Approximation of Functions of Class Lip(α, p) in L_p-Norm

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Abstract Mittal and Rhoades (Int. J. Math. Game Theory Algebra **9**(4), 259–267, 1999 [9]; J. Comput. Anal. Appl. **2**(1) 1–10, 2000 [10]) and Mittal et al. (J. Math. Anal. Appl. **326**(1) 667–676, 2007 [7]; Appl. Math. Comput. **217**(9), 4483–4489, 2011 [8]) initiated the studies of error estimates $E_n(f)$ through trigonometric-Fourier approximation (tfa) for situations in which the summability matrix T does not have monotone rows. In this paper, we extend the results of Mittal et al. (Appl. Math. Comput. **217**(9), 4483–4489, 2011 [8]) to a more general C_λ-method in view of Armitage and Maddox (Analysis **9**, 195–204, 1989 [1]), which in turn generalizes the several previous known results due to Mittal and Singh (Int. J. Math. Math. Sci., Art. ID **267383**, 1–6, 2014 [11]), Değer et al. (Proc. Jangjeon Math. Soc. **15**(2), 203–213, 2012 [4]), Leindler (J. Math. Anal. Appl. **302**, 129–136, 2005 [6]), Chandra (J. Math. Anal. Appl. **275**, 13–26, 2002 [3]) and Quade (Duke Math. J. **3**(3), 529–543, 1937 [15]).

Keywords Trigonometric Fourier approximation · C_λ-method · L_p-norm · Class $Lip(\alpha, p)$

1 Introduction

For a given function $f \in L_p := L_p[0, 2\pi],\ p \geq 1$, let

$$s_n(f) := s_n(f; x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) = \sum_{k=0}^{n} u_k(f; x) \qquad (1)$$

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denote the partial sums, called trigonometric polynomials of degree (or order) n, of the first $(n+1)$ terms of the Fourier series of f at a point x.

A positive sequence $\mathbf{c} := \{c_n\}$ is called almost monotone decreasing (increasing) if there exists a constant $K := K(\mathbf{c})$, depending on the sequence $\mathbf{c}$ only, such that for all $n \geq m$, $c_n \leq Kc_m (Kc_n \geq c_m)$. Such sequences will be denoted by $\mathbf{c} \in AMDS$ and $\mathbf{c} \in AMIS$ respectively. A sequence which is either $AMDS$ or $AMIS$ is called almost monotone sequence and will be denoted by $\mathbf{c} \in AMS$.

Let $\mathbb{F}$ be an infinite subset of $\mathbb{N}$ and $\mathbb{F}$ the range of strictly increasing sequence of positive integers, say $\mathbb{F} = \{\lambda(n)\}_{n=1}^{\infty}$. The Cesàro submethod C_λ is defined as

$$(C_\lambda x)_n = \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} x_k, \ (n = 1, 2, 3, ...),$$

where $\{x_k\}$ is a sequence of real or complex numbers. Therefore, the C_λ-method yields a subsequence of the Cesàro method C_1, and hence it is regular for any λ. Matrix-C_λ is obtained by deleting a set of rows from Cesàro matrix. The basic properties of C_λ-method can be found in [1, 14].
Define

$$\tau_n^\lambda(f) = \tau_n^\lambda(f;x) = \sum_{k=0}^{\lambda(n)} a_{\lambda(n),k} s_k(f;x), \ \forall n \geq 0.$$

The trigonometric Fourier series of the signal f is said to be T^λ-summable to s if $\tau_n^\lambda(f) \to s$ as $n \to \infty$.

Throughout $T \equiv (a_{n,k})$, a linear operator, will denote an infinite lower triangular matrix with nonnegative entries and row sums 1. Such a matrix T is said to have monotone rows if , $\forall n$, $\{a_{n,k}\}$ is either nonincreasing or nondecreasing in k, $0 \leq k \leq n$. A linear operator T is said to be regular if it is limit-preserving over the space of convergent sequences.

We write

$$s_n(f;x) = \frac{1}{\pi} \int_0^{2\pi} f(x+t) D_n(t)\,\mathrm{d}t, \quad D_n(t) = (\sin(n+1/2)t)/2\sin(t/2),$$

$$A_{\lambda(n),k} = \sum_{r=k}^{\lambda(n)} a_{\lambda(n),r}, \quad A_{\lambda(n),0} \equiv 1, \forall n \geq 0.$$

The notation $[x]$ means the greatest integer contained in x.

2 Known Results

Chandra [3] proved three theorems on the trigonometric approximation using Nörlund and Riesz matrices. Some of them give sharper estimates than the results proved by Quade [15], Mohapatra and Russell [12] and himself earlier [2]. Similar results were proved by Khan [5] for generalized N_p-mean and Mohapatra et al. [13] for Taylor mean. Leindler [6] extended the results of Chandra [3] without the assumption of monotonicity on the generating sequence $\{p_n\}$. Leindler [6] proved the following:

Theorem 1 ([6]) *If $f \in Lip(\alpha, p)$ and $\{p_n\}$ be positive. If one of the conditions*
(i) $p > 1, 0 < \alpha < 1$ and $\{p_n\} \in AMDS$,
(ii) $p > 1, 0 < \alpha < 1$ and $\{p_n\} \in AMIS$ and

$$(n+1)p_n = O(P_n)\, holds, \tag{2}$$

(iii) $p > 1, \alpha = 1$ and $\sum_{k=1}^{n-1} k|\triangle p_k| = O(P_n)$,
(iv) $p > 1, \alpha = 1, \sum_{k=0}^{n-1} |\triangle p_k| = O(P_n/n)$ and (2) holds,
(v) $p = 1, 0 < \alpha < 1$ and $\sum_{k=-1}^{n-1} |\triangle p_k| = O(P_n/n)$,
maintains, then

$$||f - N_n(f)||_p = O(n^{-\alpha}). \tag{3}$$

Theorem 2 ([6]) *Let $f \in Lip(\alpha, 1), 0 < \alpha < 1$. If the positive $\{p_n\}$ satisfies conditions (2) and $\sum_{k=0}^{n-1} |\triangle p_k| = O(P_n/n)$ hold, then*

$$||f - R_n(f)||_1 = O(n^{-\alpha}).$$

Mittal et al. [7, 8] extended the work of Chandra to general matrices. Mittal et al. [8] proved the following:

Theorem 3 ([8]) *Let $f \in Lip(\alpha, p)$ and let $T = (a_{n,k})$ be an infinite regular triangular matrix.*
(i) If $p > 1, 0 < \alpha < 1$, $\{a_{n,k}\} \in AMS$ in k and satisfies

$$(n+1)\max\{a_{n,0}, a_{n,r}\} = O(1). \tag{4}$$

where $r := [n/2]$ then

$$||f - \tau_n(f)||_p = O(n^{-\alpha}). \tag{5}$$

(ii) If $p > 1, \alpha = 1$ and $\sum_{k=0}^{n-1}(n-k)|\triangle_k a_{n,k}| = O(1)$, or
(iii)If $p > 1, \alpha = 1$ and $\sum_{k=0}^{n} |\triangle_k a_{n,k}| = O(a_{n,0})$, or
(iv) If $p = 1, 0 < \alpha < 1$ and $\sum_{k=0}^{n} |\triangle_k a_{n,k}| = O(a_{n,0})$,
and also $(n+1)a_{n,0} = O(1)$, holds then (5) is satisfied.

Recently, Değer et al. [4] extended the results of Chandra [3] to more general C_λ-method in view of Armitage and Maddox [1]. Değer et al. [4] proved:

Theorem 4 ([4]) *Let $f \in Lip(\alpha, p)$ and $\{p_n\}$ be positive such that*

$$(\lambda(n)+1)p_{\lambda(n)} = O(P_{\lambda(n)}), \tag{6}$$

If either (i) $p > 1, 0 < \alpha \leq 1$ and $\{p_n\}$ is monotonic or (ii) $p = 1, 0 < \alpha < 1$ and $\{p_n\}$ is nondecreasing then

$$||f - N_n^{\lambda}(f)||_p = O(n^{-\alpha}).$$

Theorem 5 ([4]) *Let* $f \in Lip(\alpha, 1), 0 < \alpha < 1$. *If the positive* $\{p_n\}$ *satisfies condition (6) and nondecreasing, then* $||f - R_n^{\lambda}(f)||_1 = O(n^{-\alpha})$.

Very recently, in [11], the authors of this paper generalized two theorems of Değer et al. [4], by dropping the monotonicity on the elements of the matrix rows. These results also generalize the results of Leindler [6] to more general C_{λ}-method.

Theorem 6 ([11]) *If* $f \in Lip(\alpha, p)$ *and* $\{p_n\}$ *be positive. If one of the following conditions*
(i) $p > 1, 0 < \alpha < 1$ *and* $\{p_n\} \in AMDS$,
(ii) $p > 1, 0 < \alpha < 1$ *and* $\{p_n\} \in AMIS$ *and (6) holds,*
(iii) $p > 1, \alpha = 1$ *and* $\sum_{k=1}^{\lambda(n)-1} k|\triangle p_k| = O(P_{\lambda(n)})$,
(iv) $p > 1, \alpha = 1, \sum_{k=0}^{\lambda(n)-1} |\triangle p_k| = O\left(\frac{P_{\lambda(n)}}{\lambda(n)}\right)$ *and (6) holds,*
(v) $p = 1, 0 < \alpha < 1$ *and* $\sum_{k=-1}^{\lambda(n)-1} |\triangle p_k| = O\left(\frac{P_{\lambda(n)}}{\lambda(n)}\right)$,
maintains, then

$$||f - N_n^{\lambda}(f)||_p = O\left((\lambda(n))^{-\alpha}\right). \tag{7}$$

Theorem 7 ([11]) *Let* $f \in Lip(\alpha, 1), 0 < \alpha < 1$. *If the positive* $\{p_n\}$ *satisfies (6) and the condition* $\sum_{k=0}^{\lambda(n)-1} |\triangle p_k| = O\left(\frac{P_{\lambda(n)}}{\lambda(n)}\right)$ *holds, then*

$$||f - R_n^{\lambda}(f)||_1 = O\left((\lambda(n))^{-\alpha}\right). \tag{8}$$

3 Main Results

Mittal and Rhoades [9, 10] initiated the studies of error estimates through trigonometric-Fourier approximation (tfa) for situations in which the summability matrix T does not have monotone rows. In continuation of Mittal and Singh [11], in this paper, we generalize Theorem 3 of Mittal et al. [8] using more general C_{λ}-method. We prove the following:

Theorem 8 *Let* $f \in Lip(\alpha, p)$ *and let* $T = (a_{n,k})$ *be an infinite regular triangular matrix.*
(i) If $p > 1, 0 < \alpha < 1, \{a_{n,k}\} \in AMS$ *in* k *and satisfies*

$$(\lambda(n) + 1)\max\{a_{\lambda(n),0}, a_{\lambda(n),r}\} = O(1), \tag{9}$$

where $r := [\lambda(n)/2]$ *then*

$$||f - \tau_n^{\lambda}(f)||_p = O((\lambda(n))^{-\alpha}). \tag{10}$$

(ii) If $p > 1, \alpha = 1$ *and*

$$\sum_{k=0}^{\lambda(n)-1} (\lambda(n) - k)|\triangle_k a_{\lambda(n),k}| = O(1), \text{ or} \tag{11}$$

(iii) If $p > 1, \alpha = 1$ *and*

$$\sum_{k=0}^{\lambda(n)} |\triangle_k a_{\lambda(n),k}| = O(a_{\lambda(n),0}), \text{ or} \tag{12}$$

(iv) If $p = 1, 0 < \alpha < 1$ *and*

$$\sum_{k=0}^{\lambda(n)} |\triangle_k a_{\lambda(n),k}| = O(a_{\lambda(n),0}), \tag{13}$$

and also

$$(\lambda(n) + 1)a_{\lambda(n),0} = O(1), \tag{14}$$

holds then (10) is satisfied.

Remarks (1) If $\lambda(n) = n$, then our Theorem 8 generalizes Theorem 3.
(2) If $T \equiv (a_{n,k})$ is a Nörlund N_p (or weighted R_p) matrix then-
(a) If $\lambda(n) = n$, then condition (9) (or (14)) reduces to (2) while the conditions (11), (12), (13) reduce to conditions in (iii), (iv) and (v) of Theorem 1 respectively. Thus our Theorem 8 generalizes Theorems 1 and 2.
(b) Değer et al. [4] used the monotone sequences $\{p_n\}$ in Theorems 4 and 5 while our Theorem 8 claims less than the requirement of their theorems. For example, condition (11) of Theorem 8 is automatically satisfied if $\{p_n\}$ is nonincreasing sequence, i.e., L.H.S. of (11) gives

$$\begin{aligned}\sum_{k=0}^{\lambda(n)-1} (\lambda(n) - k) \left|\frac{\triangle_k p_{\lambda(n)-k}}{P_{\lambda(n)}}\right| &= \frac{1}{P_{\lambda(n)}} \sum_{k=0}^{\lambda(n)-1} (\lambda(n) - k)|p_{\lambda(n)-k} - p_{\lambda(n)-k-1}| \\ &= \frac{P_{\lambda(n)-1} - \lambda(n)p_{\lambda(n)}}{P_{\lambda(n)}} = O(1) = R.H.S.,\end{aligned}$$

while the condition (12) is always satisfied if $\{p_n\}$ is nondecreasing, i.e.,

$$\begin{aligned}\sum_{k=0}^{\lambda(n)} \left|\frac{\triangle_k p_{\lambda(n)-k}}{P_{\lambda(n)}}\right| &= \frac{1}{P_{\lambda(n)}} \sum_{k=0}^{\lambda(n)} |p_{\lambda(n)-k} - p_{\lambda(n)-k-1}| \\ &= \frac{1}{P_{\lambda(n)}}[p_{\lambda(n)} - p_{\lambda(n)-1} + p_{\lambda(n)-1} - p_{\lambda(n)-2} + \ldots + p_0 - p_{-1}] \\ &= O\left(\frac{p_{\lambda(n)}}{P_{\lambda(n)}}\right).\end{aligned}$$

Further, condition (9) (or (14)) of Theorem 8 reduces to (6) of Theorem 4. Thus our Theorem 8 generalizes the Theorems 4 and 5 of Değer et al. [4] under weaker assumptions and gives sharper estimate because all the estimates of Değer et al. [4] are in terms of n while our estimates are in terms of $\lambda(n)$ and $(\lambda(n))^{-\alpha} \leq n^{-\alpha}$ for $0 < \alpha \leq 1$.
(c) Also, Theorem 8 extends Theorems 6 and 7 of Mittal, Singh [11] where two theorems of Değer et al. [4] were generalized by dropping the monotonicity on the elements of matrix rows.

4 Lemmas

We shall use the following lemmas in the proof of our Theorem:

Lemma 1 ([15]) *If* $f \in Lip(1, p)$, *for* $p > 1$ *then*

$$||\sigma_n(f) - s_n(f)||_p = O(n^{-1}), \ \ \forall n > 0.$$

Lemma 2 ([15]) *If* $f \in Lip(\alpha, p)$, *for* $0 < \alpha \leq 1$ *and* $p > 1$. *Then*

$$||f - s_n(f)||_p = O(n^{-\alpha}), \ \ \forall n > 0.$$

Note: We are using sums upto $\lambda(n)$ in the nth partial sums s_n and σ_n and writing these sums s_n^λ and σ_n^λ, respectively, in the above lemmas for our purpose in this paper.

Lemma 3 *Let T have AMS rows and satisfy (4). Then, for* $0 < \alpha < 1$,

$$\sum_{k=0}^{\lambda(n)} a_{\lambda(n),k} \ (k+1)^{-\alpha} = O\left((\lambda(n)+1)^{-\alpha}\right).$$

Proof Suppose that the rows of T are *AMDS*. Then there exists a $K > 0$ such that

$$\sum_{k=0}^{\lambda(n)} a_{\lambda(n),k} \ (k+1)^{-\alpha} = \sum_{k=0}^{\lambda(n)} K a_{\lambda(n),0}(k+1)^{-\alpha} = K a_{\lambda(n),0} \sum_{k=0}^{\lambda(n)} (k+1)^{-\alpha}$$
$$= O(a_{\lambda(n),0}(\lambda(n)+1)^{1-\alpha}) = O((\lambda(n)+1)^{-\alpha}).$$

A similar result can be proved if the rows of T are *AMIS*.

5 Proof of the Theorem 8

Case I. $p > 1, 0 < \alpha < 1$. We have

$$\tau_n^{\lambda}(f) - f = \sum_{k=0}^{\lambda(n)} a_{\lambda(n),k} s_k(f) - f = \sum_{k=0}^{\lambda(n)} a_{\lambda(n),k}(s_k(f) - f) \tag{15}$$

Thus in view of Lemmas 2 and 3 we have

$$\begin{aligned}||\tau_n^{\lambda}(f) - f||_p \leq \sum_{k=0}^{\lambda(n)} a_{\lambda(n),k}||s_k(f) - f||_p &= \sum_{k=0}^{\lambda(n)} a_{\lambda(n),k} O((k+1)^{-\alpha}) \\ &= O\left((\lambda(n)+1)^{-\alpha}\right).\end{aligned}$$

Case III. $p > 1, \alpha = 1$. We have

$$||\tau_n^{\lambda}(f) - f||_p \leq ||\tau_n^{\lambda}(f) - s_n^{\lambda}(f)||_p + ||s_n^{\lambda}(f) - f||_p.$$

Again using the Lemma 2, we get

$$||\tau_n^{\lambda}(f) - f||_p \leq ||\tau_n^{\lambda}(f) - s_n^{\lambda}(f)||_p + O\left((\lambda(n))^{-1}\right). \tag{16}$$

So, it remains to show that

$$||\tau_n^{\lambda}(f) - s_n^{\lambda}(f)||_p = O\left((\lambda(n))^{-1}\right). \tag{17}$$

Since $A_{\lambda(n),0} = 1$, we have

$$\tau_n^{\lambda}(f) - s_n^{\lambda}(f) = \sum_{k=1}^{\lambda(n)} (A_{\lambda(n),k} - A_{\lambda(n),0}) u_k(f) = \sum_{k=1}^{\lambda(n)} \left(\frac{A_{\lambda(n),k} - A_{\lambda(n),0}}{k}\right)(ku_k(f)).$$

Thus using Abel's transformation, we get

$$\begin{aligned}||\tau_n^{\lambda}(f;x) - s_n^{\lambda}(f;x)||_p \leq \sum_{k=1}^{\lambda(n)-1} \left|\triangle_k \left(\frac{A_{\lambda(n),k} - A_{\lambda(n),0}}{k}\right)\right| .|| \sum_{j=1}^{k} ju_j(f)||_p \\ + \left|\frac{A_{\lambda(n),\lambda(n)} - A_{\lambda(n),0}}{\lambda(n)}\right| .|| \sum_{j=1}^{\lambda(n)} ju_j(f)||_p.\end{aligned} \tag{18}$$

Let $\sigma_n(s)$ denote the nth term of the $(C, 1)$ transform of the sequence s, then

$$s_n^\lambda(f) - \sigma_n^\lambda(f) = \frac{1}{(\lambda(n)+1)} \sum_{j=1}^{\lambda(n)} j u_j(f).$$

Using Lemma 1, we get

$$|| \sum_{j=1}^{\lambda(n)} j u_j ||_p = (\lambda(n)+1) || s_n^\lambda(f) - \sigma_n^\lambda(f) ||_p = (\lambda(n)+1) O\left((\lambda(n))^{-1} \right) = O(1). \tag{19}$$

Note that

$$\left| \frac{A_{\lambda(n),0} - A_{\lambda(n),\lambda(n)}}{\lambda(n)} \right| \le (\lambda(n))^{-1} A_{\lambda(n),0} = O\left((\lambda(n))^{-1} \right).$$

Thus

$$\left| \frac{A_{\lambda(n),0} - A_{\lambda(n),\lambda(n)}}{\lambda(n)} \right| . || \sum_{j=1}^{\lambda(n)} j u_j(f) ||_p = O\left((\lambda(n))^{-1} \right). \tag{20}$$

Now

$$\begin{aligned} \triangle_k \left(\frac{A_{\lambda(n),k} - A_{\lambda(n),0}}{k} \right) &= \frac{1}{k} \triangle_k (A_{\lambda(n),k} - A_{\lambda(n),0}) + \frac{A_{\lambda(n),k+1} - A_{\lambda(n),0}}{k(k+1)} \\ &= \frac{1}{k(k+1)} \left[(k+1) \triangle_k A_{\lambda(n),k} + \sum_{r=k+1}^{\lambda(n)} a_{\lambda(n),r} - \sum_{r=0}^{\lambda(n)} a_{\lambda(n),r} \right] \\ &= \frac{1}{k(k+1)} \left[(k+1) a_{\lambda(n),k} - \sum_{r=0}^{k} a_{\lambda(n),r} \right]. \end{aligned} \tag{21}$$

Next we claim that $\forall k \in N$,

$$| \sum_{r=0}^{k} a_{\lambda(n),r} - (k+1) a_{\lambda(n),k} | \le \sum_{r=0}^{k-1} (r+1) | a_{\lambda(n),r} - a_{\lambda(n),r+1} |, \tag{22}$$

If $k = 1$, then the inequality (22) reduces to

$$| \sum_{r=0}^{1} a_{\lambda(n),r} - 2 a_{\lambda(n),1} | = | a_{\lambda(n),0} - a_{\lambda(n),1} |.$$

Thus (22) holds for $k = 1$. Now let us assume that (22) is true for $k = m$, i.e.,

$$|\sum_{r=0}^{m} a_{\lambda(n),r} - (k+1)a_{\lambda(n),m}| \leq \sum_{r=0}^{m-1}(r+1)|a_{\lambda(n),r} - a_{\lambda(n),r+1}|. \tag{23}$$

Let $k = m + 1$, using (23), we get

$$\begin{aligned}
&|\sum_{r=0}^{m+1} a_{\lambda(n),r} - (m+2)a_{\lambda(n),m+1}| \\
&= |\sum_{r=0}^{m} a_{\lambda(n),r} - (m+1)a_{\lambda(n),m} + (m+1)a_{\lambda(n),m} - (m+1)a_{\lambda(n),m+1}| \\
&\leq \sum_{r=0}^{m-1}(r+1)|a_{\lambda(n),r} - a_{\lambda(n),r+1}| + (m+1)|a_{\lambda(n),m} - a_{\lambda(n),m+1}| \\
&= \sum_{r=0}^{(m+1)-1} (r+1)|a_{\lambda(n),r} - a_{\lambda(n),r+1}|.
\end{aligned}$$

Thus (22) is true $\forall k$. Using (12), (14), (21), (22), we get

$$\begin{aligned}
\sum_{k=1}^{\lambda(n)} |\triangle_k \left(\frac{A_{\lambda(n),k} - A_{\lambda(n),0}}{k}\right)| &= \sum_{k=1}^{\lambda(n)} \frac{1}{k(k+1)} \left|(k+1)a_{\lambda(n),k} - \sum_{r=0}^{k} a_{\lambda(n),r}\right| \\
&\leq \sum_{k=1}^{\lambda(n)} \frac{1}{k(k+1)} \sum_{m=0}^{k-1}(m+1)|a_{\lambda(n),m} - a_{\lambda(n),m+1}| \\
&= \sum_{k=1}^{\lambda(n)} \frac{1}{k(k+1)} \sum_{m=1}^{k} m|a_{\lambda(n),m-1} - a_{\lambda(n),m}| \\
&\leq \sum_{m=1}^{\lambda(n)} m|\triangle_m a_{\lambda(n),m-1}| \sum_{k=m}^{\infty} \frac{1}{k(k+1)} \\
&= \sum_{k=0}^{\lambda(n)-1} |\triangle_k a_{\lambda(n),k}| = O(a_{\lambda(n),0}) = O\left((\lambda(n))^{-1}\right).
\end{aligned} \tag{24}$$

Combining (18), (19), (20) and (24) yields (17). From (17) and (16), we get

$$||\tau_n^{\lambda}(f) - f||_p = O\left((\lambda(n))^{-1}\right).$$

Case II. $p > 1, \alpha = 1$. For this we first prove that the condition $\sum_{k=0}^{\lambda(n)-1}(\lambda(n) - k)|\triangle_k a_{\lambda(n),k}| = O(1)$ implies that

$$\sum_{k=1}^{\lambda(n)} \left[\triangle_k \left(\frac{A_{\lambda(n),k} - A_{\lambda(n),0}}{k}\right)\right] = O\left((\lambda(n))^{-1}\right). \tag{25}$$

As in case (iii), using (22) and taking $r := [\lambda(n)/2]$ throughout the case, we have

$$\begin{aligned}\sum_{k=1}^{\lambda(n)} \left|\triangle_k \left(\frac{A_{\lambda(n),k} - A_{\lambda(n),0}}{k}\right)\right| &= \sum_{k=1}^{\lambda(n)} \frac{1}{k(k+1)} \left|(k+1)a_{\lambda(n),k} - \sum_{m=0}^{k} a_{\lambda(n),m}\right| \\ &= \sum_{k=1}^{\lambda(n)} \frac{1}{k(k+1)} \sum_{m=0}^{k-1} (m+1)|a_{\lambda(n),m} - a_{\lambda(n),m+1}| \\ &= \left(\sum_{k=1}^{r} + \sum_{k=r+1}^{\lambda(n)}\right) k^{-1}(k+1)^{-1} \sum_{m=1}^{k} m|\triangle_m a_{\lambda(n),m-1}| \\ &:= B_1 + B_2, \, say.\end{aligned}$$

Now interchanging the order of summation and using (11), we get

$$\begin{aligned}B_1 &= \sum_{k=1}^{r} k^{-1}(k+1)^{-1} \sum_{m=1}^{k} m|\triangle_m a_{\lambda(n),m-1}| \le \sum_{m=1}^{r} m|\triangle_m a_{\lambda(n),m-1}| \sum_{k=m}^{\infty} k^{-1}(k+1)^{-1} \\ &= \sum_{m=1}^{r} |\triangle_m a_{\lambda(n),m-1}| = \sum_{m=\lambda(n)-r+1}^{\lambda(n)} |\triangle_{\lambda(n)-m} a_{\lambda(n),\lambda(n)-m}| \\ &= \sum_{m=r-1}^{\lambda(n)} |\triangle_{\lambda(n)-m} a_{\lambda(n),\lambda(n)-m}| . \left(\frac{m}{r-1}\right) \\ &\le \frac{1}{r-1} \sum_{m=1}^{\lambda(n)} m|\triangle_{\lambda(n)-m} a_{\lambda(n),\lambda(n)-m}| = \frac{1}{r-1} \sum_{k=0}^{\lambda(n)-1} (\lambda(n)-k)|\triangle_k a_{\lambda(n),k}| \\ &= \frac{1}{r-1} O(1) = O\left((\lambda(n))^{-1}\right).\end{aligned} \tag{26}$$

$$\begin{aligned}\text{Now} \quad B_2 &= \sum_{k=r}^{\lambda(n)} k^{-1}(k+1)^{-1} \sum_{m=1}^{k} m|\triangle_m a_{\lambda(n),m-1}| \\ &\le \sum_{k=r}^{\lambda(n)} k^{-1}(k+1)^{-1} \left[\left(\sum_{m=1}^{r} + \sum_{m=r}^{k}\right) m|\triangle_m a_{\lambda(n),m-1}|\right] := B_{21} + B_{22}, \, say.\end{aligned}$$

Furthermore, using again our assumption, we get

$$\begin{aligned}
B_{21} &= \sum_{k=r}^{\lambda(n)} k^{-1}(k+1)^{-1} \sum_{m=1}^{r} m|\triangle_m a_{\lambda(n),m-1}| \\
&\le r^{-1} \sum_{k=r}^{\lambda(n)} (k+1)^{-1} \sum_{m=1}^{\lambda(n)} m|\triangle_{\lambda(n)-m} a_{\lambda(n),\lambda(n)-m}| \\
&= r^{-1} \sum_{k=r}^{\lambda(n)} (k+1)^{-1} \sum_{k=0}^{\lambda(n)-1} (\lambda(n)-k)|\triangle_k a_{\lambda(n),k}| \\
&= O(r^{-1}) \sum_{k=r}^{\lambda(n)} (k+1)^{-1} = O\left((\lambda(n))^{-1}\right). \qquad (27)
\end{aligned}$$

Again interchanging the order of summation and using (11), we get

$$\begin{aligned}
B_{22} &= \sum_{k=r}^{\lambda(n)} k^{-1}(k+1)^{-1} \sum_{m=r}^{k} m|\triangle_m a_{\lambda(n),m-1}| \le \sum_{k=r}^{\lambda(n)} (k+1)^{-1} \sum_{m=r}^{k} |\triangle_m a_{\lambda(n),m-1}| \\
&\le \sum_{m=r}^{\lambda(n)} |\triangle_m a_{\lambda(n),m-1}| \sum_{k=m}^{\lambda(n)} (k+1)^{-1} \le (r+1)^{-1} \sum_{m=r}^{\lambda(n)} |\triangle_m a_{\lambda(n),m-1}| \sum_{k=m}^{\lambda(n)} 1 \\
&= (r+1)^{-1} \sum_{m=r}^{\lambda(n)} (\lambda(n)-m+1)|\triangle_m a_{\lambda(n),m-1}| \\
&= (r+1)^{-1} \sum_{k=r-1}^{\lambda(n)-1} (\lambda(n)-k)|\triangle_k a_{\lambda(n),k}| \\
&= (r+1)^{-1} O(1) = O\left((\lambda(n))^{-1}\right). \qquad (28)
\end{aligned}$$

Summing up our partial results (26), (27), (28) we verified (25). Thus (16), (18), (19), (25) and Lemma 2, again yield

$$||f - \tau_n^{\lambda}(f)||_p = O\left((\lambda(n))^{-1}\right).$$

Case IV. $p = 1, 0 < \alpha < 1$.
Using Abel's transformation, conditions (13), (14), convention $a_{n,n+1} = 0$ and the result of Quade [15], we obtain

$$
\begin{aligned}
||\tau_n^{\lambda}(f) - f||_1 &= ||\sum_{k=0}^{\lambda(n)} a_{\lambda(n),k} s_k(f) - f||_1 = ||\sum_{k=0}^{\lambda(n)} a_{\lambda(n),k}(s_k(f) - f)||_1 \\
&= ||\sum_{k=0}^{\lambda(n)-1} \left(\triangle_k a_{\lambda(n),k}\right) \sum_{r=0}^{k}(s_r(f) - f) + (a_{\lambda(n),\lambda(n)} - a_{\lambda(n),\lambda(n)+1}) \sum_{r=0}^{\lambda(n)}(s_r(f) - f)||_1 \\
&= ||\sum_{k=0}^{\lambda(n)} \left(\triangle_k a_{\lambda(n),k}\right) \sum_{r=0}^{k}(s_r(f) - f)||_1 = ||\sum_{k=0}^{\lambda(n)} \left(\triangle_k a_{\lambda(n),k}\right)(k+1)(\sigma_k(f) - f)||_1 \\
&\le \sum_{k=0}^{\lambda(n)}(k+1)|\triangle_k a_{\lambda(n),k}|.||\sigma_k(f) - f)||_1 = O\left(\sum_{k=0}^{\lambda(n)}(k+1)^{1-\alpha}|\triangle_k a_{\lambda(n),k}|\right) \\
&= O\left(\lambda(n)^{1-\alpha}\right)\sum_{k=0}^{\lambda(n)}|\triangle_k a_{\lambda(n),k}| = O\left(\lambda(n)^{1-\alpha}\right) O\left(a_{\lambda(n),0}\right) = O\left((\lambda(n))^{-\alpha}\right).
\end{aligned}
$$

This completes the proof of case (iv) and hence the proof of Theorem 8 is complete.

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