Reconstruction of Multiply Generated Splines from Local Average Samples

P. Devaraj and S. Yugesh

Abstract We analyze the following average sampling problem: Let h be a non-negative measurable function supported in $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Given a sequence of samples $\{y_n\}_{n\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$ of polynomial growth, find a multiply generated spline f of polynomial growth such that $\int_{-\frac{1}{2}}^{\frac{1}{2}} f(n-t)h(t)dt = y_n$, $n \in \mathbb{Z}$. It is shown that the solution to this problem is unique over certain subspaces of the multiply generated spline space of polynomial growth.

Keywords Interpolation · Multiply generated splines · Average sampling

1 Introduction

The sampling theorem is one of the widely used results in the signal processing field. The well-known Shannon sampling theorem states that, any bandlimited signal f is completely determined by its samples [4, 8]. Although the Shannon sampling theorem is very useful, it has a number of problems when using it for practical applications. The bandlimited functions have analytic continuation to the entire complex plane and hence they are of infinite duration which is not always realistic. On the other hand, the sinc function has a very slow decay. Further, the measured samples are not exact in practical problems and they are the average of the signal around the sampling point and the averaging function depends on the aperture device used for capturing the samples. For these reasons, sampling and local average sampling have been investigated in several other classes of signals. In general, spline spaces yield many advantages in their generation and numerical treatment so that there are

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many practical applications in signal, image processing, and communication theory. In the literature [1–8] many authors have investigated the generalized sampling technique for multiply generated shift-invariant spaces and spline subspaces. The multiply generated spline space is defined in [5, 6] as

$$\mathscr{S} = \left\{ f : f = \sum_{i=1}^{r} \sum_{n \in \mathbb{Z}} a_i(n) \beta_{d_i}(t-n) \right\}$$

with suitable coefficients $a_i(n)$, where β_{d_i} is the cardinal central B-spline of degree d_i and is defined by,

$$\beta_{d_i} = \chi_{[-\frac{1}{2},\frac{1}{2}]} \star \chi_{[-\frac{1}{2},\frac{1}{2}]} \star \cdots \star \chi_{[-\frac{1}{2},\frac{1}{2}]} (d_i + 1 \text{ terms}),$$

where $\star$ represents the convolution (The convolution of two functions f and g is defined as $f \star g(n) = \int f(t)g(n-t)dt$). We consider the following subspace of the multiply generated spline space:

$$\mathscr{S}_N := \left\{ f : f = \sum_{n \in \mathbb{Z}} a_n \sum_{i=1}^{r} \beta_{d_i}(t-n) \right\}$$

If $M = max\{d_1, d_2, \ldots, d_r\}$ and $m = \min\{d_1, d_2, \ldots, d_r\}$, then $f \in \mathscr{S}_N$ provided that $f(x) \in C^{m-1}(\mathbb{R})$ and that the restriction of $f(x)$ to any interval between consecutive knots is identical with a polynomial of degree not exceeding M. If d_i's are distinct, then $\sum_{i=1}^{r} \beta_{d_i}(.-n), n \in \mathbb{Z}$ are globally linearly independent.

We consider the following local average sampling problem:

Problem: Let $\{y_n\}_{n\in\mathbb{Z}}$ be a given sequence of real numbers. Find a spline $f \in \mathscr{S}_N$ such that $f \star h(n) = y_n, n \in \mathbb{Z}$, where $h \in L^1(\mathbb{R})$ and $\left(h \star \sum_{i=1}^{r} \beta_{d_i}\right)(n) \neq 0$, for some $n \in \mathbb{Z}$ and

$$supp(h) \subseteq \left[-\frac{1}{2}, \frac{1}{2}\right], h(t) \geq 0, t \in \mathbb{R}, 0 < \int_{-\frac{1}{2}}^{0} h(t)dt < \infty, 0 < \int_{0}^{\frac{1}{2}} h(t)dt < \infty. \quad (1)$$

We show that this problem has infinitely many solutions. The uniqueness of solution is obtained by imposing the following growth conditions on the samples and the splines as that of Schoenberg [9]:

$$\mathscr{S}_{\mathscr{N},\gamma} = \left\{ f(t) \in \mathscr{S}_N : f(t) = O(|t|^{\gamma}) \; as \; t \to \pm\infty \right\}$$

and

$$\mathscr{D}_{\gamma} = \left\{ \{y_n\} : y_n = O(|n|^{\gamma}) \; as \; n \to \pm\infty \right\}.$$

This problem over the singly generated spline space is analyzed in [10]. It is shown in [10] that the local average sampling problem has a unique solution for $d \leq 4$ when the spline space is generated by a single central B-spline. For $d > 4$ the same problem has been posed as an open problem. The same authors have analyzed the problem for $d \geq 5$ by reducing the support of h. They have shown in [11] that the local average sampling problem for singly generated spline has a unique solution when h is supported in $\left[-\frac{l}{2}, \frac{l}{2}\right]$, $l < 1$.

Lemma 1 *Let* $\psi(x) = \sum_{i=1}^{r} \beta_{d_i}(x)$ *and let* A *be the greatest integer such that* $h \star \psi(n) = 0, \forall n < A$, *and let* N_1 *be the smallest nonnegative integer such that* $h \star \psi(n) = 0, \forall n > A + N_1$. *Then the solutions of the problem form a linear manifold in* $\mathscr{S}_N$ *of dimension* N_1. *Moreover,* $N_1 = M + 1$*, if* M *is odd and* $N_1 = M$*, if* M *is even.*

Proof When $N_1 = 0$ this problem has a unique solution. For $N_1 > 0$, we consider the linear map from $\mathbb{C}^{\mathbb{Z}}$ to $\mathscr{S}_N$ defined by

$$\{a_n\}_{n\in\mathbb{Z}} \longmapsto \sum_{n\in\mathbb{Z}} a_n \psi(t-n).$$

Since the integer translates of ψ are globally linearly independent, this map is an isomorphism from $\mathbb{C}^{\mathbb{Z}}$ onto $\mathscr{S}_N$. Therefore $h \star f(n) = y_n$ in $\mathbb{C}^{\mathbb{Z}}$ if and only if,

$$\sum_{k=0}^{N_1} h \star \psi(A+k) a_{n-A-k} = y_n, \forall n \in \mathbb{Z}.$$

This forms a linear difference equation of order N_1 with constant coefficients. Hence the solution space is an N_1 dimensional manifold in $\mathscr{S}_N$. □

2 Local Average Sampling Theorems

Theorem 1 (Main Theorem) *Let* $d_i \leq 4$ *and let* $h(t)$ *be an integrable function satisfying condition* (1). *Then for a given sequence of numbers* $\{y_n\}_{n\in\mathbb{Z}} \in \mathscr{D}_\gamma$, *there exists a unique* $f \in \mathscr{S}_{N,\gamma}$ *such that*

$$f \star h(n) = y_n, n \in \mathbb{Z}. \tag{2}$$

We define the function

$$G(z) := \sum_{i=1}^{r} G_i(z)$$

where

$$G_i(z) := \sum_{n\in\mathbb{Z}} h \star \beta_{d_i}(n) z^n.$$

The exponential Euler spline is defined as

$$\Upsilon_{z,d_i}(t) = \sum_{n\in\mathbb{Z}} z^n \beta_{d_i}(n-t).$$

In terms of the exponential Euler spline we can write $G_i(z) = \int_{-\frac{1}{2}}^{\frac{1}{2}} h(t)\Upsilon_{z,d_i}(t)dt$. Hence

$$G(z) = \int_{-\frac{1}{2}}^{\frac{1}{2}} h(t)\Upsilon_z(t)dt,$$

where $\Upsilon_z(t) = \sum_{i=1}^{r} \Upsilon_{z,d_i}(t) = \sum_{n\in\mathbb{Z}} z^n \sum_{i=1}^{r} \beta_{d_i}(n-t)$.

We need some properties of $\Upsilon_z(t)$.

Lemma 2 *For $d \in \mathbb{N}, n \in \mathbb{Z}$ and $z \in \mathbb{C} \setminus \{0\}$, we have:*

(i) $\Upsilon_{z^{-1}}(-t) = \Upsilon_z(t)$,
(ii) $\Upsilon_z(t+n) = (z)^n \Upsilon_z(t)$,
(iii) $\frac{d}{dt}(\Upsilon_{z,d_i+1}(t)) = \left(1-\frac{1}{z}\right)\Upsilon_{z,d_i}\left(t+\frac{1}{2}\right)$,
(iv) $\Upsilon_{-1,d_i}\left(\frac{1}{2}\right) = 0$ *and* $\Upsilon_{-1,d_i}(t) > 0$ *for* $t \in \left(-\frac{1}{2}, \frac{1}{2}\right)$.

Proof (i)

$$\begin{aligned}\Upsilon_{z^{-1}}(-t) &= \sum_{n\in\mathbb{Z}} z^{-n} \sum_{i=1}^{r} \beta_{d_i}(n+t) \\ &= \sum_{n\in\mathbb{Z}} z^{-n} \sum_{i=1}^{r} \beta_{d_i}(-n-t) = \sum_{n\in\mathbb{Z}} z^{n} \sum_{i=1}^{r} \beta_{d_i}(n-t) = \Upsilon_z(t).\end{aligned}$$

(ii)

$$\Upsilon_z(t+n) = \sum_{k\in\mathbb{Z}} z^k \sum_{i=1}^{r} \beta_{d_i}(k-t-n) = z^n \sum_{k\in\mathbb{Z}} z^k \sum_{i=1}^{r} \beta_{d_i}(k-t) = z^n \Upsilon_z(t).$$

(iii)

$$\begin{aligned}\frac{d}{dt}(\Upsilon_{z,d_i+1}(t)) &= \sum_{n\in\mathbb{Z}} z^n \frac{d}{dt}(\beta_{d_i+1}(n-t)) \\ &= \sum_{n\in\mathbb{Z}} z^n \left[\beta_{d_i}\left(n-\left(t+\frac{1}{2}\right)\right) - \beta_{d_i}\left(n-\left(t-\frac{1}{2}\right)\right)\right]\end{aligned}$$

$$= \Upsilon_{z,d_i}\left(t+\frac{1}{2}\right) - \sum_{n\in\mathbb{Z}} z^{n-1}\beta_{d_i}\left(n-1-t+\frac{1}{2}\right)$$
$$= \Upsilon_{z,d_i}\left(t+\frac{1}{2}\right) - \frac{1}{z}\sum_{n\in\mathbb{Z}} z^{n}\beta_{d_i}\left(n-\left(t+\frac{1}{2}\right)\right)$$
$$= \Upsilon_{z,d_i}\left(t+\frac{1}{2}\right) - \frac{1}{z}\Upsilon_{z,d_i}\left(t+\frac{1}{2}\right)$$
$$= \left(1-\frac{1}{z}\right)\Upsilon_{z,d_i}\left(t+\frac{1}{2}\right)$$

(iv)
$$\Upsilon_{-1,d_i}\left(\frac{1}{2}\right) = \sum_{n\in\mathbb{Z}}(-1)^n\beta_{d_i}\left(n-\frac{1}{2}\right) = 0.$$

We shall show that $\Upsilon_{-1,d_i}(t) > 0$ for $t \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, by using induction on d_i. For $d_i = 1$ by simple manipulation we get $\Upsilon_{-1,1}(t) > 0$ for $t \in (-\frac{1}{2}, \frac{1}{2})$. Assume that it is true for d_i and using (iii) we get

$$\frac{d}{dt}(\Upsilon_{-1,d_i+1}(t)) = 2\Upsilon_{-1,d_i}\left(t+\frac{1}{2}\right) > 0, t \in \left(-\frac{1}{2}, 0\right).$$

Using $\Upsilon_{-1,d_i+1}\left(-\frac{1}{2}\right) = 0$ and Υ_{-1,d_i+1} and being an even function, we obtain that $\Upsilon_{-1,d_i}(t) > 0$ for $t \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. □

2.1 Uniqueness Theorem

Theorem 2 *Let* $\Lambda = \{f \in \mathscr{S}_N : f \star h(n) = 0, n \in \mathbb{Z}\}$ *and* $z_1, z_2, \ldots, z_l$ *be the roots of* $G(z)$*. If the roots of* $G(z)$ *are simple, then the set of functions* $\Upsilon_{z_j^{-1}}$*, where* $j = 1, 2, \ldots, l$ *form a basis of* Λ*.*

Proof By Lemma 1, Λ is a $l = N_1$ dimensional subspace of $\mathscr{S}_N$.
Using Lemma 2, we get

$$h \star \Upsilon_{z_j^{-1}}(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} h(t)\Upsilon_{z_j^{-1}}(n-t)dt$$
$$= z_j^n \int_{-\frac{1}{2}}^{\frac{1}{2}} h(t)\Upsilon_{z_j^{-1}}(-t)dt$$
$$= z_j^n \int_{-\frac{1}{2}}^{\frac{1}{2}} h(t)\Upsilon_{z_j}(t)dt$$
$$= z_j^n G(z_j)$$

Therefore, $\Upsilon_{z_j^{-1}} \in \Lambda$ for $j = 1, 2, \ldots, l$.

Next, we have to prove that the elements of Λ are linearly independent.

$$\sum_{j=1}^{l} c_j \Upsilon_{z_j^{-1}}(t) = 0 \Leftrightarrow \sum_{j=l}^{l} c_j \left[\sum_{n\in\mathbb{Z}} z_j^{-n} \sum_{i=1}^{r} \beta_{d_i}(n-t)\right] = 0$$
$$\Leftrightarrow \sum_{n\in\mathbb{Z}} \left[\sum_{j=1}^{l} c_j z_j^{-n}\right] \left\{\sum_{i=1}^{r} \beta_{d_i}(n-t)\right\} = 0.$$

As $\left\{\sum_{i=1}^{r} \beta_{d_i}(n-t)\right\}$ are linearly independent, we obtain

$$\sum_{j=1}^{l} c_j z_j^{-n} = 0.$$

This is a linear system of equation in the variable $c_1, c_2, \ldots, c_l$ with coefficient matrix, the Vandermonde's determinant. Therefore $c_j = 0$.
Hence, the functions $\Upsilon_{z_j^{-1}}(t),\ j = 1, 2, \ldots, l$ form a basis of Λ. □

Theorem 3 *Let $d_i \in \mathbb{N}$ and $h(t)$ be an integrable function satisfying condition (1). If the roots of $G(z)$ are simple and no roots on the unit circle $|z| = 1$, then for a given sequence of numbers $\{y_n\}_{n\in\mathbb{Z}} \in \mathscr{D}_\gamma$, there exists a unique $f \in \mathscr{S}_{N,\gamma}$, such that*

$$f \star h(n) = y_n, n \in \mathbb{Z}. \tag{3}$$

Moreover, the solution can be written as

$$f(t) = \sum_{n\in\mathbb{Z}} y_n L(t-n),$$

where the reconstruction function L is given by $L(t) := \sum_{i=1}^{r} L_i(t) := \sum_{i=1}^{r} \sum_{n\in\mathbb{Z}} c_n \beta_{d_i}(t-n)$ and c_n are the coefficients of the Laurent series expansion of $G(z)^{-1}$. Further the reconstruction function L is of exponential decay.

Proof Let $C(z) = G^{-1}(z) = \sum_{n\in\mathbb{Z}} c_n z^n$. Then there exist $\mu \in (0, 1)$ such that $c_n = O\left(\mu^{|n|}\right)$. As β_{d_i} has compact support, we obtain that $O(L) = O(\mu^{|t|})$. Now for $|t| > 2$, we have

$$\begin{aligned}\frac{\sum_{n\in\mathbb{Z}} |n|^\gamma \mu^{|t-n|}}{(|t|+1)^\gamma} &\leq \frac{\sum_{n\in\mathbb{Z}} |n|^\gamma \mu^{|[t]-n|-1}}{(|[t]|)^\gamma} \\ &= \frac{\sum_{n\in\mathbb{Z}} (|[t]-n|)^\gamma \mu^{|n|-1}}{(|[t]|)^\gamma} \\ &\leq \sum_{n\in\mathbb{Z}} (1+|n|)^\gamma \mu^{|n|-1} \\ &< \infty.\end{aligned}$$

As a consequence of the above inequality we obtain that

$$f(t) = \sum_{n\in\mathbb{Z}} y_n L(t-n) = O(|t|^\gamma),$$

as $t \to \pm\infty$. Since $y_n L(t-n) = O(|n|^\gamma \mu^{|t-n|})$, it is easy to see that the series

$$\sum_{n\in\mathbb{Z}} y_n L(t-n)$$

converges uniformly and absolutely in every finite interval. Also,

$$\begin{aligned} f(t) &= \sum_{n\in\mathbb{Z}} y_n L(t-n) \\ &= \sum_{n\in\mathbb{Z}} y_n \sum_{i=1}^{r} \sum_{k\in\mathbb{Z}} c_k \beta_{d_i}(t-n-k) \\ &= \sum_{k\in\mathbb{Z}} \left(\sum_{n\in\mathbb{Z}} y_n c_{k-n} \right) \sum_{i=1}^{r} \beta_{d_i}(t-k). \end{aligned}$$

Therefore $f \in \mathscr{S}_{N,\gamma}$.
Using $C(z)G(z) = 1$, we obtain that

$$(h \star L)(n) = \sum_{i=1}^{r} \sum_{k\in\mathbb{Z}} c_k h \star \beta_{d_i}(n-k) = \delta_0(n).$$

Hence $f(t) = \sum_{n\in\mathbb{Z}} y_n L(t-n)$ satisfies

$$(h \star f)(n) = y_n, n \in \mathbb{Z}. \tag{4}$$

Now we shall show the uniqueness. Assume that $f, g \in \mathscr{S}_{N,\gamma}$ satisfy (4). Then $f - g \in \Lambda$. Using Theorem 2, there exist a constant c_j such that

$$f(t) - g(t) = \sum_{j=1}^{l} c_j \left(\Upsilon_{z_j^{-1}} \right).$$

As $f, g \in \mathscr{S}_{N,\gamma}$, we get $f(t) - g(t) = O(|t|^\gamma)$.

Using Lemma 2 and the behavior of $\left(\Upsilon_{z_j^{-1}}\right)(t)$ at $\pm\infty$, we get $c_j = 0$ and hence $f = g$.

□

For $d_i = 1, 2, 3, 4$ we shall show that the roots of $G(z)$ are simple and not on the unit circle $|z| = 1$.

Proof (*Main Theorem*) As a consequence of Theorem 1 it is sufficient to prove that, all the roots of $G(z)$ are simple and not on the unit circle $|z| = 1$ for distinct $d_i = 1, 2, 3, 4$.

We have $G(z) = \sum_{i=1}^{r} G_i(z)$. We can write

$$G(z) = \sum_{i=1}^{r} z^{\frac{-l_i}{2}} P_i(z)$$

where $l_i := \begin{cases} d_i + 1 & \text{if } d_i \text{ is odd} \\ d_i & \text{if } d_i \text{ is even} \end{cases}$ and $P_i(z)$ is a polynomial of degree l_i. Therefore,

$$G(z) = z^{\frac{-m}{2}} \sum_{i=1}^{r} z^{\frac{m-l_i}{2}} P_i(z) = z^{\frac{-m}{2}} P(z),$$

where $P(z)$ is a polynomial of degree $m = max(l_1, l_2, \ldots, l_r)$.

As d_i's are distinct, we can take $d_1 = 1$, $d_2 = 2$, $d_3 = 3$, and $d_4 = 4$. Therefore $m = 4$ and we obtain

$$\begin{aligned} P(z) &= z^2 G(z) \\ &= z^4 \left\{h \star \beta_{d_4}(2) + h \star \beta_{d_3}(2)\right\} + z^3 \left\{h \star \beta_{d_4}(1) + h \star \beta_{d_3}(1) + h \star \beta_{d_2}(1)\right. \\ &\quad \left. + h \star \beta_{d_1}(1)\right\} + z^2 \left\{h \star \beta_{d_4}(0) + h \star \beta_{d_3}(0) + h \star \beta_{d_2}(0) + h \star \beta_{d_1}(0)\right\} \\ &\quad + z \left\{h \star \beta_{d_4}(-1) + h \star \beta_{d_3}(-1) + h \star \beta_{d_2}(-1) + h \star \beta_{d_1}(-1)\right\} + \left\{h \star \beta_{d_4}(-2)\right. \\ &\quad \left. + h \star \beta_{d_3}(-2)\right\}. \end{aligned}$$

Hence $P(0) > 0$ and $P(1) > 0$.

We can write

$$P(z) = z^2 \sum_{i=1}^{4} \int_{-\frac{1}{2}}^{\frac{1}{2}} h(t) \Upsilon_{z,d_i}(t) dt. \tag{5}$$

Using Lemma 2 and Eq. (5) we get

$$P(-1) = \sum_{i=1}^{4} \int_{-\frac{1}{2}}^{\frac{1}{2}} h(t) \Upsilon_{-1,d_i}(t) dt > 0.$$

Since $\lim_{z \longrightarrow \infty} P(z) = \infty$, It is suffices to find $z_0 \in (-1, 0)$ such that

$$\sum_{i=1}^{4} \Upsilon_{z_0,d_i}(t) < 0, \text{ for all } t \in \left(-\frac{1}{2}, \frac{1}{2}\right), \tag{6}$$

since for such a z_0, we have

$$P(z_0) = z_0^2 \sum_{i=1}^{4} \int_{-\frac{1}{2}}^{\frac{1}{2}} h(t)\Upsilon_{z_0,d_i}(t)dt < 0, z_0 \in (-1, 0)$$

and $P\left(\frac{1}{z_0}\right) = \frac{1}{z_0^2} \sum_{i=1}^{4} \int_{-\frac{1}{2}}^{\frac{1}{2}} h(t)\Upsilon_{z_0^{-1},d_i}(t)dt = \frac{1}{z_0^2} \sum_{i=1}^{4} \int_{-\frac{1}{2}}^{\frac{1}{2}} h(t)\Upsilon_{z_0,d_i}(-t)dt < 0$ and $z_0^{-1} \in (-\infty, -1)$. By solving $\sum_{i=1}^{4} \Upsilon_{z_0,d_i}\left(\frac{1}{2}\right) = 0$, we get a unique $z_0 \in (-1, 0)$.

Now $\sum_{i=1}^{4} \Upsilon_{z_0,d_i}\left(\frac{1}{2}\right) = 0 \Leftrightarrow \Upsilon_{z_0,1}\left(\frac{1}{2}\right) + \Upsilon_{z_0,2}\left(\frac{1}{2}\right) + \Upsilon_{z_0,3}\left(\frac{1}{2}\right) + \Upsilon_{z_0,4}\left(\frac{1}{2}\right) = 0$

$$\Leftrightarrow z_0^3 \left\{\beta_4\left(\frac{3}{2}\right) + \beta_3\left(\frac{3}{2}\right)\right\} + z_0^2 \left\{\beta_4\left(\frac{1}{2}\right) + \beta_3\left(\frac{1}{2}\right) + \beta_2\left(\frac{1}{2}\right) + \beta_1\left(\frac{1}{2}\right)\right\}$$
$$+ z_0 \left\{\beta_4\left(-\frac{1}{2}\right) + \beta_3\left(-\frac{1}{2}\right) + \beta_2\left(-\frac{1}{2}\right) + \beta_1\left(-\frac{1}{2}\right)\right\}$$
$$+ \left\{\beta_4\left(-\frac{3}{2}\right) + \beta_3\left(-\frac{3}{2}\right)\right\} = 0$$
$$\Leftrightarrow z_0^3 \frac{3}{48} + z_0^2 \frac{93}{48} + z_0 \frac{93}{48} + \frac{3}{48} = 0.$$

The possible solutions of z_0 are -1, $-15 - 4\sqrt{14}$, $-15 + 4\sqrt{14}$. The unique solution $z_0 \in (-1, 0)$ is $-15 + 4\sqrt{14}$. For this z_0 value

$$\sum_{i=1}^{4} \Upsilon_{z_0,d_i}(t) < 0, \text{ for all } t \in \left(-\frac{1}{2}, \frac{1}{2}\right).$$

Thus we can conclude that all the roots of $G(z)$ are simple and not on the unit circle $|z| = 1$ for $d_i = 1, 2, 3, 4$. □

Remark 1 The condition that the zeros of the Laurent polynomial $G(z)$ are simple and do not lie on the unit circle $|z| = 1$ is a sufficient condition for uniqueness of solution for the local average sampling problem.

3 Conclusion

We proved local average sampling theorem over certain subspaces of the multiply generated spline spaces of polynomial growth. Let $h(t)$ be an integrable function satisfying condition (1). We have shown that if the roots of $G(z)$ are simple and no roots on the unit circle $|z| = 1$, then for a given sequence of numbers $\{y_n\}_{n\in\mathbb{Z}} \in \mathscr{D}_\gamma$, there exists a unique $f \in \mathscr{S}_{N,\gamma}$ such that $f \star h(n) = y_n$, $n \in \mathbb{Z}$, for the distinct $d_i \leq 4$. Also, we have shown that the roots of $G(z)$ are simple and not on the unit circle $|z| = 1$, for $d_i \leq 4$. We could not find a proof for $d_i \geq 5$.

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