# Modified Baskakov-Szász Operators Based on $q$-Integers

**P.N. Agrawal and Arun Kajla**

**Abstract** In the present paper we introduce the Stancu variant of certain $q$-modified Baskakov *Szász* operators. We estimate the moments of the operators and obtain some direct results in terms of the modulus of continuity. Then, we study the Voronovskaja type theorem and the rate of convergence of these operators in terms of the weighted modulus of continuity. Further, we discuss the point-wise estimation using the Lipschitz type maximal function. Finally, we investigate the rate of statistical convergence of these operators using weighted modulus of continuity.

**Keywords** q-Baskakov-Szasz operators · q-integers · Modulus of smoothness · Point-wise estimates · Statistical convergence

**Mathematics Subject Classification (2010):** 26A15 · 40A35

## 1 Introduction

In recent years, the most interesting area of research in approximation theory is the application of $q$-calculus. In 1997, Phillips [20] first considered a modification of Bernstein polynomials based on $q$-integers. He studied the rate of convergence and Voronovskaja-type asymptotic formula for these operators. Very recently, Gupta and Kim [14] considered $q$-Baskakov operators and they obtained some direct local results and the degree of approximation in terms of modulus of continuity. Subsequently, several researchers have considered the different types of operators in this direction and studied their approximation properties.

Let $\alpha$ and $\beta$ be any two real numbers satisfying the condition that $0 \leq \alpha \leq \beta$, Stancu [21] defined in the following operators:

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$$S_n^{\alpha,\beta}(f,x)=\sum_{k=0}^{n}p_{n,k}(x)\left(\frac{k+\alpha}{n+\beta}\right),\quad 0\le x\le 1,$$

where $p_{n,k}(x)$ is the Bernstein basis function.

Recently, Büyükyazici [7] considered the Stancu–Chlodowsky polynomials and investigated their convergence. In 2012, Verma et al. [22] introduced a Stancu type generalization of certain $q$-Baskakov Durrmeyer operators and discussed some local direct results of these operators. For some other research papers where Stancu type operators have been considered, we refer to [1, 3, 4, 13, 15], etc.

Now, we give some basic definitions and concepts of $q$-calculus [6, 17]. For any real number $q>0$, the $q$-integer $[n]_q$ and $q$-factorial $[n]_q!$ are defined as

$$[n]_q=\begin{cases}\dfrac{(1-q^n)}{(1-q)}, & \text{if } q\neq 1\\ n, & \text{if } q=1\end{cases}$$

and

$$[n]_q!=\begin{cases}[n]_q[n-1]_q\dots 1, & n=1,2,\dots\\ 1, & n=0.\end{cases}$$

The $q$-Pochhammer symbol is defined as

$$(x;q)_n=\begin{cases}(1+x)(1+qx)\dots(1+q^{n-1}x), & n=1,2,\dots\\ 1, & n=0.\end{cases}$$

The $q$-binomial coefficients are given by

$$\binom{n}{k}_q=\frac{[n]_q!}{[k]_q![n-k]_q!},\quad 0\le k\le n.$$

The $q$-derivative $D_q$ of a function $f$ is given by

$$(D_qf)(x)=\frac{f(x)-f(qx)}{(1-q)x},\quad \text{if } x\neq 0.$$

The $q$-Jackson integrals and $q$-improper integrals are defined as

$$\int_0^a f(x)d_q(x)=(1-q)a\sum_{n=0}^{\infty}f(aq^n)q^n,\quad a>0,$$

and

$$\int_0^{\infty/A} f(x)d_q(x)=(1-q)\sum_{n=-\infty}^{\infty}f\left(\frac{q^n}{A}\right)\frac{q^n}{A},\quad A>0.$$

The $q$-Beta integral is defined by

$$\Gamma_q(t) = \int_0^{\frac{1}{1-q}} x^{t-1} E_q(-qx) d_q x, \quad t > 0 \tag{1}$$

which satisfies the following functional equation:

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t), \quad \Gamma_q(1) = 1.$$

To approximate Lebesgue integrable functions on the interval $[0, \infty)$, Agrawal and Mohammad [2] introduced the following operators:

$$M_n(f(t); x) = n \sum_{v=1}^{\infty} p_{n,v}(x) \int_0^{\infty} q_{n,v-1}(t) f(t) dt + (1+x)^{-n} f(0). \tag{2}$$

where

$$p_{n,v}(x) = \binom{n+v-1}{v} x^v (1+x)^{-(n+v)}, \quad x \in [0, \infty)$$

and

$$q_{n,v}(t) = \frac{e^{-nt}(nt)^v}{v!}, \quad \forall\, t \in [0, \infty).$$

In [2], Agrawal et al. studied the asymptotic approximation and error estimates in terms of modulus of continuity in simultaneous approximation by (2).

In [16], Gupta and Srivastava considered a sequence of positive linear operators combining the Baskakov and Szász basis functions. Deo [8] studied the simultaneous approximation by Lupas operators with the weight functions of Szász operators.

**Definition 1** For $f \in C_\gamma[0, \infty) := \{f \in C[0, \infty) : f(t) = O(e^{\gamma t}) \text{ as } t \to \infty \text{ for some } \gamma > 0\}$ and each positive integer $n$, the $q$-Baskakov operators [5] are defined as

$$\begin{aligned} V_{n,q}(f; x) &= \sum_{k=0}^{\infty} \binom{n+k-1}{k}_q q^{\frac{k(k-1)}{2}} \frac{x^k}{(1+x)_q^{n+k}} f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right) \\ &= \sum_{k=0}^{\infty} p_{n,k}^q(x) f\left(\frac{[k]_q}{q^{k-1}[n]_q}\right). \end{aligned} \tag{3}$$

*Remark 1* The first three moments of the $q$-Baskakov operators (see [5]) are given by

$$V_{n,q}(1;x)=1,\quad V_{n,q}(t;x)=x,\quad V_{n,q}(t^2;x)=x^2+\frac{x}{[n]_q}\left(1+\frac{x}{q}\right).$$

**Definition 2** For $f\in C_\gamma[0,\infty)$, $0<q<1$ and each positive integer $n$, the $q$-Baskakov Száz operators defined as

$$B_{n,q}(f;x)=[n]_q\sum_{k=0}^{\infty}p_{n,k}^{q}(x)\int_0^{\frac{q}{(1-q^n)}}q^{-k-1}s_{n,k}^{q}(t)f\left(\frac{t}{q^k}\right)d_qt, \tag{4}$$

$$\text{where } p_{n,k}^{q}(x)=\binom{n+k-1}{k}q^{\frac{k(k-1)}{2}}\frac{x^k}{(1+x)_q^{(n+k)}}$$
$$\text{and } s_{n,k}^{q}(t)=E_q(-[n]_qt)\frac{([n]_qt)^k}{[k]_q!} \tag{5}$$

have been considered by Gupta [12].

## 2 Construction of Operators

For $f\in C_\gamma[0,\infty)$, $0<q<1$ and each positive integer $n$, the Stancu-type generalization of the operators (2) based on $q$-integers is defined as follows:

$$M_{n,q}^{(\alpha,\beta)}(f;x)=[n]_q\sum_{k=1}^{\infty}p_{n,k}^{q}(x)\int_0^{\frac{q}{(1-q^n)}}q^{-k}s_{n,k-1}^{q}(t)f\left(\frac{[n]_qtq^{-k}+\alpha}{[n]_q+\beta}\right)d_qt$$
$$+f\left(\frac{\alpha}{[n]_q+\beta}\right)p_{n,0}^{q}(x), \tag{6}$$

where $p_{n,k}^{q}(x)$ and $s_{n,k}^{q}(t)$ are as defined in (5).

If $\alpha=\beta=0$ and $q\to 1-$, the operators (6) reduce to the operators (2), which is a modification of the operator given by (4) where the value of the function at zero is considered explicitly. The aim of this paper is to study some direct results and asymptotic formula for the operators (6). We also discuss the rate of convergence and point-wise estimation. Lastly, we study the statistical approximation properties of these operators.

# 3 Basic Results

## 3.1 Moment Estimates

For $\alpha = \beta = 0$, we denote the operator $M_{n,q}^{(\alpha,\beta)}$ by $M_{n,q}$.

**Lemma 1** *For the operators $M_{n,q}(f; x)$, the following equalities hold:*

*(i)* $M_{n,q}(1; x) = 1$;
*(ii)* $M_{n,q}(t; x) = x$;
*(iii)* $M_{n,q}(t^2; x) = x^2\left(1 + \frac{1}{q[n]_q}\right) + \frac{[2]_q x}{[n]_q}$.

*Proof* First, for $f(t) = 1$, we have

$$M_{n,q}(1; x) = [n]_q \sum_{k=1}^{\infty} p_{n,k}^q(x) \int_0^{\frac{q}{(1-q^n)}} q^{-k} s_{n,k-1}(t) d_q t + p_{n,0}^q(x).$$

Substituting $[n]_q t = qy$ and using (1)

$$\begin{aligned} M_{n,q}(1; x) &= [n]_q \sum_{k=1}^{\infty} p_{n,k}^q(x) \int_0^{\frac{1}{1-q}} q^{-k+1} \frac{(qy)^{k-1}}{[k-1]_q!} \frac{E_q[-qy]}{[n]_q} d_q y + p_{n,0}^q(x) \\ &= \sum_{k=1}^{\infty} p_{n,k}^q(x) \frac{\Gamma_q k}{[k-1]_q!} + p_{n,0}^q(x) \\ &= \sum_{k=0}^{\infty} p_{n,k}^q(x) \\ &= V_{n,q}(1; x) = 1, \quad \text{in view of Remark 1.} \end{aligned}$$

Next, let $f(t) = t$, we have

$$M_{n,q}(t; x) = [n]_q \sum_{k=1}^{\infty} p_{n,k}^q(x) \int_0^{\frac{q}{(1-q^n)}} q^{-2k} t^k E_q(-[n]_q t) \frac{([n]_q t)^{k-1}}{[k-1]_q!} d_q t.$$

Again, substituting $[n]_q t = qy$ and using (1)

$$\begin{aligned} M_{n,q}(t; x) &= [n]_q \sum_{k=1}^{\infty} p_{n,k}^q(x) \int_0^{\frac{1}{1-q}} q^{-2k+1} E_q(-qy) \frac{(qy)^k}{[k-1]_q!([n]_q)^2} d_q y \\ &= \sum_{k=1}^{\infty} p_{n,k}^q(x) \frac{1}{[n]_q [k-1]_q! q^{k-1}} \int_0^{\frac{1}{1-q}} E_q(-qy) y^k d_q y \\ &= \sum_{k=1}^{\infty} p_{n,k}^q(x) \frac{\Gamma_q(k+1)}{[n]_q [k-1]_q! q^{k-1}} \end{aligned}$$

$$= \sum_{k=1}^{\infty} p_{n,k}^{q}(x) \frac{[k]_q}{[n]_q q^{k-1}}$$

$$= \sum_{k=0}^{\infty} p_{n,k}^{q}(x) \frac{[k]_q}{[n]_q q^{k-1}} = V_{n,q}(t;x) = x, \quad \text{on applying Remark 1.}$$

Finally, we give the second moment as follows:

$$M_{n,q}(t^2;x) = [n]_q \sum_{k=1}^{\infty} p_{n,k}^{q}(x) \int_0^{\frac{q}{(1-q^n)}} \left(\frac{t}{q^k}\right)^2 q^{-k} \frac{([n]_q t)^{k-1}}{[k-1]_q!} E_q(-[n]_q t) d_q t.$$

Again, substituting $[n]_q t = qy$, using (1) and $[k+1]_q = [k]_q + q^k$, we have

$$M_{n,q}(t^2;x) = [n]_q \sum_{k=1}^{\infty} p_{n,k}^{q}(x) \int_0^{\frac{1}{1-q}} q^{-k} \frac{E_q(-qy)}{q^{2k}} \frac{(qy)^2}{([n]_q)^2} \frac{(qy)^{k-1} q}{[k-1]_q! [n]_q} d_q y$$

$$= \sum_{k=1}^{\infty} p_{n,k}^{q}(x) \frac{1}{q^{2k-2}([n]_q)^2 [k-1]_q!} \int_0^{\frac{1}{1-q}} E_q(-qy) y^{k+1} d_q y$$

$$= \sum_{k=1}^{\infty} p_{n,k}^{q}(x) \frac{1}{q^{2k-2}([n]_q)^2 [k-1]_q!} \Gamma(k+2)_q$$

$$= \sum_{k=1}^{\infty} p_{n,k}^{q}(x) \frac{1}{q^{2k-2}([n]_q)^2} [k]_q ([k]_q + q^k)$$

$$= \sum_{k=1}^{\infty} p_{n,k}^{q}(x) \frac{([k]_q)^2}{([n]_q)^2 q^{2k-2}} + \frac{q}{[n]_q} \sum_{k=1}^{\infty} p_{n,k}^{q}(x) \frac{[k]_q}{q^{k-1}[n]_q}$$

$$= V_{n,q}(t^2;x) + \frac{q}{[n]_q} V_{n,q}(t;x)$$

$$= x^2 + \frac{x}{[n]_q}\left(1 + \frac{x}{q}\right) + \frac{qx}{[n]_q}$$

$$= x^2\left(1 + \frac{1}{q[n]_q}\right) + \frac{[2]_q x}{[n]q}, \quad \text{on using Remark 1.}$$

□

**Lemma 2** *For* $M_{n,q}^{(\alpha,\beta)}(t^m;x)$, $m = 0, 1, 2$ *we have*

*(i)* $M_{n,q}^{(\alpha,\beta)}(1;x) = 1$;

*(ii)* $M_{n,q}^{(\alpha,\beta)}(t;x) = \dfrac{[n]_q x + \alpha}{[n]_q + \beta}$;

*(iii)* $M_{n,q}^{(\alpha,\beta)}(t^2;x) = \dfrac{[n]_q(1 + q[n]_q)x^2}{q([n]_q + \beta)^2} + \dfrac{[n]_q([2]_q + 2\alpha)x}{([n_q] + \beta)^2} + \dfrac{\alpha^2}{([n]_q + \beta)^2}$.

*Proof* Using Lemma 1, we estimate the moments as follows:

For $f(t) = 1$, we have

$$M_{n,q}^{(\alpha,\beta)}(1;x) = [n]_q \sum_{k=1}^{\infty} p_{n,k}^{q}(x) \int_0^{\frac{q}{1-q^n}} q^{-k} s_{n,k-1}^{q}(t) d_q t + p_{n,0}^{q}(x) = M_{n,q}(1;x) = 1.$$

Next, we obtain the first-order moment

$$\begin{aligned} M_{n,q}^{(\alpha,\beta)}(t;x) &= [n]_q \sum_{k=1}^{\infty} p_{n,k}^{q}(x) \int_0^{\frac{q}{1-q^n}} q^{-k} s_{n,k-1}^{q}(t) \left(\frac{[n]_q t q^{-k} + \alpha}{[n]_q + \beta}\right) d_q t + p_{n,0}^{q}(x) \left(\frac{\alpha}{[n]_q + \beta}\right) \\ &= \frac{[n]_q}{[n]_q + \beta} M_{n,q}(t;x) + \frac{\alpha}{[n]_q + \beta} M_{n,q}(1;x) \\ &= \frac{[n]_q}{[n]_q + \beta} x + \frac{\alpha}{([n]_q + \beta)} \\ &= \frac{[n]_q x + \alpha}{[n]_q + \beta}. \end{aligned}$$

Finally, for $f(t) = t^2$ we obtain

$$\begin{aligned} M_{n,q}^{(\alpha,\beta)}(t^2;x) &= [n]_q \sum_{k=1}^{\infty} p_{n,k}^{q}(x) \int_0^{\frac{q}{1-q^n}} q^{-k} s_{n,k-1}^{q}(t) \left(\frac{[n]_q t q^{-k} + \alpha}{[n]_q + \beta}\right)^2 d_q t + p_{n,0}^{q}(x) \left(\frac{\alpha}{[n]_q + \beta}\right)^2 \\ &= \frac{([n]_q)^2}{([n]_q + \beta)^2} M_{n,q}(t^2;x) + \frac{2[n]_q \alpha}{([n]_q + \beta)^2} M_{n,q}(t,x) + \frac{\alpha^2}{([n]_q + \beta)^2} M_{n,q}(1;x) \\ &= \frac{[n]_q^2}{([n]_q + \beta)^2} \left\{ x^2 \left(1 + \frac{1}{q[n]_q}\right) + \frac{x(1+q)}{[n]_q} \right\} + \frac{2[n]_q \alpha}{([n]_q + \beta)^2} x + \frac{\alpha^2}{([n]_q + \beta)^2} \\ &= \frac{[n]_q (1 + q[n]_q)}{q([n]_q + \beta)^2} x^2 + \frac{[n]_q ([2]_q + 2\alpha)}{([n]_q + \beta)^2} x + \frac{\alpha^2}{([n]_q + \beta)^2}. \end{aligned}$$

Hence, the proof is completed. □

*Remark 2* By simple computation, we have

$$\begin{aligned} M_{n,q}^{(\alpha,\beta)}((t-x);x) &= \frac{\alpha - \beta x}{[n]_q + \beta}, \\ M_{n,q}^{(\alpha,\beta)}((t-x)^2;x) &= \frac{x^2([n]_q + q\beta^2)}{q([n]_q + \beta)^2} + \frac{x([2]_q [n]_q - 2\alpha\beta)}{([n]_q + \beta)^2} + \frac{\alpha^2}{([n]_q + \beta)^2}. \end{aligned}$$

**Lemma 3** *For every $q \in (0,1)$ we have*

$$M_{n,q}^{(\alpha,\beta)}((t-x)^2;x) \le \frac{[2]_q (1 + \beta^2)}{q([n]_q + \beta)} \left(\phi^2(x) + \frac{1}{([n]_q + \beta)}\right),$$

*where $\phi(x) = \sqrt{x(1+x)}$, $x \in [0,\infty)$.*

*Proof*

$$
\begin{aligned}
M_{n,q}^{(\alpha,\beta)}((t-x)^2;x) &= \frac{x^2([n]_q + q\beta^2)}{q([n]_q + \beta)^2} + \frac{x([2]_q[n]_q - 2\alpha\beta)}{([n]_q + \beta)^2} + \frac{\alpha^2}{([n]_q + \beta)^2} \\
&\leq \frac{([2]_q[n]_q + \beta^2)}{q([n]_q + \beta)^2}(x^2 + x) + \frac{\alpha^2}{([n]_q + \beta)^2} \\
&\leq \frac{[2]_q([n]_q + \beta^2)}{q([n]_q + \beta)^2}(x^2 + x) + \frac{\alpha^2}{([n]_q + \beta)^2} \\
&\leq \frac{[2]_q[n]_q(1 + \beta^2)}{q([n]_q + \beta)^2}\phi^2(x) + \frac{\alpha^2}{([n]_q + \beta)^2} \\
&\leq \frac{[2]_q(1 + \beta^2)}{q([n]_q + \beta)}\phi^2(x) + \frac{\alpha^2}{([n]_q + \beta)^2} \\
&\leq \frac{[2]_q(1 + \beta^2)}{q([n]_q + \beta)}\left(\phi^2(x) + \frac{1}{[n]_q + \beta}\right).
\end{aligned}
$$

This completes the proof. □

## 4 Main Results

If $q = \{q_n\}$ be a sequence in $(0, 1)$ satisfying the following conditions:

$$
\lim_{n\to\infty} q_n = 1 \text{ and } \lim_{n\to\infty} q_n^n = c, (0 \leq c < 1). \tag{7}
$$

Our first result is a basic convergence theorem for the operators $M_{n,q_n}^{(\alpha,\beta)}$.

**Theorem 1** *Let* $q_n \in (0, 1)$ *and* $\lim_{n\to\infty} q_n^n = c, (0 \leq c < 1)$. *Then the sequence* $M_{n,q_n}^{(\alpha,\beta)}(f; x)$ *converges to* $f$ *uniformly on* $[0, A]$, $A > 0$, *for each* $f \in C_\gamma[0, \infty)$ *if and only if* $\lim_{n\to\infty} q_n = 1$.

*Remark 3* If $\lim_{n\to\infty} q_n = 1$, then in view of Remark 2, $M_{n,q_n}^{(\alpha,\beta)}((t-x)^2; x) \to 0$ uniformly on $[0, A]$ as $n \to \infty$. Therefore, the well-known Korovkin theorem implies that $\{M_{n,q_n}^{(\alpha,\beta)}(f; x)\}$ converges to $f$ uniformly on $[0, A]$ for each $f \in C_\gamma[0, \infty)$. The converse part follows on proceeding in a manner similar to the proof of [3], Theorem 1.

## 4.1 Direct Theorem

Let $C_B[0, \infty)$ be the space of all continuous and bounded functions $f$ defined on the interval $[0, \infty)$, endowed with the norm $\|.\|$ on the space given by

$$\|f\| = \sup_{0 \le x < \infty} | f(x) | . \tag{8}$$

If $\delta > 0$ and $W^2 = \{g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty)\}$, then the $K$-functional is defined as

$$K_2(f, \delta) = \inf\{\|f - g\| + \delta\|g''\| : g \in W^2\}. \tag{9}$$

By ([9], p. 177, Theorem 2, 4) there exists an absolute constant $C > 0$ such that $K_2(f, \delta) \le C\omega_2(f, \sqrt{\delta})$,
where second order modulus of the smoothness of $f \in C_B[0, \infty)$ is defined as

$$\omega_2(f, \sqrt{\delta}) = \sup_{0<h\le\sqrt{\delta}} \sup_{0\le x<\infty} | f(x + 2h) - 2f(x + h) + f(x) | .$$

The first-order modulus of continuity is defined as

$$\omega(f, \delta) = \sup_{0<h\le\sqrt{\delta}} \sup_{0\le x<\infty} | f(x + h) - f(x) | .$$

The next result is a direct local approximation theorem for the operators $M_{n,q}^{(\alpha,\beta)}$.

**Theorem 2** *Let $f \in C_B[0, \infty)$ and let $\{q_n\}$ be sequence satisfying the conditions (7). Then, for every $x \in [0, \infty)$ we have*

$$| M_{n,q}^{(\alpha,\beta)}(f; x) - f(x) | \le C\omega_2\left(f, \sqrt{\frac{4(1 + \beta^2)}{q([n]_q + \beta)}\left\{\phi^2(x) + \frac{1}{([n]_q + \beta)}\right\}}\right) + \omega\left(f, \frac{| \alpha - \beta x |}{[n]_q + \beta}\right).$$

*Proof* We introduce auxiliary operator $L_{n,q}^{(\alpha,\beta)}$ as follows:

$$L_{n,q}^{(\alpha,\beta)}(f; x) = M_{n,q}^{(\alpha,\beta)}(f; x) - f\left(x + \frac{\alpha - \beta x}{([n]_q + \beta)}\right) + f(x). \tag{10}$$

These operators are linear and preserve the linear functions. Hence, we have

$$L_{n,q}^{(\alpha,\beta)}(t - x; x) = 0. \tag{11}$$

Let $g \in W^2$. From the Taylor's expansion of $g$, we get

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du, \ \ t \in [0,\infty).$$

In view of (10), we get

$$\begin{aligned}
L_{n,q}^{(\alpha,\beta)}(g;x) &= g(x) + L_{n,q}^{(\alpha,\beta)}\left(\int_x^t (t-u)g''(u)du; x\right)\\
\mid L_{n,q}^{(\alpha,\beta)}(g;x) - g(x) \mid &= \left|L_{n,q}^{(\alpha,\beta)}\left(\int_x^t (t-u)g''(u)du; x\right)\right|\\
&\leq \left|M_{n,q}^{(\alpha,\beta)}\left(\int_x^t (t-u)g''(u)du; x\right)\right|\\
&+ \left|\int_x^{x+\frac{\alpha-\beta x}{([n]_q+\beta)}} \left(x + \frac{\alpha-\beta x}{[n]_q+\beta} - u\right)g''(u)du\right|\\
&\leq M_{n,q}^{(\alpha,\beta)}\left(\left|\int_x^t (t-u)g''(u)du\right|; x\right)\\
&+ \left|\int_x^{x+\frac{\alpha-\beta x}{([n]_q+\beta)}} \left|(x + \frac{\alpha-\beta x}{([n]_q+\beta)} - u)\right| |g''(u)|du\right|\\
&\leq \left\{M_{n,q}^{(\alpha,\beta)}((t-x)^2;x) + \left(\frac{(\alpha-\beta x)}{([n]_q+\beta)}\right)^2\right\}\|g''\|.
\end{aligned} \tag{12}$$

$$\begin{aligned}
\left(\frac{\alpha-\beta x}{([n]_q+\beta)}\right)^2 &= \frac{(\alpha^2 - 2\alpha\beta x + \beta^2 x^2)}{([n]_q+\beta)^2} \leq \frac{\alpha^2 + 2\alpha\beta x + \beta^2 x^2}{([n]_q+\beta)^2} \leq \frac{\beta^2(1+2x+x^2)}{([n]_q+\beta)^2}\\
&\leq \frac{2(1+\beta^2)}{q([n]_q+\beta)}\left\{x(1+x) + \frac{1}{([n]_q+\beta)}\right\}\\
&= \frac{2(1+\beta^2)}{q([n]_q+\beta)}\left\{\phi^2(x) + \frac{1}{([n]_q+\beta)}\right\}.
\end{aligned} \tag{13}$$

On the other hand, from (6), (10) and Lemma 2, we have

$$|L_{n,q}^{(\alpha,\beta)}(f;x)| \leq |M_{n,q}^{(\alpha,\beta)}(f,x)| + 2\|f\| \leq \|f\| M_{n,q}^{(\alpha,\beta)}(1;x) + 2\|f\| \leq 3\|f\|.$$

From (12) and (13), we have

$$\mid L_{n,q}^{(\alpha,\beta)}(g;x) - g(x) \mid \ \ \leq \frac{4(1+\beta^2)}{q([n]_q+\beta)}\left\{\phi^2(x) + \frac{1}{([n]_q+\beta)}\right\}\|g''\|.$$

Hence

$$\begin{aligned} | M_{n,q}^{(\alpha,\beta)}(f;x) - f(x) | &\leq | L_{n,q}^{(\alpha,\beta)}(f-g;x) - (f-g)(x) | + | L_{n,q}^{(\alpha,\beta)}(g;x) - g(x) | \\ &\quad + \left| f\left(x + \frac{\alpha - \beta x}{([n]_q + \beta)}\right) - f(x) \right| \\ &\leq 4\|f-g\| + \frac{4(1+\beta^2)}{q([n]_q+\beta)}\left\{\phi^2(x) + \frac{1}{([n]_q+\beta)}\right\}\|g''\| \\ &\quad + \omega\left(f, \frac{|\alpha - \beta x|}{([n]_q+\beta)}\right). \end{aligned}$$

Now, taking infimum on the right-hand side over all $g \in W^2$, we get

$$\begin{aligned} | M_{n,q}^{(\alpha,\beta)}(f;x) - f(x) | &\leq C\omega_2\left(f, \sqrt{\frac{4(1+\beta^2)}{q([n]_q+\beta)}\left\{\phi^2(x) + \frac{1}{([n]_q+\beta)}\right\}}\right) \\ &\quad + \omega\left(f, \frac{|\alpha - \beta x|}{[n]_q+\beta}\right). \end{aligned}$$

Hence, the proof is completed. □

## 4.2 Rate of Convergence

Let $B_{x^2}[0,\infty)$ be the space of all functions defined on $[0,\infty)$ and satisfying the condition $|f(x)| \leq M_f(1+x^2)$, where $M_f$ is a constant depending on $f$. Let $C_{x^2}[0,\infty)$ be the subspace of all continuous functions belonging to $B_{x^2}[0,\infty)$. Also, $C^*_{x^2}[0,\infty)$ is the subspace of all functions $f \in C_{x^2}[0,\infty)$, for which $\lim_{x\to\infty}\frac{f(x)}{1+x^2}$ is finite. The norm on $C^*_{x^2}[0,\infty)$ is defined as $\|f\|_{x^2} := \sup_{x\in[0,\infty)}\frac{|f(x)|}{1+x^2}$. For any positive number $a$, the usual modulus of continuity is defined as

$$\omega_a(f,\delta) = \sup_{|t-x|\leq\delta,\ x,t\in[0,a]} |f(t) - f(x)|.$$

We observe that for a function $f \in C_{x^2}[0,\infty)$, the modulus of continuity $\omega_a(f,\delta)$ tends to zero as $\delta \to 0$. Now we give a rate of convergence theorem for the operator $M_{n,q_n}^{(\alpha,\beta)}$.

**Theorem 3** *Let* $f \in C_{x^2}[0,\infty)$, $q_n \in (0,1)$ *such that* $q_n \to 1$ *as* $n \to \infty$ *and* $\omega_{a+1}$ *be its modulus of continuity on the finite interval* $[0, a+1] \subset [0,\infty)$, *where* $a > 0$, *then we have the following inequality:*

$$|M_{n,q_n}^{(\alpha,\beta)}(f;x)-f(x)| \le \frac{K}{q_n([n]_{q_n}+\beta)}\left\{\phi^2(x)+\frac{1}{([n]_{q_n}+\beta)}\right\} + 2\omega_{a+1}\left(f,\sqrt{\frac{2(1+\beta^2)}{q_n([n]_{q_n}+\beta)}\left(\phi^2(x)+\frac{1}{[n]_{q_n}+\beta}\right)}\right),$$

*where* $K = 8M_f(1+a^2)(1+\beta^2)$.

*Proof* For $x \in [0,a]$ and $t > a+1$, since $t-x>1$, we have

$$\begin{aligned} |f(t)-f(x)| &\le M_f(2+x^2+t^2) \le M_f(2+3x^2+2(t-x)^2) \\ &\le M_f(t-x)^2(2+3x^2+2) \le M_f(t-x)^2(4+3a^2) \\ |f(t)-f(x)| &\le 4M_f(1+a^2)(t-x)^2. \end{aligned} \tag{14}$$

For $x \in [0,a]$ and $t \le a+1$, we have

$$|f(t)-f(x)| \le \omega_{a+1}(f,|t-x|) \le \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta), \quad \text{with } \delta > 0. \tag{15}$$

From (14) and (15), for all $t \in [0,\infty)$ and $x \in [0,a]$ we can write

$$|f(t)-f(x)| \le 4M_f(1+a^2)(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta). \tag{16}$$

Hence, using Schwarz inequality,

$$\begin{aligned} |M_{n,q_n}^{(\alpha,\beta)}(f;x)-f(x)| &\le M_{n,q_n}^{(\alpha,\beta)}(|f(t)-f(x)|;x) \\ &\le 4M_f(1+a^2)M_{n,q_n}^{(\alpha,\beta)}((t-x)^2;x) \\ &\quad + \omega_{a+1}(f,\delta)\left(1+\frac{1}{\delta}\{M_{n,q_n}^{(\alpha,\beta)}((t-x)^2;x)\}^{\frac{1}{2}}\right). \end{aligned}$$

In view of Lemma 3, for $x \in [0,a]$

$$|M_{n,q_n}^{(\alpha,\beta)}(f;x)-f(x)| \le \frac{8M_f(1+a^2)(1+\beta^2)}{q_n([n]_{q_n}+\beta)}\left\{\phi^2(x)+\frac{1}{([n]_{q_n}+\beta)}\right\} + \omega_{a+1}(f,\delta)\left\{1+\frac{1}{\delta}\left[\frac{2(1+\beta^2)}{q_n([n]_{q_n}+\beta)}\left(\phi^2(x)+\frac{1}{([n]_{q_n}+\beta)}\right)\right]^{\frac{1}{2}}\right\}.$$

Now, by choosing $\delta = \sqrt{\frac{2(1+\beta^2)}{q_n([n]_{q_n}+\beta)}\left(\phi^2(x)+\frac{1}{([n]_{q_n}+\beta)}\right)}$, we get the desired result. □

## 4.3 Voronovskaja Type Theorem

In this section we establish a Voronovskaja type asymptotic formula for the operators $M_{n,q}^{(\alpha,\beta)}$.

**Lemma 4** *Assume that $q_n \in (0, 1)$, $q_n \to 1$ as $n \to \infty$. Then, for every $x \in [0, \infty)$ there hold*

$$\lim_{n\to\infty} [n]_{q_n} M_{n,q_n}^{(\alpha,\beta)}(t-x; x) = \alpha - \beta x$$

*and*

$$\lim_{n\to\infty} [n]_{q_n} M_{n,q_n}^{(\alpha,\beta)}((t-x)^2; x) = x^2 + 2x.$$

*In view of Remark 2, the proof of this Lemma easily follows. Hence the details are omitted.*

**Theorem 4** *Let $0 < q_n < 1$ and $q_n \to 1$ as $n \to \infty$. Then, for all $f \in C_{x^2}[0, \infty)$ we have*

$$\lim_{n\to\infty} \|M_{n,q_n}^{(\alpha,\beta)}(f) - f\|_{x^2} = 0.$$

*Proof* Using [11], it is sufficient to verify the following conditions:

$$\lim_{n\to\infty} \|M_{n,q_n}^{(\alpha,\beta)}(t^m; x) - x^m\|_{x^2} = 0, \text{ for } m = 0, 1, 2. \tag{17}$$

Since $M_{n,q_n}^{(\alpha,\beta)}(1; x) = 1$, for $m = 0$, (17) holds. By Lemma 2, we have

$$\begin{aligned}
\|M_{n,q_n}^{(\alpha,\beta)}(t; x) - x\|_{x^2} &= \sup_{x\in[0,\infty)} \frac{|M_{n,q_n}^{(\alpha,\beta)}(t; x) - x|}{(1 + x^2)} \\
&\le \sup_{x\in[0,\infty)} \frac{\left|\frac{[n]_{q_n}x + \alpha}{([n]_{q_n} + \beta)} - x\right|}{1 + x^2} \\
&\le \frac{\beta}{([n]_{q_n} + \beta)} \sup_{x\in[0,\infty)} \frac{x}{(1 + x^2)} + \frac{\alpha}{([n]_{q_n} + \beta)} \sup_{x\in[0,\infty)} \frac{1}{1 + x^2} \\
&\le \frac{\alpha + \beta}{([n]_{q_n} + \beta)} = o(1) \quad as \quad n \to \infty.
\end{aligned}$$

Hence, the condition (17) holds for $m = 1$.

Again, by Lemma 2, we obtain

$$\begin{aligned}\|M_{n,q_n}^{(\alpha,\beta)}(t^2;x)-x^2\|_{x^2} &= \sup_{x\in[0,\infty)}\frac{|M_{n,q_n}^{(\alpha,\beta)}(t^2;x)-x^2|}{(1+x^2)}\\ &= \sup_{x\in[0,\infty)}\frac{\left|\frac{[n]_{q_n}(1+q_n[n]_{q_n})x^2}{q_n([n]_{q_n}+\beta)^2}+\frac{[n]_{q_n}(1+q_n+2\alpha)x}{([n]_{q_n}+\beta)^2}+\frac{\alpha^2}{([n]_{q_n}+\beta)^2}-x^2\right|}{1+x^2}\\ &\leq \frac{([n]_{q_n}(1+2q_n\beta)+\beta^2)}{q_n([n]_{q_n}+\beta)^2}\sup_{x\in[0,\infty)}\frac{x^2}{(1+x^2)}\\ &+\frac{[n]_{q_n}(1+q_n+2\alpha)}{([n]_{q_n}+\beta)^2}\sup_{x\in[0,\infty)}\frac{x}{(1+x^2)}\\ &+\frac{\alpha^2}{([n]_{q_n}+\beta)^2}\sup_{x\in[0,\infty)}\frac{1}{1+x^2}=o(1)\quad as\quad n\to\infty,\end{aligned}$$

which implies that the condition (17) holds for $m=2$. This completes the proof. □

**Theorem 5** *Assume that* $q_n\in(0,1)$, $q_n\to 1$ *as* $n\to\infty$. *Then, for any* $f\in C_{x^2}^*[0,\infty)$ *such that* $f',f''\in C_{x^2}^*[0,\infty)$ *we have*

$$\lim_{n\to\infty}[n]_{q_n}(M_{n,q_n}^{(\alpha,\beta)}(f;x)-f(x))=(\alpha-\beta x)f'(x)+\frac{1}{2}f''(x)(x^2+2x),$$
$$\textit{uniformly in } x\in[0,A],\quad A>0.$$

*Proof* Let $f,f',f''\in C_{x^2}^*[0,\infty)$ and $x\in[0,A]$ be fixed. By Taylor's expansion, we may write

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+r(t,x)(t-x)^2, \tag{18}$$

where $r(t,x)$ is Peano form of the remainder, $r(.,x)\in C_{x^2}^*[0,\infty)$ and $\lim_{t\to x}r(t,x)=0$.

Applying $M_{n,q_n}^{(\alpha,\beta)}$ to the above Eq. (18) we obtain

$$\begin{aligned}[n]_{q_n}(M_{n,q_n}^{(\alpha,\beta)}(f;x)-f(x)) &= f'(x)[n]_{q_n}M_{n,q}^{(\alpha,\beta)}(t-x;x)\\ &+\frac{1}{2}f''(x)[n]_{q_n}M_{n,q}^{(\alpha,\beta)}((t-x)^2;x)\\ &+[n]_{q_n}M_{n,q}^{(\alpha,\beta)}\Big(r(t,x)(t-x)^2;x\Big).\end{aligned}$$

By Cauchy Schwarz inequality, we have

$$M_{n,q_n}^{(\alpha,\beta)}\Big(r(t,x)(t-x)^2;x\Big) \le \sqrt{M_{n,q_n}^{(\alpha,\beta)}\Big(r^2(t,x);x\Big)}\sqrt{M_{n,q_n}^{(\alpha,\beta)}\Big((t-x)^4;x\Big)}. \tag{19}$$

We observe that $r^2(x,x) = 0$ and $r^2(.,x) \in C^*_{x^2}[0,\infty))$. Then, it follows from Theorem 3 that

$$\lim_{n\to\infty}[n]_{q_n}(M_{n,q_n}^{(\alpha,\beta)}(r^2(t,x),x) = r^2(x,x) = 0, \tag{20}$$

uniformly with respect to $x \in [0,A]$. Now, from (19)–(20) and in view of the fact that

$$M_{n,q_n}^{(\alpha,\beta)}((t-x)^4;x) = O\bigg(\frac{1}{[n]_{q_n}}\bigg)^2$$

we obtain

$$\lim_{n\to\infty}[n]_{q_n}M_{n,q_n}^{(\alpha,\beta)}(r(t,x)(t-x)^2,x) = 0,$$

uniformly in $x \in [0,A]$. Thus, we obtain

$$\begin{aligned}\lim_{n\to\infty}[n]_{q_n}\Big(M_{n,q_n}^{(\alpha,\beta)}(f,x)-f(x)\Big) &= \lim_{n\to\infty}[n]_{q_n}\Big(f'(x)M_{n,q_n}^{(\alpha,\beta)}((t-x);x)\\ &\quad + \frac{1}{2}f''(x)M_{n,q_n}^{(\alpha,\beta)}((t-x)^2;x)\\ &\quad + M_{n,q_n}^{(\alpha,\beta)}(r(t,x)(t-x)^2,x)\Big)\\ &= (\alpha-\beta x)f'(x) + \frac{1}{2}f''(x)(x^2+2x),\end{aligned}$$

uniformly in $x \in [0,A]$. □

**Corollary 1** *Let* $q = q_n$ *satisfy* $0 < q_n < 1$ *and let* $q_n \to 1$ *as* $n \to \infty$. *For each* $f \in C_{x^2}[0,\infty)$ *and* $p > 0$, *we have*

$$\sup_{x\in[0,\infty)}\frac{|M_{n,q_n}^{(\alpha,\beta)}(f;x)-f(x)|}{(1+x^2)^{1+p}} = 0.$$

*Proof* For any fixed $x_0 > 0$

$$\sup_{x\in[0,\infty)} \frac{|M^{\alpha,\beta}_{n,q_n}(f;x) - f(x)|}{(1+x^2)^{1+p}} \leq \sup_{x\leq x_0} \frac{|M^{(\alpha,\beta)}_{n,q_n}(f;x) - f(x)|}{(1+x^2)^{1+p}} + \sup_{x\geq x_0} \frac{|M^{(\alpha,\beta)}_{n,q_n}(f;x) - f(x)|}{(1+x^2)^{1+p}}$$
$$\leq \|M^{(\alpha,\beta)}_{n,q_n}(f) - f\|_{C[0,x_0]} + \|f\|_{x^2} \sup_{x\geq x_0} \frac{M^{(\alpha,\beta)}_{n,q_n}(1+t^2,x)}{(1+x^2)^{1+p}}$$
$$+ \sup_{x\geq x_0} \frac{|f(x)|}{(1+x^2)^{1+p}}. \tag{21}$$

Since $|f(x)| \leq M_f(1+x^2)$, we have

$$\sup_{x\geq x_0} \frac{|f(x)|}{(1+x^2)^{1+p}} \leq \sup_{x\geq x_0} \frac{M_f}{(1+x^2)^p} \leq \frac{M_f}{(1+x_0^2)^p}.$$

Let $\varepsilon > 0$ be arbitrary. Then, we can choose $x_0$ to be so large that

$$\frac{M_f}{(1+x_0^2)^p} < \frac{\varepsilon}{3} \tag{22}$$

and in view of Theorem 4, we obtain

$$\|f\|_{x^2} \lim_{n\to\infty} \frac{M^{(\alpha,\beta)}_{n,q_n}(1+t^2,x)}{(1+x^2)^{1+p}} = \frac{(1+x^2)\|f\|_{x^2}}{(1+x^2)^{1+p}} = \frac{\|f\|_{x^2}}{(1+x^2)^p} \leq \frac{\|f\|_{x^2}}{(1+x_0^2)^p} < \frac{\varepsilon}{3}. \tag{23}$$

Using Theorem 3, we see that the first term of inequality (21) implies that

$$\|M^{(\alpha,\beta)}_{n,q_n}(f) - f\|_{C[0,x_0]} < \frac{\varepsilon}{3} \quad \text{as} \quad n \to \infty. \tag{24}$$

Combining (22)–(24), we get the desired result. □

## 4.4 Point-Wise Estimates

Now, we establish some pointwise estimates of the rate of convergence of the operators (6). First, we give the relationship between the local smoothness of $f$ and local approximation.

We know that a function $f \in C_B[0,\infty)$ is in $Lip_M\gamma$ on D, $\gamma \in (0,1]$, $D \subset [0,\infty)$ if it satisfies the condition

$$|f(t) - f(x)| \leq M|t-x|^\gamma, \quad t \in [0,\infty) \text{ and } x \in D,$$

where $M$ is a constant depending only on $\gamma$ and $f$.

**Theorem 6** *Let $f \in C_B[0,\infty) \bigcap Lip_M\gamma$, $\gamma \in (0,1]$, and D be any bounded subset of the interval $[0,\infty)$. Then, for each $x \in [0,\infty)$ we have*

$$|M_{n,q}^{(\alpha,\beta)}(f;x)-f(x)| \leq M\left(\left\{\frac{[2]_q(1+\beta^2)}{q([n]_q+\beta)}\left(\phi^2(x)+\frac{1}{([n]_q+\beta)}\right)\right\}^{\frac{\gamma}{2}}+2(d(x,D))^\gamma\right),$$

*where $d(x,D)$ represents the distance between x and D.*

*Proof* Let $\overline{D}$ be the closure of the set D in $[0,\infty)$. Then, there exists at least one point $x_0 \in \overline{D}$ such that

$$d(x,D)=|x-x_0|.$$

By the definition of $Lip_M\gamma$, we get

$$\begin{aligned}|M_{n,q}^{(\alpha,\beta)}(f;x)-f(x)| &\leq M_{n,q}^{(\alpha,\beta)}(|f(t)-f(x_0)|;x)+M_{n,q}^{(\alpha,\beta)}(|f(x_0)-f(x)|;x))\\ &\leq M\left\{M_{n,q}^{(\alpha,\beta)}(|t-x_0|^\gamma;x)+|x_0-x|^\gamma\right\}\\ &\leq M\left\{M_{n,q}^{(\alpha,\beta)}(|t-x|^\gamma,x)+2|x-x_0|^\gamma\right\}.\end{aligned}$$

Now, by Holder's inequality with $p=\frac{2}{\gamma}$ and $\frac{1}{q}=1-\frac{1}{p}$, we have

$$\begin{aligned}|M_{n,q}^{(\alpha,\beta)}(f;x)-f(x)| &\leq M\left\{\left[M_{n,q}^{(\alpha,\beta)}(|t-x|^{\gamma p};x)\right]^{\frac{1}{p}}\left[M_{n,q}^{(\alpha,\beta)}(1^q,x)\right]^{\frac{1}{q}}+2(d(x,D))^\gamma\right\}\\ &\leq M\left\{\left[M_{n,q}^{(\alpha,\beta)}(|t-x|^2;x)\right]^{\frac{\gamma}{2}}+2(d(x,D))^\gamma\right\}\\ &\leq M\left(\left\{\frac{[2]_q(1+\beta^2)}{q([n]_q+\beta)}\left(\phi^2(x)+\frac{1}{([n]_q+\beta)}\right)\right\}^{\frac{\gamma}{2}}+2(d(x,D))^\gamma\right).\end{aligned}$$

Hence, the proof is completed. □

Now, we give local direct estimate for the operators $M_{n,q}^{(\alpha,\beta)}$ using the Lipschitz type maximal function of order $\gamma$ studied by Lenze [18]

$$\widetilde{\omega}_\gamma(f,x)=\sup_{t\neq x,t\in[0,\infty)}\frac{|f(t)-f(x)|}{|t-x|^\gamma},\quad x\in[0,\infty)\ \text{and}\ \gamma\in(0,1]. \tag{25}$$

**Theorem 7** *Let $\gamma \in (0,1]$ and $f \in C_B[0,\infty)$. Then, for all $x \in [0,\infty)$, we have*

$$|M_{n,q}^{(\alpha,\beta)}(f;x)-f(x)| \leq \widetilde{\omega}_\gamma(f,x)\left\{\frac{[2]_q(1+\beta^2)}{q([n]_q+\beta)}\left(\phi^2(x)+\frac{1}{([n]_q+\beta)}\right)\right\}^{\frac{\gamma}{2}}.$$

*Proof* From (25), we have

$$|f(t)-f(x)| \leq \widetilde{\omega}_{\gamma}(f,x)|t-x|^{\gamma}$$

and hence

$$|M_{n,q}^{(\alpha,\beta)}(f;x)-f(x)| \leq M_{n,q}^{(\alpha,\beta)}(|f(t)-f(x)|;x) \leq \widetilde{\omega}_{\gamma}(f,x)M_{n,q}^{(\alpha,\beta)}(|t-x|^{\gamma};x).$$

Now, applying Holder's inequality with $p=\frac{2}{\gamma}$ and $\frac{1}{q}=1-\frac{1}{p}$, we have

$$|M_{n,q}^{(\alpha,\beta)}(f;x)-f(x)| \leq \widetilde{\omega}_{\gamma}(f,x)M_{n,q}^{(\alpha,\beta)}((t-x)^2;x)^{\frac{\gamma}{2}}.$$

On using Lemma 3, we have our assertion. □

## 4.5 Statistical Approximation

A sequence $(x_n)_n$ is said to be statistically convergent to a number $L$ denoted by $st-\lim_n x_n = L$ if for every $\varepsilon > 0$,

$$\delta\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} = 0,$$

where

$$\delta(K) = \lim_n \frac{1}{n}\sum_{j=1}^{n} \chi_K(j)$$

is the natural density of $K \subseteq \mathbb{N}$ and $\chi_K$ is the characteristic function of $K$. We note that every convergent sequence is statistically convergent, but the converse need not be true.

For example, let

$$x_n = \begin{cases} \log_{10} n, & n \in \{10^k, k \in \mathbb{N}\} \\ 1, & \text{otherwise.} \end{cases}$$

It follows that the sequence $\{x_n\}$ converges statistically to 1, but $\lim_n x_n$ does not exit.

**Theorem 8** *For any $f \in C^*_{x^2}[0,\infty)$ and a sequence $(q_n)_n$ in $(0,1)$ such that*

$$st-\lim_n q_n = 1,\ st-\lim_n (q_n)^n = a, (0 \leq a < 1), st-\lim_n \frac{1}{[n]_{q_n}} = 0, \tag{26}$$

*the operator* $M_{n,q}^{(\alpha,\beta)}(f;x)$ *statistically converges to* $f(x)$, *that is*

$$st-\lim_{n} \parallel M_{n,q}^{(\alpha,\beta)}(f)-f\parallel_{x^2}=0.$$

*Proof* Let us define $e_i(x) = x^i, i = 0,1,2$. It is sufficient to prove that $st-\lim_{n}\|M_{n,q_n}^{(\alpha,\beta)}(e_i)-e_i\|_{x^2}=0$, for $i=0,1,2$. It is clear that

$$st-\lim_{n}\|M_{n,q_n}^{(\alpha,\beta)}(e_0;.)-e_0\|_{x^2}=0.$$

From Lemma 2

$$\begin{aligned}
\|M_{n,q_n}^{(\alpha,\beta)}(e_1;.)-e_1\|_{x^2} &= \sup_{x\in[0,\infty)}\frac{|M_{n,q_n}^{(\alpha,\beta)}(e_1;x)-e_1(x)|}{(1+x^2)}\\
&\leq \sup_{x\in[0,\infty)}\frac{\left|\frac{[n]_{q_n}x+\alpha}{([n]_{q_n}+\beta)}-x\right|}{1+x^2}\\
&\leq \|e_0\|_{x^2}\frac{\alpha}{([n]_{q_n}+\beta)}+\frac{\beta}{([n]_{q_n}+\beta)}\|e_1\|_{x^2}\\
&\leq \frac{\alpha}{([n]_{q_n}+\beta)}+\frac{\beta}{([n]_{q_n}+\beta)}. \qquad (27)
\end{aligned}$$

Since, by the conditions (26), we get

$$st-\lim_{n}\frac{\alpha}{([n]_{q_n}+\beta)}=0$$

and

$$st-\lim_{n}\frac{\beta}{([n]_{q_n}+\beta)}=0.$$

For $\varepsilon>0$, let us define the following sets:

$$\begin{aligned}
E &:= \left\{n\in\mathbb{N}:\parallel M_{n,q_n}^{(\alpha,\beta)}(e_1;.)-e_1\parallel_{x^2}\geq\varepsilon\right\},\\
E_1 &:= \left\{n\in\mathbb{N}:\frac{\alpha}{([n]_{q_n}+\beta)}\geq\frac{\varepsilon}{2}\right\},\\
E_2 &:= \left\{n\in\mathbb{N}:\frac{\beta}{([n]_{q_n}+\beta)}\geq\frac{\varepsilon}{2}\right\}.
\end{aligned}$$

By (27), it is clear that $E\subseteq E_1\bigcup E_2$ which implies that $\delta(E)\leq\delta(E_1)+\delta(E_2)=0$, and hence

$$st-\lim_{n}\|M_{n,q_n}^{(\alpha,\beta)}(e_1;.)-e_1\|_{x^2}=0.$$

Similarly, we can estimate

$$\begin{aligned}\|M_{n,q_n}^{(\alpha,\beta)}(e_2;.)-e_2\|_{x^2} &= \sup_{x\in[0,\infty)}\frac{|M_{n,q_n}^{(\alpha,\beta)}(e_2;x)-e_2(x)|}{(1+x^2)}\\ &= \sup_{x\in[0,\infty)}\frac{\left|\frac{[n]_{q_n}(1+q_n[n]_{q_n})x^2}{q_n([n]_{q_n}+\beta)^2}+\frac{[n]_{q_n}(1+q_n+2\alpha)x}{([n]_{q_n}+\beta)^2}+\frac{\alpha^2}{([n]_{q_n}+\beta)^2}-x^2\right|}{1+x^2}\\ &\le \frac{([n]_{q_n}(1+2q_n\beta)+\beta^2)}{q_n([n]_{q_n}+\beta)^2}\|e_2\|_{x^2}+\frac{[n]_{q_n}(1+q_n+2\alpha)}{([n]_{q_n}+\beta)^2}\|e_1\|_{x^2}\\ &+\frac{\alpha^2}{([n]_{q_n}+\beta)^2}\|e_0\|_{x^2}\\ &\le \frac{([n]_{q_n}(1+2q_n\beta)+\beta^2)}{q_n([n]_{q_n}+\beta)^2}+\frac{[n]_{q_n}(1+q_n+2\alpha)}{([n]_{q_n}+\beta)^2}+\frac{\alpha^2}{([n]_{q_n}+\beta)^2}.\end{aligned} \tag{28}$$

Again, using (26), we get

$$st-\lim_n \frac{([n]_{q_n}(1+2q_n\beta)+\beta^2)}{q_n([n]_{q_n}+\beta)^2}=0,$$

$$st-\lim_n \frac{[n]_{q_n}(1+q_n+2\alpha)}{([n]_{q_n}+\beta)^2}=0,$$

$$st-\lim_n \frac{\alpha^2}{([n]_{q_n}+\beta)^2}=0.$$

For a given $\varepsilon>0$, we consider the following sets:

$$F:=\left\{n\in\mathbb{N}:\| M_{n,q_n}^{(\alpha,\beta)}(e_2;.)-e_2\|_{x^2}\ge\varepsilon\right\},$$

$$F_1:=\left\{n\in\mathbb{N}:\frac{\left([n]_{q_n}(1+2q_n\beta)+\beta^2\right)}{q_n([n]_{q_n}+\beta)^2}\ge\frac{\varepsilon}{3}\right\},$$

$$F_2:=\left\{n\in\mathbb{N}:\frac{[n]_{q_n}(1+q_n+2\alpha)}{([n]_{q_n}+\beta)^2}\ge\frac{\varepsilon}{3}\right\},$$

$$F_3:=\left\{n\in\mathbb{N}:\frac{\alpha^2}{([n]_{q_n}+\beta)^2}\ge\frac{\varepsilon}{3}\right\}.$$

Consequently, by (28) we obtain $F\subseteq F_1\bigcup F_2\bigcup F_3$, which implies that $\delta(F)\le\delta(F_1)+\delta(F_2)+\delta(F_3)=0$. Hence, we get

$$st-\lim_n\|M_{n,q_n}^{(\alpha,\beta)}(e_2;.)-e_2\|_{x^2}=0.$$

This completes the proof of the theorem. □

### 4.5.1 Rate of Statistical Convergence

For $f \in C^*_{x^2}[0,\infty)$, following Freud [10], the weighted modulus of continuity of $f$ is defined as

$$\Omega_2(f,\delta) = \sup_{x \geq 0, 0<h\leq\delta} \frac{|f(x+h)-f(x)|}{1+(x+h)^2}.$$

**Lemma 5** *[19]. Let $f \in C^*_{x^2}[0,\infty)$. Then,*

*(i) $\Omega_2(f,\delta)$ is a monotone increasing function of $\delta$,*
*(ii) $\lim_{\delta\to 0} \Omega_2(f,\delta) = 0$,*
*(iii) For any $\lambda \in [0,\infty)$, $\Omega_2(f,\lambda\delta) \leq (1+\lambda)\Omega_2(f,\delta)$.*

**Theorem 9** *Let $f \in C^*_{x^2}[0,\infty)$ and $(q_n)_n$ be a sequence satisfying (26). Then, for sufficiently large $n$.*

$$|M^{(\alpha,\beta)}_{n,q_n}(f;x) - f(x)| \leq K\Omega_2(f,\delta_n)(1+x^{2+\lambda}), \quad x \in [0,\infty),$$

*where $\lambda \geq 1$, $\delta_n = \sqrt{\frac{[2]_{q_n}(1+\beta^2)}{q_n([n]_{q_n}+\beta)}}$ and $K$ is a positive constant independent $f$ and $n$.*

*Proof*

$$\begin{aligned}
|M^{(\alpha,\beta)}_{n,q_n}(f;x) - f(x)| &\leq M^{(\alpha,\beta)}_{n,q_n}(|f(t)-f(x)|;x) \\
&\leq M^{(\alpha,\beta)}_{n,q_n}\left\{(1+(x+|t-x|)^2)\left(1+\frac{|t-x|}{\delta}\right);x\right\}\Omega_2(f,\delta) \\
&\leq M^{(\alpha,\beta)}_{n,q_n}\left\{(1+(t+2x)^2)\left(1+\frac{|t-x|}{\delta}\right);x\right\}\Omega_2(f,\delta) \\
&\leq \left(M^{(\alpha,\beta)}_{n,q_n}(\mu_x(t);x) + \frac{1}{\delta}M^{(\alpha,\beta)}_{n,q_n}(\mu_x(t)\psi_x(t);x)\right)\Omega_2(f,\delta),
\end{aligned}$$

where $\mu_x(t) = 1+(t+2x)^2$ and $\psi_x(t) = |t-x|$.

Now, using Cauchy–Schwarz inequality to the second term on the right-hand side, we obtain

$$|M^{(\alpha,\beta)}_{n,q_n}(f;x) - f(x)| \leq \left(M^{(\alpha,\beta)}_{n,q_n}(\mu_x;x) + \frac{1}{\delta}\sqrt{M^{(\alpha,\beta)}_{n,q_n}(\psi_x^2;x)}\sqrt{M^{(\alpha,\beta)}_{n,q_n}(\mu_x^2;x)}\right)\Omega_2(f,\delta). \tag{29}$$

From Lemma 2

$$M^{(\alpha,\beta)}_{n,q_n}(1+t^2;x) = \left(1 + \frac{[n]_{q_n}(1+q_n[n]_{q_n})}{q_n([n]_{q_n}+\beta)^2}x^2 + \frac{[n]_{q_n}([2]_q+2\alpha)}{([n]_{q_n}+\beta)^2}x + \frac{\alpha^2}{([n]_{q_n}+\beta)^2}\right),$$

which implies that there exists a constant $C_1 > 0$ such that

$$\frac{1}{1+x^2}M_{n,q_n}^{(\alpha,\beta)}(1+t^2;x) = \frac{1}{1+x^2} + \frac{[n]_{q_n}(1+q_n[n]_{q_n})}{q_n([n]_{q_n}+\beta)^2}\frac{x^2}{1+x^2}$$
$$+ \frac{[n]_{q_n}([2]_q+2\alpha)}{([n]_{q_n}+\beta)^2}\frac{x}{1+x^2}$$
$$+ \frac{\alpha^2}{([n]_{q_n}+\beta)^2}\frac{1}{1+x^2},$$
$$\leq (1+C_1), \quad \text{for sufficiently large n .} \tag{30}$$

We have

$$\mu_x(t) = 1 + (2x+t)^2 \leq 1 + 2(4x^2+2t^2). \tag{31}$$

From (30) and (31), there is a positive constant $K_1$, such that

$$M_{n,q_n}^{(\alpha,\beta)}(\mu_x(t);x) \leq K_1(1+x^2), \quad \text{for sufficiently large n.}$$

Similarly, from Lemma 2

$$M_{n,q_n}^{(\alpha,\beta)}(\mu_x^2(t);x) = M_{n,q_n}^{(\alpha,\beta)}\left((1+(2x+t)^2)^2;x\right),$$
$$\leq M_{n,q_n}^{(\alpha,\beta)}\left((1+2(4x^2+2t^2))^2;x\right),$$
$$\leq 64\Bigg(M_{n,q_n}^{(\alpha,\beta)}(1+t^4;x) + (1+x^2)M_{n,q_n}^{(\alpha,\beta)}(1+t^2;x)$$
$$+ (1+x^2)M_{n,q_n}^{(\alpha,\beta)}(1;x)\Bigg).$$

Since

$$\frac{1}{1+x^4}M_{n,q_n}^{(\alpha,\beta)}(1+t^4;x) \leq (1+C_2), \text{ for some constant } C_2 > 0 \text{ when n is sufficiently large },$$

there exists a positive constant $K_2$ such that

$$\sqrt{M_{n,q_n}^{(\alpha,\beta)}(\mu_x^2(t);x)} \leq K_2(1+x^2), \quad \text{for sufficiently large n.}$$

Also, from Lemma 3 we have

$$M_{n,q_n}^{(\alpha,\beta)}(\psi_x^2(t);x) \leq \frac{[2]_{q_n}(1+\beta^2)}{q_n([n]_{q_n}+\beta)}\phi^2(x) + \frac{[2]_{q_n}(1+\beta^2)}{q_n([n]_{q_n}+\beta)^2}.$$

Now from (29), we have

$$|M_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x)| \leq \Omega_2(f,\delta)$$
$$\left(K_1(1+x^2) + K_2(1+x^2)\frac{1}{\delta}\sqrt{\frac{[2]_{q_n}(1+\beta^2)}{q_n([n]_{q_n}+\beta)}\phi^2(x) + \frac{[2]_{q_n}(1+\beta^2)}{q_n([n]_{q_n}+\beta)^2}}\right).$$

Choosing $\delta = \sqrt{\frac{[2]_{q_n}(1+\beta^2)}{q_n([n]_{q_n}+\beta)}} = \delta_n$, we obtain

$$|M_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x)| \leq \Omega_2(f,\delta_n)(1+x^2)(K_1 + K_2\sqrt{1+\phi^2(x)}), \quad \text{for sufficiently large n.}$$

Hence, for sufficiently large n

$$|M_{n,q_n}^{(\alpha,\beta)}(f;x) - f(x)| \leq K\Omega_2(f,\delta_n)(1+x^{2+\lambda}), \; x \in [0,\infty),$$

where $\lambda \geq 1$ and $K$ is a positive constant. This completes the proof of the theorem. □

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