Applications of Generalized Monotonicity to Variational-Like Inequalities and Equilibrium Problems

N.K. Mahato and R.N. Mohapatra

Abstract In this paper, we introduce the concept of relaxed (ρ-θ)-η-invariant monotonicity to establish the existence of solutions for variational-like inequality problems in reflexive Banach spaces. Again we introduce the concept of (ρ-θ)-monotonicity for bifunctions. The existence of solution for equilibrium problem with (ρ-θ)-monotonicity is established by using the KKM technique.

Keywords Variational-like inequality problem · Relaxed (ρ-θ)-η-invariant monotonicity · Equilibrium problem · (ρ-θ)-monotonicity · KKM mappping

1 Introduction

Let K be a nonempty subset of a real reflexive Banach space X, and X^* be the dual space of X. Consider the operator $T : K \to X^*$ and the bifunction $\eta : K \times K \to X$. Then the variational-like inequality problem (in short, VLIP) is to find $x \in K$, such that

$$\langle Tx, \eta(y, x)\rangle \geq 0, \forall y \in K, \tag{1}$$

where $\langle ., .\rangle$ denote the pairing between X and X^*.

If we take $\eta(x, y) = x - y$, then (1) becomes to find $x \in K$, such that

$$\langle Tx, y - x)\rangle \geq 0, \forall y \in K, \tag{2}$$

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which are variational inequality problems (VIP) [1, 2]. Variational inequalities have been studied by many authors [1–5] in both finite- and infinite-dimensional spaces. When we deal with variational inequalities, the most common assumption for the operator T is monotonicity. Recently, many authors have established the existence of solutions for variational inequalities with various types of generalized monotonicity assumptions (see [3, 5–8] and the references therein). Fang and Huang [5] defined the concept of relaxed η-α monotonicity and obtained the existence of solutions for variational-like inequalities. Bai et al. [3] extended the idea of relaxed η-α monotonicity to relaxed η-α pseudomonotonicity. Yang et al. [9] defined several kinds of invariant monotone maps and generalized invariant monotone maps. Behera et al. [10] defined various concepts of generalized $(\rho$-$\theta)$-η-invariant monotonicity to generalized concepts of Yang et al. [9]. Very recently, Mahato and Nahak [11] introduced relaxed $(\rho$-$\theta)$-η-invariant pseudomonotonicity to study variational-like inequalities and $(\rho$-$\theta)$-pseudomonotonicity to study equilibrium problems. But in [11], authors did not consider the concepts such as relaxed $(\rho$-$\theta)$-η-invariant monotone mappings, and $(\rho$-$\theta)$-monotone bifunctions. Therefore, we organized this article to consider these monotonicity concepts and study the variational-like inequality problems and equilibrium problem.

Inspired and motivated by [5, 9–11], in this paper, we introduce the concept of relaxed $(\rho$-$\theta)$-η-invariant monotone mappings to establish the existence of solutions for variational-like inequality problems. We also introduce the notion of $(\rho$-$\theta)$-monotonicity for bifunctions. By using the KKM technique we have studied the existence of solutions of equilibrium problem with $(\rho$-$\theta)$-monotone mappings in reflexive Banach spaces.

2 Preliminaries

We begin with the definition of relaxed $(\rho$-$\theta)$-η-invariant monotone mappings. For this consider the function $\theta : K \times K \to \mathbb{R}$ and $\rho \in \mathbb{R}$.

Definition 1 The operator $T : K \to X^*$ is said to be relaxed $(\rho$-$\theta)$-η-invariant monotone with respect to θ, if for any pair of distinct points $x, y \in K$, we have

$$\langle Tx, \eta(y, x)\rangle + \langle Ty, \eta(x, y)\rangle + \rho|\theta(x, y)|^2 \leq 0, \ where\ \theta(x, y) = \theta(y, x). \quad (3)$$

Remark 1 (i) If we take $\rho = 0$ then from (3) it follows that $\langle Tx, \eta(y, x)\rangle + \langle Ty, \eta(x, y)\rangle \leq 0, \forall x, y \in K$, and T is said to be invariant monotone, see [9].

(ii) If we take $\rho = 0$, and $\eta(x, y) = x - y$, then (3) reduces to $\langle Tx - Ty, x - y\rangle \geq 0, \forall x, y \in K$, and T is said to be monotone map.

From the above definitions, it is clear that **invariant monotonicity** $\Rightarrow$ **relaxed $(\rho$-$\theta)$-η-invariant monotonicity**. However, in general a relaxed $(\rho$-$\theta)$-η-invariant monotone map may not be an invariant monotone map.

Example 1 Let $K = [1, 5]$ and $T : [1, 5] \to \mathbb{R}$ be defined by $Tx = x^2 + 1$. Let the functions η and θ be defined by $\eta(x, y) = x^2 + y^2$, $\theta(x, y) = (x^2 + y^2)(x^2 + y^2 + 5)$. Now, $\langle Tx, \eta(y, x)\rangle + \langle Ty, \eta(x, y)\rangle = (x^2 + y^2)(x^2 + y^2 + 2)$, which is not less than 0. Therefore, T is not invariant monotone. But, T is relaxed $(\rho\text{-}\theta)$-η-invariant monotone with respect to θ for any $\rho < 1$.

Definition 2 [5] The operator $T : K \to X^*$ is said to be η-hemicontinuous if for any fixed $x, y \in K$, the mapping $f : [0, 1] \to \mathbb{R}$ defined by $f(t) = \langle T(x + t(y - x)), \eta(y, x)\rangle$ is continuous at 0^+.

3 Relaxed $(\rho\text{-}\theta)$-η-Invariant Monotonicity and (VLIP)

In this section, we establish the existence of the solution for (VLIP), using relaxed $(\rho\text{-}\theta)$-η-invariant monotonicity. Consider the following problems:

$$\text{find } x \in K \text{ such that } \langle Ty, \eta(x, y)\rangle + \rho|\theta(x, y)|^2 \le 0, \forall y \in K. \tag{4}$$

Theorem 1 *Let K be a closed convex subset of a reflexive Banach space X. Assume that $T : K \to X^*$ is η-hemicontinuous and relaxed $(\rho\text{-}\theta)$-η-invariant monotone with the following conditions:*

(i) $\eta(x, y) + \eta(y, x) = 0, \forall x, y \in K$;
(ii) $\lim_{t\to 0} \frac{|\theta(x, x_t)|^2}{t} = 0$, where $x_t = ty + (1 - t)x, \forall x, y \in K$;
(iii) for a fixed $z, y \in K$, the mapping $x \mapsto \langle Tz, \eta(x, y)\rangle$ is convex.

Then the Problems (1) and (4) are equivalent.

Proof Let x be a solution of (1). From the definition of relaxed $(\rho\text{-}\theta)$-η-invariant monotonicity of T, we get $\langle Ty, \eta(x, y)\rangle + \rho|\theta(x, y)|^2 \le -\langle Tx, \eta(y, x)\rangle \le 0$. Conversely, suppose that $x \in K$ is a solution of (4), i.e.,

$$\langle Ty, \eta(x, y)\rangle + \rho|\theta(x, y)|^2 \le 0, \forall y \in K. \tag{5}$$

Choose any point $y \in K$ and consider $x_t = ty + (1 - t)x, t \in (0, 1]$, then $x_t \in K$. Therefore, from (5) we have

$$\begin{aligned} &\langle Tx_t, \eta(x, x_t)\rangle + \rho|\theta(x, x_t)|^2 \le 0;\\ \Rightarrow &\langle Tx_t, \eta(x_t, x)\rangle - \rho|\theta(x, x_t)|^2 \ge 0;\\ \Rightarrow &\langle Tx_t, \eta(x_t, x)\rangle \ge \rho|\theta(x, x_t)|^2. \end{aligned} \tag{6}$$

Now, $$\langle Tx_t, \eta(x_t, x)\rangle \le t\langle Tx_t, \eta(y, x)\rangle + (1-t)\langle Tx_t, \eta(x, x)\rangle = t\langle Tx_t, \eta(y, x)\rangle. \tag{7}$$

From (6) and (7) we have
$\langle Tx_t, \eta(y,x)\rangle \geq \rho\frac{|\theta(x,x_t)|^2}{t}$.
Since T is η-hemicontinuous and taking $t \to 0$ we get
$\langle Tx, \eta(y,x)\rangle \geq 0, \forall y \in K$.

Definition 3 Let $f : K \to 2^X$ be a set-valued mapping. Then f is said to be KKM mapping if for any $\{y_1, y_2, \ldots, y_n\}$ of K we have $co\{y_1, y_2, \ldots, y_n\} \subset \bigcup_{i=1}^{n} f(y_i)$, where $co\{y_1, y_2, \ldots, y_n\}$ denotes the convex hull of $y_1, y_2, \ldots, y_n$.

Lemma 1 ([12]) *Let M be a nonempty subset of a Hausdorff topological vector space X and let $f : M \to 2^X$ be a KKM mapping. If $f(y)$ is closed in X, for all $y \in M$ and compact for some $y \in M$, then $\bigcap_{y\in M} f(y) \neq \emptyset$.*

Theorem 2 Let K be a nonempty bounded closed convex subset of a real reflexive Banach space X. Assume that $T : K \to X^*$ is η-hemicontinuous and relaxed (ρ-θ)-η-invariant monotone. Let the following hold:

(i) $\eta(x,y) + \eta(y,x) = 0, \forall x, y \in K$;
(ii) $\lim_{t\to 0} \dfrac{|\theta(x,x_t)|^2}{t} = 0$, where $x_t = ty + (1-t)x,\ \forall x, y \in K$; and θ is lower semicontinuous in the first argument;
(iii) for a fixed $z, y \in K$, the mapping $x \mapsto \langle Tz, \eta(x,y)\rangle$ is convex and lower semicontinuous.

Then the Problem (1) has a solution.

Proof Consider the set-valued mapping $F : K \to 2^X$ such that
$F(y) = \{x \in K : \langle Tx, \eta(y,x)\rangle \geq 0\}, \forall y \in K$.
It is easy to see that $\overline{x} \in K$ solves the (VLIP) if and only if $\overline{x} \in \cap_{y\in K} F(y)$. We claim that F is a KKM mapping. If possible, let F not be a KKM mapping. Then there exists $\{x_1, x_2, \ldots, x_m\} \subset K$ such that $co\{x_1, x_2, \ldots, x_m\}$ not contained in $\cup_{i=1}^m F(x_i)$, that means there exists a $x_0 \in co\{x_1, x_2, \ldots, x_m\}$, $x_0 = \sum_{i=1}^{m} t_i x_i$ where

$t_i \geq 0, i = 1, 2, \ldots, m, \sum_{i=1}^{m} t_i = 1$, but $x_0 \notin \cup_{i=1}^m F(x_i)$.

Hence, $\langle Tx_0, \eta(x_i, x_0)\rangle < 0$; for $i = 1, 2, \ldots, m$. From (i) and (iii) it follows that

$$0 = \langle Tx_0, \eta(x_0,x_0)\rangle \leq \sum_{i=1}^{m} t_i \langle Tx_0, \eta(x_i,x_0)\rangle < 0,$$

which is a contradiction. Hence F is a KKM mapping.

Assume $G : K \to 2^X$ such that $G(y) = \{x \in K : \langle Ty, \eta(x,y)\rangle + \rho|\theta(x,y)|^2 \leq 0\}, \forall y \in K$.

From the relaxed $(\rho\text{-}\theta)$-η-invariant monotonicity of T it follows that $F(y) \subset G(y), \ \forall y \in K$. Therefore, G is also a KKM mapping.

Since K is closed bounded and convex, it is weakly compact. From the assumptions, we know that $G(y)$ is weakly closed for all $y \in K$. In fact, because $x \mapsto \langle Tz, \eta(x, y)\rangle$ and $x \mapsto \rho|\theta(x, y)|^2$ are lower semicontinuous. Therefore, $G(y)$ is weakly compact in K, for each y ∈ K.
Therefore, from Lemma 1 and Theorem 1 it follows that $\bigcap_{y\in K} F(y) = \bigcap_{y\in K} G(y) \neq \emptyset$.
So there exists $\overline{x} \in K$ such that $\langle T\overline{x}, \eta(y, \overline{x})\rangle \geq 0, \forall y \in K$, i.e., the Problem (1) has a solution.

Theorem 3 Let K be a nonempty unbounded closed convex subset of a real reflexive Banach space X. Suppose that $T : K \to X^*$ is η-hemicontinuous and relaxed $(\rho\text{-}\theta)$-η-invariant monotone. Let the following hold:

(i) $\eta(x, y) + \eta(y, x) = 0, \forall x, y \in K$;
(ii) $\lim_{t\to 0} \dfrac{|\theta(x, x_t)|^2}{t} = 0$, where $x_t = ty + (1 - t)x, \ \forall x, y \in K$; and θ is lower semicontinuous in the first argument;
(iii) for a fixed $z, y \in K$, the mapping $x \mapsto \langle Tz, \eta(x, y)\rangle$ is convex and lower semicontinuous;
(iv) T is weakly η-coercive, i.e., there exits $x_0 \in K$ such that $\langle Tx, \eta(x, x_0)\rangle > 0$, whenever $\|x\| \to \infty$ and $x \in K$.

Then the Problem (1) has solution.

Proof Since the proof of this theorem is very similar to Theorem 3 in [11], hence it is omitted.

4 $(\rho\text{-}\theta)$-Monotonicity and Equilibrium Problem

The equilibrium problem (in short, EP) for the bifunction $f : K \times K \to \mathbb{R}$ is to find $\overline{x} \in K$, such that

$$f(\overline{x}, y) \geq 0, \forall y \in K. \tag{8}$$

Problems like (8) were initially studied by Fan [13]. Later on Blum and Oettli [4] discussed that equilibrium problem contains many problems as particular cases for example, mathematical programming problems, complementary problems, variational inequality problems, fixed-point problems, and minimax inequality problems. Inspired and motivated by [11, 14], we introduced the concept of $(\rho\text{-}\theta)$-monotonicity to establish the existence of solution of equilibrium problem over bounded as well as unbounded domain.

Let K be a nonempty subset of a real reflexive Banach space X. Consider the function $f : K \times K \to \mathbb{R}$ and $\theta : K \times K \to \mathbb{R}$ and $\rho \in \mathbb{R}$.

Definition 4 The function $f : K \times K \to \mathbb{R}$ is said to be (ρ-θ)-monotone with respect to $\theta : K \times K \to \mathbb{R}$ if, for all $x, y \in K$, we have

$$f(x, y) + f(y, x) \leq \rho|\theta(x, y)|^2.$$

Remark 2 In the above definition,
(i) for $\rho > 0$ and $\theta(x, y) = \|x - y\|$, f is weakly monotone;
(ii) for $\rho = 0$, f is monotone;
(iii) for $\rho < 0$ and $\theta(x, y) = \|x - y\|$, f is strongly monotone.

We now give an example to show that (ρ-θ)-monotonicity is a generalization of monotonicity.

Example 2 Let $K = [1, 10]$. Let the functions f and θ be defined by

$$f(x, y) = x^2 + y^2 \text{ and } \theta(x, y) = 2(x^2 + y^2) + 4.$$

$$\begin{aligned} f(x, y) + f(y, x) &= 2(x^2 + y^2) \\ &\leq \rho(2x^2 + 2y^2 + 4)^2, \text{ for any } \rho \geq 1. \end{aligned}$$

Therefore, f is (ρ-θ)-monotone with respect to θ. But f is not monotone.

Theorem 4 *Let K be a nonempty convex subset of a real reflexive Banach space X. Suppose $f : K \times K \to \mathbb{R}$ is (ρ-θ)-monotone with respect to θ and is hemicontinuous in the first argument with the following conditions:*
(i) $f(x, x) = 0, \ \forall x \in K$;
(ii) for fixed $z \in K$, the mapping $x \mapsto f(z, x)$ is convex;
(iii) $\lim_{t \to 0} \frac{|\theta(x, x_t)|^2}{t} = 0$, where $x_t = ty + (1 - t)x, \ \forall x, y \in K$.
Then $x \in K$ is a solution of (8) if and only if

$$f(y, x) \leq \rho|\theta(x, y)|^2, \forall y \in K. \tag{9}$$

Proof Let x is a solution of (8), i.e., $f(x, y) \geq 0, \ \forall y \in K$. Therefore, from the definition of (ρ-θ)-monotonicity of f it follows that

$$f(y, x) \leq \rho|\theta(x, y)|^2 - f(x, y) \leq \rho|\theta(x, y)|^2, \forall y \in K. \tag{10}$$

Conversely, suppose $x \in K$ satisfying (9), i.e.,

$$f(y, x) \leq \rho|\theta(x, y)|^2, \forall y \in K. \tag{11}$$

Choose any point $y \in K$ and $x_t = ty + (1-t)x$, $t \in (0, 1]$, then $x_t \in K$. Therefore, from (11) we have

$$f(x_t, x) \leq \rho|\theta(x, x_t)|^2, \forall y \in K. \tag{12}$$

Now conditions (i) and (ii) imply that,

$$0 = f(x_t, x_t) \leq tf(x_t, y) + (1-t)f(x_t, x)$$

$$\Rightarrow t[f(x_t, x) - f(x_t, y)] \leq f(x_t, x). \tag{13}$$

From (12) and (13) we have

$$f(x_t, x) - f(x_t, y) \leq \rho\frac{|\theta(x, x_t)|^2}{t}, \forall y \in K.$$

Since f is hemicontinuous in the first argument and taking $t \to 0$, it implies that $f(x, y) \geq 0, \forall y \in K$. Hence x is a solution of (8).

Theorem 5 Let K be a nonempty bounded convex subset of a real reflexive Banach space X. Suppose $f : K \times K \to \mathbb{R}$ is (ρ-θ)-monotone with respect to θ and is hemicontinuous in the first argument with the following conditions:
(i) $f(x, x) = 0, \ \forall x \in K$;
(ii) for fixed $z \in K$, the mapping $x \mapsto f(z, x)$ is convex and lower semicontunuous;
(iii) $\lim\limits_{t\to 0} \dfrac{|\theta(x, x_t)|^2}{t} = 0$, where $x_t = ty + (1-t)x, \ \forall x, y \in K$, and θ is upper semicontinuous in the first argument.
Then the Problem (8) has a solution.

Proof Consider the two set-valued mappings $F : K \to 2^X$ and $G : K \to 2^X$ such that

$$F(y) = \{x \in K : f(x, y) \geq 0\}, \forall y \in K,$$

and

$$G(y) = \{x \in K : f(y, x) \leq \rho|\theta(x, y)|^2\}, \forall y \in K.$$

It is easy to see that $\overline{x} \in K$ solves the equilibrium Problem (8) if and only if $\overline{x} \in \bigcap\limits_{y\in K} F(y)$. First to show that F is a KKM mapping. If possible, let F not be a KKM mapping. Then there exists $\{x_1, x_2, \ldots, x_m\} \subset K$ such that $co\{x_1, x_2, \ldots, x_m\}$ is not contained in $\bigcup\limits_{i=1}^{m} F(x_i)$, that means there exists a $x_0 \in co\{x_1, x_2, \ldots, x_m\}$, $x_0 = \sum\limits_{i=1}^{m} t_i x_i$ where $t_i \geq 0, i = 1, 2, \ldots, m, \sum\limits_{i=1}^{m} t_i = 1$, but $x_0 \notin \bigcup\limits_{i=1}^{m} F(x_i)$.
Hence, $f(x_0, x_i) < 0$; for $i = 1, 2, \ldots, m$. From (i) and (ii) it follows that

$$0 = f(x_0, x_0) \leq \sum_{i=1}^{m} t_i f(x_0, x_i) < 0,$$

which is a contradiction. Hence F is a KKM mapping.

From the (ρ-θ)-monotonicity of f we will show that $F(y) \subset G(y), \ \forall y \in K$. For any given $y \in K$, let $x \in F(y)$, then

$$f(x, y) \geq 0.$$

From the (ρ-θ)-monotonicity of f, it follows that

$$f(y, x) \leq \rho|\theta(x, y)|^2 - f(x, y) \leq \rho|\theta(x, y)|^2.$$

Therefore $x \in G(y)$, i.e., $F(y) \subset G(y)$, $\forall y \in K$. This implies that G is also a KKM mapping.

Since K is closed bounded and convex, it is weakly compact. From the assumptions, we know that $G(y)$ is weakly closed for all $y \in K$. In fact, because $x \mapsto f(z, x)$ is lower semicontinuous and $x \mapsto \rho|(\theta(x, z)|^2$ is upper semicontinuous. Therefore, $G(y)$ is weakly compact in K, for each y $\in$ K.

Therefore from Lemma 1 and Theorem 4 it follows that $\bigcap_{y \in K} F(y) = \bigcap_{y \in K} G(y) \neq \emptyset$.

So there exists $\overline{x} \in K$ such that $f(\overline{x}, y) \geq 0, \forall y \in K$, i.e., (8) has a solution.

Theorem 6 Let K be a nonempty unbounded closed convex subset of a real reflexive Banach space X. Suppose $f : K \times K \to \mathbb{R}$ is (ρ-θ)-monotone with respect to θ and is hemicontinuous in the first argument and satisfy the following assumptions:

(i) $f(x, x) = 0$, $\forall x \in K$;
(ii) for fixed $z \in K$, the mapping $x \mapsto f(z, x)$ is convex and lower semicontinuous;
(iii) $\lim_{t \to 0} \frac{|\theta(x, x_t)|^2}{t} = 0$, where $x_t = ty + (1 - t)x$, $\forall x, y \in K$, and is upper semicontinuous in the first argument;
(iv) f is weakly coercive, that is there exists $x_0 \in K$ such that $f(x, x_0) < 0$, whenever $\|x\| \to +\infty$ and $x \in K$.

Then (8) has a solution.

Proof Since the proof of this theorem is very similar to Theorem 4.9. in [11], hence it is omitted.

5 Application to Fixed-Point Problems

Let $X = X^*$ be a Hilbert space. Let $T : K \to K$ be a given mapping. Then the fixed-point problem states that find $\overline{x} \in K$ such that

$$T\overline{x} = \overline{x}.$$

Now, by the setting $f(x, y) = \langle x - Tx, y - x\rangle$ we can show that if $\overline{x}$ solves the equilibrium problem (8) then $\overline{x}$ is also a solution of the above fixed-point problem.

Indeed, let $\overline{x}$ is a solution of the equilibrium problem, i.e., $f(\overline{x}, y) \geq 0,\ \forall y \in K$. Let us choose $y = T\overline{x}$, then

$$f(\overline{x}, y) = f(\overline{x}, T\overline{x}) = -\|T\overline{x} - \overline{x}\| \geq 0 \quad \Rightarrow \quad T\overline{x} = \overline{x},$$

which shows that $\overline{x}$ is a fixed point of T.

In this case, notice that $f(x, y)$ is (ρ-θ)-monotone if and only if T is (ρ-θ)-monotone. Since by Theorems 5 and 6, the equilibrium problem has solution, hence by the above result the fixed-point problem also has solution.

6 Conclusions

In this study the existence of solutions for variational-like inequality problems under a new concept relaxed (ρ-θ)-η-invariant monotone maps in reflexive Banach spaces have been established. We have also obtained the existence of solutions of variational inequality and equilibrium problems with (ρ-θ)-monotone mappings. This leads to the natural question of making sensitivity analysis and obtaining results using ε-efficiency conditions as in [15, 16]. We plan to pursue these as our subsequent research works.

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