Numerical Solution of Highly Oscillatory Nonlinear Integrals Using Quasi-Monte Carlo Methods

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Abstract Highly oscillating integrals occur in many engineering applications. This paper discusses the quasi-Monte Carlo methods for calculation of the highly oscillating integrals using a low discrepancy sequence. We evaluated the highly oscillating integrals using a low discrepancy sequence known as Vander Corput sequence. The theoretical error bounds are calculated and are compared with analytical results. The reliability of the quasi-Monte Carlo methods is compared with He's homotopy perturbation method.

Keywords Highly oscillatory integrals · Quasi Monte-Carlo methods · Low discrepancy sequence

1 Introduction

Highly oscillatory functions arise in wide range of applications in science and engineering. The integration of high oscillatory functions is a challenging task from several years. Most of the techniques or analysis for integration of highly oscillatory functions are problem-oriented or technique-oriented. For example, integration of these functions occurs in solving the problems modeling of wave phenomena like diffraction of light, scattering of acoustic waves [8], scattering of electromagnetic waves [11], etc. The boundary element method also requires the evaluation of highly oscillatory integrals [3]. Explicit solution exists only for a few cases. So one needs to go for numerical methods.

The main goal of the present paper is on the analysis and computation of the integrals of the form

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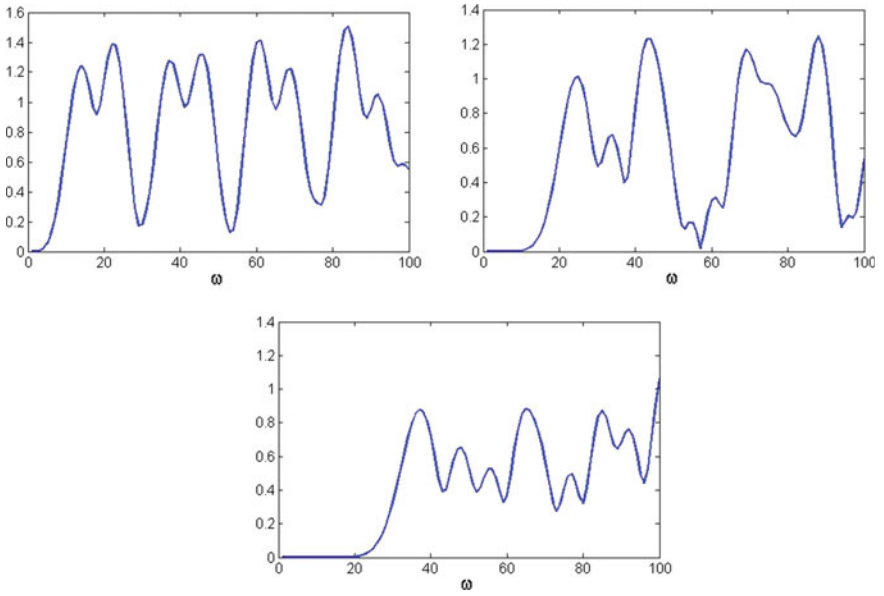

Fig. 1 Error in Gaussian quadrature with $f(x) = \cos(x)$, $g(x) = x^2$ for the quadrature points 5, 10, and 16

$$I[f, \Omega] = \int_{\Omega} f(x) e^{i\omega g(x)} dV, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is bounded and open domain with piecewise smooth boundary. The functions $f, g \in C^{\infty}$ are smooth. For large values of $|\omega|$, the integral (1) oscillates rapidly as a function of ω.

A classical technique to compute (1) is *Gaussian quadrature* [6] method. If the integrand oscillates rapidly (for large values of ω), the Gaussian quadrature methods are not appropriate. For example, let us consider the following integral

$$\int_0^1 \cos(x) e^{i\omega x^2} dx. \tag{2}$$

The integral is evaluated using Gauss–Legendre quadrature rule at different quadrature points. One may observe that the Gauss–Legendre quadrature rule gives good results for small values of ω. As ω becomes large in comparison with quadrature points, high oscillations set in and the error becomes $\mathcal{O}(1)$. The absolute error with respect to ω is plotted in the Fig. 1.

There exists few techniques to calculate the highly oscillating integrals. Among all *Asymptotic expansion methods*, *Filon-type methods*, and *Levin-type methods* are most popular. The Asymptotic method in a straightforward manner is nothing but

repeatedly applying the integration by parts. But the accuracy of the asymptotic expansion is limited due to the divergence of the series.

An even better approach is Filon [2]-type method. In this method instead of approximating whole integral, we approximate $f(x)$ of Eq. (1) at a set of quadrature nodes $c_1, \ldots, c_\nu$, by a polynomial $\tilde{f}$. Evaluation of the moments makes Filon-type methods difficult to certain type of applications.

In Levin [9]-type method we collocate the integrand at specific points. The Levin-type method is advantageous over Filon-type method and it is due to the fact that Levin-type method works easily on all types of domains and nonlinear oscillators. Most of the methods used in the current research are either Filon- or Levin-type methods or a modified form of these methods [1, 14]. He's homotopy Perturbation Method (HPM) is used in [10] for the numerical solution of the highly oscillating integrals.

In spite of all these methods, new applications continuously give rise to situations where straightforward application of these formulas are either inefficient or simply not possible. For example, if the function to be integrated contains critical point, then both Filon-type method and Levin collocation method are not accurate. So there is a need to device new methods for the integration of highly oscillatory functions.

The Monte Carlo method can be used to approximate the definite integral. This method gives the accuracy $\mathcal{O}\left(\frac{1}{\sqrt{n}}\right)$, which is not at all competitive with good algorithms, such as the Romberg method [6]. The present paper proposes the application of quasi-Monte Carlo methods for the numerical integration of highly oscillatory functions. In these methods selection of abscissas are based on Vander Corput sequence [7], which is a low discrepancy sequence.

In Sect. 2, an introduction to quasi-Monte Carlo methods, low discrepancy sequences, and Vander Corput sequence are presented. Section 3 gives the error bounds for quasi-Monte Carlo integration of the highly oscillating integrals. In Sect. 4, the quasi-Monte Carlo method with a Vander Corput sequence is applied to various problems. The efficiency of the QMC method is compared with other methods. Conclusions are drawn and are discussed in Sect. 5.

2 Quasi-Monte Carlo Methods

The only difference between the Monte Carlo and quasi-Monte Carlo methods is the selection of abscissa set $\{x_i\}$ (grid points). In Monte Carlo methods the abscissa are generated as a set of random number, whereas in quasi-Monte Carlo methods the quadrature nodes are calculated from deterministic algorithms.

2.1 Low Discrepancy Sequences

Now we introduce a quantity (the so-called discrepancy of the sequence) that measures the deviation of the sequence from an ideal distribution. This measure enables us to distinguish between *good* and *bad* sequences.

Definition 1 [7] For a nonempty set $\mathscr{M}$ of measurable subsets of C_N^+, the discrepancy $D_n^{\mathscr{M}}$ of the finite sequence $x_1, x_2, \ldots, x_n \in C_N^+$ with respect to $\mathscr{M}$ is defined by

$$D_n^{\mathscr{M}}(x_1, x_2, \ldots, x_n) := \sup_{E \in \mathscr{M}} \left| \frac{A(E; n)}{n} - \int_{C_N^+} C_E(x) dx \right|.$$

where $A(E; n) := \sum_{i=1}^{n} C_E(x_i)$ counts the number of points $x_i \in E$ and $E \subseteq C_N^+$.

2.2 Vander Corput Sequence

In this subsection, we are going to describe a low discrepancy sequence known as *Vander Corput sequence* [7]. In fact, this is the only infinite sequence having a uniformly smaller discrepancy than any other sequence exists upto now. Therefore, we have chosen this sequence for our numerical integration.

We define the so-called Vander Corput sequence $\{x_n\}$ as follows: For $n \geq 1$, let $n - 1 = \sum_{j=0}^{s} a_j 2^j$ be the dyadic expansion of $n - 1$. Then we set $x_n = \sum_{j=0}^{s} a_j 2^{-j-1}$. The sequence $\{x_n\}$ is then clearly contained in the unit interval. The following theorem gives the bounds for discrepancy of the Vander Corput sequence.

Theorem 1 *The discrepancy $D_N(\zeta)$ of the Vander Corput sequence $\zeta = \{x_N\}$ satisfies*

$$D_N(\zeta) \leq \frac{\ln(N+1)}{N \ln 2}$$

for N grid points.

Proof See [7].

3 Error Bounds for Quasi-Monte Carlo Methods

The selection of a numerical scheme is generally based on its accuracy (error bound), convergence, and computational cost. In this section, we are going to analyze these characters for quasi-Monte Carlo methods.

3.1 Variation of a Function

The variation of a univariate real function $f : [a, b] \to \mathbb{R}$ characterizes the regularity of f on the interval $[a, b]$. For a partition $\mathscr{P}$ of the interval $[a, b]$ into n subintervals,

$$\mathscr{P} : \{x_i : a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b\},$$

the sum

$$V(f; \mathscr{P}) := \sum_{i=1}^{n} \mid f(x_i) - f(x_{i-1}) \mid$$

measures the discrete variation of f with respect to the specific partition $\mathscr{P}$. The continuous variation of f can be characterized by the supremum of all such discrete variations $V(f; \mathscr{P})$.

Definition 2 [12] **Variation of a univariate function**. The variation of a univariate function $f : [a, b] \to \mathbb{R}$ is defined as

$$V(f) := \sup\{V(f; \mathscr{P})\} = \sup_{\mathscr{P}} \left\{ \sum_{i=1}^{n} \mid f(x_i) - f(x_{i-1}) \mid \right\}.$$

If $V(f)$ is finite, f is said to be of bounded variation on $[a, b]$. If f is continuously differentiable then the relationship holds.

$$V(f) = \int_a^b \mid f'(x) \mid dx$$

The variation of nonlinear oscillator is obtained as follows:

Proposition 1 *Suppose $f(x), g(x)$ are two continuously differentiable real valued functions with a finite bounded variation on $[a, b]$ and let*

$$I = \int_a^b f(x) e^{i\omega g(x)} dx,$$

then the bounded variation of the integrand is calculated as

$$BV\left[f(x)e^{i\omega g(x)}\right] \leq \int_a^b \left|i\omega g'(x) f(x) + f'(x)\right| dx$$

Proof Let the integrand be denoted as

$$\phi(x) = f(x)e^{i\omega g(x)}$$

Since $\phi(x)$ is piecewise smooth then the bounded variation is calculated as

$$
\begin{aligned}
BV(\phi) &= \int_a^b |\phi'(x)| dx \\
&= \int_a^b \left| i\omega g'(x) f(x) e^{i\omega g(x)} + f'(x) e^{i\omega g(x)} \right| dx \\
\therefore \quad BV(\phi) &\leq \int_a^b \left| i\omega g'(x) f(x) + f'(x) \right| \left| e^{i\omega g(x)} \right| dx \\
&\leq \int_a^b \left| i\omega g'(x) f(x) + f'(x) \right| dx
\end{aligned}
$$

Corollary 1 *Suppose $g(x) = Constant$, then*

$$
BV\left[f(x) e^{i\omega g(x)} \right] \leq \int_a^b \left| f'(x) \right| dx
$$

which corresponds to normal integration.

Corollary 2 *Suppose $g(x) = x$, i.e., linear oscillator, then*

$$
BV\left[f(x) e^{i\omega g(x)} \right] \leq \int_a^b \left| i\omega f(x) + f'(x) \right| dx
$$

But it is not always possible to calculate the variation of the functions.

3.2 Error Bounds

Now we discuss the error bounds for quasi-Monte Carlo approximation for more general integration domains. All these bounds depend on the variation of the integrand which involves the oscillatory parameter ω. A classical result is the following inequality of Koksma [7].

Theorem 2 *If f has bounded variation $V(f)$ on $[0, 1]$, then, for any sequence $x_1, x_2, \ldots, x_N \in [0, 1]$, we have*

$$
\left| \frac{1}{N} \sum_{n=1}^{N} f(x_n) - \int_0^1 f(u) du \right| \leq V(f) D_N(x_1, x_2, \ldots, x_N). \tag{3}
$$

Proof We can assume that $x_1 \leq x_2 \leq \cdots \leq x_N$. Put $x_0 = 0$ and $x_{N+1} = 1$. Using summation by parts and integration by parts, we obtain

$$\frac{1}{N}\sum_{n=1}^{N} f(x_n) - \int_0^1 f(u)du = -\sum_{n=0}^{N} \frac{n}{N}(f(x_{n+1}) - f(x_n)) + \int_0^1 u df(u)$$
$$= \sum_{n=0}^{N} \int_{x_n}^{x_{n+1}} \left(u - \frac{n}{N}\right) df(u).$$

For fixed n with $0 \le n \le N$, we have

$$\left|u - \frac{n}{N}\right| \le D_N(x_1, x_2, \ldots, x_N) \quad \text{for } x_n \le u \le x_{n+1}$$

$$\therefore \quad \left|\frac{1}{N}\sum_{n=1}^{N} f(x_n) - \int_0^1 f(u)du\right| \le D_N(x_1, x_2, \ldots, x_N) \sum_{n=0}^{N} \int_{x_n}^{x_{n+1}} |df(u)|$$
$$\le D_N(x_1, x_2, \ldots, x_N) \sum_{n=1}^{N} |f(x_{n+1}) - f(x_n)|$$
$$\le D_N(x_1, x_2, \ldots, x_N) V(f).$$

Hence we get the desired inequality.

The Koksma's inequality is applicable for C^∞ functions also.

4 Numerical Experiments

In this section, we consider two different example problems. We evaluated the integrals using quasi-Monte Carlo methods with Vander Corput sequence and error bounds are calculated using Koksma's inequality.

Example 1 In this example, we consider the highly oscillating integrals of the form

$$I = \int_a^b e^{i\omega g(x)} dx$$

where $g'(0) = g''(0) = \cdots = g^{(r-1)}(0) = 0$ and $g^{(r)}(x) \neq 0$, for all $x \in [0, 1]$. This oscillator is known as irregular oscillator, where g is real.

In particular let us consider the integral

$$\int_0^1 e^{i\omega x^2} dx = \frac{\text{erf}(\sqrt{-i\omega})\sqrt{\pi}}{2\sqrt{-i\omega}}$$

It can be observed that the above integral has a unique critical point 0 of $g(x)$ in [0, 1]. Therefore, quasi-Monte Carlo method can be applied to evaluate the integral by using Vander Corput sequence.

Now we calculate the error bound for the given integral as follows: As we know $f(x) = 1$, and $g(x) = x^2$ and is continuous and differentiable. Therefore, the variation of $\phi(x)$ can be calculated as

$$\begin{aligned} V(\phi) &= \int_0^1 |\phi'(x)|dx = \int_0^1 \left|2i\omega x e^{i\omega x^2}\right| dx \\ \therefore V(\phi) &= 2\omega \int_0^1 \left|x e^{i\omega x^2}\right| dx \\ &\leq 2\omega \end{aligned}$$

By Koksma inequality (3), we get

$$\left|\frac{1}{N}\sum_{n=1}^{N} f(x_n) - \int_0^1 f(u)du\right| \leq \omega D_N(x_1, x_2, \ldots, x_N).$$

From Theorem 1, we know that

$$D_N(x_1, x_2, \ldots, x_N) \leq \frac{\ln(N+1)}{N \ln 2}$$

Therefore, the error bound for this integral is obtained as

$$\text{Error Bound} \leq \omega \frac{\ln(N+1)}{N \ln 2} \tag{4}$$

From this Eq. (4), we can observe that the error bound is dependent on both oscillating parameter ω and number of quadrature points N. The absolute error of the numerical scheme is plotted in Fig. 2 for $N = \omega$. We can observe that the error is of order $\ln(N)$.

Example 2 Consider the highly oscillatory integral [13]:

$$I(f) = \int_0^1 f(x) e^{i\omega \sin x} dx \tag{5}$$

where $f(x) = \cos(\sin x)\cos(x)$ and $g(x) = \sin x$. The bounded variation of the above integral is obtained as follows:

$$BV(f) \leq \int_0^1 \left| i\omega g'(x) f(x) + f'(x) \right| dx$$
$$\leq \int_0^1 |i\omega \cos(x) \cos(\sin x) \cos(x) - \cos(x) \sin(\sin(x)) \cos(x)$$
$$- \cos(\sin(x)) \sin(x)| \, dx \leq \omega$$

The exact solution of the above integral (5) is obtained as

$$I(Q) = \frac{e^{i\omega \sin 1}}{1+\omega^2} [i\omega \cos(\sin 1) + \sin(\sin 1)] - \frac{1}{1+\omega^2} i\omega.$$

The approximate solution of the above integral (5) is calculated using homotopy perturbation method (HPM) in [10]. It is given as

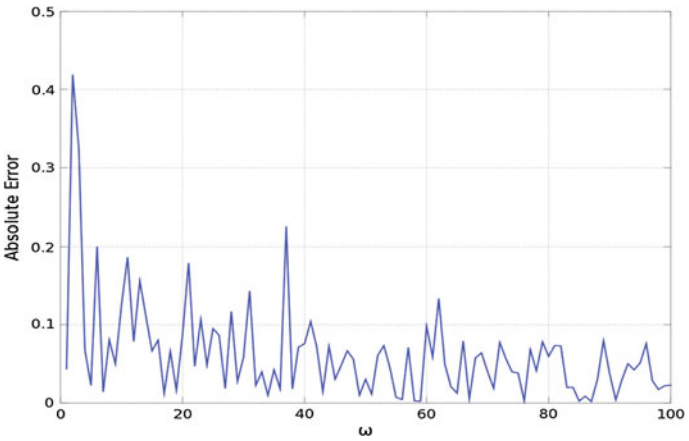

Fig. 2 Absolute error for $N = \omega$ and $0 \leq \omega \leq 100$

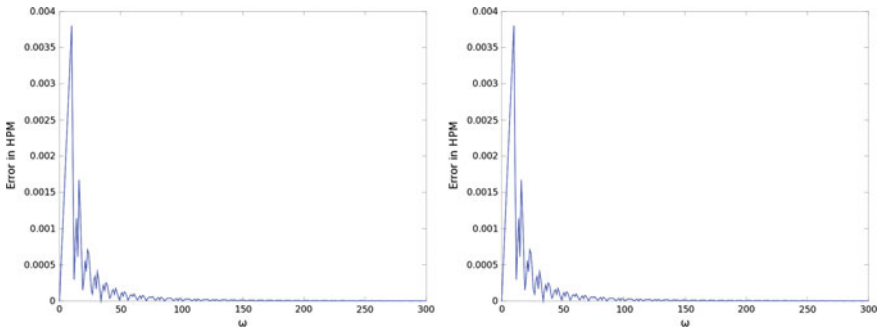

Fig. 3 The Absolute error of the integral (5) using HPM and QMC methods

$$I_{10}(Q) = \frac{1}{\omega^{10}}\left[i\omega(457 + 37\omega^2 + 5\omega^4 + \omega^6 + \omega^8)\right] + e^{0.8414709848i\omega}\,(968.8821733 + 169.63521533i\omega + 37.55809703\omega^2 + 17.513719298i\omega^3 - 2.7632852366\omega^4 + 2.3069869759i\omega^5 - 1.4293570493\omega^6 + 0.4328909146i\omega^7 - 0.4028624431\omega^8 - 0.6663667454i\omega^9)$$

and the absolute error between exact and HPM is plotted in Fig. 3.The numerical solution of the above integral (5) is calculated using quasi-Monte Carlo method and the corresponding absolute error with respect to the exact integral is plotted in Fig. 3.

We can observe that the numerical solution is same as the exact solution with very small difference as ω increases. The relative absolute error for the two methods are calculated. Corresponding to HPM method we have relative error 0.002522606 and corresponding quasi-Monte Carlo method gives the relative error 0.068065001. The relative error corresponding to HPM is less compared to quasi-Monte Carlo method due to the fact that HPM is a semiquantitive method and is applicable to few specific problems. This shows that the quasi-Monte Carlo methods are reliable and can be applied to a wide range of problems.

5 Conclusions

In this paper we are able to find the numerical solutions of highly nonlinear oscillatory integrals using quasi-Monte Carlo Methods. It is observed that the numerical solution satisfies the analytical error bounds. This shows the good agreement of results between analytical and numerical calculations. The work is under progress for the application of quasi-Monte Carlo methods to higher dimensional problems. This is due to the fact that as dimension of the problem increases the computational cost of traditional Gaussian-type methods increases. Therefore, Monte Carlo and quasi-Monte Carlo methods are the best choices.

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