Approximation of Solutions of a Stochastic Differential Equation

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Abstract The existence, uniqueness, and convergence of approximate solutions of a stochastic differential equation with deviated argument is studied using analytic semigroup theory and fixed point method. Then we consider Faedo-Galerkin approximation of solution and prove some convergence results. We also study an example to illustrate our result.

Keywords Analytic semigroup · Deviated argument · Stochastic Fractional differential equation

Mathematics Subject Classification (2010): 34G10, 34G20, 34K30, 35K90, 47D20

1 Introduction

Fractional differential equations appear abundantly in the theory of fractals, visco-elasticity, seismology, polymers, etc. Stochastic evolution equations are natural generalizations of ordinary differential equations incorporating the random noise which causes fluctuations in deterministic models. For details refer [1]. In certain real-world problems, delay depends not only on the time but also on the unknown quantity as we can see in [2]. Das et al. [3, 4] can be referred for related work with deviated argument. Bahuguna et. al. [5] discussed the Faedo-Galerkin approximation of solution.

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So far the Faedo-Galerkin approximation of solution stochastic fractional differential equation with deviated argument is neglected in the literature. In an attempt to fill this gap we study the following stochastic fractional differential equation with deviated argument in a separable Hilbert space $(H, (.,.))$.

$$
\begin{aligned}
{}^cD_t^{\beta}u(t) + Au(t) &= f(u(t), u(h(u(t))))\frac{dw(t)}{dt}, \quad t \in [0, T] \\
u(0) &= u_0 \in H
\end{aligned} \tag{1}
$$

where $0 < \beta < 1$ and $0 < T < \infty$. ${}^cD_t^{\beta}$ denotes the Caputo fractional derivative of order β and $A : D(A) \subset X \to H$ is a linear operator. A and the functions f, h are defined in the hypotheses $(H1) - (H3)$ of Sect. 2.

2 Preliminaries

Here we deal with two separable Hilbert spaces H and K.

(H1) A is a closed, densely defined, self-adjoint operator with pure point spectrum $0 \le \lambda_0 \le \lambda_1 \le \cdots \le \lambda_m \le \cdots$ with $\lambda_m \to \infty$ and $m \to \infty$ and corresponding complete orthonormal system of eigenfunctions ϕ_j such that

$$A\phi_j = \lambda_j \phi_j \; and \; < \phi_i, \phi_j >= \delta_{i,j}$$

(H2) The function $f : [O, T] \times H_\alpha \times H_{\alpha-1} \to L(K, H)$ is continuous and $\exists$ constant L_f such that

$$\|f(u, u_1) - f(v, v_1)\|_Q^2 \le L_f[+\|u - v\|_\alpha + \|u_1 - v_1\|_{\alpha-1}]$$

(H3) The map $h : H_\alpha \times \mathcal{R}_+ \to \mathcal{R}_+$ satisfies $\|h(u,) - h(v,)\| \le L_h(\|u - v\|_\alpha)$

If $(H1)$ is satisfied then $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA} : t \ge 0\}$ in H. We also note that $\exists$ constant C such that $\|S(t)\| \le Ce^{\omega t}$ and constants C_i 's such that $\|\frac{d^i}{dt^i}S(t)\| \le C_i,\ t > 0,\ i = 1, 2$. Also $\|AS(t)\| \le Ct^{-1}$ and $\|A^\alpha S(t)\| \le C_\alpha t^{-\alpha}$.

We define the space H_α as $D(A^\alpha)$ endowed with the norm $\|.\|_\alpha$. Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space endowed with complete family of right continuous increasing sub σ—algebras $\{\mathfrak{F}_t, t \in J\}$ such that $\mathfrak{F}_t \subset \mathfrak{F}$. A H—valued random variable is a $\mathcal{F}$—measurable process. We also assume that W is a Wiener process on K with covariance operator Q. Suppose Q is symmetric, positive, linear and bounded operator with $TrQ < \infty$. Let $K_0 = Q^{\frac{1}{2}}(K)$. The space $L_2^0 = L_2(K_0, H_\alpha)$ is a separable Hilbert space with norm $\|\psi\|_{L_2^0} = \|\psi Q^{\frac{1}{2}}\|_{L_2(K,H_\alpha)}$. Let $L_2(\Omega, \mathfrak{F}, P; H_\alpha) \equiv L_2(\Omega; H_\alpha)$ be the Banach space of all strongly measurable, square integrable, H_α—valued random variables equipped with the norm

$\|u(.)\|^2_{L_2} = E\|u(.;w)\|^2_{H_\alpha}$. C^α_T denotes the Banach space of all continuous maps from $J = (0, T]$ into $L_2(\Omega; H_\alpha)$ which satisfy $sup_{t\in J} E\|u(t)\|^2_{C^\alpha} < \infty$. $L^0_2(\Omega, H_\alpha) = \{f \in L_2(\Omega, H_\alpha) : f\ is\ \mathcal{F}_0 - measurable\}$ denotes an important subspace. For $0 \le \alpha < 1$ define

$$C^{\alpha-1}_T = \{u \in C^\alpha_T : \|u(t) - u(s)\|_{\alpha-1} \le L|t - s|, \forall t, s \in [0, T]\}.$$

Now let us define mild solution of (1):

Definition 1 The mild solution of (1) is a continuous $\mathfrak{F}_t$ adapted stochastic process $u \in C^\alpha_T \cap C^{\alpha-1}_T$ which satisfies the following:

1. $u(t) \in H_\alpha$ has $C\grave{a}dl\grave{a}g$ paths on $t \in [0, T]$.
2. $\forall t \in [0, T],\ u(t)$ is the solution of the integral equation

$$u(t) = T_\beta(t)u_0 + \int_0^t (t - s)^{\beta-1} S_\beta(t - s) f(u(s), u(h(u(s), s)))dw(s),\ t \in [0, T] \tag{2}$$

where $S_\beta(t) = \int_0^\infty \zeta_\beta(\theta)S(t^\beta\theta)d\theta$; and $T_\beta(t) = q\int_0^\infty \theta\zeta_\beta(\theta)S(t^\beta\theta)d\theta$; ζ_β is a probability density function defined on $(0,\infty)$, i.e. $\zeta_\beta(\theta) \ge 0,\ \theta \in (0,\infty)$ and $\int_0^\infty \zeta_\beta(\theta)d\theta = 1$. Also $\|T_\beta(t)u\| \le C\|u\|,\ \|S_\beta(t)u\| \le \frac{\beta C}{\Gamma(1+\beta)}\|u\|,\ \|A^\alpha S_\beta(t)u\| \le \frac{\beta C_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))} t^{-\alpha\beta}\|u\|$.

Lemma 1 *Let $f : J \times \Omega \times \Omega \to L^0_2$ be a strongly measurable mapping with $\int_0^T E\|f(t)\|^p_{L^0_2}dt < \infty$. Then*

$$E\|\int_0^t f(s)dw(s)\|^p \le l_s \int_0^t E\|f(s)\|^p_{L^0_2}ds$$

$\forall t \in [0, T]$ *and* $p \ge 2$ *where* l_s *is a constant containing* p *and* T.

l_s is incorporated into the constants in the following sections.

3 Existence and Uniqueness of Approximate Solutions

In this section we consider a sequence of approximate integrals and establish the existence and uniqueness of solution for each of the approximate integral equations. For $0 \le \alpha < 1$ and $u \in C^\alpha_{T_0}$, the hypotheses $(H2) - (H3)$, imply that $f(u(s), u(h(u(s), s)))$ is continuous on $[0, T_0]$. Therefore, $\exists$ a positive constant

$$N = 2L_f[T_0^{\theta_1} + 2R(1 + LL_h) + LL_hT_0^{\theta_2}] + 2N_0, \quad N_0 = E\|f(u_0, u_0)\|^2$$

such that $\|f(s, u(s), u(h(u(s), s)))\| \le N,\ \ t \in [0, T]$. Choose T_0, $0 < T_0 \le T$ such that

$$\left(\frac{\beta C_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\right)^2 N \frac{T_0^{\beta(1-\alpha)-1}}{\beta(1-\alpha)-1} \leq \frac{R}{4},$$

$$D = \left(\frac{\beta C_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\right)^2 2L_f \frac{T_0^{\beta(1-\alpha)-1}}{2\beta(1-\alpha)-1} \leq 1 \tag{3}$$

Let

$$B_R = \{u \in \mathcal{C}_{T_0}^{\alpha} \cap \mathcal{C}_{T_0}^{\alpha-1} : u(0) = u_0, \quad \|u - u_0\|_{T_0,\alpha} \leq R\}$$

It is easy to see that B_R is a closed and bounded subset of $\mathcal{C}_{T_0}^{\alpha-1}$ and complete. Let us define the operator $\mathcal{F}_n : B_R :\rightarrow B_R$ by

$$(\mathcal{F}_n u)(t) = T_\beta(t)u_0 + \int_0^t (t-s)^{\beta-1} S_\beta(t-s) f_n(u(s), u(h(u(s), s)))dw(s). \tag{4}$$

Theorem 1 *If the hypotheses* $(H1)$, $(H2)$ *and* $(H3)$ *are satisfied and* $u_0 \in L_2^0(\Omega, X_\alpha)$, $0 \leq \alpha < 1$, *then* $\exists$ *a unique* $u_n \in B_R$ *such that* $\mathcal{F}_n u_n = u_n$, $\forall\, n = 0, 1, 2, \cdots$, *i.e.,* u_n *satisfies the approximate integral equation*

$$u_n(t) = T_\beta(t)u_0 + \int_0^t (t-s)^{\beta-1} S_\beta(t-s) f_n(s, u_n(s), u_n(h(u_n(s), s)))dw(s),$$
$$t \in [0, T] \tag{5}$$

Proof Step1 : We need to show that $\mathcal{F}_n u \in \mathcal{C}_{T_0}^{\alpha-1}$, $\forall u \in \mathcal{C}_{T_0}^{\alpha-1}$. It is easy to check that $\mathcal{F}_n : \mathcal{C}_T^{\alpha} \rightarrow \mathcal{C}_T^{\alpha}$. If $u \in \mathcal{C}_{T_0}^{\alpha-1}$, $0 < t_1 < t_2 < T_0$ and $0 \leq \alpha < 1$ then

$$\begin{aligned}
& E\|\mathcal{F}_n u(t_2) - \mathcal{F}_n u(t_1)\|_{\alpha-1}^2 \\
& \leq 3E\|[T_\beta(t_2) - T_\beta(t_1)]u_0\|_{\alpha-1}^2 \\
& \quad + 3E\|\int_{t_1}^{t_2} (t_2-s)^{\beta-1} A^{\alpha-1} S_\beta(t_2-s) f_n(u(s), u(h(u(s)))dw(s)\|_Q^2 \\
& \quad + 3E\|\int_0^{t_1} A[(t_2-s)^{\beta-1} S_\beta(t_2-s) - (t_1-s)^{\beta-1} S_\beta(t_1-s)] \\
& \quad A^{\alpha-2} \times f_n(u(s), u(h(u(s))))dw(s)\|_Q \\
& \leq 3E\|[T_\beta(t_2) - T_\beta(t_1)]u_0\|_{\alpha-1}^2 + 3\frac{\beta^2 C_\alpha^2 \Gamma^2(2-\alpha)}{\Gamma^2(1+\beta(1-\alpha))} \int_{t_1}^{t_2} \|(t_2-s)^{2\beta(1-\alpha)-2}\| \\
& \quad \times \|A^{-1}\|^2 E\|f_n(u(s), u(h(u(s),)))\|^2 ds \\
& \quad + 3\int_0^{t_1} \|A[(t_2-s)^{\beta-1} S_\beta(t_2-s) - (t_1-s)^{\beta-1} S_\beta(t_1-s)] \\
& \quad \times \|A^{\alpha-2}\|^2 E\|f_n(u(s), u(h(u(s))))\|^2 ds
\end{aligned} \tag{6}$$

$\forall u \in H$, we can write

$$[S(t_2^\beta \theta) - S(t_1^\beta \theta)]u = \int_{t_1}^{t_2} \frac{d}{dt} S(t^\beta \theta) u dt = \int_{t_1}^{t_2} \theta \beta t^{\beta-1} A S(t^\beta \theta) dt.$$

The first term of (6) can be estimated as follows:

$$\begin{aligned}
\|[T_\beta(t_2) - T_\beta(t_1)]u_0\|_{\alpha-1}^2 &\le \left(\int_0^\infty \zeta_\beta(\theta) \| S(t_2^\beta \theta) - S(t_1^\beta \theta) \| \| A^{\alpha-1} u_0 \| d\theta \right)^2 \\
&\le \left(\int_0^\infty \zeta_\beta(\theta) [\int_{t_1}^{t_2} \| \frac{d}{dt} S(t^\beta \theta) \| dt] \| u_0 \|_\alpha d\theta \right)^2 \\
&\le C_1^2 \| u_0 \|_{\alpha-1}^2 (t_2 - t_1)^2 \qquad (7)
\end{aligned}$$

For the second term of (6) we get the following estimate

$$\begin{aligned}
&\int_{t_1}^{t_2} (t_2 - s)^{2\beta(1-\alpha)-2} E \| f_n(u(s), u(h(u(s)))) \|^2 ds \\
&\qquad \le \frac{N(t_2 - t_1)^{2\beta(1-\alpha)-1}}{2\beta(1-\alpha) - 1} \qquad (8)
\end{aligned}$$

For the third term we will use the following estimate

$$\begin{aligned}
&\int_0^{t_1} \| A[(t_2 - s)^{\beta-1} S_\beta(t_2 - s) - (t_1 - s)^{\beta-1} S_\beta(t_1 - s)] \|^2 \\
&\quad \times \| A^{\alpha-2} \|^2 E \| f_n(u(s), u(h(u(s)))) \|^2 ds \\
&\quad \le \int_0^{t_1} \left(\int_0^\infty \zeta_\beta(\theta) \| [\frac{d}{dt} S((t-s)^\beta \theta)|_{t=t_2} - \frac{d}{dt} S((t-s)^\beta \theta)|_{t=t_1}] \| d\theta \right)^2 \\
&\quad \times E \| f(u(s), u(h(u(s)))) \|^2 ds \\
&\quad \le \int_0^{t_1} \left(\int_0^\infty \zeta_\beta(\theta) [\int_{t_1}^{t_2} \| A^{\alpha-2} \frac{d^2}{dt^2} S((t-s)^\beta \theta) \| dt] d\theta \right)^2 N ds \\
&\quad \le C_2^2 \| A^{\alpha-2} \|^2 (t_2 - t_1)^2 N T_0 \qquad (9)
\end{aligned}$$

Hence from inequalities (7)–(9) we see that the map $\mathcal{F}_n : \mathcal{C}_{T_0}^{\alpha-1} \to \mathcal{C}_{T_0}^{\alpha-1}$ is well-defined. Now we prove that $\mathcal{F}_n : B_R \to B_R$. So for $t \in [0, T_0]$ and $u \in B_R$.

$$\begin{aligned}
&E \| (\mathcal{F}_n u)(t) - u_0 \|_\alpha^2 \\
&\quad \le 2E \| (T_\beta(t) - I) u_0 \|_\alpha^2 \\
&\qquad + 2E \| \int_0^t (t-s)^{\beta-1} S_\beta(t-s) f(u(s), u(h(u(s)))) dw(s) \|_Q^2
\end{aligned}$$

$$\leq 2E\|(T_\beta(t)-I)u_0\|_\alpha^2 + 2\left(\frac{\beta C_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\right)^2 \int_0^t \|(t_2-s)^{2\beta(1-\alpha)-2}\|^2$$
$$\times\ E\|f_n(u(s),u(h(u(s))))\|^2 ds$$
$$\leq \frac{R}{2} + 2\left(\frac{\beta C_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\right)^2 N\frac{T_0^{\beta(1-\alpha)-1}}{\beta(1-\alpha)-1} \leq \frac{R}{2}+\frac{R}{2}=R$$

Now we show that $\mathcal{F}_n$ is a contraction map by using (3) in last but one inequality. $\forall u, v \in B_R$

$$E\|(\mathcal{F}_n u)(t)-(\mathcal{F}_n v)(t)\|_\alpha^2 = E\|\int_0^t (t-s)^{\beta-1}A^\alpha S_\beta(t-s)$$
$$\times [f(u(s),u(h(u(s)))) - f(s,v(s),v(h(v(s),s)))dw(s)]\|_Q^2$$
$$\leq \left(\frac{\beta C_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\right)^2 \int_0^t (t_2-s)^{2\beta(1-\alpha)-2}$$
$$\times\ E\|f(u(s),u(h(u(s)))) - f(v(s),v(h(v(s))))\|^2 ds$$
$$\leq \left(\frac{\beta C_\alpha \Gamma(2-\alpha)}{\Gamma(1+\beta(1-\alpha))}\right)^2 2L_f(1+2LLh)\|u-v\|_\alpha^2 \frac{T^2\beta(1-\alpha)-1}{2\beta(1-\alpha)-1}$$
$$\leq \|u-v\|_\alpha^2.$$

This implies that $\exists$ a unique fixed point u_n of $\mathcal{F}_n$. Thus there a unique mild approximate solution of (1)

Lemma 2 *Let* $(H1)-(H3)$ *hold. If* $u_0 \in L_2^0(\Omega, D(A^\alpha))$, $\forall 0<\alpha<\eta<1$, *then* $u_n(t) \in D(A^\gamma)$ *for all* $t \in [0, T_0]$ *with* $0<\gamma<\eta<1$. *Also if* $u_0 \in D(A)$, *then* $u_n(t) \in D(A^\gamma)$ $\forall t \in [0, T_0]$, *where* $0<\gamma<\eta<1$.

Proof By Theorem (1) we get the existence of a unique $u_n \in B_R$, satisfying (5). Theorem 2.6.13 of [6] implies for $t>0,\ 0\leq\gamma<1,\ S(t): H \to D(A^\gamma)$ and for $0\leq\gamma<\eta<1,\ D(A^\eta)\subset D(A^\gamma)$. It is easy to see that Holder continuity of u_n can be proved using the similar arguments from (6) to (9). Also from Theorem 1.2.4 in [6], we have $S(t)u \in D(A)$ if $u \in D(A)$. The result follows from these facts and that $D(A)\subset D(A^\gamma)$ for $0\leq\gamma<1$.

Lemma 3 *Let* $(H1)-(H3)$ *hold and* $u_0 \in L_2^0(\Omega, X_\alpha)$. *Then for any* $t_0 \in (0, T_0]$ $\exists$ *a constant* U_{t_0}, *independent of n such that* $E\|u_n(t)\|_\gamma^2 \leq U_{t_0}$ $\forall t \in [t_0, T_0]$, $n = 1, 2, \cdots$. *Also if* $u_0 \in L_2^0(\Omega, D(A))$ *then* $\exists$ *constant* U_0 *independent of n such that* $E\|u_n(t)\|_\gamma^2 \leq U_0$ $\forall t \in [t_0, T_0]$, $n = 1, 2, \cdots$, $\forall\, 0<\gamma\leq 1$.

Proof Let $u_0 \in L_2^0(\Omega, H_\alpha)$. Applying A^γ on both sides of (4)

$$
\begin{aligned}
&E\|u_n(t)\|_\gamma^2 \\
&\le 2E\|T_\beta(t)u_0\|_\gamma^2 + 2\|\int_0^t (t-s)^{\beta-1} S_\beta(t-s) f_n(u(s), u(h(u(s))))dw(s)\|_Q^2 \\
&\le 2C_\gamma^2 t_0^{-2\gamma\beta}\|u_0\|^2 + \left(\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))}\right)^2 \frac{N(T_0)^{2\beta(1-\gamma)-1}}{2\beta(1-\gamma)-1} = U_{t_0}.
\end{aligned}
$$

Also if $u_0 \in L_2^0(\Omega, D(A))$, then we have that $u_0 \in L_2^0(\Omega, D(A^\gamma))$ for $0 \le \gamma < 1$. Hence,

$$
\begin{aligned}
&E\|u_n(t)\|_\gamma^2 \\
&\le 2E\|T_\beta(t)u_0\|_\gamma^2 + 2\|\int_0^t (t-s)^{\beta-1} S_\beta(t-s) f_n(u(s), u(h(u(s))))dw(s)\|_Q^2 \\
&\le 2C^2\|u_0\|^2 + \left(\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))}\right)^2 \frac{N(T_0)^{2\beta(1-\gamma)-1}}{2\beta(1-\gamma)-1} = U_0.
\end{aligned}
$$

Hence proved.

4 Convergence of Solutions

In this section the convergence of the solution $u_n \in H_\alpha$ of the approximate integral equation (5) to a unique solution u of (2), is discussed.

Theorem 2 *Let the hypotheses* $(H1)-(H3)$ *hold and if* $u_0 \in L_2^0(\Omega, H_\alpha)$ *then* $\forall t_0 \in (0, T]$,

$$\lim_{m\to\infty} \sup_{\{n\ge M, t_0\le t\le T_0\}} \|u_n(t) - u_m(t)\|_\alpha = 0.$$

Proof Let $0 < \alpha < \gamma < \eta$. For $t_0 \in (0, T_0]$

$$
\begin{aligned}
&E\|f_n(u_n(t), u_n(h(u_n(t)))) - f_m(t, u_m(t), u_m(h(u_m(t))))\|^2 \\
&\quad\le 2E\|f_n(u_n(t), u_n(h(u_n(t)))) - f_n(t, u_m(t), u_m(h(u_m(t))))\|^2 \\
&\quad\le 2E\|f_n(u_m(t), u_m(h(u_m(t)))) - f_m(t, u_m(t), u_m(h(u_m(t))))\|^2 \\
&\quad\le 2(2L_f(1+2LL_h)[E\|u_n - u_m\|_\alpha^2 + E\|(P^n - P^m)u_m(t)\|_\alpha^2]) \qquad (10)
\end{aligned}
$$

Now,

$$E\|(P^n - P^m)u_m(t)\|^2 \le E\|A^{\alpha-\gamma}(P^n - P^m)A^\gamma u_m(t)\|^2 \le \frac{1}{\lambda_m^{2(\gamma-\alpha)}} E\|A^\gamma u_m(t)\|^2$$

Then we have

$$
\begin{aligned}
&E\|f_n(t,u_n(t),u_n(h(u_n(t)))) - f_m(t,u_m(t),u_m(h(u_m(t))))\|^2\\
&\quad \leq 2\left(2L_f(1+2LL_h)\left[E\|u_n-u_m\|_\alpha^2 + \frac{1}{\lambda_m^{2(\gamma-\alpha)}}E\|A^\gamma u_m(t)\|^2\right]\right)
\end{aligned}
$$

For $0 < t_0' < t_0$

$$
\begin{aligned}
E\|u_n(t)-u_m(t)\|_\alpha^2 &\leq 2\left(\int_0^{t_0'} + \int_{t_0'}^{t}\right)\|(t-s)^{\beta-1}A^\alpha S_\beta(t-s)\|^2\\
&\quad \times\ E\|f_n(u_n(t),u_n(h(u_n(t)))) - f_m(u_m(t),u_m(h(u_m(t))))\|^2 ds \qquad (11)
\end{aligned}
$$

The estimate of first integral of the above inequality is

$$
\begin{aligned}
&E\|u_n(t)-u_m(t)\|_\alpha^2\\
&\quad\leq \int_0^{t_0'} \|(t-s)^{\beta-1}A^\alpha S_\beta(t-s)\|^2\\
&\qquad \times\ E\|f_n(u_n(t),u_n(h(u_n(t)))) - f_m(u_m(t),u_m(h(u_m(t))))\|^2 ds\\
&\quad\leq \left(\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))}\right)^2 \frac{2N(t_0-\delta_1 t_0')^{2\beta(1-\gamma)-2}}{2\beta(1-\gamma)-1}t_0', \quad 0<\delta<1 \qquad (12)
\end{aligned}
$$

The estimate of second integral is

$$
\begin{aligned}
E\|u_n(t)-u_m(t)\|_\alpha^2 &\leq \int_{t_0'}^{t} \|(t-s)^{\beta-1}A^\alpha S_\beta(t-s)\|^2\\
&\quad \times\ E\|f_n(u_n(t),u_n(h(u_n(t)))) - f_m(u_m(t),u_m(h(u_m(t))))\|^2 ds\\
&\leq \left(\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))}\right)^2 \int_{t_0'}^{t}(t-s)^{2\beta(\alpha-1)-2}\\
&\quad \times\ 4L_f(1+2LL_h)\left[E\|u_n-u_m\|_\alpha^2 + \frac{E\|A^\gamma u_m(s)\|^2}{\lambda^2(\gamma-\alpha)}\right]ds\\
&\leq 4L_f(1+2LL_h)\left(\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))}\right)^2 [\int_{t_0'}^{t}(t-s)^{2\beta(\alpha-1)-2}\\
&\quad \times\ E\|u_n-u_m\|_\alpha^2 ds + \frac{U_{t_0}}{\lambda_m^{2(\gamma-\alpha)}}\frac{T_0^{2\beta(1-\alpha)-1}}{2\beta(1-\alpha)-1}] \qquad (13)
\end{aligned}
$$

Substituting inequalities (12), (13) into (11) we get

$$
\begin{aligned}
&E\|u_n(t)-u_m(t)\|_\alpha^2 \\
&\quad\leq \left(\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))}\right)^2 \frac{4N(t_0-\delta_1 t_0')^{2\beta(1-\gamma)-2}}{2\beta(1-\gamma)-1} t_0' \\
&\quad + 8L_f(1+2LL_h)\left(\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))}\right)^2 [\int_{t_0'}^t (t-s)^{2\beta(\alpha-1)-2} \\
&\quad \times E\|u_n-u_m\|_\alpha^2 ds + \frac{U_{t_0}}{\lambda_m^{2(\gamma-\alpha)}} \frac{T_0^{2\beta(1-\alpha)-1}}{2\beta(1-\alpha)-1}]
\end{aligned}
$$

By using Gronwall's inequality, $\exists$ a constant D such that

$$
\begin{aligned}
E\|u_n(t)-u_m(t)\|_\alpha^2 \leq &\left[\left(\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))}\right)^2 \frac{4N(t_0-\delta_1 t_0')^{2\beta(1-\gamma)-2}}{2\beta(1-\gamma)-1} t_0'\right. \\
&\left. + 8L_f(1+2LL_h)\left(\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))}\right)^2 \frac{U_{t_0}}{\lambda_m^{2(\gamma-\alpha)}} \frac{T_0^{2\beta(1-\alpha)-1}}{2\beta(1-\alpha)-1}\right] \times D
\end{aligned}
$$

Let $m \to \infty$. Taking supremum over $[t_0, T_0]$ we get the following inequality:

$$
E\|u_n(t)-u_m(t)\|_\alpha^2 \leq \left[\left(\frac{\beta C_\gamma \Gamma(2-\gamma)}{\Gamma(1+\beta(1-\gamma))}\right)^2 \frac{4N(t_0-\delta_1 t_0')^{2\beta(1-\gamma)-2}}{2\beta(1-\gamma)-1} t_0'\right] \times D
$$

Since t_0' is arbitrary, the right-hand side can be made infinitesimally small by choosing t_0' sufficiently small. Thus the lemma is proved.

Corollary 1 *If* $u_0 \in D(A)$, *then* $\lim_{m\to\infty} \sup_{\{n\geq m,\ 0\leq t\leq T_0\}} E\|u_n(t)-u_m(t)\|_\alpha^2 = 0$

Proof By using Lemmas (2) and (3) we can take $t_0 = 0$ in the proof of Theorem (2) and hence the corollary follows.

Theorem 3 *Let us assume that* $(H1)-(H3)$ *are satisfied and suppose* $u_0 \in L_2^0(\Omega, X_\alpha)$. *Then for* $t \in [0, T_0]$, $\exists$ *a unique function* $u_n \in B_R$ *where*
$u_n(t) = T_\beta u_0 + \int_0^t (t-s)^{\beta-1} S_\beta(t-s) f_n(u_n(s), u_n(h_n(u_n(s))))dw(s)$,
and $u(t) \in B_R$, *where*
$u(t) = T_\beta u_0 + \int_0^t (t-s)^{\beta-1} S_\beta(t-s) f(u(s), u(h(u(s))))dw(s)$, $t \in [0, T_0]$, *such that* $u_n \to u$ *as* $n \to \infty$ *in* B_R *and* u *satisfies (2) on* $[0, T_0]$.

Proof By using the above Corollary, Theorems 1 and 2 it is to see that $\exists\, u(t) \in B_R$ such that
$\lim_{n\to\infty} E\|u_n(t)-u(t)\|_\alpha^2 = 0$ on $[0, T_0]$. Now

$$E\|u_n(t) - T_\beta u_0 + \int_{t_0}^{t} (t-s)^{\beta-1} S_\beta(t-s) f_n(u_n(s), u_n(h_n(u_n(s))))dw(s)\|^2$$
$$\leq E\| \int_0^{t_0} (t-s)^{\beta-1} S_\beta(t-s) f_n(u_n(s), u_n(h_n(u_n(s))))dw(s)\|^2$$
$$\leq \left(\frac{\beta C}{\Gamma(1+\beta)}\right)^2 N \frac{T_0^{2\beta-2}}{2\beta-2} t_0 \tag{14}$$

Let $n \to \infty$ then
$E\|u_n(t) - T_\beta u_0 + \int_{t_0}^{t} (t-s)^{\beta-1} S_\beta(t-s) f_n(u_n(s), u_n(h_n(u_n(s))))dw(s)\|^2$
$\leq \left(\frac{\beta C}{\Gamma(1+\beta)}\right)^2 N \frac{T_0^{2\beta-2}}{2\beta-2} t_0$ and since t_0 is arbitrary we conclude $u(t)$ satisfies (2). Uniqueness follows easily from Theorems 1, 2 and Gronwall's inequality.

4.1 Faedo-Galerkin Approximations

We know from the previous sections that for any $0 \leq T_0 \leq T$, we have a unique $u \in C^\alpha_{T_0}$ satisfying the integral equation
$u(t) = T_\beta u_0 + \int_0^t (t-s)^{\beta-1} S_\beta(t-s) f(u(s), u(h(u(s))))dw(s)$, $t \in [0, T_0]$ Also, $\exists$ a unique solution $u_n \in C^\alpha_{T_0}$ of the approximate integral equation
$u_n(t) = T_\beta u_0 + \int_0^t (t-s)^{\beta-1} S_\beta(t-s) f_n(u_n(s), u_n(h(u_n(s))))dw(s)$, $t \in [0, T_0]$.
Faedo-Galerkin approximation $\bar{u}_n = P^n u_n$ is given by
$P^n u_n(t) = \bar{u}_n(t) = T_\beta(t) P^n u_0$
$+ \int_0^t (t-s)^{\beta-1} S_\beta(t-s) P^n f(u_n(s), u_n(h(u_n(s))))dw(s), t \in [0, T_0]$. If the solution $u(t)$ to (2) exists on $[0, T_0]$ then it has the representation

$$u(t) = \sum_{i=0}^{\infty} \alpha_i(t)\phi_i, \text{ where } \alpha_i(t) = (u(t), \phi_i) \text{ for } i = 0, 1, 2, 3, \cdots \text{ and}$$

$$\bar{u}_n(t) = \sum_{i=0}^{n} \alpha_i^n(t)\phi_i, \text{ where } \alpha_i^n(t) = (\bar{u}_n(t), \phi_i) \text{ for } i = 0, 1, 2, 3, \cdots.$$

As a consequence of Theorems 1 and 2, we have the following result.

Theorem 4 *Let us assume that* $(H1) - (H3)$ *are satisfied and suppose* $u_0 \in L_2^0(\Omega, X_\alpha)$. *Then for* $t \in [0, T_0]$, $\exists$ *a unique function* $u_n \in B_R$ *where*
$u_n(t) = T_\beta P^n u_0 + \int_0^t (t-s)^{\beta-1} S_\beta(t-s) P^n f_n(u_n(s), u_n(h(u_n(s))))dw(s)$,
and $u(t) \in B_R$, *where*
$u(t) = T_\beta u_0 + \int_0^t (t-s)^{\beta-1} S_\beta(t-s) f(u(s), u(h(u(s))))dw(s), t \in [0, T_0]$, *such that* $u_n \to u$ *as* $n \to \infty$ *in* B_R *and* u *satisfies (2) on* $[0, T_0]$.

Now the convergence of $\alpha_i^n(t) \to \alpha_i(t)$ is shown. It is easily seen that

$$A^\alpha [u(t) - \bar{u}_n(t)] = A^\alpha \Big[\sum_{i=0}^{n} \{\alpha_i(t) - \alpha_i^n(t)\}\phi_i \Big] + A^\alpha \sum_{i=n+1}^{\infty} \alpha_i(t)\phi_i$$

$$= \sum_{i=0}^{n} \lambda_i^{\alpha}\{\alpha_i(t) - \alpha_i^n(t)\}\phi_i + \sum_{i=n+1}^{\infty} \lambda_i^{\alpha}\alpha_i(t)\phi_i. \text{ Thus we have}$$

$E\|A^{\alpha}[u(t) - \bar{u}_n(t)\|^2 \geq \sum_{i=0}^{n} \lambda_i^{2\alpha} E|\alpha_i(t) - \alpha_i^n(t)|^2$.

Theorem 5 *Let us assume* $(H1) - (H3)$ *hold.*

(i) *If* $u_0 \in L_2^0(\Omega, X_\alpha)$ *then* $\lim_{n\to\infty} \sup_{t\in[t_0,T_0]} \left[\sum_{i=0}^{n} \lambda_i(t)^{2\alpha} E\|\alpha_i(t) - \alpha_i^n(t)\|^2\right] = 0$

(ii) *If* $u_0 \in L_2^0(\Omega, D(A))$ *then* $\lim_{n\to\infty} \sup_{t\in[0,T_0]} \left[\sum_{i=0}^{n} \lambda_i(t)^{2\alpha} E\|\alpha_i(t) - \alpha_i^n(t)\|^2\right] = 0$

Theorem 5 follows from the facts mentioned above the theorem.

Corollary 2 *Let us assume* $(H1) - (H3)$ *hold.*
(i) *If* $u_0 \in L_2^0(\Omega, X_\alpha)$ *then* $\lim_{n\to\infty} \sup_{t\in[t_0,T_0],n\geq m} E\|A^{\alpha}[\bar{u}_n(t) - \bar{u}_m(t)]\|^2 = 0$
(ii) *If* $u_0 \in L_2^0(\Omega, D(A))$ *then* $\lim_{n\to\infty} \sup_{t\in[0,T_0],n\geq m} E\|A^{\alpha}[\bar{u}_n(t) - \bar{u}_m(t)]\|^2 = 0$

Proof

$$\begin{aligned} E\|A^{\alpha}[\bar{u}_n(t) - \bar{u}_m(t)]\|^2 &= E\|P^n u_n(t) - P^m u_m(t)\|_\alpha^2 \\ &\leq 2E\|P^n[u_n(t) - u_m(t)]\|_\alpha^2 + 2E\|(P^n - P^m)y_m(t)\|_\alpha^2 \\ &\leq 2E\|[u_n(t) - u_m(t)]\|_\alpha^2 + 2\frac{1}{\lambda_m^{\gamma-\alpha}} E\|A^{\gamma}u_m(t)\|^2 \end{aligned}$$

Then the result (i) follows from Theorem 2 and result (ii) follows from Corollary 1.

5 Example

Suppose for $t \geq 0,\ x \in (0,1), 0 < \beta \leq 1$

$$\begin{aligned} &{}^cD^{\beta}v_t(t,x) = v_{xx}(t,x) + F(v(t,x), v(h(t,v(x))))\frac{dw(t)}{dt}, \\ &\qquad v(t,x) = v_0,\ t = 0,\ x \in (0,1) \quad and \quad v(t,0) = v(t,1) = 0,\ t \geq 0 \quad (15) \end{aligned}$$

Let F be an appropriate Holder continuous function satisfying $(H2)$ in $L_2^0(K,(0,1))$. w is a standard $L_2(0,1)$ valued Weiner process. Let us define $A = -\frac{d^2}{dx^2},\ f := F,\ v(x) = u(t)$ and let $D(A) = H_0^1(0,1) \cap H^2(0,1),\ D(A^{1/2}) = H_0^1(0,1)$. Then (15) can be reformulated into (1). Now from Theorems (1), (2) we can similarly prove the existence, uniqueness, and approximation of the mild solution of (15).

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