Degree of Approximation by Certain Genuine Hybrid Operators

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Abstract This paper is in continuation of our work on certain genuine hybrid operators in (Positivity (Under review)) [3]. First, we discuss some direct results in simultaneous approximation by these operators, e.g. pointwise convergence theorem, Voronovskaja-type theorem and an error estimate in terms of the modulus of continuity. Next, we estimate the rate of convergence for functions having a derivative that coincides a.e. with a function of bounded variation.

Keywords Rate of convergence · Modulus of continuity · Simultaneous approximation · Bounded variation

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1 Introduction

Recently, Gupta and Rassias [5] introduced the Lupaş-Durrmeyer operators based on Polya distribution and discussed some local and global direct results. Also, Gupta [2] studied some other hybrid operators of Durrmeyer type. Păltănea [11] (see also [10]) considered a Durrmeyer-type modification of the genuine Szász-Mirakjan operators based on two parameters $\alpha, \rho > 0$. Inspired by his work, in [3] Gupta et al. introduced certain genuine hybrid operators as follows:

For $c \in \{0, 1\}$ and $f \in C_\gamma[0, \infty) := \{f \in C[0, \infty) : |f(t)| \leq M_f e^{\gamma t},$ for some $\gamma > 0, M_f > 0\}$, we define

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$$B_\alpha^\rho(f,x) = \sum_{k=1}^{\infty} p_{\alpha,k}(x,c) \int_0^\infty \theta_{\alpha,k}^\rho(t) f(t)dt \ + \ p_{\alpha,0}(x,c) f(0), \tag{1}$$

$$= \int_0^\infty K_\alpha^\rho(x,t) f(t)dt, \tag{2}$$

where

$$p_{\alpha,k}(x,c) = \frac{(-x)^k}{k!}\phi_{\alpha,c}^{(k)}(x), \theta_{\alpha,k}^\rho(t) = \frac{\alpha\rho}{\Gamma(k\rho)} e^{-\alpha\rho t}(\alpha\rho t)^{k\rho-1}$$

$$\text{and} \quad K_\alpha^\rho(x,t) = \sum_{k=1}^{\infty} p_{\alpha,k}(x,c)\theta_{\alpha,k}^\rho(t) \ + \ p_{\alpha,0}(x,c)\delta(t); \ \ x \in (0,\infty).$$

It is observed that the operators $B_\alpha^\rho(f,x)$ are well-defined for $\alpha\rho > \gamma$. We assume that

$$\phi_{\alpha,c}(x) = \begin{cases} e^{-\alpha x}, & \text{for } c = 0, \\ (1 \ + \ x)^{-\alpha}, & \text{for } c = 1. \end{cases}$$

As shown in paper [3], the operators (1) include several linear positive operators as special cases. Further, we note that the operators (1) preserve the linear functions. In [3], we studied some direct results, e.g. Voronovskaja-type theorems in ordinary and simultaneous approximation for first-order derivatives as well as results in local and weighted approximation. In this paper, we continue this work by discussing simultaneous approximation for $f^{(r)}(x), r \in \mathbb{N}$ and the rate of convergence of the operators (1) for the functions with derivatives of bounded variation on each finite subinterval of $(0, \infty)$. The paper is organized as follows:

In Sect. 2, we discuss some auxiliary results and then in Sect. 3, we obtain the main results of this paper.

2 Auxiliary Results

For $f : [0,\infty) \to R$, we define

$$S_\alpha(f;x) = \sum_{k=0}^{\infty} p_{\alpha,k}(x,c) f\left(\frac{k}{\alpha}\right) \tag{3}$$

such that (3) makes sense for all $x \geq 0$.

For $m \in \mathbb{N}^0 = \mathbb{N} \cup \{0\}$, the mth order central moment of the operators S_α is given by

$$\upsilon_{\alpha,m}(x) := S_\alpha((t-x)^m; x) = \sum_{k=0}^{\infty} p_{\alpha,k}(x,c)\left(\frac{k}{\alpha} - x\right)^m.$$

Lemma 1 *For the function $\upsilon_{\alpha,m}(x)$, we have*

$$\upsilon_{\alpha,0}(x) = 1,\ \ \upsilon_{\alpha,1}(x) = 0$$

and

$$x(1+cx)[\upsilon'_{\alpha,m}(x) + m\upsilon_{\alpha,m-1}(x)] = \alpha\upsilon_{\alpha,m+1}(x).$$

Thus,

(i) $\upsilon_{\alpha,m}(x)$ *is a polynomial in x of degree* $[m/2]$;
(ii) *for each* $x \in [0,\infty)$, $\upsilon_{\alpha,m}(x) = O(\alpha^{-[(m+1)/2]})$, *where* $[\beta]$ *denotes the integral part of* β.

Proof For the cases $c = 0$ and 1, the proof of this lemma can be found in [8, 12] respectively.

Lemma 2 *For the mth order ($m \in \mathbb{N}^0$) moment of the operators (1) defined as* $u_{\alpha,m}(x) := B_\alpha^\rho(t^m; x)$, *we have*

$$u_{\alpha,0}(x) = 1,\ \ u_{\alpha,1}(x) = x,\ \ u_{\alpha,2}(x) = x^2 + \frac{x}{\alpha}\left(\frac{1}{\rho} + (1+cx)\right)$$

and

$$x(1+cx)u'_{\alpha,m}(x) = \alpha u_{\alpha,m+1}(x) - \left(\frac{m}{\rho} + \alpha x\right)u_{\alpha,m}(x),\ m \in \mathbb{N}.$$

Consequently, for each $x \in (0,\infty)$ *and* $m \in \mathbb{N}$, $u_{\alpha,m}(x) = x^m + \alpha^{-1}(p_m(x,c) + o(1))$,
where $p_m(x,c)$ *is a rational function of x depending on the parameters m and c.*

Lemma 3 *[3] For* $m \in \mathbb{N}^0$, *if the mth order central moment* $\mu_{\alpha,m}(x)$ *for the operators* B_α^ρ *is defined as*

$$\mu_{\alpha,m}(x) := B_\alpha^\rho((t-x)^m, x) = \sum_{k=1}^{\infty} p_{\alpha,k}(x,c)\int_0^\infty \theta_{\alpha,k}^\rho(t)(t-x)^m dt + p_{\alpha,0}(x,c)(-x)^m,$$

then we have the following recurrence relation:

$$\alpha\mu_{\alpha,m+1}(x) = x(1+cx)\mu'_{\alpha,m}(x) + mx\left[\frac{1}{\rho} + (1+cx)\right]\mu_{\alpha,m-1}(x) + \frac{m}{\rho}\mu_{\alpha,m}(x).$$

Consequently,

(i) $\mu_{\alpha,0}(x)=1,\ \mu_{\alpha,1}(x)=0,\ \mu_{\alpha,2}(x)=\dfrac{\{1+\rho(1+cx)\}x}{\alpha\rho}$;

(ii) $\mu_{\alpha,m}(x)$ *is a polynomial in* x *of degree atmost* m;

(iii) *for every* $x\in(0,\infty)$, $\mu_{\alpha,m}(x)=O\left(\alpha^{-[(m+1)/2]}\right)$;

(iv) *the coefficients of* α^{-m} *in* $\mu_{\alpha,2m}(x)$ *and* $\mu_{\alpha,2m-1}(x)$ *are* $(2m-1)!!\left\{x\left(\dfrac{1}{\rho}+(1+cx)\right)\right\}^m$ *and* $\dfrac{(2m-1)!!(m-1)}{3}x^{m-1}\left(\dfrac{1}{\rho}+(1+cx)\right)^{m-2}\left\{(1+cx)\left(\dfrac{1}{\rho}+(1+2cx)\right)+\dfrac{2}{\rho}\left(\dfrac{1}{\rho}+(1+cx)\right)\right\}$ *respectively.*

Corollary 1 *For* $x\in[0,\infty)$ *and* $\alpha>0$, *it is observed that*

$$\mu_{\alpha,2}(x)\le\frac{\lambda x(1+cx)}{\alpha},\ \text{where } \lambda=1+\frac{1}{\rho}>1.$$

Corollary 2 *[3] Let* γ *and* δ *be any two positive real numbers and* $[a,b]\subset(0,\infty)$ *be any bounded interval. Then, for any* $m>0$ *there exists a constant* M' *independent of* α *such that*

$$\left\|\sum_{k=1}^{\infty}p_{\alpha,k}(x,c)\int\limits_{|t-x|\ge\delta}\theta^{\rho}_{\alpha,k}(t)e^{\gamma t}dt\right\|\le M'\alpha^{-m},$$

where $\|.\|$ *is the sup-norm over* $[a,b]$.

Lemma 4 *For every* $x\in(0,\infty)$ *and* $r\in\mathbb{N}^0$, *there exist polynomials* $q_{i,j,r}(x)$ *in* x *independent of* α *and* k *such that*

$$\frac{d^r}{dx^r}p_{\alpha,k}(x,c)=p_{\alpha,k}(x,c)\sum_{\substack{2i+j\le r\\ i,j\ge0}}\alpha^i(k-\alpha x)^j\frac{(q_{i,j,r}(x,c))}{(p(x,c))^r},$$

where $p(x,c)=x(1+cx)$.

Proof For the cases $c=0,1$, the proof of this lemma can be seen in [8, 12] respectively.

3 Main Results

3.1 Simultaneous Approximation

Throughout this section, we assume that $0 < a < b < \infty$.

In the following theorem, we show that the derivative $B_\alpha^{\rho(r)}(f;.)$ is also an approximation process for $f^{(r)}$.

Theorem 1 (Basic convergence theorem) *Let $f \in C_\gamma[0,\infty)$. If $f^{(r)}$ exists at a point $x \in (0,\infty)$, then we have*

$$\lim_{\alpha\to\infty}\left(\frac{d^r}{d\omega^r}B_\alpha^\rho(f;\omega)\right)_{\omega=x} = f^{(r)}(x). \tag{4}$$

Further, if $f^{(r)}$ is continuous on $(a-\eta, b+\eta), \eta > 0$, then the limit in (4) holds uniformly in $[a,b]$.

Proof By our hypothesis, we have

$$f(t) = \sum_{\nu=0}^{r}\frac{f^{(\nu)}(x)}{\nu!}(t-x)^\nu + \psi(t,x)(t-x)^r,\ \ t \in [0,\infty),$$

where the function $\psi(t,x) \to 0$ as $t \to x$. From the above equation, we may write

$$\begin{aligned}\left(\frac{d^r}{d\omega^r}B_\alpha^\rho(f(t);\omega)\right)_{\omega=x} &= \sum_{\nu=0}^{r}\frac{f^{(\nu)}(x)}{\nu!}\left(\frac{d^r}{d\omega^r}B_\alpha^\rho((t-x)^\nu;\omega)\right)_{\omega=x}\\ &\quad + \left(\frac{d^r}{d\omega^r}B_\alpha^\rho(\psi(t,x)(t-x)^r;\omega)\right)_{\omega=x}\\ &=: I_1 + I_2,\ \text{say}.\end{aligned}$$

First, we estimate I_1.

$$\begin{aligned}I_1 &= \sum_{\nu=0}^{r}\frac{f^{(\nu)}(x)}{\nu!}\left\{\frac{d^r}{d\omega^r}\left(\sum_{j=0}^{\nu}\binom{\nu}{j}(-x)^{\nu-j}B_\alpha^\rho(t^j;\omega)\right)_{\omega=x}\right\}\\ &= \sum_{\nu=0}^{r}\frac{f^{(\nu)}(x)}{\nu!}\sum_{j=0}^{\nu}\binom{\nu}{j}(-x)^{\nu-j}\left(\frac{d^r}{d\omega^r}B_\alpha^\rho(t^j;\omega)\right)_{\omega=x}\\ &= \sum_{\nu=0}^{r-1}\frac{f^{(\nu)}(x)}{\nu!}\sum_{j=0}^{\nu}\binom{\nu}{j}(-x)^{\nu-j}\left(\frac{d^r}{d\omega^r}B_\alpha^\rho(t^j;\omega)\right)_{\omega=x}\end{aligned}$$

$$+\frac{f^{(r)}(x)}{r!}\sum_{j=0}^{r}\binom{r}{j}(-x)^{r-j}\left(\frac{d^r}{d\omega^r}B_\alpha^\rho(t^j;\omega)\right)_{\omega=x}$$
$$:= I_3 + I_4, \text{ say.}$$

First, we estimate I_4.

$$I_4 = \frac{f^{(r)}(x)}{r!}\sum_{j=0}^{r-1}\binom{r}{j}(-x)^{r-j}\left(\frac{d^r}{d\omega^r}B_\alpha^\rho(t^j;\omega)\right)_{\omega=x} + \frac{f^{(r)}(x)}{r!}\left(\frac{d^r}{d\omega^r}B_\alpha^\rho(t^r;\omega)\right)_{\omega=x}$$
$$:= I_5 + I_6, \text{ say.}$$

Using Lemma 2, we get
$I_6 = f^{(r)}(x) + O\left(\frac{1}{\alpha}\right)$, $I_3 = O\left(\frac{1}{\alpha}\right)$ and $I_5 = O\left(\frac{1}{\alpha}\right)$, as $\alpha \to \infty$.
Combining the above estimates, for each $x \in (0, \infty)$ we obtain $I_1 \to f^{(r)}(x)$ as $\alpha \to \infty$.

Next, we estimate I_2. By making use of Lemma 4, we have

$$|I_2| \leq \sum_{k=1}^{\infty}\frac{p_{\alpha,k}(x,c)}{(p(x,c))^r}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\alpha^i|k-\alpha x|^j|q_{i,j,r}(x,c)|\int_0^\infty\theta_{\alpha,k}^\rho(t)|\psi(t,x)||(t-x)^r|dt$$
$$+\left|\left(\frac{d^r}{d\omega^r}p_{\alpha,0}(\omega,c)\right)_{\omega=x}\right||\psi(0,x)(-x)^r|$$
$$:= I_7 + I_8, \text{ say.}$$

Since $\psi(t,x) \to 0$ as $t \to x$, for a given $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\psi(t,x)| < \varepsilon$ whenever $|t-x| < \delta$. For $|t-x| \geq \delta$, $|(t-x)^r\psi(t,x)| \leq Me^{\gamma t}$, for some constant $M > 0$.

Again, using Lemma 4, we have

$$|I_7| \leq \sum_{k=1}^{\infty}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}\alpha^i|k-\alpha x|^j\frac{|q_{i,j,r}(x,c)|}{(p(x,c))^r}p_{\alpha,k}(x,c)\left(\varepsilon\int_{|t-x|<\delta}\theta_{\alpha,k}^\rho(t)|t-x|^r dt\right.$$
$$\left.+M\int_{|t-x|\geq\delta}\theta_{\alpha,k}^\rho(t)e^{\gamma t}dt\right) := I_9 + I_{10}, \text{ say.}$$

Let $K = \sup\limits_{\substack{2i+j\leq r\\ i,j\geq 0}}\dfrac{|q_{i,j,r}(x,c)|}{(p(x,c))^r}$. By applying the Schwarz inequality, Lemmas 1 and 3, we get

$$
\begin{aligned}
|I_9| &\leq \varepsilon K \sum_{k=1}^{\infty} \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} \alpha^i |k-\alpha x|^j p_{\alpha,k}(x,c) \left(\int_0^{\infty} \theta_{\alpha,k}^{\rho}(t)(t-x)^{2r} dt \right)^{\frac{1}{2}} \\
&\leq \varepsilon K \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} \alpha^{i+j} \left(\sum_{k=1}^{\infty} \left(\frac{k}{\alpha} - x \right)^{2j} p_{\alpha,k}(x,c) \right)^{\frac{1}{2}} \\
&\quad \left(\sum_{k=1}^{\infty} p_{\alpha,k}(x,c) \int_0^{\infty} \theta_{\alpha,k}^{\rho}(t)(t-x)^{2r} dt \right)^{\frac{1}{2}} \\
&\leq \varepsilon K \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} \alpha^{i+j} \left(\upsilon_{\alpha,2j}(x) - x^{2j} \phi_{\alpha,c}(x) \right)^{\frac{1}{2}} \\
&\quad \left(B_{\alpha}^{\rho}((t-x)^{2r};x)) - x^{2r} \phi_{\alpha,c}(x) \right)^{\frac{1}{2}} \\
&= \varepsilon \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} \alpha^{i+j} \{ O(\alpha^{-j}) + O(\alpha^{-s_1}) \}^{1/2} \\
&\quad \times \{ O(\alpha^{-r}) + O(\alpha^{-s_2}) \}^{1/2}, \text{ for any } s_1, s_2 > 0.
\end{aligned}
$$

Choosing s_1, s_2 such that $s_1 > j$ and $s_2 > r$, we have $|I_9| = \varepsilon$ $\sum_{\substack{2i+j \leq r \\ i,j \geq 0}} \alpha^{i+j} O(\alpha^{-j/2}) O(\alpha^{-r/2}) = \varepsilon . O(1)$.

Since $\varepsilon > 0$ is arbitrary, $I_9 \to 0$ as $\alpha \to \infty$.
Now, we estimate I_{10}. By applying Cauchy–Schwarz inequality, Lemma 1 and Corollary 2, we obtain

$$
\begin{aligned}
|I_{10}| &\leq MK \sum_{k=1}^{\infty} \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} \alpha^i |k-\alpha x|^j p_{\alpha,k}(x,c) \int_{|t-x| \geq \delta} \theta_{\alpha,k}^{\rho}(t) e^{\gamma t} dt \\
&\leq M_1 \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} \alpha^{i+j} \left(\sum_{k=1}^{\infty} \left(\frac{k}{\alpha} - x \right)^{2j} p_{\alpha,k}(x,c) \right)^{1/2} \\
&\quad \times \left(\sum_{k=1}^{\infty} p_{\alpha,k}(x,c) \int_{|t-x| \geq \delta} \theta_{\alpha,k}^{\rho}(t) e^{2\gamma t} dt \right)^{1/2}, \text{ where } M_1 = MK \\
&\leq M_1 \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} \alpha^{i+j} \left(\upsilon_{\alpha,2j}(x) - x^{2j} \phi_{\alpha,c}(x) \right)^{1/2}
\end{aligned}
$$

$$\times\left(\sum_{k=1}^{\infty} p_{\alpha,k}(x,c) \int\limits_{|t-x|\geq\delta} \theta_{\alpha,k}^{\rho}(t)e^{2\gamma t}dt\right)^{1/2}$$
$$= \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} \alpha^{i+j}\{O(\alpha^{-j}) + O(\alpha^{-m_1})\}^{1/2}$$
$$\times\{O(\alpha^{-m_2})\}^{1/2}, \text{ for any } m_1, m_2 > 0.$$

Choosing $m_1 > j$, we get

$$|I_{10}| = \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} \alpha^{i+j} O(\alpha^{-j/2})O(\alpha^{-m_2/2}) = O(\alpha^{(r-m_2)/2}),$$

which implies that $I_{10} = o(1)$, as $\alpha \to \infty$, on choosing $m_2 > r$. Next, we estimate I_8. We may write

$$|I_8| = \left|\left(\frac{d^r}{d\omega^r} p_{\alpha,0}(\omega,c)\right)_{\omega=x}\right| |\psi(0,x)|x^r$$
$$= |\phi_{\alpha,c}^{(r)}(x)|\ |\psi(0,x)|x^r.$$

Now, we observe that $\phi_{\alpha,0}^{(r)}(x) = e^{-\alpha x}(-\alpha)^r$ and $\phi_{\alpha,1}^{(r)}(x) = \dfrac{(-1)^r(\alpha)_r}{(1+x)^{\alpha+r}}$, which implies that $I_8 = O(\alpha^{-p})$ for any $p > 0$, in view of the fact that $|\psi(0,x)x^r| \leq N_1$, for some $N_1 > 0$.

By combining the estimates $I_7 - I_{10}$, we obtain $I_2 \to 0$ as $\alpha \to \infty$.

To prove the uniformity assertion, it is sufficient to remark that $\delta(\varepsilon)$ in the above proof can be chosen to be independent of $x \in [a,b]$ and also that the other estimates hold uniformity in $x \in [a,b]$. This completes the proof of the theorem.

Next, we establish an asymptotic formula.

Theorem 2 (Voronovskaja type result) *Let $f \in C_\gamma[0,\infty)$. If f admits a derivative of order $(r+2)$ at a fixed point $x \in (0,\infty)$, then we have*

$$\lim_{\alpha\to\infty} \alpha\left(\left(\frac{d^r}{d\omega^r} B_\alpha^\rho(f;\omega)\right)_{\omega=x} - f^{(r)}(x)\right) = \sum_{\nu=1}^{r+2} Q(\nu,r,c,a,x) f^{(\nu)}(x), \quad (5)$$

where $Q(\nu,r,c,a,x)$ are certain rational functions of x independent of α. Further, if $f^{(r+2)}$ is continuous on $(a-\eta, b+\eta)$, $\eta > 0$, then the limit in (5) holds uniformly in $[a,b]$.

Proof From the Taylor's theorem, for $t \in [0, \infty)$ we may write

$$f(t) = \sum_{\nu=0}^{r+2} \frac{f^{(\nu)}(x)}{\nu!}(t-x)^{\nu} + \psi(t,x)(t-x)^{r+2}, \tag{6}$$

where the function $\psi(t,x) \to 0$ as $t \to x$.
Now, from Eq. (6), we have

$$\begin{aligned}
\left(\frac{d^r}{d\omega^r} B_\alpha^\rho(f(t);\omega)\right)_{\omega=x} &= \sum_{\nu=0}^{r+2} \frac{f^{(\nu)}(x)}{\nu!}\left(\frac{d^r}{dw^r}(B_\alpha^\rho((t-x)^\nu;\omega)\right)_{\omega=x} \\
&\quad + \left(\frac{d^r}{d\omega^r} B_\alpha^\rho(\psi(t,x)(t-x)^{r+2};\omega)\right)_{\omega=x} \\
&= \sum_{\nu=0}^{r+2} \frac{f^{(\nu)}(x)}{\nu!} \sum_{j=0}^{\nu} \binom{\nu}{j} (-x)^{\nu-j} \left(\frac{d^r}{d\omega^r} B_\alpha^\rho(t^j;\omega)\right)_{\omega=x} \\
&\quad + \left(\frac{d^r}{d\omega^r} B_\alpha^\rho(\psi(t,x))(t-x)^{r+2};\omega\right)_{\omega=x} \\
&:= J_1 + J_2, \ say.
\end{aligned}$$

Proceeding in a manner similar to the estimate of I_2 in Theorem 1, for each $x \in (0, \infty)$ we get $\alpha J_2 \to 0$ as $\alpha \to \infty$.

Next, we estimate J_1.

$$\begin{aligned}
J_1 &= \sum_{\nu=0}^{r-1} \frac{f^{(\nu)}(x)}{\nu!} \sum_{j=0}^{\nu} \binom{\nu}{j} (-x)^{\nu-j} \left(\frac{d^r}{d\omega^r} B_\alpha^\rho(t^j;\omega)\right)_{\omega=x} \\
&\quad + \frac{f^{(r)}(x)}{r!} \sum_{j=0}^{r} \binom{r}{j} (-x)^{r-j} \left(\frac{d^r}{d\omega^r} B_\alpha^\rho(t^j;\omega)\right)_{\omega=x} \\
&\quad + \frac{f^{(r+1)}(x)}{(r+1)!} \sum_{j=0}^{r+1} \binom{r+1}{j} (-x)^{r+1-j} \left(\frac{d^r}{d\omega^r} B_\alpha^\rho(t^j;\omega)\right)_{\omega=x} \\
&\quad + \frac{f^{(r+2)}(x)}{(r+2)!} \sum_{j=0}^{r+2} \binom{r+2}{j} (-x)^{r+2-j} \left(\frac{d^r}{d\omega^r} B_\alpha^\rho(t^j;\omega)\right)_{\omega=x}.
\end{aligned}$$

Making use of Lemma 2, we have

$$J_1 = f^{(r)}(x) + \alpha^{-1}\left(\sum_{\nu=1}^{r+2} Q(\nu, r, c, a, x) f^{(\nu)}(x) + o(1)\right).$$

Thus, from the estimates of J_1 and J_2, the required result follows.

The uniformity assertion follows as in the proof of Theorem 1. This completes the proof.

The next result provides an estimate of the degree of approximation in $B_{\alpha}^{\rho(r)}(f;x) \to f^{(r)}(x), r \in \mathbb{N}$.

Theorem 3 (Degree of approximation) *Let* $r \leq q \leq r+2$, $f \in C_{\gamma}[0,\infty)$ *and* $f^{(q)}$ *exist and be continuous on* $(a-\eta, b+\eta)$ *where* $\eta > 0$ *is sufficiently small. Then, for sufficiently large* α

$$\left\| \left(\frac{d^r}{d\omega^r} B_{\alpha}^{\rho}(f;\omega) \right)_{\omega=x} - f^{(r)}(x) \right\|_{C[a,b]} \leq \max\{C_1 \alpha^{-(q-r)/2} \omega_{f^{(q)}}(\alpha^{-1/2}, (a-\eta, b+\eta)), C_2\, \alpha^{-1}\},$$

where $C_1 = C_1(r,c)$ *and* $C_2 = C_2(r,f,c)$.

Proof By our hypothesis we have,

$$f(t) = \sum_{i=0}^{q} \frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{f^{(q)}(\xi) - f^{(q)}(x)}{q!}(t-x)^q \chi(t) + \phi(t,x)(1-\chi(t)), \tag{7}$$

where ξ lies between t and x and $\chi(t)$ is the characteristic function of $(a-\eta, b+\eta)$. The function $\phi(t,x)$ for $t \in [a,b]$ is bounded by $Me^{\gamma t}$ for some constant $M > 0$.

We operate $\frac{d^r}{d\omega^r} B_{\alpha}^{\rho}(.;\omega)$ on the equality (7) and break the right-hand side into three parts E_1, E_2 and E_3, say, corresponding to the three terms on the right-hand side of Eq. (7).

Now, treating E_1 in a manner similar to the treatment of J_1 of Theorem 2, we get $E_1 = f^{(r)}(x) + O(\alpha^{-1})$, uniformly in $x \in [a,b]$.
Making use of the inequality

$$|f^{(q)}(\xi) - f^{(q)}(x)| \leq \left(1 + \frac{|t-x|}{\delta}\right) \omega_{f^{(q)}}(\delta),\ \delta > 0,$$

and Lemma 4, we get

$$|E_2| \leq \frac{\omega_{f^{(q)}}(\delta)}{q!} \left\{ \sum_{k=1}^{\infty} \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} \frac{\alpha^i |k-\alpha x|^j |q_{i,j,r}(x,c)|}{(p(x,c))^r} p_{\alpha,k}(x,c) \right.$$
$$\times \int_0^{\infty} \theta_{\alpha,k}^{\rho}(t) \left(1 + \frac{|t-x|}{\delta}\right) |t-x|^q \chi(t) dt$$

$$
\begin{aligned}
&+\left(x^q + \frac{x^{q+1}}{\delta}\right)\phi_{\alpha,c}^{(r)}(x)\Bigg\} \\
&= E_4 + E_5.
\end{aligned}
$$

Finally, let

$$
S_1 = \sup_{x\in[a,b]} \sup_{\substack{2i+j\leq r \\ i,j\geq 0}} \frac{|q_{i,j,r}(x,c)|}{(p(x,c))^r},
$$

then by applying Schwarz inequality, Lemmas 1 and 3, we obtain

$$
\begin{aligned}
E_4 &\leq \frac{\omega_{f^{(q)}}(\delta)S_1}{q!} \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} \alpha^{i+j}\left(\sum_{k=1}^{\infty}\left(\frac{k}{\alpha}-x\right)^{2j} p_{\alpha,k}(x,c)\right)^{1/2} \\
&\quad\times\left\{\left(\sum_{k=1}^{\infty} p_{\alpha,k}(x,c)\int_0^{\infty}\theta_{\alpha,k}^{\rho}(t)(t-x)^{2q}dt\right)^{1/2}\right. \\
&\quad\left.+\frac{1}{\delta}\left(\sum_{k=1}^{\infty} p_{\alpha,k}(x,c)\int_0^{\infty}\theta_{\alpha,k}^{\rho}(t)(t-x)^{2q+2}dt\right)^{1/2}\right\} \\
&\leq \omega_{f^{(q)}}(\delta)S_1 \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} \alpha^{i+j}\left(\upsilon_{\alpha,2j}(x) - x^{2j}\phi_{\alpha,c}(x)\right)^{1/2} \\
&\quad\times\left\{\left(B_{\alpha}^{\rho}((t-x)^{2q};x) - x^{2q}\phi_{\alpha,c}(x)\right)^{1/2}\right. \\
&\quad\left.+\frac{1}{\delta}\left(B_{\alpha}^{\rho}((t-x)^{2q+2};x) - x^{2q+2}\phi_{\alpha,c}(x)\right)^{1/2}\right\} \\
&= \omega_{f^{(q)}}(\delta) \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} \alpha^{i+j}\{O(\alpha^{-j}) + O(\alpha^{-s_1})\}^{1/2} \\
&\quad\times\{(O(\alpha^{-q}) + O(\alpha^{-s_2})\}^{1/2} + \frac{1}{\delta}\{(O(\alpha^{-(q+1)}) \\
&\quad+ O(\alpha^{-s_3}))\}^{1/2}, \quad for\ any\ s_1,s_2,s_3 > 0.
\end{aligned}
$$

Choosing s_1, s_2, s_3 such that $s_1 > j, s_2 > q, s_3 > q + 1$, we have

$$
|E_4| = \omega_{f^{(q)}}(\delta) \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} \alpha^{i+j} O\left(\frac{1}{\alpha^{j/2}}\right)\left\{O\left(\frac{1}{\alpha^{q/2}}\right) + \frac{1}{\delta}O\left(\frac{1}{\alpha^{(q+1)/2}}\right)\right\}.
$$

Now, on choosing $\delta = \alpha^{-1/2}$, we get

$$|E_4| \leq C_1 \alpha^{-(q-r)/2} \omega_{f^{(q)}}(\alpha^{-1/2}, (a-\eta, b+\eta)).$$

Next, proceeding in a manner similar to the estimate of I_8 in Theorem 1, we have $E_5 = O(\alpha^{-p})$, for any $p > 0$. Choosing $p > 1$, we have $E_5 = O(\alpha^{-1})$, as $\alpha \to \infty$. Finally, proceeding along the lines of the estimate of I_{10} of Theorem 2, we obtain $E_3 = o(\alpha^{-1})$ as $\alpha \to \infty$.

On combining the estimates of $E_1 - E_5$, we get the required result.

3.2 Rate of Convergence

In this section, we shall estimate the rate of convergence for the generalized hybrid operators B_α^ρ for functions with derivatives of bounded variation. In recent years, several researchers have obtained results in this direction for different sequences of linear positive operators. We refer the reader to some of the related papers (cf. [1, 4, 6, 7, 9], etc.).

Let $f \in DBV_\gamma[0,\infty)$, $\gamma \geq 0$ be the class of all functions defined on $[0,\infty)$, having a derivative that coincides, a.e. with a function of bounded variation on every finite subinterval of $[0,\infty)$ and $|f(t)| \leq Mt^\gamma$, $\forall \ t > 0$.
It turns out that for $f \in DBV_\gamma[0,\infty)$, we may write

$$f(x) = \int_0^x g(t)dt + f(0),$$

where $g(t)$ is a function of bounded variation on each finite subinterval of $[0,\infty)$.

Lemma 5 *For all $x \in (0,\infty)$, $\lambda > 1$ and α sufficiently large, we have*

$$\text{(i)}\ \ \lambda_\alpha^\rho(x,t) = \int_0^t K_\alpha^\rho(x,u)du \leq \frac{1}{(x-t)^2}\frac{\lambda x(1+cx)}{\alpha},\ \ 0 \leq t < x;$$

$$\text{(ii)}\ \ 1-\lambda_\alpha^\rho(x,z) = \int_z^\infty K_\alpha^\rho(x,u)du \leq \frac{1}{(z-x)^2}\frac{\lambda x(1+cx)}{\alpha},\ \ x < z < \infty.$$

Proof First we prove (i).

$$\begin{aligned}\lambda_\alpha^\rho(x,t) &= \int_0^t K_\alpha^\rho(x,u)du \le \int_0^t \left(\frac{x-u}{x-t}\right)^2 K_\alpha^\rho(x,u)du\\ &\le \frac{1}{(x-t)^2} B_\alpha^\rho((u-x)^2;x)\\ &\le \frac{1}{(x-t)^2}\frac{\lambda x(1+cx)}{\alpha}.\end{aligned}$$

The proof of (ii) is similar.

Theorem 4 *Let $f \in DBV_\gamma[0,\infty), \gamma \ge 0$. Then for every $x \in (0,\infty)$, $r(\in \mathbb{N}) > 2\gamma$ and sufficiently large α, we have*

$$\begin{aligned}|B_\alpha^\rho(f;x) - f(x)| \le{}& \left|\frac{f'(x+)-f'(x-)}{2}\right|\left\{\frac{\lambda x(1+cx)}{\alpha}\right\}^{1/2}\\ &+\frac{x}{\sqrt{\alpha}}\bigvee_{x-\frac{x}{\sqrt{\alpha}}}^{x+\frac{x}{\sqrt{\alpha}}}(f_x') + \frac{\lambda(1+cx)}{\alpha}\sum_{m=1}^{\sqrt{[\alpha]}}\bigvee_{x-\frac{x}{m}}^{x+\frac{x}{m}}(f_x')\\ &+|f'(x+)|\left\{\frac{\lambda x(1+cx)}{\alpha}\right\}^{1/2}\\ &+|f(2x)-f(x)-xf'(x+)|\frac{\lambda(1+cx)}{\alpha x}\\ &+M'\frac{A(r,x)}{\alpha^{\gamma/2}} + |f(x)|\frac{\lambda(1+cx)}{\alpha x},\end{aligned}$$

where

$$f_x'(t) = \begin{cases} f'(t)-f'(x+), & x<t<\infty\\ 0 & t=x\\ f'(t)-f'(x-), & 0\le t<x\end{cases},$$

$\bigvee_a^b(f'(x))$ is the total variation of f_x' on $[a,b]$, $A(r,x)$ is a constant depending on r and x and M' is a constant depending on f and γ.

Proof By the hypothesis, we may write

$$\begin{aligned}f'(t) ={}& \frac{1}{2}\Big(f'(x+)+f'(x-)\Big) + f_x'(t)\\ &+\frac{1}{2}\Big(f'(x+)-f'(x-)\Big)sgn(t-x)\\ &+\delta_x(t)\Big(f'(t)-\frac{1}{2}\Big(f'(x+)+f'(x-)\Big)\Big),\end{aligned}\tag{8}$$

where

$$\delta_x(t) = \begin{cases} 1 & t = x \\ 0 & t \neq x. \end{cases}$$

From Eqs. (2) and (8), we have

$$\begin{aligned} B_\alpha^\rho(f;x) - f(x) &= \int_0^\infty K_\alpha^\rho(x,t)f(t)dt - f(x) = \int_0^\infty (f(t)-f(x))K_\alpha^\rho(x,t)dt \\ &= \int_0^x (f(t)-f(x))K_\alpha^\rho(x,t)dt + \int_x^\infty (f(t)-f(x))K_\alpha^\rho(x,t)dt \\ &= -\int_0^x \Big(\int_t^x f'(u)du\Big)K_\alpha^\rho(x,t)dt + \int_x^\infty \Big(\int_x^t f'(u)du\Big)K_\alpha^\rho(x,t)dt \\ &= I_1(x) + I_2(x), \ say. \end{aligned}$$

Using Eq. (8), we get

$$\begin{aligned} I_1(x) = \int_0^x \Big\{ \int_t^x \frac{1}{2}\Big(f'(x+) + f'(x-)\Big) + f_x'(u) + \frac{1}{2}\Big(f'(x+) - f'(x-)\Big) sgn(u-x) \\ + \delta_x(u)\Big(f'(u) - \frac{1}{2}\Big(f'(x+) + f'(x-)\Big)\Big)du \Big\} K_\alpha^\rho(x,t)dt. \end{aligned}$$

Since $\int_x^t \delta_x(u)du = 0$, we have

$$\begin{aligned} I_1(x) &= \frac{1}{2}\Big(f'(x+) + f'(x-)\Big)\int_0^x (x-t)K_\alpha^\rho(x,t)dt + \int_0^x \Big(\int_x^t f_x'(u)du\Big)K_\alpha^\rho(x,t)dt \\ &\quad -\frac{1}{2}\Big(f'(x+) - f'(x-)\Big)\int_0^x |x-t|K_\alpha^\rho(x,t)dt. \end{aligned} \tag{9}$$

Proceeding similarly, we find that

$$\begin{aligned} I_2(x) &= \frac{1}{2}\Big(f'(x+) + f'(x-)\Big)\int_x^\infty (t-x)K_\alpha^\rho(x,t)dt + \int_x^\infty \Big(\int_x^t f_x'(u)du\Big)K_\alpha^\rho(x,t)dt \\ &\quad +\frac{1}{2}\Big(f'(x+) - f'(x-)\Big)\int_x^\infty |t-x|K_\alpha^\rho(x,t)dt. \end{aligned} \tag{10}$$

By combining (9) and (10), we get

$$\begin{aligned}
B_\alpha^\rho(f;x)-f(x) &= \frac{1}{2}\Big(f'(x+)+f'(x-)\Big)\int_0^\infty (t-x)K_\alpha^\rho(x,t)dt\\
&+\frac{1}{2}\Big(f'(x+)-f'(x-)\Big)\int_0^\infty |t-x|K_\alpha^\rho(x,t)dt\\
&-\int_0^x\left(\int_t^x f_x'(u)du\right)K_\alpha^\rho(x,t)dt+\int_x^\infty\left(\int_x^t f_x'(u)du\right)K_\alpha^\rho(x,t)dt.
\end{aligned}$$

Hence

$$\begin{aligned}
|B_\alpha^\rho(f;x)-f(x)| &\le \left|\frac{f'(x+)+f'(x-)}{2}\right||B_\alpha^\rho(t-x;x)|+\left|\frac{f'(x+)-f'(x-)}{2}\right|B_\alpha^\rho(|t-x|;x)\\
&+\left|\int_0^x\left(\int_t^x f_x'(u)du\right)K_\alpha^\rho(x,t)dt\right|+\left|\int_x^\infty\left(\int_x^t f_x'(u)du\right)K_\alpha^\rho(x,t)dt\right|.
\end{aligned}\tag{11}$$

On application of Lemma 5 and integration by parts, we obtain

$$\int_0^x\left(\int_t^x f_x'(u)du\right)K_\alpha^\rho(x,t)dt=\int_0^x\left(\int_t^x f_x'(u)du\right)\frac{\partial}{\partial t}\lambda_\alpha^\rho(x,t)dt=\int_0^x f_x'(t)\lambda_\alpha^\rho(x,t)dt.$$

Thus,

$$\begin{aligned}
\left|\int_0^x\left(\int_t^x f_x'(u)du\right)K_\alpha^\rho(x,t)dt\right| &\le \int_0^x |f_x'(t)|\lambda_\alpha^\rho(x,t)dt\\
&\le \int_0^{x-\frac{x}{\sqrt{\alpha}}}|f_x'(t)|\lambda_\alpha^\rho(x,t)dt+\int_{x-\frac{x}{\sqrt{\alpha}}}^x |f_x'(t)|\lambda_\alpha^\rho(x,t)dt.
\end{aligned}$$

Since $f_x'(x)=0$ and $\lambda_\alpha^\rho(x,t)\le 1$, we get

$$\int_{x-\frac{x}{\sqrt{\alpha}}}^{x} |f_x'(t)|\lambda_\alpha^\rho(x,t)dt = \int_{x-\frac{x}{\sqrt{\alpha}}}^{x} |f_x'(t) - f_x'(x)|\lambda_\alpha^\rho(x,t)dt \leq \int_{x-\frac{x}{\sqrt{\alpha}}}^{x} \bigvee_t^x (f_x')dt$$
$$\leq \bigvee_{x-\frac{x}{\sqrt{\alpha}}}^{x} (f_x') \int_{x-\frac{x}{\sqrt{\alpha}}}^{x} dt = \frac{x}{\sqrt{\alpha}} \bigvee_{x-\frac{x}{\sqrt{\alpha}}}^{x} (f_x').$$

Similarly, using Lemma 5 and putting $t = x - \frac{x}{u}$, we get

$$\int_0^{x-\frac{x}{\sqrt{\alpha}}} |f_x'(t)|\lambda_\alpha^\rho(x,t)dt \leq \frac{\lambda x(1+cx)}{\alpha} \int_0^{x-\frac{x}{\sqrt{\alpha}}} |f_x'(t)|\frac{dt}{(x-t)^2}$$
$$\leq \frac{\lambda x(1+cx)}{\alpha} \int_0^{x-\frac{x}{\sqrt{\alpha}}} \bigvee_t^x (f_x')\frac{dt}{(x-t)^2}$$
$$= \frac{\lambda(1+cx)}{\alpha} \int_1^{\sqrt{\alpha}} \bigvee_{x-\frac{x}{u}}^{x} (f_x')du \leq \frac{\lambda(1+cx)}{\alpha} \sum_{m=1}^{[\sqrt{\alpha}]} \bigvee_{x-\frac{x}{m}}^{x} (f_x').$$

Consequently,

$$\left| \int_0^x \left(\int_t^x f_x'(u)du \right) K_\alpha^\rho(x,t)dt \right| \leq \frac{x}{\sqrt{\alpha}} \bigvee_{x-\frac{x}{\sqrt{\alpha}}}^{x} (f_x') + \frac{\lambda(1+cx)}{\alpha} \sum_{m=1}^{[\sqrt{\alpha}]} \bigvee_{x-\frac{x}{m}}^{x} (f_x'). \tag{12}$$

Also, we have

$$\left| \int_x^\infty \left(\int_x^t f_x'(u)du \right) K_\alpha^\rho(x,t)dt \right| \leq \left| \int_x^{2x} \left(\int_x^t f_x'(u)du \right) \frac{\partial}{\partial t}(1-\lambda_\alpha^\rho(x,t))dt \right|$$
$$+ \left| \int_{2x}^\infty \left(\int_x^t f_x'(u)du \right) K_\alpha^\rho(x,t)dt \right|$$
$$\leq \left| \int_{2x}^\infty (f(t)-f(x))K_\alpha^\rho(x,t)dt \right|$$
$$+ |f'(x+)| \left| \int_{2x}^\infty (t-x)K_\alpha^\rho(x,t)dt \right|$$

$$+\left|\int_x^{2x} f_x'(u)du\right||1-\lambda_\alpha^\rho(x,2x)|$$
$$+\int_x^{2x}|f_x'(t)|(1-\lambda_\alpha^\rho(x,t))dt.$$

Applying Lemma 5, we get

$$\begin{aligned}\left|\int_x^\infty\left(\int_x^t f_x'(u)du\right)K_\alpha^\rho(x,t)dt\right| &\le M\int_{2x}^\infty t^\gamma K_\alpha^\rho(x,t)dt+|f(x)|\int_{2x}^\infty K_\alpha^\rho(x,t)dt\\ &+|f'(x+)|\left\{\frac{\lambda x(1+cx)}{\alpha}\right\}^{1/2}\\ &+\frac{\lambda(1+cx)}{\alpha x}|f(2x)-f(x)-xf'(x+)|\\ &+\frac{x}{\sqrt{\alpha}}\bigvee_x^{x+\frac{x}{\sqrt{\alpha}}}(f_x')+\frac{\lambda(1+cx)}{\alpha}\sum_{m=1}^{[\sqrt{\alpha}]}\bigvee_x^{x+\frac{x}{m}}(f_x').\end{aligned} \tag{13}$$

We note that we can choose $r\in\mathbb{N}$ such that $2r>\gamma$.

Since $t\le 2(t-x)$ and $x\le t-x$ when $t\ge 2x$, using Hölder's inequality and Lemma 3, we obtain

$$M\int_{2x}^\infty t^\gamma K_\alpha^\rho(x,t)dt+|f(x)|\int_{2x}^\infty K_\alpha^\rho(x,t)dt$$

$$\begin{aligned}&\le 2^\gamma M\int_{2x}^\infty (t-x)^\gamma K_\alpha^\rho(x,t)dt+\frac{|f(x)|}{x^2}\int_{2x}^\infty (t-x)^2K_\alpha^\rho(x,t)dt\\ &\le 2^\gamma M\left(\int_0^\infty (t-x)^{2r}K_\alpha^\rho(x,t)dt\right)^{\gamma/2r}+|f(x)|\frac{\lambda(1+cx)}{\alpha x}\\ &\le M'\frac{A(r,x)}{\alpha^{\gamma/2}}+|f(x)|\frac{\lambda(1+cx)}{\alpha x},\ \text{where } M'=2^\gamma M.\end{aligned} \tag{14}$$

Using Lemma 3 and combining (11), (12), (13) and (14), we get the required result.

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