Approximate Controllability of Semilinear Stochastic System with State Delay

Anurag Shukla, N. Sukavanam and D.N. Pandey

Abstract The objective of this paper is to present some sufficient conditions for approximate controllability of semilinear stochastic system with state delay. Sufficient conditions are obtained by separating the given semilinear system into two systems namely a semilinear deterministic system and a linear stochastic system. To prove our results, the Schauder fixed-point theorem is applied. At the end, an example is given to show the effectiveness of the result.

Keywords Approximate controllability · State delay · Stochastic system · Schauder fixed point theorem

1 Introduction

Controllability concepts play a vital role in deterministic control theory. It is well known that controllability of deterministic equation is widely used in many fields of science and technology. But in many practical problems such as fluctuating stock prices or physical system subject to thermal fluctuations, population dynamics, etc., some randomness appear, so the system should be modelled stochastic form.

In setting of deterministic systems: Kalman [1] introduced the concept of controllability for finite-dimensional deterministic linear control systems. Then Barnett [2] and Curtain [3] introduced the concepts of deterministic control theory in finite and infinite-dimensional spaces. Naito [4] established sufficient conditions for approximate controllability of deterministic semilinear control system dominated by the

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linear part using Schauder's fixed-point theorem. In [5, 6], Wang extended the results of [4] and established sufficient conditions for delayed deterministic semilinear systems using same Schauder's fixed-point theorem. In [7] author provided more applications of Schauder's fixed-point theorem in nonlinear controllability problems.

In setting of stochastic systems: In [8, 9] Mahmudov established some results for controllability of linear stochastic systems in finite-dimensional and infinite-dimensional spaces, respectively. Sukavanam et al. in [10] obtained some sufficient conditions for s-controllability of an abstract first-order semilinear control system using Schauder's fixed-point theorem. Recently, Anurag et al. [11] obtained some sufficient conditions for approximate controllability of retarded semilinear stochastic system with nonlocal conditions using Banach fixed-point theorem.

The present paper is generalized form of the system taken in [10]. In this paper system is taken with finite delay in state which is not discussed up to now in the literature in best of my knowledge. The technique is adopted similar to discussed in [10, 12] with suitable modifications.

Let X and U be the Hilbert spaces and $Z = L_2[0, b; X]$, $Z_h = L_2[-h, b; X]$, $0 < h < b$, and $Y = L_2[0, b; U]$ be function spaces. $\mathbf{R}^k$ denotes k-dimensional real Euclidean space. Let (Ω, ζ, P) be the probability space with a probability measure P on Ω and a filtration $\{\zeta_t | t \in [0, b]\}$ generated by a Wiener Process $\{\omega(s) : 0 \leq s \leq t\}$.

We consider the semilinear stochastic control system of the form:

$$\begin{aligned} dx(t) &= [Ax(t) + Bu(t) + f(t, x_t)]dt + d\omega(t), \ t > 0 \\ x(t) &= \xi(t), \quad t \in [-h, 0] \end{aligned} \tag{1}$$

where the state function $x \in Z$; $A : D(A) \subseteq X \to X$ is a closed linear operator which generates a strongly continuous semigroup $S(t)$; $B : Y \to Z$ is a bounded linear operator; function $f : [0, b] \times X \to X$ is a nonlinear operator such that, f is measurable with respect to t, for all $x \in Z$ and continuous with respect to x for almost all $t \in [0, b]$; $x_t \in L_2([-h, 0], X) = \mathbf{C}$ (let)-valued stochastic processes and defined as $x_t(s) = \{x(t + s)| -h \leq s \leq 0|\}$; Control $u(t)$ takes values in U for each $t \in [0, b]$.

By splitting the system (1), we get the following pair of coupled systems

$$\begin{aligned} \frac{dy(t)}{dt} &= [Ay(t) + Bv(t) + f(t, (y + z)_t]; \quad 0 \leq t \leq b \\ y(t) &= \psi(t), \quad t \in [-h, 0] \end{aligned} \tag{2}$$

and

$$\begin{aligned} dz(t) &= [Az(t) + Bw(t)]dt + d\omega(t); \quad 0 \leq t \leq b \\ z(t) &= \xi(t) - \psi(t), \quad t \in [-h, 0] \end{aligned} \tag{3}$$

The system represented by (3) is linear stochastic system and for each realization $z(t)$ of system (3), the system given by (2) is a deterministic system. Thus the solution $y(t)$

of the semilinear system (2) depends on the solution $z(t)$ of linear stochastic system (3). The functions v and w are Y-valued control function, such that $u = v + w$.

It can be easily seen that, the solution $x(t)$ of the semilinear stochastic system (1) is given by $y(t) + z(t)$ where $y(t)$ and $z(t)$ are the solutions of the systems (2) and (3), respectively.

2 Preliminaries

In this section, some definitions are discussed which will be used in proof of main results.

The mild solution of the systems (1) can be written as

$$x(t) = \begin{cases} S(t)\xi(0) + \int_0^t S(t-s)\{Bu(s) + f(s, x_s)\}ds + \int_0^t S(t-s)d\omega(s), \ t > 0 \\ \xi(t) \qquad -h \leq t \leq 0 \end{cases} \tag{4}$$

the mild solution of the semilinear system (2) can be written as

$$y(t) = \begin{cases} S(t)\psi(0) + \int_0^t S(t-s)\{Bv(s) + f(s, (y+z)_s\}ds, \ t > 0 \\ \psi(t) \qquad -h \leq t \leq 0 \end{cases} \tag{5}$$

and the mild solution of the linear stochastic system (3) can be written as

$$z(t) = \begin{cases} S(t)(\xi(0) - \psi(0)) + \int_0^t S(t-s)Bw(s)ds + \int_0^t S(t-s)d\omega(s), \ t > 0 \\ \xi(t) - \psi(t) \qquad -h \leq t \leq 0 \end{cases} \tag{6}$$

Consider the linear system corresponding to the system (2) given by

$$\begin{aligned} \frac{dp(t)}{dt} &= Ap(t) + Br(t), \quad t > 0 \\ p(t) &= \psi(t) \quad t \in [-h, 0] \end{aligned} \tag{7}$$

The mild solution of the above linear system is expressed as

$$p(t) = \begin{cases} S(t)\psi(0) + \int_0^t S(t-s)Br(s)ds \ t > 0 \\ \psi(t) \qquad -h \leq t \leq 0 \end{cases} \tag{8}$$

Definition 1 The set given by $K_T(f) = \{x(T) \in X : x \in Z_h\}$ where x is a mild solution of (1) corresponding to control $u \in Y$ is called Reachable set of the system (1).

Definition 2 The system (1) is said to be approximately controllable if $K_T(f)$ is dense in X, means $\overline{K_T(f)} = X$.

3 Basic Assumptions

In this section, some basic conditions and lemmas are assumed and discussed for obtaining the main results. Throughout this paper $D(A)$, $R(A)$, and $N_0(A)$ denote the domain, range, and null space of operator A, respectively.
The following conditions are assumed:
(H_1) For every $p \in Z$ there exists a $q \in \overline{R(B)}$ such that $Lp = Lq$ where the operator $L : Z \to X$ is defined as

$$Lx = \int_0^b S(b-s)x(s)ds$$

(H_2) The semigroup $\{S(t), t \geq 0\}$ generated by A is compact on X and there is a constant $M \geq 0$ such that $||S(t)|| \leq M$.
(H_3) $f(t,x)$ satisfies Lipschitz continuity on Z. i.e

$$||f(t,x_1) - f(t,x_2)|| \leq l_p||x_1 - x_2||, \quad l_p > 0$$

(H_4) $f(t,x)$ satisfies linear growth condition, that is,

$$||f(t,x)|| \leq a_1 + b_1||x||,$$

where a_1 and b_1 are constants.
(H_5) $Mbb_1(1+c) < 1$
where the constants b and b_1 appear in the above conditions. The constant c is defined in Lemma 1.
Let $G : N_0^{\perp}(L) \to \overline{R(B)}$ be an operator defined as follows:

$$Ga = a_0$$

where $a \in N_0^{\perp}(L)$ and a_0 is the unique minimum norm element in the set $\{a + N_0(L)\} \bigcap \overline{R(B)}\}$ satisfying the following condition

$$||Ga|| = ||a_0|| = \min\left[||e|| : e \in \{a + N_0(L)\} \bigcap \overline{R(B)}\}\right] \tag{9}$$

The operator G is well defined, linear, and continuous (see [4], Lemma 1). From continuity of G, it follows that $||Ga|| \leq c||a||_Z$, for some constant $c \geq 0$.

Since $Z = N_0(L) + \overline{R(B)}$ as is evident from condition $(H1)$, any element $z \in Z$ can be expressed as

$$z = n + q : n \in N_0(L), q \in \overline{R(B)}$$

Lemma 1 *In [12], for $z \in Z$ and $n \in N_0(L)$, the following inequality holds*

$$||n||_Z \leq (1 + c)||z||_Z \tag{10}$$

where c is such that $||G|| \leq c$.

Let us introduce some operators in the following way:
$K : Z \to Z$ defined by

$$(Kz)(t) = \int_0^t S(t - s)z(s)ds$$

Now, let M_0 be the subspace of Z_h (see [11]) such that

$$M_0 = \begin{cases} m \in Z_h : m(t) = (Kn)(t), \; n \in N_0(L) & 0 \leq t \leq b \\ m(t) = 0, & -h \leq t \leq 0 \end{cases}$$

It can be noted that $m(b) = 0$, for all $m \in M_0$.

For each solution $p(t)$ of the system (7) with control r and for each realization $z(t)$ of the system (3), define the random operator $f_p : \overline{M_0} \to M_0$ as

$$f_p = \begin{cases} Kn, & 0 < t < b \\ 0, & -h \leq t \leq 0 \end{cases} \tag{11}$$

where n is given by the unique decomposition

$$F(p + z + m) = n + q : n \in N_0(L), \; q \in \overline{R(B)}, \tag{12}$$

where $F: L_2([0, b], \mathbf{C}) \to X$ given by

$$(Fx)(t) = f(t, x_t(.)); \qquad 0 \leq t \leq b$$

It is easy to see that F satisfies Lipschitz continuity (H_3) and linear growth conditions (H_4).

4 Main Results

In this section, approximate controllability of systems (2), (3) is proved. System (1) is splitted in systems (2), (3), so if systems (2), (3) are approximately controllable then system (1) is also approximately controllable.

The linear system (7) corresponding to system (2) is approximately controllable under the condition (H_1) (see [5]).
For approximate controllability of (3)

$$\begin{aligned} dz(t) &= [Az(t) + Bw(t)]dt + d\omega(t); \quad 0 \leq t \leq b \\ z(t) &= \xi(t) - \psi(t), \quad t \in [-h, 0] \end{aligned} \tag{13}$$

The mild solution of above system is

$$z(t) = \begin{cases} S(t)(\xi(0) - \psi(0)) + \int_0^t S(t-s)Bw(s)ds + \int_0^t S(t-s)d\omega(s), \ t > 0 \\ \xi(t) - \psi(t) \qquad\qquad -h \leq t \leq 0 \end{cases} \tag{14}$$

Define the operator $L_0^b : L_2[0, b; U] \to L_2[\Omega, \zeta_t, X]$, the controllability operator $\Pi_s^b : L_2[\Omega, \zeta_t, X] \to L_2[\Omega, \zeta_t, X]$ associated with (14), and the controllability operator $\Gamma_s^b : X \to X$ associated with the corresponding deterministic system of (14) as

$$L_0^b = \int_0^b S(b-s)Bw(s)ds \tag{15}$$

$$\Pi_s^b\{.\} = \int_s^b S(b-t)BB^*S^*(b-t)\mathbf{E}\{.|\zeta_t\}dt \tag{16}$$

$$\Gamma_s^b = \int_s^b S(b-t)BB^*S^*(b-t)dt \tag{17}$$

It is easy to see that the operators $L_0^b, \Pi_s^b, \Gamma_s^b$ are linear-bounded operators, and the adjoint $(L_0^b)^* : L_2[\Omega, \zeta_t, X] \to L_2[0, b; U]$ of L_0^b is defined by

$$(L_0^b)^* = B^*S^*(b-t)\mathbf{E}\{z|\zeta_t\}\Pi_0^b = L_0^b(L_0^b)^*.$$

Before studying the approximate controllability of system (3), let us first investigate the relation between Π_s^b and Γ_s^b; $s \leq r < b$ and resolvent operator $R(\lambda, \Pi_s^b) = (\lambda I + \Pi_s^b)^{-1}$ and $R(\lambda, \Gamma_r^b) = (\lambda I + \Gamma_r^b)^{-1}$, $s \leq r < b$ for $\lambda > 0$, respectively.

Lemma 2 *For every $z \in L_2[\Omega, \zeta_t, X]$ there exists $\varphi(.) \in L_2^\zeta(0, b; \boldsymbol{L}(\boldsymbol{R}^k, X))$ such that*

1. $\boldsymbol{E}\{z|\zeta_t\} = \boldsymbol{E}\{z\} + \int_0^t \varphi(s)d\omega(s)$,
2. $\Pi_s^b z = \Gamma_s^b \boldsymbol{E}z + \int_s^b \Gamma_r^b \varphi(r)d\omega(r)$,
3. $R(\lambda, \Pi_s^b)z = R(\lambda, \Gamma_s^b)\boldsymbol{E}\{z|\zeta_t\} + \int_s^b \Gamma_r^b \varphi(r)d\omega(r)$.

Proof The proof is straightforward adaption of the proof of [10, Lemma 2.3]. □

Theorem 1 *The control system (3) is approximately controllable on* $[0, b]$ *if and only if one of the following conditions holds.*

1. $\Pi_0^b > 0$.
2. $\lambda R(\lambda, \Pi_0^b)$ *converges to the zero operator as* $\lambda \to 0^+$ *in the strong operator topology.*
3. $\lambda R(\lambda, \Pi_0^b)$ *converges to the zero operator as* $\lambda \to 0^+$ *in the weak operator topology.*

Proof The proof is straightforward adaption of the proof of [9, Theorem 2]. □

Lemma 3 *Under the conditions* (H_2), (H_4), *and* (H_5), *the operator* f_p *has a fixed point* $m_0 \in M_0$ *for each realization* $z(t)$ *of the system (3).*

Proof From the compactness of $S(t)$ the integral operator K is compact and hence f_p is compact for each p, (see [1]). Now let $||m|| \leq \tilde{r}$. Then from the condition (H_4) and from the inequality (10) and (12), we have

$$\begin{aligned} ||f_p(m)||^2 &\leq \left|\left| \int_0^t S(t-s)n(s)ds \right|\right|^2 \\ &\leq \int_0^b \left|\left| \int_0^t S(t-s)n(s)ds \right|\right|^2 dt \\ &\leq M^2 b^2 (1+c)^2 ||F(p+z+m||_Z^2 \\ &\leq M^2 b^2 (1+c)^2 \{a_1 + b_1 ||p+z+m||_Z\}^2 \\ &\leq M^2 b^2 (1+c)^2 \{a_1 + b_1 ||p+z|| + b_1 \tilde{r}\}^2 \end{aligned} \tag{18}$$

Using Schauder's fixed-point theorem, it is clear from the compactness of f_p and (18) that f_p has a fixed point in M_0 in a ball of radius $\tilde{r} > 0$, if

$$\tilde{r} > \frac{Mb(1+c)(a_1 + b_1||p+z||)}{1 - Mb(1+c)b_1}$$

Thus $f_p(m_0) = m_0$

The approximate controllability of the semilinear system (2) is proved in following manner using the above lemma.

Lemma 4 *For each realization* $z(t)$ *of the system (3), the semilinear control system (2) is approximately controllable under the conditions* (H_1)–(H_4).

Proof From the Eq. (12), we have

$$F(p+z+m) = n+q$$

Operating K on both the sides at $m = m_0$ (fixed point of f_p) and using (11), we get

$$\begin{aligned} KF(p+z+m_0) &= Kn + Kq \\ &= m_0 + Kq \end{aligned}$$

Adding p on both sides, we get

$$p + KF(p + z + m_0) = p + m_0 + Kq$$

Let $p + m_0 = y^*$, then the above equation is equivalent to

$$p + KF(y^* + z) = y^* + Kq$$

Since, from the Eq. (8)

$$p = S(t)\psi(0) + KBr$$

we have

$$S(t)\psi(0) + KBr + KF(y^* + z) = y^* + Kq$$
$$S(t)\psi(0) + K(Br - q) + KF(y^* + z) = y^*$$

Thus, it follows that $y^*(t)$ is a solution of the semilinear system

$$\begin{aligned} \frac{dy^*(t)}{dt} &= Ay^*(t) + f(t, (y^* + z)_t) + Br(t) - q(t), \\ y^*(0) &= \psi(0) \end{aligned} \tag{19}$$

with control $(Br - q)$.
Moreover, since $y^*(t) = p(t) + m_0(t)$, it follows that

$$y^*(b) = p(b) + m_0(b),$$

as $m_0(b) = 0$ it follows that

$$y^*(b) = p(b) \tag{20}$$

From the Eqs. (19) and (20), it is clear that the reachable set of (19) is a superset of the reachable set of the system (7), which is dense in X.

Further $q \in \overline{R(B)}$ implies that for any given $\varepsilon_1 > 0$, there exists $v_1 \in Y$ such that $||q - Bv_1|| \leq \varepsilon_1$.

Now consider the equation

$$\begin{aligned} \frac{dy(t)}{dt} &= Ay(t) + f(t, (y + z)_t) + B(r(t) - v_1(t)), \\ y(0) &= \psi(0) \end{aligned} \tag{21}$$

Let $y(t)$ be the solution of the system (21), corresponding to control $v = r - v_1$. Then $||y^*(b) - y(b)||$ can be made arbitrary small by choosing a suitable v_1, which implies that the reachable set of the system (21) is dense in the reachable set of the system (19), which in turn is dense in X. This proves that the system (2) is approximately controllable. □

5 Example

Consider the stochastic control system with delay governed by the semilinear heat equation

$$\partial y(t,x) = \left[\frac{\partial^2 y(t,x)}{\partial x^2} + Bu(t,x) + f(t, y(t+v,x))\right]\partial t + \partial\omega(t)$$
$$\text{for} \quad 0 < t < \tau;\ v \in [-h, 0]; \quad 0 < x < \pi$$
$$\text{with conditions} \quad y(t,0) = y(t,\pi) = 0, \quad 0 \le t \le \tau$$
$$y(t,x) = \xi(t,x), \qquad -h \le t \le 0,\ 0 \le x \le \pi \tag{22}$$

The system (22) can be written in the abstract form (1), by setting $X = L_2(0,\pi)$ and $A = \frac{d^2}{dx^2}$, with domain consisting of all $y \in X$ with $\left(\frac{d^2 y}{dx^2}\right) \in X$ and $y(0) = 0 = y(\pi)$. Take $\phi_(x) = (2/\pi)^{1/2} sin(nx),\ 0 \le x \le \pi,\ n = 1, 2, 3, ...,$ then $\{\phi_n(x)\}$ is an orthonormal basis for X and ϕ_n ia an eigenfunction corresponding to the eigenvalue $\lambda_n = -n^2$ of the operator $A, n = 1, 2, 3, ...$ Then the C_0-semigroup $T(t)$ generated by A has $e^{\lambda_n t}$ as the eigenvalues and ϕ_n as their corresponding eigenfunctions.

Define an infinite-dimensional space U by

$$U = \left\{u : u = \sum_{n=2}^{\infty} u_n \phi_n \text{ with } \sum_{n=2}^{\infty} u_n^2 < \infty\right\}$$

The norm defined by

$$||u||_U = \left(\sum_{n=2}^{\infty} u_n^2\right)^{1/2}$$

$\xi(t,x)$ is known function.

Let B be a continuous linear operator from U to X defined as

$$Bu = 2u_2\phi_1 + \sum_{n=2}^{\infty} u_n\phi_n, \quad u = \sum_{n=2}^{\infty} u_n\phi_n \in U$$

The nonlinear operator f is assumed to satisfy conditions (H_3) and (H_4).

The approximate controllability of the corresponding semilinear deterministic heat equation of (22) was considered by Naito [4] and proved under the conditions

(H_1)–(H_4). Here approximate controllability of the stochastic semilinear heat control system (22) is considered.

The system (22) can be associated with two control systems under the initial and boundary conditions, as given below

$$\frac{\partial y(t,x)}{\partial t} = \frac{\partial^2 y(t,x)}{\partial x^2} + y(t-h,x) \\ +Bv(t,x) + f(t, y(t-h,x) + z(t-h,x)) \; t \in [0,b] \; x \in [0,\pi] \quad (23)$$

$$y(t,x) = \xi(t,x), \qquad -h \le t \le 0, \; 0 \le x \le \pi \\ \partial z(t,x) = \left[\frac{\partial^2 z(t,x)}{\partial x^2} + z(t-h,x) + Bw(t)\right]\partial t + \partial \omega(t) \quad (24)$$

The system (24) is a linear stochastic system and for each realization $z(t)$ of the system (24), the system (23) is a deterministic system.

From Lemma 4 and using the conditions (H_1)–(H_4), it is clear that for each realization $z(t)$ of the system (24), the system (23) is approximately controllable. The linear stochastic system (24) is approximately controllable from Lemma 3 corresponding to (23) and linear system corresponding to system (23) is approximately controllable from [4].

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References

1. Kalman, R.E.: Controllability of linear systems. Contribut. Differ. Equ. **1**, 190–213 (1963)
2. Barnett, S.: Introduction to Mathematical Control Theory. Clarendon Press, Oxford (1975)
3. Curtain, R.F., Zwart, H.J.: An Introduction to Infinite Dimensional Linear Systems Theory. Springer, New York (1995)
4. Naito, K.: Controllability of semilinear control systems dominated by the linear part. SIAM J. Control Optim. **25**, 715–722 (1987)
5. Wang, L.W.: Approximate controllability of integrodifferential equations with multiple delays. J. Optim. Theory Appl. **143**, 185–206 (2009)
6. Wang, L.W.: Approximate controllability of delayed semilinear control of control system. J. Appl. Math. Stoch. Anal. **1**, 67–76 (2005)
7. Klamka, J.: Schuder fixed point theorem in nonlinear controllability problems. Control Cybernet. **29**(1), 153–165 (2000)
8. Mahmudov, N.I., Denker, A.: On controllability of linear stochastic systems. Int. J. Control **73**(2), 144–151 (2000)
9. Mahmudov, N.I.: Controllability of linear stochastic systems in hilbert spaces. J. Math. Anal. Appl. **259**(1), 64–82 (2001)
10. Sukavanam, N., Kumar, M.: S-controllability of an abstract first order semilinear control system. Numer. Funct. Anal. Optim. **31**(7–9), 1023–1034 (2010)

11. Shukla, A., Arora, U., Sukavanam, N.: Approximate controllability of retarded semilinear stochastic system with non local conditions. J. Appl. Math. Comput. doi:10.1007/s12190-014-0851-9
12. Sukavanam, N.: Approximate controllability of semilinear control systems with growing nonlinearit. In: Mathematical Theory of Control Proceedings of International Conference, pp. 353–357. Marcel Dekker, New York (1993)