A Sequence Space and Uniform (A, φ)—Statistical Convergence

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Abstract In this, we introduce and study some properties of the new sequence space that is defined using the φ—function and de la Valée-Poussin mean. We also study some connections between $V_\lambda((A, \varphi))$—strong summability of sequences and λ—strong convergence with respect to a modulus.

Keywords Modulus function · φ-function · λ—strong convergence · Matrix transformations · Sequence spaces · Statistical convergence

1 Introduction and Background

Let s denote the set of all real and complex sequences $x = (x_k)$. By l_∞ and c, we denote the Banach spaces of bounded and convergent sequences $x = (x_k)$ normed by $||x|| = \sup_n |x_n|$, respectively. A sequence $x \in l_\infty$ is said to be almost convergent if all of its Banach limits coincide. Let $\hat{c}$ denote the space of all almost convergent sequences. Lorentz [6] has shown that

$$\hat{c} = \left\{ x \in l_\infty : \lim_m t_{m,n}(x) \text{ exists uniformly in } n \right\}$$

where

$$t_{m,n}(x) = \frac{x_n + x_{n+1} + x_{n+2} + \cdots + x_{n+m}}{m+1}.$$

The space $[\hat{c}]$ of strongly almost convergent sequences was introduced by Maddox [7] and also independently by Freedman et al. [3] as follows:

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R.N. Mohapatra et al. (eds.), *Mathematics and Computing*,
Springer Proceedings in Mathematics & Statistics 139,
DOI 10.1007/978-81-322-2452-5_34

$$[\hat{c}] = \left\{ x \in l_\infty : \lim_m t_{m,n}(|x - L|) = 0, \text{ uniformly in } n, \text{ for some } L \right\}.$$

Let $\lambda = (\lambda_i)$ be a nondecreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{i+1} \leq \lambda_i + 1, \lambda_1 = 1.$$

The collection of such sequence λ will be denoted by Δ.

The generalized de la Valée-Poussin mean is defined as

$$T_i(x) = \frac{1}{\lambda_i} \sum_{k \in I_i} x_k$$

where $I_i = [i - \lambda_i + 1, i]$. A sequence $x = (x_n)$ is said to be (V, λ)—summable to a number L, if $T_i(x) \to L$ as $i \to \infty$ (see [9]).

Recently, Malkowsky and Savaş [9] introduced the space $[V, \lambda]$ of λ—strongly convergent sequences as follows:

$$[V, \lambda] = \left\{ x = (x_k) : \lim_i \frac{1}{\lambda_i} \sum_{k \in I_i} |x_k - L| = 0, \text{ for some } L \right\}.$$

Note that in the special case where $\lambda_i = i$, the space $[V, \lambda]$ reduces the space w of strongly Cesàro summable sequences which is defined as

$$w = \left\{ x = (x_k) : \lim_i \frac{1}{i} \sum_{k=1}^{i} |x_k - L|) = 0, \text{ for some } L \right\}.$$

More results on λ- strong convergence can be seen from [12, 20–24].

Ruckle [16] used the idea of a modulus function f to construct a class of FK spaces

$$L(f) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} f\,(|x_k|) < \infty \right\}.$$

The space $L(f)$ is closely related to the space l_1, which is an $L(f)$ space with $f(x) = x$ for all real $x \geq 0$.

Maddox [8] introduced and examined some properties of the sequence spaces $w_0(f)$, $w(f)$, and $w_\infty(f)$ defined using a modulus f, which generalized the well-known spaces w_0, w and w_∞ of strongly summable sequences.

Recently, Savas [19] generalized the concept of strong almost convergence using a modulus f and examined some properties of the corresponding new sequence spaces.

Waszak [26] defined the lacunary strong (A, φ)—convergence with respect to a modulus function.

Following Ruckle [16], a modulus function f is a function from $[0, \infty)$ to $[0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$,
(ii) $f(x + y) \leq f(x) + f(y)$ for all $x, y \geq 0$,
(iii) f increasing,
(iv) f is continuous from the right at zero.

Since $|f(x) - f(y)| \leq f(|x - y|)$, it follows from condition (iv) that f is continuous on $[0, \infty)$.

If $x = (x_k)$ is a sequence and $A = (a_{nk})$ is an infinite matrix, then Ax is the sequence whose nth term is given by $A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k$. Thus we say that x is A-summable to L if $\lim_{n\to\infty} A_n(x) = L$. Let X and Y be two sequence spaces and $A = (a_{nk})$ an infinite matrix. If for each $x \in X$ the series $A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k$ converges for each n and the sequence $Ax = A_n(x) \in Y$ we say that A maps X into Y. By (X, Y) we denote the set of all matrices which maps X into Y, and in addition if the limit is preserved then we denote the class of such matrices by $(X, Y)_{reg}$.

A matrix A is called regular , i.e., $A \in (c, c)_{reg}$. if $A \in (c, c)$ and $lim_n A_n(x) = lim_k x_k$ for all $x \in c$.

In 1993, Nuray and Savas [14] defined the following sequence spaces:

Definition 1 Let f be a modulus and A a nonnegative regular summability method. We let

$$w(\hat{A}, f) = \left\{ x : lim_n \sum_{k=1}^{\infty} a_{nk} f(|x_{k+m} - L|) = 0, \text{ for some L, uniformly in } m \right\}$$

and

$$w(\hat{A}, f)_0 = \left\{ x : lim_n \sum_{k=1}^{\infty} a_{nk} f(|x_{k+m}|) = 0, \text{ uniformly in } m \right\}.$$

If we take $A = (a_{nk})$ as

$$a_{nk} := \begin{cases} \frac{1}{n}, & \text{if } \quad n \geq k, \\ 0, & \text{otherwise.} \end{cases}$$

Then the above definitions are reduced to $[\hat{c}(f)]$ and $[\hat{c}(f)]_0$ which were defined and studied by Pehlivan [15].

If we take $A = (a_{nk})$ is a de la Valée poussin mean, i.e.,

$$a_{nk} := \begin{cases} \frac{1}{\lambda_n}, & \text{if } \ k \in I_n = [n - \lambda_n + 1, n], \\ 0, & \text{otherwise.} \end{cases}$$

Then these definitions are reduced to the following sequence spaces which were defined and studied by Malkowsky and Savas [9].

$$w(\hat{V}, \lambda, f) = \left\{ x : lim_j \frac{1}{\lambda_j} \sum_{k \in I_j} f(|x_{k+m} - L|) = 0, \text{ for some L, uniformly in } m \right\}$$

and

$$w(\hat{V}, \lambda, f)_0 = \left\{ x : lim_j \frac{1}{\lambda_j} \sum_{k \in I_j} f(|x_{k+m}|) = 0, \text{ uniformly in } m \right\}$$

When $\lambda_j = j$ the above sequence spaces become $[\hat{c}(f)]_0$ and $[\hat{c}(f)]$.

By a φ-function we understand a continuous nondecreasing function $\varphi(u)$ defined for $u \geq 0$ and such that $\varphi(0) = 0$, $\varphi(u) > 0$, for $u > 0$ and $\varphi(u) \to \infty$ as $u \to \infty$, (see, [26]).

A φ-function φ is called non-weaker than a φ-function ψ if there are constants $c, b, k, l > 0$ such that $c\psi(lu) \leq b\varphi(ku)$, (for all large u) and we write $\psi \prec \varphi$.

A φ-function φ and ψ are called equivalent and we write $\varphi \sim \psi$ if there are positive constants b_1, b_2, c, k_1, k_2, l such that $b_1\varphi(k_1u) \leq c\psi(lu) \leq b_2\varphi(k_2u)$, (for all large u), (see, [26]).

A φ-function φ is said to satisfy (Δ_2)-condition, (for all large u) if there exists constant $K > 1$ such that $\varphi(2u) \leq K\varphi(u)$.

In this paper, we introduce and study some properties of the following sequence space that is defined using the φ- function and de la Valée-Poussin mean and some known results are also obtained as special cases.

2 Main Results

Let $\Lambda = (\lambda_j)$ be the same as above, φ be given φ-function, and f be given modulus function, respectively. Moreover, let $\mathbf{A} = (a_{nk}(i))$ be the generalized three-parametric real matrix. Then we define

$$V^0_\lambda((A, \varphi), f) = \left\{ x = (x_k) : \lim_j \frac{1}{\lambda_j} \sum_{n \in I_j} f\left(\left| \sum_{k=1}^{\infty} a_{nk}(i)\varphi(|x_k|) \right| \right) = 0, \text{ uniformly in } i \right\}.$$

If $\lambda_j = j$, we have

$$V^0_\lambda((A, \varphi), f) = \left\{ x = (x_k) : \lim_j \frac{1}{j} \sum_{n=1}^{j} f\left(\left| \sum_{k=1}^{\infty} a_{nk}(i)\varphi(|x_k|) \right| \right) = 0, \text{ uniformly in } i \right\}.$$

If $x \in V_\lambda^0((A,\varphi), f)$, the sequence x is said to be λ—strong (A,φ)—convergent to zero with respect to a modulus f. When $\varphi(x) = x$ for all x, we obtain

$$V_\lambda^0((A), f) = \left\{ x = (x_k) : \lim_j \frac{1}{\lambda_j} \sum_{n \in I_j} f\Big(\Big| \sum_{k=1}^{\infty} a_{nk}(i)(|x_k|) \Big| \Big) = 0, \text{ uniformly in } i \right\}.$$

If $f(x) = x$, we write

$$V_\lambda^0(A, \varphi) = \left\{ x = (x_k) : \lim_j \frac{1}{\lambda_j} \sum_{n \in I_j} \Big(\Big| \sum_{k=1}^{\infty} a_{nk}(i)\varphi(|x_k|) \Big| \Big) = 0, \text{ uniformly in } i \right\}.$$

If we take $A = I$ and $\varphi(x) = x$ respectively, then we have

$$V_\lambda^0(I, f) = \left\{ x = (x_k) : \lim_j \frac{1}{\lambda_j} \sum_{k \in I_j} f\Big(|x_k| \Big) = 0 \right\}.$$

If we take $A = I$, $\varphi(x) = x$ and $f(x) = x$ respectively, then we have

$$V_\lambda^0((I)) = \left\{ x = (x_k) : \lim_j \frac{1}{\lambda_j} \sum_{k \in I_j} |x_k| = 0 \right\},$$

which was defined and studied by Savaş and Savaş [18].

If we define the matrix $A = (a_{nk}(i))$ as follows: for all i

$$a_{nk}(i) := \{ \begin{matrix} \frac{1}{n}, & \text{if} \quad n \geq k, \\ 0, & \text{otherwise.} \end{matrix}$$

then we have,

$$V_\lambda^0(\mathbf{C}, \varphi, f) = \left\{ x = (x_k) : \lim_j \frac{1}{\lambda_j} \sum_{n \in I_j} f\Big(\Big| \frac{1}{n} \sum_{k=1}^{n} \varphi(|x_k|) \Big| \Big) = 0 \right\}.$$

If we define

$$a_{nk}(i) := \{ \begin{matrix} \frac{1}{n}, & \text{if } i \le k \le i+n-1, \\ 0, & \text{otherwise.} \end{matrix}$$

then we have,

$$V_\lambda^0(\hat{c}, \varphi, f) = \left\{ x = (x_k) : \lim_j \frac{1}{\lambda_j} \sum_{n \in I_j} f\left(\left| \frac{1}{n} \sum_{k=i}^{i+n} \varphi(|x_k|) \right| \right) = 0, \text{ uniformly in } i \right\}.$$

We now have:

Theorem 1 *Let $\boldsymbol{A} = (a_{nk}(i))$ be the generalized three parametric real matrix and let the φ—function $\varphi(u)$ satisfy the condition (Δ_2). Then the following conditions are true:*

(a) If $x = (x_k) \in w((\boldsymbol{A}, \varphi), f)$ and α is an arbitrary number, then $\alpha x \in w((\boldsymbol{A}, \varphi), f)$.
(b) If $x, y \in w((\boldsymbol{A}, \varphi), f)$ where $x = (x_k)$, $y = (y_k)$ and α, β are given numbers, then $\alpha x + \beta y \in w((\boldsymbol{A}, \varphi), f)$.

The proof is a routine verification by using standard techniques and hence is omitted.

Theorem 2 *Let f be a modulus function.*

$$V_\lambda^0(A, \varphi) \subseteq V_\lambda^0((A, \varphi), f).$$

Proof Let $x \in V_\lambda^0(A, \varphi)$. For a given $\varepsilon > 0$ we choose $0 < \delta < 1$ such that $f(x) < \varepsilon$ for every $x \in [0, \delta]$. We can write for all i

$$\frac{1}{\lambda_j} \sum_{n \in I_j} f\left(\left| \sum_{k=1}^{\infty} a_{nk}(i) \varphi(|x_k|) \right| \right) = S_1 + S_2,$$

where $S_1 = \frac{1}{\lambda_j} \sum_{n \in I_j} f\left(\left| \sum_{k=1}^{\infty} a_{nk}(i) \varphi(|x_k|) \right| \right)$ and this sum is taken over

$$\sum_{k=1}^{\infty} a_{nk}(i) \varphi(|x_k|) \le \delta$$

and

$$S_2 = \frac{1}{\lambda_j} \sum_{n \in I_j} f\left(\left| \sum_{k=1}^{\infty} a_{nk}(i)\varphi(|x_k|) \right| \right)$$

and this sum is taken over

$$\sum_{k=1}^{\infty} a_{nk}(i)\varphi(|x_k|) > \delta.$$

By definition of the modulus f we have $S_1 = \frac{1}{\lambda_j} \sum_{n \in I_j} f\Big(\delta\Big) = f(\delta) < \varepsilon$ and moreover

$$S_2 = f(1)\frac{1}{\delta}\frac{1}{\lambda_j} \sum_{n \in I_j} \sum_{k=1}^{\infty} a_{nk}(i)\varphi(|x_k|).$$

Thus we have $x \in V_{\lambda}^{0}((A, \varphi), f)$.
This completes the proof.

3 Uniform (A, φ)—Statistical Convergence

The idea of convergence of a real sequence was extended to statistical convergence by Fast [2] (see also Schoenberg [25]) as follows: If $\mathbb{N}$ denotes the set of natural numbers and $K \subset \mathbb{N}$ then $K(m, n)$ denotes the cardinality of the set $K \cap [m, n]$, the upper and lower natural densities of the subset K are defined as

$$\overline{d}(K) = \lim_{n \to \infty} \sup \frac{K(1, n)}{n} \text{ and } \underline{d}(K) = \lim_{n \to \infty} \inf \frac{K(1, n)}{n}.$$

If $\overline{d}(K) = \underline{d}(K)$ then we say that the natural density of K exists and it is denoted simply by $d(K)$. Clearly $d(K) = \lim_{n \to \infty} \frac{K(1, n)}{n}$.

A sequence (x_k) of real numbers is said to be statistically convergent to L if for arbitrary $\epsilon > 0$, the set $K(\epsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$ has natural density zero. Statistical convergence turned out to be one of the most active areas of research in summability theory after the work of Fridy [4] and Šalát [17].

In another direction, a new type of convergence called λ-statistical convergence was introduced in [13] as follows.

A sequence (x_k) of real numbers is said to be λ- statistically convergent to L (or, S_λ-convergent to L) if for any $\epsilon > 0$,

$$\lim_{j \to \infty} \frac{1}{\lambda_j} |\{k \in I_j : |x_k - L| \geq \epsilon\}| = 0$$

where $|A|$ denotes the cardinality of $A \subset \mathbb{N}$. In [13] the relation between λ-statistical convergence and statistical convergence was established among other things.

Recently, Savas [20] defined almost λ-statistical convergence using the notion of (V, λ)-summability to generalize the concept of statistical convergence.

Assume that A is a nonnegative regular summability matrix. Then the sequence $x = (x_n)$ is called statistically convergent to L provided that, for every $\varepsilon > 0$, (see, [5])

$$lim_j \sum_{n:|x_n - L| \geq \varepsilon} a_{jn} = 0.$$

Let $\mathbf{A} = (a_{nk}(i))$ be the generalized three parametric real matrix and the sequence $x = (x_k)$, the φ-function $\varphi(u)$ and a positive number $\varepsilon > 0$ be given. We write, for all i

$$K_\lambda^j((A, \varphi), \varepsilon) = \{n \in I_j : \sum_{k=1}^{\infty} a_{nk}(i)\varphi(|x_k|) \geq \varepsilon\}.$$

The sequence x is said to be uniform (A, φ)—statistically convergent to a number zero if for every $\varepsilon > 0$

$$lim_j \frac{1}{\lambda_j} \mu(K_\lambda^j((A, \varphi), \varepsilon)) = 0, \text{ uniformly in } i$$

where $\mu(K_\lambda^j((A, \varphi), \varepsilon))$ denotes the number of elements belonging to $K_\lambda^j((A, \varphi), \varepsilon)$. We denote by $S_\lambda^0((A, \varphi))$, the set of sequences $x = (x_k)$ which are uniform (A, φ)—statistical convergent to zero.

If we take $A = I$ and $\varphi(x) = x$ respectively, then $S_\lambda^0((A, \varphi))$ reduce to S_λ^0 which was defined as follows, (see, Mursaleen [13]).

$$S_\lambda^0 = \left\{ x = (x_k) : lim_j \frac{1}{\lambda_j} |\{k \in I_j : |x_k| \geq \varepsilon\}| = 0 \right\}.$$

Remark 1 (i) If for all i,

$$a_{nk} := \{ \begin{matrix} \frac{1}{n}, & \text{if} \quad n \geq k, \\ 0, & \text{otherwise.} \end{matrix}$$

then $S_\lambda((A, \varphi))$ reduce to $S_\lambda^0((C, \varphi))$, i.e., uniform (C, φ)— statistical convergence. (ii) If for all i, (see, [1]),

$$a_{nk} := \{ \begin{matrix} \frac{p_k}{P_n}, & \text{if} \quad n \geq k, \\ 0, & \text{otherwise.} \end{matrix}$$

then $S_\lambda((A, \varphi))$ reduce to $S_\lambda^0((N, p), \varphi))$, i.e., uniform $((N, p), \varphi)$— statistical convergence, where $p = p_k$ is a sequence of nonnegative numbers such that $p_0 > 0$ and

$$P_i = \sum_{k=0}^{n} p_k \to \infty (n \to \infty).$$

We are now ready to state the following theorem.

Theorem 3 *If $\psi \prec \varphi$ then $S_\lambda^0((A, \psi)) \subset S_\lambda^0((A, \varphi))$.*

Proof By our assumptions we have $\psi(|x_k|) \le b\varphi(c|x_k|)$ and we have for all i,

$$\sum_{k=1}^{\infty} a_{nk}(i)\psi(|x_k|) \le b \sum_{k=1}^{\infty} a_{nk}(i)\varphi(c|x_k|) \le K \sum_{k=1}^{\infty} a_{nk}(i)\varphi(|x_k|)$$

for $b, c > 0$, where the constant K is connected with properties of φ. Thus, the condition $\sum_{k=1}^{\infty} a_{nk}(i)\psi(|x_k|) \ge \varepsilon$ implies the condition $\sum_{k=1}^{\infty} a_{nk}(i)\varphi(|x_k|) \ge \varepsilon$ and in consequence we get

$$\mu(K_\lambda^j((A, \varphi), \varepsilon)) \subset \mu(K_\lambda^j((A, \psi), \varepsilon))$$

and

$$lim_j \frac{1}{\lambda_j} \mu\Big(K_\lambda^j((A, \varphi), \varepsilon)) \le lim_j \frac{1}{\lambda_j} \mu(K_\lambda^j((A, \psi), \varepsilon))\Big).$$

This completes the proof.

Theorem 4 *(a) If the matrix A, functions f, and φ are given, then*

$$V_\lambda^0((A, \varphi), f) \subset S_\lambda^0(A, \varphi).$$

(b) If the φ- function $\varphi(u)$ and the matrix A are given, and if the modulus function f is bounded, then

$$S_\lambda^0(A, \varphi) \subset V_\lambda^0(A, \varphi), f).$$

(c) If the φ- function $\varphi(u)$ and the matrix A are given, and if the modulus function f is bounded, then

$$S_\lambda^0(A, \varphi) = V_\lambda^0(A, \varphi), f).$$

Proof (a) Let f be a modulus function and let ε be a positive number. We write the following inequalities:

$$\begin{aligned}\frac{1}{\lambda_j}\sum_{n\in I_j} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}(i)\varphi(|x_k|)\Big|\Big)
&\geq \frac{1}{\lambda_j}\sum_{n\in I_j^1} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}(i)\varphi(|x_k|)\Big|\Big)\\
&\geq \frac{1}{\lambda_j} f(\varepsilon)\sum_{n\in I_j^1} 1\\
&\geq \frac{1}{\lambda_j} f(\varepsilon)\mu(K_\lambda^j(A,\varphi),\varepsilon),\end{aligned}$$

where

$$I_j^1 = \left\{ n \in I_j : \sum_{k=1}^{\infty} a_{nk}(i)\varphi(|x_k|) \geq \varepsilon \right\}.$$

Finally, if $x \in V_\lambda^0((A,\varphi), f)$ then $x \in S_\lambda^0(A,\varphi)$.

(b) Let us suppose that $x \in S_\lambda^0(A,\varphi)$. If the modulus function f is a bounded function, then there exists an integer M such that $f(x) < M$ for $x \geq 0$. Let us take

$$I_j^2 = \left\{ n \in I_j : \sum_{k=1}^{\infty} a_{nk}(i)\varphi(|x_k|) < \varepsilon \right\}.$$

Thus we have

$$\begin{aligned}\frac{1}{\lambda_j}\sum_{n\in I_j} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}(i)\varphi(|x_k|)\Big|\Big)
&\leq \frac{1}{\lambda_j}\sum_{n\in I_j^1} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}(i)\varphi(|x_k|)\Big|\Big)\\
&+ \frac{1}{\lambda_j}\sum_{n\in I_j^2} f\Big(\Big|\sum_{k=1}^{\infty} a_{nk}(i)\varphi(|x_k|)\Big|\Big)\\
&\leq \frac{1}{\lambda_j} M\mu(K_\lambda^j((A,\varphi),\varepsilon) + f(\varepsilon).\end{aligned}$$

Taking the limit as $\varepsilon \to 0$, we obtain that $x \in V_\lambda^0(A,\varphi, f)$.
The proof of (c) follows from (a) and (b).
This completes the proof.

In the next theorem we prove the following relation.

Theorem 5 *If a sequence $x = (x_k)$ is $S(A, \varphi)$—convergent to L and*

$$\lim inf_j \left(\frac{\lambda_j}{j}\right) > 0$$

then it is $S_\lambda(A, \varphi)$ convergent to L, where

$$S(A, \varphi) = \{x = (x_k) : lim_j \frac{1}{j}\mu(K(A, \varphi, \varepsilon)) = 0\}.$$

Proof For a given $\varepsilon > 0$, we have, for all i

$$\{n \in I_j : \sum_{k=0}^{\infty} a_{nk}(i)\varphi(|x_k - L|) \geq \varepsilon\} \subseteq \{n \leq j : \sum_{k=0}^{\infty} a_{nk}(i)\varphi(|x_k - L|) \geq \varepsilon\}.$$

Hence we have,

$$K_\lambda(A, \varphi, \varepsilon) \subseteq K(A, \varphi, \varepsilon).$$

Finally the proof follows from the following inequality:

$$\frac{1}{j}\mu(K(A, \varphi, \varepsilon)) \geq \frac{1}{j}\mu(K_\lambda(A, \varphi, \varepsilon)) = \frac{\lambda_j}{j}\frac{1}{\lambda_j}\mu(K_\lambda(A, \varphi, \varepsilon)).$$

This completes the proof.

Theorem 6 *If $\lambda \in \triangle$ be such that $lim_j \frac{\lambda_j}{j} = 1$ and the sequence $x = (x_k)$ is $S_\lambda(A, \varphi)$—convergent to L then it is $S(A, \varphi)$ convergent to L,*

Proof Let $\delta > 0$ be given. Since $lim_j \frac{\lambda_j}{j} = 1$, we can choose $m \in N$ such that $|\frac{\lambda_j}{j} - 1| < \frac{\delta}{2}$, for all $j \geq m$. Now observe that, for $\varepsilon > 0$

$$\begin{aligned}
&\frac{1}{j}\left|\left\{n \leq j : \sum_{k=0}^{\infty} a_{nk}(i)\varphi(|x_k - L|) \geq \varepsilon\right\}\right| \\
&= \frac{1}{j}\left|\left\{k \leq j - \lambda_j : \sum_{k=0}^{\infty} a_{nk}(i)\varphi(|x_k - L|) \geq \epsilon\right\}\right| \\
&+ \frac{1}{j}\left|\left\{n \in I_j : \sum_{k=0}^{\infty} a_{nk}(i)\varphi(|x_k - L|) \geq \epsilon\right\}\right| \\
&\leq \frac{j - \lambda_j}{j} + \frac{1}{j}\left|\left\{n \in I_j : \sum_{k=0}^{\infty} a_{nk}(i)\varphi(|x_k - L|) \geq \epsilon\right\}\right|
\end{aligned}$$

$$\leq 1-(1-\frac{\delta}{2})+\frac{1}{j}\left|\left\{n \in I_j : \sum_{k=0}^{\infty} a_{nk}(i)\varphi(|x_k - L|) \geq \epsilon\right\}\right|$$
$$=\frac{\delta}{2}+\frac{1}{j}\left|\left\{n \in I_j : \sum_{k=0}^{\infty} a_{nk}(i)\varphi(|x_k - L|) \geq \epsilon\right\}\right|,$$

This completes the proof.

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