Controllability of Nonlinear Fractional Neutral Stochastic Dynamical Systems with Poisson Jumps

T. Sathiyaraj and P. Balasubramaniam

Abstract This paper is concerned with the controllability of fractional neutral stochastic dynamical systems with Poisson jumps in the finite dimensional space. Sufficient conditions for controllability results are obtained by using Krasnoselskii's fixed point theorem. The controllability Grammian matrix is defined by Mittag-Leffler matrix function.

Keywords Controllability · Fractional differential equation · Mittag-Leffler function · Neutral stochastic system · Poisson jumps

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1 Introduction

Fractional differential equations have recently been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. It draws a great application in nonlinear oscillations of earthquakes, many physical phenomena such that seepage flow in porous media and in fluid dynamic traffic model. There has been a significant development in fractional differential equations in recent years (see [6, 9, 11, 12]).

It is well known that the concept of controllability plays an important role in engineering and control theory. The controllability results for linear and nonlinear integral order dynamical systems in finite-dimensional space have discussed extensively (see [4]). Local null controllability of nonlinear functional differential systems

T. Sathiyaraj · P. Balasubramaniam (✉)
Gandhigram Rural Institute-Deemed University,
Gandhigram 624 302, Tamil Nadu, India
e-mail: balugru@gmail.com

T. Sathiyaraj
e-mail: sathiyaraj133@gmail.com

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in Banach space has been studied in [1]. Approximate controllability of fractional order semilinear systems with bounded delay has been studied (see [8]).

In recent years, the controllability problems for stochastic differential equations have become a field of increasing interest (see [2, 7, 10] and references therein). Stochastic differential equations have many applications in ecology, finance, and economics. The extensions of deterministic controllability concepts to stochastic system have been discussed only in a limited number of publications.

The Poisson jumps have become very popular in recent years, because it is extensively used to model many of the phenomena arising in areas such as economics, finance, physics, biology, medicine, and other science. For example, if a system jumps from a "normal state" to a "bad state," the strength of systems is random. It is natural and necessary to include a jump term in any dynamical system to make more realistic sysrems. Complete controllability of stochastic evolution equations with jumps has been studied in [13].

However, to the best of authors' knowledge, there are no relevant reports on the controllability of fractional neutral stochastic dynamical systems with Poisson jumps in the finite-dimensional space. Motivated by the above, in this article the controllability of fractional neutral stochastic dynamical systems is studied with Poisson jumps in finite-dimensional spaces. Sufficient conditions for controllability results are obtained by using Krasnoselskii's fixed point theorem with a Grammian matrix defined by Mittag-Leffler matrix function.

The paper is organized as follows: In Sect. 2, some well-known fractional operators and the solution representation of linear fractional stochastic differential equation with Poisson jumps are discussed. In Sect. 3, the linear and nonlinear fractional neutral stochastic differential equation with Poisson jumps are considered and the controllability conditions are established by using the controllability Grammian matrix which is defined by means of the Mittag-Leffler matrix function. Finally, concluding remarks are given in Sect. 4.

2 Preliminaries

Let p and q are some positive constants satisfying $n-1 < q < n, n-1 < p < n$ and $n \in \mathbb{N}$. Let $\mathbb{R}^m$ be the m-dimensional Euclidean space. The following notations and definitions are well known, for a suitable function $f \in L_1(\mathbb{R}_+), \mathbb{R}_+ = [0, \infty)$ for more details, (see [6]).

(a) Riemann–Liouville fractional operator:

$$(I_{0+}^q f)(x) = \frac{1}{\Gamma(q)} \int_0^x (x-t)^{q-1} f(t) dt$$

(b) Mittag-Leffler Function:
The most interesting properties of the Mittag-Leffler function are associated with

their Laplace integral

$$\int_0^{\infty} e^{-st} t^{p-1} E_{q,p}(\pm a t^q) dt = \frac{s^{q-p}}{(s^q \mp a)},$$

That is,

$$\mathscr{L}\{t^{p-1} E_{q,p}(\pm a t^q)\}(s) = \frac{s^{q-p}}{(s^q \mp a)},$$

(see [12]) for more details.

(c) Solution representation:

Consider the linear fractional stochastic differential equation with Poisson jumps represented in the following form:

$$\begin{aligned} d\Big[J_t^{1-q}(x(t) - x_0)\Big] &= \Big[Ax(t) + Bu(t) + \int_0^t \sigma(s) dw(s)\Big] dt + \int_{-\infty}^{+\infty} h(t, \eta) \lambda(dt, d\eta), \\ & s, t \in J := [0, T], \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where $0 < q < 1$, J_t^{1-q} is the $(1-q)-$ order Riemann–Liouville fractional integral operator $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, A, B are matrices of dimensions $n \times n$, $n \times m$ respectively and $\sigma : J \longrightarrow \mathbb{R}^{n \times n}$, $h : J \times J \longrightarrow \mathbb{R}^n$ are given functions.

Let $\{\overline{\lambda}(dt, d\eta),\ t, \eta \in J\}$ is a centered Poisson random measure with parameter $\pi(d\eta)dt$. Let $\int_{-\infty}^{+\infty} \pi(d\eta) < \infty$ and $\lambda(dt, d\eta) = \overline{\lambda}(dt, d\eta) - \pi(d\eta)dt$ is compensated Poisson random measure which is independent of $w(s)$.
Now applying the Riemann–Liouville fractional integral operator on both sides, we get

$$\begin{aligned} x(t) = x_0 &+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Ax(s) ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Bu(s) ds \\ &+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \int_0^s \sigma(\theta) dw(\theta) ds \\ &+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \int_{-\infty}^{+\infty} h(s, \eta) \lambda(ds, d\eta). \end{aligned}$$

Taking the Laplace Transformation on both sides, we obtain

$$\hat{x}(s) = \frac{1}{s} x_0 + \frac{1}{s^q} A \hat{x}(s) + \frac{1}{s^q} B \hat{u}(s) + \frac{1}{s^q} \hat{\sigma}(s) + \frac{1}{s^q} \hat{h}(s).$$

Taking inverse Laplace Transformation on both sides, we get

$$x(t) = E_{q,1}(At^q)x_0 + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q)\Big(Bu(s) + \int_0^s \sigma(\theta)dw(\theta)\Big)ds$$
$$+ \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) \int_{-\infty}^{+\infty} h(s,\eta)\lambda(ds, d\eta). \quad (2)$$

Let $(\Omega, \mathscr{F}, P)$ be the complete probability space with a probability measure P on Ω and $w(t) = (w_1(t), w_2(t), \ldots, w_n(t))^T$ be an $n-$dimensional Wiener process defined on the probability space. Let $\{\mathscr{F}_t | t \in J\}$ be the filtration generated by $\{w(s) : 0 \le s \le t\}$ defined on the probability space $(\Omega, \mathscr{F}, P)$. Let $L_2(\Omega, \mathscr{F}_T, \mathbb{R}^n)$ denotes the Hilbert space of all $\mathscr{F}_T$ measurable square integrable random variables with values in $\mathbb{R}^n$. Let $L_2^{\mathscr{F}}(J, \mathbb{R}^n)$ be the Hilbert space of all square integrable and $\mathscr{F}_t$-measurable processes with values in $\mathbb{R}^n$. Let $\mathscr{B}$ is the Banach space of all square integrable and $\mathscr{F}_t$-adapted process $x(t)$ with norm

$$\|x\|^2 = \sup_{t\in J}\{\mathbb{E}\|x(t)\|^2\},$$

where $\mathbb{E}(\cdot)$ denotes the mathematical expectation operator of stochastic process with respect to the given probability measure P. Let $\mathscr{L}(\mathbb{R}^n, \mathbb{R}^m)$ be the space of all linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Further, we assume that the set of admissible controls $\mathscr{U}_{ad} := L_2^{\mathscr{F}}(J, \mathbb{R}^m)$. Now let us introduce the following operators and sets. The linear bounded operator

$$\mathbb{L} \in \mathscr{L}(L_2^{\mathscr{F}}(J, \mathbb{R}^m), L_2(\Omega, \mathscr{F}_t, \mathbb{R}^n))$$

is defined by

$$\mathbb{L}u = \int_0^T (T-s)^{q-1} E_{q,q}(A(T-s)^q)Bu(s)ds$$

and its adjoint linear bounded operator

$$\mathbb{L}^* : L_2(\Omega, \mathscr{F}_T, \mathbb{R}^n) \longrightarrow L_2^{\mathscr{F}}(J, \mathbb{R}^m)$$

is defined by

$$(\mathbb{L}^* z)(t) = B^* E_{q,q}(A^*(T-t)^q)\mathbb{E}\{z|\mathscr{F}_t\},$$

and the set of all states attainable from x_0 in time $t > 0$ using admissible controls is defined by

$$\mathscr{R}_t(\mathscr{U}_{ad}) = \{x(t; x_0, u) \in L_2(\Omega, \mathscr{F}_T, \mathbb{R}^n) : u(\cdot) \in \mathscr{U}_{ad}\}.$$

The linear controllability operator $W_0^T \in \mathscr{L}(L_2(\Omega, \mathscr{F}_T, \mathbb{R}^n), L_2(\Omega, \mathscr{F}_T, \mathbb{R}^n))$ which is associated with the operator $\mathbb{L}$ is defined by

$$W_0^T = \mathbb{L}\mathbb{L}^*\{\cdot\} = \int_0^T (T-\tau)^{q-1}[E_{q,q}(A(T-\tau)^q)B][E_{q,q}(A(T-\tau)^q)B]^*\mathbb{E}\{(\cdot)|\mathscr{F}_t\}d\tau,$$

and the deterministic matrix $\Gamma_s^T \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^n)$ is

$$\Gamma_s^T = \int_s^T (T-\tau)^{q-1}[E_{q,q}(A(T-\tau)^q)B][E_{q,q}(A(T-\tau)^q)B]^*d\tau,\ s \in J.$$

Definition 1 The system (1) is said to be controllable on J if for every $x_0, x_1 \in \mathbb{R}^n$ there exists a stochastic control $u(t) \in \mathscr{U}_{ad}$ such that the solution of $x(t)$ of system (1) satisfies the conditions $x(0) = x_0$ and $x(T) = x_1$.

Definition 2 The system (1) is completely controllable on J if

$$\mathscr{R}_T(x_0) = L_2(\Omega, \mathscr{F}_T, \mathbb{R}^n),$$

that is, all points in $L_2(\Omega, \mathscr{F}_T, \mathbb{R}^n)$ can be exactly reached from an arbitrary initial condition $x_0 \in L_2(\Omega, \mathscr{F}_T, \mathbb{R}^n)$ at time T.

3 Controllability Results

In this section, we discuss the controllability criteria of linear and nonlinear stochastic system with Poisson jumps.

Lemma 1 ([10]) *If the linear system (1) is completely controllable, then for some* $\gamma > 0$,

$$\mathbb{E}\langle W_0^T z, z\rangle \geq \gamma \mathbb{E}\|z\|^2,$$

for all $z \in L_2(\Omega, \mathscr{F}_t, \mathbb{R}^n)$
and, consequently,

$$\mathbb{E}\|(W_0^T)^{-1}\|^2 \leq \frac{1}{\gamma} = l_2.$$

Lemma 2 ([10]) *Assume that the operator* W_0^T *is invertible. Then, for arbitrary* $x_1 \in L_2(\Omega, \mathscr{F}_T, \mathbb{R}^n)$, *the control*

$$u(t) = B^*E_{q,q}(A^*(T-t)^q)\mathbb{E}\Big\{(W_0^T)^{-1}\Big(x_1 - E_{q,1}(AT^q)x_0 - \int_0^T (T-s)^{q-1}E_{q,q}(A(T-s)^q) \times \Big[\int_0^s \sigma(\theta)dw(\theta)\Big]ds - \int_0^T (T-s)^{q-1}E_{q,q}(A(T-s)^q)\int_{-\infty}^{+\infty} h(s,\eta)\lambda(ds,d\eta)\Big)\Big|\mathscr{F}_t\Big\}$$

transfers the system (1) form $x_0 \in \mathbb{R}^n$ *to* $x_1 \in \mathbb{R}^n$ *at time* T.

Proof Substituting the control $u(t)$ into the solution $x(t)$ in (2) and substituting $t = T$, one can easily verify that the control $u(t)$ steers the linear system $x(t)$ from x_0 to x_1.

Let us consider the nonlinear fractional neutral stochastic dynamical systems with Poisson jumps represented in the following form

$$\begin{aligned} d\Big[J_t^{1-q}(x(t) - g(t, x(t)) - x_0 - g(0, x_0))\Big] &= \Big[A\Big(x(t) - g(t, x(t))\Big) + Bu(t) \\ &\quad + J_t^{1-q} f(t, x(t)) + \int_0^t \sigma(s, x(s))dw(s)\Big]dt \\ &\quad + \int_{-\infty}^{+\infty} h(t, x(t), \eta)\lambda(dt, d\eta), \ s, t \in J, \\ x(0) &= x_0, \end{aligned} \tag{3}$$

where $0 < q < 1$, J_t^{1-q} is the $(1-q)$–order Riemann–Liouville fractional integral operator, A, B are the matrices of dimensions $n \times n, n \times m$ respectively and $f : J \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\sigma : J \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times n}$ and $h : J \times \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^n$, are given functions. Then the solution (3) is given by (see [3, 5])

$$\begin{aligned} x(t) = E_{q,1}(At^q)[x_0 + g(0, x_0)] + g(t, x(t)) + \int_0^t E_{q,1}(A(t-s)^q) f(s, x(s))ds \\ + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q)\Big(Bu(s) + \int_0^s \sigma(\theta, x(\theta))dw(\theta)\Big)ds \\ + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) \int_{-\infty}^{+\infty} h(s, x(s), \eta)\lambda(ds, d\eta). \end{aligned}$$

Lemma 3 (Krasnoselskii's fixed point theorem) *Let E be a Banach space, let B be a bounded closed and convex subset of E and let Φ_1, Φ_2 be maps of B into E such that $\Phi_1 x, \Phi_2 y \in B$ for every pair $x, y \in B$. If Φ_1 is a contraction and Φ_2 is completely continuous, then the equation $\Phi_1 x + \Phi_2 x = x$ has a solution of B.*

In order to prove the main results we assume the following conditions hold:

(H1) The functions g, f, σ and h satisfy the following Lipschitz conditions and there exist some positive constants K, L, M and N such that

- (i) $\|g(t, x) - g(t, y)\|^2 \le K\|x - y\|^2$
- (ii) $\|f(t, x) - f(t, y)\|^2 \le L\|x - y\|^2$
- (iii) $\|\sigma(t, x) - \sigma(t, y)\|^2 \le M\|x - y\|^2$
- (iv) $\int_{-\infty}^{+\infty} \|h(t, x, \eta) - h(t, y, \eta)\|^2\lambda(d\eta) \le N\|x - y\|^2$

(H2) The functions g, f, σ and h are continuous and satisfy the following linear growth conditions. That is, there exist some positive constants $\overline{K}, \overline{L}, \overline{M}$ and $\overline{N}$ such that

(i) $\|g(t,x)\|^2 \le \overline{K}(1+\|x\|^2)$
(ii) $\|f(t,x)\|^2 \le \overline{L}(1+\|x\|^2)$
(iii) $\|\sigma(t,x)\|^2 \le \overline{M}(1+\|x\|^2)$
(iv) $\int_{-\infty}^{+\infty} \|h(t,x,\eta)\|^2 \lambda(d\eta) \le \overline{N}(1+\|x\|^2)$

(H3) The linear system (1) is completely controllable on J.

Now, define the nonlinear operator Φ from $\mathscr{B}$ to $\mathscr{B}$ as follows

$$\begin{aligned}(\Phi x)(t) &= E_{q,1}(At^q)[x_0+g(0,x_0)] + g(t,x(t)) + \int_0^t E_{q,1}(A(t-s)^q) f(s,x(s))ds \\ &\quad + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q)\Big(Bu(s) + \int_0^s \sigma(\theta,x(\theta))dw(\theta)\Big)ds \\ &\quad + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) \int_{-\infty}^{+\infty} h(s,x(s),\eta)\lambda(ds,d\eta)\end{aligned}$$

$$\begin{aligned}\mathbb{E}\|(\Phi x)(t)\|^2 \le \Delta &:= 36 l_1 l_2 \|x_1\|^2 + 6(1+6l_1 l_2)\Big(2S_1(\|x_0\|^2 + \|g(0,x_0)\|^2) \\ &\quad + \Big[\overline{K} + T^2 S_2 \overline{L} + \frac{T^{2q+1}}{q^2} S_3 M_\sigma \overline{M} + \frac{T^{2q}}{q^2} S_3 \overline{N}\Big](1+\mathbb{E}\|x\|^2)\Big)\end{aligned}$$

where

$$\begin{aligned}u_x(t) &= B^* E_{q,q}(A^*(T-t)^q)\mathbb{E}\Big\{(W_0^T)^{-1}\big[x_1 - E_{q,1}(AT^q)[x_0+g(0,x_0)]\big] - g(T,x(T)) \\ &\quad - \int_0^T E_{q,1}(A(T-s)^q) f(s,x(s))ds - \int_0^T (T-s)^{q-1} E_{q,q}(A(T-s)^q) \\ &\quad \times \Big(\int_0^s \sigma(\theta,x(\theta))dw(\theta)\Big)ds - \int_0^T (T-s)^{q-1} E_{q,q}(A(T-s)^q) \\ &\quad \times \int_{-\infty}^{+\infty} h(s,x(s),\eta)\lambda(ds,d\eta)\Big]\Big|\mathscr{F}_t\Big\}.\end{aligned}$$

Applying Lemma 3 we need to construct two mapping Φ_1 and Φ_2 such that

$$(\Phi x)(t) = (\Phi_1 x)(t) + (\Phi_2 x)(t)$$

where

$$\begin{aligned}(\Phi_1 x)(t) &= \int_0^t E_{q,1}(A(t-s)^q) f(s,x(s))ds + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q)\Big(Bu(s) \\ &\quad + \int_0^s \sigma(\theta,x(\theta))dw(\theta)\Big)ds + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) \\ &\quad \times \int_{-\infty}^{+\infty} h(s,x(s),\eta)\lambda(ds,d\eta),\end{aligned}$$

and

$$(\Phi_2 x)(t) = E_{q,1}(At^q)[x_0+g(0,x_0)] + g(t,x(t)).$$

For convenience, let us introduce the following notations:

$$l_1 = \max\{\|\Gamma_s^T\|^2\},\ S_1 = \|E_{q,1}(At^q)\|^2,\ S_2 = \|E_{q,1}(A(T-s)^q)\|^2,\ S_3 = \|E_{q,q}(A(T-s)^q)\|^2.$$

Theorem 1 *Assume that the conditions **(H1)–(H3)** are hold and if $\Delta < 1$ are satisfied, then the nonlinear system (3) is completely controllable on J.*

Proof In order to make more clear presentations, we divide the proof into the following three several steps.

Step I: For $t \in J$ and any $x, y \in \mathscr{B}$, we have

$$\begin{aligned}\mathbb{E}\|(\Phi_1 x)(t)\|^2 \le\ & 4\mathbb{E}\left\|\int_0^t E_{q,1}(A(t-s)^q)f(s,x(s))ds\right\|^2 + 4\mathbb{E}\left\|\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)\right. \\ & \left.\times Bu_x(s)ds\right\|^2 + 4\mathbb{E}\left\|\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)\int_0^s \sigma(\theta, x(\theta))dw(\theta)ds\right\|^2 \\ & + 4\mathbb{E}\left\|\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)\int_{-\infty}^{+\infty} h(s,x(s),\eta)\lambda(ds,d\eta)\right\|^2.\end{aligned}$$

Now, we have the following estimate

$$\begin{aligned}\mathbb{E}\left\|\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)Bu_x(s)ds\right\|^2 \le\ & 6l_1l_2\Big[\|x_1\|^2 + 2S_1(\|x_0\|^2 + \|g(0,x_0)\|^2) \\ & + \Big(\overline{K} + T^2S_2\overline{L} + \frac{T^{2q+1}}{q^2}S_3M_\sigma\overline{M} + \frac{T^{2q}}{q^2}S_3\overline{N}\Big) \\ & \times (1+\mathbb{E}\|x\|^2)\Big].\end{aligned}$$

Thus

$$\begin{aligned}\mathbb{E}\|(\Phi_1 x)(t)\|^2 \le\ & 4\Big[T^2S_2\mathbb{E}\|f(t,x(t))\|^2 + 6l_1l_2\Big(\|x_1\|^2 + 2S_1(\|x_0\|^2 + \|g(0,x_0)\|^2) \\ & + \Big(\overline{K} + T^2S_2\overline{L} + \frac{T^{2q+1}}{q^2}S_3M_\sigma\overline{M} + \frac{T^{2q}}{q^2}S_3\overline{N}\Big)(1+\mathbb{E}\|x\|^2)\Big) \\ & + \frac{T^{2q+1}}{q^2}S_3M_\sigma\mathbb{E}\|\sigma(t,x(t))\|^2 + \frac{T^{2q}}{q^2}S_3\int_{-\infty}^{+\infty}\mathbb{E}\|h(t,x(t),\eta)\|^2\lambda(d\eta)\Big] \\ \le\ & 4\Big[6l_1l_2[\|x_1\|^2 + 2S_1(\|x_0\|^2 + \|g(0,x_0)\|^2)] + \Big(6l_1l_2\overline{K} + (1+6l_1l_2) \\ & \times\Big(T^2S_2\overline{L} + \frac{T^{2q+1}}{q^2}S_3M_\sigma\overline{M} + \frac{T^{2q}}{q^2}S_3\overline{N}\Big)\Big)(1+\mathbb{E}\|x\|^2)\Big]\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\|(\Phi_2 y)(t)\|^2 &\le 2\|E_{q,1}(At^q)[x_0 + g(0,x_0)]\|^2 + 2\mathbb{E}\|g(t,y(t))\|^2 \\ &\le 4S_1[\|x_0\|^2 + \|g(0,x_0)\|^2] + 2\overline{K}(1+\mathbb{E}\|y\|^2).\end{aligned}$$

By the condition $\Delta < 1$, we can find a $r > 0$ such that

$$x, y \in \mathscr{B}_r = \{x \in \mathscr{B} : \mathbb{E}\|x\|^2 \leq r\},\ \mathbb{E}\|\Phi_1 x + \Phi_2 y\|^2 \leq r$$

that is $\Phi_1 x + \Phi_2 y \in \mathscr{B}_r$.

Step II: Φ_1 is a contraction mapping on $\mathscr{B}_r$. For any $x, y \in \mathscr{B}_r$ and $t \in J$, we have

$$\begin{aligned}
\mathbb{E}\|(\Phi_1 x)(t) - (\Phi_1 y)(t)\|^2 \leq\ & 4\mathbb{E}\left\| \int_0^t E_{q,1}(A(t-s)^q)[f(s, x(s)) - f(s, y(s))]ds \right\|^2 \\
& + 4\,\mathbb{E}\left\| \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) B[u_x(s) - u_y(s)]ds \right\|^2 \\
& + 4\,\mathbb{E}\Big\| \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) \\
& \times \left(\int_0^s [\sigma(\theta, x(\theta)) - \sigma(\theta, y(\theta))]dw(\theta) \right) ds \Big\|^2 \\
& + 4\,\mathbb{E}\Big\| \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) \\
& \times \left(\int_{-\infty}^{+\infty} [h(s, x(s), \eta) - h(s, y(s), \eta)]\lambda(ds, d\eta) \right) \Big\|^2 \\
\leq\ & 4\left[4l_1 l_2 K + (1 + 4l_1 l_2)\left(T^2 S_2 L + \frac{T^{2q+1}}{q^2} S_3 M_\sigma M + \frac{T^{2q}}{q^2} S_3 N \right) \right] \\
& \times \mathbb{E}\|x(t) - y(t)\|^2 =: \Upsilon \mathbb{E}\|x(t) - y(t)\|^2.
\end{aligned}$$

From the condition $\Delta < 1$, we obtain $\Upsilon < 1$, which implies that Φ_1 is a contraction mapping.

Step III: Φ_2 is a completely continuous operator.

Due to continuity of A and continuity of g, the operator is Φ_2 is continuous. Next, we will show that $\{\Phi_2 x, x \in \mathscr{B}_r\}$ is relatively compact. It suffices to show that the family of function $\{\Phi_2 x, x \in \mathscr{B}_r\}$ is uniformly bounded and equicontinuous for any $t \in J$ and $\{(\Phi_2 x)(t), x \in \mathscr{B}_r\}$ is relatively compact. For any $x \in \mathscr{B}_r$, we have $\mathbb{E}\|\Phi_2 x\|^2 \leq r$ which implies that $\{\Phi_2 x, x \in \mathscr{B}_r\}$ is uniformly bounded. In the following, we will show that $\{\Phi_2 x, x \in \mathscr{B}_r\}$ is a family of equicontinuous functions. For any $x \in \mathscr{B}_r$ and $0 \leq t_1 < t_2 \leq T$, we have

$$\begin{aligned}
\mathbb{E}\|(\Phi_2 x)(t_2) - (\Phi_2 x)(t_1)\|^2 \leq\ & 4\|E_{q,1}(At_2^q) - E_{q,1}(At_1^q)\|^2(\|x_0\|^2 + \|g(0, x_0)\|^2) \\
& + 2\mathbb{E}\|g(t_2, x(t_2)) - g(t_1, x(t_1))\|^2.
\end{aligned}$$

The right side of the above equation is independently of $x \in \mathscr{B}_r$ as $(t_2 - t_1) \longrightarrow 0$ which means that $\{\Phi_2 x, x \in \mathscr{B}_r\}$ is equicontinuous. Therefore $\{\Phi_2 x, x \in \mathscr{B}_r\}$ is relatively compact by Arzela–Ascoli theorem. The continuity of Φ_2 and relative compactness of $\{\Phi_2 x, x \in \mathscr{B}_r\}$ imply that Φ_2 is a completely continuous operator. By using Krasnoseskii's fixed point theorem we obtain that $\Phi_1 + \Phi_2$ has a fixed point on $\mathscr{B}_r$. Therefore the system (3) has atleast one fixed point on J.

4 Conclusion

This paper deal with the controllability of fractional neutral stochastic dynamical systems with Poisson jumps in the finite-dimensional space. Sufficient conditions for controllability results have been obtained by using Krasnoseskii's fixed point theorem . The controllability Grammian matrix is defined by Mittag-Leffler matrix function.

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