Mild Solutions for Impulsive Functional Differential Equations of Order $\alpha \in (1, 2)$

Ganga Ram Gautam and Jaydev Dabas

Abstract In this research paper, first we develop the definition of mild solutions for impulsive fractional differential equations of order $\alpha \in (1, 2)$. Second, we study the uniqueness result of mild solutions for impulsive fractional differential equation with state-dependent delay by applying fixed point theorem and solution operator. At last, we present an example to illustrate the uniqueness result using fractional partial derivatives.

Keywords Fractional order differential equation · Functional differential equations · Impulsive conditions · Fixed point theorem

1 Introduction

In this research paper, we consider the following impulsive fractional differential equation with state-dependent delay of the form

$$ {}^{C}D_t^{\alpha} u(t) = Au(t) + f(t, u_{\rho(t,u_t)}), \ t \in J = [0, T], \ t \neq t_k, \tag{1} $$

$$ u(t) = \phi(t), \ t \in (-\infty, 0], \ u'(0) = u_1 \in X, \tag{2} $$

$$ \Delta u(t_k) = I_k(u(t_k^-)), \ \Delta u'(t_k) = Q_k(u(t_k^-)), \quad k = 1, 2, ...m, \tag{3} $$

where ${}^{C}D_t^{\alpha}$ is the Caputo's fractional derivative of order $\alpha \in (1, 2)$, u' is ordinary derivative with respect to t and J is operational interval. $A : D(A) \subset X \to X$ is the sectorial operator defined on a complex Banach space X. The functions

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$f : J \times \mathscr{B}_h \rightarrow X$, $\rho : J \times \mathscr{B}_h \rightarrow (-\infty, T]$ and $\phi \in \mathscr{B}_h$ are given and satisfies some assumptions, where $\mathscr{B}_h$ is introduced in Sect. 2. The history function $u_t : (-\infty, 0] \rightarrow X$ is defined by $u_t(\theta) = u(t+\theta)$, $\theta \in (-\infty, 0]$ belongs to $\mathscr{B}_h$. Here $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T < \infty$ and the functions $I_k, Q_k \in C(X, X)$, $k = 1, 2, ...m$, are bounded. We have $\triangle u(t_k) = u(t_k^+) - u(t_k^-)$ where $u(t_k^+)$ and $u(t_k^-)$ represent the right- and left-hand limits of $u(t)$ at $t = t_k$, also we take $u(t_k^-) = u(t_k)$. Furthermore, $\triangle u'(t_k) = u'(t_k^+) - u'(t_k^-)$ where $u'(t_k^+)$ and $u'(t_k^-)$ represent the right- and left-hand limits of $u'(t)$ at $t = t_k$, also we take $u'(t_k^-) = u'(t_k)$, respectively.

Impulsive differential equations with fractional order (see for fractional calculus [15, 16, 18–20]) are paying attention by many researchers because the model processes which are subjected to abrupt changes cannot described by ordinary differential equations, so such type equations are modeled in term of impulse. The most important applications of these equations are in the ecology, mechanics, electrical, and medicine biology. On the other hand, functional differential equations originate in several branches of engineering, applied mathematics, and science. Recently, fractional functional differential equations with state-dependent delay seems frequently in many fields as modeling of equations, panorama of natural phenomena, and porous media. See for more details of the relevant development theory in the cited papers [1, 2, 4–9, 11, 13].

In our survey, we found that Feckan et al. [12] gave the new concept of solution for impulsive nonlinear fractional differential equation order $\alpha \in (0, 1)$. Wang et al. [22] defined the definition of mild solution using the probability density function for the impulsive fractional evolution equation of order $\alpha \in (0, 1)$. By Motivated work [22], Dabas and Chauhan [10] defined the mild solution for neutral impulsive fractional functional differential equation of order $\alpha \in (0, 1)$ using analytic operator theory. Wang et al. [23] extended the problem, consider in paper [12] for of order $\alpha \in (1, 2)$. Shu et al. [21] introduced the definition of mild solution for fractional differential equations with nonlocal conditions of order $\alpha \in (1, 2)$ without impulse. We found that there is no literature available on mild solution for impulsive fractional functional differential equation of order $\alpha \in (1, 2)$.

To fill this gap and inspired by the above-mentioned work [10, 12, 21–23], we develop the definition of mild solution for the problem (1)–(3) and show the existence result. For further details, this work has four sections, Sect. 2 provides some basic definitions, preliminaries, theorems, and lemmas. The Sect. 3 is equipped with main results for the considered problem (1)–(3) and in Sect. 4 an example is considered.

2 Preliminaries and Background Martials

Let $(X, \|\cdot\|_X)$ be a complex Banach space of functions with the norm $\|u\|_X = \sup_{t\in J}\{|u(t)| : u \in X\}$ and $L(X)$ denotes the Banach space of bounded linear operators from X into X equipped with norm is denoted by $\|\cdot\|_{L(X)}$.

For the analysis of the infinite delay, we shall use abstract phase space $\mathscr{B}_h$ as defined in [14] details are as follow:

Let $h : (-\infty, 0] \to (0, \infty)$ be a continuous function with $l = \int_{-\infty}^{0} h(s)ds < \infty, s \in (-\infty, 0]$. For any $a > 0$, we define space

$$\mathscr{B} = \{\psi : [-a, 0] \to X \text{ such that } \psi(t) \text{ is bounded and measurable}\},$$

equipped with the norm $\|\psi\|_{[-a,0]} = \sup_{s\in[-a,0]} \|\psi(s)\|_X, \forall\, \psi \in \mathscr{B}$. Let us define abstract space as

$$\mathscr{B}_h = \left\{\psi : (-\infty, 0] \to X, \text{ s.t. for any } a \geq c > 0, \psi \mid_{[-c,0]} \in \mathscr{B} \int_{-\infty}^{0} h(s)\|\psi\|_{[s,0]}ds < \infty\right\}.$$

If $\mathscr{B}_h$ is endowed with the norm $\|\psi\|_{\mathscr{B}_h} = \int_{-\infty}^{0} h(s)\|\psi\|_{[s,0]}ds,\ \forall\, \psi \in \mathscr{B}_h$, then it is clear that $(\mathscr{B}_h, \|\cdot\|_{\mathscr{B}_h})$ is a complete Banach space. Let

$$C_t^1([0, T], X) = C^1([0, t]; X),\ 0 < t \leq T < \infty,$$

be a Banach space of all functions $u : [0, T] \to X$ such that u is continuously differentiable on $[0, T]$ endowed with the norm

$$\|u\|_{C_t^1} = \sup_{t\in[0,T]} \left\{\sum_{j=0}^{1} \|u^j(t)\|_X, u \in C_t^1\right\}.$$

To use the impulsive condition with infinite delay, we consider a Banach space

$$\mathscr{B}_h' := PC^1((-\infty, T]; X),\ T < \infty,$$

formed by all functions $u : (-\infty, T] \to X$ such that u is continuously differentiable on $[0, T]$ except for a finite number of points $t_i \in (0, T),\ i = 1, 2, \ldots,$ N, at which $u'(t_i^+)$ and $u'(t_i^-) = u'(t_i)$ exist and endowed with the seminorm $\|\cdot\|_{B_h'}$ in $\mathscr{B}_h'$

$$\|u\|_{\mathscr{B}_h'} = \sup\{\|u\|_{C_t^1} : 0 \leq t \leq T\} + \|\phi\|_{\mathscr{B}_h}, u \in \mathscr{B}_h'.$$

For a function $u \in \mathscr{B}_h'$ and $i \in \{0, 1, \ldots, N\}$, we introduce the function $\bar{u}_i \in C^1((t_i, t_{i+1}]; X)$ given by

$$\bar{u}_i(t) = \begin{cases} u'(t), & \text{for } t \in (t_i, t_{i+1}], \\ u'(t_i^+), & \text{for } t = t_i. \end{cases}$$

Let $u : (-\infty, T] \to X$ be the function such that $u_0 = \phi$, $u \mid_{J_k} \in C^1(J_k, X)$ then for all $t \in J_k$, the following conditions hold:

(C_1) $u_t \in \mathscr{B}_h$.
(C_2) $\|u(t)\|_X \leq H\|u_t\|_{\mathscr{B}_h}$.
(C_3) $\|u_t\|_{\mathscr{B}_h} \leq K(t)\sup\{\|u(s)\| : 0 \leq s \leq t\} + M(t)\|\phi\|_{\mathscr{B}_h}$, where $H > 0$ is constant; $K, M : [0,\infty) \to [0,\infty)$, $K(\cdot)$ is continuous, $M(\cdot)$ is locally bounded and K, M are independent of $u(t)$.
(C_{4_ϕ}) The function t $\to \phi_t$ is well-defined and continuous from the set

$$\Re(\rho^-) = \{\rho(s,\psi) : (s,\psi) \in [0,T] \times \mathscr{B}_h\}$$

into $\mathscr{B}_h$ and there exist a continuous and bounded function $J^\phi : \Re(\rho^-) \to (0,\infty)$ such that $\|\phi_t\|_{\mathscr{B}_h} \leq J^\phi(t)\|\phi\|_{\mathscr{B}_h}$ for every $t \in \Re(\rho^-)$.

Lemma 1 ([5]) *Let* $u : (-\infty, T] \to X$ *be function such that* $u_0 = \phi, u \mid_{J_k} \in C^1(J_k, X)$ *and if* (C_{4_ϕ}) *hold, then*

$$\|u_s\|_{\mathscr{B}_h} \leq (M_b + J^\phi)\|\phi\|_{\mathscr{B}_h} + K_b \sup\left\{\|u(\theta)\|;\ \theta \in [0, \max\{0,s\}]\right\},\ s \in \Re(\rho^-) \cup J_k,$$

where $J^\phi = \sup_{t\in\Re(\rho^-)} J^\phi(t),\ M_b = \sup_{s\in[0,T]} M(s)$ *and* $K_b = \sup_{s\in[0,T]} K(s)$.

Definition 1 Caputo's derivative of order $\alpha > 0$ with lower limit a, for a function $f : [a,\infty) \to \mathrm{R}$ such that $f \in C^n([a,\infty), \mathrm{R})$ is defined as

$${}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds =_a J_t^{n-\alpha} f^{(n)}(t),$$

where $a \geq 0,\ n-1 < \alpha < n,\ n \in \mathrm{N}$.

Definition 2 The Riemann–Liouville fractional integral operator of order $\alpha > 0$ with lower limit a, for a continuous function $f : [a,\infty) \to \mathrm{R}$ such that $f \in L^1_{loc}([a,\infty), \mathrm{R})$ is defined by

$${}_aJ_t^0 f(t) = f(t),\ {}_aJ_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad t > 0,$$

where $a \geq 0$ and $\Gamma(\cdot)$ is the Euler gamma function.

Definition 3 ([21]) Let $A : D(A) \subseteq X \to X$ be a densely defined, closed, and linear operator in X. A is said to be sectorial of the type (M, θ, α, μ) if there exist $\mu \in \mathrm{R},\ \theta \in (\frac{\pi}{2}, \pi),\ M > 0$, such that such that the α-resolvent of A exists outside the sector and following two conditions are satisfied

(1) $\mu + S_\theta = \{\mu + \lambda^\alpha : \lambda \in \mathscr{C},\ |Arg(-\lambda^\alpha)| < \theta\}$,
(2) $\|(\lambda^\alpha I - A)^{-1}\|_{L(X)} \leq \frac{M}{|\lambda^\alpha - \mu|},\ \lambda \notin \mu + S_\theta$,

where X is the complex Banach space with norm denoted $\|\cdot\|_X$.

Definition 4 ([19]) A two parameter function of the Mittag-Leffler type is defined by the series expansion and integral form

$$E_{\alpha,\beta}(y) = \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(\alpha k + \beta)} = \frac{1}{2\pi\iota} \int_c \frac{\mu^{\alpha-\beta} e^{\mu}}{\mu^{\alpha} - y} d\mu, \ \alpha, \ \beta > 0, \ y \in \mathrm{C},$$

where c is a contour which starts and ends at $-\infty$ and encircles the disk $|\mu| \le |y|^{\frac{1}{\alpha}}$ counter clockwise.

The Laplace integral of this function given by

$$\int_0^{\infty} e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(\omega t^{\alpha}) dt = \frac{\lambda^{\alpha-\beta}}{\lambda^{\alpha} - \omega}, \ Re\lambda > \omega^{\frac{1}{\alpha}}, \ \omega > 0.$$

From paper [17], putting $\beta = 1$, $\omega = A$ and using the sign $\div$ for the juxtaposition of a function depending on t with its Laplace transform depending on λ, we get the following Laplace transform pairs

$$S_{\alpha}(t) = E_{\alpha}(At^{\alpha}) \div \frac{\lambda^{\alpha-1}}{\lambda^{\alpha} I - A}, \ Re\lambda > A^{\frac{1}{\alpha}}.$$

More general Laplace transform pairs with integral

$${}_0J_t^j S_{\alpha}(t) \div \frac{\lambda^{\alpha-j-1}}{\lambda^{\alpha} I - A}, \ j = 0, 1.$$

Definition 5 ([2]) Let A be a closed and linear operator with the domain $D(A)$ defined in a Banach space X and $\alpha > 0$. We say that A is the generator of a solution operator if there exist $\omega \ge 0$ and a strongly continuous function $S_{\alpha} : \mathrm{R}^+ \to L(X)$, such that $\{\lambda^{\alpha} : Re\lambda > \omega\} \subset \rho(A)$ and

$$\frac{\lambda^{\alpha-1}}{\lambda^{\alpha} I - A} x = \int_0^{\infty} e^{\lambda t} S_{\alpha}(t) x dt, \quad Re\lambda > \omega, x \in X.$$

In this case, $S_{\alpha}(t)$ is called the solution operator generated by A.

Definition 6 ([3]) Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X. Let $\rho(A)$ be the resolvent set of A, we call A is the generator of an α-resolvent family if there exists $\omega \ge 0$ and a strongly continuous function $T_{\alpha} : \mathrm{R}^+ \to L(X)$ such that $\{\lambda^{\alpha} : Re\lambda > \omega\} \subset \rho(A)$ and

$$(\lambda^{\alpha} I - A)^{-1} x = \int_0^{\infty} e^{-\lambda t} T_{\alpha}(t) x dt, \ Re\lambda > \omega, \ x \in X.$$

In this case, $T_{\alpha}(t)$ is called α-resolvent family generated by A.

Lemma 2 *Let f be a continuous function and A be a sectorial operator of the type (M, θ, α, μ). Consider following differential equation of order $\alpha \in (1, 2)$*

$$
\begin{aligned}
{}^C D_t^{\alpha} u(t) &= Au(t) + f(t),\ \ t \in J = [0, T], t \neq t_k, &(4)\\
u(0) &= u_0 \in X,\ \ u'(0) = u_1 \in X, &(5)\\
\Delta u(t_k) &= I_k(u(t_k^-)), \Delta u'(t_k) = Q_k(u(t_k^-)),\ \ t \neq t_k, \quad k = 1, 2, ...m. &(6)
\end{aligned}
$$

Then a function $u(t) \in PC^1([0, T], X)$ is a solution of the system (4)–(6) if it satisfies following integral equation

$$
u(t) = \begin{cases} S_\alpha(t)u_0 + u_1 \int_0^t S_\alpha(s)ds + \int_0^t T_\alpha(t-s)f(s)ds, t \in (0, t_1] \\ S_\alpha(t)u_0 + K_\alpha(t)u_1 + \sum_{i=1}^k S_\alpha(t - t_i)I_i(u(t_i^-)) \\ + \sum_{i=1}^k Q_i\left(u(t_i^-)\right) \int_{t_i}^t S_\alpha(s - t_i)ds + \int_0^t T_\alpha(t-s)f(s)ds, t \in (t_k, t_{k+1}], \end{cases}
$$

where $S_\alpha(t)$ and $T_\alpha(t)$ are operators generated by A and defined as

$$
S_\alpha(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha I - A)^{-1} d\lambda;\ \ T_\alpha(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda^\alpha I - A)^{-1} d\lambda,
$$

and Γ is a suitable path such that $\lambda^\alpha \notin \mu + S_\theta$ for $\lambda \in \Gamma$.

Proof If $t \in (0, t_1]$, we have following problem

$$
\begin{aligned}
{}^C D_t^{\alpha} u(t) &= Au(t) + f(t), &(7)\\
u(0) &= u_0,\ \ u'(0) = u_1. &(8)
\end{aligned}
$$

By Lemma 3.1 in [23], the solution of Eqs. (7)–(8), we get

$$
u(t) = u_0 + u_1 t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} Au(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds. \tag{9}
$$

If $t \in (t_k, t_{k+1}], k = 1, 2, ...m$, we have the following problem

$$
\begin{aligned}
{}^C D_t^{\alpha} u(t) &= Au(t) + f(t), &(10)\\
u(t_k^+) &= u(t_k^-) + I_k(u(t_k^-)), &(11)\\
u'(t_k^+) &= u'(t_k^-) + Q_k(u(t_k^-)). &(12)
\end{aligned}
$$

By Lemma 3.1 in [23] the solution of Eqs. (10)–(12), we get

$$
\begin{aligned}
u(t) = u_0 + u_1 t &+ \sum_{i=1}^k I_i(u(t_i^-)) + \sum_{i=1}^k Q_i(u(t_i^-))(t - t_i) \\
&+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma\alpha} Au(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds. \qquad (13)
\end{aligned}
$$

Summarizing Eqs. (9) and (13) to $t \in (0, T]$, we get

$$u(t) = u_0 + u_1 t + \sum_{i=1}^{m} \chi_{t_i}(t) I_i(u(t_i^-)) + \sum_{i=1}^{m} \chi_{t_i}(t) Q_i(u(t_i^-))(t - t_i) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma\alpha} Au(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds, \tag{14}$$

where

$$\chi_{t_i}(t) = \begin{cases} 0 & t \le t_i \\ 1 & t > t_i. \end{cases}$$

By taking the Laplace transformation on Eq. (14), we have

$$L\{u(t)\} = \frac{u_0}{\lambda} + \frac{u_1}{\lambda^2} + \sum_{i=1}^{m} \frac{e^{-\lambda t_i}}{\lambda} I_i(u(t_i^-)) + \sum_{i=1}^{m} \frac{e^{-\lambda t_i}}{\lambda^2} Q_i(u(t_i^-)) + \frac{A}{\lambda^\alpha} L\{u(t)\} + \frac{1}{\lambda^\alpha} L\{f(t)\}. \tag{15}$$

On simplifying Eq. (15), we get

$$L\{u(t)\} = \frac{\lambda^{\alpha-1}(u_0)}{(\lambda^\alpha I - A)} + \frac{\lambda^{\alpha-2}(u_1)}{(\lambda^\alpha I - A)} + \sum_{i=1}^{m} \frac{\lambda^{\alpha-1}}{(\lambda^\alpha I - A)} e^{-\lambda t_i} I_i(u(t_i^-)) + \sum_{i=1}^{m} \frac{\lambda^{\alpha-2}}{(\lambda^\alpha I - A)} e^{-\lambda t_i} Q_i(u(t_i^-)) + \frac{1}{(\lambda^\alpha I - A)} L\{f(t)\}. \tag{16}$$

Now, taking the inverse Laplace transformation of Eq. (16), we have

$$u(t) = S_\alpha(t)u_0 + u_1 \int_0^t S_\alpha(s)ds + \sum_{i=1}^{m} \chi_{t_i}(t) I_i(u(t_i^-)) S_\alpha(t - t_i) + \sum_{i=1}^{m} \chi_{t_i}(t) Q_i(u(t_i^-)) \int_{t_i}^t S_\alpha(s - t_i)ds + \int_0^t T_\alpha(t-s) f(s)ds, \quad t \in J.$$

This complete the proof of the lemma.

Now, we state the definition of mild solutions of problem (1)–(3) by Lemma 2.

Definition 7 A function $u : (-\infty, T] \to X$ such that $u \in \mathscr{B}'_h$, $u(0) = \phi(0)$, $u'(0) = u_1$, is called a mild solution of problem (1)–(3) if it satisfies the following integral equation

$$u(t)=\begin{cases} S_\alpha(t)\phi(0)+u_1\int_0^t S_\alpha(s)ds+\int_0^t T_\alpha(t-s)f(s,u_{\rho(s,u_s)})ds, & t\in(0,t_1]\\ S_\alpha(t)\phi(0)+u_1\int_0^t S_\alpha(s)ds+\sum_{i=1}^k I_i(u(t_i^-))S_\alpha(t-t_i) & \\ +\sum_{i=1}^k Q_i(u(t_i^-))\int_{t_i}^t S_\alpha(s-t_i)ds+\int_0^t T_\alpha(t-s)f(s,u_{\rho(s,u_s)})ds, & t\in(t_k,t_{k+1}].\end{cases}$$

3 Uniqueness Result of Mild Solution

In this section, we prove the existence of mild solutions for the problem (1)–(3) with a non-convex valued right-hand side . If A sectorial operator of the type (M,θ,α,μ) then the strongly continuous functions $\|S_\alpha(t)\|\le M$, $\|T_\alpha(t)\|\le M$. To prove our results, we shall assume the function ρ is continuous. Our result is based on contraction fixed point theorem, for this we have following assumptions

(H_1) The function f is continuous and there exists $l_f\in L^1(J,\mathrm{R}^+)$ such that

$$\|(f(t,\psi)-f(t,\xi))\|_X\le l_f(t)\|\psi-\xi\|_{\mathscr{B}_h}\text{ for every }\psi,\xi\in\mathscr{B}_h.$$

(H_2) The functions I_k, Q_k are continuous and there exist $l_i,l_j\in L^1(J,\mathrm{R}^+)$ such that

$$\|I_k(x)-I_k(y)\|_X\le l_i(t)\|x-y\|_X;\ \|Q_k(x)-Q_k(y)\|_X\le l_j(t)\|x-y\|_X,$$

for all $x,y\in X$ and $k=1,\dots,m$.

Theorem 1 *Let the assumption (H_1) and (H_2) hold and the constant*

$$\Delta=M\left[m\|l_i\|_{L^1(J,\mathrm{R}^+)}+mT\|l_j\|_{L^1(J,\mathrm{R}^+)}+K_b\int_0^T l_f(s)ds\right]<1.$$

Then problem (1)–(3) has a unique mild solutions u on J.

Proof We convert the problem (1)–(3) in to fixed point problem. Let $\bar{\phi}:(-\infty,T)\to X$ be the extension of ϕ to $(-\infty,T]$ such that $\bar{\phi}(t)=\phi(0)$ on J. Consider the space Banach $\mathscr{B}_h''=\{u\in\mathscr{B}_h':u(0)=\phi(0),u'(0)=u_1\}$ and define the operator $P:\mathscr{B}_h''\to\mathscr{B}_h''$ as

$$Pu(t)=\begin{cases} S_\alpha(t)\phi(0)+u_1\int_0^t S_\alpha(s)ds+\int_0^t T_\alpha(t-s)f(s,\bar{u}_{\rho(s,\bar{u}_s)})ds, & t\in(0,t_1]\\ S_\alpha(t)\phi(0)+u_1\int_0^t S_\alpha(s)ds+\sum_{i=1}^k I_i(\bar{u}(t_i^-))S_\alpha(t-t_i) & \\ +\sum_{i=1}^k Q_i(\bar{u}(t_i^-))\int_{t_i}^t S_\alpha(s-t_i)ds+\int_0^t T_\alpha(t-s)f(s,\bar{u}_{\rho(s,\bar{u}_s)})ds, & t\in(t_k,t_{k+1}],\end{cases}$$

where $\bar{u}:(-\infty,T]\to X$ is such that $\bar{u(0)}=\phi$ and $\bar{u}=u$ on J. It is clear that u is unique mild solution of the problem (1)–(3) if and only if u is a solution of the operator equation $Pu=u$. Let $u,u^*\in\mathscr{B}_h''$, for $t\in(0,t_1]$ we have

$$\|Pu - Pu^*\|_X \le \int_0^t \|T_\alpha(t-s)\|_{L(X)} \|f\left(s, \bar{u}_{\rho(s,\bar{u}_s)}\right) - f\left(s, \bar{u}^*_{\rho(s,\bar{u}^*_s)}\right)\|_X ds$$

$$\|Pu - Pu^*\|_{\mathscr{B}''_h} \le MK_b \left[\int_0^T l_f(s)ds\right] \|u - u^*\|_{\mathscr{B}''_h}.$$

Now, without lose of generality we consider the subinterval $(t_k, t_{k+1}]$ to prove our result. Let $u, u^* \in \mathscr{B}''_h$ for $(t_k, t_{k+1}]$, we have

$$\begin{aligned}
\|Pu - Pu^*\|_X &\le \sum_{i=1}^{k} \|S_\alpha(t-t_i)\|_{L(X)} \|I_i\left(\bar{u}\left(t_i^-\right)\right) - I_i\left(\bar{u}^*\left(t_i^-\right)\right)\|_X \\
&\quad + \sum_{i=1}^{k} \int_{t_i}^{t} \|S_\alpha(s-t_i)\|_{L(X)} ds \|Q_i\left(\bar{u}\left(t_i^-\right)\right) - Q_i\left(\bar{u}^*\left(t_i^-\right)\right)\|_X \\
&\quad + \int_0^t \|T_\alpha(t-s)\|_{L(X)} \|f\left(s, \bar{u}_{\rho(s,\bar{u}_s)}\right) - f\left(s, \bar{u}^*_{\rho(s,\bar{u}^*_s)}\right)\|_X ds \\
\|Pu - Pu^*\|_{\mathscr{B}''_h} &\le M\left[m\|l_i\|_{L^1(J,\mathrm{R}^+)} + mT\|l_j\|_{L^1(J,\mathrm{R}^+)} + K_b \int_0^T l_f(s)ds\right] \|u - u^*\|_{\mathscr{B}''_h} \\
&\le \Delta \|u - u^*\|_{\mathscr{B}''_h}.
\end{aligned}$$

Since $\Delta < 1$, which implies that P is contraction map. Hence P has a unique fixed point, which is the mild solutions of problem (1)–(3) on J. This completes the proof of the theorem.

4 Application

Consider the following impulsive fractional partial differential equation of the form

$$\frac{\partial^\alpha}{\partial t^\alpha} u(t,x) = \frac{\partial^2}{\partial y^2} u(t,x) + \int_{-\infty}^{t} e^{2(s-t)} \frac{u(s - \rho_1(s)\rho_2(\|u\|), x)}{81} ds, \; t \ne \frac{1}{2}, \quad (17)$$

$$u(t,0) = u(t,\pi) = 0; u'(t,0) = u'(t,\pi) = 0 \;\; t \ge 0, \quad (18)$$

$$u(t,x) = \phi(t,x), u'(t,x) = 0, t \in (-\infty, 0], x \in [0,\pi], \quad (19)$$

$$\Delta u|_{t=\frac{1}{2}} = \frac{\|u\left(\frac{1^-}{2}\right)\|}{36 + \|u\left(\frac{1^-}{2}\right)\|}, \;\; \Delta u'|_{t=\frac{1}{2}} = \frac{\|u\left(\frac{1^-}{2}\right)\|}{49 + \|u\left(\frac{1^-}{2}\right)\|}, \quad (20)$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is Caputo's fractional derivative of order $\alpha \in (1,2)$, $0 < t_1 < t_2 < \cdots < t_n < T$ are prefixed numbers and $\phi \in \mathscr{B}_h$. Let $X = L^2[0,\pi]$ and define the operator $A : D(A) \subset X \to X$ by $Aw = w''$ with the domain $D(A) := \{w \in X : \; w, w'$ are absolutely continuous, $w'' \in X, w(0) = 0 = w(\pi)\}$. Then

$$Aw = \sum_{n=1}^{\infty} n^2 (w, w_n) w_n, \;\; w \in D(A),$$

where $w_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx), \;\; n \in \mathrm{N}$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t \geq 0}$ in X given by

$$T(t)\omega = \sum_{n=1}^{\infty} e^{-n^2 t} (\omega, \omega_n)\omega_n, \;\text{ for all } \omega \in X, \;\text{ and every } t > 0.$$

By subordination principle of solution operator, we have $\|S_\alpha(t)\|_{L(X)} \leq M$ for $t \in J$. Let $h(s) = e^{2s}, \;\; s < 0$ then $l = \int_{-\infty}^{0} h(s)ds = \frac{1}{2} < \infty$, for $t \in (-\infty, 0]$ and define

$$\|\phi\|_{\mathscr{B}_h} = \int_{-\infty}^{0} h(s) \sup_{\theta \in [s,0]} \|\phi(\theta)\|_{L^2} ds.$$

Hence for $(t, \phi) \in [0, 1] \times \mathscr{B}_h$, where $\phi(\theta)(x) = \phi(\theta, x), \;\; (\theta, x) \in (-\infty, 0] \times [0, \pi]$. We assume that $\rho_i : [0, \infty) \to [0, \infty), \;\; i = 1, 2,$ are continuous functions.

Set $u(t)(x) = u(t, x)$, and $\rho(t, \phi) = \rho_1(t)\rho_2(\|\phi(0)\|)$, we have

$$f(t, \phi)(x) = \frac{\phi}{81}, \; I_k(u) = \frac{\|u\|}{36 + \|u\|}, \; J_k(u) = \frac{\|u\|}{49 + \|u\|},$$

then with these settings the problem (17)–(20) can be written in the abstract form of Eqs. (1)–(3). It is obvious that the maps f, I_k, J_k following the assumption H_1, H_2. This implies that there exists a unique mild solutions of problem (17)–(20) on J.

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