{mn} resonant mode (cut off frequency), k propagation constant. The generation of modes or characteristics frequencies $\omega(mnp)$ due to electromagnetic fields oscillations inside the cavity resonator has been described. Orthogonal Fourier basis function $u_{m,n}(x,y) = \frac{2}{\sqrt{ab}}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$; $\omega(mnp)$ is the characteristic frequency and $\psi(mnp)$ is the phase of current applied. The rectangular cavity resonator is excited at the center with an antenna probe carrying current $i(t)$ of some known

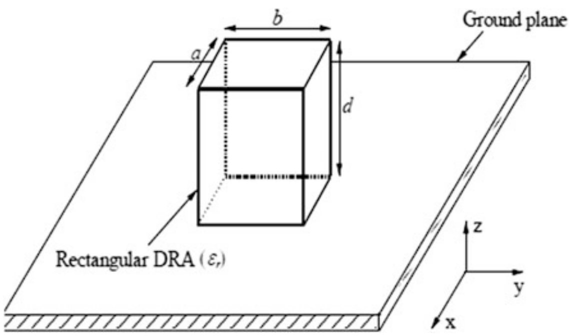


Fig. 5.3 RDRA with ground plane

frequency $\omega(mnp)$. This generates the field E_z inside the cavity of the form given below:

$$k^2 + \gamma_{mn}^2 = h_{mn}^2$$

hence,

$$k^2 = h_{mn}^2 + \frac{\pi^2 p^2}{d^2}$$

$$E_z(x, y, z, t) = \sum_{m,n,p} \mathrm{Re} \int C_{mnp}\, e^{j\omega(mnp)t}\, u_{mnp}(x, y, z);$$

or $\sum_{m,n,p} \left|C_{mnp}\right| u_{mnp}(x, y, z) \cos(\omega(mnp)t + \psi(mnp));$

$$E_z\ (x, y, \delta, t) = \int G(x, y) \frac{j\omega\mu I \mathrm{d}l(x^2 + y^2)}{4\pi\left(x^2 + y^2 + \delta^2\right)^{3/2}} e^{(j\omega t - \frac{\omega}{c}\sqrt{x^2+y^2+\delta^2})} \times I(\omega) e^{j\omega t} \mathrm{d}\omega$$

where $G(x,y)$ are the constant terms associated with the current.

Equating RDRA probe current fields with the antenna-radiated current fields at $z = \delta$;

Radiated currents:

$$= \sum_p \left|C_{mnp}\right| \sqrt{\frac{2}{d}} \sin\left(\frac{p\pi\delta}{d}\right) \cos\left(\omega(mnp)t + \phi(mnp)\ u_{m,n}(x, y)\right);$$

Due to orthonormality, probe currents will be equal to radiated fields.

Probe currents:

$$= \int G(x, y) \frac{j\omega\mu I \mathrm{d}l(x^2 + y^2)}{4\pi\left(x^2 + y^2 + \delta^2\right)^{3/2}} I(\omega) e^{jkt} \mathrm{d}\omega \left(e^{\left(j\omega t - \frac{\omega}{c}\sqrt{x^2+y^2+\delta^2} + \psi_{mnp}\right)} u_{m,n}(x, y)\right) \mathrm{d}x\mathrm{d}y$$

It is clear that these two expressions have to be equal due to energy conservation.

The probe current can be defined as:

$$I(\omega) = \frac{1}{2} \sum_{mnp} |I(mnp)| \left[\delta(\omega - \omega(mnp) e^{j\varnothing(mnp)}) + e^{j\varnothing(mnp)} \delta(\omega - \omega(mnp))\right]$$

The antenna probe current must contain only the resonator characteristics frequencies $\omega(mnp)$. The radiated and input currents are equated as:

$$\sum_{p} = |C_{mnp}|\sqrt{\frac{2}{d}}\sin\left(\frac{p\pi\delta}{d}\right)\cos((\omega(mnp)t + \phi(mnp))u_{m,n}(x,y))$$
$$= \int G(x,y)\frac{j\omega\mu I\mathrm{d}l(x^2+y^2)}{4\pi\left(x^2+y^2+\delta^2\right)^{3/2}}.I(\omega)e^{jkt}\mathrm{d}\omega(e^{\left(j\omega t-\frac{\omega}{c}\sqrt{x^2+y^2+\delta^2}+\psi_{mnp}\right)}u_{m,n}(x,y)\mathrm{d}x\mathrm{d}y;$$

probe current = radiated current; thus C_{mnp} can be completely determined.

Hence, we can conclude that modes generation is due to the dipole moment in cavity resonator, mostly depend on size, dimensions of device, excitation type, coupling, and point of excitation.

5.7 RDRA Modes with Different Feed Position

Let us take $z = \delta$, i.e., very small probe length inserted into RDRA resonator at point of insertion $\left({}^{a}/_{2}, {}^{b}/_{2}, \delta\right)$ or $\left(x - {}^{a}/_{2}, y - {}^{b}/_{2}, \delta\right)$; where δ—length of insertion.

(H_x, H_y, E_x, E_y), transverse fields; (E_z and H_z) longitudinal fields

$$E_z = \sum_{mnp} u_{mnp}(x,y,z)R_e\left(C_{mnp}e^{j\omega_{mnp}t}\right)$$

where $u_{mnp}(x,y,z) = \frac{2^{3/2}}{\sqrt{abd}}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi z}{d}\right) = E_z$, (when top and bottom walls are PMC, rest all four walls are PEC).

Appling boundary conditions on transparent sidewalls (on all four sides of RDRA or resonator) and top and bottom planes as electrical walls, we get $H_z = 0$, for magnetic walls; and $E_z = 0$, for electrical walls; fields to be computed are

(E_z, H_z)—longitudinal fields;

$$E_z(x,y,z,t) = \sum_{mnp} \mathrm{Re}\int C_{mnp}e^{j\omega(mnp)t}u_{mnp}(x,y,z)$$

At $z = 0$; E_z, E_x, E_y all will be zero

$E_z = \sum_{mnp}\mathrm{Re}\left[C_{mnp}e^{j\omega(mnp)t}\right]\sqrt{\frac{2}{d}}\sin\left(\frac{p\pi\delta}{d}\right)u_{mn}(x,y)$; this is the E_z field in the resonator at $z = \delta$. It must be equated to the corresponding field generated by the antenna probe, i.e., for the above two expressions to be equal, the antenna probe currents must contain frequencies only from the set $\{\omega(mnp)\}$.

Where $u_{mn}(x,y) = \frac{2}{\sqrt{ab}}\sin\left(\frac{m\pi}{a}\right)\sin\left(\frac{n\pi}{b}\right)$

E_z will exist little above from $z = 0$ plane; $E_z = \int \frac{j\omega\mu I \mathrm{d}l e^{-jk\sqrt{x^2+y^2+\delta^2}}}{4\pi\sqrt{x^2+y^2+\delta^2}} I(\omega)\mathrm{d}\omega$; where $I(\omega)$ is the Fourier transform of $i(t)$

$$E_z = -j\omega A_z - \frac{\partial \phi}{\partial z}$$

$$\mathrm{div} \cdot A = \frac{\mu I \mathrm{d}l}{4\pi}(-jk\cos\theta)e^{-jkr} \tag{5.50}$$

$$\frac{kc^2}{\omega} = -\frac{j\omega}{c^2}\phi$$

Hence, scalar potential $\phi = \frac{\mu I \cos}{4\pi r} e^{jkr}$

$$\begin{aligned} \frac{\partial \phi}{\partial z} &= \frac{\mu I \cos\theta}{4\pi r}(-jk\cos\theta)e^{-jkr} \\ &= \frac{jk\mu I \cos 2\theta}{4\pi r} e^{-jkr} \end{aligned} \tag{5.51}$$

$$E_z = -\frac{j\omega\mu I \mathrm{d}l}{4\pi r}e^{-jkr} + \frac{j\omega c\mu I dl \cos 2\theta}{4\pi r}e^{-jkr}$$

Hence,

$$E_z \text{ will be } \frac{j\omega\mu I \mathrm{d}l \sin^2\theta}{4\pi r} \; e^{-jkr}$$

where

$$\cos\theta = \frac{\delta}{\sqrt{x^2 + y^2 + \delta^2}}$$

$$\sin^2\theta = \frac{x^2 + y^2}{x^2 + y^2 + \delta^2}$$

$$E_z|_{z=} = \delta \sum_{mnp} \mathrm{Re}[C(mnp)e^{j\omega(mnp)t} \times \sqrt{\frac{2}{d}} \sin\left(\frac{p\pi\delta}{d}\right) u_{m,n}(x, y) \tag{5.52}$$

$$E_z = \int \frac{j\omega\mu I \mathrm{d}l(x^2 + y^2)}{4\pi\left(x^2 + y^2 + \delta^2\right)^{3/2}} e^{-j\frac{\omega}{c}\sqrt{x^2+y^2+\delta^2}} \times I(\omega)e^{jkt}\mathrm{d}\omega \tag{5.53}$$

Here, $I(\omega)$ is the Fourier transform of source current, i.e., $I(t)$ probe current

$$I(\omega) = \frac{1}{2}\sum_{mn\rho} |I(mnp)|[\delta(\omega - \omega(mnp))e^{j\o}(mnp) + e^{j\o}(mnp)\delta(\omega - \omega(mnp))]$$

$$I(\omega) = \frac{1}{2}\sum_{mn\rho} |I(mnp)|[\delta(\omega - \omega(mnp))e^{j\o}(mnp) + e^{j\o}(mnp)\delta(\omega - \omega(mnp))]$$

$$I(\omega) = \int \cos(\omega(mnp)t)e^{-j\omega t}\mathrm{d}t$$

When $\omega(mnp) = \pi^2\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{p^2}{d^2}}$, probe current magnitude and phase $I(\omega) = \sum_{m,np}|I(mnp)|\cos(\omega(mnp)t + \phi(mnp))$ $\o(mnp)$ is the phase of current at frequency $\omega(mnp)$.

$$\begin{aligned} &E_z(x, y, z, t) \\ &= \frac{\mu \mathrm{d}l(x^2 + y^2)}{4\pi(x^2 + y^2 + \delta^2)^{3/2}}\omega(mnp)|I(mnp)|\sin\left(\omega(mnp)\left(t - \frac{\sqrt{x^2 + y^2 + \delta^2}}{c} + \phi(mnp)\right)\right) \\ &= |C_{mn}|u_{mn}(x, y)\cos\omega((mnp)t + \psi(mnp))\sqrt{\frac{2}{d}}\sin\left(\frac{p\pi\delta}{d}\right). \end{aligned} \tag{5.54}$$

5.8 *R*, *L*, *C* Circuits and Resonant Modes

The information contained in eigenvalue or eigenvector of modes can impart the knowledge of antenna radiation behavior, surface current distribution, input impedance, and its feeding point location. Combinations of feeding configuration and dimensions can generate or excite various modes. Thus, modes can be effectively used in design control of an antenna. Surface current and geometry of an antenna give eigenfunctions or eigenvectors. Closed-loop currents of eigenvectors that present inductive nature are the magnetic fields. Horizontal and vertical eigenvectors are noninductive are electric fields. These electric fields are produced by supplied probe currents. Number of lobes in radiation pattern gets increased if mode number or order of mode is increased and vice versa. The modal excitation coefficients shall depend on position, magnitude, and phase of the applied probe current. The effective current is superposition of all modes excited. The eigenvalue is most important because its magnitude tells effectiveness of radiation or reactive power and modes are the solution of characteristics equation. Smaller magnitude of

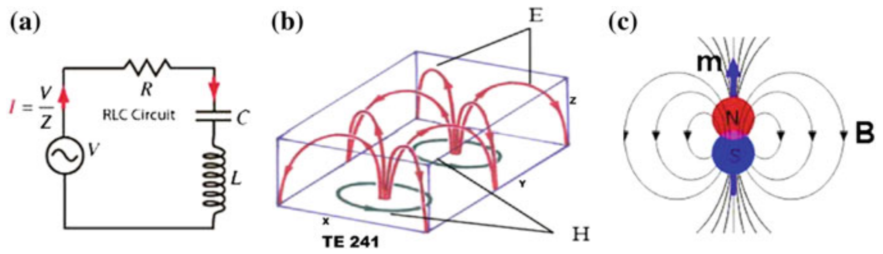


Fig. 5.4 **a** *RLC* circuit, **b** resonance higher modes, **c** magnetic dipoles

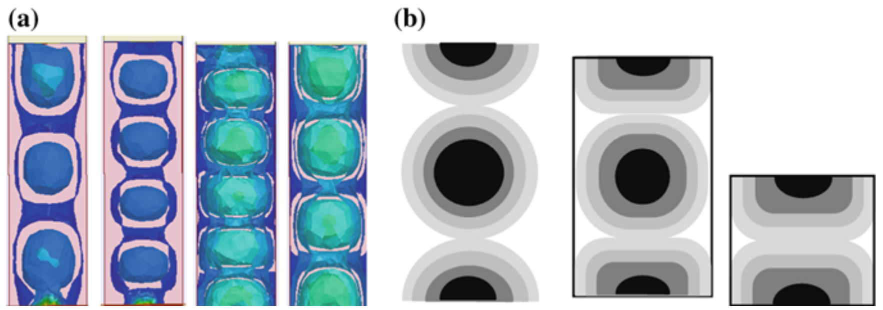


Fig. 5.5 Higher-order even and odd modes

eigenvalue is more efficient. Positive eigenvalue is the magnetic energy storing mode, and if modes are negative, it stores electric energy. The eigenvalue variation versus frequency gives information about resonance and radiation nature. Excitation angle can have impact on antenna quality factor. The excited mode will adjust the phase of the reflected currents. Orthogonality of modes can be used to produce circular polarization in the RDRA. Figure 5.4 represents the equivalent *RLC* circuit of RDRA, resonant modes excited, and corresponding magnetic dipoles. Figure 5.5 depicts the even and odd modes generation. Figure 5.6 presents RDRA HFF model along with its equivalent *RLC* circuit. Figures 5.7 and 5.8 are *RLC* circuits which are used for derivation of resonant frequency and impedance.

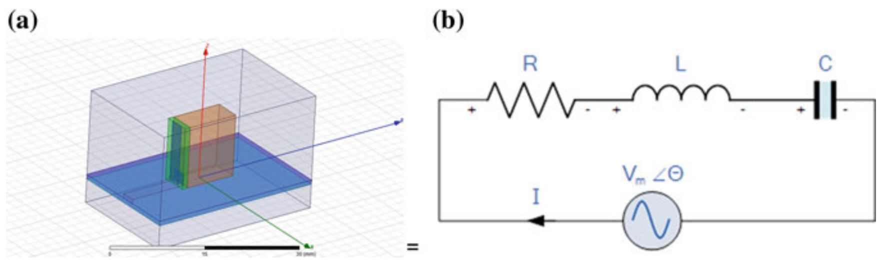


Fig. 5.6 **a** RDRA model and **b** equivalent *RLC* circuit

Fig. 5.7 *RLC* circuit

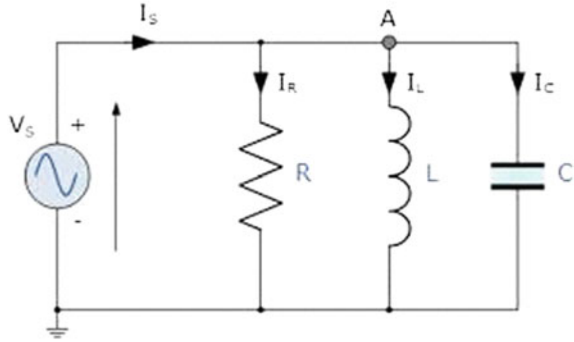
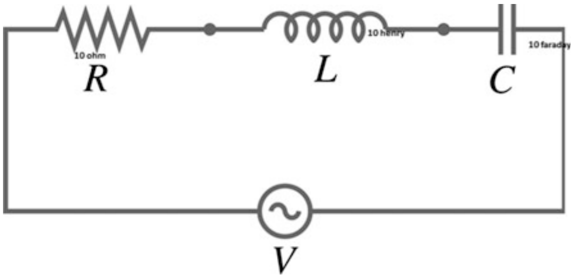


Fig. 5.8 Series *RLC* circuit



R, *L*, *C* equivalent circuit: An antenna can be represented as *R*, *L*, *C* circuitry with natural frequencies ω_c and forced resonance due to excitation eigen-valued $\left(\omega_{mnp}\right)$ has been determined along with eigenvector J_{mnp}. Separation of all frequencies will be the out come of modes. The second-order differential equation is the general solution of equivalent antenna (*R*, *L*, *C*) circuit. Fourier solution will provide a discrete solution of resonance. J_e is excitation current or probe current and γ is an propagation constant $(\gamma = \alpha + j\beta)$. *L*, *C* circuit will introduce non-homogeneous or inhomogeneous matter, ω^2 will be replaced in this case by $\omega^2\mu\epsilon \times -\gamma$ is replaced by $\tilde{\gamma}$ introducing decay. $H_z^{(f)}$ represents forced resonance mode.

$$L\ddot{q} + \frac{q}{C} + R\dot{q} = v_s(t)$$

where $\ddot{q} = \frac{d^2q}{\mathrm{d}t^2}$

$$X_L = j\omega L, \quad X_C = \frac{1}{j\omega C}$$

Taking Fourier Transform

$$\left((j\omega L)^2 + \frac{1}{C} + j\omega R\right)Q(\omega) = V_s(\omega)$$

$$Q\,(\omega) = \frac{V_s(\omega)}{(j\omega L)^2 + \frac{1}{c} + j\omega R}$$

$$q(t) = \int_{-\infty}^{+\infty} \frac{Q(\omega)e^{j\omega t}\mathrm{d}\omega}{2\pi}$$

$$R = 0$$

$$(\omega L)^2 = \frac{1}{\omega c}$$

$$\omega = \frac{1}{\sqrt{LC}}$$

$$\underline{J}_{se}(x,y,z) = J_{sx}(x,y)\delta(z-d_0)\hat{x} + J_{sy}(x,y)\delta(z-d_0)\hat{y}$$

where J_s is the current surface density, and J_e is the electron current

$$\int J_s \mathrm{d}z = J_{sx}(x,y,\omega)\hat{x} + J_{sy}(x,y,\omega)$$

From Maxwell's equation,

$$\underline{\nabla} \times \underline{H} = \underline{J}_e + (\sigma + j\omega\epsilon)E$$
$$\underline{\nabla} \times \underline{E} = -j\omega\mu\underline{H}$$
$$-\nabla^2 E = -j\omega\mu(\sigma + j\omega\epsilon)E + \underline{J}_e$$
$$\nabla^2 E_z = \gamma^2(\omega)E_z$$

When

$$\gamma(\omega) = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)} = \alpha(\omega) + j\beta(\omega)$$

Similarly, we can compute

$$\nabla^2 H_z = \gamma^2(\omega)H_z$$

Boundary conditions are applied

$$H_z = 0,\quad x = 0, a,\quad \text{or}\quad y = 0, b,\quad z = 0, d$$
$$E_x = 0,\quad x = 0, a,\quad \text{when}\quad z = 0, d, E_y = 0, y = 0,\quad b, z = 0, d;$$

Fields propagating is H_z for TE mode

$$H_z(x,y,z,\omega) = \sum \frac{2\sqrt{2}}{\sqrt{abd}} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{p\pi d}{d}\right) \mathrm{Re}\left(C(mnp)\exp^{(j\omega(nmp)t)}\right)$$

$$\left(\nabla^2 - \gamma^2(\omega)\right)E_z = 0$$

γ and $\tilde{\gamma}$ are two propagation constants

$$\begin{aligned}
&(\nabla^2 - \gamma^2(\omega))E_z = 0\\
&(\nabla^2 - \gamma^2(\omega))\underline{E}_\perp = \underline{J}_e\\
&-\nabla^2\underline{H} = \underline{\nabla} \times \underline{J}_e + (\sigma + j\omega\epsilon)(-j\omega\mu j\underline{H})\\
&\left(\nabla^2 - \gamma^2(\omega)\right)\underline{H} = -\underline{\nabla} \times \underline{J}_e\\
&(\nabla^2 - \gamma^2(\omega))\underline{E}_\perp = \underline{J}_e\\
&\left(\nabla^2 - \tilde{\gamma}^2\right)H_x = J_{sy}\delta'(z - d_0)\\
&\left(\nabla^2 - \tilde{\gamma}^2\right)H_y = -J_{sx}\delta'(z - d_0)\\
&\tilde{\gamma}^2(\omega) + \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{p^2}{d^2}\right) = 0\\
&j\omega\mu(\sigma + j\omega\epsilon) + \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{p^2}{d^2}\right) = 0\\
&-\tilde{\gamma}^2\ (\omega_{(mnp)}) = \omega^2\mu\epsilon - j\omega\mu\sigma = \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{p^2}{d^2}\right)\\
&\omega(mnp) = \omega_{\mathrm{Real}}(mnp) + j\omega_{\mathrm{Img}}(mnp)\\
&e^{j\omega(mnp)t} = e^{j\omega_{\mathrm{Real}}(mnp)t}e^{-\omega_{\mathrm{Img}}(mnp)t}\\
&\omega^2\mu\epsilon \to \omega^2\mu\epsilon - j\omega\mu\sigma = -\tilde{\gamma}^2(\omega)\\
&(J_{s_{xy}}(x, y\ \omega) - J_{s_{yx}}(x, y\ \omega)\delta(z - d_0) = \sum J_z(n,m,p,\omega)u_{nmp}(x,y,z)
\end{aligned}$$

$$J[n,m,p,\omega] = \int_0^a \int_0^b (J_{s_{xy}}(x,y,\omega) - J_{s_{yx}}(x,y,\omega) \times \frac{2\sqrt{2}}{\sqrt{abd}} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{d_0\pi p}{d}\right) \mathrm{d}x\mathrm{d}y \tag{5.55}$$

$$\left(\nabla^2 - \tilde{\gamma}^2(\omega)\right)H_z = \sum_{mnp} J_z[mnp,\omega]u_{mnp}(x,y,z)$$

$$H_z^{(f)} = \sum_{mnp} H_z[mnp,\omega]u_{mnp}(x,y,z);$$

When $H_z^{(f)}$ is the forced resonant mode, then

$$J_z[nmp,\omega] = [\pi^2\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}+\frac{p^2}{d^2}\right)+\tilde{\gamma}^2(\omega)]H_z[nmp,\omega]$$

$$H_z^{(f)}(x,y,z\ \omega) = \sum_{mnp}\frac{J_z[nmp,\omega]u_{mnp}(x,y)}{\tilde{\gamma}^2(\omega)-\tilde{\gamma}^2(\omega(nmp))}$$

where

$$v_{mnp} = \frac{2\sqrt{2}}{\sqrt{abd}}\cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right)\cos\left(\frac{p\pi z}{d}\right) = H_z$$

$$J_{sy}(x,y,\omega)\delta'(z-d_0) = \sum J_y[mnp,\omega]v_{mnp}(x,y,z)$$

Hence, current density

$$J_y[n,m,p,\omega] = \int_0^a\int_0^b\int_0^d J_{Sy}(x,y,\omega)\delta'(z-d_0)v_{mnp}(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

This completes the general solution of *R*, *L*, *C* circuit.

5.9 Resonant Modes Based on *R*, *L*, *C* Circuits

$$\underline{\nabla}_\perp H_z\times\hat{z}-\gamma\ \hat{z}\times\underline{H}_\perp = J_e+(\sigma+j\omega\epsilon)E_\perp \tag{5.56}$$

$$\underline{\nabla}_\perp E_z\times\hat{z}-\gamma\ \hat{z}\times\underline{E}_\perp = -j\omega\mu\underline{H}_\perp \tag{5.57}$$

$$\hat{z}\times(\underline{\nabla}_\perp E_z\times\hat{z}-\gamma\ \hat{z}\times\underline{E}_\perp) = \hat{z}\times(-j\omega\mu\,\underline{H}_\perp)$$

$$\underline{\nabla}_\perp E_z+\gamma\underline{E}_\perp = -j\omega\mu\ \hat{z}\times\underline{H}_\perp \tag{5.58}$$

Eliminate $\hat{z}\times\underline{H}_\perp$ from Eqs. (5.56) and (5.58)

$$\underline{\nabla}_\perp E_z+\gamma\underline{E}_\perp = \frac{-j\omega\mu}{\gamma}(\underline{\nabla}_\perp H_z\times\hat{z}-J_e-(\sigma+j\omega\varepsilon)E_\perp).$$

$$\left(\gamma^2-\tilde{\gamma}^2(\omega)\right)E_\perp = -j\omega\mu\ \underline{\nabla}_\perp H_z\times\hat{z}+j\omega\mu\ J_e-\gamma\ \underline{\nabla}_\perp E_z$$

Hence,

$$E_{\perp} = \frac{-j\omega\mu}{\gamma^2 - \tilde{\gamma}^2(\omega)}\ \underline{\nabla}_{\perp} H_z \times\ \hat{z} + \frac{j\omega\mu}{\gamma^2 - \tilde{\gamma}^2(\omega)}\ J_e - \frac{\gamma\, \underline{\nabla}_{\perp} E_z}{\gamma^2 - \tilde{\gamma}^2(\omega)}$$

Parallel *RLC* Circuits solution:

$$Ii + L\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{c}\int i\ \mathrm{d}t = V$$

On differentiating

$$L\ \frac{\mathrm{d}i^2}{\mathrm{d}^2 t} + R\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{c} = 0$$

Second-order linear, homogeneous differential equation dividing by L both sides gives the following:

$$\frac{\mathrm{d}i^2}{\mathrm{d}^2 t} + \frac{R}{L}\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{LC} = 0$$

Taking Laplace transform

$$S^2 + \frac{R}{L}S + \frac{1}{LC} = 0$$

$$S = \ -\frac{R}{L} \pm \frac{\sqrt{\left(\frac{R}{L}\right)^2 - 4 \times \frac{1}{LC}}}{2}$$

$$S_1 = -\frac{R}{L} + \frac{\sqrt{\left(\frac{R}{L}\right)^2 - \left(\frac{2}{\sqrt{LC}}\right)^2}}{2}$$

$$S_2 = -\frac{R}{L} - \frac{\sqrt{\left(\frac{R}{L}\right)^2 - \left(\frac{2}{\sqrt{LC}}\right)^2}}{2}$$

Series *RLC* circuit

Let

$$\lambda = -\frac{R}{2L}$$
$$\omega_2 = \sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{Lc}\right)^2}$$
$$S_1 = \lambda + \omega_2$$
$$S_2 = \lambda - \omega_1$$

Hence, solution of differential equation can be written as:

$$I = A_1 e^{s_1 t} + A_z e^{s_2 t}$$

Here, A_1 and A_2 are the magnitude of currents

Now

$$\text{Care 1} \quad \left(\frac{R}{2L}\right)^2 > \frac{1}{Lc}$$

$$\text{Care 2} \quad \left(\frac{R}{2L}\right)^2 < \frac{1}{Lc}$$

$$\text{Care 3} \quad \left(\frac{R}{2L}\right)^2 = \frac{1}{Lc}$$

$$V = Ri + \frac{L\mathrm{d}i}{\mathrm{d}t} + \frac{1}{C}\int i \,\mathrm{d}t$$

Taking Laplace transformation

$$\frac{V}{s} = RI(s) + LsI(s) + \frac{I(s)}{sC}$$
$$I(s) = \frac{V}{s\left[R + Ls + \frac{1}{sC}\right]}$$
$$I(s) = \frac{V}{Rs + Ls^2 + \frac{1}{C}}$$

Since,

$$s_{1,2} = \frac{-R/L \pm \sqrt{\left(\frac{R}{L}\right)^2 - 4/LC}}{2}$$

$$I(s) = \frac{V}{L\left[s^2 + \frac{Rs}{L} + \frac{1}{LC}\right]} = \frac{V}{L(s_1 - s_2)}\left[\frac{1}{s - s_1} - \frac{1}{s - s_2}\right]$$

Taking Laplace inverse of equation

$$I(t) = \frac{1}{L\sqrt{\frac{R}{L^2} - \frac{4}{LC}}} e^{s_1 t} - \frac{1}{L\sqrt{\frac{R}{L^2} - \frac{4}{LC}}} e^{s_2 t}$$

Example 5.1 Series *RLC* circuit solution

$$v = R \times i(t) + L\frac{\mathrm{d}i(t)}{\mathrm{d}t}\frac{1}{C}\int_0^t i(t) \times \mathrm{d}t$$

Taking Laplace transform on both the sides gives

$$\frac{v}{s} = I(s) \times R + L[s \times I(\mathrm{s}) - i(0)] + \frac{1}{C \times s} I(\mathrm{s}) \quad [i(0) = 0]$$

$$\frac{v}{s} = I(s)\left[R + L \times s + \frac{1}{C \times s}\right]$$

$$v = I(s)\left[R \times s + L \times s^2 + \frac{1}{C}\right]$$

Roots of the equation are as follows:

$$L \times s^2 + R \times s + C^{-1} = 0$$

$$S = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$\text{Let} \quad s_1 = s = \frac{-R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$\text{and} \quad s_2 = \frac{-R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$\text{Now,} \quad s_1 + s_2 = \frac{-R}{L} \;\; \text{and} \;\; s_1 s_2 = \frac{1}{LC}$$

$$v = I(s) \times L[s^2 - s \times (s_1 + s_2) + s_1 s_2]$$

$$v = I(s) \times L[s(s - s_1) - s_2(s - s_1)]$$

$$v = I(s) \times L[(s - s_1)(s - s_2)]$$

$$I(s) = \frac{v}{L} \times \frac{1}{(s - s_1)(s - s_2)}$$

Using partial fraction solution, we get

$$I(s) = \frac{v}{L(s_1 - s_2)} \times \left[\frac{1}{(s - s_1)} - \frac{1}{(s - s_2)}\right]$$

Taking inverse Laplace transform on both the sides

$$i(t) = \frac{v}{L(s_1 - s_2)} \times [e^{s_1 t} - e^{s_2 t}]$$

$$s_1 - s_2 = \frac{R}{L}\sqrt{1 - \frac{4L}{C}}$$

$$i(t) = \frac{v}{R\sqrt{1 - \frac{4L}{C}}} \times [e^{s_1 t} - e^{s_2 t}]$$

$$\text{Let,} \quad A_1 = -A_2 = \frac{v}{R\sqrt{1 - \frac{4L}{C}}}$$

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$