Degree of Approximation of $f \in L[0, \infty)$ by Means of Fourier–Laguerre Series

Soshal Saini and Uaday Singh

Abstract In this paper, we determine the degree of approximation of functions belonging to $L[0, \infty)$ by the Hausdorff means of its Fourier–Laguerre series at $x = 0$. Our theorem extends some of the recent results of Nigam and Sharma [A study on degree of approximation by $(E, 1)$ summability means of the Fourier–Laguerre expansion, Int. J. Math. Math. Sci. (2010), Art. ID 351016, 7], Krasniqi [On the degree of approximation of a function by $(C, 1)(E, q)$ means of its Fourier–Laguerre series, International Journal of Analysis and Applications 1 (2013), 33–39] and Sonker [Approximation of Functions by $(C, 2)(E, q)$ means of its Fourier–Laguerre series, Proceeding of ICMS-2014 ISBN 978-93-5107-261-4:125–128.] in the sense that the summability methods used by these authors have been replaced by the Hausdorff matrices.

Keywords Degree of approximation · Hausdorff means · Fourier–Laguerre series

1 Introduction

Let f be a function belonging to $L[0, \infty)$ in the sense that f is Labesgue integrable in the interval $[0, \infty)$. The Fourier–Laguerre expansion of f is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(x), \tag{1}$$

S. Saini (✉) · U. Singh
Department of Mathematics, Indian Institute of Technology Roorkee,
Roorkee 247667, India
e-mail: sainisoshal25@gmail.com

U. Singh
e-mail: usingh2280@yahoo.co.in

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where

$$\Gamma(\alpha+1)\binom{n+\alpha}{n}a_n = \int_0^\infty e^{-x}x^\alpha f(x)L_n^{(\alpha)}(x)dx \tag{2}$$

and $L_n^{(\alpha)}(x)$ denotes the nth Laguerre polynomial of order $\alpha > -1$, defined by the generating function

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\omega^n = (1-\omega)^{-\alpha-1}\exp\left(\frac{-x\omega}{1-\omega}\right). \tag{3}$$

When $x = 0$,

$$L_n^{(\alpha)}(0) = \binom{n+\alpha}{n} \text{ [9].}$$

The nth partial sums of (1) are defined by

$$s_n(f;x) = \sum_{k=0}^{n} a_k L_k^{(\alpha)}(x). \tag{4}$$

The Cesàro means of order λ of the Fourier–Laugerre series are defined by

$$C_n^\lambda(f;x) = \frac{1}{\binom{n+\lambda}{n}}\sum_{k=0}^{n}\binom{\lambda+n-k-1}{n-k}s_k(f;x).$$

The Euler means of the Fourier–Laugerre series are defined by

$$E_n^q(f;x) = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}s_k(f;x), \quad q > 0.$$

The Hausdorff matrix $H \equiv (h_{n,k})$ is an infinite lower triangular matrix defined by

$$h_{n,k} = \begin{cases} \binom{n}{k}\triangle^{n-k}\mu_k & , 0 \le k \le n, \\ 0 & , \ k > n, \end{cases}$$

where $\triangle$ is the forward difference operator defined by $\triangle\mu_n = \mu_n - \mu_{n+1}$ and $\triangle^{k+1}\mu_n = \triangle^k(\triangle\mu_n)$. If H is regular, then $\{\mu_n\}$, known as moment sequence, has the representation

$$\mu_n = \int_0^1 u^n d\gamma(u),$$

where $\gamma(u)$, known as mass function, is continuous at $u = 0$ and belongs to $BV[0, 1]$ such that $\gamma(0) = 0, \gamma(1) = 1$; and for $0 < u < 1$, $\gamma(u) = [\gamma(u+0) + \gamma(u-0)]/2$ [11].

The Hausdorff means of the Fourier–Laugerre series are defined by

$$H_n(f; x) := \sum_{k=0}^{n} h_{n,k} s_k(f; x), \quad n = 0, 1, 2, ... \tag{5}$$

The Fourier–Laugerre series is said to be summable to s by the Hausdorff means, if $H_n(f; x) \to s \text{ as } n \to \infty$, [3].

For the examples of Hausdorff matrices, one can see [7, 8, 11] and references therein.

In this paper, the class of all regular Hausdorff matrices with moment sequence $\{\mu_n\}$ associated with mass function $\gamma(u)$ having constant derivative, is denoted by H_1.

We also write

$$\varphi(y) = \frac{e^{-y} y^{\alpha}(f(y) - f(0))}{\Gamma(\alpha+1)},$$

and

$$g(u, y) = \sum_{k=0}^{n} \binom{n}{k} u^k (1-u)^{n-k} L_k^{(\alpha+1)}(y).$$

2 Known Results

Gupta [2] obtained the degree of approximation of $f \in L[0, \infty)$ by Cesàro means of order k of the Fourier–Laguerre series at the point $x = 0$, where $k > \alpha + 1/2$. Nigam and Sharma [5] have used $(E, 1)$ means of the Fourier–Laguerre series for $-1 < \alpha < 1/2$ which is more appropriate range from the application point of view. The authors have proved the following result:

Theorem A *If*

$$E_n^1 = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} s_k \quad \to \infty \text{ as } n \to \infty,$$

then the degree of approximation of Fourier–Laguerre expansion at the point $x = 0$ *by* $(E, 1)$ *means* E_n^1 *is given by*

$$E_n^1(0) - f(0) = o(\xi(n)), \tag{6}$$

provided that

$$\Phi(t) = \int_0^t |\varphi(y)|dy = o\left(t^{\alpha+1}\xi(1/t)\right), \quad t \to 0, \tag{7}$$

$$\int_\delta^n e^{y/2} y^{-((2\alpha+3)/4)} |\varphi(y)|dy = o\left(n^{-((2\alpha+1)/4)}\xi(n)\right), \tag{8}$$

$$\int_n^\infty e^{y/2} y^{-1/3} |\varphi(y)|dy = o(\xi(n)), \quad n \to \infty, \tag{9}$$

where δ is a fixed positive constant and $\alpha \in (-1, -1/2)$, and $\xi(t)$ is a positive monotonic increasing function of t such that $\xi(n) \to \infty$ as $n \to \infty$.

Following, Nigam and Sharma [5], Krasniqi [4] has used the $(C, 1)(E, q)$ means of the Fourier–Laguerre series to obtain the degree of approximation of $f \in L[0, \infty)$ at point $x = 0$ and has proved the following result:

Theorem B *The degree of approximation of the Fourier–Laguerre expansion at the point $x = 0$ by the $[(C, 1)(E, q)]_n$ means is given by*

$$[(C, 1)(E, q)]_n(0) - f(0) = o(\xi(n)), \tag{10}$$

provided that the conditions (7)–(9) given in Theorem A are satisfied.

Recently, Sonker [10] has also proved the same result using $[(C, 2)(E, q)]_n$ means of the Fourier–Laguerre series for the same range of α as follows:

Theorem C *The degree of approximation of the Fourier–Laguerre expansion at the point $x = 0$ by the $[(C, 2)(E, q)]_n$ means is given by*

$$[(C, 2)(E, q)]_n(0) - f(0) = o(\xi(n)), \tag{11}$$

provided that the conditions (7)–(9) given in Theorem A are satisfied.

Remark 1 We observe that Krasniqi [4, p. 37] has optimized $\sum_{k=0}^{\nu} \binom{\nu}{k} q^k k^{(2\alpha+1)/4}$ by its maximum value $(1+q)^\nu \nu^{(2\alpha+1)/4}$. This is possible only when $\alpha > -1/2$. But the author has used $-1 < \alpha < 1/2$ [4, p. 35, Theorem 2.1]. Similar error can also be seen in [5, 10].

3 Main Results

In this paper, we extend the above results using the Hausdorff means, which is a more general summability method, for an appropriate range of α. More precisely, we prove the following:

Theorem 1 *The degree of approximation of $f \in L[0, \infty)$ at the point $x = 0$ by the Hausdorff means of the Fourier–Laguerre series generated by $H \in H_1$ is given by*

$$H_n(f; 0) - f(0) = o(\xi(n)) \tag{12}$$

where $\xi(t)$ is a positive increasing function such that $\xi(t) \to \infty$ as $t \to \infty$ and satisfies the following conditions

$$\Phi(y) = \int_0^t |\varphi(y)| dy = o\left(t^{\alpha+1}\xi(1/t)\right), \quad t \to 0, \tag{13}$$

$$\int_\delta^n e^{y/2} y^{-((2\alpha+3)/4)} |\varphi(y)| dy = o\left(n^{-((2\alpha+1)/4)}\xi(n)\right), \tag{14}$$

and

$$\int_n^\infty e^{y/2} y^{-1/3} |\varphi(y)| dy = o(\xi(n)), \quad n \to \infty, \tag{15}$$

where δ is a fixed positive constant and $\alpha > -1/2$.

For the proof of our theorem, we need the following lemmas:

Lemma 1 [9, *p*. 177]. *Let α be an arbitrary real number, c and δ be fixed positive constants. Then*

$$L_n^{(\alpha)}(x) = \begin{cases} O\left(n^{(\alpha)}\right), & \text{if } 0 \le x \le \frac{c}{n}, \\ O\left(x^{-(2\alpha+1)/4} n^{(2\alpha-1)/4}\right), & \text{if } \frac{c}{n} \le x \le \delta, \end{cases} \tag{16}$$

as $n \to \infty$.

Lemma 2 [9, *p*. 240]. *Let α be an arbitrary real number, $\delta > 0$ and $0 < \eta < 4$. Then*

$$\max e^{-x/2} x^{(\alpha/2+1/4)} |L_n^{(\alpha)}(x)| = \begin{cases} O\left(n^{(\alpha/2-1/4)}\right), & \text{if } \delta \le x \le (4-\eta)n, \\ O\left(n^{(\alpha/2-1/12)}\right), & \text{if } x \ge \delta, \end{cases} \tag{17}$$

as $n \to \infty$.

Lemma 3 *For $0 < u < 1$ and $0 \le y \le \delta$,*

$$\left|\int_0^1 g(u, y) d\gamma(u)\right| = \begin{cases} O\left(n^{(\alpha+1)}\right), & \text{if } 0 \le y \le \frac{1}{n}, \\ O\left(y^{-(2\alpha+3)/4} n^{(2\alpha+1)/4}\right), & \text{if } \frac{1}{n} \le y \le \delta, \end{cases} \tag{18}$$

as $n \to \infty$.

Proof The $g(u, y)$ can be written as

$$g(u, y) = (1-u)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{u}{1-u}\right)^k L_k^{(\alpha+1)}(y).$$

Then

$$\begin{aligned}
\left|\int_0^1 g(u, y)d\gamma(u)\right| &= \left|\int_0^1 (1-u)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{u}{1-u}\right)^k L_k^{(\alpha+1)}(y)d\gamma(u)\right| \\
&= \left|M\int_0^1 (1-u)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{u}{1-u}\right)^k L_k^{(\alpha+1)}(y)du\right|
\end{aligned}$$

Now, using Lemma 1 for $0 \le y \le \frac{1}{n}$ (taking $\alpha+1$ for α and $c = 1$), we have

$$\begin{aligned}
\left|\int_0^1 g(u, y)d\gamma(u)\right| &= \int_0^1 (1-u)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{u}{1-u}\right)^k O(k^{\alpha+1})du \\
&= O\left(n^{\alpha+1}\int_0^1 (1-u)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{u}{1-u}\right)^k du\right) \\
&= O\left(n^{\alpha+1}\int_0^1 (1-u)^n du\right) \\
&= O\left(n^{\alpha+1}\right). \qquad (19)
\end{aligned}$$

Again, using Lemma 1 for $\frac{1}{n} \le y \le \delta$, we have

$$\begin{aligned}
\left|\int_0^1 g(u, y)d\gamma(u)\right| &= \int_0^1 (1-u)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{u}{1-u}\right)^k O\left(y^{-(2\alpha+3)/4}k^{(2\alpha+1)/4}\right) du \\
&= O\left(y^{-(2\alpha+3)/4}n^{(2\alpha+1)/4}\int_0^1 (1-u)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{u}{1-u}\right)^k du\right) \\
&= O\left(y^{-(2\alpha+3)/4}n^{(2\alpha+1)/4}\right). \qquad (20)
\end{aligned}$$

Collecting (19) and (20), the proof of Lemma 3 is completed.

Lemma 4 *For* $0 < u < 1$,

$$\left|\int_0^1 g(u, y)d\gamma(u)\right| = \begin{cases} O\left(e^{y/2}y^{-(2\alpha+3)/4}n^{(2\alpha+1)/4}\right), & \text{if } \delta \le y \le n, \\ O\left(e^{y/2}y^{-(3\alpha+5)/6}n^{(\alpha+1)/2}\right), & \text{if } y \ge \delta, \end{cases} \qquad (21)$$

as $n \to \infty$.

Proof Following the Lemma 3, we have

$$\left|\int_0^1 g(u, y)d\gamma(u)\right| = \left|\int_0^1 (1-u)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{u}{1-u}\right)^k L_k^{(\alpha+1)}(y)du\right|$$

Now, using Lemma 2 for $\delta \le y \le n$ (taking $\alpha + 1$ for α and $\eta = 3$), we have

$$\begin{aligned}\left|\int_0^1 g(u, y)d\gamma(u)\right| &= \left|\int_0^1 e^{(y/2)} y^{-(2\alpha+3)/4} (1-u)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{u}{1-u}\right)^k e^{-(y/2)} y^{(2\alpha+3)/4} L_k^{(\alpha+1)}(y)du\right| \\ &= \int_0^1 e^{y/2} y^{-(2\alpha+3)/4} (1-u)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{u}{1-u}\right)^k O\left(k^{(2\alpha+1)/4}\right) du \\ &= O\left(e^{y/2} y^{-(2\alpha+3)/4} n^{(2\alpha+1)/4}\right). \end{aligned} \tag{22}$$

Again, using Lemma 2 for $y \ge n$, we have

$$\begin{aligned}\left|\int_0^1 g(u, y)d\gamma(u)\right| &= \left|\int_0^1 e^{(y/2)} y^{-(3\alpha+5)/6} (1-u)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{u}{1-u}\right)^k e^{-(y/2)} y^{(3\alpha+5)/6} L_k^{(\alpha+1)}(y)du\right| \\ &= \int_0^1 e^{y/2} y^{-(3\alpha+5)/6} (1-u)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{u}{1-u}\right)^k O\left(k^{(\alpha+1)/2}\right) du \\ &= O\left(e^{y/2} y^{-(3\alpha+5)/6} n^{(\alpha+1)/2}\right). \end{aligned} \tag{23}$$

Collecting (22) and (23), the proof of Lemma 4 is completed.

Proof of Theorem 1 We have

$$\begin{aligned} s_n(0) &= \sum_{k=0}^{n} a_k L_k^{(\alpha)}(0) \\ &= \sum_{k=0}^{n} \frac{1}{\Gamma(\alpha+1)\binom{n+\alpha}{n}} \left(\int_0^\infty e^{-y} y^\alpha f(y) L_k^{(\alpha)}(y)dy\right) L_k^{(\alpha)}(0) \\ &= \frac{1}{\Gamma(\alpha+1)} \int_0^\infty e^{-y} y^\alpha f(y) \sum_{k=0}^{n} L_k^{(\alpha)}(y)dy \\ &= \frac{1}{\Gamma(\alpha+1)} \int_0^\infty e^{-y} y^\alpha f(y) L_n^{(\alpha+1)}(y)dy, \end{aligned}$$

so that

$$\begin{aligned} H_n(f; 0) &= \sum_{k=0}^{n} h_{n,k} s_k(0) \\ &= \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k} \mu_k \frac{1}{\Gamma(\alpha+1)} \int_0^\infty e^{-y} y^\alpha f(y) L_k^{(\alpha+1)}(y)dy. \end{aligned}$$

Thus

$$\begin{aligned}
H_n(f;0)-f(0) &= \sum_{k=0}^{n}\binom{n}{k}\Delta^{n-k}\mu_k\left(\frac{1}{\Gamma(\alpha+1)}\int_0^\infty e^{-y}y^\alpha f(y)L_k^{(\alpha+1)}(y)dy - f(0)\right)\\
&= \sum_{k=0}^{n}\binom{n}{k}\Delta^{n-k}\mu_k\frac{1}{\Gamma(\alpha+1)}\int_0^\infty e^{-y}y^\alpha (f(y)-f(0))L_k^{(\alpha+1)}(y)dy\\
&= \sum_{k=0}^{n}\binom{n}{k}\Delta^{n-k}\mu_k\int_0^\infty \varphi(y)L_k^{(\alpha+1)}(y)dy\\
&= \int_0^\infty \varphi(y)\left(\sum_{k=0}^{n}\binom{n}{k}\Delta^{n-k}\mu_k L_k^{(\alpha+1)}(y)\right)dy\\
&= \int_0^\infty \varphi(y)\left(\sum_{k=0}^{n}\binom{n}{k}\int_0^1 u^k(1-u)^{n-k}d\gamma(u)L_k^{(\alpha+1)}(y)\right)dy\\
&= \int_0^\infty \varphi(y)\left(\int_0^1\sum_{k=0}^{n}\binom{n}{k} u^k(1-u)^{n-k}L_k^{(\alpha+1)}(y)d\gamma(u)\right)dy\\
&= \int_0^\infty \varphi(y)\left(\int_0^1 g(u,y)d\gamma(u)\right)dy
\end{aligned}$$

and

$$\begin{aligned}
|H_n(f;0)-f(0)| &= \left|\int_0^\infty \varphi(y)\left(\int_0^1 g(u,y)d\gamma(u)\right)dy\right|\\
&\le \int_0^\infty |\varphi(y)|\left|\int_0^1 g(u,y)d\gamma(u)\right|dy\\
&= \left(\int_0^{1/n}+\int_{1/n}^{\delta}+\int_{\delta}^{n}+\int_n^\infty\right)\left(|\varphi(y)|\left|\int_0^1 g(u,y)d\gamma(u)\right|dy\right)\\
&= I_1+I_2+I_3+I_4. \qquad (24)
\end{aligned}$$

Now, using Lemma 3 for $0 \le y \le \frac{1}{n}$, we have

$$\begin{aligned}
I_1 &= \int_0^{1/n}|\varphi(y)|\left|\int_0^1 g(u,y)d\gamma(u)\right|dy\\
&= O\left(n^{\alpha+1}\right)\int_0^{1/n}|\varphi(y)|dy\\
&= O(n^{(\alpha+1)})o\left(\left(\frac{1}{n}\right)^{\alpha+1}\xi(n)\right)\\
&= o(\xi(n)), \qquad (25)
\end{aligned}$$

in view of condition (13).

Further, using Lemma 3 for $\frac{1}{n} \leq y \leq \delta$, we have,

$$\begin{aligned} I_2 &= \int_{1/n}^{\delta} |\varphi(y)| O\left(y^{-(2\alpha+3)/4} n^{(2\alpha+1)/4}\right) dy \\ &= O\left(n^{(2\alpha+1)/4}\right)\left(\int_{1/n}^{\delta} y^{-(2\alpha+3)/4} |\varphi(y)| dy\right). \end{aligned}$$

Following [5, p. 6], we have

$$I_2 = o(\xi(n)), \tag{26}$$

in view of condition (13).

Now, using Lemma 4 for $\delta \leq y \leq n$, we have

$$\begin{aligned} I_3 &= \int_{\delta}^{n} |\varphi(y)| \left|\int_0^1 g(u, y) d\gamma(u)\right| dy \\ &= \int_{\delta}^{n} O\left(e^{y/2} y^{-((2\alpha+3)/4)} n^{(2\alpha+1)/4}\right) |\varphi(y)| dy \\ &= O\left(n^{(2\alpha+1)/4}\right)\left(\int_{\delta}^{n} e^{y/2} y^{-((2\alpha+3)/4)} |\varphi(y)| dy\right) \\ &= O\left(n^{(2\alpha+1)/4}\right) o\left((n^{-(2\alpha+1)/4}) \xi(n)\right) \\ &= o(\xi(n)), \end{aligned} \tag{27}$$

in view of condition (14).

Further, using Lemma 4, we have

$$\begin{aligned} I_4 &= \int_{n}^{\infty} |\varphi(y)| \left|\int_0^1 g(u, y) d\gamma(u)\right| dy \\ &= \int_{n}^{\infty} |\varphi(y)| O\left(e^{y/2} y^{-(3\alpha+5)/6} n^{(\alpha+1)/2}\right) dy \\ &= O\left(n^{(\alpha+1)/2}\right)\left(\int_{n}^{\infty} \frac{e^{y/2} y^{-1/3} |\varphi(y)|}{y^{(\alpha+1)/2}} dy\right) \\ &= o\left((\xi(n)) n^{(\alpha+1)/2}\left(n^{-(\alpha+1)/2}\right)\right) \\ &= o(\xi(n)), \end{aligned} \tag{28}$$

in view of condition (15).

Collecting (24)–(28), we have

$$H_n(f; 0) - f(0) = o(\xi(n)).$$

Hence the proof of Theorem 1 is completed.

4 Corollaries

The following corollaries can be derived from our Theorem 1.

Corollary 1 *As discussed in* [7, *p.* 306, *Lemma* 1] *and* [11, *p.* 38]*, if we take the mass function* $\gamma(u)$ *given by*

$$\gamma(u) = \begin{cases} 0, & 0 \leq u \leq a, \\ 1, & a \leq u \leq 1, \end{cases}$$

where $a = \frac{1}{(1+q)}, q > 0$*, the Hausdorff matrix H reduces to Euler matrix* $(E, q), q > 0$ *and defines the corresponding* (E, q) *means given by*

$$E_q^n(f; x) = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k(f; x), \quad q > 0.$$

Hence the Theorem 1 *reduces to Theorem A (result proved by Nigam and Sharma* [5, *p.* 3, *Theorem* 2.1]*).*

Corollary 2 *As discussed in* [1, *p.* 400] *and* [6, *p.* 2747]*, the Cesàro matrix of order* λ*, is also a Hausdorff matrix obtained by mass function* $\gamma(u) = 1 - (1-u)^{\lambda}$ *and the corresponding Cesàro means are given by*

$$C_n^{\lambda}(f; x) = \frac{1}{\binom{n+\lambda}{n}} \sum_{k=0}^{n} \binom{\lambda + n - k - 1}{n-k} s_k(f; x).$$

Further, Rhoades [7, *p.* 308] *and Rhoades et al.* [8, *p.* 6869] *has mentioned that the product of two Hausdorff matrices is again a Hausdorff matrix. Hence the Theorem B and Theorem C (results proved by Krasniqi* [4, *p.* 35, *Theorem* 2.1] *and Sonker* [10, *p.* 126, *Theorem* 1]*) are also particular cases of our Theorem* 1.

Remark 2 This is an open problem to associate the above discussed results with the L^p-spaces.

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