L(4, 3, 2, 1)-Labeling for Simple Graphs

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Abstract An $L(4, 3, 2, 1)$-*labeling* of a graph is a function which assigns label to each vertex of the graph such that if two vertices are one, two, three and four distance apart then assigned labels must have a difference of at least 4, 3, 2 and 1 respectively between them. This paper presents $L(4, 3, 2, 1)$-*labeling number* for simple graphs such as complete graphs, complete bipartite graphs, stars, paths and cycles. This paper also presents an $L(4, 3, 2, 1)$-labeling algorithm for paths which is optimal for paths on $n \geq 7$ vertices.

Keywords $L(4, 3, 2, 1)$-labeling · Labeling number · Graph labeling · Channel assignment problem

1 Introduction

Channel Assignment Problem (*CAP*) is a problem of assigning radio channels to radio transmitters such that interferences among different radio stations could be avoided. Two stations which are at certain distance apart must be assigned channels with some predefined separation. Separation between two channels depends on the distance between the corresponding stations and is inversely proportional to the distance between stations. So, *CAP* can be considered as a *graph labeling problem* where vertices of the graph represent the radio stations and the edges represent geographical adjacency among the radio stations. *CAP* has been considered as *vertex coloring problem* by Hale [1]. Roberts [2] has considered the problem of

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assigning channels or frequencies to the transmitters which are "very close" and "close". The "very close" transmitters receive frequencies with at least two separations and "close" transmitters receive frequencies with at least one separation. Based on this simple model, Griggs and Yeh [3] has defined the $L(2, 1)$-*labeling* of a graph. An $L(2, 1)$-*labeling* of a graph $G = (V, E)$ is a mapping f which assigns integer labels $f(x)$ and $f(y)$ to two vertices x and y such that $|f(x) - f(y)| \geq 2$ if $d(x, y) = 1$ and $|f(x) - f(y)| \geq 1$ if $d(x, y) = 2$, where $d(x, y)$ represents the shortest distance between the vertices x and y [3]. The $L(2, 1)$-*labeling* problem is known to be NP-complete [3]. More details on $L(2, 1)$-*labeling* of graphs can be found in [4]. In practice interference may go beyond two levels. So, Liu and Shao [5] generalized the $L(2, 1)$-*labeling* problem and introduced the $L(3, 2, 1)$-*labeling* problem of graph. An $L(3, 2, 1)$-*labeling* of graph $G = (V, E)$ is a mapping $f : V \rightarrow \mathbb{N}$ such that $|f(x) - f(y)| \geq 3$ if $d(x, y) = 1$, $|f(x) - f(y)| \geq 2$ if $d(x, y) = 2$ and $|f(x) - f(y)| \geq 1$ if $d(x, y) = 3$, $\forall x, y \in V$. The $L(3, 2, 1)$-*labeling number* of a graph G is the smallest number λ such that G has an $L(3, 2, 1)$-*labeling* with λ as the maximum label and the $L(3, 2, 1)$-*labeling number* is denoted as $\lambda(G)$ [5]. The $L(3, 2, 1)$-*labeling numbers* for paths, cycles, caterpillars, n-ary trees, complete graphs and complete bipartite graphs are determined by Clipperton et al. [6]. The $L(3, 2, 1)$-*labeling numbers* for the Cartesian product of paths, cycles and powers of paths are determined by Chia et al. [7]. The upper bounds of the $L(3, 2, 1)$-*labeling numbers* for a graph with the maximum degree Δ and trees can also be found in [7].

It is very natural to consider interference beyond three levels. That is why we have considered interference up to four levels in this paper. If the underlying network topology is considered as a simple graph then this type of labeling can be used to solve *CAP*.

The organization of the paper is as follows: Section 2 provides some basic definitions of graph theory and notations used in this paper. In Sect. 3 we present the $L(4, 3, 2, 1)$-*labeling numbers* for complete graphs, complete bipartite graphs, stars, paths and cycles. Section 4 presents an $L(4, 3, 2, 1)$-labeling algorithm for paths and we prove that this algorithm is optimal for paths on $n \geq 7$ vertices. Section 5 concludes the paper.

2 Definitions and Notation

In this section we give some definitions and introduce some notations used in this paper.

Definition 1 Let $G = (V, E)$ be a connected undirected graph and the minimum distance between any two vertices $x, y \in V$ be denoted as $d(x, y)$. An $L(4, 3, 2, 1)$-labeling of a graph G is a mapping $f : V \rightarrow \mathbb{N}$ if and only if $\forall x, y \in V$, the following inequalities hold

$$|f(x) - f(y)| \geq \begin{cases} 4, & \textit{if } d(x,y) = 1 \\ 3, & \textit{if } d(x,y) = 2 \\ 2, & \textit{if } d(x,y) = 3 \\ 1, & \textit{if } d(x,y) = 4. \end{cases}$$

The L(4, 3, 2, 1)-labeling number $\lambda(G)$ of a graph G is the smallest number $\lambda \in \mathbb{N}$ such that G has an L(4, 3, 2, 1)-labeling with λ as its maximum label. An L(4, 3, 2, 1)-labeling of a graph G is said to be a minimal L(4, 3, 2, 1)-labeling of G if the highest label used in any vertex of G is λ. An L(4, 3, 2, 1)-labeling of a graph G with λ as its maximum label is often denoted as λ-L(4, 3, 2, 1)-labeling.

Definition 2 A simple connected graph $G = (V, E)$ with $|V| = n$ is said to be complete graph if $\forall x, y \in V$, $(x, y) \in E$, i.e. every pair of vertices of the graph are adjacent to each other. This graph is denoted as K_n [8].

Definition 3 A simple connected graph $G = (V, E)$ is said to be a complete bipartite graph if there exists two sets A and B such that (i) $A \cup B = V$ and $A \cap B = \phi$ with $|A| = m$, $|B| = n$ and $|V| = m + n$, (ii) $\forall a_i, a_j \in A, (a_i, a_j) \notin E$ and $\forall b_i, b_j \in B, (b_i, b_j) \notin E$, (iii) $\forall a_i \in A$ and $b_j \in B$, $(a_i, b_j) \in E$. This graph is denoted as $K_{m,n}$ [8].

Definition 4 A star graph can be defined as a $K_{1,n}$ complete bipartite graph. It is generally denoted as S_n [8].

Definition 5 A graph $G = (V, E)$ is said to be a path if $(v_i, v_{i+1}) \in E, 1 \leq i < n$ where $|V| = n$ and it is denoted as P_n [8].

Definition 6 A graph $G = (V, E)$ is said to be a cycle if $(v_i, v_{i+1}) \in E, 1 \leq i < n$ and $(v_n, v_1) \in E$ where $|V| = n$ and it is denoted as C_n [8].

3 *L*(4, 3, 2, 1)-Labeling Numbers for Simple Graphs

3.1 Complete Graphs

In this section we find the minimal L(4, 3, 2, 1)-labeling number $\lambda(K_n)$ for complete graphs.

Theorem 1 *For any complete graph, K_n with n vertices, the minimal* L(4, 3, 2, 1)-*labeling number $\lambda(K_n)$ is* $4n - 3$.

Proof Let $K_n = (V, E)$ be a complete graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and also let f be a minimal L(4, 3, 2, 1)-labeling of K_n. Without loss of generality, we can assume that $f(v_i) < f(v_j)$ when $i < j$ and $f(v_1) = 1$. As K_n is a complete graph, therefore, $d(x, y) = 1, \forall x, y \in V, x \neq y$. This implies that $|f(x) - f(y)| \geq 4, \forall x, y \in V$. Again $f(v_1) = 1$. Therefore,

$$
\begin{aligned}
&f(v_2) \geq f(v_1) + 4 = 1 + 4 = 5;\\
&f(v_3) \geq f(v_2) + 4 = 5 + 4 = 9;\\
&\vdots\\
&f(v_n) \geq f(v_{n-1}) + 4 = f(v_1) + (n-1)4 = 1 + (n-1)4 = 4n - 3.
\end{aligned}
$$

Therefore, $\lambda(K_n) = 4n - 3$. □

3.2 Complete Bipartite Graphs

In this section we find the minimal $L(4, 3, 2, 1)$-labeling number $\lambda(K_{m,n})$ for complete bipartite graphs.

Theorem 2 *For any complete bipartite graph, $K_{m,n}$ with $(m+n)$ vertices, the minimal $L(4, 3, 2, 1)$-labeling number $\lambda(K_{m,n})$ is $3(m+n) - 1$.*

Proof Let $K_{m,n} = (V, E)$ be a complete bipartite graph with $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ such that $A \cup B = V$ and $A \cap B = \phi$ and also let f be a minimal $L(4, 3, 2, 1)$-labeling of $K_{m,n}$.

Without loss of generality, we can assume that $f(a_i) < f(a_j)$ when $i<j$ and $f(b_i) < f(b_j)$ when $i<j$ and $f(a_1) = 1$. Now $d(a_i, a_j) = 2, \forall a_i, a_j \in A$ and $i \neq j$. Therefore, $|f(a_i) - f(a_j)| \geq 3, \forall a_i, a_j \in A$ and $i \neq j$. Since $f(a_1) = 1$, we have

$$
\begin{aligned}
f(a_2) &\geq f(a_1) + 3 = 1 + 3 = 4;\\
f(a_3) &\geq f(a_2) + 3 = 4 + 3 = 7;\\
&\vdots\\
f(a_m) &\geq f(a_{m-1}) + 3 = f(a_1) + (m-1)3 = 1 + (m-1)3 = 3m - 2.
\end{aligned}
$$

Again $d(a_i, b_j) = 1$, $\forall a_i \in A, b_j \in B$. Therefore, $|f(a_i) - f(b_j)| \geq 4$. Since f is minimal labeling, we have $f(b_1) = f(a_m) + 4 = (3m-2) + 4 = 3m + 2$.

Moreover, $d(b_i, b_j) = 2, \forall b_i, b_j \in B$ and $i \neq j$. Therefore, $|f(b_i) - f(b_j)| \geq 3$, $\forall b_i, b_j \in B$ and $i \neq j$. Since $f(b_1) = 3m + 2$, we have

$$
\begin{aligned}
&f(b_2) \geq f(b_1) + 3 = (3m+2) + 3 = 3m + 5;\\
&f(b_3) \geq f(b_2) + 3 = (3m+5) + 3 = 3m + 8;\\
&\vdots\\
&f(b_n) \geq f(b_{n-1}) + 3 = f(b_1) + (n-1)3 = (3m+2) + (n-1)3 = 3(m+n) - 1.
\end{aligned}
$$

Therefore, $\lambda(K_{m,n}) = 3(m+n) - 1$. □

Corollary 1 *For a star, S_n, the minimal $L(4, 3, 2, 1)$-labeling number $\lambda(S_n)$ is $3n + 2$.*

Proof According to the definition of star, S_n is $K_{1,n}$. Therefore, using Theorem 2, we can write $\lambda(S_n) = 3(1+n) - 1 = 3n+2$. □

3.3 Paths

In this section we find the *minimal* $L(4, 3, 2, 1)$-*labeling number* $\lambda(P_n)$ for paths.

Lemma 1 *For a path on n vertices,* P_n*, with* $n \geq 7$*, the minimal* $L(4, 3, 2, 1)$-*labeling number* $\lambda(P_n)$ *is at least* 13.

Proof We prove this lemma using method of contradiction. Let f be a *minimal* $L(4, 3, 2, 1)$-*labeling* for a path on n vertices, P_n and v_1 be the vertex with label 1. Now, suppose that $\lambda(P_n) < 13$ for $n \geq 7$. Obviously keeping v_1 as an end vertex, there exists an induced sub-path of at least three vertices. Let $\{v_1, v_2, v_3\}$ be this path. Now, $f(v_2)$ will have the following possibilities.

Case-I: $f(v_2) = 5$:
Then $f(v_3) = 9$ and $f(v_4) = 13$, which contradicts our assumption.
Case-II: $f(v_2) = 6$:
Then $f(v_3) = 10$ and $f(v_4) = 14$, which contradicts our assumption.
Case-III: $f(v_2) = 7$:
Then $f(v_3) = 11$ and $f(v_4) = 4, 3$. Either possibilities of $f(v_4)$ forces $f(v_5) \geq 14$, which contradicts our assumption.

Therefore we can conclude that $\lambda(P_n)$ is at least 13 for $n \geq 7$. □

Theorem 3 *For a path,* P_n *on n vertices, the minimal* $L(4, 3, 2, 1)$-*labeling number* $\lambda(P_n)$ *is*

$$\lambda(P_n) = \begin{cases} 1 & \textit{if } \mathrm{n} = 1 \\ 5 & \textit{if } \mathrm{n} = 2 \\ 8 & \textit{if } \mathrm{n} = 3 \\ 11 & \textit{if } \mathrm{n} = 4, 5, 6 \\ 13 & \textit{if } \mathrm{n} \geq 7 \end{cases}$$

Proof Here, we prove each of the cases one by one.

Case-I: $n = 1$:
For a path, P_n with $n = 1$, the *minimal* $L(4, 3, 2, 1)$-*labeling number* $\lambda(P_n)$ is 1. This is trivially true. Therefore, $\lambda(P_1) = 1$.
Case-II: $n = 2$:
For a path, P_n with $n = 2$, the *minimal* $L(4, 3, 2, 1)$-*labeling number* $\lambda(P_n)$ is 5, as we don't have any other choice of labeling except either $\{1, 5\}$ or $\{5, 1\}$. Therefore, $\lambda(P_2) = 5$.

Case-III: $n = 3$:
The labeling pattern $\{5, 1, 8\}$ shows that $\lambda(P_n) \leq 8$ for $n = 3$. Suppose $\lambda(P_n) < 8$ for $n = 3$. If $f(v_1) = 1$, then $f(v_2) \geq 5$ and $f(v_3) \geq 9$, which is a contradiction. Again if $f(v_2) = 1$, then either $f(v_1) \geq 5, f(v_3) \geq 8$ or $f(v_1) \geq 8$, $f(v_3) \geq 5$. In either case we have a contradiction. Therefore, $\lambda(P_3) = 8$.
Case-IV: $n = 4, 5, 6$:
The labeling pattern $\{9, 5, 1, 11, 7, 3\}$ shows that $\lambda(P_n) \leq 11$ for $n = 4, 5, 6$. Suppose $\lambda(P_n) < 11$ for $n = 4, 5, 6$. As f is a minimal $L(4, 3, 2, 1)$-labeling, a vertex must have the label 1. If $f(v_1) = 1$ then $f(v_4) \geq 13$, a contradiction. If $f(v_2) = 1$ then either $f(v_1) = 5$ or $f(v_3) = 5$. When $f(v_1) = 5$, then $f(v_4) = 12$, a contradiction. When $f(v_3) = 5$, then $f(v_5) = 13$, a contradiction. Therefore, $\lambda(P_n) = 11$ for $n = 4, 5, 6$.
Case-V: $n \geq 7$:
We can find a labeling pattern $f(\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}) = \{5, 9, 13, 3, 7, 11, 1\}$. It can be defined that two vertices with indices i and j will get same label if $i \equiv j \pmod 7$ and the maximum natural number used to label the first seven vertices starting from index $i = 1$ to index $i = 7$ is 13. Therefore, we can conclude that $\lambda(P_n) \leq 13$ for $n \geq 7$. Again using Lemma 3.3 we get $\lambda(P_n) \geq 13$ for $n \geq 7$. So, combining these two results we can finally conclude that $\lambda(P_n) = 13$ for $n \geq 7$. □

3.4 Cycles

Lemma 2 *For a cycle, C_n on n vertices, the minimal $L(4, 3, 2, 1)$-labeling number $\lambda(C_n)$ is*

$$\lambda(C_n) = \begin{cases} 1 & \textit{if } n = 1 \\ 5 & \textit{if } n = 2 \\ 9 & \textit{if } n = 3 \end{cases}$$

Proof

Case-I: $n = 1$:
This is trivially true. Therefore, $\lambda(C_n) = 1$ if $n = 1$.
Case-II: $n = 2$:
Here we have only two choices for labeling the vertices v_1 and v_2. The two possibilities of $\{f(v_1), f(v_2)\}$ is either $\{1, 5\}$ or $\{5, 1\}$. Therefore, $\lambda(C_n) = 5$ if $n = 2$.
Case-III: $n = 3$:
Clearly, a cycle with 3 vertices is nothing but a complete graph with 3 vertices. Using Theorem 1 we can compute $\lambda(K_3) = 9$. Therefore, $\lambda(C_n) = 9$ if $n = 3$.□

Observation 1 *For a cycle, C_n on n vertices, the distance between any two vertices is at most* $\left\lfloor \frac{n}{2} \right\rfloor$.

Observation 2 $\lambda(C_n) \geq \lambda(P_n)$.

This is true because P_n is a subgraph of C_n.

4 *L*(4, 3, 2, 1)-Labeling Algorithm for Path

In this section we present an algorithm of *L*(4, 3, 2, 1)-labeling for path, P_n on n vertices. The label of any vertex of P_n obtained from our proposed algorithm depends only on the index of the corresponding vertex.

Algorithm: *L*(4, 3, 2, 1)-labeling algorithm for Path

Input: Path, P_n on n vertices.
Output: $L(4,3,2,1)$-labeling for P_n.
Let the label assigned to the vertex of index i be denoted as $L(i)$ where $1 \leq i \leq n$.
begin
for every vertex of index i

$L(i) = (4i \bmod 14) + 1$

end

An example of label assignment for P_n with $n = 20$ using the above algorithm is as follows: $\{5, 9, 13, 3, 7, 11, 1, 5, 9, 13, 3, 7, 11, 1, 5, 9, 13, 3, 7, 11\}$.

Claim 1 *The proposed* L(4, 3, 2, 1)*-labeling algorithm for Path, P_n on n vertices is optimal for* $n \geq 7$.

Proof In Theorem 3 we have already proved that $\lambda(P_n) = 13$ for $n \geq 7$. The label obtained for any vertex with index *i* from the proposed algorithm is at most 13. This proves that our proposed algorithm is optimal. □

5 Conclusion

This paper presents *L*(4, 3, 2, 1)-labeling for different types of graphs. It may be worthy to find out the *labeling number* for different complex graphs.

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