Approximation of Periodic Functions Belonging to $W(L^r, \xi(t), (\beta \geq 0))$-Class By $(C^1 \cdot T)$ Means of Fourier Series

Smita Sonker

Abstract Various investigators such as Khan [3], Qureshi [8–10], Qureshi and Nema [11], Leindler [6] and Chandra [1] have determined the degree of approximation of functions belonging to the classes $W(L^r, \xi(t))$, $Lip(\xi(t), r)$, $Lip(\alpha, r)$ and $Lip\alpha$ using different summability methods with monotonocity conditions. Recently, Lal [5] has determined the degree of approximation of the functions belonging to $Lip\alpha$ and $W(L^r, \xi(t))$ classes by using Cesàro-Nörlund $(C^1 \cdot N_p)$—summability with non-increasing weights $\{p_n\}$. In this paper, we shall determine the degree of approximation of 2π-periodic function f belonging to the function classes $Lip\alpha$ and $W(L^r, \xi(t))$ by $C^1 \cdot T$—means of Fourier series of f. Our theorems generalize the results of Lal [5], and we also improve these results in the light of [7, 12, 13]. From our results, we derive some corollaries also.

Keywords Trigonometric fourier series · $W(L^r, \xi(t), (\beta \geq 0))$-class · Fourier series · Matrix means · Signals · Trigonometric polynomials

1 Introduction

For a given signal $f \in L^r := L^r[0, 2\pi]$, $r \geq 1$, let

$$s_n(f) := s_n(f; x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) = \sum_{k=0}^{n} u_k(f; x), \qquad (1)$$

denote the partial sums, called trigonometric polynomial of degree (or order) n, of the first $(n+1)$ terms of the Fourier series of f. The matrix means of (1) are defined by

S. Sonker (✉)
Department of Mathematics, National Institute of Technology Kurukshetra, Kurukshetra 136119, India
e-mail: smita.sonker@gmail.com

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$$t_n(f) := t_n(f;x) = \sum_{k=0}^{n} a_{n,k} s_k, \quad n = 0, 1, 2, ...,$$

where $T \equiv (a_{n,k})$ is a lower triangular matrix with non-negative entries such that $a_{n,-1} = 0$, $A_{n,k} = \sum_{r=k}^{n} a_{n,r}$ so that $A_{n,0} = 1$, $\forall n \geq 0$. The Fourier series of f is said to be T-summable to s, if $t_n(f) \to s$ as $n \to \infty$.

By superimposing C^1 summability upon T summability, we get the $C^1 \cdot T$ summability. Thus the $C^1 \cdot T$ means of $\{s_n(f)\}$ denoted by $t_n^{C^1 \cdot T}(f)$ are given by

$$t_n^{C^1 \cdot T}(f) := (n+1)^{-1} \sum_{r=0}^{n} \left(\sum_{k=0}^{r} a_{r,k} s_k(f) \right).$$

If $t_n^{C^1 \cdot T} \to s_1$ as $n \to \infty$, then the Fourier series of f is said to be $C^1 \cdot T$—summable to the sum s_1. The regularity of methods C^1 and T implies regularity of method $C^1 \cdot T$. A function $f \in Lip\alpha$ if $|f(x+t) - f(x)| = O(|t|^\alpha)$, for $0 < \alpha \leq 1$, $f \in Lip(\alpha, r)$ if $\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx \right)^{1/r} = O(|t|^\alpha)$, $0 < \alpha \leq 1$, $r \geq 1$,
$f \in Lip(\xi(t), r)$ if $\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx \right)^{1/r} = O(\xi(t))$ and
$f \in W(L^r, \xi(t))$ if $\left(\int_0^{2\pi} |(f(x+t) - f(x)) \sin^\beta(x/2)|^r dx \right)^{1/r} = O(\xi(t))$,
$\beta \geq 0$, $r \geq 1$, where $\xi(t)$ is a positive increasing function of t.
If $\beta = 0$, $W(L^r, \xi(t),) \equiv Lip(\xi(t), r)$ and for $\xi(t) = t^\alpha (\alpha > 0)$, $Lip(\xi(t), r) \equiv Lip(\alpha, r)$. $Lip(\alpha, r) \to Lip\alpha$ as $r \to \infty$. Thus

$$Lip\alpha \subseteq Lip(\alpha, r) \subseteq Lip(\xi(t), r) \subseteq W(L^r, \xi(t)).$$

The L^r-norm of $f \in L^r[0, 2\pi]$ is defined by
$\|f\|_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^r dx \right\}^{1/r}$ $(1 \leq r < \infty)$ and $\|f\|_\infty = \sup_{x \in [0,2\pi]} |f(x)|$.
The degree of approximation of $f \in L^r$ denoted by $E_n(f)$ is given by

$$E_n(f) = \min_{T_n} \| f(x) - T_n(x) \|_r,$$

in terms of n , where $T_n(x)$ is a trigonometric polynomial of degree n.

This method of approximation is called trigonometric Fourier approximation. We also write

$$\phi(t) = f(x+t) + f(x-t) - 2f(x),$$

$$(C^1 \cdot T)_n(t) = \frac{1}{2\pi(n+1)} \sum_{r=0}^{n} \sum_{k=0}^{r} a_{r,r-k} \frac{\sin(r-k+1/2)t}{\sin(t/2)},$$

and $\tau = [1/t]$, the integral part of $1/t$.

2 Known Results

Various investigators such as Khan [3], Qureshi [8–10], Qureshi and Nema [11], Leindler [6] and Chandra [1] have determined the degree of approximation of functions belonging to the classes $W(L^r, \xi(t))$, $Lip(\xi(t), r)$, $Lip(\alpha, r)$ and $Lip\alpha$ with $r \geq 1$ and $0 < \alpha \leq 1$ using different summability methods with monotonocity conditions on the rows of summability matrices. Recently, Lal [5] has determined the degree of approximation of the functions belonging to $Lip\alpha$ and $W(L^r, \xi(t))$ classes by using Cesáro-Nörlund $(C^1 \cdot N_p)$—summability with non-increasing weights $\{p_n\}$. He proved:

Theorem 1 *Let N_p be a regular Nörlund method defined by a sequence $\{p_n\}$ such that*

$$P_\tau \sum_{v=\tau}^{n} P_v^{-1} = O(n+1). \tag{2}$$

Let $f \in L^1[0, 2\pi]$ be a 2π-periodic function belonging to $Lip\,\alpha$ $(0 < \alpha \leq 1)$, then the degree of approximation of f by $C^1 \cdot N_p$ means of its Fourier series is given by

$$\sup_{0 \leq x \leq 2\pi} |t_n^{CN}(x) - f(x)| = \|t_n^{CN} - f\|_\infty = \begin{cases} O((n+1)^{-\alpha}), & 0 < \alpha < 1, \\ O\left(\log(n+1)\pi e/(n+1)\right), & \alpha = 1. \end{cases}$$

Theorem 2 *If f is a 2π-periodic function and Lebesgue integrable on $[0, 2\pi]$ and is belonging to $W(L^r, \xi(t))$ class then its degree of approximation by $C^1 \cdot N_p$ means of its Fourier series is given by*

$$\|t_n^{CN} - f\|_r = O\left((n+1)^{\beta + 1/r} \xi\left(1/(n+1)\right)\right),$$

provided $\xi(t)$ satisfies the following conditions:

$$\{\xi(t)/t\} \text{ be a decreasing sequence}, \tag{3}$$

$$\left(\int_0^{\pi/(n+1)} \left(t|\phi(t)| \sin^\beta(t)/\xi(t)\right)^r dt\right)^{1/r} = O((n+1)^{-1}), \tag{4}$$

$$\left(\int_{\pi/(n+1)}^{\pi} \left(t^{-\delta}|\phi(t)|/\xi(t)\right)^r dt\right)^{1/r} = O((n+1)^{\delta}), \tag{5}$$

where δ is an arbitrary number such that $s(1-\delta) - 1 > 0$, $r^{-1} + s^{-1} = 1$, $r \geq 1$, conditions (4) and (5) hold uniformly in x.

The improved version of above theorems with their generalization to non-monotone weights $\{p_n\}$ can be seen in [13].

3 Main Results

In this paper, we generalize Theorems 1 and 2 by replacing matrix N_p with matrix T in the light of Remarks 2.3 and 2.4 of [13, pp. 3–4]. More precisely, we prove:

Theorem 3 *If $T \equiv (a_{n,k})$ is a lower triangular regular matrix with non-negative and non-decreasing (with respect to k) entries which satisfy*

$$\sum_{r=\tau}^{n} A_{r,r-\tau} = O(n+1), \tag{6}$$

hold uniformly in $\tau = [1/t]$, then the degree of approximation of a 2π-periodic function $f \in Lip\alpha$ $(0 < \alpha \leq 1) \subset L^1[0, 2\pi]$ by $C^1 \cdot T$ means of its Fourier series is given by

$$\|t_n^{C^1 \cdot T}(f) - f(x)\|_\infty = \begin{cases} O((n+1)^{-\alpha}), & 0 < \alpha < 1, \\ O\left((\log(n+1))/(n+1)\right), & \alpha = 1. \end{cases} \tag{7}$$

Theorem 4 *If $T \equiv (a_{n,k})$ be a lower triangular regular matrix with non-negative and non-decreasing (with respect to k) entries which satisfy condition (6), then the degree of approximation of a 2π-periodic function with $r > 1$ and $0 < \beta s < 1$ by $C^1 \cdot T$ means of its Fourier series is given by*

$$\|t_n^{C^1 \cdot T}(f) - f(x)\|_r = O\left((n+1)^{\beta+1/r} \xi\left(1/(n+1)\right)\right), \tag{8}$$

provided positive increasing function $\xi(t)$ satisfies the conditions:

$$\xi(t)/t \text{ be a decreasing function}, \tag{9}$$

$$\left(\int_0^{\pi/(n+1)} \left(|\phi(t)| \sin^\beta(t/2)/\xi(t)\right)^r dt\right)^{1/r} = O((n+1)^{-1/r}), \tag{10}$$

$$\left(\int_{\pi/(n+1)}^{\pi} \left(t^{-\delta}|\phi(t)| \sin^\beta(t/2)/\xi(t)\right)^r dt\right)^{1/r} = O((n+1)^{\delta-1/r}), \tag{11}$$

where δ is a real number such that $\beta + 1/r > \delta > r^{-1}$, $r^{-1} + s^{-1} = 1$, $r > 1$. Also, conditions (10) and (11) hold uniformly in x.

Remark 1 If we take $a_{n,k} = p_{n-k}/P_n$ for $k \leq n$ and $a_{n,k} = 0$ for $k > n$ such that $P_n (= \sum_{k=0}^{n} p_k \neq 0) \to \infty$ as $n \to \infty$ and $P_{-1} = 0 = p_{-1}$, then $C^1 \cdot T$ means reduce to $C^1 \cdot N_p$ means and

$$\sum_{r=\tau}^{n} A_{r,r-\tau} = \sum_{r=\tau}^{n} \sum_{k=r-\tau}^{r} a_{r,k} = \sum_{r=\tau}^{n} \sum_{k=r-\tau}^{r} (p_{r-k}/P_r) = \sum_{r=\tau}^{n} (P_\tau/P_r) = P_\tau \sum_{r=\tau}^{n} P_r^{-1}.$$

Therefore, condition (6) reduces to condition (2) and $t_n^{C^1 \cdot T}$ means reduce to t_n^{CN} *means*. Hence our Theorems 3 and 4 are generalization of Theorems 1 and 2, respectively.

4 Lemmas

We need the following lemmas for the proof of our theorems.

Lemma 1 *Let* $\{a_{r,k}\}$ *be a non-negative sequence of real numbers, then*

$$(C^1 \cdot T)_n(t) = O(n+1), \textit{ for } 0 < t \le \pi/(n+1).$$

Proof Using $\sin nt \le nt$ and $\sin(t/2) \ge t/\pi$ for $0 < t \le \pi/(n+1)$, we have

$$\begin{aligned}
\left|(C^1 \cdot T)_n(t)\right| &= (2\pi(n+1))^{-1} \left| \sum_{r=0}^{n} \sum_{k=0}^{r} a_{r,r-k} \, \sin((r-k+1/2)t)/\sin(t/2) \right| \\
&= (2\pi(n+1))^{-1} \sum_{r=0}^{n} \sum_{k=0}^{r} a_{r,r-k} \, |\sin((r-k+1/2)t)/\sin(t/2)| \\
&\le (2\pi(n+1))^{-1} \sum_{r=0}^{n} \sum_{k=0}^{r} a_{r,r-k} (r-k+1/2)t/(t/\pi) \\
&\le (4(n+1))^{-1} \sum_{r=0}^{n} \sum_{k=0}^{r} a_{r,r-k} (2r-2k+1) \\
&\le (4(n+1))^{-1} \sum_{r=0}^{n} (2r+1) \sum_{k=0}^{r} a_{r,r-k} \\
&= (4(n+1))^{-1} \sum_{r=0}^{n} (2r+1) A_{r,0} = O(n+1).
\end{aligned}$$

Lemma 2 *[4] If* $\{a_{r,k}\}$ *is a non-negative and non-decreasing (with respect to* k*) sequence, then for* $0 \le a < b \le \infty$, $0 < t \le \pi$ *and for every* r

$$\left| \sum_{k=a}^{b} a_{r,r-k} e^{i(r-k)t} \right| = O(A_{r,r-\tau}).$$

Lemma 3 *If* $\{a_{r,k}\}$ *is non-negative and non-decreasing (with respect to* k*) sequence, then for* $0 < t \leq \pi$

$$\left|\sum_{r=0}^{n}\sum_{k=0}^{r} a_{r,r-k}e^{i(r-k)t}\right| = O(t^{-1}) + O\left(\sum_{r=\tau}^{n} A_{r,r-\tau}\right),$$

holds uniformly in $\tau = [1/t]$.

Proof For $0 < t \leq \pi$, we have

$$\begin{aligned}
\left|\sum_{r=0}^{n}\sum_{k=0}^{r} a_{r,r-k}e^{i(r-k)t}\right| &\leq \left|\sum_{r=0}^{\tau-1}\sum_{k=0}^{r} a_{r,r-k}e^{i(r-k)t} + \sum_{r=\tau}^{n}\sum_{k=0}^{r} a_{r,r-k}e^{i(r-k)t}\right| \\
&\leq \sum_{r=0}^{\tau-1}\sum_{k=0}^{r} a_{r,r-k}|e^{i(r-k)t}| + \left|\sum_{r=\tau}^{n}\sum_{k=0}^{r} a_{r,r-k}e^{i(r-k)t}\right| \\
&\leq \sum_{r=0}^{\tau-1}\sum_{k=0}^{r} a_{r,r-k} + \sum_{r=\tau}^{n}\left|\sum_{k=0}^{r} a_{r,r-k}e^{i(r-k)t}\right| \\
&\leq \sum_{r=0}^{\tau-1} 1 + \sum_{r=\tau}^{n} O(A_{r,r-\tau}) = (\tau - 1 + 1) + O\left(\sum_{r=\tau}^{n} A_{r,r-\tau}\right) \\
&= O(t^{-1}) + O\left(\sum_{r=\tau}^{n} A_{r,r-\tau}\right),
\end{aligned}$$

in view of Lemma 2.

Lemma 4 *If* $\{a_{r,k}\}$ *is non-negative and non-decreasing (with respect to* k*) sequence and satisfies the condition (6), then*

$$|(C^1 \cdot T)_n(t)| = O\left(t^{-2}/(n+1)\right) + O(t^{-1}), \textit{ for } \pi/(n+1) < t \leq \pi.$$

Proof Using $\sin(t/2) \geq t/\pi$, for $\pi/(n+1) < t \leq \pi$ and Lemma 3, we have

$$\begin{aligned}
|(C^1 \cdot T)_n(t)| &= (2\pi(n+1))^{-1}\left|\sum_{r=0}^{n}\sum_{k=0}^{r} a_{r,r-k}\sin((r-k+1/2)t)/\sin(t/2)\right| \\
&\leq (2\pi(n+1))^{-1}\left|\sum_{r=0}^{n}\sum_{k=0}^{r} a_{r,r-k}\sin((r-k+1/2)t)/(t/\pi)\right| \\
&= (2t(n+1))^{-1}\left|\sum_{r=0}^{n}\sum_{k=0}^{r} a_{r,r-k}\sin(r-k+1/2)t\right| \\
&\leq (2t(n+1))^{-1}\left|\sum_{r=0}^{n}\sum_{k=0}^{r} a_{r,r-k}e^{i(r-k+1/2)t}\right|
\end{aligned}$$

$$= (2t(n+1))^{-1}\left|e^{it/2}\sum_{r=0}^{n}\sum_{k=0}^{r}a_{r,r-k}e^{i(r-k)t}\right|$$
$$= (2t(n+1))^{-1}\left|\sum_{r=0}^{n}\sum_{k=0}^{r}a_{r,r-k}e^{i(r-k)t}\right|$$
$$= (2t(n+1))^{-1}\left|O(t^{-1})+O\left(\sum_{r=\tau}^{n}A_{r,r-\tau}\right)\right| = O\left(t^{-2}/(n+1)\right)+O(t^{-1}),$$

in view of condition (6).

5 Proof of Theorem 3

Following Titchmarsh [14], we have

$$s_n(f;x)-f(x)=\frac{1}{2\pi}\int_0^{\pi}\phi(t)(\sin(n+1/2)t/\sin(t/2))dt$$

Denoting $C^1\cdot T$ means of $\{s_n(f;x)\}$ by $t_n^{C^1\cdot T}(f)$, we write

$$\begin{aligned} t_n^{C^1\cdot T}(f)-f(x) &= \int_0^{\pi}\phi(t)(2\pi(n+1))^{-1}\sum_{r=0}^{n}\sum_{k=0}^{r}a_{r,r-k}\sin((r-k+1/2)t)/\sin(t/2)dt \\ &= \int_0^{\pi/(n+1)}\phi(t)(C^1\cdot T)_n(t)dt+\int_{\pi/(n+1)}^{\pi}\phi(t)(C^1\cdot T)_n(t)dt \\ &= I_1+I_2, \text{ say.} \end{aligned} \tag{12}$$

Using Lemma 1 and the fact that $f\in Lip\,\alpha \Rightarrow \phi\in Lip\,\alpha$ {[2], Lemma 5.27}, we have

$$\begin{aligned} |I_1| &\le \int_0^{\pi/(n+1)}|\phi(t)||(C^1\cdot T)_n(t)|dt = O(n+1)\int_0^{\pi/(n+1)}t^{\alpha}dt \\ &= O(n+1)((n+1)^{-\alpha-1}) = O((n+1)^{-\alpha}). \end{aligned} \tag{13}$$

Now, using Lemma 4 and the fact that $f\in Lip\,\alpha \Rightarrow \phi\in Lip\,\alpha$,

$$\begin{aligned} |I_2| &\le \int_{\pi/(n+1)}^{\pi}|\phi(t)|\ \left|(C^1\cdot T)_n(t)\right|dt \le \int_{\pi/(n+1)}^{\pi}|\phi(t)|O\left[(t^{-2}/(n+1))+t^{-1}\right]dt \\ &= O(I_{21})+O(I_{22}), \text{ say,} \end{aligned} \tag{14}$$

where

$$I_{21}=(n+1)^{-1}\int_{\pi/(n+1)}^{\pi}t^{\alpha-2}dt=\begin{cases} O((n+1)^{-\alpha}), & 0<\alpha<1, \\ O\left(\log(n+1)/(n+1)\right), & \alpha=1. \end{cases} \tag{15}$$

and

$$I_{22} = O\left(\int_{\pi/(n+1)}^{\pi} t^{\alpha-1} dt\right) = O((n+1)^{-\alpha}). \tag{16}$$

Collecting (12)–(16), we get

$$t_n^{C^1 \cdot T}(f) - f(x) = \begin{cases} O((n+1)^{-\alpha}), & 0 < \alpha < 1, \\ O(\log(n+1)/(n+1)), & \alpha = 1. \end{cases}$$

Thus

$$\|t_n^{C^1 \cdot T}(f) - f\|_\infty = \sup_{0 \le x \le 2\pi} \{|t_n^{C^1 \cdot T}(x) - f(x)|\} = \begin{cases} O((n+1)^{-\alpha}), & 0 < \alpha < 1, \\ O((\log(n+1))/(n+1)), & \alpha = 1. \end{cases}$$

6 Proof of Theorem 4

Following the proof of Theorem 3,

$$\begin{aligned} t_n^{C^1 \cdot T}(f) - f(x) &= \int_0^{\pi/(n+1)} \phi(t)(C^1 \cdot T)_n(t) dt + \int_{\pi/(n+1)}^{\pi} \phi(t)(C^1 \cdot T)_n(t) dt \\ &= I_1' + I_2', \text{ say.} \end{aligned} \tag{17}$$

Using Hölder's inequality, $\phi(t) \in W(L^r, \xi(t))$, condition (10), Lemma 1 and mean value theorem for integrals, we have

$$\begin{aligned} |I_1'| &= \left| \lim_{\varepsilon \to 0} \int_\varepsilon^{\pi/(n+1)} \left[(\phi(t) \sin^\beta(t/2)/\xi(t)) \cdot (\xi(t)(C^1 \cdot T)_n(t))/(\sin^\beta(t/2)) \right] dt \right| \\ &\le \left[\int_0^{\pi/(n+1)} \left(|\phi(t)| \sin^\beta(t/2)/\xi(t) \right)^r dt \right]^{1/r} \\ &\cdot \left[\lim_{\varepsilon \to 0} \int_\varepsilon^{\pi/(n+1)} \left(\xi(t) |(C^1 \cdot T)_n(t)|/(\sin^\beta(t/2)) \right)^s dt \right]^{1/s} \\ &= O((n+1)^{-1/r}) \left[\lim_{\varepsilon \to 0} \int_\varepsilon^{\pi/(n+1)} \left| \xi(t)(n+1)/(\sin^\beta(t/2)) \right|^s dt \right]^{1/s} \\ &= O(n+1)^{1-1/r} (\xi(\pi/(n+1)) \left[\lim_{\varepsilon \to 0} \int_\varepsilon^{\pi/(n+1)} t^{-\beta s} dt \right]^{1/s} \\ &= O(\xi(1/(n+1)(n+1)^{\beta+1-1/r-1/s}) = O((n+1)^\beta \xi(1/(n+1)), \end{aligned} \tag{18}$$

in view of condition (9), i.e. $(\xi(\pi/(n+1))/(\pi/(n+1))) \leq (\xi(1/(n+1))/(1/(n+1)))$.

Using Lemma 4, we have

$$\begin{aligned}
|I_2^{'}| &= \left[\int_{\pi/(n+1)}^{\pi} |\phi(t)| \left[O\left(t^{-2}/(n+1)\right) + O\left(t^{-1}\right)\right] dt\right] \\
&= O\left[\int_{\pi/(n+1)}^{\pi} t^{-2}|\phi(t)|/(n+1)dt\right] + O\left[\int_{\pi/(n+1)}^{\pi} t^{-1}|\phi(t)|dt\right] \\
&= O(I_{21}^{'}) + O(I_{22}^{'}). \qquad (19)
\end{aligned}$$

Using Hölder's inequality, $|\sin t| \leq 1$, $\sin(t/2) \geq (t/\pi)$ and condition (11), we have

$$\begin{aligned}
|I_{21}^{'}| &= (n+1)^{-1}\left[\int_{\pi/(n+1)}^{\pi} \left\{(t^{-\delta}|\phi(t)|\sin^{\beta}(t/2)/\xi(t)) \cdot (\xi(t)/(t^{-\delta+2}\sin^{\beta}(t/2)))\right\} dt\right] \\
&\leq ((n+1)^{-1})\left[\int_{\pi/(n+1)}^{\pi} \left|t^{-\delta}|\phi(t)|\sin^{\beta}(t/2)/\xi(t)\right|^{r} dt\right]^{1/r} \\
&\quad \cdot \left[\int_{\pi/(n+1)}^{\pi} \left|\xi(t)/\left(t^{-\delta+2}\sin^{\beta}(t/2)\right)\right|^{s} dt\right]^{1/s} \\
&= O((n+1)^{-1})\left[\int_{\pi/(n+1)}^{\pi} \left|t^{-\delta}|\phi(t)|\sin^{\beta}(t/2)/\xi(t)\right|^{r} dt\right]^{1/r} \\
&\cdot \left[\int_{\pi/(n+1)}^{\pi} \left|\xi(t)/\left(t^{-\delta+2}\sin^{\beta}(t/2)\right)\right|^{s} dt\right]^{1/s} \\
&= O((n+1)^{-1})O\left((n+1)^{\delta-1/r}\right)\left[\int_{\pi/(n+1)}^{\pi} \left|\xi(t)/\left(t^{-\delta+2}\sin^{\beta}(t/2)\right)\right|^{s} dt\right]^{1/s} \\
&= O((n+1)^{\delta-1-1/r})\left[\int_{\pi/(n+1)}^{\pi} \left(\xi(t)/t^{-\delta+2+\beta}\right)^{s} dt\right]^{1/s} \\
&= O((n+1)^{\delta-1/r})\xi(\pi/(n+1))\left[\int_{\pi/(n+1)}^{\pi} t^{-(-\delta+1+\beta)s} dt\right]^{1/s} \\
&= O((n+1)^{\delta-1/r})\xi(\pi/(n+1))\left[(n+1)^{(-\delta+1+\beta)-1/s} dt\right] \\
&= O(\xi(1/(n+1)(n+1)^{\beta}) \qquad (20)
\end{aligned}$$

in view of decreasing nature of $\xi(t)/t$ and $r^{-1} + s^{-1} = 1$.

Similarly, as above, we have

$$\begin{aligned}
|I_{22}^{'}| &= \int_{\pi/(n+1)}^{\pi} t^{-1}|\phi(t)|dt = \int_{\pi/(n+1)}^{\pi} \left(t^{-\delta}|\phi(t)|\sin^{\beta}(t/2)/\xi(t)\right)\left(\xi(t)/(t^{1-\delta}\sin^{\beta}(t/2))\right) dt \\
&\leq \left[\int_{\pi/(n+1)}^{\pi} \left|t^{-\delta}|\phi(t)|\sin^{\beta}(t/2)/\xi(t)\right|^{r} dt\right]^{1/r}\left[\int_{\pi/(n+1)}^{\pi} \left|\xi(t)/\left(t^{1-\delta}\sin^{\beta}(t/2)\right)\right|^{s} dt\right]^{1/s} \\
&= O\left((n+1)^{\delta-1/r}\right)\left[\int_{\pi/(n+1)}^{\pi} \left(\xi(t)/t^{1-\delta+\beta}\right)^{s} dt\right]^{1/s}
\end{aligned}$$

$$= O\left((n+1)^{\delta+1-1/r}\right)\xi(1/(n+1))\left[\int_{\pi/(n+1)}^{\pi} t^{(\delta-\beta)s}dt\right]^{1/s}$$
$$= O\left((n+1)^{\delta+1-1/r}\right)\xi(1/(n+1))(n+1)^{(-\delta+\beta)-1/s}$$
$$= O(\xi(1/(n+1))(n+1)^{\beta+1-1/r-1/s}$$
$$= O(\xi(1/(n+1))(n+1)^{\beta}. \quad (21)$$

Collecting (17)–(21), we have

$$|t_n^{C^1\cdot T}(f) - f(x)| = O\left((n+1)^{\beta}\xi(1/(n+1))\right).$$

Hence,

$$\|t_n^{C^1\cdot T}(f) - f(x)\|_r = \left(1/2\pi\int_0^{2\pi}|t_n^{C^1\cdot T}(f) - f(x)|^r dx\right)^{1/r} = O\left((n+1)^{\beta}\xi\left(1/(n+1)\right)\right).$$

Remark 2 The proof of Theorem 3, for $r = 1$, *i.e.* $s = \infty$ can be written by using sup norm while using Hölder's inequality.

7 Corollaries

The following corollaries can be derived from Theorem 4
1. If $\beta = 0$, then for $f \in Lip(\xi(t), r)$, $\|t_n^{C^1\cdot T}(f) - f(x)\|_r = O\left(\xi(1/n)\right)$.
2. If $\beta = 0$, $\xi(t) = t^{\alpha}(0 < \alpha \leq 1)$, then for $f \in Lip(\alpha, r)(\alpha > 1/r)$,

$$\|t_n^{C^1\cdot T}(f) - f(x)\|_r = O\left(n^{-\alpha}\right). \quad (22)$$

3. If $r \to \infty$ in Corollary 2, then for $f \in Lip\alpha(0 < \alpha < 1)$, (22) gives

$$\|t_n^{C^1\cdot T}(f) - f(x)\|_{\infty} = O(n^{-\alpha}).$$

Remark 3 In view of Remark 2, corollaries of Lal [5, p. 350] are particular cases of our Corollaries 2 and 3, respectively.

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