kth Order Kantorovich Modification of Linking Baskakov-Type Operators

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Abstract In 1957 Baskakov introduced a general method for the construction of positive linear operators depending on a real parameter c. The so-called genuine Baskakov–Durrmeyer-type operators form a class of operators reproducing the linear functions, interpolating at (finite) endpoints of the interval, and having other nice properties. In this paper we consider a nontrivial link between Baskakov-type operators and genuine Baskakov–Durrmeyer-type operators. We establish explicit representations for the images of monomials and for the moments; they are useful, e.g., in studying asymptotic formulas.

Keywords Baskakov-and-Durrmeyer-type operators · Linking operators · Kantorovich-type modifications · Moments · Images of monomials

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1 Introduction and Definition of the Operators

In 1957 Baskakov [1] introduced a general method for the construction of positive linear operators depending on a real parameter c including the classical Bernstein, Szász-Mirakjan, and Baskakov operators as special cases. All these Baskakov-type operators preserve linear functions and interpolate at (finite) endpoints of the corresponding interval. The so-called Bernstein–Durrmeyer operators were introduced by Durrmeyer in [2] and independently developed by Lupaş [9]. Afterwards, this construction was carried over to many other classical operators; for instance see

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[10, 16] and in the general setting for so-called Baskakov–Durrmeyer-type operators [6]. These operators have a lot of nice properties; they commute, they commute with certain differential operators, they are self-adjoint but they only reproduce constants.

The consideration of so-called genuine Baskakov–Durrmeyer-type operators leads to a class of operators again reproducing the linear functions and interpolating at (finite) endpoints of the corresponding interval. These operators are related to the Baskakov–Durrmeyer-type operators in the same way as the Baskakov-type operators to their corresponding Kantorovich variants.

In [11, 12] Păltănea introduces operators depending on a parameter $\rho \in \mathbb{R}^+$, which constitute a nontrivial link between the Bernstein and Szász-Mirakjan operators, respectively, and their genuine Durrmeyer modifications. Further results can also be found in [3, 4, 13].

In this paper we consider a nontrivial link between Baskakov-type operators and genuine Baskakov–Durrmeyer-type operators. Moreover, we investigate the kth order Kantorovich modification of them; for $k = 1$ this means a link between the Kantorovich modification of Baskakov-type and Baskakov–Durrmeyer-type operators.

In what follows for $c \in \mathbb{R}$ we use the notations

$$a^{c,\overline{j}} := \prod_{l=0}^{j-1}(a + cl),\ a^{c,\underline{j}} := \prod_{l=0}^{j-1}(a - cl),\ j \in \mathbb{N}; \quad a^{c,\overline{0}} = a^{c,\underline{0}} := 1$$

which can be considered as a generalization of rising and falling factorials. Note that $a^{-c,\overline{j}} = a^{c,\underline{j}}$ and $a^{c,\overline{j}} = a^{-c,\underline{j}}$. This notation enables us to state the results for the different operators in a unified form.

In a recent paper [8] we already considered the linking operators between the kth order Kantorovich modification of the Bernstein and the genuine Bernstein–Durrmeyer operators. Comparison of the results in [8] with the outcomes of the present paper shows that all the representations for the moments and the images of monomials are also valid for the Bernstein case by setting $c = -1$ in the subsequent theorems.

In the following definitions of the operators we omit the parameter c in the notations in order to reduce the necessary sub- and superscripts.

Let $c \in \mathbb{R}$, $c \geq 0$, $n \in \mathbb{R}$, $n > c$, $\rho \in \mathbb{R}^+$, $j \in \mathbb{N}_0$, $x \in [0, \infty)$. Then the basis functions are given by

$$p_{n,j}(x) = \begin{cases} \frac{n^j}{j!} x^j e^{-nx} & , \ c = 0, \\ \frac{n^{c,\overline{j}}}{j!} x^j (1 + cx)^{-(\frac{n}{c}+j)} & , \ c > 0. \end{cases}$$

In the following definition we assume that $f : [0, \infty) \longrightarrow \mathbb{R}$ is given in such a way that the corresponding integrals and series are convergent.

Definition 1 The operators of Baskakov type are defined by

$$B_n(f,x) = \sum_{j=0}^{\infty} p_{n,j}(x) f\left(\frac{j}{n}\right), \tag{1}$$

the genuine Baskakov–Durrmeyer-type operators are denoted by

$$B_{n,1}(f,x) = f(0)p_{n,0}(x) + \sum_{j=1}^{\infty} p_{n,j}(x) \int_0^{\infty} p_{n+2c,j-1}(t) f(t)dt, \tag{2}$$

and for $\rho \in \mathbb{R}^+$ the linking operators are given by

$$\begin{aligned} B_{n,\rho}(f,x) &= \sum_{j=0}^{\infty} F_{n,j}^{\rho}(f) p_{n,j}(x) && (3)\\ &= f(0)p_{n,0}(x) + \sum_{j=1}^{\infty} p_{n,j}(x)(n+c) \int_0^{\infty} \mu_{n,j}^{\rho}(t) f(t)dt, && (4) \end{aligned}$$

where

$$\mu_{n,j}^{\rho}(t) = \begin{cases} \frac{(n\rho)^{j\rho}}{\Gamma(j\rho)} t^{j\rho-1} e^{-n\rho t} & , \ c = 0, \\ \frac{c^{j\rho}}{B(j\rho, \frac{n}{c}\rho+1)} t^{j\rho-1}(1+ct)^{-(\frac{n}{c}+j)\rho-1} & , \ c > 0. \end{cases}$$

Setting $c = 0$ in (2) leads to the Phillips operators [14], $c > 0$ was investigated in [18]. To the best of our knowledge the case $c = 0$ in (3) was first considered in [12].

As in [8] for the Bernstein case we also consider the kth order Kantorovich modification of the operators $B_{n,\rho}$, i.e.,

$$B_{n,\rho}^{(k)} := D^k \circ B_{n,\rho} \circ I_k \tag{5}$$

where D^k denotes the kth order ordinary differential operator and

$$I_k f = f, \text{ if } k = 0, \text{ and } I_k(f,x) = \int_0^x \frac{(x-t)^{k-1}}{(k-1)!} f(t)dt, \text{ if } k \in \mathbb{N}.$$

For $k = 0$ we omit the superscript (k) as indicated by the definition above.

This general definition contains many known operators as special cases. For $c = 0$ we get the linking operators considered in [13]. For $\rho = 1$ we get the genuine Baskakov–Durrmeyer-type operators $B_{n,1}$, for $\rho = 1$, $k \in \mathbb{N}$ the Baskakov–Durrmeyer-type operators $B_{n,1}^{(1)}$ (see [6, (1.3)], named M_{n+c} there) and the auxiliary operators $B_{n,1}^{(k)}$ considered in [7, (3.5)], (named $M_{n+c,k-1}$ there) with the explicit representation

$$(B_{n,1}^{(k)} f)(x) = \frac{n^{c,\overline{k}}}{n^{c,\underline{k-1}}} \sum_{j=0}^{\infty} p_{n+ck,j}(x) \int_0^{\infty} p_{n-c(k-2),j+k-1}(t) f(t) dt.$$

For an arbitrary sequence of linear operators, the images of monomials and the moments are important, e.g., in studying the asymptotic behavior. In this paper we establish explicit representations for the images of the monomials and for the moments of the investigated operators. Corresponding recursion formulas and further results will be given in a forthcoming paper.

Below we will use the following basic formulas.

$$\int_0^{\infty} \mu_{n,j}^{\rho}(t) dt = B\left(j\rho, \frac{n}{c}\rho + 1\right), \tag{6}$$

$$\sum_{j=0}^{\infty} p_{n,j}(x) = 1, \tag{7}$$

$$\frac{j}{n} p_{n,j}(x) = x p_{n+c,j-1}(x), \tag{8}$$

$$x(1+cx) p_{n,j}'(x) = (j - nx) p_{n,j}(x), \tag{9}$$

with the convention $p_{n,l}(x) = 0$, if $l < 0$. As usual, empty products are defined to be one.

2 Explicit Formulas for the Images of Monomials

In this section we prove general explicit formulas for the images of the monomials of the operators $B_{n,\rho}^{(k)}$. In what follows we denote by $e_\nu(t) = t^\nu$, $\nu \in \mathbb{N}_0$, the monomials and by

$$\boldsymbol{\Delta}_h^l f(x) = \sum_{\kappa=0}^{l} (-1)^{l-\kappa} \binom{l}{\kappa} f(x + \kappa h) \tag{10}$$

the lth order forward difference of a function f with step h and define

$$p_\nu^\rho(\xi) := \prod_{l=1}^{\nu-1} \left(\xi + \frac{l}{\rho}\right), \ \nu \in \mathbb{N}.$$

This can be rewritten as

$$p_\nu^\rho(\xi) = \sum_{i=0}^{\nu-1} \frac{\boldsymbol{\Delta}_1^i p_\nu^\rho(1)}{i!} \prod_{l=1}^{i}(\xi - l), \tag{11}$$

which can be derived by using the Newton representation of the interpolation polynomial of p_ν^ρ for the equidistant knots $1, 2, \ldots, \nu$.

We first consider the images of the monomials for the case $k = 0$, i.e., for the operators $B_{n,\rho}$.

Theorem 1 *Let $n \in \mathbb{R}$, $n\rho > c(\nu - 1)$, $\rho \in \mathbb{R}_+$, $\nu \in \mathbb{N}_0$, $\nu \leq n$. Then*

$$(B_{n,\rho} e_0)(x) = 1, \tag{12}$$

$$(B_{n,\rho} e_\nu)(x) = \frac{\rho^\nu}{(n\rho)^{c,\underline{\nu}}} \sum_{i=1}^{\nu} \frac{n^{c,\overline{i}}}{(i-1)!} \left(\boldsymbol{\Delta}_1^{i-1} p_\nu^\rho(1)\right) x^i, \ \nu \in \mathbb{N}. \tag{13}$$

Proof (12) follows immediately from (6) and (7).
In order to prove (13) we take into account that for $c = 0$

$$\frac{(n\rho)^{j\rho}}{\Gamma(j\rho)} \int_0^\infty t^\nu t^{j\rho-1} e^{-n\rho t} dt = \frac{1}{(n\rho)^\nu} \cdot \frac{\Gamma(j\rho + \nu)}{\Gamma(j\rho)} = \frac{1}{n^\nu} \prod_{l=0}^{\nu-1} \left(j + \frac{l}{\rho}\right)$$

and for $c > 0$

$$\begin{aligned} &\frac{c^{j\rho}}{B\left(j\rho, \frac{n}{c}\rho + 1\right)} \int_0^\infty t^\nu t^{j\rho-1} (1 + ct)^{-(\frac{n}{c}+j)\rho-1} dt \\ &= c^{-\nu} \frac{\Gamma(j\rho + \nu)\Gamma\left(\frac{n}{c}\rho + 1 - \nu\right)}{\Gamma(j\rho)\Gamma\left(\frac{n}{c}\rho + 1\right)} = \frac{\rho^\nu}{(n\rho)^{c,\underline{\nu}}} \prod_{l=0}^{\nu-1} \left(j + \frac{l}{\rho}\right). \end{aligned}$$

Thus we get for $\nu \geq 1$ with (8) and (11)

$$\begin{aligned} (B_{n,\rho} e_\nu)(x) &= \frac{\rho^\nu}{(n\rho)^{c,\underline{\nu}}} \sum_{j=1}^{\infty} p_{n,j}(x) \prod_{l=0}^{\nu-1} \left(j + \frac{l}{\rho}\right) \\ &= \frac{\rho^\nu}{(n\rho)^{c,\underline{\nu}}} nx \sum_{j=1}^{\infty} p_{n+c,j-1}(x) p_\nu^\rho(j) \\ &= \frac{\rho^\nu}{(n\rho)^{c,\underline{\nu}}} nx \sum_{j=1}^{\infty} p_{n+c,j-1}(x) \sum_{i=0}^{\nu-1} \frac{\boldsymbol{\Delta}_1^i p_\nu^\rho(1)}{i!} \prod_{l=1}^{i}(j - l) \\ &= \frac{\rho^\nu}{(n\rho)^{c,\underline{\nu}}} nx \sum_{i=0}^{\nu-1} \frac{\boldsymbol{\Delta}_1^i p_\nu^\rho(1)}{i!} \sum_{j=i+1}^{\infty} p_{n+c,j-1}(x) \prod_{l=1}^{i}(j - l). \end{aligned} \tag{14}$$

Applying (8) for $j \geq i+1$ we have

$$p_{n+c,j-1}(x)\prod_{l=1}^{i}(j-l) = p_{n+c(i+1),j-i-1}(x)x^i\prod_{l=1}^{i}(n+cl).$$

Hence with (7)

$$(B_{n,\rho}e_\nu)(x) = \frac{\rho^\nu}{(n\rho)^{c,\underline{\nu}}}\sum_{i=1}^{\nu}\frac{n^{c,\bar{i}}}{(i-1)!}\left(\mathbf{\Delta}_1^{i-1}p_\nu^\rho(1)\right)x^i. \qquad \square$$

Remark 1 Using (10), the representation (13) can be rewritten as

$$(B_{n,\rho}e_\nu)(x) = \frac{\rho^\nu}{(n\rho)^{c,\underline{\nu}}}\sum_{i=1}^{\nu}n^{c,\bar{i}}x^i\sum_{\kappa=0}^{i-1}(-1)^{i-1-\kappa}\frac{1}{\kappa!(i-1-\kappa)!}p_\nu^\rho(1+\kappa).$$

Now we consider the special cases $\rho = 1$ and $\rho \to \infty$.
$\underline{\rho = 1}$: Then with [5, (3.48)] (see [8, p. 323])

$$\mathbf{\Delta}_1^{i-1}p_\nu^1(1) = (\nu-1)!\binom{\nu}{i}.$$

Thus

$$(B_{n,1}e_\nu)(x) = \frac{1}{n^{c,\underline{\nu}}}\sum_{i=1}^{\nu}n^{c,\bar{i}}\frac{(\nu-1)!}{(i-1)!}\binom{\nu}{i}x^i,$$

which coincides with the formula given in [18, Lemma 1.11] and [7, (4.3)] with $s=-1$ and taking $n+c$ instead of n there.
$\underline{\rho \to \infty}$: Then $\dfrac{\rho^\nu}{(n\rho)^{c,\underline{\nu}}} \to \dfrac{1}{n^\nu}$, and (see [8, p. 323])

$$\mathbf{\Delta}_1^{i-1}p_\nu^\infty(1) = (i-1)!\sigma_\nu^i,$$

where σ_ν^j denote the Stirling numbers of second kind. Thus

$$(B_{n,\infty}e_\nu)(x) = \frac{1}{n^\nu}\sum_{i=1}^{\nu}n^{c,\bar{i}}\sigma_\nu^i x^i,$$

which coincides with the corresponding result for the classical Baskakov-type operators which can be calculated directly from the definition of the operators by using (8).

Next, we consider the images of the monomials for the case $k \in \mathbb{N}$.

Theorem 2 *Let $n \in \mathbb{R}$, $k \in \mathbb{N}$, $\rho \in \mathbb{R}_+$, $\nu \in \mathbb{N}_0$, $n\rho > c(\nu + k - 1)$. Then*

$$(B_{n,\rho}^{(k)} e_\nu)(x) = \frac{\nu!}{(\nu+k)!} \frac{\rho^{\nu+k}}{(n\rho)^{c,\underline{\nu+k}}} \sum_{i=0}^{\nu} \frac{n^{c,\overline{i+k}}}{i!} (i+k) \left(\mathbf{\Delta}_1^{i+k-1} p_{\nu+k}^{\rho}(1) \right) x^i. \quad (15)$$

Proof By using $B_{n,\rho}^{(k)} e_\nu = \frac{\nu!}{(\nu+k)!} D^k B_{n,\rho} e_{\nu+k}$ we get from (13) for $k \in \mathbb{N}$

$$\begin{aligned}(B_{n,\rho}^{(k)} e_\nu)(x) &= \frac{\nu!}{(\nu+k)!} \frac{\rho^{\nu+k}}{(n\rho)^{c,\underline{\nu+k}}} \sum_{i=k}^{\nu+k} \frac{n^{c,\bar{i}}}{(i-1)!} \left(\mathbf{\Delta}_1^{i-1} p_{\nu+k}^{\rho}(1) \right) \frac{i!}{(i-k)!} x^{i-k} \\ &= \frac{\nu!}{(\nu+k)!} \frac{\rho^{\nu+k}}{(n\rho)^{c,\underline{\nu+k}}} \sum_{i=0}^{\nu} \frac{n^{c,\overline{i+k}}}{i!} (i+k) \left(\mathbf{\Delta}_1^{i+k-1} p_{\nu+k}^{\rho}(1) \right) x^i. \qquad \square\end{aligned}$$

Remark 2 Using again (10), the representation (15) can be rewritten as

$$\begin{aligned}(B_{n,\rho}^{(k)} e_\nu)(x) = {} & \frac{\nu!}{(\nu+k)!} \frac{\rho^{\nu+k}}{(n\rho)^{c,\underline{\nu+k}}} \sum_{i=0}^{\nu} n^{c,\overline{i+k}} \frac{(i+k)!}{i!} x^i \\ & \times \sum_{\kappa=0}^{i+k-1} (-1)^{i+k-1-\kappa} \frac{1}{\kappa!(i+k-1-\kappa)!} p_{\nu+k}^{\rho}(1+\kappa).\end{aligned}$$

Again we consider the special cases $\rho = 1$ and $\rho \to \infty$.
$\underline{\rho = 1}$: Then again with [5, (3.48)]

$$\mathbf{\Delta}_1^{i+k-1} p_{\nu+k}^{1}(1) = (\nu+k-1)! \binom{\nu+k}{i+k}.$$

Thus

$$(B_{n,1}^{(k)} e_\nu)(x) = \frac{1}{n^{c,\underline{\nu+k}}} \sum_{i=0}^{\nu} n^{c,\overline{i+k}} \frac{(\nu+k-1)!}{(i+k-1)!} \binom{\nu}{i} x^i.$$

This coincides with the corresponding result in [7, Satz 4.2] for the auxiliary operators with the notation $B_{n,\rho}^{(k)} = M_{n+c,k-1}$ there.
$\underline{\rho \to \infty}$: Then $\dfrac{\rho^{\nu+k}}{(n\rho)^{c,\underline{\nu+k}}} \to \dfrac{1}{n^{\nu+k}}$ and

$$\mathbf{\Delta}_1^{i+k-1} p_{\nu+k}^{\infty}(1) = (i+k-1)! \sigma_{\nu+k}^{i+k}.$$

Thus

$$(B_{n,\infty}^{(k)}e_\nu)(x) = \frac{\nu!}{(\nu+k)!}\frac{1}{n^{\nu+k}}\sum_{i=0}^{\nu}\frac{n^{c,\overline{i+k}}}{i!}(i+k)!\sigma_{\nu+k}^{i+k}x^i.$$

From the explicit representations of the images of the monomials we can deduce the following result concerning the limit of the operators $B_{n,\rho}^{(k)}$ when $\rho \to \infty$.

Corollary 1 *For each polynomial p we have*

$$\lim_{\rho\to\infty} B_{n,\rho}^{(k)}p(x) = B_n^{(k)}p(x)$$

uniformly on every compact subinterval of $[0,\infty)$.

For the evaluation of $B_{n,\rho}^{(k)}e_\nu$, $k \in \mathbb{N}$, for special values of ν, we use the representation

$$p_{\nu+k}^{\rho}(\xi) = \sum_{l=0}^{\nu+k-1}\rho^{-l}\sigma_l(1,2,\ldots,\nu+k-1)\xi^{\nu+k-1-l},$$

with the notation $\sigma_j(x_0,x_1,\ldots,x_n)$, $j \in \mathbb{N}$, for the symmetric function which is the sum of all products of j distinct values from the set $\{x_0,x_1,\ldots x_n\}$ and $\sigma_0(x_0,x_1,\ldots,x_n) := 1$.

For the monomial e_m, it is known (see, e.g., [15, Theorem 1.2.1]) that

$$\mathbf{\Delta}_1^{j+k-1}e_m(1) = \begin{cases} 0, & m < j+k-1, \\ (j+k-1)!\tau_{m-(j+k-1)}(1,2,\ldots,j+k), & 0 \le j+k-1 \le m, \end{cases}$$

with the complete symmetric function $\tau_j(x_0,x_1,\ldots,x_n)$ which is the sum of all products of $x_0,x_1,\ldots,x_n$ of total degree j, $j \in \mathbb{N}$, and $\tau_0(x_0,x_1,\ldots,x_n) := 1$.

Thus we can rewrite $(B_{n,\rho}^{(k)}e_\nu)$ as

$$(B_{n,\rho}^{(k)}e_\nu)(x) = \frac{\nu!}{(\nu+k)!}\frac{\rho^{\nu+k}}{(n\rho)^{c,\underline{\nu+k}}}\sum_{i=0}^{\nu}\frac{n^{c,\overline{i+k}}(i+k)!}{i!}x^i \qquad (16)$$
$$\times\sum_{l=0}^{\nu-i}\rho^{-l}\sigma_l(1,2,\ldots,\nu+k-1)\tau_{\nu-l-j}(1,2,\ldots,i+k).$$

As a corollary we present the results for $\nu = 0,\ 1,\ 2$.

Corollary 2 *For $k \in \mathbb{N}_0$ the images for the first monomials are given by*

$$
\begin{aligned}
(B_{n,\rho}^{(k)} e_0)(x) &= \frac{\rho^k}{(n\rho)^{c,\underline{k}}} \cdot n^{c,\overline{k}}, \\
(B_{n,\rho}^{(k)} e_1)(x) &= \frac{\rho^{k+1}}{(n\rho)^{c,\underline{k+1}}} \cdot n^{c,\overline{k}} \left[\frac{1}{2} k \left(1 + \frac{1}{\rho}\right) + (n+ck)x \right], \\
(B_{n,\rho}^{(k)} e_2)(x) &= \frac{\rho^{k+2}}{(n\rho)^{c,\underline{k+2}}} \cdot n^{c,\overline{k}} \left[\frac{1}{2} k \left(\frac{3k+1}{6} + \frac{k+1}{\rho} + \frac{3k+5}{6\rho^2} \right) \right. \\
&\qquad \left. + (n+ck) \left((k+1) \left(1 + \frac{1}{\rho}\right) x + (n + c(k+1)) x^2 \right) \right].
\end{aligned}
$$

Proof For $k = 0$ the identities follow from Theorem 1. For $k \in \mathbb{N}$ we derive the proposition by using the representation (16) and the fact that for $m \in \mathbb{N}$

$$
\begin{aligned}
\sigma_0(1, \ldots, m) = \tau_0(1, \ldots, m) &= 1, \\
\sigma_1(1, \ldots, m) = \tau_1(1, \ldots, m) &= \frac{1}{2} m(m+1), \\
\sigma_2(1, \ldots, m) &= \frac{1}{24}(m-1)m(m+1)(3m+2), \\
\tau_2(1, \ldots, m) &= \frac{1}{24} m(m+1)(m+2)(3m+1).
\end{aligned}
$$

□

In the following theorem we state a representation of $B_{n,\rho}^{(k)} e_\nu$ in terms of the images of monomials of the operators $B_n^{(k)}$. This underlines the close relationship beween the linking operators $B_{n,\rho}^{(k)}$ and the kth order Kantorovich modification of the classical operators B_n.

Theorem 3 *The images of the monomials under $B_{n,\rho}^{(k)}$ can be expressed as*

$$
(B_{n,\rho}^{(k)} e_\nu)(x) = \frac{\nu!}{(\nu+k)!} \frac{1}{(n\rho)^{c,\underline{\nu+k}}} \sum_{i=0}^{\nu} s_{\nu+k}^{i+k} (\rho n)^{i+k} \frac{(i+k)!}{i!} (B_n^{(k)} e_i)(x), \ \ k \in \mathbb{N}_0,
$$

where $s_{\nu+k}^{i+k}$ denote the Stirling numbers of first kind.

Proof For $\nu \in \mathbb{N}$ and $k = 0$ we derive from (14)

$$
\begin{aligned}
(B_{n,\rho} e_\nu)(x) &= \frac{1}{(n\rho)^{c,\underline{\nu}}} \sum_{j=1}^{\infty} p_{n,j}(x) \prod_{l=0}^{\nu-1} (j\rho + l) \\
&= \frac{1}{(n\rho)^{c,\underline{\nu}}} \sum_{i=0}^{\nu} s_\nu^i (\rho n)^i \sum_{j=1}^{\infty} p_{n,j}(x) \left(\frac{j}{n}\right)^i \\
&= \frac{1}{(n\rho)^{c,\underline{\nu}}} \sum_{i=0}^{\nu} s_\nu^i (\rho n)^i (B_n e_i)(x).
\end{aligned}
$$

For $k \in \mathbb{N}$ the conclusion follows by using $(B_{n,\rho}^{(k)} e_\nu) = \frac{\nu!}{(\nu+k)!} D^k(B_{n,\rho} e_{\nu+k})$ and $D^k(B_n e_i) = \frac{i!}{(i-k)!}(B_n^{(k)} e_{i-k})$, respectively. □

For the case $k = 0$ a corrresponding result for the Bernstein operators can be found in [17, Theorem 3.2.1].

3 Explicit Formulas for the Moments

Next, we consider the moments of $B_{n,\rho}$ and $B_{n,\rho}^{(k)}$. For abbreviation, we use the notation

$$M_{n,\rho,m}^{(k)}(x) = \left[B_{n,\rho}^{(k)}(e_1 - xe_0)^m\right](x),\ m \in \mathbb{N}_0,\ x \in [0,\infty) \tag{17}$$

where we again omit the superscript (k) in case $k = 0$. We use the fact that $M_{n,\rho,m}^{(k)}(x) = \sum_{\nu=0}^{m} \binom{m}{\nu} (-x)^{m-\nu} (B_{n,\rho}^{(k)} e_\nu)(x)$.
Again, we first treat the case $k = 0$.

Theorem 4 *Let $n \in \mathbb{R}$, $\rho \in \mathbb{R}_+$, $m \in \mathbb{N}_0$, $n\rho > c(m-1)$. Then*

$$M_{n,\rho,0}(x) = 1, \tag{18}$$

$$M_{n,\rho,1}(x) = 0, \tag{19}$$

$$\begin{aligned} M_{n,\rho,m}(x) = (-x)^m + \sum_{i=1}^{m} (-x)^i \sum_{\nu=1}^{i} \frac{\rho^{\nu+m-i}}{(n\rho)^{c,\underline{\nu+m-i}}} (-1)^\nu \binom{m}{i-\nu} \\ \times \frac{n^{c,\overline{\nu}}}{(\nu-1)!} \mathbf{\Delta}_1^{\nu-1} p_{\nu+m-i}^{\rho}(1),\ m \geq 2. \end{aligned} \tag{20}$$

Proof Equations (18) and (19) follow immediately from Corollary 2.

In order to prove (20) we apply Theorem 1. With the index transform $i \to i - m + \nu$, changing the order of summation and applying the index transform $\nu \to \nu + m - i$, we derive

$$\begin{aligned} &M_{n,\rho,m}(x) \\ &= (-x)^m + \sum_{\nu=1}^{m} \binom{m}{\nu} (-x)^{m-\nu} \frac{\rho^\nu}{(n\rho)^{c,\underline{\nu}}} \sum_{i=1}^{\nu} \frac{n^{c,\overline{i}}}{(i-1)!} \left(\mathbf{\Delta}_1^{i-1} p_\nu^\rho(1)\right) x^i \\ &= (-x)^m + \sum_{\nu=1}^{m} \binom{m}{\nu} (-1)^{m-\nu} \frac{\rho^\nu}{(n\rho)^{c,\underline{\nu}}} \\ &\quad \times \sum_{i=m-\nu+1}^{m} \frac{n^{c,\overline{i-m+\nu}}}{(i-m+\nu-1)!} \left(\mathbf{\Delta}_1^{i-m+\nu-1} p_\nu^\rho(1)\right) x^i \end{aligned}$$

$$
\begin{aligned}
&= (-x)^m + \sum_{i=1}^{m} x^i \sum_{\nu=m+1-i}^{m} \binom{m}{\nu} (-1)^{m-\nu} \frac{\rho^\nu}{(n\rho)^{c,\underline{\nu}}} \\
&\qquad \times \frac{n^{c,\overline{i-m+\nu}}}{(i-m+\nu-1)!} \left(\mathbf{\Delta}_1^{i-m+\nu-1} p_\nu^\rho(1) \right) \\
&= (-x)^m + \sum_{i=1}^{m} (-x)^i \sum_{\nu=1}^{i} \binom{m}{i-\nu} (-1)^\nu \frac{\rho^{\nu+m-i}}{(n\rho)^{c,\underline{\nu+m-i}}} \\
&\qquad \times \frac{n^{c,\overline{\nu}}}{(\nu-1)!} \left(\mathbf{\Delta}_1^{\nu-1} p_{\nu+m-i}^\rho(1) \right). \qquad \square
\end{aligned}
$$

Remark 3 Analogously as for the images of monomials, (20) can be rewritten as

$$
\begin{aligned}
M_{n,\rho,m}(x) &= (-x)^m + \sum_{i=1}^{m} (-x)^i \sum_{\nu=1}^{i} \frac{\rho^{\nu+m-i}}{(n\rho)^{\nu,\underline{\nu+m-i}}} n^{c,\overline{\nu}} \binom{m}{i-\nu} \\
&\qquad \times \sum_{\kappa=0}^{\nu-1} (-1)^{\kappa+1} \frac{1}{\kappa!(\nu-1-\kappa)!} p_{\nu+m-i}^\rho(1+\kappa).
\end{aligned}
$$

Next, we consider the special cases $\rho = 1$ and $\rho \to \infty$.
$\underline{\rho = 1}$: With [5, (3.48)]

$$
\mathbf{\Delta}_1^{\nu-1} p_{\nu+m-i}^1(1) = (\nu+m-i-1)! \binom{\nu+m-i}{\nu}.
$$

we get

$$
\begin{aligned}
M_{n,\rho,m}(x) &= (-x)^m + \sum_{i=1}^{m} (-x)^i \frac{m!}{i!} \sum_{\nu=1}^{i} (-1)^\nu \frac{n^{c,\overline{\nu}}}{n^{c,\underline{\nu+m-i}}} \\
&\qquad \times \binom{i}{\nu} \binom{\nu+m-i-1}{\nu-1},
\end{aligned}
$$

which coincides with the result in [18, Korollar 1.12] and with [7, Korollar 4.4] with $s = -1$ and $n + c$ instead of n there.
$\underline{\rho \to \infty}$: Then $\dfrac{\rho^{\nu+m-i}}{(n\rho)^{c,\underline{\nu+m-i}}} \to \dfrac{1}{n^{\nu+m-i}}$ and

$$
\mathbf{\Delta}_1^{\nu-1} p_{\nu+m-i}^\infty(1) = (\nu-1)! \sigma_{\nu+m-i}^\nu.
$$

Thus

$$M_{n,\infty,m}(x) = (-x)^m + \sum_{i=1}^{m}(-x)^i \sum_{\nu=1}^{i} \frac{n^{c,\overline{\nu}}}{n^{\nu+m-i}} \binom{m}{i-\nu} (-1)^\nu \sigma_{\nu+m-i}^{\nu}.$$

In our next theorem, we evaluate the moments for the case $k \in \mathbb{N}$.

Theorem 5 *Let $n \in \mathbb{R}$, $\rho \in \mathbb{R}_+$, $k \in \mathbb{N}$, $m \in \mathbb{N}_0$, $n\rho > c(m+k-1)$. Then*

$$M_{n,\rho,m}^{(k)}(x) = \sum_{i=0}^{m}(-x)^i \sum_{\nu=0}^{i} \frac{\rho^{\nu+m-i+k}}{(n\rho)^{c,\underline{\nu+m-i+k}}} (-1)^\nu \binom{m}{i-\nu} \times \frac{(\nu+m-i)!}{(\nu+m-i+k)!} \frac{(\nu+k)}{\nu!} n^{c,\overline{k+\nu}} \boldsymbol{\Delta}_1^{\nu+k-1} p_{\nu+m-i+k}^{\rho}(1). \tag{21}$$

Proof The result can be proved by using Theorem 2 and carrying out the same steps as in the proof of Theorem 4. □

Remark 4 With (10), we can rewrite the representation (21) as

$$M_{n,\rho,m}^{(k)}(x) = \sum_{i=0}^{m}(-x)^i \sum_{\nu=0}^{i} \frac{\rho^{\nu+m-i+k}}{(n\rho)^{c,\underline{\nu+m-i+k}}} \binom{m}{i-\nu} \frac{(\nu+m-i)!}{(\nu+m-i+k)!} \times n^{c,\overline{\nu+k}} \frac{(\nu+k)!}{\nu!} \sum_{\kappa=0}^{\nu+k-1} (-1)^{k+1+\kappa} \frac{1}{\kappa!(\nu+k-1-\kappa)!} p_{\nu+m-i+k}^{\rho}(1+\kappa).$$

From Theorem 5 we derive the following identity for the special cases $\rho = 1$ and $\rho \to \infty$.
$\underline{\rho = 1}$: With [5, (3.48)] we have

$$\boldsymbol{\Delta}_1^{\nu+k-1} p_{\nu+m-i+k}^{1}(1) = (\nu+m-i+k-1)! \binom{\nu+m-i+k}{\nu+k}.$$

Thus

$$M_{n,1,m}^{(k)}(x) = \sum_{i=0}^{m}(-x)^i \frac{m!}{i!} \sum_{\nu=0}^{i}(-1)^\nu \frac{n^{c,\overline{\nu+k}}}{n^{c,\underline{\nu+m-i+k}}} \binom{i}{\nu} \binom{\nu+m-i+k-1}{\nu+k-1}.$$

This coincides with the result [7, Korollar 4.4] for the moments of the auxiliary operators named $M_{n+c,k-1}$ there.
$\underline{\rho \to \infty}$: Then $\dfrac{\rho^{\nu+m-i+k}}{(n\rho)^{c,\underline{\nu+m-i+k}}} \to \dfrac{1}{n^{\nu+m-i+k}}$ and

$$\Delta_1^{\nu+k-1} p_{\nu+m-i+k}^{\infty}(1) = (\nu+k-1)!\sigma_{\nu+m-i+k}^{\nu+k}.$$

Thus

$$M_{n,\infty,m}^{(k)}(x) \\ = \sum_{i=0}^{m}(-x)^i \sum_{\nu=0}^{i} \frac{n^{c,\overline{\nu+k}}}{n^{\nu+m-i+k}} \binom{m}{i-\nu} (-1)^{\nu} \frac{(\nu+m-i)!(\nu+k)!}{(\nu+m-i+k)!\nu!} \sigma_{\nu+m-i+k}^{\nu+k}.$$

With the same notations and arguments used for Corollary 2, the moments (20) and (21) can be computed by using

$$\Delta_1^{\nu+k-1} p_{\nu+m-j+k}^{\rho}(1) \\ = (\nu+k-1)! \sum_{l=0}^{m-j} \rho^{-l} \sigma_l(1,2,\ldots,\nu+m-j+k-1)\tau_{m-j-l}(1,2,\ldots,\nu+k).$$

Corollary 3 *For $k \in \mathbb{N}_0$ the first moments are given by*

$$M_{n,\rho,0}^{(k)}(x) = \frac{\rho^k}{(n\rho)^{c,\underline{k}}} n^{c,\overline{k}}, \quad M_{n,\rho,1}^{(k)}(x) = \frac{\rho^{k+1}}{(n\rho)^{c,\underline{k+1}}} n^{c,\overline{k}} \frac{1}{2} k \left(1+\frac{1}{\rho}\right)(1+2cx),$$

$$M_{n,\rho,2}^{(k)}(x) = \frac{\rho^{k+2}}{(n\rho)^{c,\underline{k+2}}} n^{c,\overline{k}} \left(1+\frac{1}{\rho}\right) \left\{ \left[n + c\left(1+\frac{1}{\rho}\right) k(k+1) \right] x(1+cx) \right. \\ \left. + \frac{k}{12}\left[(3k+1)\left(1+\frac{1}{\rho}\right) + \frac{3k+5}{\rho} \right] \right\}.$$

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