Subcentral Automorphisms

R.G. Ghumde and S.H. Ghate

Abstract A concept of subcentral automorphisms of group G with respect to a characterstic subgroup M of $Z(G)$ along with relevant mathematical paraphernalia has been introduced. With the help of this, a number of results on central automorphisms have been generalized.

Keywords Central automorphisms · Subcentral automorphisms · Purely nonabelian group

1 Introduction

Let G be a group. We shall denote the commutator, center, group of automorphisms, and group of inner automorphisms of G by G', $Z(G)$, $\mathrm{Aut}(G)$, and $\mathrm{Inn}(G)$, respectively. Let $\exp(G)$ denote the exponent of G.

For any group H and abelian group K, let $\mathrm{Hom}(H, K)$ denote the group of all homomorphisms from H to K. This is an abelian group with binary operation $fg(x) = f(x)g(x)$ for $f, g \in \mathrm{Hom}(H, K)$.

An automorphism α of G is called central if $x^{-1}\alpha(x) \in Z(G)$ for all $x \in G$. The set of all central automorphisms of G, which is here denoted by $\mathrm{Aut}_c(G)$, is a normal subgroup of $\mathrm{Aut}(G)$. Notice that $\mathrm{Aut}_c(G) = C_{\mathrm{Aut}(G)}(\mathrm{Inn}(G))$, the centralizer of the subgroup $\mathrm{Inn}(G)$ in the group Aut(G). The elements of $\mathrm{Aut}_c(G)$ act trivially on G'.

There have been number of results on the central automorphisms of a group. M.J. Curran [2] proved that, "For any non abelian finite group G, $\mathrm{Aut}_z^z(G)$ is isomorphic with Hom $(G/G'Z(G), Z(G))$, where $\mathrm{Aut}_z^z(G)$ is group of all those central

R.G. Ghumde (✉)
Department of Mathematics, Ramdeobaba College of Engineering and Management, Nagpur 440013, India
e-mail: ranjitghumde@gmail.com

S.H. Ghate
Department of Mathematics, R.T.M. Nagpur University, Nagpur 440013, India
e-mail: sureshghate@gmail.com

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automorphisms which preserve the centre $Z(G)$ elementwise." In [3], Franciosi et al. showed that, If "$Z(G)$ is torsion free and $Z(G)/G' \cap Z(G)$ is torsion, then $\mathrm{Aut}_c(G)$ acts trivially on $Z(G)$. It is an abelian and torsion free group". They further proved that,"$\mathrm{Aut}_c(G)$ is trivial when $Z(G)$ is torsion free and G/G' is torsion." In [5], Jamali et al. proved that, "For a finite group G in which $Z(G) \leq G'$, $\mathrm{Aut}_c(G) \cong \mathrm{Hom}(G/G', Z(G))$." They also proved that, "If G is a purely nonabelian finite p-group of class two (p odd), then $\mathrm{Aut}_c(G)$ is elementary abelian if and only if $\Omega_1(Z(G)) = \phi(G)$, and $exp(Z(G)) = p$ or $exp(G/G') = p$," where $\phi(G)$ is Frattini subgroup of G and $\Omega_1(Z(G)) = \langle x \in Z(G) | x^p = 1\rangle$. Note that, a group G is called purely nonabelian if it has no nontrivial abelian direct factor. Adney [1] proved that, "If a finite group G has no abelian direct factor, then there is a one-one and onto map between $\mathrm{Aut}_c(G)$ and $\mathrm{Hom}(G, Z(G))$."

In this article, we generalize the above results to subcentral automorphisms.

2 Subcentral Automorphisms

Let M and N be two normal subgroups of G.

By $\mathrm{Aut}^N(G)$, we mean the subgroup of $\mathrm{Aut}(G)$ consisting of all automorphisms which induce identity on G/N.

By $\mathrm{Aut}_M(G)$, we mean the subgroup of $\mathrm{Aut}(G)$ consisting of all automorphisms which induce identity on M.

Let $\mathrm{Aut}_M^N(G) = \mathrm{Aut}^N(G) \cap \mathrm{Aut}_M(G)$. From now onward, M will be a characteristic central subgroup, and elements of $\mathrm{Aut}^M(G)$ will be called as subcentral automorphisms of G (with respect to subcentral subgroup M). It can be seen that, $\mathrm{Aut}^M(G)$ is a normal subgroup of $\mathrm{Aut}_c(G)$.

We further, let $C^* = \{\alpha \in \mathrm{Aut}_M(G) | \alpha\beta = \beta\alpha, \forall \beta \in \mathrm{Aut}^M(G)\}$.

Clearly, C^* is a normal subgroup of $\mathrm{Aut}(G)$. Since every inner automorphism commutes with elements of $\mathrm{Aut}_c(G)$, $\mathrm{Inn}(G) \leq C^*$. If we take $M = Z(G)$, then C^* is same as $\mathrm{Inn}(G)$.

Let $K =< \{[g, \alpha] | g \in G, \alpha \in C^*\} >$, where $[g, \alpha] \equiv g^{-1}\alpha(g)$.

If $M = Z(G)$ then $K = G'$. However, in general, G' is a subgroup of K for every central subgroup M.

In the following, K and C^* will always correspond to a central subgroup of M of G as in the above definitions.

Our main results are given by the following theorems.

Theorem 1 *For a finite group G,* $\mathrm{Aut}_M^M(G) \cong \mathrm{Hom}\left(\frac{G}{KM}, M\right)$.

Theorem 2 *Let G be a group with M torsion free and $M/M \cap K$ torsion. Then* $\mathrm{Aut}^M(G)$ *is a torsion-free abelian group which acts trivially on M.*

Theorem 3 *Let G be a purely nonabelian finite group, then* $|\mathrm{Aut}^M(G)| = |\mathrm{Hom}(G, M)|$.

Theorem 4 *Let G be a purely nonabelian finite p-group (p odd), then* $\mathrm{Aut}^M(G)$ *is an elementary abelian p-group if and only if $exp(M) = p$ or $exp(G/K) = p$.*

Following proposition shows that each element of K is invariant under the natural action of $\mathrm{Aut}^M(G)$.

Proposition 1 $\mathrm{Aut}^M(G)$ *acts trivially on K.*

Proof Consider an automorphism $\alpha \in \mathrm{Aut}^M(G)$. This implies $x^{-1}\alpha(x) \in M$, for all $x \in G$. So $\alpha(x) = xm$ for some $m \in M$. Let $\beta \in C^*$. By definition of C^*, we have $\alpha([x, \beta]) = \alpha(x^{-1}\beta(x)) = (\alpha(x))^{-1}\beta(\alpha(x)) = m^{-1}x^{-1}\beta(xm) = m^{-1}x^{-1}\beta(x)m = x^{-1}\beta(x) = [x, \beta]$. Hence the results follows. □

Proof of Theorem 1 For any $\mu \in \mathrm{Aut}_M^M(G)$, define the map $\psi_\mu \in \mathrm{Hom}\left(\frac{G}{KM}, M\right)$ as $\psi_\mu(gKM) = g^{-1}\mu(g)$.
We first show that ψ_μ is well defined.
Let $gKM = hKM$, i.e., $gh^{-1} \in KM$.
$\therefore\ \mu(gh^{-1}) = gh^{-1} \Rightarrow g^{-1}\mu(g) = h^{-1}\mu(h) \Rightarrow \psi_\mu(gKM) = \psi_\mu(hKM)$.

For proving ψ_μ is a homomorphism, consider $\psi_\mu(gKMhKM) = \psi_\mu(ghKM) = (gh)^{-1}\mu(gh) = h^{-1}g^{-1}\mu(g)\mu(h) = g^{-1}\mu(g)h^{-1}\mu(h) = \psi_\mu(gKM).\psi_\mu(hKM)$

Now define a map $\psi : \mathrm{Aut}_M^M(G) \longrightarrow \mathrm{Hom}\left(\frac{G}{KM}, M\right)$, as $\psi(\mu) = \psi_\mu$.

We show that ψ is the required isomorphism.
For $f, g \in \mathrm{Aut}_M^M(G)$ and $h \in G$, $\psi(fg)(hKM) = \psi_{fg}(hKM) = h^{-1}fg(h) = h^{-1}f(hh^{-1}g(h)) = h^{-1}f(h)h^{-1}g(h) = \psi_f(hKM)\psi_g(hKM) = \psi_f.\psi_g(hKM)$.
Hence $\psi(fg) = \psi(f)\psi(g)$.

Consider $\psi(\mu_1) = \psi(\mu_2)$, i.e., $\psi_{\mu_1}(gKM) = \psi_{\mu_2}(gKM), g \in G$. This implies $g^{-1}\mu_1(g) = g^{-1}\mu_2(g) \Rightarrow \mu_1 = \mu_2$, as g is an arbitrary element of G. Thus ψ is a monomorphism.

We next show that ψ is onto. For any $\tau \in \mathrm{Hom}\left(\frac{G}{KM}, M\right)$, define a map $\mu : G \to G$ as $\mu(g) = g\tau(gKM), g \in G$.
Now we show that $\mu \in \mathrm{Aut}_M^M(G)$. For $g_1, g_2 \in G, \mu(g_1g_2) = g_1g_2\tau(g_1g_2KM) = g_1\tau(g_1KM)g_2\tau(g_2KM) = \mu(g_1)\mu(g_2)$. $\therefore\ \mu$ is a homomorphism on G.
Further, let $\mu(g) = 1$. This implies $g\tau(gKM) = 1 \Rightarrow \tau(gKM) = g^{-1} \Rightarrow g^{-1} \in M$ $\therefore\ gKM = KM \Rightarrow \tau(gKM) = 1 \Rightarrow g = 1$. Hence μ is one-one.
As G is finite, μ must be onto. So $\mu \in \mathrm{Aut}(G)$. Further, as $g^{-1}\mu(g) = g^{-1}g\tau(gKM) = \tau(gKM) \in M$, so $\mu \in \mathrm{Aut}^M(G)$. Also if $g \in M$, then $\mu(g) = g(\tau(gKM)) = g\tau(KM) = g$. Thus, $\mu \in \mathrm{Aut}_M^M(G)$ and $\psi(\mu) = \tau$.
Hence the theorem follows. □

Corollary 1 *Let G be finite group with $M \leq K$, then* $\mathrm{Aut}^M(G) \cong \mathrm{Hom}(G/K, M)$.

Proof Since $M \leq K$, $\frac{G}{KM} = G/K$. The result follows directly from Theorem 1 and Proposition 1. □

Proof of Theorem 2 Let $\alpha \in \mathrm{Aut}^M(G)$. If x is an element of M, then by the hypothesis $x^n \in M \cap K$ for some positive integer n. By Proposition 1, we have $x^n = \alpha(x^n) = (\alpha(x))^n$, and hence $x^{-n}(\alpha(x))^n = 1$. Since $x^{-1}\alpha(x) \in M$, this implies $(x^{-1}\alpha(x))^n = 1$. As M is torsion free, this implies that $x^{-1}\alpha(x) = 1$, i.e., $\alpha(x) = x$. Therefore, $\mathrm{Aut}^M(G)$ acts trivially on M.
Let $\alpha, \beta \in \mathrm{Aut}^M(G)$ and $x \in G$. So $\alpha\beta(x) = \alpha(\beta(x)) = \alpha(xx^{-1}\beta(x)) = \alpha(x)x^{-1}\beta(x) = xx^{-1}\alpha(x)x^{-1}\beta(x) = \beta(x)x^{-1}\alpha(x) = \beta(x)\beta(x^{-1}\alpha(x)) = \beta\alpha(x)$. Thus, $\mathrm{Aut}^M(G)$ is an abelian group.
Now, consider $\alpha \in \mathrm{Aut}^M(G)$, and suppose there exists $k \in N$ such that $\alpha^k = 1$. Since $x^{-1}\alpha(x) \in M$ for all $x \in G$, there exists $g \in M$ such that $\alpha(x) = xg$. Further, $\alpha^2(x) = \alpha(\alpha(x)) = \alpha(xg) = \alpha(x)\alpha(g) = xg^2 (\because \alpha$ acts trivially on M). Hence, by induction, $\alpha^n(x) = xg^n$. But $\alpha^k = 1 \Rightarrow x = xg^k$, i.e., $g^k = 1$. As M is torsion free, we must have $g = 1$. Thus $\alpha(x) = x$ for every x, i.e., $\alpha = 1$.
Therefore, $\mathrm{Aut}^M(G)$ is torsion free, and the theorem follows. □

Proposition 2 *Let G be a group in which M is torsion free and G/K is torsion, then* $\mathrm{Aut}^M(G) = 1$.

Proof Let $\alpha \in \mathrm{Aut}^M(G)$ and $x \in G$. Then by the assumption, $x^n \in K$ for some $n \in N$. As α fixes K elementwise, we have $(\alpha(x))^n = \alpha(x^n) = x^n$. So $x^{-n}(\alpha(x))^n = 1$. But $\alpha \in \mathrm{Aut}^M(G)$ and hence $x^{-1}\alpha(x) \in M \leq Z(G)$. This implies that $(x^{-1}\alpha(x))^n = 1$. Since M torsion free, it follows that $x^{-1}\alpha(x) = 1$, i.e., $\alpha(x) = x, \forall x \in G$. So $\mathrm{Aut}^M(G) = 1$. □

Proof of Theorem 3 For $f \in \mathrm{Aut}^M(G)$, we let $\alpha(f) \equiv \alpha_f$ defined as $\alpha(f)(g) \equiv \alpha_f(g) = g^{-1}f(g), g \in G$. It can be shown that $\alpha_f \in \mathrm{Hom}(G, M)$. We thus have $\alpha : \mathrm{Aut}^M(G) \to \mathrm{Hom}(G, M)$.
One can easily see that α is injective.
It just remains to show that α is onto.
For $\sigma \in \mathrm{Hom}(G, M)$, consider the map $f : G \to G$ given by $f(g) = g\sigma(g)$. f is an endomorphism and also $g^{-1}f(g) = \sigma(g) \in M$, which implies that f is subcentral endomorphism of G, and hence f is normal endomorphism(i.e., f commutes with all inner automorphisms). So, clearly $\mathrm{Im}(f)$ is a normal subgroup of G.
It is easy to see that f^n is also normal endomorphism and hence $\mathrm{Im} f^n$ is a normal subgroup of G, for all $n \geq 1$. Since G is a finite group, the two series

$$\mathrm{Ker} f \leq \mathrm{Ker} f^2 \leq \ldots\ldots$$

$$\mathrm{Im}\ f \geq \mathrm{Im}\ f^2 \geq \ldots\ldots$$

will terminate.

So there exists $k \in N$ such that

$$\text{Ker} f^k = \text{Ker} f^{k+1} = \ldots = A$$

$$\text{Im} f^k = \text{Im} f^{k+1} = \ldots\ldots = B$$

Now, we prove that $G = AB$.

Let $g \in G$, $f^k(g) \in \text{Im} f^k = \text{Im} f^{2k}$, and so $f^k(g) = f^{2k}(h)$, for some $h \in G$. Therefore $f^k(g) = f^k(f^k(h))$. This implies $f^k(g^{-1})f^k(g) = f^k(g^{-1})f^k(f^k(h))$. Thus $(f^k(h))^{-1}g \in \text{Ker} f^k = A$. Thus $g \in AB$ and hence $G = AB$.
Clearly $A \cap B =< 1 >$ and therefore $G = A \times B$. If $f(g) = 1$, then $g^{-1}\sigma(g) = 1$. This implies $\text{Ker} f \leq M$. Similarly, if $f^2(g) = 1$, i.e., $f(f(g)) = 1$. Thus $f(g) \in \ker f \leq M$. Therefore, $g\sigma(g) \in M \Rightarrow g \in M$. Hence $\ker f^2 \leq M$. Repetition of this argument gives, $A \equiv \ker f^k \leq M \leq Z(G)$. This implies A is an abelian group. By assumption, G is purely nonabelian and hence, we must have $A \equiv \text{Ker} f^k = 1$. This further implies $\text{Ker} f = 1$, i.e., f is injective. So $G = B \equiv \text{Im } f^k = \text{Im} f$. Thus f surjective. Hence, $f \in \text{Aut}^M(G)$. From the definition of α, it follows that $\alpha(f) = \sigma$. α is thus surjective. Therefore, α is the required bijection. Hence the result follows. □

Proposition 3 *Let G be a purely nonabelian finite group, then for each $\alpha \in \text{Hom}(G, M)$ and each $x \in K$, we have $\alpha(x) = 1$. Further* $\text{Hom}(G/K, M) \cong \text{Hom}(G, M)$.

Proof Whenever G is purely nonabelian group, then by Theorem 3, $|\text{Aut}^M(G)| = |\text{Hom}(G, M)|$. For every $\sigma \in \text{Aut}^M(G)$, it follows that $f_\sigma : x \rightarrow x^{-1}\sigma(x)$ is a homomorphism from G to M. Further the map $\sigma \rightarrow f_\sigma$ is one–one and thus a bijection because $|\text{Aut}^M(G)| = |\text{Hom}(G, M)|$. So every homomorphism from G to M can be considered as an image of some element of $\text{Aut}^M(G)$ under this bijection. Let $\alpha \in \text{Hom}(G, M)$. Since $K = \{[g, \alpha] | g \in G, \alpha \in C^*\}$, a typical generator of K is given by $g^{-1}\beta(g)$ for some $g \in G$, and $\beta \in C^*$. So $\alpha(g^{-1}\beta(g)) = f_\sigma(g^{-1}\beta(g)) = (g^{-1}\beta(g))^{-1}\sigma(g^{-1}\beta(g)) = \beta^{-1}(g)gg^{-1}\beta(g) = 1(\because g^{-1}\beta(g) \in K)$. It follows that $\alpha(x) = 1$, for every $x \in K$.

Now consider the map $\phi : \text{Hom}(G, M) \longrightarrow \text{Hom}(G/K, M)$ such that $\phi(f) = \bar{f}$, where $\bar{f}(gK) = f(g)$ for all $g \in G$. Clearly this map ϕ is an isomorphism. □

Proposition 4 *Let G be a purely nonabelian finite group, then* $|\text{Aut}^M(G)| = |\text{Hom}(G/K, M)|$.

Proof Proof follows directly from Theorem 3 and Proposition 3. □

Proposition 5 *Let p be a prime number. If G is a purely nonabelian finite p-group then* $\text{Aut}^M(G)$ *is a p-group.*

Proof By the assumption, the subgroup M and hence Hom $(G/K, M)$ are finite p-groups. Hence the result follows directly from Proposition 4. □

Proposition 6 *Let G be a purely nonabelian finite group*
(i) If $gcd(|G/K|, |M|) = 1$, *then* $\mathrm{Aut}^M(G) = 1$.
(ii) If $\mathrm{Aut}^M(G) = 1$, *then* $M \leq K$.

Proof (i) Follows from Proposition 4.
(ii) Let $|G/K| = a$ and $|M| = b$. Since $\mathrm{Aut}^M(G) = 1$, hence by Proposition 4, $(a, b) = 1$. So there exist integers λ and μ such that $\lambda a + \mu b = 1$. Let $x \in M$. Thus $xK = (xK)^1 = (xK)^{\lambda a+\mu b} = (xK)^{\lambda a}(xK)^{\mu b} = K \Rightarrow x \in K$. □

Remark 1 From Corollary 1, and Proposition 3, we can say that, whenever $M \leq K$, $\mathrm{Aut}^M(G) \cong \mathrm{Hom}(G, M)$. Even when $\mathrm{Im} f \leq K$, for all $f \in \mathrm{Hom}(G, M)$, this result holds. Thus, if G is a purely nonabelian finite group and if for all $f \in \mathrm{Hom}(G, M)$, $\mathrm{Im} f \leq K$, then $\mathrm{Aut}^M(G) \cong \mathrm{Hom}(G/K, M)$.

Remark 2 For every $f \in \mathrm{Hom}(G, M)$, the map $\sigma_f : x \to xf(x)$ is a subcentral endomorphism of G. This endomorphism is an automorphism if and only if $f(x) \neq x^{-1}$ for all $1 \neq x \in G$ (G is finite).

Following lemma has been proved in [4], we shall use it to prove Theorem 4.

Lemma 1 *Let x be an element of a finite p-group G and N a normal subgroup of G containing* G' *such that* $o(x) = o(xN) = p$. *If the cyclic subgroup* $< x >$ *is normal in G such that* $ht(xN) = 1$, *then* $< x >$ *is a direct factor of G.*

In the above statement ht denotes height. Height of an element a of a group G is defined as the largest positive integer n such that for some x in G, $x^n = a$.

Proof of Theorem 4 For the odd prime p, let $\mathrm{Aut}^M(G)$ be an elementary abelian p-group. Assume that the exponent of M and G/K are both strictly greater than p. Since G/K is finite abelian, it has a cyclic direct summand $\langle xK\rangle$ say, of order $p^n (n \geq 2)$ and hence $G/K \cong \langle xK\rangle \times L/K$. For $f \in \mathrm{Hom}(G, M)$, consider $f(x) = a$ for any $x \in G$. So $\bar{f}(xK) = a$. Since exp (M) is strictly greater than p, the order of a is p^m, for some m, $2 \leq m \leq n$.
We can use the homomorphism $\bar{f}$ to get corresponding homomorphism (also denoted by same notation) $\bar{f}$ as $\bar{f} : \langle xK\rangle \times L/K \to M$ with $(x^i K, lK) \to a^i$. The map $\bar{f}$ on $< xK > \times L/K$ is well defined, since $o(a)|o(xK)$ (as $m \leq n$).
If $aK = (x^s K, lK)$ then we show that $p|s$. Assume $p \not| s$, then $< xK >=< x^s K >$ and hence $G/K =< aK > L/K$. Now we have $o(a) \geq o(aK) \geq o(x^s K) = o(xK) \geq o(\bar{f}(xK)) = o(a)$. This implies that $o(a) = o(aK)$. Thus $< a > \cap K = 1$. As $o(aK) = o(xK)$, we get $G/K \cong< aK > \times L/K$ and hence $G \cong< a > \times L$. This is a contradiction, as G is a purely nonabelian group. Thus $p|s$.
By Remark 2 and Theorem 3, $\sigma_f \in \mathrm{Aut}^M(G)$ and by assumption $o(\sigma_f) = p$.

Now, we have $\sigma_f(x) = xf(x) = xa$. Since $f(a) = \bar{f}((xK)^s, lK) = a^s$, we have $\sigma_f^2(x) = xa^{s+2} = xa^{\frac{(s+1)^2-1}{s}}$.

Also, $\sigma_f^3(x) = xa^{\left(\frac{(s+1)^3-1}{s}\right)}$.
Generalizing this,
we get $\sigma_f^t(x) = xa^{\left(\frac{(s+1)^t-1}{s}\right)}$, for every $t \in N$.

As the order of σ_f is p, we have $a^{\frac{(s+1)^p-1}{s}} = 1$. Since p is odd and $p|s$, we have $p^2|(\left(\frac{(s+1)^p-1}{s}\right) - p)$.
$\therefore qp^2 + p = \frac{(s+1)^p-1}{s}$ for some $q \in Z$. Thus $(a^p)^{qp+1} = 1$. But $o(a) = p^m \Rightarrow o(a^p) = p^{m-1}$.
Now
(1) if $a^p \neq 1$, then $p^{m-1}|(qp+1)$. But this is impossible as $m \geq 2$.
(2) $a^p = 1$ is also not possible as $o(a) = p^m$ and $m \geq 2$.

So, the assumption that $exp(M)$ and $exp(G/M)$ are stricly greater than p is wrong. Conversly, assume that $exp(G/K) = p$ and $f \in \text{Hom}(G, M)$. Then by proposition 3, $\bar{f} \in \text{Hom}(G/K, M)$. So for $x \in G$, put $\bar{f}(xK) = a$. If $aK \neq 1$, then it follows that $o(aK(G)) = o(a) = p$. Clearly $< a > \leq M(G) \leq Z(G)$ and hence the cyclic subgroup $< a >$ is normal in G. We also have $ht(aK) = 1$. Now by the Lemma 1, the cyclic subgroup $< a >$ is an abelian direct factor of G, and this contradicts the assumption. Therefore $a \in K$. This implies that $\text{Im}(f) \leq K$. Hence by Remark 1 $\text{Aut}^M(G) \cong \text{Hom}(G/K, M)$. But as M is abelian, $\text{Hom}(G/K, M)$ is abelian. Thus $\text{Aut}^M(G)$ is abelian. Since $exp(G/M) = p$, this implies that $\text{Aut}^M(G)$ is an elementary abelian p-group.

Now assume that $exp(M) = p$. Consider $f, g \in \text{Hom}(G, M)$. We first show that $g \circ f(x) = 1$, for all $x \in G$. Assume that $\bar{f}(xK) = b \in M$, for $x \in G$. Since $exp(M) = p$, it implies that $o(b)|p$. If $b = 1$ then $g \circ f(x) = g(\bar{f}(xK(G))) = 1$. Now take, $o(b) = p$. If $b \in K$ then we have $g(f(x)) = g(\bar{f}(xK(G))) = g(b) = 1$. Assume b does not belong to K. As $b^p = 1$, it follows that $o(bK) = p$. Also, as $b \in M \leq Z(G), < b >$ is normal in G. Now if $ht(bK(G)) = 1$, then by the Lemma 1, the cyclic subgroup $< b >$ is an abelian direct factor of G, giving a contradiction. So assume $ht(bk(G)) = p^m$ for some $m \in N$. By the definition of height, there exists an element yK in G/K such that $bK = (yK)^{p^m}$. But $exp(M) = p$. Therefore $g \circ f(x) = g(b) = \bar{g}(bK) = \bar{g}(yK)^{p^m} = 1$. Thus, for all $f, g \in \text{Hom}(G, M)$ and each $x \in G, g(f(x)) = 1$. We can similarly show that $f(g(x)) = 1$ and hence $f \circ g = g \circ f$. From Remark 2, $\sigma_f \circ \sigma_g = \sigma_g \circ \sigma_f$. This shows that $\text{Aut}^M(G)$ is abelian.

Now we show that each nontrivial element of $\text{Aut}^M(G)$ has order p. So if $\alpha \in \text{Aut}^M(G)$, then by Remark 2, there exists a homomorphism $f \in \text{Hom}(G, M)$ such that $\alpha = \sigma_f$. Therefore, we have to show that $o(\sigma_f)|p$. Clearly, taking $f = g$ and using $f(f(x)) = 1, x \in G$, we have $x \in G$, we have $\sigma_f^2(x) = \sigma_f(xf(x))$

$= x(f(x))^2$. In general for $n \geq 1$, $\sigma_f^n(x) = x(f(x))^n$. As exp $(M) = p$ and $f(x) \in M$ we have, $\sigma_f^p(x) = x$ which implies $\sigma_f^p = 1_{\mathrm{Aut}^M(G)}$.
Hence $o(\sigma_f)|p$. Thus, $o(\alpha)|p\ \forall \alpha \in \mathrm{Aut}_M(G)$. $\therefore$ $\mathrm{Aut}^M(G)$ is an elementary abelian group.

□

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