Chapter 3
Mathematical Analysis of Rectangular DRA

Abstract This chapter, mathematical analysis of electromagnetic fields in rectangular dielectric resonator antenna (RDRA) has been introduced. The investigations are based on the first applying waveguide theory, then converting it to resonator by replacing $-\gamma$ to d/dz. Initially, these fields are exploited using the Maxwell curl equations, then manipulating them to express the transverse components of the fields in terms of partial derivatives of the longitudinal components of the fields with respect to x and y axis (i.e., the transverse coordinates). Waveguide models of four rectangular DRAs with specified boundary conditions with linear permittivity have been realized.

Keywords Electromagnetic fields mathematical modeling · Resonator · Waveguide · Homogeneous medium · Boundary conditions · Surface interface

In this chapter, mathematical analysis of electromagnetic fields in rectangular dielectric resonator antenna (RDRA) has been introduced. The investigations are based on first applying waveguide theory, then converting it to resonator by replacing $-\gamma$ to d/dz. Through out this book, electromagnetic field propagation has been taken along z-axis, i.e., $\exp(-\gamma z)$. Initially, these fields are exploited using the Maxwell curl equations, then manipulating them to express the transverse components of the fields in terms of partial derivatives of the longitudinal components of the fields with respect to x and y axis (i.e., the transverse coordinates). Waveguide models of four different rectangular DRAs with specified boundary conditions with homogeneous material having linear permittivity have been mathematically modeled. The fields are realized to determine TE and TM modes of propagating fields. These have resulted into different sine–cosine combinations. Propagation of these fields have been split as inside the RDRA and outside RDRA. The interfacing surface is having two different dielectrics. The solution is developed as transcendental equation, which purely characterized rectangular DRA frequency and propagating fields in terms of propagation constants and dominant resonant frequency. TE modes generation required H_z as longitudinal fields and E_x, E_y, H_x, and H_y as transverse fields. Excitation is applied along x-axis as partial fields, y-axis will have fixed variation, and z-axis will have desired variation in propagating

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fields, for example, TE δ_{13} and TE δ_{43}. Similar cases can be developed for other modes, so as to propagate E_z fields as longitudinal and E_x, E_y, H_x, and H_y as transverse fields. In this case, H_z shall get vanished because of boundary conditions. Resonant modes, i.e., amplitude coefficient of these fields C_{mnp} and D_{mnp} inside the DRA can be determined by comparing magnetic energies equal to electrical energies based on principle orthonormality or law of conservation. The derivation for the quality factor and radiation pattern have been developed for deeper antenna analysis.

3.1 Rectangular DRA with Homogeneous Medium

In Rectangular DRA as shown in Fig. 3.1, top and bottom walls of RDRA are PMC and rest of the other walls are PEC. On magnetic walls (PMC), $n \cdot E = 0$, where E denotes the electric field intensity and n denotes the normal to the surface of the resonator. Similarly, $n \times H = 0$ is not necessarily satisfied at all the surfaces of the DRA by all the modes. Different resonant modes shall have different electromagnetic

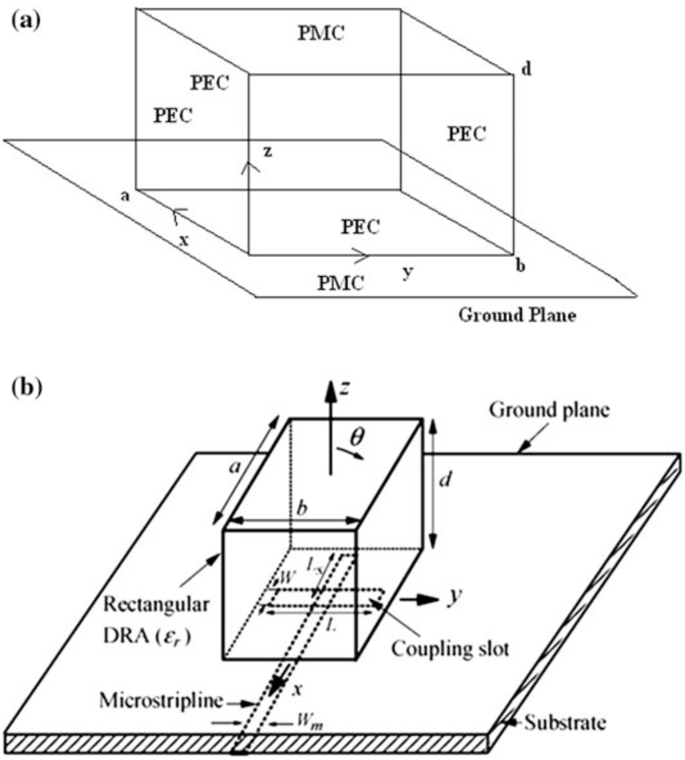

Fig. 3.1 **a** Rectangular DRA with aperture-coupled feed. **b** RDRA with input excitation

field distribution inside the RDRA, and each mode may provide a different resonant frequency and radiation pattern, i.e., eigen vector and eigen frequency. Excitation-based resonant modes can generate desired radiation pattern for different coverage requirements. By making use of this mechanism, internal as well as associated external fields distribution can be obtained.

Rectangular DRA is better choice due to flexible aspect ratio, i.e., *b*/*a* or *d*/*a* options can generate different modes. The existence of two independent aspect ratios in a rectangular DRA offers better design flexibility. Assuming the ground plane to be infinitely large, image theory is applied to replace the isolated RDRA by a grounded resonator of half-size. In this RDRA, two of the six surfaces of the resonator are assumed to be perfect magnetic walls, while the remaining four are assumed to be perfect electric walls. Electromagnetic theory is then applied to study its theoretical analysis, and later three more cases have been developed based on the different boundary conditions. For example, the fields undergo one half-wave variation along the dimension '*a*' and remains constant along dimension '*b*'. They undergo less than a half-wave variation along *z*-axis, i.e., variation along DRA height '*d*'. The resonant mode is therefore identified as $TE^{z}{}_{10}\delta_{.}$ The propagation direction has been assumed in *z*-direction. $TE^{z}{}_{310}$ resonant fields undergo three half-wave variations along length of DRA '*a*' and one half-wave variation along breadth '*b*', and no variation along height '*d*'. To adapt these formulae to an DRA, we note that the propagation constants along *z* can be $\pm\gamma$ with the linear combinations of coefficients chosen, so as to meet the boundary conditions at $z = 0$ and $z = d$, i.e., the top and bottom surfaces of the RDRA, which have been taken as PEC (permanent electrical conducting) walls. On a PEC, the tangential components ($n \times E = 0$) of the electric field and the normal component ($n \cdot H = 0$) of the magnetic fields get vanished. While on a PMC wall, by directly, the normal component of the electric field ($n \cdot E = 0$) and the tangential components ($n \times H = 0$) of the magnetic field get vanished.

To compute resonant modes, vector principle of orthonormality on half-wave Fourier analysis has been applied, i.e., radiated magnetic energies are compared with applied electrical energies in RDRA. More number of modes along z-axis in RDRA can be generated either by increasing electrical height '*d*' of RDRA or by increasing excitation resonant frequency. Given below are the two rectangular DRAs with different configurations shown in Fig. 3.1.

In Fig. 3.1, PMC and PEC walls' configuration is labeled. The mathematical solution is developed based on this configuration. The boundary conditions of interface walls shall form linear combinations of sine–cosine terms. Accordingly, they will decide whether transverse electric fields or magnetic fields will vanish. Propagation of longitudinal fields shall depend on the direction of excitation. Excitation of resonant modes in rectangular boundaries are easier as compared to cylindrical. Transcendental equation and characteristics equations have been developed for rectangular DRA. This has provided complete solution of resonant frequency and propagation constants.

3.2 Rectangular DRA Mathematical Modeling

In this chapter, four different solutions are presented, each RDRA is associated with different boundaries. The resultant field formed the resonant modes of different kinds.

Figure 3.2 described E and H fields pattern forming resonant modes, i.e., dominant or higher-order excited modes inside the RDRA.

3.2.1 Model-1

(a) Here, top and bottom walls are assumed as PMC and rest of the other four walls are PEC as per Fig. 3.1.

Given top and bottom surfaces of RDRA as PMC at $z = 0, d$;

$$\begin{aligned} \therefore n \times H &= 0 \\ n \cdot E &= 0; \\ H_y = H_x &= 0; \\ E_z &= 0; \end{aligned}$$

Rest of the other four walls are PEC.

$$\begin{aligned} & n \times E = 0; \\ & n \cdot H = 0; \\ & x = 0, a; \\ & E_y = E_z = 0, \\ & H_x = 0; \end{aligned}$$

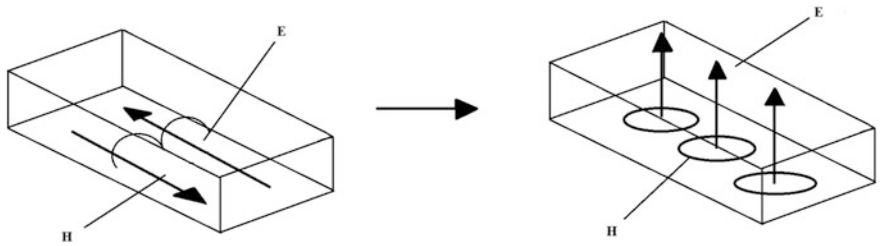

Fig. 3.2 E and H fields pattern inside RDRA

At,

$$y = 0, b;$$
$$E_x = E_z = 0,$$
$$H_y = 0;$$

From separation of variables (Refer Chap. 2),

$$E_x = \frac{1}{j\omega\epsilon\left(1 + \frac{\gamma^2}{k^2}\right)}\left[\frac{\partial H_z}{\partial y} - \frac{1}{j\omega\mu}\frac{\partial^2 E_z}{\partial z \partial x}\right] \tag{3.1}$$

$$E_y = \frac{1}{j\omega\epsilon\left(1 + \frac{\gamma^2}{k^2}\right)}\left[-\frac{1}{j\omega\mu}\frac{\partial^2 E_z}{\partial z \partial y} - \frac{\partial H_z}{\partial x}\right] \tag{3.2}$$

$$H_x = \frac{-1}{j\omega\mu\left(1 + \frac{\gamma^2}{k^2}\right)}\left[\frac{\partial E_z}{\partial y} - \frac{1}{j\omega\epsilon}\frac{\partial^2 H_z}{\partial z \partial x}\right] \tag{3.3}$$

$$H_y = \frac{-1}{j\omega\mu\left(1 + \frac{\gamma^2}{k^2}\right)}\left[\frac{1}{j\omega\epsilon}\frac{\partial^2 H_z}{\partial z \partial y} - \frac{\partial E_z}{\partial x}\right] \tag{3.4}$$

Solution of second-order differential equation is given as,

$$\psi_z = X(x)Y(y)Z(z)$$

where

$$X(x) = A_1 \sin k_x x + A_2 \cos k_x x \tag{3.5}$$

$$Y(y) = A_3 \sin k_y y + A_4 \cos k_y y \tag{3.6}$$

$$Z(z) = A_5 \sin k_z z + A_6 \cos k_z z \tag{3.7}$$

TE mode $(E_z = 0$ and $H_z \neq 0)$

$$\psi_{H_z} = X(x)Y(y)Z(z)$$

substituting $E_z = 0$ in Eq. (3.2) to get, E_y

$$E_y = C'\left[-\frac{\partial H_z}{\partial x}\right];$$

or,

$$E_y = C'X'(x)Y(y)Z(z);$$

Now

$$X'(x) = A_1 \cos k_x x - A_2 \sin k_x x;$$

But at

$$x = 0, a;\ E_y = 0;$$

$$\therefore \quad 0 = A_1 \cos k_x 0 - A_2 \sin k_x 0;$$

or,

$$A_1 = 0 \quad \text{and} \quad k_x = \frac{m\pi}{a};$$

Similarly from Eq. (3.1)

$$E_x = C'\left[\frac{\partial H_z}{\partial y}\right];$$

or,

$$E_x = C'X(x)Y'(y)Z(z).$$

Now

$$Y'(y) = A_3 \cos k_y y - A_4 \sin k_y y;$$

At,

$$y = 0, b; \quad E_x = 0;$$

$$\therefore \quad 0 = A_3 \cos k_y 0 - A_4 \sin k_y 0;$$

or,

$$A_3 = 0 \quad \text{and} \quad k_y = \frac{n\pi}{b};$$

From above equation,

$$H_x = C'\left[-\frac{1}{j\omega\epsilon}\frac{\partial^2 H_z}{\partial z \partial x}\right];$$

or,

$$H_x = C'X'(x)Y(y)Z'(z);$$

Now

$$Z'(z) = A_5 \cos k_z z - A_6 \sin k_z z;$$

At,

$$z = 0, d \quad H_x = 0;$$

$$\therefore \quad A_5 \cos k_z 0 - A_6 \sin k_z 0 = 0;$$

$$A_5 = 0 \quad \text{and} \quad k_z = \frac{p\pi}{d};$$

Hence,

$$H_z = A_2 A_4 A_6 \cos\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) \cos\left(\frac{p\pi}{d}z\right) \tag{3.8}$$

Using Eqs. (3.1)–(3.4), and (3.8), we get

$$\begin{aligned}
H_x &= C'' A_2 A_4 A_6 \left(\frac{m\pi}{a}\right)\left(\frac{p\pi}{d}\right) \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) \sin\left(\frac{p\pi}{d}z\right);\\
H_y &= C'' A_2 A_4 A_6 \left(\frac{n\pi}{b}\right)\left(\frac{p\pi}{d}\right) \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \sin\left(\frac{p\pi}{d}z\right);\\
E_y &= C'' A_2 A_4 A_6 \left(\frac{m\pi}{a}\right) \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) \cos\left(\frac{p\pi}{d}z\right);\\
E_x &= C'' A_2 A_4 A_6 \left(\frac{n\pi}{b}\right) \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \cos\left(\frac{p\pi}{d}z\right);
\end{aligned}$$

Similarly, for TM mode ($\boldsymbol{H_z = 0}$ and $\boldsymbol{E_z \neq 0}$)

$$\psi_{E_z} = X(x)Y(y)Z(z);$$

At,

$$\begin{aligned}
&x = 0, a;\\
&E_z = 0;\\
&A_1 \sin k_x 0 + A_2 \cos k_x 0 = 0;\\
&\therefore \quad A_2 = 0 \quad \text{and} \quad k_x = \frac{m\pi}{a};
\end{aligned}$$

Also, at

$$\begin{aligned}
&y = 0, b;\\
&E_z = 0;\\
&A_3 \sin k_y 0 + A_4 \cos k_y 0 = 0;\\
&A_4 = 0; \quad \text{and} \quad k_y = \frac{n\pi}{b}
\end{aligned}$$

At,

$$\begin{aligned}
&z = 0, d;\\
&E_z = 0;\\
&\therefore \quad A_5 \sin k_z 0 + A_6 \cos k_z 0 = 0;\\
&A_6 = 0 \quad \text{and} \quad k_z = \frac{p\pi}{d};
\end{aligned}$$

Hence,

$$E_z = A_1 A_3 A_5 \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \sin\left(\frac{p\pi}{d}z\right) \tag{3.9}$$

Using Eqs. (1.1)–(1.4), and (1.9), we get

$$\begin{aligned}
E_x &= C'' A_1 A_3 A_5 \left(\frac{m\pi}{a}\right)\left(\frac{p\pi}{d}\right) \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \cos\left(\frac{p\pi}{d}z\right);\\
E_y &= C'' A_1 A_3 A_5 \left(\frac{n\pi}{b}\right)\left(\frac{p\pi}{d}\right) \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) \cos\left(\frac{p\pi}{d}z\right);\\
H_y &= C'' A_1 A_3 A_5 \left(\frac{m\pi}{a}\right) \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \sin\left(\frac{p\pi}{d}z\right);\\
H_x &= C'' A_1 A_3 A_5 \left(\frac{n\pi}{b}\right) \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) \sin\left(\frac{p\pi}{d}z\right);
\end{aligned}$$

3.2.2 Model-2

(b) Top and bottom walls are PEC and rest of the other walls are PMC: Assuming the top and bottom surface plane be at $z = 0, d$;

$$n \times E = 0;$$
$$n \cdot H = 0;$$
$$E_y = E_x = 0;$$
$$H_z = 0;$$

Rest of the other walls are PMC

$$n \times H = 0;$$
$$n \cdot E = 0;$$

At,

$$x = 0, a;$$
$$H_y = H_z = 0;$$
$$E_x = 0;$$

At,

$$y = 0, b;$$
$$H_x = H_z = 0;$$
$$E_y = 0;$$

We also know that

$$E_x = \frac{1}{j\omega\epsilon\left(1 + \frac{\gamma^2}{k^2}\right)}\left[\frac{\partial H_z}{\partial y} - \frac{1}{j\omega\mu}\frac{\partial^2 E_z}{\partial z \partial x}\right]; \tag{3.10}$$

$$E_y = \frac{1}{j\omega\epsilon\left(1 + \frac{\gamma^2}{k^2}\right)}\left[-\frac{1}{j\omega\mu}\frac{\partial^2 E_z}{\partial z \partial y} - \frac{\partial H_z}{\partial x}\right]; \tag{3.11}$$

$$H_x = \frac{-1}{j\omega\mu\left(1 + \frac{\gamma^2}{k^2}\right)}\left[\frac{\partial E_z}{\partial y} - \frac{1}{j\omega\epsilon}\frac{\partial^2 H_z}{\partial z \partial x}\right]; \tag{3.12}$$

$$H_y = \frac{-1}{j\omega\mu\left(1+\frac{\gamma^2}{k^2}\right)}\left[\frac{1}{j\omega\epsilon}\frac{\partial^2 H_z}{\partial z \partial y} - \frac{\partial E_z}{\partial x}\right]; \tag{3.13}$$

Now, the solution of second-order differential equation is given as

$$\psi_z = X(x)Y(y)Z(z);$$

where

$$X(x) = A_1 \sin k_x x + A_2 \cos k_x x; \tag{3.14}$$

$$Y(y) = A_3 \sin k_y y + A_4 \cos k_y y; \tag{3.15}$$

$$Z(z) = A_5 \sin k_z z + A_6 \cos k_z z; \tag{3.16}$$

TE mode ($E_z = 0$ and $H_z \neq 0$)

$$\psi_{H_z} = X(x)Y(y)Z(z);$$

At,

$$\begin{aligned} &x = 0, a; \\ &H_z = 0; \\ &A_1 \sin k_x 0 + A_2 \cos k_x 0 = 0; \\ &A_2 = 0; \end{aligned}$$

and

$$k_x = \frac{m\pi}{a};$$

Also, at

$$\begin{aligned} &y = 0, b; \\ &H_z = 0; \\ &A_3 \sin k_y 0 + A_4 \cos k_y 0 = 0; \\ &A_4 = 0; \end{aligned}$$

and

$$k_y = \frac{n\pi}{b};$$

At,

$$\begin{aligned} &z = 0, d; \\ &H_z = 0; \\ &A_5 \sin k_z 0 + A_6 \cos k_z 0 = 0; \\ &A_6 = 0; \end{aligned}$$

and

$$k_z = \frac{p\pi}{d};$$

Hence,

$$H_z = A_1 A_3 A_5 \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \sin\left(\frac{p\pi}{d}z\right); \tag{3.17}$$

Using Eqs. (1.1)–(1.4), and (1.8), we get

$$\begin{aligned} H_x &= C'' A_1 A_3 A_5 \left(\frac{m\pi}{a}\right)\left(\frac{p\pi}{d}\right) \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \cos\left(\frac{p\pi}{d}z\right), \\ H_y &= C'' A_1 A_3 A_5 \left(\frac{n\pi}{b}\right)\left(\frac{p\pi}{d}\right) \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) \cos\left(\frac{p\pi}{d}z\right), \\ E_y &= C'' A_1 A_3 A_5 \left(\frac{m\pi}{a}\right) \cos\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) \sin\left(\frac{p\pi}{d}z\right); \\ E_x &= C'' A_1 A_3 A_5 \left(\frac{n\pi}{b}\right) \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right) \sin\left(\frac{p\pi}{d}z\right); \end{aligned}$$

TM mode ($\boldsymbol{H_z = 0}$ and $\boldsymbol{E_z \neq 0}$)

$$\psi_{E_z} = X(x)Y(y)Z(z);$$

From Eq. (3.2) after substituting $H_z = 0$, we get

$$E_y = C'\left[-\frac{1}{j\omega\mu}\frac{\partial^2 E_z}{\partial z \partial y}\right];$$

$$E_y = C' X(x) Y'(y) Z'(z);$$

Now

$$Y'(y) = A_3 \cos k_y y - A_4 \sin k_y y;$$

At,

$$\begin{aligned}
&y = 0, b; \quad E_y = 0; \\
&0 = A_3 \cos k_y 0 - A_4 \sin k_y 0; \\
&A_3 = 0 \quad \text{and} \quad k_y = \frac{n\pi}{b};
\end{aligned}$$

Similarly, from the above equations,

$$\begin{aligned}
E_x &= C' \left[-\frac{1}{j\omega\mu} \frac{\partial^2 E_z}{\partial z \partial x} \right]; \\
E_x &= C' X'(x) Y(y) Z'(z); \\
X'(x) &= A_1 \cos k_x x - A_2 \sin k_x x; \\
x &= 0, a; \\
E_x &= 0; \\
0 &= A_1 \cos k_x 0 - A_2 \sin k_x 0; \\
A_1 &= 0;
\end{aligned}$$

and

$$k_x = \frac{m\pi}{a};$$

Also, from above equations,

$$Z'(z) = A_5 \cos k_z z - A_6 \sin k_z z;$$

At,

$$\begin{aligned}
&z = 0, d; \quad E_x = 0; \\
\therefore \quad &0 = A_5 \cos k_z 0 - A_6 \sin k_z 0;
\end{aligned}$$

or,

$$A_5 = 0 \quad \text{and} \quad k_z = \frac{p\pi}{d};$$

Hence,

$$E_z = A_2 A_4 A_6 \cos\left(\frac{m\pi}{a} x\right) \cos\left(\frac{n\pi}{b} y\right) \cos\left(\frac{p\pi}{d} z\right); \tag{3.18}$$

Using Eqs. (3.11–3.14) and (3.18), we get

$$E_x = A_2A_4A_6\left(\frac{m\pi}{a}\right)\left(\frac{p\pi}{d}\right)\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\sin\left(\frac{p\pi}{d}z\right);$$
$$E_y = A_2A_4A_6\left(\frac{n\pi}{b}\right)\left(\frac{p\pi}{d}\right)\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\sin\left(\frac{p\pi}{d}z\right);$$
$$H_x = A_2A_4A_6\left(\frac{n\pi}{b}\right)\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\cos\left(\frac{p\pi}{d}z\right);$$
$$H_y = A_2A_4A_6\left(\frac{m\pi}{a}\right)\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\cos\left(\frac{p\pi}{d}z\right);$$

3.2.3 Model-3

(c) Solution of RDRA, when all six walls are PEC (perfect electrical walls):
Using Maxwell equations:

$$\nabla \times E = -j\omega B = -j\omega\mu H;$$
$$\nabla \times H = j\omega\epsilon E;$$
$$\nabla \times E = -j\omega\mu H;$$
$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -j\omega\mu H;$$
$$\hat{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{y}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = -j\omega\mu H;$$

On comparing (x, y, z) components both the sides

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu H_x; \tag{3.19}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y; \tag{3.20}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z; \tag{3.21}$$

Similarly, using $\nabla \times H = j\omega\epsilon E$; We get

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\epsilon E_x; \tag{3.22}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y; \tag{3.23}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\epsilon E_z; \tag{3.24}$$

Comparing above equations,

$$E_x = \frac{1}{j\omega\epsilon}\left[\frac{\partial H_z}{\partial y} + \frac{1}{j\omega\mu}\left(\frac{\partial^2 E_x}{\partial z^2} - \frac{\partial^2 E_z}{\partial x \partial z}\right)\right]; \tag{3.25}$$

$$E_x + \frac{1}{k^2}\frac{\partial^2 E_x}{\partial z^2} = \frac{1}{j\omega\epsilon}\left[\frac{\partial H_z}{\partial y} - \frac{1}{j\omega\mu}\frac{\partial^2 E_z}{\partial x \partial z}\right]; \tag{3.26}$$

$$E_x\left(1 + \frac{\gamma^2}{k^2}\right) = \frac{1}{j\omega\epsilon}\left[\frac{\partial H_z}{\partial y} - \frac{\gamma}{j\omega\mu}\frac{\partial E_z}{\partial x}\right];$$

E_y, H_x, and H_y are expressed in E_z and H_z fields:

$$E_y\left(1 + \frac{\gamma^2}{k^2}\right) = \frac{1}{j\omega\epsilon}\left[\frac{-\gamma}{j\omega\mu}\frac{\partial E_z}{\partial y} - \frac{\partial H_z}{\partial x}\right];$$

$$H_x\left(1 + \frac{\gamma^2}{k^2}\right) = -\frac{1}{j\omega\mu}\left[\frac{\partial E_z}{\partial y} - \frac{\gamma}{j\omega\epsilon}\frac{\partial H_z}{\partial x}\right];$$

$$H_y\left(1 + \frac{\gamma^2}{k^2}\right) = -\frac{1}{j\omega\mu}\left[\frac{\gamma}{j\omega\mu}\frac{\partial H_z}{\partial y} - \frac{\partial E_z}{\partial x}\right]$$

Separation of variables with given boundary conditions, solution is obtained.

$$\begin{aligned}\psi &= X(x)Y(y)Z(z);\\ &= (A_1 \sin k_x x + A_2 \cos k_x x)\left(A_3 \sin k_y y + A_y \sin k_y y\right)(A_5 \sin k_z z + A_6 \cos k_z z);\end{aligned}$$

TM mode of propagation, $\boldsymbol{H_z = 0}$;

Boundary conditions

Electrical walls $\rightarrow E_{\tan} = 0 = n \times E$;

$\rightarrow H_n = 0 = n \cdot H$;

At, $x = 0$;

$$E_z = X(x) = A_z \cos(0); \quad \text{so } A_2 \text{ must be zero.}$$
$$y = 0, Y(y) = A_y \cos(0); \quad A_4 \text{ must be zero.}$$

For standing wave in direction of z,
Therefore,

$$\frac{\partial}{\partial z} Z(z) = 0;$$

$$A_5 \cos(k_z z) - A_6 \sin(k_z z) = 0;$$

Therefore, at

$$z = 0, d;$$
$$A_5 \text{ must be zero;}$$

Hence, we are left with

$$E_z = A_1, A_3, A_5, \sin k_x x \sin k_y y \cos k_z z;$$

Next, boundary conditions are
At,

$$x = a,$$
$$X(x) = A_1 \sin k_x a = 0;$$

$$k_x = \frac{m\pi}{a};$$

At,

$$Y = b; \quad Y(y) = A_2 \sin k_y d = 0;$$

$$k_y = \frac{n\pi}{b};$$

At,

$$z = 0 \, z(z) \, A_4 \sin k_z d = 0;$$

$$k_z = \frac{p\pi}{d};$$

As, we know that

$$k_0^2 = k_x^2 + k_y^2 + k_z^2;$$

$$E_x = \frac{1}{j\omega\epsilon\left(1+\frac{y^2}{k^2}\right)}\left(\frac{\partial H_z}{\partial y} - \frac{1}{j\omega\mu}\frac{\partial^2 E_z}{\partial x \partial z}\right);$$

$$E_z = 0;$$

$$E_x = \frac{1}{k^2\left(1+\frac{y^2}{k^2}\right)}\left(A_1 A_3 A_5 k_x k_z \cos k_x x \sin k_y y \sin k_z z\right);$$

$$E_y = \frac{1}{j\omega\epsilon\left(1+\frac{y^2}{k^2}\right)}\left(-\frac{\partial H_z}{\partial x} - \frac{1}{j\omega\mu}\frac{\partial^2 E_z}{\partial y \partial z}\right);$$

$$E_y = \frac{-A_1 A_3 A_5 k_y}{k^2\left(1+\frac{y^2}{k^2}\right)} k_z\left(\sin k_x x \cos k_y y \sin k_z z\right);$$

$$\begin{aligned} H_x &= \frac{-1}{j\omega\mu\left(1+\frac{y^2}{k^2}\right)}\left(\frac{\partial E_z}{\partial y} - \frac{1}{j\omega\epsilon}\frac{\partial^2 H_z}{\partial x \partial z}\right); \\ &= \frac{-k_y A_1 A_3 A_5}{\omega^2\mu\epsilon\left(1+\frac{y^2}{k^2}\right)}\left(\sin k_x x \cos k_y y \cos k_z z\right); \end{aligned}$$

$$\begin{aligned} H_y &= \frac{-1}{j\omega\mu\left(1+\frac{y^2}{k^2}\right)}\left(\frac{1}{j\omega\epsilon}\frac{\partial^2 H_z}{\partial y \partial z} - \frac{\partial E_z}{\partial x}\right); \\ &= \frac{-k_z k_x A_1 A_3 A_5}{\omega^2\mu\epsilon\left(1+\frac{y^2}{k^2}\right)}\left(\cos k_x x \sin k_y y \cos k_z z\right); \\ H_y &= \frac{k_x A_1 A_3 A_5}{j\omega\mu\left(1+\frac{y^2}{k^2}\right)}\left(\cos k_x x \sin k_y y \cos k_z z\right); \end{aligned}$$

For, TE mode

$$\psi = A_1\left(\sin k_x x + A_2 \cos kx\right)\left(A_3 \sin k_y y + A_y \sin ky\right)\left(A_5 \sin k_z z + A_6 \cos k_z z\right);$$

For PEC walls, electric field components are assumed to be varying with H_z *in* direction of (x, y, z)

$$\begin{aligned}
E_x &= C' \frac{\partial}{\partial y} H_z \\
&= C' X(x) Y'(y) Z(z); \\
y &= 0, b; \\
Y'(y) &= A_3 \cos k_y y - A_4 \sin k_y y = 0; \\
A_3 &= 0; \\
k_y &= \frac{n\pi}{b};
\end{aligned}$$

Similarly, $E_y = C'' \frac{\partial}{\partial x} H_z;$

$$\begin{aligned}
A_1 &= 0; \\
k_x &= \frac{m\pi}{a}; \\
Z(z) &= A_5 \sin k_z z + A_6 \cos k_z z;
\end{aligned}$$

At,

$$\begin{aligned}
z &= 0, d; \\
A_6 &= 0; \\
k_z &= \frac{p\pi}{d}; \\
H_z &= A_2 A_4 A_5 \cos k_x x \cos k_y y \sin k_z z;
\end{aligned}$$

Therefore,

$$\begin{aligned}
E_x &= \frac{1}{j\epsilon\omega\left(1 + \frac{y^2}{k^2}\right)} (A_2 A_4 A_5 \cos k_x x \sin ky \sin kz); \\
E_y &= \frac{-A_2 A_4 A_5 k_x}{j\epsilon\omega\left(1 + \frac{y^2}{k^2}\right)} \left(\sin k_x x \cos k_y y \sin k_z z\right); \\
H_x &= \frac{k_x k_z A_1 A_3 A_5}{k^2\left(1 + \frac{y^2}{k^2}\right)} \left(\sin k_x x \cos k_y y \cos k_z z\right); \\
H_y &= \frac{-k_z k_y A_1 A_3 A_5}{\omega^2 \mu\epsilon\left(1 + \frac{y^2}{k^2}\right)} \left(\cos k_x x \sin k_y y \cos k_z z\right);
\end{aligned}$$

3.2.4 Model-4

(d) When all the six walls of RDRA are assumed to be PMC (permanent magnetic walls),

$\psi_z = X(x)Y(y)Z(z)$ where ψ_z is wave function in x, y, and z direction as space.

$$\text{Or} = (A_1 \sin k_x x + A_2 \cos k_x x)\left(A_3 \sin k_y y + A_4 \cos k_y y\right)(A_5 \sin k_z z + A_6 \cos k_z z) \tag{3.27}$$

where $A1$–$A6$ are constants and $(A_1 \sin k_x x + A_2 \cos k_x x)$ is solution of second-order differential equation in x direction, i.e., $X(x)$.

When all six walls are PMC, the rectangular DRA solution is

$$H_{\tan} = n \times H = 0;$$
$$H_{\text{nor}} = n \cdot E = 0;$$

Applying boundaries,
At,

$$x = 0, a \Rightarrow H_y \text{ and } H_z = 0; E_x = 0;$$

At,

$$y = 0, b \Rightarrow H_x \text{ and } H_z = 0; E_y = 0;$$

At,

$$z = 0, d \Rightarrow H_x \text{ and } H_y = 0; E_z = 0;$$

TE mode of propagation $(\boldsymbol{E_z} = 0; \boldsymbol{H_z} \neq 0)$
Using boundary conditions
At,

$$x = 0, a; \quad H_z = 0 \Rightarrow A_2 = 0 \quad \text{and} \quad k_x = \frac{m\pi}{a};$$

At,

$$y = 0, b; \quad H_z = 0 \Rightarrow A_4 = 0 \quad \text{and} \quad k_y = \frac{n\pi}{b};$$

Now,

$$H_x = C'' \frac{\partial^2 H_z}{\partial x \partial z} = C'' X'(x) Y(y) z'(z)$$

$$z'(z) = A_5 \cos k_z z - A_6 \sin k_z z$$

At,

$$\begin{gathered} z = 0, d \Rightarrow d \Rightarrow H_x = 0; \\ \Rightarrow H_x = 0 \Rightarrow A_5 = 0; k_z = \frac{p\pi}{d}; \end{gathered}$$

Hence,

$$H_z = A_1 A_3 A_6 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \cos\left(\frac{p\pi z}{d}\right) \tag{3.28}$$

TM mode of propagation $(\boldsymbol{E}_z \neq 0, \boldsymbol{H}_z = 0)$
We again look for the conditions, when H_z = 0, i.e., to get the value of E_z

$$\begin{aligned} H_z &= \frac{C' \partial E_z}{\partial y} \\ &= C' X(x) Y(y') Z(z); \\ Y'(y) &= A_3 \cos k_y y - A_4 \sin k_y y; \\ H_x &= 0 \text{ at } Y = 0, b; \\ &\Rightarrow A_3 = 0 \text{ at } k_y = \frac{n\pi}{b}; \end{aligned} \tag{3.29}$$

Similarly,

$$H_y = C'' \frac{\partial E_z}{\partial_x};$$

$$\begin{aligned} &C'' X'(x) Y(y') Z(z); \\ &X'(x) = A_1 \cos k_x x - A_2 \sin k_x x; \\ &\Rightarrow H_y = 0 \text{ at } x = 0, a, \\ &\Rightarrow A_1 = 0, \\ &k_x = \frac{m\pi}{a}; \end{aligned}$$

At,

$$z = 0,\, d \Rightarrow E_z = 0;$$
$$\Rightarrow A_5 = 0 \text{ and } k_z = \frac{p\pi}{d};$$

At,

$$z = 0,\, d \Rightarrow E_z = 0;$$
$$\Rightarrow A_5 = 0 \text{ and } k_z = \frac{p\pi}{d};$$

Hence,

$$E_z = A_2 A_4 A_5 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi z}{d}\right). \tag{3.30}$$

3.2.5 Basic Theory

Depending on the nature of the surfaces, different linear combinations of the $\pm\gamma$ modes are formed. The propagation constant (γ) itself is taking discrete values. This forces the natural frequencies of the field oscillations to take discrete values (mnp), indexed by three positive integers m, n, and p. The solutions of the waveguide problem yield discrete values of γ, i.e., $\gamma(m, n, \omega)$ for a given frequency ω by applying boundary conditions to the electromagnetic fields on the side walls. The corresponding field amplitudes are solutions to the 2-D Helmholtz equations corresponding to the transverse Laplacian $\nabla_\perp^2$. These amplitudes are called “the waveguide modes” and are of the form

$$\mathcal{L}\oint\left\{\cos\left(\frac{n\pi x}{a}\right), \sin\left\{\frac{n\pi x}{a}\right\}\right\} \otimes \mathcal{L}\oint\left\{\cos\left(\frac{m\pi y}{b}\right), \sin\left\{\frac{m\pi y}{b}\right\}\right\}$$

where $\mathcal{L}$ denotes linear components. It turns out that, depending on the nature of wall surfaces (PEC or PMC), four possible linear combinations can appear $(\cos\otimes\sin, \sin\otimes\cos, \sin\otimes\sin,\ \text{and}\ \cos\otimes\cos)$.

In rectangular DRA, we have got to applying in additional boundary conditions on top and bottom surfaces to be the linear combinations as compared to the waveguide.

$$C_1 \exp\{(-\gamma(m, n, \omega)z\} + C_2 \exp\{+\gamma(m, n, \omega)z\}$$

and these cases are $\gamma(m, n, \omega) = \frac{\pi p}{d}$, when $p = 1, 2, 3\ldots$ and have two possible linear combinations of $\sin\left(\frac{\pi p z}{d}\right)$ and $\cos\left(\frac{\pi p z}{d}\right)$.

Thus, the possible frequencies ω obtained by solving $\gamma(m, n, \omega) = \frac{\pi p}{d}$ and then comes out to be

$$\omega(m, n, p) = \pi \left[\frac{m^2}{a^2} + \frac{n^2}{b^2} + \frac{p^2}{d^2} \right]^{1/2}$$

An equivalent but computationally simpler way to pass on from waveguide physics to resonator physics is to just replace γ by $-\frac{\partial}{\partial z}$ in all the waveguide formulae that express the tangential field components in terms of the longitudinal components. This is done after solving the full 3-D Helmholtz equations using separation of variable in x, y, and z.

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\begin{pmatrix} E_z \\ H_z \end{pmatrix} = 0$$

The discrete modes $\omega(mnp)$ enable us to visualize the resonator as collection of L, C oscillators with different L, C values. The outcome of all this analysis enables us to write down the $\underline{E}$ and $\underline{H}$ fields inside the resonator, as superposition of four or three vector-valued basis functions.

$$\begin{aligned} \underline{E}(x, y, z, t) = & \sum_{m,n,p=1}^{\infty} \mathrm{Re}\left\{ C(mnp) e^{j\omega(mnp)t} \underline{\psi}^{E}_{mnp}(x, y, z) \right\} \\ & + \sum_{m,n,p=1}^{\infty} \mathrm{Re}\left\{ D(mnp) e^{j\omega(mnp)t} \underline{\bar{\phi}}^{E}_{mnp}(x, y, z) \right\} \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} \underline{H}(x, y, z, t) = & \sum_{m,n,p=1}^{\infty} \mathrm{Re}\left\{ C(mnp) e^{j\omega(mnp)t} \underline{\psi}^{H}_{mnp}(x, y, z) \right\} \\ & + \sum_{m,n,p=1}^{\infty} \mathrm{Re}\left\{ D(mnp) e^{j\omega(mnp)t} \underline{\bar{\phi}}^{H}_{mnp}(x, y, z) \right\} \end{aligned} \tag{3.32}$$

We note that there are only two sets of amplitude coefficients $\{C(mnp)\}$ and $\{D(mnp)\}$ of linear combination of coefficients using from the E_z and H_z expansions. The vector-valued complex functions are $\underline{\psi}^{E}_{mnp}$, $\underline{\bar{\phi}}^{E}_{mnp}$, $\underline{\psi}^{H}_{mnp}$, $\underline{\bar{\phi}}^{H}_{mnp} \epsilon R^3$ (where R is autocorrelation) and contains components $\{\cos, \sin\} \otimes \{\cos, \sin\} \otimes \{\cos, \sin\}$, functions and hence for $(m'n'p') \neq (mnp)$, each function of the set

$$\left\{ \underline{\psi}^{E}_{mnp}, \underline{\bar{\phi}}^{E}_{mnp}, \underline{\psi}^{H}_{mnp}, \underline{\bar{\phi}}^{H}_{mnp} \right\};$$

is orthogonal to each functions of the set

$$\left\{ \underline{\psi}^{E}_{m'n'p'}, \ \underline{\bar{\phi}}^{E}_{m'n'p'}, \underline{\psi}^{H}_{mnp}, \underline{\bar{\phi}}^{H}_{m'n'p'} \right\};$$

w.r.t. The measure of dx dy dz over [0, a] × [0, b] × [0, d];

The exact form of the function $\underline{\bar{\phi}}^{E}, \underline{\bar{\phi}}^{H}, \underline{\psi}^{E}, \underline{\psi}^{H}$ depends on the nature of the boundaries. The next problem addressed can be on excitations of RDRA. To calculate the amplitude coefficients $\{C(mnp)\}$ and $\{D(mnp)\}$, we assume that at $z = 0$, an excitation $E_x^{(e)}(x, y, t)$ or $E_y^{(e)}(x, y, t)$ is applied for some time say $t \in [0, T]$ and then removed. Then, the Fourier components in this excitation corresponding to the frequencies $\{\omega(mnp)\}$ are excited and their solutions are the oscillations, while the waveguide for $t > T$. The other Fourier components decay within the resonator.

$\{C(mnp), \ D(mnp)\}$ are components of the form,

$$\begin{aligned} E_x^{(e)}(x, y, t) = & \sum_{m,n,p} \mathrm{Re}(C(mnp))e^{j\omega(mnp)t}\psi^{E}_{mnp\,x}(x, y, 0) \\ & + \mathrm{Re}\left\{ D(mnp)e^{j\omega(mnp)t}\underline{\bar{\phi}}^{E}_{mnp\,x}x, y, 0 \right\} \end{aligned} \tag{3.33}$$

and

$$\begin{aligned} E_y^{(e)}(x, y, t) = & \sum_{m,n,p} \mathrm{Re}\ (C(mnp))e^{j\omega(mnp)t}\psi^{E}_{mnp\,y}(x, y, 0) \\ & + \mathrm{Re}\left\{ D(mnp)e^{j\omega(mnp)t}\underline{\bar{\phi}}^{E}_{mnp\,y}(x, y, 0) \right\} \end{aligned} \tag{3.34}$$

By using orthogonality of $\left\{ \psi^{E}_{mnp\,x}(x, y, 0), \underline{\bar{\phi}}^{E}_{mnp\,x}(x, y, 0) \right\}$, for different (m, n), we can write p to be fixed and likewise of $\left\{ \psi^{E}_{mnp\,y}(x, y, 0), \underline{\bar{\phi}}^{E}_{mnp\,y}(x, y, 0) \right\}$;

In addition, we need to use KAM (Kolmogorov–Arnold–Moser) type of time averaging to yield

$$\begin{aligned} & C(mnp)\psi^{E}_{mnp\,x}(x, y, 0) + D(mnp)\underline{\bar{\phi}}^{E}_{mnp\,x}(x, y, 0) \\ & = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E_x^{(e)}(x, y, t)e^{-j\omega(mnp)t}\mathrm{d}t \end{aligned}$$

and likewise

$$C(mnp)\psi^{E}_{mnp\ y}(x,y,0) + D(mnp)\underline{\bar{\phi}}^{E}_{mnp\ y}(x,y,0) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E_y^{(e)}(x,y,t) e^{j\omega(mnp)t} \mathrm{d}t$$