Frames in Semi-inner Product Spaces

N.K. Sahu and Ram N. Mohapatra

Abstract The objective of this paper is to study the theory of frames in semi-inner product spaces. Several researchers have studied frames in Banach spaces by using the bounded linear functionals. Application of semi-inner product is a new approach to investigate the theory of frames. The notion of semi-frame is introduced in this new aspect.

Keywords Frames · Semi-frames · Semi-inner product

1 Introduction

The theory of frames plays a fundamental role in signal processing, image processing, data compression, sampling theory and has found considerable applications in many more fields. Mathematically, the frame is equivalent to a spanning set in a vector space, but it may not be minimal. It may have more number of vectors than a basis. One of the main advantages in using frames in signal transmission over a basis is that if in the process of transmission, signal along a frame is lost, it is possible to reconstruct completely due to the built in redundancy which is not possible while using a basis. In applications one determines "optimal frames with erasers" (see Han and Sun [12], Pehlivan et al. [15] and the references there in).

The main objective of this paper is to describe Frames in Hilbert space, Banach space, Hilbert C^*-module and then define semi-inner product space, and Frames and semi-frames in that context. The advantage of using semi-inner product is to facilitate

N.K. Sahu (✉)
Dhirubhai Ambani Institute of Information and Communication Technology, Gandhinagar, India
e-mail: nabindaiict@gmail.com

R.N. Mohapatra
University of Central Florida, Orlando, FL 32816, USA
e-mail: ramm@mail.ucf.edu

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calculations in uniformly convex smooth Banach spaces and obtain results that pose difficulties.

Frames in Hilbert Spaces

Frames for a Hilbert space were formally defined by Duffin and Schaeffer [5] in 1952. Frames in Hilbert space have been well investigated. For Hilbert space frames one can refer to Christensen [3] and the references there in.

Definition 1 Let H be a Hilbert space and I be an index set. A sequence $\{f_i\}_{i\in I}$ of elements in H is called a Bessel sequence for H if there exists a real constant $B > 0$ such that

$$\sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2 \text{ for all } f \in H. \tag{1}$$

B is called the Bessel bound for the Bessel sequence $\{f_i\}$.

Definition 2 A sequence $\{f_i\}_{i\in I}$ of elements in a Hilbert space H is called a frame for H, if there exist real constants A, B with $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2 \text{ for all } f \in H. \tag{2}$$

Here A and B are called lower and upper frame bounds, respectively.

The largest number A and the smallest number B satisfying the frame inequality (2) for all $f \in H$ are called optimal frame bounds. If $A = B$, we call the frame $\{f_i\}$, a tight frame. When $A = B = 1$, the frame is called a Parseval frame. If all the frame elements have the same norm, then the frame is called equal norm frame, and if all the frame elements are of unit norm, then it is called unit norm frame. A frame is exact if it ceases to be a frame when any one of its element is removed. A frame is exact if and only if it is a Riesz basis. A non-exact frame is called over complete in the sense that if at least one vector is removed, the remaining ones still constitute a frame.

Frames in Banach Spaces

While constructing frames in Hilbert space we need the sequence space l^2. Similarly, while constructing frames in Banach space one needs a Banach space of scalar-valued sequences (BK-space).

A Banach space of scalar valued sequences (BK-space) is a linear space of sequences with a norm which makes it a Banach space and for which the coordinate functionals are continuous. In a BK-space, the unit vectors are defined by $e_i(j) = \delta_{ij}$ (Kronecker delta). Gröchenig [7] first generalized the concept of frames to Banach spaces and called them atomic decompositions.

Definition 3 Let X be a Banach space with norm $\|.\|_X$ and X_d be an associated BK-space with norm $\|.\|_{X_d}$. Let $\{f_i\}$ be a sequence of elements in X^*, the dual space of X, and $\{x_i\}$ be a sequence of elements in X. If

(i) $\{(x, f_i)\} \in X_d$, for all $x \in X$, where (x, f_i) denotes the value of the functional f_i at the point x,
(ii) the norms $\|x\|_X$ and $\|\{(x, f_i)\}\|_{X_d}$ are equivalent,
(iii) $x = \sum_i (x, f_i) x_i$, for all $x \in X$, then the pair $\big(\{f_i\}, \{x_i\}\big)$ is called an atomic decomposition of X with respect to X_d.

With a more general setting Gröchenig defined Banach frames as follows:

Definition 4 Let X be a Banach space with norm $\|.\|_X$ and X_d be an associated BK-space with norm $\|.\|_{X_d}$. Let $\{f_i\}$ be a sequence of elements in X^* and an operator $S : X_d \to X$ be given. If

(i) $\{(x, f_i)\} \in X_d$, for all $x \in X$,
(ii) the norms $\|x\|_X$ and $\|\{(x, f_i)\}\|_{X_d}$ are equivalent,
(iii) S is bounded and linear, and $S((x, f_i)) = x$ for each $x \in X$, then $\big(\{f_i\}, S\big)$ is a Banach frame for X with respect to X_d.

There is considerable research on frames in Banach spaces and for details on frames in Banach spaces one may refer to Christensen and Heil [4], Stoeva [17], Casazza and Christensen [2], Koushik [13].

Frames in Hilbert C^*-module
In recent years, many mathematicians generalized the frame theory in Hilbert spaces to frame theory in Hilbert C^*-modules and got significant results which enrich the theory of frames.

Definition 5 Let $\mathscr{A}$ be a unital C^*-algebra and $\mathbb{J}$ be a finite or countable index set. A sequence $\{x_j\}_{j\in\mathbb{J}}$ of elements in a Hilbert $\mathscr{A}$-module $\mathscr{H}$ is said to be a frame if there exists two real constants $A, B > 0$ such that

$$A\langle x, x\rangle \le \sum_{j\in\mathbb{J}} \langle x, x_j\rangle\langle x_j, x\rangle \le B\langle x, x\rangle \tag{3}$$

for every $x \in \mathscr{H}$. The optimal constants (maximal for A and minimal for B) are called frame bounds.

The frame $\{x_j\}_{j\in\mathbb{J}}$ is said to be a tight frame if $A = B$, and said to be a Parseval frame if $A = B = 1$.

Wu Jing in his doctoral dissertation to University of Central Florida gave an equivalent formulation of modular frames, and derived many interesting results. Details about these can be found in Han et al. [8–11].

Due to lack of inner product structure in general Banach spaces, people studied the theory of frames by taking the help of bounded linear functionals, that is by taking the help of the dual space. Many of the results on classical frame theory have been generalized to Banach spaces, in this way. The use of arbitrary bounded linear functionals is not always a convenient way to study these notions. It is also difficult to construct examples to verify the established theoretical results. So, in this paper we have put some effort to study the theory of frames in Banach spaces in a different way. We have seen frames being defined in semi-inner product spaces (see Zhang and Zhang [18]). We use the notion of semi-inner product to study more into the theory of frames. In the next section, we give a brief introduction of semi-inner product space. It is worth mentioning that this approach will result for frames in l^p and L^p spaces for $1 < p < \infty$.

2 Semi-inner Product Space

Definition 6 (Lumer [14]) Let X be a vector space over the field F of real or complex numbers. A functional $[.,.] : X \times X \to F$ is called a semi-inner product if it satisfies the following:

1. $[x + y, z] = [x, z] + [y, z], \quad \forall x, y, z \in X$;
2. $[\lambda x, y] = \lambda[x, y], \quad \forall \lambda \in F$ and $x, y \in X$;
3. $[x, x] > 0$, for $x \neq 0$;
4. $|[x, y]|^2 \leq [x, x][y, y]$.
   The pair $(X, [.,.])$ is called a semi-inner product space.

We observe that $\|x\| = [x, x]^{\frac{1}{2}}$ is a norm on X. Hence every semi-inner product space is a normed linear space. On the other hand, in a normed linear space, one can generate semi-inner product in infinitely many different ways. Giles [6] had proved that if the underlying space X is a uniformly convex smooth Banach space then it is possible to define a semi-inner product, uniquely. Also the unique semi-inner product has the following nice properties:

(i) $[x, y] = 0$ if and only if y is orthogonal to x, that is if and only if $\|y\| \leq \|y + \lambda x\|$, for all scalars λ.
(ii) Generalized Riesz representation theorem: If f is a continuous linear functional on X then there is a unique vector $y \in X$ such that $f(x) = [x, y]$, for all $x \in X$.
(iii) The semi-inner product is continuous, that is for each $x, y \in X$, we have $Re[y, x + \lambda y] \to Re[y, x]$ as $\lambda \to 0$.

The sequence space $l^p,\ p > 1$ and the function space $L^p,\ p > 1$ are uniformly convex smooth Banach spaces. So one can define semi-inner product on these spaces, uniquely.

Example 1 The real sequence space l^p for $1 < p < \infty$ is a semi-inner product space with the semi-inner product defined by

$$[x, y] = \frac{1}{\|y\|_p^{p-2}} \sum_i x_i y_i |y_i|^{p-2}, \quad x, y \in l^p.$$

Example 2 (Giles [6]) The real Banach space $L^p(X, \mu)$ for $1 < p < \infty$ is a semi-inner product space with the semi-inner product defined by

$$[f, g] = \frac{1}{\|g\|_p^{p-2}} \int_X f(x)|g(x)|^{p-1}\, sgn(g(x)) d\mu, \quad f, g \in L^p.$$

3 Frames in Semi-inner Product Spaces

Recently, Zhang and Zhang [18] investigated the theory of frames in Banach spaces by applying the notion of semi-inner product. They generalized the classical theory on frames and Riesz bases in this new perspective. They have defined frames in the following way:

Definition 7 Let X be a Banach space with a compatible semi-inner product $[.,.]$ and norm $\|.\|_X$. Let X_d be an associated BK-space (sequence space with continuous coordinate linear functionals) with norm $\|.\|_{X_d}$. A sequence of elements $\{f_j\} \subseteq X$ is called an X_d-frame for X if $\{[f, f_j]\} \in X_d$, for all $f \in X$ and there exist two positive constants A, B such that

$$A\|f\|_X \le \|\{[f, f_j]\}\|_{X_d} \le B\|f\|_X \text{ for all } f \in X.$$

They have also defined frames for the dual space X^* of the Banach space X.

Definition 8 Let X be a Banach space with a compatible semi-inner product $[.,.]$ and norm $\|.\|_X$. Let X^* be the dual space of X. Let X_d be an associated BK-space with norm $\|.\|_{X_d}$, and X_d^* be the dual space of X_d. A sequence of elements $\{f_j^*\} \subseteq X^*$ is an X_d^*-frame for X^* if $\{[f_j, f]\} \in X_d^*$, for all $f \in X$ and there exist two positive constants A, B such that

$$A\|f\|_X \le \|\{[f_j, f]\}\|_{X_d^*} \le B\|f\|_X \text{ for all } f \in X.$$

The notion of frame is too restrictive, in the sense that one cannot satisfy both upper and lower frame bounds simultaneously. Thus there is a scope for two natural generalizations, named as upper semi-frame and lower semi-frame. The notion of semi-frame in Hilbert space was studied by Antoine and Balazs [1]. In this paper we define the notion of semi-frame in Banach spaces by using the semi-inner product.

Definition 9 Let X be a Banach space with a compatible semi-inner product $[.,.]$ and norm $\|.\|_X$. Let X_d be an associated BK-space with norm $\|.\|_{X_d}$. A sequence of elements $\{f_j\} \subseteq X$ is called upper semi-X_d-frame for X if

(i) $\{f_j\}$ is total in X;
(ii) $\{[f, f_j]\} \in X_d$, for all $f \in X$;
(iii) there exists a positive constant B such that

$$0 \leq \|\{[f, f_j]\}\|_{X_d} \leq B\|f\|_X \text{ for all } f \in X.$$

Definition 10 Let X be a Banach space with a compatible semi-inner product $[.,.]$ and norm $\|.\|_X$. Let X_d be an associated BK-space with norm $\|.\|_{X_d}$. A sequence of elements $\{f_j\} \subseteq X$ is called lower semi-X_d-frame for X if

(i) $\{f_j\}$ is total in X;
(ii) $\{[f, f_j]\} \in X_d$, for all $f \in X$;
(iii) there exists a positive constant A such that

$$A\|f\|_X \leq \|\{[f, f_j]\}\|_{X_d} \text{ for all } f \in X.$$

Similarly, we define upper semi-X_d^*-frame and lower semi-X_d^*-frame for the dual space X^*.

Definition 11 Let X be a Banach space with a compatible semi-inner product $[.,.]$ and norm $\|.\|_X$. Let X^* be the dual space of X. Let X_d be an associated BK-space with norm $\|.\|_{X_d}$, and X_d^* be the dual space of X_d. A sequence of elements $\{f_j^*\} \subseteq X^*$ is upper semi-X_d^*-frame for X^* if

(i) $\{f_j^*\}$ is total in X^*;
(ii) $\{[f_j, f]\} \in X_d^*$, for all $f \in X$;
(iii) there exists a positive constant B such that

$$0 \leq \|\{[f_j, f]\}\|_{X_d^*} \leq B\|f\|_X \text{ for all } f \in X.$$

Definition 12 Let X be a Banach space with a compatible semi-inner product $[.,.]$ and norm $\|.\|_X$. Let X^* be the dual space of X. Let X_d be an associated BK-space with norm $\|.\|_{X_d}$, and X_d^* be the dual space of X_d. A sequence of elements $\{f_j^*\} \subseteq X^*$ is lower semi-X_d^*-frame for X^* if

(i) $\{f_j^*\}$ is total in X^*;
(ii) $\{[f_j, f]\} \in X_d^*$, for all $f \in X$;
(iii) there exists a positive constant A such that

$$A\|f\|_X \leq \|\{[f_j, f]\}\|_{X_d^*} \text{ for all } f \in X.$$

Zhang and Zhang [18] established the reconstruction property for X_d-frame and X_d^*-frame in a semi-inner space X. They defined the operator (so-called analysis operator) $U : X \to X_d$ by $U(f) = \{[f, f_j]\}$. They proved that

Theorem 1 *([18]) If $\{f_j\}$ is an X_d-frame for X and $Rng(U)$ has an algebraic complement in X_d, then there exists an X_d^*-frame $\{g_j^*\}$ for X^* such that*

$$f = \sum_{j \in I} [f, f_j] g_j \text{ for all } f \in X$$

and

$$f^* = \sum_{j \in I} [g_j, f] f_j^* \text{ for all } f \in X.$$

Based on the above theorem, we formulate the following definition.

Definition 13 Let $\{f_j\}$ be an X_d-frame for X. If there exists an X_d^*-frame $\{g_j^*\}$ for X^* such that

$$f = \sum_{j \in I} [f, f_j] g_j \text{ for all } f \in X$$

and

$$f^* = \sum_{j \in I} [g_j, f] f_j^* \text{ for all } f \in X.$$

Then $\{f_j\}$ and $\{g_j^*\}$ are called dual frame pair.

Now we are in a position to propose the following theorem.

Theorem 2 *Let $\{f_j\}$ be an upper semi-X_d-frame for X with bound B. If $\{g_j^*\}$ is a sequence of element in X^* such that $\{f_j\}$ and $\{g_j^*\}$ are dual frame pair, then $\{g_j^*\}$ is a lower semi-X_d^*-frame for X^* with bound $\frac{1}{B}$.*

Proof Since $\{f_j\}$ is an upper semi-X_d-frame for X with bound B, we have

$$0 \leq \|\{[f, f_j]\}\|_{X_d} \leq B\|f\|_X \text{ for all } f \in X.$$

Now

$$\begin{aligned}
\|f\|_X^2 = [f, f] &= \left[\sum [f, f_j] g_j, f\right] \\
&= \sum [[f, f_j] g_j, f] \\
&= \sum [f, f_j][g_j, f] \\
&\leq \|\{[f, f_j]\}\|_{X_d} \|\{[g_j, f]\}\|_{X_d^*} \\
&\leq B\|f\|_X \|\{[g_j, f]\}\|_{X_d^*}
\end{aligned}$$

$$\Rightarrow \frac{1}{B}\|f\|_X \leq \|\{[g_j, f]\}\|_{X_d^*} \text{ for all } f \in X.$$

That is, $\{[g_j, f]\}$ is a lower semi-X_d^*-frame for X^*.

Similarly, we can easily prove the following theorem.

Theorem 3 *Let $\{g_j\}$ be an upper semi-X_d^*-frame for X^* with bound B. If $\{f_j\}$ is a sequence of element in X such that $\{f_j\}$ and $\{g_j^*\}$ are dual frame pair, then $\{f_j\}$ is a lower semi-X_d-frame for X with bound $\frac{1}{B}$.*

Proof Since $\{g_j\}$ is an upper semi-X_d^*-frame for X^* with bound B, we have

$$0 \le \|\{[g_j, f]\}\|_{X_d^*} \le B\|f\|_X \text{ for all } f \in X.$$

Now for any $f \in X$, we have

$$\begin{aligned} \|f\|_X^2 = [f, f] &= \left[\sum [f, f_j]g_j, f\right] \\ &= \sum [[f, f_j]g_j, f] \\ &= \sum [f, f_j][g_j, f] \\ &\le \|\{[f, f_j]\}\|_{X_d} \|\{[g_j, f]\}\|_{X_d^*} \\ &\le \|\{[f, f_j]\}\|_{X_d} \, B\|f\|_X \\ \Rightarrow \frac{1}{B}\|f\|_X &\le \|\{[f, f_j]\}\|_{X_d} \text{ for all } f \in X. \end{aligned}$$

That is, $\{[f, f_j]\}$ is a lower semi-X_d-frame for X.

4 Frames for $l^p(1 < p < \infty)$ Spaces

In this section, we define frames in $l^p(1 < p < \infty)$ spaces.

We know that l^p $(1 < p < \infty)$ spaces are uniformly convex smooth Banach spaces. It is seen that those spaces are semi-inner product spaces with uniquely defined semi-inner product (see Giles [6]). For the rest of this section, we assume that X is the real sequence space $l^p(1 < p < \infty)$ with norm $\|.\|_p$ and semi-inner product $[., .]$. The following definitions of Bessel sequence and frame can be found in Sahu and Nahak [16].

Definition 14 A set of elements $f = \{f_i\}_{i=1}^{\infty} \subseteq X$ is called a Bessel sequence if there exists a constant $B > 0$ such that

$$\sum_{i=1}^{\infty} |[f_i, x]|^q \le B(\|x\|_p)^q, \ \forall x \in X,$$

where $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. The number B is called Bessel bound.

Definition 15 A sequence of elements $\{f_i\}_{i=1}^{\infty}$ in X is called a frame if there exist positive constants A and B such that

$$A(\|x\|_p)^q \le \sum_{i=1}^{\infty} |[f_i, x]|^q \le B(\|x\|_p)^q, \ \ \forall x \in X,$$

where $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. A and B are called lower and upper frame bounds, respectively.

If $A = B$ then the frame is called a tight frame, and if $A = B = 1$ then the frame is called a Parseval frame. A frame is called a normalized frame if each frame element has unit norm.

Example 3 Consider the set $\{e_i\}_{i=1}^{\infty} \in l^p$, where $e_i = (0, 0, ..., 1, 0, 0..)$, where 1 is at the ith coordinate and 0 at the other coordinates.

(i) $\{e_1, 0, e_2, 0, e_3, 0, ...\}$ is a Parseval frame.
(ii) $\{e_1, e_1, e_2, e_2, ...\}$ is a tight frame with bound 2.
(iii) $\left\{\frac{e_1}{\sqrt{2}}, \frac{e_1}{\sqrt{2}}, \frac{e_2}{\sqrt{2}}, \frac{e_2}{\sqrt{2}}, ...\right\}$ is a tight frame with bound $\frac{2}{(\sqrt{2})^{\frac{p}{p-1}}}$.
(iv) $\{ne_n\}_{n=1}^{\infty}$ is a lower semi-frame but not a frame.
(v) $\left\{\frac{1}{n}e_n\right\}_{n=1}^{\infty}$ is an upper semi-frame but not a frame.

Some of the classical frame theory results in Hilbert spaces can be generalized to l^p spaces in this new approach. The reconstruction formula naturally holds true for Parseval frames and tight frames. In this connection, we state the following two theorems.

Theorem 4 *A set of elements $\{f_i\}_{i=1}^{\infty}$ is a Parseval frame for X if and only if*

$$x = \sum_{i=1}^{\infty} \frac{|[f_i, x]|^{q-2}}{\|\{[f_i, x]\}\|^{q-2}} [f_i, x] f_i, \quad \forall x \in X. \tag{4}$$

Theorem 5 *A set of elements $\{f_i\}_{i=1}^{\infty}$ is a tight frame with bound A for X if and only if*

$$x = \sum_{i=1}^{\infty} \frac{1}{A^{\frac{2}{q}}} \frac{|[f_i, x]|^{q-2}}{\|\{[f_i, x]\}\|^{q-2}} [f_i, x] f_i \quad \forall x \in X. \tag{5}$$

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