Trends in Mathematics

Santosh Joshi
Michael Dorff
Indrajit Lahiri
Editors

Current Topics in Pure and Computational Complex Analysis

Birkhäuser

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Preface

This volume contains a collection of eight survey articles and five research papers, most of which were presented at the international workshop on Complex Analysis and Its Applications held at Walchand College of Engineering, Sangli, India, during June 11–15, 2012. The topics of the articles include geometric function theory, planar harmonic mappings, and entire and meromorphic functions. The main aim of the workshop was to present the new developments and techniques in different branches of complex analysis, and the articles in this volume reflect that aim.

We hope that this volume will contribute to the research being done on complex analysis and will motivate the young generation of mathematicians to continue the research in these areas.

We thank the organizers of the workshop, the speakers, and the authors of the articles for their efforts. The workshop was supported by Walchand College of Engineering, Sangli, through the Technical Education Quality Improvement Programme (TEQIP) grant, and National Board for Higher Mathematics (NBHM), Mumbai. We are grateful to these organizations for their generous support. We would also like to thank Springer for kindly publishing this volume, and especially we appreciate Shamim Ahmad and his team at New Delhi for their friendly and efficient cooperation throughout this project. Finally, Professor Santosh Joshi would like to thank his wife Dr. Sayali Joshi for her help and support at each stage of this project.

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Chapter 1
Boundary Behavior of Univalent Harmonic Mappings
A Survey of Recent Boundary Behavior Results of Univalent Harmonic Mappings

Daoud Bshouty and Abdallah Lyzzaik

1.1 Introduction

A *harmonic mapping* f of a complex region G is a complex-valued function that satisfies Laplace's equation

$$\Delta f \equiv f_{xx} + f_{yy} = 0.$$

This function can be written as

$$f(z) = u(x, y) + iv(x, y), \quad z = x + iy,$$

where u and v are real-valued harmonic functions, and as

$$f(z) = h(z) + \overline{g(z)}, \tag{1.1}$$

where h and g are analytic functions, called respectively the *analytic* and *coanalytic* parts of f, are single-valued if G is simply-connected and possibly multiple-valued if G is otherwise. In either case, the *second complex dilatation* $a = g'/h'$ is defined which is either a meromorphic function or identical to infinity in G. It is known that $|a| < 1$ in G if, and only if, f is open and sense-preserving, and $|a| > 1$ in G if, and only if, f is open and sense-reversing.

Throughout this chapter, we denote by $\mathbb{C}$, $\mathbb{D}$, and $\mathbb{T}$ the complex plane, the open unit disc, and the unit circle, respectively. We may identify for a given function f^* of $\mathbb{T}$ a value $f^*(e^{i\theta})$ by $f^*(\theta)$ for any $\theta \in [-\pi, \pi[$.

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Harmonic mappings of $\mathbb{D}$ may be constructed as follows. For a Lebesgue integrable function f^* of $\mathbb{T}$, the function

$$f(z) = P[f^*] = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r, \varphi - \theta) f^*(\varphi)\, d\varphi; \quad z = re^{i\theta} \in \mathbb{D}, \tag{1.2}$$

where $P(r,t)$ is the *Poisson kernel* of $\mathbb{D}$, is a harmonic mapping of $\mathbb{D}$ whose unrestricted limit at every continuity point $e^{i\theta_0}$ of f^* is $f^*(\theta_0)$. The Rado–Kneser–Choquet theorem [11, pp. 29–30] asserts that if $|f^*| = 1$ and $\arg f^*(e^{i\theta})$ is a monotone increasing function of θ with $\Delta_{\partial D} \arg f^*(e^{i\theta}) \leq 2\pi$, then f is a univalent sense-preserving harmonic mapping of $\mathbb{D}$ onto the interior of the convex hull of $f^*(\mathbb{T})$.

Sense-preserving harmonic mappings of $\mathbb{D}$ also arise as solutions of linear elliptic partial differential equations of the form

$$\overline{f_{\bar{z}}}(z) = a(z) f_z(z); \quad z \in \mathbb{D}, \tag{1.3}$$

where a is an analytic function from $\mathbb{D}$ into itself; a is indeed the dilatation of f.

Univalent harmonic mappings may also be introduced through minimal surfaces. Let $S = (u, v, s)$, $s = G(u, v)$, be a nonparametric surface that spreads over a simply connected domain $\Omega \neq \mathbb{C}$. Then S is a minimal surface if, and only if, there exists a univalent harmonic mapping $f = u + iv$ from the unit disc $\mathbb{D}$ onto Ω such that $(s_z)^2 = -f_z\overline{f_{\bar{z}}} = af_z^2$. Note that f is uniquely expressed in the form (1.1) where h and g, with $g(0) = 0$, are analytic functions of $\mathbb{D}$. Without loss of generality, we may assume that f is sense-preserving, or else we consider $f(\bar{z})$. It follows that the dilatation $a = \overline{f_{\bar{z}}}/f_z = g'/h'$ is a square of an analytic function of $\mathbb{D}$ that satisfies $|a| < 1$ on $\mathbb{D}$. The function is known as the Weierstrass parameter of the minimal surface and the Gauss map of S is given by the normal vector

$$\mathbf{N} = \frac{(2\Im\sqrt{a}, 2\Re\sqrt{a}, 1 - |a|)}{1 + |a|}.$$

The study of nonparametric minimal surfaces over Ω with a given Gauss map leads to the problem of finding univalent harmonic maps from $\mathbb{D}$ onto Ω which are solutions of (1.3).

For the special case where $|a| < k < 1$ in $\mathbb{D}$, it is classical that the existence part of the *Riemann mapping theorem (RMT)* of (1.3) holds; namely, for a given bounded simply connected domain Ω and a fixed $w_0 \in \Omega$, there is a univalent solution f of (1.3) that satisfies $f(0) = w_0$ and $f_z(0) > 0$ and maps $\mathbb{D}$ onto Ω. In addition, if Ω is a Jordan domain, then f extends to a homeomorphism from $\overline{\mathbb{D}}$ onto $\overline{\Omega}$. However, in the case where $\|a(z)\|_\infty = 1$ the following theorem holds ([18], Theorems 4.2 and 4.3).

Theorem 1 *Let Ω be a bounded simply connected domain whose boundary $\partial\Omega$ is locally connected. Suppose that $a(\mathbb{D}) \subset \mathbb{D}$ and w_0 is a fixed point of Ω. Then there exists a univalent solution f of (1.3) having the following properties:*

(a) $f(0) = w_0$, $f_z(0) > 0$ *and* $f(\mathbb{D}) \subset \Omega$.

(b) *There is a countable set* $E \subset \partial\mathbb{D}$ *such that the unrestricted limits* $f^*(e^{it}) = \lim_{z \to e^{it}} f(z)$ *exist on* $\partial\mathbb{D} \setminus E$ *and belong to* $\partial\Omega$.

(c) *The functions*

$$f_-^*(e^{it}) = \text{ess} \lim_{s \to t-} f^*(e^{is}) \quad \textit{and} \quad f_+^*(e^{it}) = \text{ess} \lim_{s \to t+} f^*(e^{is})$$

exist on $\partial\mathbb{D}$, *belong to* $\partial\Omega$, *and are equal on* $\partial\mathbb{D} \setminus E$.

(d) *The cluster set of* f *at* $e^{it} \in E$ *is the straight line segment joining* $f_-^*(e^{it})$ *to* $f_+^*(e^{it})$.

The mapping f is termed a *generalized Riemann mapping (GRM) from* $\mathbb{D}$ *onto* Ω. It is immediate that the boundary function f^* is continuous at every point in $\partial\mathbb{D} \setminus E$ and has a jump discontinuity at every point in E. We will use the term *jump* to describe the behavior of f^* at every point of E.

Note that f^* is continuous on $(f^*)^{-1}(V)$ on every concave part $V \subset \partial\Omega$. Thus, the study of the continuity of f^* is of interest on convex intervals $V \subset \mathbb{T}$ and where $|a|$ tends to 1 on $(f^*)^{-1}(V)$ almost everywhere.

The cluster set of f on an interval $I \subset \partial\mathbb{D}$ induces a boundary positively directed arc in $\partial f(\mathbb{D})$; this arc is denoted, with abuse of notation, by $f^*(I)$.

It is of interest to note that the uniqueness of the RMT is still an open question. Nonetheless, by a consequence of a paper by Gergen and Dressel [16] (see also [15]), the question was shown to be true by Bshouty, Hengartner, and Hossian [6] for symmetric domains with symmetric dilatations. Moreover, it was noted by Kühnau that it also holds true for starlike domains.

On the other hand, it is interesting to know that the uniqueness of GRMs has been established recently for strictly starlike domains Ω with respect to some interior point [7, 4]; that is, every ray through the point meets $\partial\Omega$ at one point only. Obviously, every convex domain is strictly starlike relative to any interior point.

The purpose of this chapter is to study the interplay between the behavior of the boundary function f^* of a harmonic mapping f of form (1.1) on one hand and the boundary function of the dilatation and the analytic and coanalytic parts of f on the other.

The chapter is organized as follows. In Sect. 2, we address the results relating univalent harmonic mappings of the unit disc and Hardy spaces. In Sects. 3 and 4, we present the boundary behavior results of global and local nature of univalent harmonic mappings respectively. In Sect. 5, we introduce four open questions regarding the subject of this chapter.

1.2 Univalent Harmonic Mappings and Hardy Spaces

The purpose of this section is to establish a relationship between univalent harmonic mappings and H^p spaces.

Let S_H denote the class of all univalent, sense-preserving, harmonic mappings f of $\mathbb{D}$ normalized by

$$f(0) = 0 \quad \text{and} \quad f_z(0) = 1.$$

Then the analytic and coanalytic parts of f are respectively $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$. Clunie and Sheil-Small [10] proved that S_H is normal with respect to the topology of uniform convergence on compact subsets of $\mathbb{D}$. Set

$$A = \sup_{S_H} |a_2|. \tag{1.4}$$

An analytic function k of $\mathbb{D}$ is called *Bloch* if

$$\sup_{z\in\mathbb{D}} (1 - |z|^2)|k'(z)| < \infty.$$

Using the Koebe transform of f and the compactness of S_H in the topology of almost uniform convergence, Abu Muhanna and Lyzzaik [1] proved the following result:

Theorem 2 *Let* $f = h(z) + \overline{g(z)} \in S_H$. *Then* $\log h'$ *is a Bloch function, that is,*

$$\left| \frac{h''(z)}{h'(z)} \right| \le \frac{2A+2}{1-r} \qquad \left(z = re^{i\theta}\right), \tag{1.5}$$

where A *is as defined in* (1.4). *Moreover, if* $\alpha > 0$, *then*

$$\lim_{r\to 1^-} (1-r)^\alpha h'(re^{i\theta}) = 0$$

for almost all θ.

As a consequence of this result, Abu Muhanna and Lyzzaik [1] concluded that the boundary functions of h, g, and f exist almost everywhere:

Theorem 3 *Let* $f = h(z) + \overline{g(z)} \in S_H$. *Then the integrals*

$$\int_0^1 |h'(re^{i\theta})|dr, \quad \int_0^1 |g'\left(re^{i\theta}\right)|dr, \quad \text{and} \quad \int_0^1 |f_r\left(re^{i\theta}\right)|dr$$

converge for almost all θ, *and the boundary function*

$$\hat{f}\left(e^{i\theta}\right) = \lim_{r\to 1^-} f\left(re^{i\theta}\right)$$

exists almost everywhere.

More generally, Abu Muhanna and Lyzzaik [1] established that h, g, and f belong to Hardy spaces.

Theorem 4 *Let* $f = h(z) + \overline{g(z)} \in S_H$. *Then* $h, g \in H^p$ *and* $f \in h^p$ *for every* $p, 0 < p < (2A+2)^{-2}$, *where* A *is as defined in* (1.4).

This result was subsequently improved by Nowak [24] by showing that $h, g \in H^p$ and $f \in h^p$ for every $p, 0 < p < A^{-2}$.

Let K_H and C_H denote the subclasses of S_H of univalent convex and close-to-convex harmonic mappings, respectively. It has been shown by Clunie and Sheil-Small [10] that if $f = h(z) + \overline{g(z)} \in K_H$, then h is close-to-convex and $|g(z)| < |h(z)|$ for $z \in \mathbb{D} \setminus \{0\}$. Using these facts, Nowak [24] showed the following:

Theorem 5 *Let* $f = h(z) + \overline{g(z)} \in S_H$.

(a) *If* $f \in K_H$, *then* $h, g \in H^p$ *and* $f \in h^p$ *for every* $p, 0 < p < 1/2$.
(b) *If* $f \in C_H$, *then* $h, g \in H^p$ *and* $f \in h^p$ *for every* $p, 0 < p < 1/3$.

1.3 Boundary Behavior in the Large Versus Dilatation

The purpose of this section is to present the results that affirm the interplay between the global boundary behavior of a univalent harmonic mapping of $\mathbb{D}$ and its dilatation.

The next result characterizes every univalent harmonic mapping whose dilatation is a finite Blaschke product and which is a GRM onto a bounded convex domain.

Theorem 6 *Let* Ω *be a bounded convex domain. Then a univalent harmonic mapping* f *of* $\mathbb{D}$ *is a GRM onto* Ω *whose dilatation is a finite Blaschke product of order* n, $n = 1, 2, \cdots$, *if, and only if,* f^* *is a step function given by* $f^*(e^{it}) = c_j \in \partial\Omega$, $t_{j-1} < t < t_j, 1 \leq j \leq n+2$, $t_{n+2} = t_0 + 2\pi$ *and* $[c_1, c_2, \cdots, c_{n+2}, c_1]$ *is a positively-directed convex* $(n+2)$*-gon. In this case,* $f(\mathbb{D})$ *is the inner domain of the convex polygon whose vertices are the points* c_j.

The "if" part of this result is due to Sheil-Small [26] and "only if" part is due to Hengartner and Schober [17].

The next theorem of Laugesen [21] demonstrates the global nature of the relationship between the boundary values of a univalent harmonic mapping and its dilatation.

Theorem 7 *Let* $f = P[f^*]$, *where* $f^*(\theta)$ *is a Lebesgue integral function on* $[0, 2\pi[$ *be a sense-preserving harmonic mapping of* $\mathbb{D}$ *whose dilatation is* a. *Assume that* $f^*(\theta)$ *has bounded variation on a subinterval* I *of* $[0, 2\pi]$.

(a) *For almost every* $\theta \in I$, *we have*

$$\frac{df^*(\theta)}{d\theta} = 0 \Rightarrow |a(e^{i\theta}| = 1. \tag{1.6}$$

(b) *Suppose that for almost all* $\theta \in I$ *whenever* $df^*(\theta)/d\theta \neq 0$, $f(\mathbb{D})$ *lies strictly to the left of the directed tangent line through* $f^*(\theta)$ *in the direction of* $df^*(\theta)/d\theta$, *then for almost all* $\theta \in I$ *we have*

$$|a(e^{i\theta})| = 1 \Rightarrow \frac{df^*(\theta)}{d\theta} = 0. \tag{1.7}$$

The proof of this theorem involves showing that, under the given assumptions the nontangential limits of h' and g' exist and are finite for almost all $e^{i\theta} \in I$. Note also that the proof of (b) depends on a formula for the Jacobian on the boundary which may be found in Martio [23, p. 6].

The hypothesis in (b) holds at once if $f(\mathbb{D})$ is a bounded convex region. Note that in this case, $df^*(\theta)/d\theta$ cannot be absolutely continuous on I or else it is identically zero and consequently constant there; see Bshouty and Hengartner [3]. Duren and Khavinson [12] have presented a geometric proof of a version of (1.7) where they require C^1-smoothness of $f^*(\theta)$ to get a result for all θ instead of the same result for almost all θ.

Laugesen [21] also showed that part (a) of the theorem holds under other assumptions which allow us to examine dilatations of sense-preserving harmonic mappings that do not admit Poisson integral representations.

Theorem 8 *Suppose that f is a sense-preserving harmonic mapping of $\mathbb{D}$ that extends continuously to $\mathbb{D} \cup I$ for a subinterval I of $\mathbb{T}$ and that $f^*(\theta)$ has bounded variation on I, then Theorem 7 (a) holds.*

This chapter uses Blaschke products and, more generally, inner functions. Let $\{\zeta_n\}$ be an infinite sequence of points in $\mathbb{D}$ with the first m terms, say $\zeta_1, \zeta_2, \ldots, \zeta_m$, assume the value zero. Then the *Blaschke product* associated with the values $\{\zeta_n\}$ is defined as the infinite product

$$B(z) = e^{i\alpha} z^m \prod_{n=m+1}^{\infty} \frac{|\zeta_n|}{\zeta_n} \frac{\zeta_n - z}{1 - \overline{\zeta}_n z}. \tag{1.8}$$

It is well known that B belongs to the class $\mathcal{B}$ of analytic functions on $\mathbb{D}$ bounded by one there if, and only if, $\sum(1 - |\zeta_n|)$ converges.

An *inner function* is a function $h \in \mathcal{B}$ for which the radial limits exist and have modulus 1 almost everywhere. A striking relationship between Blaschke products and inner functions is the following result of Frostman [14].

Theorem 9 *If h is an inner function, then for all $k \in \mathbb{D}$ but possibly a set of zero capacity the function*

$$h_k(z) = \frac{h(z) - k}{1 - \overline{k} h(z)}$$

is a Blaschke product.

Laugesen [21] raised a problem to study the boundary behavior of a GRM f from $\mathbb{D}$ onto a bounded convex domain Ω whose dilatation a is an inner function. This problem becomes more relevant knowing that such a mapping is unique [4]. By using Theorem 9, it was concluded by the authors [8, p. 264] that the study of this problem can be reduced to the case where the dilatation a of f is a Blaschke product. Another conclusion on the same page answers in the negative a problem of Laugesen [21] which was motivated by Theorem 5 of his paper; Laugesen's problem may be formulated here as follows: "Must the boundary function f^* of a GRM f from $\mathbb{D}$ onto a bounded convex domain have a jump if the dilatation of f is an infinite Blaschke product?"

Laugesen [21] used Theorem 7 to characterize inner dilatations at once as follows.

Theorem 10 *Let $f = P[f^*]$, where $f^*(\theta)$ is a function of bounded variation in a neighborhood of almost every $\theta \in [0, 2\pi[$, be a sense-preserving harmonic mapping of $\mathbb{D}$ whose dilatation is a. Then*

$$\frac{df^*(\theta)}{d\theta} = 0 \text{ a.e.} \Rightarrow a \text{ is inner.}$$

Moreover, if $f(\mathbb{D})$ is convex, then

$$a \text{ is inner} \Rightarrow \frac{df^*(\theta)}{d\theta} = 0.$$

The next result of Laugesen [21] illustrates the previous theorem and provides a sufficient condition on the boundary values of a univalent harmonic mapping of $\mathbb{D}$ for its dilatation to be a Blaschke product.

Theorem 11 *Let $E \subset \mathbb{T}$ be a closed and countable set, and let $f = P[f^*]$, where $f^*(\theta)$ is a function of bounded variation on $[0, 2\pi[$ that is constant on each connected component of $\mathbb{T} \setminus E$ and jumps at every point of E, is a sense-preserving harmonic mapping of $\mathbb{D}$ whose dilatation is a. Then a is a Blaschke product whose radial limit at every θ exists and has modulus* 1.

Moreover, if E is infinite, f is univalent, and $f(\mathbb{D})$ is convex, then a must be an infinite Blaschke product.

The relationship between the dilatation and the boundary value function of a univalent harmonic function of $\mathbb{D}$ seen above in Theorems 7, 8, and 10 was further enhanced by Bshouty and Hengartner [3] for bounded univalent harmonic mappings of $\mathbb{D}$ for which the dilatation extends analytically across a subinterval of $\mathbb{T}$ on which it attains modulus 1. Using Theorem 2 and Helly's selection theorem, they established the following:

Theorem 12 *Let a be an analytic function of $\mathbb{D}$ that admits an analytic extension across an open interval $I = \{e^{it} : \gamma < t < \delta\}$, $\gamma < \delta < \gamma + 2\pi$, such that $|a| \equiv 1$ on I. Let f be a bounded sense-preserving harmonic mapping of $\mathbb{D}$ whose dilatation is a and which is univalent on a one-sided open neighborhood V of I defined by*

$$V = \{z = re^{i\theta} : r_0 < r < 1 \text{ and } \gamma < \theta < \delta\}$$

for some positive r_0. Then for almost all $e^{i\theta} \in I$, we have

$$f^*(e^{i\theta}) - \overline{a(e^{i\theta}) f^*(e^{i\theta})} + \overline{\int^{\theta} f^*(e^{it})\, da(e^{it})} = \text{ const.} \tag{1.9}$$

Moreover, if f is a univalent harmonic mapping of $\mathbb{D}$ onto a Jordan domain, then a variant of (1.9), where $f^*(e^{i\theta})$ is substituted by any cluster point of f at $e^{i\theta}$, holds.

Note that this theorem generalizes an earlier result of Lyzzaik [22] which relates the behavior of a harmonic mapping of a Jordan arc I and its dilatation a in the case when $|a| \equiv 1$ on I.

Remark 1 Theorem 12 also holds if the univalence of f in V is replaced by the condition that f^* is of bounded variation on I since, as mentioned earlier, the former condition yields that the nontangential limits of h' and g' exist and are finite for almost all $e^{i\theta} \in I$.

Remark 2 Equation (1.9) can be expressed in the differential form

$$df^*(e^{i\theta}) - \overline{a(e^{i\theta})df^*(e^{i\theta})} = 0, \quad f^* - \text{a.e.} \quad \text{in } I \tag{1.10}$$

or equivalently by

$$\Im\left\{\sqrt{a(e^{i\theta})}df^*(e^{i\theta})\right\} = 0, \quad f^* - \text{a.e.} \quad \text{in } I. \tag{1.11}$$

Hence, unless $df^*(e^{i\theta}) = 0$, we have

$$\arg df^*(e^{i\theta}) = -\frac{1}{2}\arg a(e^{i\theta}) \mod \pi. \tag{1.12}$$

Bshouty and Hengartner [3] deduced from Theorem 12 the following two corollaries:

Corollary 1 *Under the assumptions of Theorem 12, for each $e^{it} \in I$, there is a branch of $\sqrt{a}$ such that*

(a) $\operatorname{esslim}_{s\to t+}\Im\left\{\sqrt{a(e^{it})}[f^*(e^{is}) - f^*(e^{it-})]\right\}$ *exists for all* $e^{it} \in I$.

(b) $\operatorname{esslim}_{h\to 0}\Im\left\{\sqrt{a(e^{it})}[f^*(e^{i(t+h)}) - f^*(e^{it})]/h\right\} = 0$ *exists for almost all* $e^{it} \in I$.

If, in addition, f is a univalent harmonic mapping from $\mathbb{D}$ onto a Jordan domain Ω, then either f^* is continuous or has a jump at e^{it}.

Corollary 2 *Let f be a univalent harmonic mapping from $\mathbb{D}$ onto a Jordan domain Ω whose dilatation a is given as in Theorem 12, and let $e^{it} \in I$.*

(a) *If f^* has a jump at e^{it} (which can happen only if $f^*(I)$ contains a line segment), then*

$$\arg\left[f^*(e^{it+}) - f^*(e^{it-})\right] = -\frac{1}{2}\arg a(e^{it}) \mod \pi. \tag{1.13}$$

(b) *If f^* is continuous at e^{it}, then*

$$\lim_{h\to 0}\Im\left\{\sqrt{a(e^{it})}[f^*(e^{i(t+h)}) - f^*(e^{it})]/h\right\} = 0. \tag{1.14}$$

(c) *If* f^* *is not constant on any interval* $[t, t+\delta]$ *for some* $\delta > 0$, *then the right limit*

$$\psi_R(t) = \lim_{h\to 0+} \arg \left[f^*(e^{i(t+h)}) - f^*(e^{it-}) \right] \tag{1.15}$$

$$= -\frac{1}{2} \arg a(e^{it}) \mod \pi \tag{1.16}$$

exists. Analogously, If f^* *is not constant on any interval* $[t-\delta, t]$ *for some* $\delta > 0$, *then the left limit*

$$\psi_L(t) = \lim_{h\to 0+} \arg \left[f^*(e^{it+}) - f^*(e^{i(t-h)}) \right] \tag{1.17}$$

$$= -\frac{1}{2} \arg a(e^{it}) \mod \pi \tag{1.18}$$

exists.

In particular, Corollary 2 states that at every point of the arc $f(I)$, except possibly its endpoints, there exist right and left tangent lines which consequently yield interior and exterior angles.

Definition 1 Let Ω be a simply connected domain of $\mathbb{C}$. We say that a point ω of the boundary $\partial\Omega$ of Ω is a *point of concavity* (with respect to Ω) if there is a line segment L containing ω as an interior point such that $L \setminus \{\omega\}$ is in Ω. Also, we say that ω is *point of convexity* (with respect to Ω) if there is a line segment L containing ω as an interior point such that $L \setminus \{\omega\}$ lies in the exterior of Ω.

With a as in Theorem 3.7, since $a(e^{it})$ is a strictly increasing function on I, Bshouty and Hengartner [3], see also [28], concluded at once:

Corollary 3 *Let* a *and* f *be as in Theorem 3.7. Then there is no open nonempty subinterval* I_1 *of* I *on which all* $f^*(e^{it})$ *are points of convexity. In particular, there is no univalent sense-preserving harmonic mapping of* $\mathbb{D}$ *which maps* I *onto a strictly convex domain. On the other hand, if* $f^*(I)$ *contains a straight line segment* J, *then* f^* *connects the endpoints of* J *through a number of jumps.*

Remark 3 Corollary 3 need not hold if the analyticity of a on I is replaced by the condition that $|a(e^{it})| = 1$ almost everywhere on I; Laugesen [21] gave examples of univalent harmonic mappings from $\mathbb{D}$ onto itself with inner function dilatations.

As a consequence of Theorem 12 and Corollary 3, we have:

Corollary 4 *Let* f *be a GRM from the unit disk* $\mathbb{D}$ *onto a bounded convex domain* Ω *whose dilatation* a *admits an analytic extension across an open interval* $I = \{e^{it} : \gamma < t < \delta\}$, $\gamma < \delta < \gamma + 2\pi$, *such that* $|a| \equiv 1$ *on* I. *Then the following hold:*

(a) f^* *has a jump at* $e^{i\theta} \in I$ *if, and only if,* $\arg\{\sqrt{a(e^{i\theta})} df^*(e^{i\theta})\} = 0 \mod \pi$.
(b) *If* f *has no jumps in* I, *then* f^* *is constant on* I.

The question of the behavior of a univalent harmonic mapping of $\mathbb{D}$ whose boundary function is constant on a subinterval I of $\mathbb{T}$ was tackled by Abu Muhanna and Lyzzaik [1, Theorem 3] and subsequently more generally by Bshouty and Hengartner [3].

Before stating the related results, there is a need to examine the "preimage" of each point in the image boundary of the mapping. let us note that the boundary function of a sense-preserving harmonic mapping may have jumps in as much as intervals of constancy or else continuity points. The "preimage" of a boundary point could be a point or a segment. The following definition describes what is meant by "preimage" of a boundary point.

Definition 2 Let f be a univalent sense-preserving harmonic mapping of $\mathbb{D}$ onto a Jordan domain Ω and let $q \in \partial\Omega$.

(a) If q does not belong to a jump of f^*, then define $\gamma(q)$ and $\delta(q)$ by $(f^*)^{-1}(q) = \{e^{it} : \gamma(q) \le t \le \delta(q)\}$, where $\gamma(q) \le \delta(q) < \gamma(q) + 2\pi$.
(b) If q is an interior point of a jump of f^*, i.e. $q = \lambda f^*(e^{i(t+)}) + (1-\lambda) f^*(e^{i(t-)})$, $0 < \lambda < 1$, then define $\gamma(q) = \delta(q) = t$.
(c) If q is the endpoint $f^*(e^{i(t-)})$ of a jump, then define $\gamma(q)$ as in (a) and put $\delta(q) = t$.
(d) If q is the endpoint $f^*(e^{i(t+)})$ of a jump, then put $\gamma(q) = t$ and define $\delta(q)$ as in (a).

Note that the cluster set $C(f^*, e^{i\gamma(q)})$ contains points other than q if a jump occurs at $e^{i\gamma(q)}$ and likewise for $C(f^*, e^{i\delta(q)})$.

The following result is due to Bshouty and Hengartner [3].

Theorem 13 *Let f be a univalent sense-preserving harmonic mapping of $\mathbb{D}$ onto a Jordan domain Ω whose dilatation a admits an analytic extension across an open interval $I_1 = \{e^{it} : \gamma_1 < t < \delta_1\}$, $\gamma_1 < \delta_1 < \gamma_1 + 2\pi$, such that $|a| \equiv 1$ on I_1. Let q be an interior point of $f^*(I_1)$ and let $\gamma(q)$ and $\delta(q)$ be as in Definition 2. Finally, let $\alpha(q)$ be the opening angle at q as seen from the inside of Ω. Then the following holds:*

1. *If $\alpha(q) = 0$, then $\gamma(q) = \delta(q)$. For $0 < \alpha(q) < \pi$, then $\gamma(q) < \delta(q)$ and*

$$\frac{1}{2}\left[\arg\ a(e^{i\delta(q)}) - \arg\ a(e^{i\gamma(q)})\right] = \alpha(q). \tag{1.19}$$

2. *If $\alpha(q) = \pi$, then either $\gamma(q) = \delta(q)$ or $\gamma(q) < \delta(q)$ and (1.19) holds. Both cases are possible.*
3. *If $\pi < \alpha(q) < 2\pi$, then $\gamma(q) < \delta(q)$ and either (1.19) holds or*

$$\frac{1}{2}[\arg\ a(e^{i\delta(q)}) - \arg\ a(e^{i\gamma(q)})] = \alpha(q) - \pi. \tag{1.20}$$

Both cases are possible.

1.4 Boundary Behavior in the Small Versus Dilatation

The purpose of this section is to study the local behavior of the boundary function f^* of a univalent harmonic mappings f of $\mathbb{D}$ versus its dilatations a; that is, the behavior of the boundary function at individual points of $\mathbb{T}$. Here, three kinds of results are addressed: The first describes the behavior of f^* in deleted neighborhoods of points in $\mathbb{T}$; the second and third consider those results involving the differentiability and continuity of f^* at points in $\mathbb{T}$.

The first result in this section is due to Bshouty, Lyzzaik, and Weitsman [8] and provides a sufficient condition for f^* to be nonconstant on any *right* or *left interval* of the point $e^{i\theta_0}$; i.e., on any interval of form $\{e^{it} : \theta_0 < t < \gamma\}$ or $\{e^{it} : \gamma < t < \theta_0\}$ respectively.

Theorem 14 *Let f be a GRM from $\mathbb{D}$ onto a Jordan domain Ω with rectifiable boundary, and let the dilatation a of f be a Blaschke product of form* (1.8). *Fix $e^{i\theta_0} \in \partial\mathbb{D}$. If*

$$\sum_{n=1}^{\infty} \frac{1 - |\zeta_n|}{|e^{i\theta_0} - \zeta_n|} = \infty, \tag{1.21}$$

then f^* is nonconstant on any right or left interval of $e^{i\theta_0}$.

The main idea behind the proof of this theorem is that (1.21) holds if, and only if, $\Delta_I \arg\{\sqrt{a(z)}\} = \infty$ on any right or left interval I of $e^{i\theta_0}$; $\Delta_I \arg\{\sqrt{a(z)}\}$ denote the net variation over the circular arc I of a continuous single-valued branch of $\arg\sqrt{a}$.

In the same direction, Bshouty, Lyzzaik, and Weitsman [8] established the following result.

Theorem 15 *Let f be a GRM from the unit disk $\mathbb{D}$ onto a Jordan domain Ω with rectifiable boundary whose dilatation a is a Blaschke product of form* (1.8). *Let there be a left (right) interval I of $e^{i\theta_0}$ across which a continues analytically with $|a| = 1$ on I.*

(a) *A sufficient condition for f^* to be nonconstant on any left (right) interval of $e^{i\theta_0}$ is one of the following:*

  (i) *for every left (right) interval $K \subset I$ of $e^{i\theta_0}$ $\Delta_K \arg\{\sqrt{a(z)}\} = \infty$.*

  (ii) *There exists a subsequence $\{\xi_n\}$ of $\{\zeta_n\}$ converging to $e^{i\theta_0}$ such that*

$$\sum_{\arg \xi_n \le \theta_0} 1 + \sum_{\arg \xi_n \ge \theta_0} \frac{1 - |\xi_n|}{|e^{i\theta_0} - \xi_n|} = \infty, \tag{1.22}$$

$$\left(\sum_{\arg \xi_n \ge \theta_0} 1 + \sum_{\arg \xi_n \le \theta_0} \frac{1 - |\xi_n|}{|e^{i\theta_0} - \xi_n|} = \infty \right).$$

(b) *If Ω is a bounded convex domain, then each of the above sufficient conditions is also necessary for f^* to be nonconstant on any left (right) interval of $e^{i\theta_0}$.*

As a result of this theorem, the following three corollaries hold [8]:

Corollary 5 *Let f be a GRM from the unit disk $\mathbb{D}$ onto a bounded convex domain Ω whose dilatation a is a Blaschke product of form (1.8) such that there exists a left (right) interval I of θ_0 across which a continues analytically with $|a| = 1$ on I. Then a necessary and sufficient condition for f^* to be nonconstant on any left (right) interval of $e^{i\theta_0}$ is that one of the following two conditions holds:*

(i) $\Delta_I \arg\{\sqrt{a(z)}\} = \infty$;
(ii) *there exists a subsequence $\{\xi_n\}$ of $\{\zeta_n\}$ converging to $e^{i\theta_0}$ such that (1.22) holds.*

The following corollary follows at once from Theorem 15(b) and shows that the "nonconstancy" result of Theorem 14 is sharp.

Corollary 6 *Let f be a GRM from the unit disk $\mathbb{D}$ onto a bounded convex domain Ω whose dilatation a is a Blaschke product of form (1.8), where $m = 0, \Im\left\{e^{-i\theta_0}\zeta_n\right\} > 0$ ($\Im\{e^{-i\theta_0}\zeta_n\} < 0$) for all n, and*

$$\sum_{n=1}^{\infty} \frac{1 - |\zeta_n|}{|e^{i\theta_0} - \zeta_n|} < \infty. \tag{1.23}$$

Then f^* is constant on some left (right) interval of $e^{i\theta_0}$.

The third corollary examines the stability of a function f given in Theorem 15 upon the addition of finitely many zeros to its dilatation a. Indeed, it asserts that f is stable if a satisfies (1.21) and unstable otherwise.

Corollary 7 *Let a be a Blaschke product whose zeros $\{\zeta_n\}_{n=1}^{\infty}$ accumulate at $e^{i\theta_0}$, and let A be the Blaschke product whose zeros are those of a and an additional value ζ_0. Let f and F be the GRMs of the unit disk onto a bounded convex domain Ω whose dilatations are a and A respectively.*

(a) *If f^* is nonconstant on any left or right interval of $e^{i\theta_0}$, then so is F^*.*
(b) *If f^* is constant on some left (right) interval I of $e^{i\theta_0}$, then for a suitable choice of ζ_0 the function F admits a jump in I.*

Next, we address the recent boundary behavior results of a univalent harmonic mapping f of $\mathbb{D}$ that relate the differentiability of the boundary function at a given point with the behavior of the dilatation a near the point.

A requisite for these results is the following result due to Fatou [27, pp. 132–135].

Theorem 16 *Let u^* be an integrable real-valued function of $\mathbb{T}$ and let $u = P[u^*]$. Then*

(i) *If $(du^*/d\theta)(e^{i\theta_0})$ exists and is finite, then the angular limit*

$$\lim_{z \to e^{i\theta_0}} \frac{\partial u}{\partial \theta}(z) = \frac{du^*}{d\theta}\left(e^{i\theta_0}\right)$$

is uniform in any Stolz angle with vertex at $e^{i\theta_0}$.

(ii) *If* $(du^*/d\theta)(e^{i\theta_0}) = +\infty$, *then*

$$\lim_{r\to 1^-} \frac{\partial u}{\partial \theta}(re^{i\theta_0}) = +\infty.$$

If, in addition, u^* *is monotone increasing in a neighborhood of* θ_0, *then the angular limit*

$$\lim_{z\to e^{i\theta_0}} \frac{\partial u}{\partial \theta}(z) = +\infty$$

is uniform in any Stolz angle with vertex at $e^{i\theta_0}$.

(iii) *If* $(du^*/d\theta)(e^{i\theta})$ *is continuous on an arc* $\theta \in [\alpha, \beta]$ *and* $\alpha < \alpha_1 < \beta_1 < \beta$, *then*

$$\lim_{z\to e^{i\theta}} \frac{\partial u}{\partial \theta}(z) = \frac{du^*}{d\theta}(e^{i\theta})$$

uniformly for $\theta \in [\alpha_1, \beta_1]$ *as* $z \to e^{i\theta}$ *from* $\mathbb{D}$.

Evidently, in (ii) $-\infty$ may replace $+\infty$, and (i) and (iii) hold true for integrable complex-valued functions f^* of $\mathbb{T}$.

By virtue of this theorem and an extension of a theorem of Heinz [19] due to Kalaj [20, Theorem 2.5], Bshouty, Lyzzaik, and Weitsman [9] obtained the following result:

Theorem 17 *Let* f *be the GRM from the unit disc* $\mathbb{D}$ *onto a bounded convex domain with boundary function* f^* *and dilatation* a, *and let* $(df^*/d\theta)(e^{i\theta_0}) = 0$. *Then the angular limit of* $|a|$ *at* $e^{i\theta_0}$ *is 1; in particular,* a *has at most a finite number of zeros in any Stolz angle with vertex* $e^{i\theta_0}$.

As a special case of this this theorem we have:

Corollary 8 *Let* f *of form* (1.1) *be the GRM from the unit disc* $\mathbb{D}$ *onto a bounded convex domain with boundary function* f^* *and dilatation* a. *If* $(df^*/d\theta)(e^{i\theta_0}) = 0$ *and the angular limit* $\lim_{z\to e^{i\theta_0}} \arg a(z) = \alpha$, *then the angular limits*

$$\lim_{z\to e^{i\theta_0}} \arg h'(z) = -\theta_0 - \frac{1}{2}\alpha \pmod{\pi}$$

and

$$\lim_{z\to e^{i\theta_0}} \arg g'(z) = -\theta_0 + \frac{1}{2}\alpha \pmod{\pi}$$

hold.

The next result [9] deals with the case where $(df^*/d\theta)(e^{i\theta_0})$ is nonzero and finite.

Corollary 9 *Let* f^* *be an integrable complex-valued function of* $\mathbb{T}$ *and let* $f = P[f^*]$ *be the harmonic mapping of* $\mathbb{D}$ *of form* (1.1) *and whose dilatation is* a. *Suppose that* $(df^*/d\theta)(e^{i\theta_0}) = \alpha \neq 0, \infty$ *and the angular limit* $\lim_{z\to e^{i\theta_0}} h'(z) = \beta$ *exists. Then* $\beta \neq 0$ *and the angular limit* $\lim_{z\to e^{i\theta_0}} a(z)$ *exists.*

In the special case of GRM's onto convex domains, we have the interesting result [9]:

Theorem 18 *Let f be the GRM from the unit disc $\mathbb{D}$ onto a bounded convex domain with boundary function f^* and dilatation a, and let $(df^*/d\theta)(e^{i\theta_0}) = \alpha \neq 0$. If $|\alpha|$ is sufficiently small, then a has at most a finite number of zeros in any Stolz angle with vertex $e^{i\theta_0}$.*

In the remaining part of this section, we consider the recent boundary behavior results of a univalent harmonic mapping f of $\mathbb{D}$ that relate the continuity of the boundary function at a given point with the dilatation a near the point.

We begin by a fact obtained in the proof of Theorem 5 in Laugesen [21] regarding harmonic mappings f of $\mathbb{D}$ of form (1.3) where $f^*(e^{i\theta})$ is a function of bounded variation of $[0, 2\pi[$. If f^* has a jump at $e^{i\theta_0}$, then the angular limit of the dilatation a at $e^{i\theta_0}$ satisfies

$$\lim_{z \to e^{i\theta_0}} a(z) = \frac{\overline{df^*(\theta_0)}}{df^*(\theta_0)}.$$

In particular, this limit has magnitude 1. In [8], this observation was formulated as follows:

Theorem 19 *Let f be a GRM from the unit disk $\mathbb{D}$ onto a Jordan domain Ω with rectifiable boundary, let the dilatation a of f be a Blaschke product of form (1.8), and let $e^{i\theta_0} \in \partial\mathbb{D}$. Then the following holds:*

If $\lim_{r \to 1^-} a(re^{i\theta_0})$ does not exist, or if otherwise

$$\lim_{r \to 1^-} a\left(re^{i\theta_0}\right) \neq \alpha, \quad |\alpha| = 1, \tag{1.24}$$

then f^* is continuous at $e^{i\theta_0}$.

In the next result, Bshouty, Lyzzaik, and Weitsman [9] gave a characterization of the continuity and the jump of a GRM at a boundary point in terms of the behavior of h and g.

Theorem 20 *Let f of form (1.1) be a GRM from $\mathbb{D}$ onto a bounded Jordan domain with a rectifiable boundary, and let a and f^* be dilatation and boundary function of f respectively. Then $\lim_{r \to 1^-} (1-r)\underline{h'(re^{i\theta_0})} = c$ and $\lim_{r \to 1^-} (1-r)g'(re^{i\theta_0}) = d$ exist, are finite, and satisfy $ce^{i\theta_0} = \overline{de^{i\theta_0}}$; moreover, either limit is zero if, and only if, f^* is continuous at $e^{i\theta_0}$.*

We conclude at once that f^* is continuous at $e^{i\theta_0}$ if f satisfies in particular the condition that either one of the cluster sets of h', or g', at $e^{i\theta_0}$ is away from infinity.

Another consequence of Theorem 20 [9] is the following result:

Corollary 10 *Under the assumptions of Theorem 20, suppose that there exists a sequence $\{z_n\}$ of complex numbers in $\mathbb{D}$ that satisfies the following properties:*

(a) $\lim_{n \to \infty} z_n = e^{i\theta_0}$ *for some* $\theta_0 \in \mathbb{R}$;
(b) $\lim_{n \to \infty} |z_n - z_{n-1}|/|1 - z_n z_{n-1})| = 0$;
(c) $\lim_{n \to \infty} g'(z_n) = \alpha$ *exists and is finite.*

Then f^* is continuous at $e^{i\theta_0}$.

Note that $|z_n - z_{n-1}|/|1 - z_n z_{n-1})|$ is the pseudo hyperbolic distance between z_n and z_{n-1}.

A special case of this corollary is when the values z_n are zeros of g' which are the same as the zeros of a; see [9].

Theorem 21 *Let f of form (1.1) be a GRM from $\mathbb{D}$ onto a bounded Jordan domain with a rectifiable boundary, and let a and f^* be dilatation and boundary function of f respectively. If $\{z_n\}$ is a sequence of complex numbers satisfying (a) and (b) of Corollary 10, and $a(z_n) = 0$ for all $n = 1, 2, \cdots$, then f^* is continuous at $e^{i\theta_0}$.*

In the next result [9] which uses Corollaries 3, 4, and 5, we illustrate the significance of Theorem 20 by showing the surprising fact that the set of zeros of the dilatation of the GRM f from $\mathbb{D}$ onto $\mathbb{D}$ need not determine the behavior of the boundary function f^*.

Theorem 22 *Suppose the following:*

(a) *The numbers $\zeta_n \in \mathbb{D}$, $n = 1, 2, \ldots$, such that $\Im\left(e^{-i\theta_0}\zeta_n\right) > 0$ and*

$$\sum_{n=1}^{\infty} \frac{1 - |\zeta_n|}{|e^{i\theta_0} - \zeta_n|} < \infty. \tag{1.25}$$

(b) *The Blaschke subproducts*

$$B_N(z) = \prod_{n=N}^{\infty} \frac{|\zeta_n|}{\zeta_n} \frac{\zeta_n - z}{1 - \overline{\zeta}_n z}.$$

(c) *The GRMs $f_{\lambda,N}$, where $\lambda \in \mathbb{T}$, from $\mathbb{D}$ onto $\mathbb{D}$, with $f_{\lambda,N}(0) = 0$, whose dilatation is λB_N.*

Then for sufficiently large N, there exist two distinct values λ such that the boundary function $f^*_{\lambda,N}$ of $f_{\lambda,N}$ has a jump discontinuity at $e^{i\theta_0}$ for one value and is continuous at $e^{i\theta_0}$ for the other value.

The condition "$\Im(e^{-i\theta_0}\zeta_n) > 0$" may be replaced by the condition "$\Im(e^{-i\theta_0}\zeta_n) < 0$"; in this case the desired GRM has form $\overline{f(\overline{z})}$, where f is the desired GRM whose Blaschke product is $\overline{B(\overline{z})}$, with zeros $\overline{\zeta_n}$ satisfying $\Im\left(e^{i\theta_0}\overline{\zeta_n}\right) > 0$ and $\overline{\zeta_n} \to e^{-i\theta_0}$. In view of this, either condition becomes unnecessary for the validity of Theorem 22 provided that the Blaschke products B_j^N are chosen as subproducts of B with a common set of zeros, say $\{\zeta_{n_k}\}$; namely a set that satisfies for all k either $\Im\left(e^{-i\theta_0}\zeta_{n_k}\right) > 0$ or $\Im(e^{-i\theta_0}\zeta_{n_k}) < 0$.

Next, we present a result of Bshouty, Lyzzaik, and Weitsman [9] that gives a sufficient condition for the boundary function of a GRM f from $\mathbb{D}$ onto a bounded convex domain Ω whose dilatation is an infinite Blaschke product to be continuous at a given boundary point. This result is based on the following result of Protas [25]:

Theorem 23 *Let B be an infinite Blaschke product with zeros $\{\zeta_n\}$ and let $\zeta = e^{i\theta_0}$, $\gamma \geq 1$, and $m > 0$. Then (1.23) holds if, and only if,*

$$\int_{\Gamma_{\zeta,\gamma,m}} \frac{1-|B(z)|^2}{1-|z|^2}|dz| \quad < \infty.$$

where $\Gamma_{\zeta,\gamma,m} = \Gamma$ is the "curve" in $\mathbb{D}$ defined by $\Gamma(\theta) = (1-m|\theta|^\gamma)e^{i\theta}$ for $0 < |\theta| < \min\{\pi, m^{-1/\gamma}\}$.

Theorem 24 *Let f of form (1.1) be a GRM from $\mathbb{D}$ onto a bounded convex domain Ω, whose dilatation is a Blaschke product a with zeros ζ_n, $n = 1, 2, \ldots$, and whose boundary function f^*. If*

$$\sum_{n=1}^{\infty} \frac{1-|\zeta_n|}{|e^{i\theta_0}-\zeta_n|} \quad = \infty,$$

then f^* is continuous at $e^{i\theta_0}$.

Remark 4 Let $\phi = z^n B(z)s(z)$ be an inner function, where B is a Blaschke product and s is a singular inner function associated with the measure σ. Using Protas [25, Theorem 2], an adaptation of the proof of Theorem 24 yields the following result [9]:

Let f be the GRM from $\mathbb{D}$ onto a bounded convex domain Ω associated with the above-mentioned inner function ϕ. If

$$\sum_{n=1}^{\infty} \frac{1-|\zeta_n|}{|e^{i\theta_0}-\zeta_n|} + \int_0^{2\pi} |1-e^{i(t-\theta_0)}|d\sigma(t) \quad = \infty,$$

then f^* is continuous at $e^{i\theta_0}$.

In [8], the authors exhibited the following example of a GRM f from $\mathbb{D}$ onto itself whose dilatation a is a Blaschke product having zeros that satisfy (1.23) for $\theta_0 = 0$ such that the jumps of the boundary function f^* are generally not entirely dependent on the zeros of the dilatation. This is done by showing that upon merely applying a rigid rotation to a, continuity of f^* at a point may turn into a jump there.

Example 4.1 Consider the following:

(a) The infinite partition $\pi > t_0 > t_1 > \cdots > \pi/4$, where $\lim_{k\to\infty} t_k = \pi/4$;
(b) The open circular arcs $J_k = \{e^{it} : \pi/2^{k+1} < t < \pi/2^k\}$, $k = 0, 1, \cdots$;
(c) The function f^* defined by $f^*(J_k) = e^{it_k}$, $k = 0, 1, \cdots$, and the symmetry property $f^*(e^{-it}) = \overline{f^*(e^{it})}$;
(d) The function f is Poisson integral of f^*.

Then, by the Radó–Kneser–Choquet Theorem, f is a univalent harmonic mapping of $\mathbb{D}$ into itself that satisfies $f(0) = w_0 \in \mathbb{R}$ and $f_z(0) > 0$. In view of (c) and (d), the symmetry property $f(\overline{z}) = \overline{f(z)}$ holds for f. Consequently, the dilatation a of f is likewise symmetric and its zeros are symmetric about the real axis. Moreover, by Theorem 10 [21], the dilatation a is an infinite Blaschke product having the property that for every θ, the radial limit $\lim_{r\to 1^-} a(re^{i\theta})$ exists and has modulus 1. Hence there exists only finitely many zeros of a on every radius of the unit disc. Since f^* is constant on every arc J_k, a continues analytically across J_k with $|a| \equiv 1$ there. Hence, the zeros of a accumulate in the set $\{e^{\pm i\pi/2^k} : k = 0, 1, \cdots\}$. But f^* is constant on intervals left and right of every point $\nu_k = e^{\pm i\pi/2^k}$, $k \neq 0$, which, by Corollary 5, neither side of the diameter of $\mathbb{D}$ ending at ν_k contains a sequence of zeros of a converging to ν_k. Hence, every ν_k is not an accumulation point of the zeros of a and, consequently, the zeros of a accumulate only at 1.

Because $\lim_{r\to 1^-} a(r)$ exists and has modulus 1 and a has the symmetry property, we conclude that either $a(z) = a_1(z)$ or $a(z) = a_{-1}(z)$, where

$$a_\eta(z) = \eta \prod_{n=1}^{\infty} \frac{(z-\zeta_n)(z-\overline{\zeta_n})}{(1-\overline{\zeta_n}z)(1-\zeta_n z)}.$$

Thus $a_\eta(1) = \eta$. But since $df^*(1) = \sqrt{2}\, i$, Theorem B(a) yields

$$\Im\left\{\sqrt{a_\eta(1)}\, df^*(1)\right\} = \pm\Im\left\{\eta^{1/2}\sqrt{2}\, i\right\} = \pm\sqrt{2}\,\Re\,\eta^{1/2} = 0.$$

Hence $\eta = -1$. Finally, by Theorem 4.11,

$$\sum_{n=1}^{\infty} \frac{1-|\zeta_n|}{|1-\zeta_n|} < \infty.$$

We have thus exhibited a GRM from $\mathbb{D}$ onto $\mathbb{D}$ with dilatation a_{-1}.

Next, let F be the GRM from the open unit disc onto itself whose dilatation is a_1 and satisfies $F(0) = 0$ and $F_z(0) > 0$. It may be easily verified that $\overline{F(\overline{z})}$ is also a GRM from the open unit disc onto itself whose dilatation is $\overline{a_1(\overline{z})} = a_1(z)$ and which is normalized at the origin exactly like F. Then, by [4, Theorem 1], F satisfies the symmetry property $F(\overline{z}) = \overline{F(z)}$. Because infinitely many zeros of a lie on either side of the real axis, by Theorem 15, F^* is nonconstant on any left or right interval of 1. Moreover, F^* is continuous at 1 or else it has a jump of pure imaginary size iC, where $C > 0$. Then, by invoking Theorem 12, we obtain

$$0 = \Im\left\{\sqrt{a(1)}\, dF^*(1)\right\} = \pm\Im\{iC\} = \pm C = 0.$$

Thus F^* is continuous at 1.

1.5 Some Open Questions

We end the chapter with the following relevant open questions.

Question 1 (A. Weitsman) Is there a univalent harmonic self mapping f of $\mathbb{D}$ whose dilatation a is an infinite Blaschke product?

The answer to this question is yes if $f(\mathbb{D})$ is the inner region of an ellipse [8].

Laugesen [21, Theorem 3.6] gave a class of univalent harmonic mappings f of $\mathbb{D}$ whose dilatations a are infinite Blaschke products. In particular, in the proof of Theorem 11, it was shown that if E is a closed countable subset of $\mathbb{D}$ and if f is a univalent harmonic mapping of $\mathbb{D}$ with jumps at each point of E and is constant on each component of $\mathbb{D} \setminus E$, then a is a Blaschke product. In the same paper above, Laugesen asked if one could find conditions on univalent harmonic self mappings of $\mathbb{D}$ that would yield Blaschke product dilatations. This question is unstable: for if f is a univalent harmonic self mapping f of $\mathbb{D}$ whose dilatation a is inner, then by a mere affine transformation of the form $f + \alpha \overline{f}$ for almost all values α in a neighborhood of the origin the dilatation is a Blaschke product and the image is an elliptic domain close to $\mathbb{D}$. However, the following question is relevant.

Question 2 (R. S. Laugesen) Is there a univalent harmonic self mapping f of $\mathbb{D}$ whose dilatation a is a singular inner function.

Question 3 (A. Lyzzaik, D. Bshouty, and A. Weitsman) Let f be the GRM from $\mathbb{D}$ onto $\mathbb{D}$ with dilatation a, and let $f^*(e^{it})$ denote the radial boundary values of f. If $(df^*/dt)(e^{i\theta})$ exists, is it true that a has finitely many zeros in any Stolz angle at $e^{i\theta}$?

For related results to this question see [7].

Questions 1, 2, and 3 appear in [5].

It is known by Fatou's theorem [13, p. 12] that if f is a bounded harmonic mapping of $\mathbb{D}$, then f has radial limits almost everywhere and the exceptional set can be any F_σ set of measure zero. However, if f is univalent analytic function, then by Beurling's theorem [2] the exceptional set have capacity zero. In view of this, the following question arises:

Question 4 If f is a univalent harmonic mapping of $\mathbb{D}$, then can the exceptional set have positive capacity? Can it have positive Hausdorf dimension? Can it have Hausdorf dimension 1?

The authors would like to thank the referee for carefully reading this chapter and raising Question 4.

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Chapter 2
Harmonic Univalent Mappings and Minimal Graphs

Zach Boyd and Michael Dorff

2.1 Introduction

In this chapter we will discuss some topics about planar harmonic mappings. These functions can be thought of as a generalization of analytic maps, and so we will first present a brief background of analytic univalent mappings. Then we will discuss harmonic mappings with an emphasis on three topics: the shearing technique, inner mapping radius, and convolutions. Finally, we will discuss the connection between planar harmonic mappings and minimal surfaces.

2.1.1 Analytic Univalent Maps

Harmonic maps naturally generalize analytic functions by relaxing the requirement of analyticity while still retaining some important features. We begin with an overview of the relevant properties of analytic functions to make clear the analogy with harmonic maps. In both cases, we focus entirely on functions which are *univalent*, or one-to-one, although much interesting work has been done on multivalent functions as well.

Definition 1.1 Let $F : D \subset \mathbb{C} \to \mathbb{C}$. The function $F(x, y) = u(x, y) + iv(x, y)$ is *analytic* if:

- F is continuous;
- u and v are real harmonic in D; and
- u and v are harmonic conjugates (that is, $u_x = v_y$ and $u_y = -v_x$).

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In this context, a function $u(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is called *real harmonic* if $u_{xx} + u_{yy} = 0$.

While analytic functions may map from any open, connected set in general, the following theorem allows us to restrict attention to the unit disk in many cases.

Theorem 1.2 (Riemann Mapping Theorem) *Let $G \neq \mathbb{C}$ be a simply-connected domain with $a \in G$. Then there exists a unique univalent, onto analytic function $F : G \to \mathbb{D}$ such that $F(a) = 0$ and $F'(a) > 0$.*

Thus, if D is a simply-connected, proper subset of the complex plane, we may replace the function $f : D \to \mathbb{C}$ by the function $f \circ \phi : \mathbb{D} \to \mathbb{C}$, where the existence of $\phi : \mathbb{D} \to D$ is guaranteed. Therefore, in the study of univalent (one-to-one) analytic functions, we may restrict our attention to the following class of functions.

Definition 1.3 The family of analytic, normalized, and univalent functions denoted by S is

$$S = \{F : \mathbb{D} \to \mathbb{C} \mid F \text{ is analytic, univalent with } F(0) = 0, F'(0) = 1\}.$$

This family of functions is also known as *schlicht* functions. Note that $F \in S$ implies $F(z) = z + a_2 z^2 + a_3 z^3 + \cdots$. The following are two essential examples that will be used throughout the chapter.

Example 1.4 (The Analytic Right Half-Plane Mapping)

$$F_h(z) = \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n = z + z^2 + z^3 + \cdots \in S.$$

Example 1.5 (The Koebe Function)

$$F_k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n = z + 2z^2 + 3z^3 + \cdots \in S.$$

Observe that F_k maps to the entire complex plane minus a slit from $-1/4$ to ∞ (Fig. 2.1).

Some important properties of the family S include

- The uniqueness condition in the Riemann Mapping Theorem.
- (de Branges' Theorem) For $F \in S$, $|a_n| \leq n$, for all n.
- (Koebe $\frac{1}{4}$-Theorem) If $F \in S$, then $F(\mathbb{D})$ contains the disk $G = \{w : |w| < \frac{1}{4}\}$.

See [14] for more background in univalent analytic functions.

2.1.2 Harmonic Univalent Maps

Complex-valued harmonic functions are a generalization of the analytic functions in which one of the requirements is relaxed.

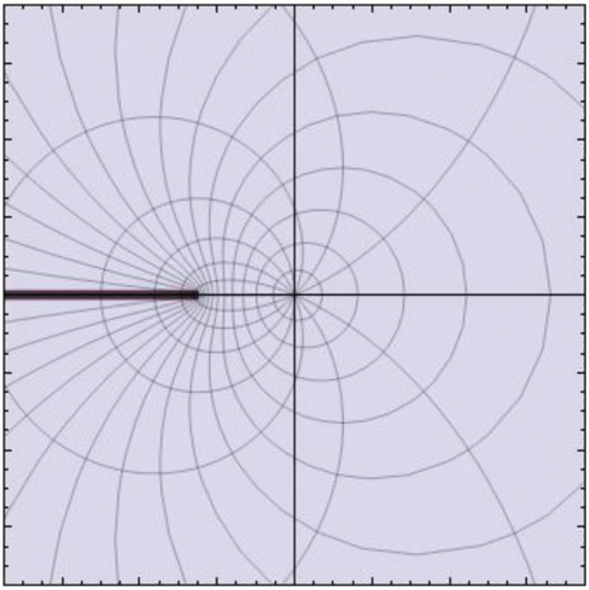

Fig. 2.1 The image of $\mathbb{D}$ under $F_k(z) = \frac{z}{(1-z)^2} \in S$

Definition 1.6 Let $f : D \subset \mathbb{C} \to \mathbb{C}$. The function $f(x, y) = u(x, y) + iv(x, y)$ is a (complex-valued) *harmonic function* if:

- f is continuous; and
- u and v are real harmonic in D.

This definition views harmonic functions as being composed of real and imaginary parts. If D is simply-connected, we have a more useful characterization ([3]).

Theorem 1.7 *If $f = u + iv$ is harmonic in a simply-connected domain G, then $f = h + \overline{g}$, where h and g are analytic.*

Note that $f = h + \overline{g}$ is equivalent to $f = \text{Re}\{h + g\} + i\text{Im}\{h - g\}$. Also, one consequence of this theorem is that a harmonic function f is represented by a power series of the form

$$f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n.$$

In particular, every harmonic function with domain $\mathbb{D}$ is just the sum of analytic and coanalytic parts, represented by h and g, respectively. To see the geometric effect of including $\overline{g}$, we recall that an analytic map is called *conformal* if its derivative never vanishes. The conformal property means that intersecting curves in the domain are mapped to intersecting curves in the image, and the angle of intersection is preserved. A harmonic map is the sum of two maps, one which preserves angles, and another which reverses them. After some reflection, it should be clear that if $|h'(z_0)| > |g'(z_0)|$, then the map is *sense-preserving* at z_0, meaning that positive angles

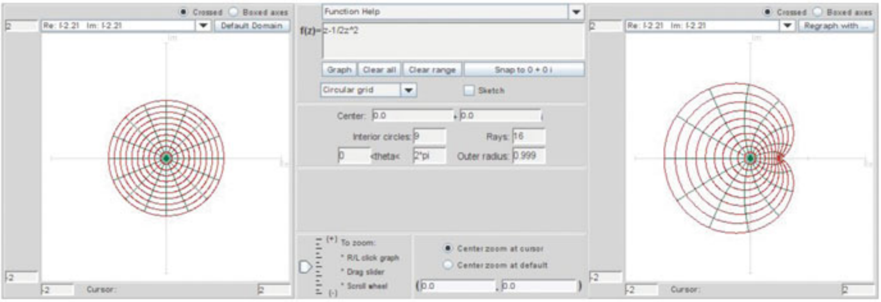

Fig. 2.2 The image of $\mathbb{D}$ under F_p

remain positive, and negative angles remain negative under the map f. Equivalently, we say that a function is sense-preserving if the left-hand side of a curve is mapped to the left-hand side of its image. The following theorem formalizes this intuition.

Theorem 1.8 **(Lewy [22])** *$f(z) = h(z) + \overline{g(z)}$ is locally univalent and sense-preserving if and only if $|\omega(z)| = |g'(z)/h'(z)| < 1$, for all $z \in \mathbb{D}$.*

The function $\omega = g'/h'$ is known as the *dilatation* of $f = h + \overline{g}$.

Observe that in the harmonic case, terms involving $\overline{z}$ are permissible, but terms involving $z\overline{z}$ are not. Also, the graphics highlight the fact that the images of radial and circular lines intersect at right angles in the conformal case, but not in the harmonic case.

The boundary of $f_p(\mathbb{D})$ in Fig. 2.3 consists of concave arcs and the boundary of $f_h(\mathbb{D})$ in Fig. 2.5 gets mapped to just two points, $w = -\frac{1}{2}$ and $w = \infty$. These examples illustrate a difference between analytic and harmonic maps and an important fact about the boundary behavior of certain harmonic functions.

Theorem 1.9 *Let $f = h + \overline{g}$ be a sense-preserving harmonic map with dilatation $\omega = g'/h'$. If $|\omega(z)| = 1$ for almost all z in an arc γ of $\partial\mathbb{D}$, then the image of γ under f is either a concave arc or a stationary point.*

Example 1.10 In the following pages, graphs of functions are usually the image of the unit disk under the function in question. Also, many of these images have been created by the online applet *ComplexTool* [9] (Figs. 2.2–2.5)

Example 1.11 The uniqueness part of the Riemann mapping theorem fails in the harmonic case, since both maps, F_h and f_h, send the disk to the same right half-plane.

Open Problem 1 *What is the analogue of the Riemann mapping theorem for harmonic mappings?*

As a final point in this section, we note that, in analogy to S, we define the classes S_H and S_H^O as follows.

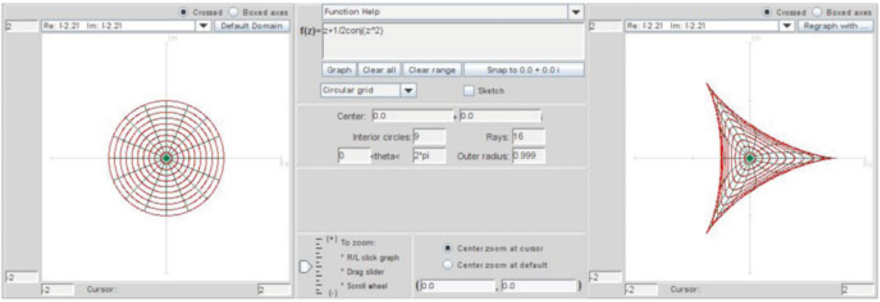

Fig. 2.3 The image of $\mathbb{D}$ under f_p

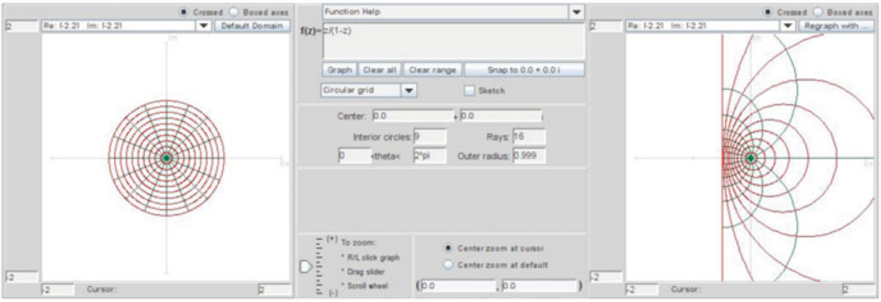

Fig. 2.4 The image of $\mathbb{D}$ under F_h

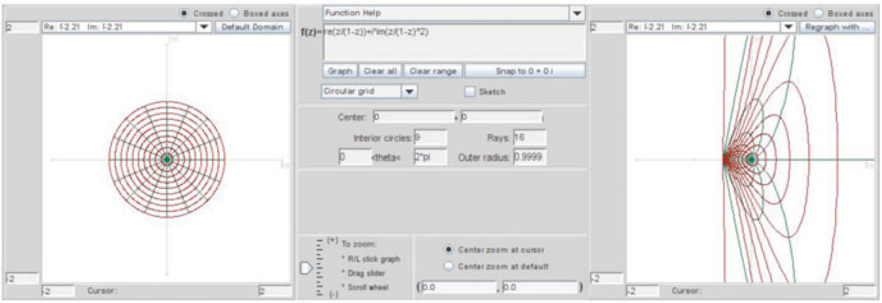

Fig. 2.5 The image of $\mathbb{D}$ under f_h

- **Analytic polynomial map:** $F_p(z) = z - \frac{1}{2}z^2$
- **Harmonic polynomial map:** $f_p(z) = z + \frac{1}{2}\bar{z}^2$
- **Analytic right half-plane map:** $F_h(z) = \frac{z}{1-z}$
- **Harmonic right half-plane map:** $f_h(z) = \text{Re}(\frac{z}{1-z}) + i\,\text{Im}(\frac{z}{(1-z)^2})$

Definition 1.12 Let S_H be the family of complex-valued harmonic, univalent mappings that are normalized on the unit disk, that is,

$$S_H = \{f : \mathbb{D} \to \mathbb{C} \mid f \text{ is harmonic, univalent with } f(0) = a_0 = 0, f_z(0) = a_1 = 1\}.$$

$$S_H^O = \{f \in S_H \mid f_{\bar{z}}(0) = b_1 = 0\}.$$

Thus, $S \subset S_H^O \subset S_H$. Other important classes include K, K_H, and K_H^O, which are the subclasses of S, S_H, and S_H^O containing only the *convex* functions, which are exactly those whose image is a convex domain in $\mathbb{C}$.

We now introduce some major unsolved problems in the field that have obvious analogues in the theory of analytic functions. For years, the biggest problem in the theory of univalent analytic functions was the *Bieberbach Conjecture*, now known as DeBrange's Theorem. Solving this problem allows us to know the sharp bounds on growth and distortion of harmonic maps, among other things. In the nonanalytic case, we have the following.

Conjecture 1 (Harmonic Bieberbach Conjecture) Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \bar{z}^n \in S_H^O.$$

Then

- $|a_n| \leq \frac{1}{6}(n+1)(2n+1)$,
- $|b_n| \leq \frac{1}{6}(n-1)(2n-1)$,
- $||a_n| - |b_n|| \leq n$.

Currently, the best bound is that for all functions $f \in S_H^O$, $|a_2| < 49$ ([15]). The conjecture is that $|a_2| \leq \frac{5}{2}$.

Open Problem 2 *Prove a bound on* $|a_2|$ *that is lower than 49.*

Recall that for analytic functions we have the Koebe $\frac{1}{4}$-Theorem, which expresses bounds on the distortion of the unit disk under normalized analytic maps. In the harmonic case, we have

Conjecture 2 If $f \in S_H^O$, then $f(\mathbb{D})$ contains the disk $G = \{w : |w| < \frac{1}{6}\}$.
Currently, the best result is that the range of $f \in S_H^O$ contains the disk $\{w : |w| < \frac{1}{16}\}$.

Open Problem 3 *Prove that the radius can be increased to* K *where* $\frac{1}{16} < K \leq \frac{1}{6}$.

2.2 Shearing

In their paper, Clunie and Sheil-Small introduced the shearing technique that provides a procedure for constructing harmonic maps $f = h + \overline{g}$ that are univalent. Before describing the shearing technique, we need the following definition.

Definition 2.1 A domain Ω is convex in the horizontal direction (CHD) if every line parallel to the real axis has a connected intersection with Ω.

We can now state the shearing theorem.

Theorem 2.2 (Clunie and Sheil-Small, [3]) *Let $f = h + \overline{g}$ be a harmonic function that is locally univalent in $\mathbb{D}$ (i.e., $|\omega(z)| < 1$ for all $z \in \mathbb{D}$). The function $F = h - g$ is an analytic univalent mapping of $\mathbb{D}$ onto a CHD domain if and only if $f = h + \overline{g}$ is a univalent mapping of $\mathbb{D}$ onto a CHD domain.*

Summary of the Shearing Technique: To use the shearing technique we start with

- an analytic function F that is CHD, and
- an analytic function ω such that $|\omega(z)| < 1$ for all $z \in \mathbb{D}$.

Then we

- write F as $F = h - g$ and ω as $\omega = g'/h'$, and
- explicitly solve for h and g.

The resulting harmonic function $f = h + \overline{g}$ is guaranteed to be univalent.

Notice that it is easy to reformulate Clunie and Sheil-Small's shearing theorem for functions which are convex in other directions. In particular, consider the case of convex in the vertical direction (CVD) which we will use in this chapter.

Definition 2.3 A domain Ω is CVD if every line parallel to the imaginary axis has a connected intersection with Ω.

Theorem 2.4 *Let $f = h + \overline{g}$ be a harmonic function that is locally univalent in $\mathbb{D}$ (i.e., $|\omega(z)| < 1$ for all $z \in \mathbb{D}$). The function $F = h + g$ is an analytic univalent mapping of $\mathbb{D}$ onto a CVD domain if and only if $f = h + \overline{g}$ is a univalent mapping of $\mathbb{D}$ onto a CVD domain.*

Example 2.5 Consider the analytic function

$$F_p(z) = z - \tfrac{1}{2}z^2.$$

This is the analytic polynomial map F_p given in Example 2.10. It is CHD. Now choose a dilatation. We will choose

$$\omega(z) = g'(z)/h'(z) = z.$$

Note that $|\omega(z)| < 1 \ \forall z \in \mathbb{D}$. Next, set $h(z) - g(z) = F_p(z) = z - \frac{1}{2}z^2$. Taking the derivative of both sides, yields $h'(z) - g'(z) = 1 - z$. Since $g'(z) = zh'(z)$, we substitute $g'(z)$ into the previous equation to get $h'(z) = 1$. Integrating this and normalizing it so that $h(0) = 0$, yields $h(z) = z$. Because $g'(z) = zh'(z)$, we can solve for g to get $g(z) = \frac{1}{2}z^2$. Hence, by the Shearing Theorem

$$f_p(z) = h(z) + \overline{g(z)} = z + \tfrac{1}{2}\,\overline{z}^2 \in S_H^O.$$

Thus, we have constructed a harmonic function f_p that is univalent and CHD. Note that this is the harmonic polynomial function f_p in Example 2.10.

Example 2.6 Consider

$$F_k(z) = h(z) - g(z) = \frac{z}{(1-z)^2} \qquad \text{with} \qquad \omega(z) = z.$$

Using the same approach as above, we get

$$f_k(z) = h(z) + \overline{g(z)} = \text{Re}\Big(\frac{z + \frac{1}{3}z^3}{(1-z)^3}\Big) + i\,\text{Im}\Big(\frac{z}{(1-z)^2}\Big) \in S_H^O.$$

The harmonic function f_k is a slit mapping which maps $\mathbb{D}$ onto $\mathbb{C}$ minus a slit on the negative real axis with the tip of the slit at $-\frac{1}{6}$. There is considerable evidence that f_k can fill a role in harmonic function theory similar to that of the Koebe function in analytic function theory, and for this reason, f_k is called the *harmonic Koebe function*.

To help explore how shearing affects the geometry between analytic and harmonic mappings, one can use the online applet *ShearTool* [9]. The image below demonstrates the functionality of this applet, which simultaneously plots both $h - g$ and $h + \overline{g}$ (Fig. 2.6).

Almost all examples of shearing have used dilatations that are finite Blaschke products. One important type of mappings that are not finite Blaschke products is a singular inner function. We give a brief description of this topic. For more details, see [21].

Definition 2.7 A bounded analytic function f is called an *inner function* if $|\lim_{r\to 1^-} f(re^{i\theta})| = 1$ almost everywhere with respect to Lebesgue measure on $\partial\mathbb{D}$. If f has no zeros on $\mathbb{D}$, then f is called a *singular inner function*.

Every inner function can be expressed in the form

$$f(z) = e^{i\alpha} B(z) \exp\left(-\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(e^{i\theta})\right),$$

where $\alpha, \theta \in R$, μ is a positive measure on $\partial\mathbb{D}$, and $B(z)$ is a Blaschke product, i.e., $B(z) = e^{i\theta}\prod_{j=1}^{\infty}(\frac{z-a_j}{1-\overline{a_j}z})^{m_j}$, for some series of constants $|a_j| < 1$ satisfying $\sum_{n=1}^{\infty}(1 - |a_n|) < \infty$.

The function $f(z) = e^{\frac{z+1}{z-1}}$ is an example of a singular inner function. Weitsman [29] provided the following example.

Example 2.8 Shear

$$h(z) - g(z) = \frac{z}{1-z} + \frac{1}{2}e^{\frac{z+1}{z-1}} \qquad \text{with} \qquad \omega(z) = e^{\frac{z+1}{z-1}}.$$

By a result by Pommenke [27], it can be shown that $h-g$ is convex in the direction of the real axis. Shearing $h - g$ with $\omega(z) = e^{\frac{z+1}{z-1}}$ and normalizing yields

$$h(z) = \int \frac{1}{(1-z)^2}\, dz = \frac{z}{1-z}.$$

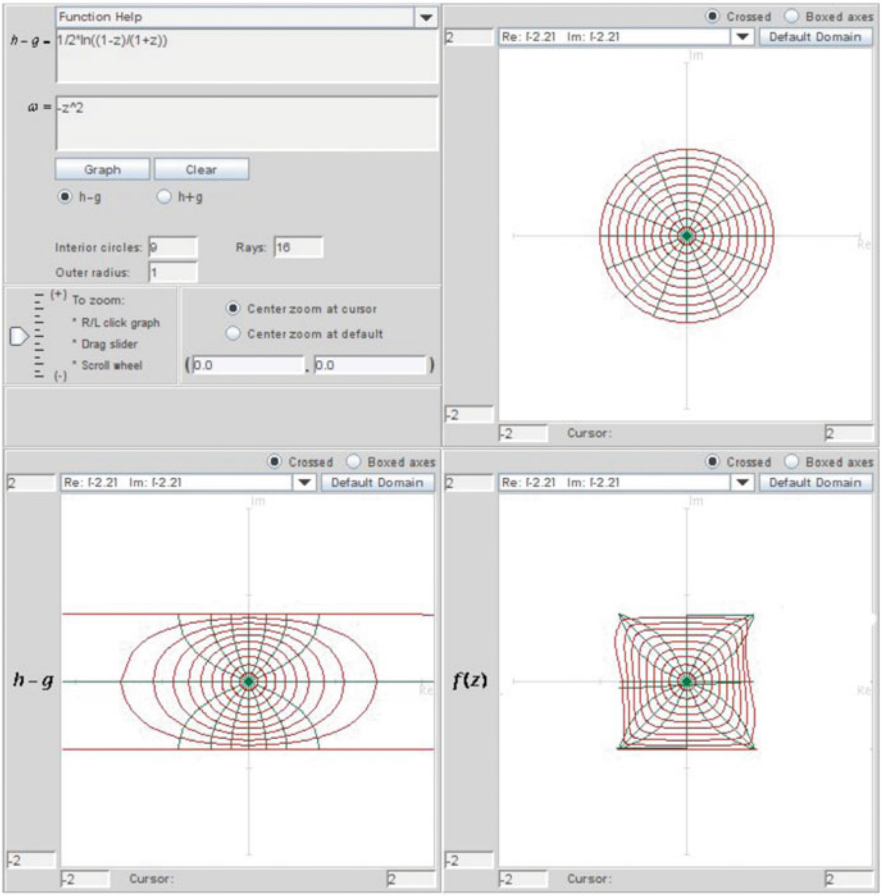

Fig. 2.6 The image of $\mathbb{D}$ under the $f = h + \overline{g}$ is shown in the *bottom right*, where f is constructed from shearing $h(z) - g(z) = \frac{1}{2} \log\left(\frac{1+z}{1-z}\right)$ with $\omega(z) = -z^2$

Solving for g we get

$$g(z) = -\frac{1}{2} e^{\frac{z+1}{z-1}}.$$

The image given by the map is similar to the image given by the right half-plane map $\frac{z}{1-z}$ except that there are an infinite number of cusps (Fig. 2.7).

A technique to find harmonic mappings whose dilatations are singular inner functions involves using a theorem by Clunie and Sheil-Small [3].

Theorem 2.9 *Let $f = h + \overline{g}$ be locally univalent in $\mathbb{D}$ and suppose that $h + \epsilon g$ is convex for some $|\epsilon| \leq 1$. Then f is univalent.*

To develop the technique, we let $\epsilon = 0$ in Theorem 9. This means that if h is analytic convex and if ω is analytic with $|\omega(z)| < 1$, then $f = h + \overline{g}$ is a harmonic

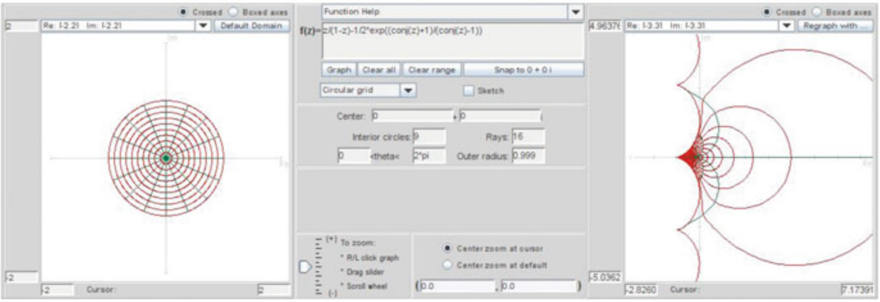

Fig. 2.7 Image of $\mathbb{D}$ under $f(z) = \frac{z}{1-z} - \frac{1}{2}e^{\frac{\bar{z}+1}{\bar{z}-1}}$

univalent mapping. To establish that a function f is convex, we will use the following theorem ([14]).

Theorem 2.10 *Let f be analytic in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$. Then f is univalent and maps onto a convex domain if and only if*

$$\mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] \geq 0, \text{ for all } z \in \mathbb{D}.$$

Example 2.11 Let

$$h(z) = z + 2\log(z+1) \quad \text{with} \quad \omega(z) = g'(z)/h'(z) = e^{\frac{z-1}{z+1}}.$$

Using Theorem 10, we can show that h is convex. Then solving for g we get $g(z) = (z+1)e^{(z-1)/(z+1)}$.

Hence,

$$f(z) = h(z) + \overline{g(z)} = z + 2\log(z+1) + (\bar{z}+1)e^{\frac{\bar{z}-1}{\bar{z}+1}}.$$

By Theorem 9, $f = h + \bar{g}$ is univalent. The image of $\mathbb{D}$ under f is shown in Fig. 2.8.

Open Problem 4 *Construct examples of harmonic univalent functions whose dilatation is a singular inner function and determine properties of these functions.*

2.3 Inner Mapping Radius

The analytic Koebe function F_k is an important function. It is extremal (or maximal) in several important senses. It is the function in S that gives equality for the coefficient bounds in deBranges' Theorem. It is the function that maps the unit disk to a domain that contains the largest possible disk centered at the origin as described in the Koebe $\frac{1}{4}$-Theorem. It is the function that exhibits both the largest and smallest possible

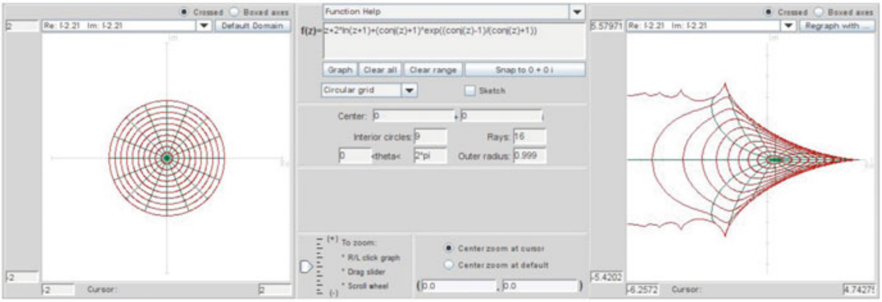

Fig. 2.8 Image of $\mathbb{D}$ under $f(z) = h + \overline{g}$ in Example 2.11

growth possible. It is the function for which the complement of its image is closest to the origin. It is conjectured that the harmonic Koebe function f_k from Example 2.6 has analogous properties in the class S_H^O although these properties have not been proven (see Conjectures 1 and 2).

Recall that the tip of the slit of the harmonic Koebe function is at $-\frac{1}{6}$ while the tip of the slit for the analytic Koebe function is at $-\frac{1}{4}$. Notice that if we multiply the analytic Koebe function by $\frac{2}{3}$, then the images of the unit disk under $\frac{2}{3}F_k$ and under f_k, the harmonic Koebe function, would be the same. That is,

$$\tfrac{2}{3}F_k(\mathbb{D}) = f_k(\mathbb{D}).$$

This multiplier factor of $\frac{2}{3}$ is known as the *inner mapping radius* for $f_k(\mathbb{D})$. For other functions in S_H^O, the inner mapping radius may be different. For example, using the analytic and harmonic versions of the right half-plane maps from Example 1.11, the inner mapping radius for $f_h(\mathbb{D})$ is 1 since $f_h(\mathbb{D}) = F_h(\mathbb{D})$.

Let's define this idea of inner mapping radius precisely.

Definition 3.1 For $f \in S_H^O$, the *inner mapping radius*, $\rho_O(f)$, of the domain $f(\mathbb{D})$ is the real number $F'(0)$, where

- F is the analytic function that maps $\mathbb{D}$ onto $f(\mathbb{D})$
- $F(0) = 0$
- $F'(0) > 0$.

Notice that the existence of such a function F is guaranteed by the Riemann Mapping Theorem. The functions in S are normalized by requiring that $F'(0) = 1$. The Riemann Mapping Theorem does not guarantee that there is a schlicht mapping to any simply-connected domain but does guarantee that we can multiply a schlicht function by some positive real number in order to map onto that domain. This positive real number is the inner mapping radius.

In the example above with the Koebe functions, $F(z) = \frac{2}{3}F_k(z)$, and the inner mapping radius $\rho_O(k_0) = F'(0) = \frac{2}{3}$. Because of the extremal nature of the analytic Koebe function, it was conjectured that $\frac{2}{3} \leq \rho_O(f) \leq 1$. This conjecture was shown not to be true in the following examples by Dorff and Suffrdge [10].

Example 3.2 This example demonstrates that the conjectured upper bound of $\rho_O < 1$ was too low. Consider the family of functions $f_\alpha = h + \overline{g}$ constructed by shearing

$$h(z) - g(z) = \frac{z}{1-z} \qquad \text{with} \qquad \omega(z) = \frac{z^2 + \alpha z}{\alpha z + 1},$$

where $\alpha \in \mathbb{R}$. It can be shown that if $|\alpha| \leq 1$, then $f_\alpha(z) \in S_H^O$.

Let $|\alpha| \leq 1$ and $\alpha \neq -1$, then $f_\alpha(\mathbb{D})$ is a slit domain consisting of the complex plane minus a slit along the negative real axis with the tip of the slit at $\frac{1}{6}\alpha - \frac{1}{3}$. Hence, the tip can vary from $-\frac{1}{6}$ to $-\frac{1}{2} + \epsilon$. If $\alpha = -1$, then $f_{-1}(\mathbb{D})$ is the half plane $\text{Re}(w) > -\frac{1}{2}$. Thus, for this family of functions,

$$\tfrac{2}{3} \leq \rho_O(f_\alpha) < 2.$$

Example 3.3 This example demonstrates that the conjectured lower bound of $\rho_O < 1$ was too high. Consider the family of functions $f_t = h + \overline{g}$ constructed by shearing

$$F_t(z) = h(z) - g(z) = \frac{z - tz^2}{(1-z)^2} \qquad \text{with} \qquad \omega(z) = z,$$

where $t \in [0, 1]$. For $0 < t < 1$, $F_t(\mathbb{D})$ is the exterior of the parabola $u > -\frac{1-t}{t^2}v^2 - \frac{t+1}{4}$ while $f_t(\mathbb{D})$ is the exterior of the parabola $\widetilde{u} > -\frac{1}{t}\widetilde{v}^{\,2} - \frac{1}{6} - \frac{t}{12}$. It can be computed that when $t = \frac{1}{4}$, $\rho_O(f_t)$ is smallest, and we obtain that for $0 < t < 1$,

$$\tfrac{1}{2} \leq \rho_O(f_t) \leq \tfrac{2}{3}.$$

It has been proven that

$$\tfrac{1}{4} \leq \rho_O(f) \leq \tfrac{8\pi\sqrt{3}}{9} < 4.837.$$

Because of the way these bounds were determined, they are probably not the tightest bounds, and it is likely they can be improved. There are no known functions in S_H^O that have an inner mapping radius equal to either of these extreme values. On the other hand, from the previous two examples, we know there are specific functions that have $\rho_O(f) = \frac{1}{2}$ and $\rho_O(f) = 2$. The result of $\rho_O(f) = \frac{1}{2}$ in Example 3.3 was very surprising because this value did not come from a slit mapping. It is not known if there is a function in S_H^O whose inner mapping radius is less than $\frac{1}{2}$ or larger than 2.

Open Problem 5 *Prove $\frac{1}{2} \leq \rho_O(f) \leq 2$ or find a harmonic map $f \in S_H^O$ such that $\rho_O(f) < \frac{1}{2}$ or $\rho_O(f) > 2$.*

The definition of the inner mapping radius can be extended to functions in S_H. Let us denote the inner mapping radius of $f \in S_H$ by $\rho(f)$. It is known that

$$0 < \rho(f) \leq 2\pi$$

([4]). In [10] an example is constructed for which $0 < \rho(f) \leq 4$.

Open Problem 6 *Prove* $\rho(f) \leq 4$ *or find a harmonic map* $f \in S_H$ *such that* $\rho(f) > 4$.

2.4 Convolutions

The shearing technique given in Theorem 2.2 provides a way to construct harmonic functions that are univalent. This approach requires certain conditions in order to apply the technique. Convolutions is another approach to construct harmonic univalent functions. It also requires certain conditions in order to guarantee that the resulting functions are univalent. In addition, the study of convolutions is an interesting topic on its own.

The convolution of harmonic functions is a generalization of the convolution of analytic functions which is an important area in the study of schlicht functions ([28] for more information about the convolution of analytic functions). However, many of the nice theorems in the analytic case do not carry over to the harmonic case. For example, the Polya–Schoenberg conjecture which was proved by Ruscheweyh and Sheil-Small states that convexity is preserved under analytic convolution. This convexity preserving property does not hold for harmonic convolutions. But there are several open areas related to harmonic convolutions to investigate. In this section we will explore some of these. For more details about harmonic convolutions, see [6].

Let's begin with the definition of the convolution for analytic functions.

Definition 4.1 (Analytic Convolution) Given $F_1, F_2 \in S$ represented by

$$F_1(z) = \sum_{n=0}^{\infty} A_n z^n \quad \text{and} \quad F_2(z) = \sum_{n=0}^{\infty} B_n z^n,$$

their convolution is defined as

$$F_1(z) * F_2(z) = \sum_{n=0}^{\infty} A_n B_n z^n.$$

As mentioned above, the analytic convolution preserves convexity since $F_1, F_2 \in K \Rightarrow F_1 * F_2 \in K$. The algebra of convolutions is also simplified by viewing certain functions as operators. For instance, $F(z) = \frac{z}{1-z}$ is the convolution identity because its power series is $z + z^2 + z^3 + \cdots$.

We define an analogous operation for harmonic functions as follows:

Definition 4.2 Given

$$f_1 = h_1 + \overline{g_1} = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \, \overline{z}^n \quad \text{and}$$

$$f_2 = h_2 + \overline{g_2} = z + \sum_{n=2}^{\infty} c_n z^n + \sum_{n=1}^{\infty} \overline{d}_n \, \overline{z}^n,$$

define *harmonic convolution* as

$$f_1 * f_2 = h_1 * h_2 + \overline{g_1 * g_2} = z + \sum_{n=2}^{\infty} a_n c_n z^n + \sum_{n=1}^{\infty} \overline{b_n d_n}\, \overline{z}^n.$$

Harmonic convolutions involve difficulties not present in the analytic case. For instance, it is not difficult to find $f_1, f_2 \in K_H^O$ such that $f_1 * f_2 \notin K_H^O$. In fact, $f_1 * f_2$ may even fail to be univalent. The example below illustrates this.

Example 4.3 Let $f_h = h_1 + \overline{g}_1 \in K_H^O$ be the harmonic right half-plane map in Example 1.11, where

$$h_1(z) = \frac{z - \frac{1}{2}z^2}{(1-z)^2}, \quad g_1(z) = \frac{-\frac{1}{2}z^2}{(1-z)^2},$$

and let $f_2 = h_2 + \overline{g}_2 \in K_H^O$ be the canonical regular 6-gon map, where

$$h_2(z) = z + \sum_{n=1}^{\infty} \tfrac{1}{6n+1} z^{6n+1}, \quad g_2(z) = \sum_{n=1}^{\infty} \tfrac{-1}{6n-1} z^{6n-1}.$$

Then $f_h * f_2$ is not univalent, because

$$|(g_1(z) * g_2(z))'/(h_1(z) * h_2(z))'| = |z^4(2+z^6)/(1+2z^6)| \not< 1, \forall z \in \mathbb{D}.$$

Open Problem 7 *Let $f_1, f_2 \in K_H^O$. Since $f_1 * f_2$ is not necessarily univalent, what additional conditions can we impose upon f_1, f_2 so that $f_1 * f_2 \in S_H^O$?*

Several researchers have recently published results related to this question [2, 5, 12, 17, 23, 24]. Let's look at some of these results. Theorem 4.4 ([5]) gives conditions under which local univalence of the convolution is enough to establish global univalence.

Theorem 4.4 *Let $f_1 = h_1 + \overline{g}_1, f_2 = h_2 + \overline{g}_2 \in S_H^O$ such that $h_i(z) + g_i(z) = \frac{z}{1-z}$. Let $\widetilde{\omega}$ be the dilatation of $f_1 * f_2$. If $|\widetilde{\omega}(z)| < 1$ for all $z \in \mathbb{D}$, then $f_1 * f_2 \in S_H^O$ and is CHD.*

Theorem 4.4 has been used to determine specific cases in which harmonic convolutions preserve univalence. In [12], the following result is proved.

Theorem 4.5 *Consider the right half-plane map*

$$f_h(z) = h_1(z) + \overline{g_1(z)} = \frac{z - \frac{1}{2}z^2}{(1-z)^2} - \overline{\frac{\frac{1}{2}z^2}{(1-z)^2}},$$

and let $f = h + \overline{g} \in K_H^O$ with $h(z) + g(z) = \frac{z}{1-z}$ and $\omega = g'/h' = e^{i\theta} z^n$ ($n \in \mathbb{Z}^+, \theta \in \mathbb{R}$). If $n = 1, 2$, then $f_h * f \in S_H^O$ and is CHD.

The proof of this theorem relies on properties on analytic convolutions and results about the location of zeros of symmetric polynomials. If $n > 2$ in the above theorem, then $f_h * f$ fails to be univalent. In [2], we get the next theorem.

Theorem 4.6 *Let* $f_\theta = h_\theta + \overline{g_\theta}$, $f_\rho = h_\rho + \overline{g_\rho} \in S_H^O$ *such that* $h_\theta(z) + g_\theta(z) = h_\rho(z)+g_\rho(z) = \frac{z}{1-z}$, $g_\theta'/h_\theta' = e^{i\theta}z$, *and* $g_\rho'/h_\rho' = e^{i\rho}z\,(\theta, \rho \in \mathbb{R})$. *Then* $f_\theta * f_\rho \in S_H^O$ *is CHD.*

The following theorem was proved in [24]

Theorem 4.7 *Let* $f = h + \overline{g} \in S_H^O$ *with* $h(z) + g(z) = \frac{z}{1-z}$ *and* $\omega(z) = \frac{z+\overline{a}}{1+\overline{a}z}$ *with* $|a| < 1$. *Then* $f_h * f \in S_H^O$ *and is CHD if and only if*

$$(\text{Re } a)^2 + 9\,(\text{Im } a)^2 \leq 1.$$

There are other convolution problems that remain to be investigated. In many theorems, the canonical harmonic right half-plane function f_h is convoluted with other harmonic functions. Can similar theorems be proven if f_h is replaced with a different function? For example, consider the harmonic mapping f_1 formed by shearing $h_1(z) + g_1(z) = \frac{z}{1-z}$ with other dilatations such as $\omega(z) = e^{i\theta}\frac{z+\overline{a}}{1+\overline{a}z}$ with $|a| < 1$ or $\omega(z) = z$.

Open Problem 8 *Let* $f = h + \overline{g} \in S_H^O$ *with* $h(z) + g(z) = \frac{z}{1-z}$ *and* $\omega = g'/h' = e^{i\theta}z^n$ $(n \in \mathbb{Z}^+, \theta \in \mathbb{R})$. *Determine the values of n for which* $f_1 * f$ *is univalent.*

Many of the harmonic convolution results, given above, require that one of the functions be a sheared half-plane. In [12] and [17], results are proven about the harmonic convolutions of strip mappings and polygons.

Open Problem 9 *Determine more results about the convolutions of harmonic functions that are shears of vertical strips or polygons.*

2.5 Harmonic Maps and Minimal Surfaces

Planar harmonic mappings with certain properties are related to minimal surfaces in $\mathbb{R}^3$, and it is possible to use results from one area to prove new results in the other area. Before discussing this further, we need to present some background material about minimal surfaces.

Minimal surfaces are one solution to the problem of finding the minimal surface area required to span a given curve. Minimal surfaces are guaranteed to minimize area only locally but often they provide the globally-minimal solution as well. One consequence of the area-minimizing property is that all minimal surfaces look like saddle surfaces at each point, and the bending upward in one direction is matched by the downward bending in the orthogonal direction (This equal-but-opposite bending property will be defined later as “zero mean curvature.”).

2.5.1 Background

In order to explore minimal surfaces more fully, we introduce three important concepts from differential geometry, which is the study of differentiable surfaces in space. For more details on the material from this section, [7].

A surface, $M \in \mathbb{R}^3$, can be *parametrized* by a smooth function $\mathbf{x} : D \to \mathbb{R}^3$ if $\mathbf{x}(D) = M$ and $\mathbf{x}$ is one-to-one. Parameterizing a surface with smooth functions allows us to do calculus with the surface and gives us a way to translate geometric concepts into rigorous analytic language. Isothermal parameterizations are essential for the study of minimal surfaces. Basically, such parametrizations map small squares to small squares. Every minimal surface in $\mathbb{R}^3$ has an isothermal parametrization.

Next, we need to discuss the idea of *normal curvature*. At each point p on the surface M, there is a unit normal $\mathbf{n}$. The normal curvature measures how much the surface bends toward $\mathbf{n}$ as you travel in the direction of the tangent vector $\mathbf{w}$ at p. Specifically, given the normal vector $\mathbf{n}$ at each point $p \in M$, we can find a plane $\mathcal{P}$ containing $\mathbf{n}$ that intersects M in some curve $\mathbf{c}$, which has a curvature value k. As the plane $\mathcal{P}$ revolves around the unit normal $\mathbf{n}$ at p, we get a continuous function of curvature values $k(\theta)$. Let k_1 and k_2 be the maximum and minimum curvature values at p. The *mean curvature* of a surface M at p is $H = \frac{1}{2}(k_1 + k_2)$.

Definition 5.1 A *minimal surface* is a surface M with $H = 0$ at all $p \in M$.

Recall that the intuition behind vanishing mean curvature is that M is a saddle surface with positive curvature in one direction being matched by negative curvature in the orthogonal direction.

Just as the shearing theorem links analytic function theory to harmonic function theory, the Weierstrass Representation links harmonic function theory to minimal surface theory.

Theorem 5.2 ***(General Weierstrass Representation)*** *If we have analytic functions* $\boldsymbol{\varphi}_k$ *(*$k = 1, 2, 3$*) such that*

- $\phi^2 = (\varphi_1)^2 + (\varphi_2)^2 + (\varphi_3)^2 = 0$
- $|\boldsymbol{\phi}|^2 = |\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2 \neq 0$ *and is finite,*

then the parametrization

$$\boldsymbol{x} = \left(\mathrm{Re}\int \varphi_1(z)dz, \mathrm{Re}\int \varphi_2(z)dz, \mathrm{Re}\int \varphi_3(z)dz\right)$$

defines a minimal surface.

We also have the following converse.

Theorem 5.3 *Let M be a surface with parametrization $\boldsymbol{x} = (x_1, x_2, x_3)$ and let $\boldsymbol{\phi} = (\varphi_1, \varphi_2, \varphi_3)$, where $\varphi_k = \frac{\partial x_k}{\partial z}$.*

$$\boldsymbol{x} \textit{ is isothermal} \iff \boldsymbol{\phi}^2 = (\varphi_1)^2 + (\varphi_2)^2 + (\varphi_3)^2 = 0.$$

If $\boldsymbol{x}$ is isothermal, then

$$M \textit{ is minimal if and only if each } \varphi_k \textit{ is analytic.}$$

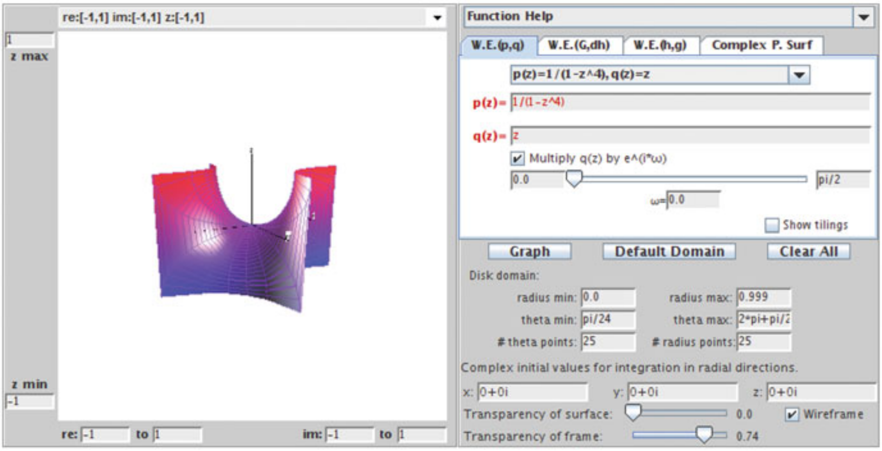

Fig. 2.9 The *MinSurfTool* applet

We can apply the above theorems to planar harmonic mappings. First, recall $f = h + \overline{g} = \mathrm{Re}(h+g) + i\mathrm{Im}(h-g)$. In Theorem 5.2, choose $\varphi_1 = h' + g'$ and $\varphi_2 = -i(h' - g')$. Then we find φ_3 that will satisfy the requirements of the Weierstrass representation. That is,

$$\begin{aligned} 0 &= (\varphi_1)^2 + (\varphi_2)^2 + (\varphi_3)^2 \\ &= \left(h' + g'\right)^2 + \left[-i(h' - g')\right]^2 + (\phi_3)^2. \end{aligned}$$

Solving for φ_3 yields $(\varphi_3)^2 = -4h'g'$, so $\varphi_3 = -2i\sqrt{h'g'}$.

Notice that $\sqrt{h'g'}$ may not always exist as an analytic function, but whenever it does, the Weierstrass representation applies. Since $\sqrt{h'g'} = h'\sqrt{\omega}$, it is enough for the dilatation to have an analytic square root. Thus, we have the following result.

Theorem 5.4 *(**Weierstrass Representation - (h,g)**) Let the harmonic mapping $f = h + \overline{g}$ be univalent with g'/h' being the square of an analytic function. Then the parametrization*

$$X = \left(\mathrm{Re}\,(h+g)\,, \mathrm{Im}\,(h-g)\,, 2\;\mathrm{Im}\int \sqrt{h'g'}\right)$$

defines a minimal graph whose projection is $f(\mathbb{D})$.

MinSurfTool [9] is another applet available online that allows for quick and easy visualization of minimal surfaces (Fig. 2.9).

Example 5.5 Consider the harmonic map

$$f(z) = h(z) + \overline{g(z)} = \mathrm{Re}\left[\frac{i}{2}\log\left(\frac{i+z}{i-z}\right)\right] + i\mathrm{Im}\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right].$$

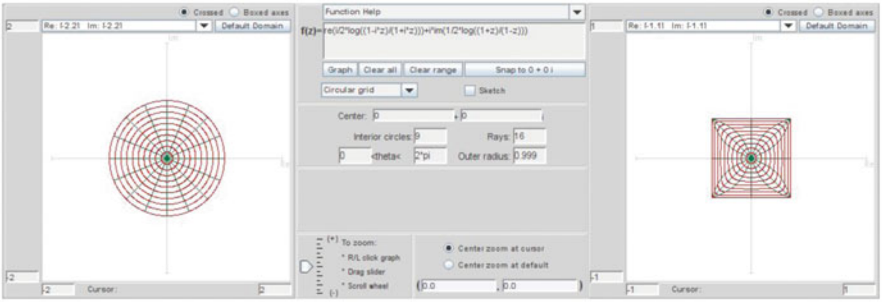

Fig. 2.10 The image of $\mathbb{D}$ under f, the harmonic square map

It can be constructed by shearing $h(z) - g(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ with $g'(z)/h'(z) = -z^2$ and is therefore univalent. Note that $f(\mathbb{D})$ is a square region (Fig. 2.10).

Since the dilatation is the square of an analytic function, we can apply Theorem 5.4. Then $x_3(z) = 2\text{ Im}\int\sqrt{h'g'} = \frac{1}{2}\text{Im}[i\log(\frac{1+z^2}{1-z^2})]$.

By the Weierstrass representation, we have the parametrization of a minimal graph given by

$$\begin{aligned}\mathbf{x} &= \left(\text{Re}\,(h+g)\,,\text{Im}\,(h-g)\,,2\text{ Im}\int\sqrt{h'g'}\right)\\ &= \left(\text{Re}\left[\frac{i}{2}\log\left(\frac{i+z}{i-z}\right)\right],\text{Im}\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right],\text{Im}\left[\frac{i}{2}\log\left(\frac{1+z^2}{1-z^2}\right)\right]\right).\end{aligned}$$

This minimal surface is Scherk's doubly periodic surface. In Fig. 2.11 Scherk's doubly periodic surface is shown along with the corresponding harmonic map (it is the projection of the minimal surface onto the complex plane).

We might wonder if the integrals found in the Weierstrass representations are well-defined. In certain cases, they may indeed be multi-valued. But in such cases, the ill-definedness reflects the fact that surface is periodic in one or more of the coordinates, as is the case with the Scherk surfaces.

With the background we just discussed, we are ready to explore applications of harmonic maps to minimal surface theory. Our goal is to help the reader get a sense of some important techniques and to suggest some research areas.

2.5.2 Connecting Harmonic Maps to Specific Minimal Graphs

The Weierstrass Representation allows us to take an harmonic univalent function with an appropriate dilatation and lift it to a minimal graph. Several recent papers have used this technique [11, 13, 16, 18, 25, 26]. However, it is often difficult to identify the resulting minimal graphs. One approach to recognizing the minimal surface is to use a change of variable ([8]).

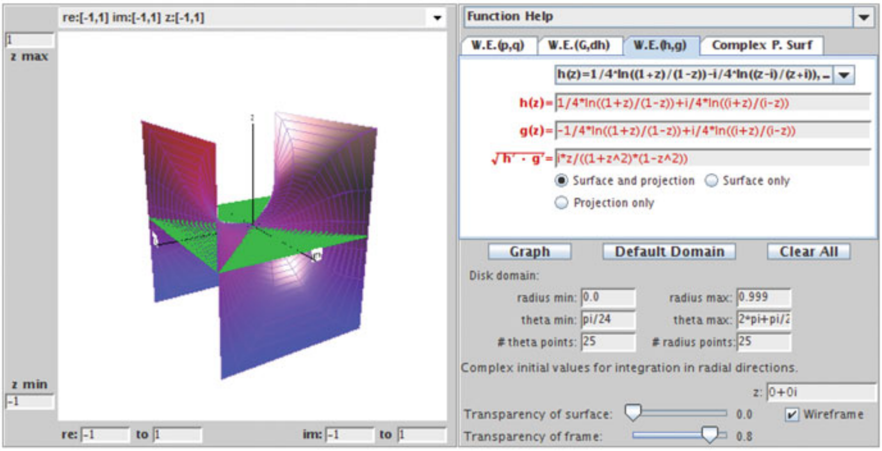

Fig. 2.11 Scherk's doubly periodic minimal surface

Example 5.6 Shearing $h(z) - g(z) = \frac{z}{(1-z)^2}$ with $\omega(z) = z^2$ yields the univalent harmonic slit-map

$$f(z) = \frac{z - z^2 + \frac{1}{3}z^3}{(1-z)^3} + \overline{\frac{\frac{1}{3}z^3}{(1-z)^3}}.$$

The parametrization of the corresponding minimal graph is

$$\mathbf{x} = \left(\mathrm{Re}\left\{\frac{z - z^2 + \frac{2}{3}z^3}{(1-z)^3}\right\}, \mathrm{Im}\left\{\frac{z}{(1-z)^2}\right\}, \mathrm{Im}\left\{\frac{2z^2 - \frac{2}{3}z^3}{(1-z)^3}\right\}\right).$$

This is not a standard form for a known minimal surface. However, using the substitution $z \to \frac{\tilde{z}+1}{\tilde{z}-1}$ and interchanging the second and third coordinate functions, we derive the parametrization

$$\tilde{\mathbf{x}} = \left(-\tfrac{1}{4}\mathrm{Re}\left\{\tilde{z} + \tfrac{1}{3}\tilde{z}^3\right\}, \tfrac{1}{4}\mathrm{Im}\left\{\tilde{z} - \tfrac{1}{3}\tilde{z}^3\right\}, \tfrac{1}{4}\mathrm{Im}\left\{\tilde{z}^2\right\}\right).$$

This is Ennepers surface. Thus, the original surface $\mathbf{x}$ is the part of Ennepers surface formed by using a right half-plane as the domain instead of the standard unit disk.

Open Problem 10 *Determine the minimal graphs formed by lifting harmonic univalent mappings in any of the following papers* [11, 13, 16, 25, 26].

Open Problem 11 *Use the shearing technique to generate a univalent harmonic map with a dilatation that is a perfect square and use the Weierstrass representation to construct the minimal graph. Then determine what surface it is.*

2.5.3 *Using Harmonic Maps to Find Curvature Bounds on Minimal Graphs*

Geometric function theory and Clunie and Sheil-Small's shearing theorem allow us to find sharp bounds on growth and other important properties of harmonic maps. Using the Weierstrass representation we can translate these bounds to minimal graphs. In particular, recall that at each point p of a surface S, we let k_1 and k_2 be the maximum and minimum curvature values and defined the mean curvature to be $H = \frac{1}{2}(k_1 + k_2)$. H is useful for characterizing minimal surfaces, but in other connections we use K, the Gauss curvature.

The *Gaussian curvature* at p is given by

$$K = k_1 k_2.$$

The *theorema eggregium* of Gauss states that K is invariant under any deformation without stretching and is thus a good intrinsic measure of curvature. The Gaussian curvature may be put in terms of the dilatation of harmonic maps. If we denote the dilatation by $\omega(z) = g'(z)/h'(z)$, we can express the Gaussian curvature of a minimal graph with $\omega^2(z) = b(z)$ by

$$K(z) = \frac{-4|b'(z)|^2}{(1 + |b(z)|^2)^4 |h'(z)|^2}.$$

We can find a bound for K in terms of h and g. By the Schwarz–Pick lemma,

$$|b'(z)| \leq \frac{1 - |\omega|^2}{1 - |z|^2}.$$

Hence

$$|K(z)| \leq \frac{4}{(|g'(z)| + |h'(z)|)^2 (1 - |z|^2)^2}.$$

This last inequality can be used to find bounds over the origin of minimal graphs over specific planner domains. In particular,

$$|K(0)| \leq \frac{4}{(|h'(0)| + |g'(0)|)^2} \leq \frac{4}{|h'(0)|^2 + |g'(0)|^2}.$$

If M is a minimal graph above the unit disk $\mathbb{D}$ and $f(0) = 0$, then Hall showed that

$$|h'(0)| + |g'(0)| \geq \frac{27}{4\pi^2}.$$

Thus for any minimal graph above the unit disk,

$$|K(0)| \leq \frac{16\pi^2}{27}.$$

Several papers have considered such a situation for arbitrary points on minimal graphs over various domains. In [18], the authors considered minimal graphs over half-planes, strips, and 1-slit domains. Papers considering minimal graphs over other domains include [19, 20], and [26].

Open Problem 12 *Find curvature bounds over arbitrary points for minimal graphs over domains not investigated in* [18–20, 26].

2.5.4 Connecting Results About Harmonic Maps with Results About Minimal Surfaces

Since certain types of harmonic univalent functions are related to minimal graphs, it should be true that theorems and concepts from one field should relate to theorems and concepts from the other field.

One example of this concerns a harmonic convolution theorem and Krust Theorem about conjugate minimal surfaces.

Definition 5.7 Let $\mathbf{x}$ and $\mathbf{y}$ be isothermal parametrizations of two minimal surfaces such that their component functions are pairwise harmonic conjugates. Then, $\mathbf{x}$ and $\mathbf{y}$ are called *conjugate minimal surfaces*.

The helicoid and the catenoid are conjugate surfaces. Any two conjugate minimal surfaces can be joined through a one-parameter family of *associated minimal surfaces* by the equation

$$\mathbf{z} = (\cos t)\mathbf{x} + (\sin t)\mathbf{y},$$

where $t \in \mathbb{R}$.

An important theorem in minimal surface theory is Krust Theorem.

Theorem 5.8 (Krust) *If an embedded minimal surface $X : \mathbb{D} \to \mathbb{R}^3$ can be written as a graph over a convex domain in $\mathbb{C}$, then all associated minimal surfaces $Z : \mathbb{D} \to \mathbb{R}^3$ are graphs.*

Now consider the following less well known theorem about harmonic convolutions [3].

Theorem 5.9 (Clunie and Sheil-Small) *If $f = h + \overline{g} \in K_H$ and $\varphi \in K$, then the functions*

$$h * \varphi + \alpha \overline{g * \varphi}$$

are univalent and close-to-convex, where ($|\alpha| \leq 1$) and $*$ denotes harmonic convolution.

Open Problem 13 *Determine theorems and properties of harmonic maps that relate to theorems and properties of minimal surfaces.*

As a second example, we will prove a result about minimal surfaces using results from harmonic univalent mappings. In particular, we will consider a family of minimal surfaces known as Scherks dihedral surfaces and determine the parameter values for which these surfaces are embedded. First, some background information.

While minimal surfaces can be parametrized by the Weierstrass representation, there is no guarantee the surface will not have self-intersections. Minimal surfaces that have no self-intersections are known as embedded minimal surfaces, and they

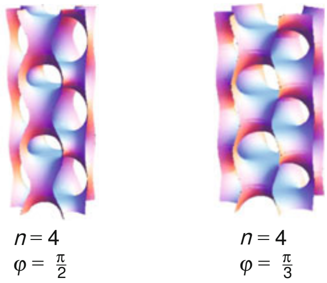

Fig. 2.12 Two examples from the family of Scherks dihedral surface

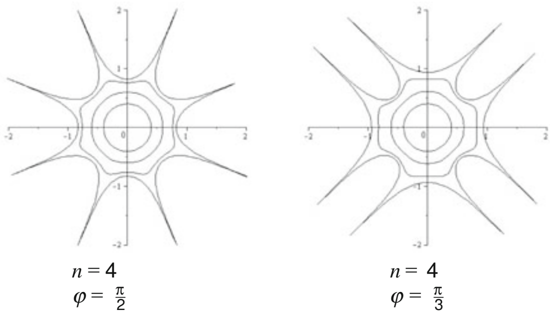

Fig. 2.13 The projection onto $\mathbb{C}$ of one piece from each examples in Fig. 2.13

are a major interest in minimal surface theory. The family $\mathcal{F}_n(\varphi)$ of singly periodic Scherk surfaces with higher dihedral symmetry have n number of vertical planes that extend to infinity. The smallest angle, φ, between these symmetric planes varies (Fig. 2.12).

We can look at the projection of one piece of these surfaces onto $\mathbb{C}$ which is also the image of the unit disk under the corresponding harmonic univalent mappings (Fig. 2.13).

These minimal surfaces are embedded, provided that

$$\frac{\pi}{2} - \frac{\pi}{n} < \frac{n-1}{n}\varphi < \frac{\pi}{2}.$$

We can prove this inequality using results planar harmonic mappings. We summarize the proof below.

Proof Consider the following family of harmonic maps: $f_n(z) = h_n(z) + \overline{g_n(z)}$, $n \geq 2$, $\varphi \in [0, \frac{\pi}{2}]$, where

$$h_n'(z) = \frac{1}{(z^n - e^{i\varphi})(z^n - e^{-i\varphi})}, \qquad g_n'(z) = \frac{z^{2n-2}}{(z^n - e^{i\varphi})(z^n - e^{-i\varphi})}$$

(Fig. 2.14).

It is known that $f_n = h_n + \overline{g_n}$ maps $\mathbb{D}$ onto a $2n$-gon, and in [25] it was shown that f_n is univalent and convex for every $\varphi \in (\frac{n}{n-1}(\frac{\pi}{2} - \frac{\pi}{n}), \frac{\pi}{2}]$. Using the Weierstrass representation, we can lift f_n to an embedded minimal surface X. Since X is over a

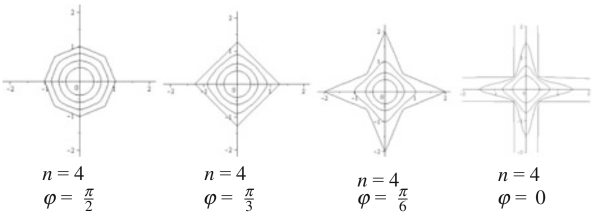

Fig. 2.14 Images of the unit disk under $f = h_n + \overline{g}_n$

convex domain, Krust theorem guarantees that the conjugate surfaces Y are embedded. These conjugate surfaces Y are Scherk surfaces with higher dihedral symmetry and this establishes the inequality.

Open Problem 14 *Use theorems and properties about harmonic univalent mappings to prove results about minimal surfaces.*

2.5.5 Using Harmonic Maps to Construct New Minimal Surfaces

In this section we show an example in which a harmonic univalent function is lifted to form a minimal graph that appears to be new. The construction is outlined below. Complete details are found in [1].

Let $f = h + \overline{g}$, where

$$h(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

and let $\omega = (e^{\frac{z+1}{z-1}})^2$. Since $g' = h'\omega = \frac{1}{1-z^2}e^{2\frac{z+1}{z-1}}$, we know that

$$g(z) = -\frac{1}{2}E_1\left(\frac{z+1}{-z+1}\right) + \frac{1}{2}E_1(1),$$

where $E_1(z)$ is the exponential integral function. By a result by Clunie and Sheil-Small, $f = h + \overline{g}$ is univalent. The image of $f(\mathbb{D})$ is shown in Fig. 2.15.

By the Weierstrass representation $f = h + \overline{g}$ lifts to an embedded minimal surface (Fig. 2.16).

This surface is constructed from a harmonic univalent map that has a dilatation being a *singular inner function* (i.e., a function which never equals zero and which has modulus equal to one on the unit disk). One consequence of having such a dilatation is that there is no (finite) point where the function is approximately analytic. This corresponds to the idea that the minimal surface never has zero Gauss curvature. The surface also has an infinite number of cusps and a singularity with unusual behavior.

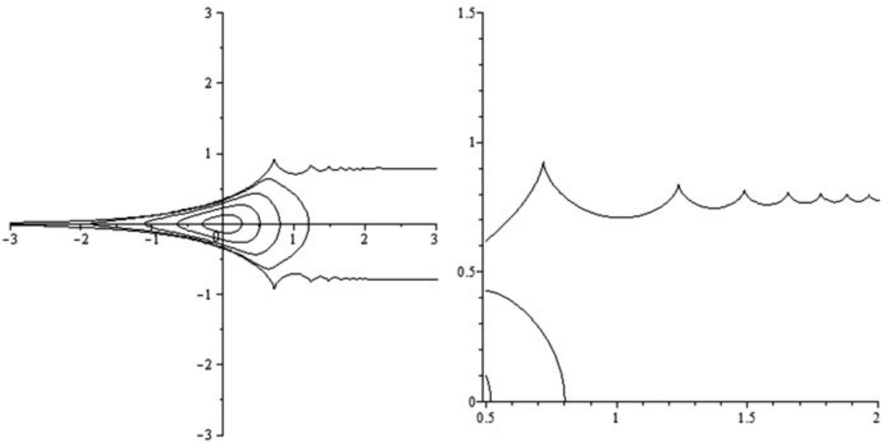

Fig. 2.15 The image of $f(\mathbb{D})$ and a close up of that image

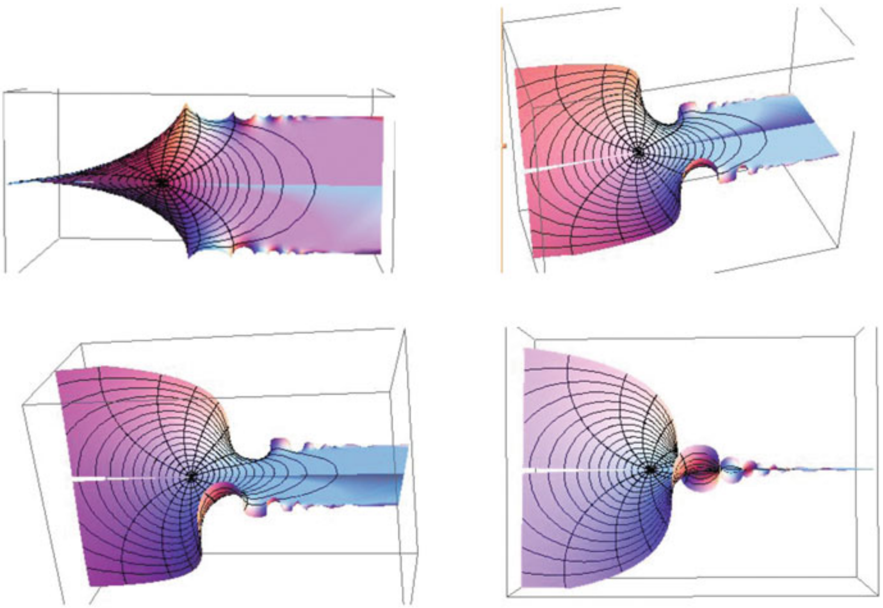

Fig. 2.16 Images of the minimal surface constructed from f

Open Problem 15 *Construct other minimal surfaces from harmonic univalent maps with dilatations that are singular inner functions.*

Open Problem 16 *Determine the necessary and sufficient conditions for a harmonic function to have a singular inner function as its dilatation. Specifically, determine the kind of growth and boundary behavior exhibited by such harmonic functions.*

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Chapter 3
The Minimal Surfaces Over the Slanted Half-Planes, Vertical Strips and Single Slit

Liulan Li, Saminathan Ponnusamy and Matti Vuorinen

3.1 Introduction

A planar harmonic mapping in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ is a complex-valued harmonic function $f(z)$, defined on $\mathbb{D}$. The mapping f has a canonical decomposition $f = h + \overline{g}$, where h and g are analytic on $\mathbb{D}$ and $g(0) = 0$. The mapping f is locally univalent in $\mathbb{D}$ if and only if its Jacobian $J_f(z) = |h'(z)|^2 - |g'(z)|^2$ does not vanish in $\mathbb{D}$. It is said to be sense-preserving on $\mathbb{D}$ if and only if $J_f(z) > 0$, or equivalently if $h'(z) \neq 0$ in $\mathbb{D}$ and f satisfies the elliptic partial differential equation

$$\overline{f_{\overline{z}}(z)} = \omega(z) f_z(z)$$

in $\mathbb{D}$, where the dilatation $\omega(z) = g'(z)/h'(z)$ has the property that $|\omega(z)| < 1$ in $\mathbb{D}$.

Planar univalent harmonic mappings are used in the study of the Gaussian curvature of nonparametric minimal surfaces over simply connected domains (for example [6, 7, 9, 10, 14]). After the publication of landmark paper of Clunie and Sheil-Small [1], considerable interest in the function theoretic properties of harmonic functions,

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quite apart from this connection, was generated. Since then, the study of univalent harmonic mappings has gained much attention. The case where $\omega(z)$ is a finite Blaschke product is of special interest, since this case arises in many different contexts [9, 16]. In this chapter, we shall explicitly study a connection between certain classes of harmonic univalent mappings and minimal surfaces.

Let S be a nonparametric minimal surface over a simply connected domain Ω in $\mathbb{C}$ given by

$$S = \{(u, v, F(u, v)) : u + iv \in \Omega\},$$

where we have identified $\mathbb{R}^2$ with the complex plane in describing the domain of F. The Weierstrass–Enneper representation provides the close link between harmonic univalent mappings and the corresponding minimal graphs. Then S is a minimal surface if and only if S has the representation of the form

$$S = \left\{\left(\mathrm{Re}\int_0^z \phi_1(t)\,dt + c_1, \mathrm{Re}\int_0^z \phi_2(t)\,dt + c_2, \mathrm{Re}\int_0^z \phi_3(t)\,dt + c_3\right) : z \in \mathbb{D}\right\},$$

where $\phi_1,\ \phi_2,\ \phi_3$ are analytic in $\mathbb{D}$,

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = 0, \text{and} f = u + iv = \mathrm{Re}\int_0^z \phi_1(t)\,dt + i\mathrm{Re}\int_0^z \phi_2(t)\,dt + c \quad (3.1)$$

is a sense-preserving univalent harmonic mapping from $\mathbb{D}$ onto Ω. For this case, we call S a minimal graph over Ω with the projection $f = u + iv$.

Further basic information about harmonic mappings and their relation to minimal surfaces may be found in [4] and [6]. For instance, the following formulation is well-known (see for instance [6, Sect. 10.2]).

Theorem A *If $f = h + \overline{g}$ is a harmonic mapping of the form (3.1) with the dilatation $\omega = b^2$, where $b(z) = \pm z$, then we have*

$$\phi_1 = h' + g',\ \phi_2 = -i\left(h' - g'\right),\ \phi_3 = 2ibh'.$$

Using this, Jun [11] has considered the minimal surfaces associated with the harmonic mappings, especially when $\Omega = \{w : \mathrm{Im}\, w > 0\}$. The author's main result, which is easy to prove, will now be recalled for the sake of convenient reference.

Theorem B ([11]) *Let $\Omega = \{w : \mathrm{Im}\, w > 0\}$ and $p = p_1 + ip_2$ be a fixed point in Ω, where $p_1,\ p_2 \in \mathbb{R}$. If S is a minimal surface over Ω with the projection $f = h + \overline{g}$, where $\omega(z) = \frac{g'(z)}{h'(z)} = b^2(z) = z^2$, $b(z) = \pm z$ and $f(0) = p$, then $S = \{(u, v, F(u, v)) : u + iv \in \Omega\}$, where*

$$u = \mathrm{Re}\, f(z) = p_1 + \frac{ip_2}{2}\left[\left(\frac{1}{2}\log\frac{1+z}{1-z} + \frac{z}{(1-z)^2}\right) - \overline{\left(\frac{1}{2}\log\frac{1+z}{1-z} + \frac{z}{(1-z)^2}\right)}\right],$$

$$v = \mathrm{Im}\, f(z) = \frac{p_2}{2}\left[\frac{1+z}{1-z} + \overline{\left(\frac{1+z}{1-z}\right)}\right],$$

$$F = \pm p_2 \mathrm{Re}\left(\frac{z}{(1-z)^2} - \frac{1}{2}\log\frac{1+z}{1-z}\right).$$

The class $\mathcal{S}_H$ of sense-preserving harmonic univalent mappings $f = h + \overline{g}$ (normalized so that $f(0) = 0 = h(0)$ and $f_z(0) = 1$) together with its many geometric subclasses have been extensively studied [1, 6]. Let $\mathcal{S}_H^0$ be the subset of all $f \in \mathcal{S}_H$ in which $b_1 = f_{\overline{z}}(0) = 0$. We remark that the familiar class $\mathcal{S}$ of normalized analytic univalent functions is contained in $\mathcal{S}_H^0$. Every $f \in \mathcal{S}_H$ admits the complex dilatation ω of f which satisfies $|\omega(z)| < 1$ in $\mathbb{D}$. When $f \in \mathcal{S}_H^0$, we also have $\omega(0) = 0$.

In this chapter, we discuss the minimal surfaces over the slanted half-planes, vertical strips, and single slit whose slit lies on the negative real axis. Slanted half-plane mappings are well suited in the study of convolution of harmonic mappings [5]. Since the slanted half-planes and vertical strips are convex domains, the following result of Clunie and Sheil-Small is applicable for these cases.

Lemma C [1] *If $f = h + \overline{g}$ is a sense-preserving univalent mapping such that $f(\mathbb{D})$ is a convex domain, then the function $h + e^{i\beta}g$ is univalent for each β, $0 \le \beta < 2\pi$.*

3.2 Slanted Half-Plane Mappings

Throughout this chapter, we let $H_\gamma := \{w : \operatorname{Re}(e^{i\gamma}w) > -1/2\}$ be a slanted half-plane with the parameter γ, where $0 \le \gamma < 2\pi$.

Theorem 1 *Let S be a minimal surface over H_γ with the projection $f = h + \overline{g}$, whose dilatation $\omega = g'/h' = b^2$, where $b(z) = \pm z$. Then*

$$S = \{(u, v, F(u, v)) : u + iv \in H_\gamma\} = \{(u(z), v(z), F(u(z), v(z))) : z \in \mathbb{D}\},$$

where

$$\begin{aligned}
u &= \frac{\pi \sin\gamma}{4} - \cos\gamma - \operatorname{Im}\left(\frac{\sin\gamma}{4}\log\frac{z - e^{-i\gamma}}{z + e^{-i\gamma}} + \frac{\sin 2\gamma}{4(z - e^{-i\gamma})}\right)\\
&\quad - \operatorname{Re}\left(\frac{\cos\gamma}{2(z - e^{-i\gamma})^2} + \frac{3}{4(z - e^{-i\gamma})}\right),\\
v &= \frac{\pi \cos\gamma}{4} + \sin\gamma - \operatorname{Im}\left(\frac{\cos\gamma}{4}\log\frac{z - e^{-i\gamma}}{z + e^{-i\gamma}} - \frac{3}{4(z - e^{-i\gamma})}\right)\\
&\quad + \operatorname{Re}\left(\frac{\sin 2\gamma}{4(z - e^{-i\gamma})} - \frac{\sin\gamma}{2(z - e^{-i\gamma})^2}\right),\\
F &= \pm \operatorname{Re}\left[\frac{\sin 2\gamma}{4}\log\frac{z + e^{-i\gamma}}{z - e^{-i\gamma}} + \frac{1}{2}e^{i(\gamma + \frac{\pi}{2})}\frac{1}{z - e^{-i\gamma}} + \frac{i}{2}\frac{1}{(z - e^{-i\gamma})^2}\right] + c,
\end{aligned}$$

if $\gamma \in \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$;

$$u = \mathrm{Im}\left(\frac{\sin\gamma}{2(1-\sin 2\gamma)}\log\frac{z+ie^{i\gamma}}{ie^{i\gamma}} + \frac{\sin\gamma}{2(1+\sin 2\gamma)}\log\frac{z-ie^{i\gamma}}{-ie^{i\gamma}} - \frac{\sin\gamma}{\cos^2 2\gamma}\log\frac{z-e^{-i\gamma}}{-e^{-i\gamma}}\right) - \frac{\cos\gamma}{\cos 2\gamma} - \frac{\cos\gamma}{\cos 2\gamma}\mathrm{Re}\,\frac{e^{-i\gamma}}{z-e^{-i\gamma}},$$

$$v = \mathrm{Im}\left(\frac{\cos\gamma}{2(1-\sin 2\gamma)}\log\frac{z+ie^{i\gamma}}{ie^{i\gamma}} + \frac{\cos\gamma}{2(1+\sin 2\gamma)}\log\frac{z-ie^{i\gamma}}{-ie^{i\gamma}} - \frac{\cos\gamma}{\cos^2 2\gamma}\log\frac{z-e^{-i\gamma}}{-e^{-i\gamma}}\right) - \frac{\sin\gamma}{\cos 2\gamma} - \frac{\sin\gamma}{\cos 2\gamma}\mathrm{Re}\,\frac{e^{-i\gamma}}{z-e^{-i\gamma}},$$

$$F = \pm\mathrm{Re}\left[\frac{\log(z+ie^{i\gamma})}{2(1-\sin 2\gamma)} - \frac{\log(z-ie^{i\gamma})}{2(1+\sin 2\gamma)} - \frac{\sin 2\gamma}{\cos^2 2\gamma}\log(z-e^{-i\gamma}) - \frac{ie^{-i\gamma}}{(z-e^{-i\gamma})\cos 2\gamma}\right] + c,$$

if $\gamma \notin \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$.

Proof Let $f = h + \overline{g} \in \mathcal{S}_H^0$ and $f(\mathbb{D}) = H_\gamma$. Then, we have

$$\mathrm{Re}\,(e^{i\gamma} f(z)) = \mathrm{Re}\,\left[e^{i\gamma}(h(z) + e^{-2i\gamma} g(z))\right] > -\frac{1}{2}, \quad z \in \mathbb{D},$$

so that $\left(h + e^{-2i\gamma} g\right)(\mathbb{D}) = H_\gamma$ and by Lemma C, $h + e^{-2i\gamma} g$ is a conformal (univalent) mapping from $\mathbb{D}$ onto H_γ.

We now consider the function $h + e^{-2i\gamma} g$. We may conveniently normalize it in such a way that $f(0) = h(0) = g(0) = 0$. Then $h(0) + e^{-2i\gamma} g(0) = 0$. We further assume that

$$h(e^{-i\gamma}) + e^{-2i\gamma} g(e^{-i\gamma}) = \infty \text{ and } h\left(e^{-i(\pi+\gamma)}\right) + e^{-2i\gamma} g\left(e^{-i(\pi+\gamma)}\right) = -\frac{1}{2}e^{-i\gamma}.$$

By the uniqueness of the Riemann mapping theorem, these observations lead to the representation (see also [5, Lemma 1])

$$h(z) + e^{-2i\gamma} g(z) = \frac{z}{1 - e^{i\gamma} z} \tag{3.2}$$

from which we obtain

$$g(z) = -\frac{1}{z - e^{-i\gamma}} - e^{2i\gamma} h(z) - e^{i\gamma} \tag{3.3}$$

and

$$h'(z) + e^{-2i\gamma} g'(z) = \frac{1}{(1 - e^{i\gamma} z)^2}.$$

Solving this together with $g'(z) = z^2\, h'(z)$ gives

$$h'(z) = \frac{1}{(z^2 + e^{2i\gamma})(z - e^{-i\gamma})^2} \text{ and } g'(z) = \frac{z^2}{(z^2 + e^{2i\gamma})(z - e^{-i\gamma})^2}.$$

It is convenient to write $h'(z)$ in the form

$$h'(z) = \frac{1}{(z - e^{i(\gamma+\pi/2)})(z - e^{i(\gamma-\pi/2)})(z - e^{-i\gamma})^2}. \tag{3.4}$$

In order to determine $h(z)$ explicitly, we need to decompose it into partial fractions, and it is also clear that we need to deal with the cases where

$$\gamma \in \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\} \text{ and } \gamma \notin \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}.$$

Case 1 Let $\gamma = \frac{\pi}{4}$.

In this case, $h'(z)$ given by (3.4) takes the form

$$h'(z) = \frac{1}{\left(z + e^{-\frac{i\pi}{4}}\right)\left(z - e^{-\frac{i\pi}{4}}\right)^3}$$

so that $h'(z)$ has a simple pole at $z = -e^{-\frac{i\pi}{4}}$ and a pole of order 3 at $z = e^{-\frac{i\pi}{4}}$. We see that

$$h'(z) = \frac{i}{8}e^{\frac{i\pi}{4}}\left(\frac{1}{z - e^{-\frac{i\pi}{4}}} - \frac{1}{z + e^{-\frac{i\pi}{4}}}\right) - \frac{i}{4}\frac{1}{\left(z - e^{-\frac{i\pi}{4}}\right)^2} + \frac{1}{2}e^{\frac{i\pi}{4}}\frac{1}{\left(z - e^{-\frac{i\pi}{4}}\right)^3}.$$

Integration from 0 to z gives

$$h(z) = \left[\frac{1}{8}e^{\frac{3i\pi}{4}}\log\frac{z - e^{-\frac{i\pi}{4}}}{z + e^{-\frac{i\pi}{4}}} + \frac{i}{4}\frac{1}{z - e^{-\frac{i\pi}{4}}} - \frac{1}{4}e^{\frac{i\pi}{4}}\frac{1}{(z - e^{-\frac{i\pi}{4}})^2}\right] - \frac{1}{2}e^{-\frac{i\pi}{4}} + \frac{\pi}{8}e^{\frac{i\pi}{4}}. \tag{3.5}$$

Eq. (3.3) for $\gamma = \frac{\pi}{4}$ gives

$$g(z) = -\frac{1}{z - e^{-\frac{i\pi}{4}}} - ih(z) - e^{\frac{i\pi}{4}}$$

so that

$$h(z) + g(z) = -\frac{1}{z - e^{-\frac{i\pi}{4}}} + \sqrt{2}e^{-\frac{i\pi}{4}}h(z) - e^{\frac{i\pi}{4}}$$

and thus, substituting the expression for $h(z)$ defined by (3.5) yields that

$$h(z)+g(z) = \frac{\sqrt{2}\pi}{8} - \frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{8}\log\frac{z - e^{-\frac{i\pi}{4}}}{z + e^{-\frac{i\pi}{4}}} - \frac{3-i}{4}\frac{1}{z - e^{-\frac{i\pi}{4}}} - \frac{\sqrt{2}}{4}\frac{1}{\left(z - e^{-\frac{i\pi}{4}}\right)^2}$$

and similarly

$$h(z)-g(z) = \frac{i\sqrt{2}\pi}{8} + \frac{i\sqrt{2}}{2} - \frac{\sqrt{2}}{8}\log\frac{z - e^{-\frac{i\pi}{4}}}{z + e^{-\frac{i\pi}{4}}} + \frac{3+i}{4}\frac{1}{z - e^{-\frac{i\pi}{4}}} - \frac{i\sqrt{2}}{4}\frac{1}{\left(z - e^{-\frac{i\pi}{4}}\right)^2}.$$

As $u = \text{Re}\, f(z) = \text{Re}\,(h(z) + g(z))$ and $v = \text{Im}\, f(z) = \text{Im}\,(h(z) - g(z))$, the last two equalities give

$$u = \frac{\sqrt{2}\pi}{8} - \frac{\sqrt{2}}{2} - \text{Im}\left(\frac{\sqrt{2}}{8}\log\frac{z - e^{-\frac{i\pi}{4}}}{z + e^{-\frac{i\pi}{4}}} + \frac{1}{4}\frac{1}{z - e^{-\frac{i\pi}{4}}}\right)$$
$$- \text{Re}\left(\frac{\sqrt{2}}{4}\frac{1}{\left(z - e^{-\frac{i\pi}{4}}\right)^2} + \frac{3}{4}\frac{1}{z - e^{-\frac{i\pi}{4}}}\right),$$

and

$$v = \frac{\sqrt{2}\pi}{8} + \frac{\sqrt{2}}{2} - \text{Im}\left(\frac{\sqrt{2}}{8}\log\frac{z - e^{-\frac{i\pi}{4}}}{z + e^{-\frac{i\pi}{4}}} - \frac{3}{4}\frac{1}{z - e^{-\frac{i\pi}{4}}}\right)$$
$$+ \text{Re}\left(\frac{1}{4}\frac{1}{z - e^{-\frac{i\pi}{4}}} - \frac{\sqrt{2}}{4}\frac{1}{\left(z - e^{-\frac{i\pi}{4}}\right)^2}\right).$$

Finally, as $b(z) = \pm z$, Theorem A gives

$$\phi_3(z) = 2ibh'(z) = \pm 2i\frac{z}{\left(z + e^{-\frac{i\pi}{4}}\right)\left(z - e^{-\frac{i\pi}{4}}\right)^3}$$
$$= \pm 2i\left[\frac{i}{8}\frac{1}{z + e^{-\frac{i\pi}{4}}} - \frac{i}{8}\frac{1}{z - e^{-\frac{i\pi}{4}}} + \frac{1}{4}e^{\frac{i\pi}{4}}\frac{1}{\left(z - e^{-\frac{i\pi}{4}}\right)^2} + \frac{1}{2}\frac{1}{\left(z - e^{-\frac{i\pi}{4}}\right)^3}\right]$$

and therefore,

$$F(z) = \text{Re}\int_0^z \phi_3(z)\,dz + c$$
$$= \mp\text{Re}\left[\frac{1}{4}\log\frac{z + e^{-\frac{i\pi}{4}}}{z - e^{-\frac{i\pi}{4}}} + \frac{1}{2}e^{\frac{3i\pi}{4}}\frac{1}{z - e^{-\frac{i\pi}{4}}} + \frac{i}{2}\frac{1}{\left(z - e^{-\frac{i\pi}{4}}\right)^2}\right] + c.$$

Case 2–4 Using the same approach as in Case 1, we get

Case 2 For $\gamma = \frac{3\pi}{4}$,

$$F(z) = \mp\text{Re}\left[-\frac{1}{4}\log\frac{z - e^{\frac{i\pi}{4}}}{z + e^{\frac{i\pi}{4}}} - \frac{1}{2}e^{\frac{i\pi}{4}}\frac{1}{z + e^{\frac{i\pi}{4}}} + \frac{i}{2}\frac{1}{\left(z + e^{\frac{i\pi}{4}}\right)^2}\right] + c.$$

Case 3 For $\gamma = \frac{5\pi}{4}$,

$$F(z) = \mp \mathrm{Re}\left[\frac{1}{4}\log\frac{z - e^{\frac{-i\pi}{4}}}{z + e^{-\frac{i\pi}{4}}} + \frac{e^{-\frac{i\pi}{4}}}{2}\frac{1}{z + e^{-\frac{i\pi}{4}}} + \frac{i}{2}\frac{1}{\left(z + e^{-\frac{i\pi}{4}}\right)^2}\right] + c.$$

Case 4 For $\gamma = \frac{7\pi}{4}$,

$$F(z) = \mp \mathrm{Re}\left[-\frac{1}{4}\log\frac{z + e^{\frac{i\pi}{4}}}{z - e^{\frac{i\pi}{4}}} + \frac{1}{2}e^{\frac{i\pi}{4}}\frac{1}{z - e^{\frac{i\pi}{4}}} + \frac{i}{2}\frac{1}{\left(z - e^{\frac{i\pi}{4}}\right)^2}\right] + c.$$

Case 5 Let $\gamma \notin \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$.

In this case, $h'(z)$ given by (3.4) has simple poles at $ie^{i\gamma}$ and $-ie^{i\gamma}$, and a pole of order 2 at $e^{-i\gamma}$. Thus, we may rewrite $h'(z)$ as

$$h'(z) = \frac{A}{z + ie^{i\gamma}} + \frac{B}{z - ie^{i\gamma}} + \frac{C}{z - e^{-i\gamma}} + \frac{D}{\left(z - e^{-i\gamma}\right)^2},$$

where $A, B, C,$ and D can be easily computed using a standard procedure from residue calculus or otherwise. Indeed

$$A = \frac{e^{-i\gamma}}{4(1 - \sin 2\gamma)},\ B = \frac{e^{-i\gamma}}{4(1 + \sin 2\gamma)},\ C = -\frac{e^{-i\gamma}}{2\cos^2 2\gamma},\ \text{and } D = \frac{1}{2\cos 2\gamma}.$$

We observe that $A + B + C = 0$. Integration from 0 to z leads to

$$h(z) = A\log\frac{z + ie^{i\gamma}}{ie^{i\gamma}} + B\log\frac{z - ie^{i\gamma}}{-ie^{i\gamma}} + C\log\frac{z - e^{-i\gamma}}{-e^{-i\gamma}} - \frac{D}{z - e^{-i\gamma}} - De^{i\gamma}. \tag{3.6}$$

Note that g defined by (3.3) gives

$$h(z) + g(z) = -\frac{1}{z - e^{-i\gamma}} - 2ie^{i\gamma}h(z)\sin\gamma - e^{i\gamma}$$

and

$$h(z) - g(z) = \frac{1}{z - e^{-i\gamma}} + 2e^{i\gamma}h(z)\cos\gamma + e^{i\gamma},$$

where h is defined by (3.6). By computation, we know that

$$u = \mathrm{Im}\left[\frac{\sin\gamma}{2(1-\sin 2\gamma)}\log\frac{z+ie^{i\gamma}}{ie^{i\gamma}} + \frac{\sin\gamma}{2(1+\sin 2\gamma)}\log\frac{z-ie^{i\gamma}}{-ie^{i\gamma}} - \frac{\sin\gamma}{\cos^2 2\gamma}\log\frac{z-e^{-i\gamma}}{-e^{-i\gamma}}\right] - \frac{\cos\gamma}{\cos 2\gamma}\mathrm{Re}\left(\frac{z}{z-e^{-i\gamma}}\right),$$

$$v = \mathrm{Im}\left[\frac{\cos\gamma}{2(1-\sin 2\gamma)}\log\frac{z+ie^{i\gamma}}{ie^{i\gamma}} + \frac{\cos\gamma}{2(1+\sin 2\gamma)}\log\frac{z-ie^{i\gamma}}{-ie^{i\gamma}} - \frac{\cos\gamma}{\cos^2 2\gamma}\log\frac{z-e^{-i\gamma}}{-e^{-i\gamma}}\right] - \frac{\sin\gamma}{\cos 2\gamma}\mathrm{Re}\left(\frac{z}{z-e^{-i\gamma}}\right).$$

In the final case, by Theorem A, we find that

$$\begin{aligned}\phi_3(z) = 2ibh'(z) &= \pm\frac{2iz}{(z^2+e^{2i\gamma})(z-e^{-i\gamma})^2}\\ &= \pm 2i\left[-\frac{i}{4(1-\sin 2\gamma)(z+ie^{i\gamma})} + \frac{i}{4(1+\sin 2\gamma)(z-ie^{i\gamma})}\right.\\ &\quad\left. + \frac{i\sin 2\gamma}{2(z-e^{-i\gamma})\cos^2 2\gamma} + \frac{e^{-i\gamma}}{2(z-e^{-i\gamma})^2\cos 2\gamma}\right].\end{aligned}$$

Integration from 0 to z gives

$$F(z) = \pm\mathrm{Re}\left[\frac{\log(z+ie^{i\gamma})}{2(1-\sin 2\gamma)} - \frac{\log(z-ie^{i\gamma})}{2(1+\sin 2\gamma)} - \frac{\sin 2\gamma}{\cos^2 2\gamma}\log(z-e^{-i\gamma}) - \frac{ie^{-i\gamma}}{(z-e^{-i\gamma})\cos 2\gamma}\right] + c.$$

The proof is completed. □

3.3 Vertical Strips

Hengartner and Schober [8] investigated the family of functions from $\mathcal{S}_H$ that map $\mathbb{D}$ onto the horizontal strip domain $\{w : |\mathrm{Im}\, w| < \pi/4\}$. As an analogous result, Dorff [2] considered the family $\mathcal{S}_H(\mathbb{D}, \Omega_\alpha)$ of functions from $\mathcal{S}_H$ which map $\mathbb{D}$ onto the asymmetric vertical strip domains

$$\Omega_\alpha = \left\{w : \frac{\alpha-\pi}{2\sin\alpha} < \mathrm{Re}\, w < \frac{\alpha}{2\sin\alpha}\right\},$$

where $\frac{\pi}{2} \le \alpha < \pi$. Set $\mathcal{S}_H^0(\mathbb{D}, \Omega_\alpha) = \mathcal{S}_H(\mathbb{D}, \Omega_\alpha) \cap \mathcal{S}_H^0$. Note that $\Omega_{\pi/2} = \{w : |\mathrm{Re}\, w| < \pi/4\}$ and so, the class discussed by Hengartner and Schober [8] follows by using a suitable rotation.

Lemma 1 *Each $f = h + \overline{g} \in \mathcal{S}_H^0(\mathbb{D}, \Omega_\alpha)$ has the form*

$$h(z) + g(z) = \psi(z), \quad \psi(z) = \frac{1}{2i \sin\alpha} \log\left(\frac{1 + ze^{i\alpha}}{1 + ze^{-i\alpha}}\right). \tag{3.7}$$

Moreover,

$$h'(z) = \frac{\psi'(z)}{1 + \omega(z)},\ g'(z) = \frac{\omega(z)\psi'(z)}{1 + \omega(z)} \text{ and } \psi'(z) = \frac{1}{(1 + ze^{-i\alpha})(1 + ze^{i\alpha})}. \tag{3.8}$$

Here $\omega(z) = g'(z)/h'(z)$ denotes the dilatation of f.

Proof The representation (3.7) is well-known, whereas (3.8) follows if we solve the pair of equations $h'(z) + g'(z) = \psi'(z)$ and $\omega(z)h'(z) - g'(z) = 0$. The proof is complete. □

Theorem 2 *Let S be a minimal surface over Ω_α with the projection $f = h + \overline{g} \in \mathcal{S}_H^0(\mathbb{D}, \Omega_\alpha)$, which satisfies (3.1) and whose dilatation $\omega = b^2$, where $b(z) = \pm z$. Then $S = \{(u, v, F(u, v)) :\ u + iv \in \Omega_\alpha\}$, where*

$$u = \frac{1}{2\sin\alpha}\text{Im}\left[\log\left(\frac{1 + ze^{i\alpha}}{1 + ze^{-i\alpha}}\right)\right],$$

$$v = \begin{cases} \text{Im}\left(\frac{z}{z^2+1}\right) & \text{if } \alpha = \frac{\pi}{2}, \\ \frac{1}{2\cos\alpha}\text{Im}\left[\log\left(\frac{(1 + ze^{i\alpha})(1 + ze^{-i\alpha})}{z^2 + 1}\right)\right] & \text{if } \frac{\pi}{2} < \alpha < \pi \end{cases}$$

and

$$F = \begin{cases} \pm\,\text{Im}\left(\frac{1}{z^2 + 1}\right) + c & \text{if } \alpha = \frac{\pi}{2}, \\ \pm\,\text{Re}\left[\frac{1}{2\cos\alpha}\log\left(\frac{z+i}{z-i}\right) - \frac{1}{\sin 2\alpha}\log\left(\frac{z+e^{i\alpha}}{z+e^{-i\alpha}}\right)\right] + c & \text{if } \frac{\pi}{2} < \alpha < \pi. \end{cases}$$

Proof Let $f = h + \overline{g} \in \mathcal{S}_H^0(\mathbb{D}, \Omega_\alpha)$ with $\omega(z) = z^2$. Then by Lemma 1, we have

$$h'(z) = \begin{cases} \frac{1}{(z+i)^2(z-i)^2} & \text{if } \alpha = \frac{\pi}{2}, \\ \frac{1}{(z+i)(z-i)(z+e^{i\alpha})(z+e^{-i\alpha})} & \text{if } \frac{\pi}{2} < \alpha < \pi. \end{cases} \tag{3.9}$$

Case (i) Let $\frac{\pi}{2} < \alpha < \pi$. The partial fraction expansion of $h'(z)$ in (3.9) yields

$$h'(z) = -\frac{1}{4\cos\alpha}\left(\frac{1}{z+i} + \frac{1}{z-i}\right) + \frac{1}{(e^{-i\alpha} - e^{3i\alpha})(z + e^{i\alpha})} + \frac{1}{(e^{i\alpha} - e^{-3i\alpha})(z + e^{-i\alpha})}.$$

Integration from 0 to z gives

$$h(z) = -\frac{1}{4\cos\alpha}\log(z^2 + 1) + \frac{1}{e^{-i\alpha} - e^{3i\alpha}}\log\left(1 + ze^{-i\alpha}\right)$$

$$+\frac{1}{e^{i\alpha}-e^{-3i\alpha}}\log\left(1+ze^{i\alpha}\right),$$

which simplifies to

$$h(z)=-\frac{1}{4\cos\alpha}\log(z^2+1)+\frac{ie^{-i\alpha}}{2\sin 2\alpha}\log(1+ze^{-i\alpha})-\frac{ie^{i\alpha}}{2\sin 2\alpha}\log(1+ze^{i\alpha}). \tag{3.10}$$

By using (3.7), we obtain that

$$u=\text{Re}\,(h(z)+g(z))=\frac{1}{2\sin\alpha}\text{Im}\left(\log\left(\frac{1+ze^{i\alpha}}{1+ze^{-i\alpha}}\right)\right).$$

Writing $h(z)-g(z)=2\,h(z)-(h(z)+g(z))$ and using (3.7) and (3.10), we can easily find that

$$h(z)-g(z)=-\frac{1}{2\cos\alpha}\log(z^2+1)+\frac{1}{2\cos\alpha}\log\left(1+ze^{-i\alpha}\right)+\frac{1}{2\cos\alpha}\log\left(1+ze^{i\alpha}\right)$$

which gives

$$v=\text{Im}\,(h(z)-g(z))=\frac{1}{2\cos\alpha}\text{Im}\left(\log\frac{(1+ze^{i\alpha})(1+ze^{-i\alpha})}{z^2+1}\right).$$

In this case, ϕ_3 given by Theorem A takes the form

$$\begin{aligned}\phi_3(z)&=\pm\frac{2iz}{(z+i)(z-i)(z+e^{i\alpha})(z+e^{-i\alpha})}\\&=\pm 2i\left[\frac{i}{4\cos\alpha}\left(\frac{1}{z+i}-\frac{1}{z-i}\right)+\frac{1}{2i\sin 2\alpha}\left(\frac{1}{z+e^{i\alpha}}-\frac{1}{z+e^{-i\alpha}}\right)\right].\end{aligned}$$

Integration from 0 to z gives

$$F=\pm\text{Re}\left[\frac{1}{2\cos\alpha}\log\left(\frac{z+i}{z-i}\right)-\frac{1}{\sin 2\alpha}\log\left(\frac{z+e^{i\alpha}}{z+e^{-i\alpha}}\right)\right]+c.$$

Case (ii) Let $\alpha=\frac{\pi}{2}$. Using the same approach as in Case (i), we derive

$$u=\frac{1}{2}\text{Im}\left(\log\left(\frac{i-z}{i+z}\right)\right),\quad v=\text{Im}\left(\frac{z}{z^2+1}\right)$$

and thus,

$$F=\pm\text{Im}\left(\frac{1}{z^2+1}\right)+c.$$

The proof is completed. □

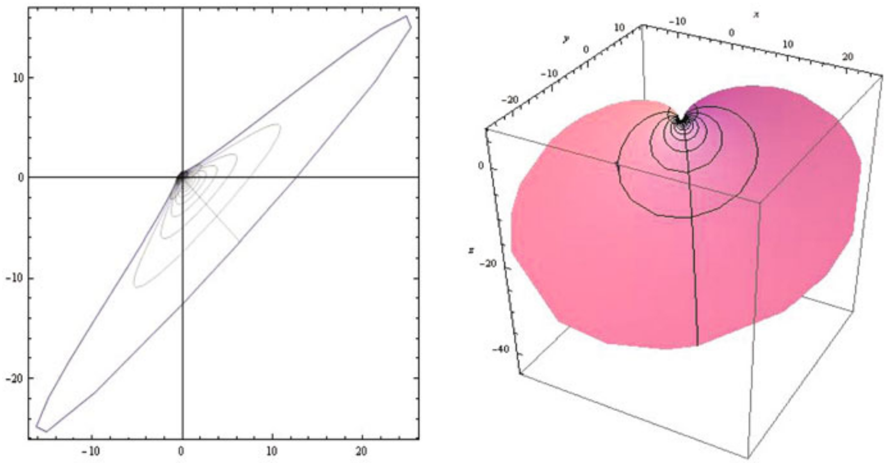

Fig. 3.1 Case 1: $\gamma = \pi/4$ of Theorem 1

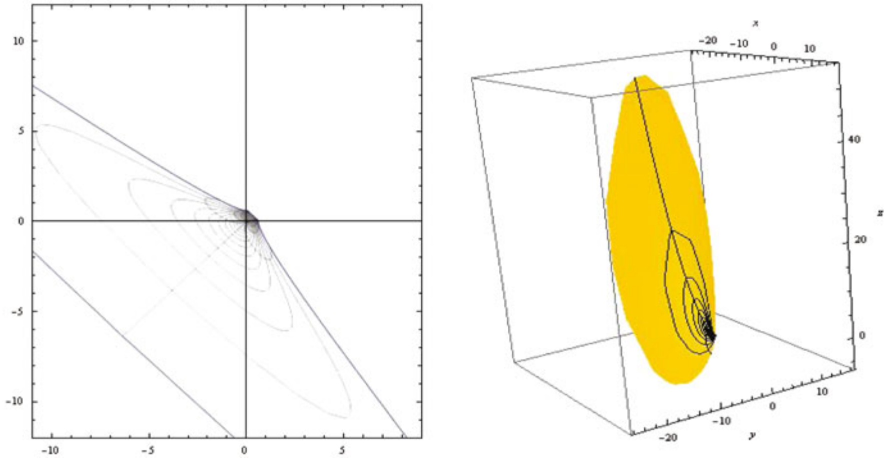

Fig. 3.2 Case 2: $\gamma = 3\pi/4$ of Theorem 1

3.4 Single Slit

Finally, we consider single slit domain L whose slit lies on the negative real axis. Moreover, by the result of Livingston [12] (see also [13] and Dorff [2, Corollary 2]) it follows that if $f = h + \overline{g} \in \mathcal{S}_H^0$ is a slit mapping whose slit lies on the negative real axis, then one has

$$h(z) - g(z) = \frac{z}{(1-z)^2}. \tag{3.11}$$

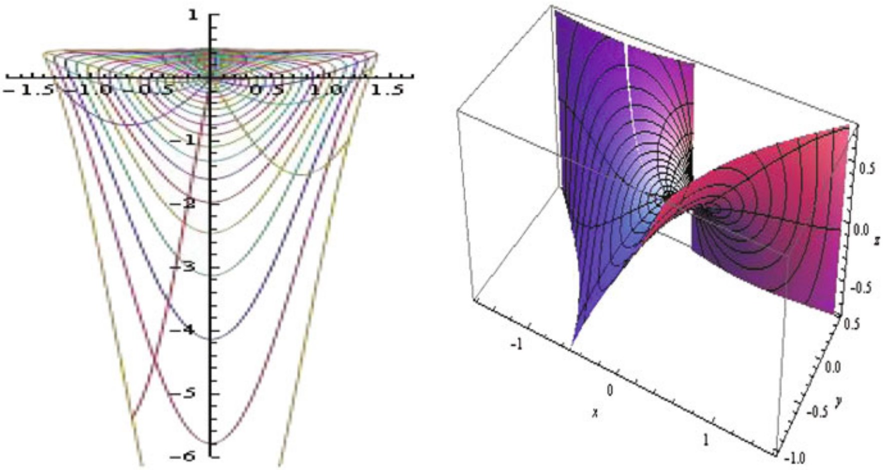

Fig. 3.3 **Case 5** with $\gamma = \pi/2$ of Theorem 1

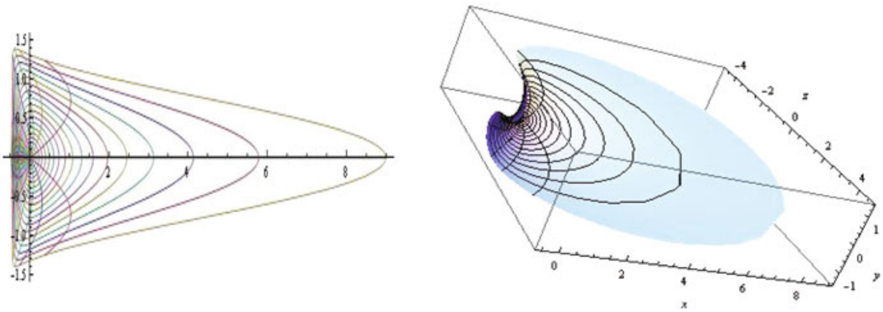

Fig. 3.4 **Case 5** with $\gamma = 0$ of Theorem 1

Theorem 3 *Let S be a minimal surface over L with the projection $f = h + \overline{g} \in \mathcal{S}_H^0$, which satisfies (3.11) and whose dilatation $\omega = b^2$, where $b(z) = \pm z$. Then $S = \{(u, v, F(u, v)) :\ u + iv \in L\}$, where*

$$u = \mathrm{Re}\left(\frac{2z^3 - 3z^2 + 3z}{3(1-z)^3}\right),\ v = \mathrm{Im}\left(\frac{z}{(1-z)^2}\right),$$

and

$$F = \pm\mathrm{Im}\left(\frac{1}{(z-1)^2} + \frac{2}{3(z-1)^3}\right) + c.$$

Proof By assumption, $f = h + \overline{g} \in \mathcal{S}_H^0$ is a single slit mapping whose slit lies on the negative real axis with $\omega(z) = z^2$. Then (3.11) holds and therefore, we have

$$h'(z) - g'(z) = \frac{1+z}{(1-z)^3} \text{ and } g'(z) = z^2\, h'(z).$$

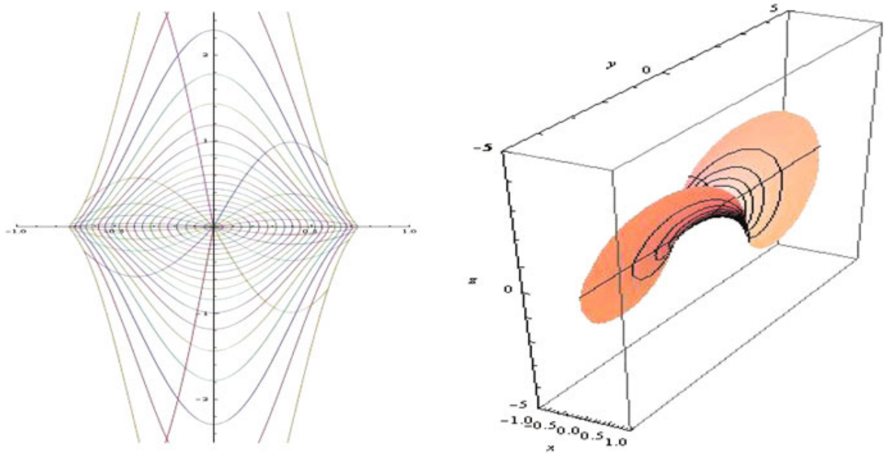

Fig. 3.5 Illustration for $\alpha = \pi/2$ of Theorem 2

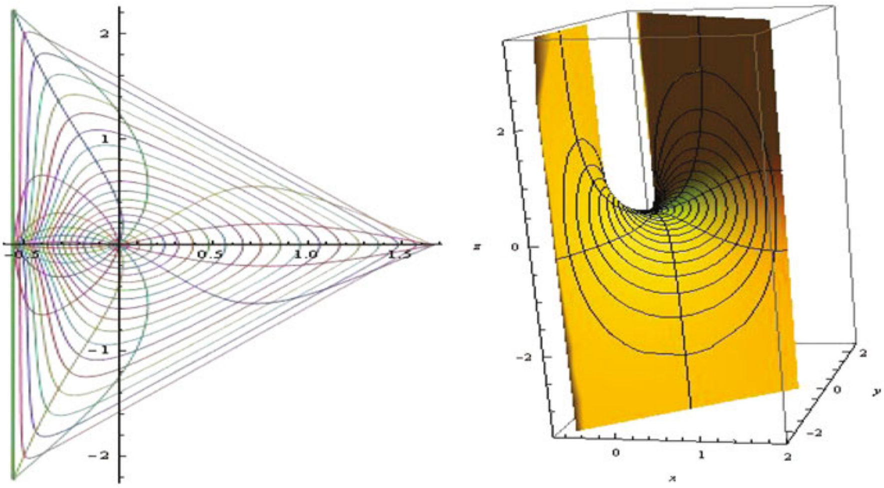

Fig. 3.6 Illustration for $\alpha = 3\pi/4$ of Theorem 2

Solving these two equations, we obtain

$$h'(z) = \frac{1}{(1-z)^4}.$$

Integrating from 0 to z yields

$$h(z) = -\frac{1}{3} + \frac{1}{3(1-z)^3}$$

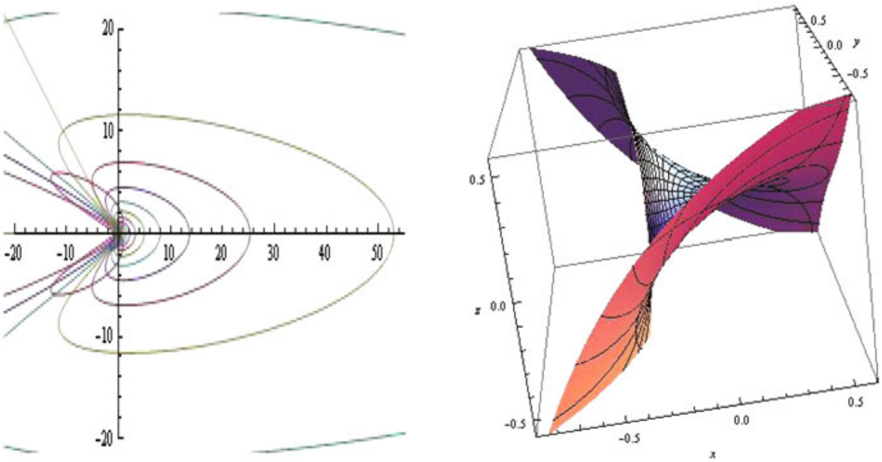

Fig. 3.7 Illustration for Theorem 3

and so

$$g(z) = h(z) - \frac{z}{(1-z)^2} = -\frac{1}{3} + \frac{1}{3(1-z)^3} - \frac{z}{(1-z)^2},$$

which, by using the previous equation, gives

$$h(z) + g(z) = \frac{2z^3 - 3z^2 + 3z}{3(1-z)^3}.$$

The desired representations for $u = \text{Re}\,(h(z)+g(z))$ and $v = \text{Im}\,(h(z)-g(z))$ follow easily.

Finally, since

$$\phi_3(z) = \pm 2izh'(z) = \pm\frac{2iz}{(1-z)^4} = \pm 2i\left(\frac{1}{(1-z)^4} - \frac{1}{(1-z)^3}\right),$$

integrating this from 0 to z yields

$$F = \pm\text{Im}\left(\frac{1}{(z-1)^2} + \frac{2}{3(z-1)^3}\right) + c.$$

The proof is completed. □

Independently, Theorem 3 has been proved by Dorff and Muir in [3].

3.5 Illustration Using Mathematica

In [3], the authors showed that in Theorem 1 with $\gamma = 0$, the corresponding minimal surface is the wavy plane, and in Theorem 3 the corresponding minimal surface is an Enneper surface. Also in [3] and [9], it was shown that in Theorem 2 with $\alpha = \frac{\pi}{2}$, the corresponding minimal surface is a part of the helicoid. It would be interesting to determine the various minimal surfaces for other values of γ and α in Theorems 1 and 2.

The images of the disk $|z| < r$ for r closer to 1 under $f = h + \overline{g}$ for various cases of Theorem 1 and the corresponding minimal surfaces associated with f are illustrated in Figs. 3.1, 3.2, 3.3, 3.4. Similar illustrations for Theorem 2 (Figs. 3.5 and 3.6) and Theorem 3 (Fig. 3.7) are also provided. These figures are drawn using Mathematica (see for example [15]).

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Chapter 4
A Survey On Some Special Classes of Bazilevič Functions and Related Function Classes

Pravati Sahoo and R. N. Mohapatra

4.1 Introduction

In this chapter we are mainly concerned with the the functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{4.1}$$

that are analytic in the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. The family $\mathcal{A}$ of all functions of the form (4.1) is the class of normalized functions $f(z)$ in Δ such that $f(0) = f'(0) - 1 = 1$. $\mathcal{S}$ is a subfamily of $\mathcal{A}$ consisting of functions univalent in Δ; that is, $f \in \mathcal{S}$, if and only if $f \in \mathcal{A}$ and $f(z_1) \neq f(z_2)$ for $z_1 \neq z_2$, whenever, $z_1, z_2 \in \Delta$. The Koebe function,

$$k(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} n z^n, \quad z \in \Delta$$

and its rotations $k_\varepsilon(z) = \frac{z}{(1-\varepsilon z)^2}$, $|\varepsilon| = 1$, are extremal for many problems in $\mathcal{S}$. The Koebe function $k(z)$ maps Δ onto the whole complex plane with a slit along the negative real axis from $-1/4$ to $-\infty$ and $k_\varepsilon(z)$ maps Δ onto the complement of the ray $\{z : z = -t\bar{\varepsilon}, t \geq 1/4\}$. In 1916, Bieberbach state a conjecture that $|a_n| \leq n$ for $f \in \mathcal{S}$, with equality only for Koebe function. Many powerful new methods, criterions were developed and a large number of problems were generated in attempts to prove this conjecture. At last, this was settled by Louis De Branges in 1985 [7]. The proof also opened a new era of application of special functions in the study of univalent function theory.

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Failure to settle the Biberbach conjecture for about 70 years led to the introduction and investigation of several subclasses of $\mathcal{S}$. An important subclass is $\mathcal{S}^*$, the set of functions that maps Δ onto a star-shaped domain with respect to the origin. A domain D is said to be star shaped if the line segment joining the origin to any other point of D lies completely in D. Clearly, Koebe function belongs to $\mathcal{S}^*$. It is well-known that $f \in \mathcal{S}^*$, if and only if $\text{Re}(\frac{zf'(z)}{f(z)}) > 0$, $z \in \Delta$. Another important subclass of $\mathcal{S}$ is the class $\mathcal{C}$ of functions which maps Δ onto a convex domain. A domain is said to be convex, if the line joining any two points on D lies completely in D. It is well-known that $f \in \mathcal{C}$, if and only if $\text{Re}(1 + \frac{zf''(z)}{f'(z)}) > 0$, $z \in \Delta$, from which it follows that $zf'(z) \in \mathcal{S}^*$. Koebe function does not belong to $\mathcal{C}$. Hence, it follows that $\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{S}$. Similarly, the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ are the classes of starlike functions of order α and convex functions of order α, respectively. It is also well-known that $f \in \mathcal{S}^*(\alpha)$, if and only if $\text{Re}(\frac{zf'(z)}{f(z)}) > \alpha$ and $f \in \mathcal{C}(\alpha)$, if and only if $\text{Re}(1 + \frac{zf''(z)}{f'(z)}) > \alpha$, $z \in \Delta$. The later two classes are subclasses of $\mathcal{S}$, only if $\alpha > 0$ and if $\alpha > 1$, then the classes contains only the trivial function $f(z) = z$.

Many extremal properties of $\mathcal{S}$ and $\mathcal{S}^*$ are same because the Koebe functions are often extremal in the whole class $\mathcal{S}$. Since $\mathcal{S}^*$ is a proper subclass of $\mathcal{S}$, the largest disk $|z| < r < 1$ that is mapped by all $f(z) \in \mathcal{S}$ onto a star-shaped domain, certainly can not be found by considering only functions in $\mathcal{S}^*$. It is therefore of interest to find subclass M of $\mathcal{S}$ for which $\mathcal{S}^* \subset M \subset \mathcal{S}$. One such class is the class of $\theta - spirallikefunctions$, which was introduced by Špaček [66]. He showed, if for $f(z) \in \mathcal{A}$, $\text{Re}(\frac{zf'(z)}{f(z)}) > 0$ then $f \in \mathcal{S}$. Without loss of generality, η can be replaced by $e^{i\theta}$, $-\pi/2 \leq \theta \leq \pi/2$. Let $\mathcal{S}p$ denote the set of all $\theta - spirallikefunctions$, that is

$$\mathcal{S}p = \left\{ f \in \mathcal{A} : \text{Re}\left(e^{i\theta} \frac{zf'(z)}{f(z)} \right) > 0,\ z \in \Delta,\ -\pi/2 < \theta < \pi/2 \right\}$$

For $\theta = \pm\pi/2$, $\mathcal{S}p$ will contain only the identity functions $f(z) = z$. The function called $\theta - spiral\ Koebe\ function\ k_\theta(z) = \frac{z}{(1-z)^{2e^{-i\theta}\cos\theta}} \in \mathcal{S}p$ but $k_\theta \notin \mathcal{S}^*$. This shows that $\mathcal{S}^0 \equiv \mathcal{S}^*$ and $\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{S}p \subset \mathcal{S}$.

In 1955, Kaplan [16], defined the close-to-convex function as: A function $f(z) \in \mathcal{A}$ is said to be close-to-convex in Δ, if there is a function $\phi(z)$ in $\mathcal{C}$ and a real β such that $\text{Re}\{e^{-i\beta}\frac{f'(z)}{\phi'(z)}\} > 0$ for $z \in \Delta$. $\mathcal{K}$ denotes the set of all close-to-convex functions. Many other subclasses of $\mathcal{S}$ have been introduced but we are not going to include them here, it will take us away from our desired goal.

In 1955, Bazilevič [2], was able to obtain a structural formula for a sufficiently large class of functions from $\mathcal{S}$ which contains the subclass $\mathcal{S}p$ and $\mathcal{K}$.

A function $f(z) \in \mathcal{A}$ is said to be Bazilevič of type (α, β), if $f(z)$ is of the form

$$f(z) = z\left\{(\alpha + i\beta)z^{-(\alpha+i\beta)} \int_0^z \left(\frac{g(t)}{t}\right)^\alpha h(t)t^{(\alpha+i\beta-1)}\,dt\right\}^{1/(\alpha+i\beta)},$$

where $z \in \Delta - \{0\}$, $g \in \mathcal{S}^*$, and $\text{Re}(e^{i\gamma}h(z)) > 0$ in Δ for some $\alpha > 0$, γ, $\beta \in \mathbb{R}$. More precisely, a function $f(z) \in \mathcal{A}$ is said to be Bazilevič of type $\mu = \alpha + i\beta$

($\alpha \geq 0,\ \beta \in \mathbb{R}$), if $f(z)$ satisfies the differential equation

$$f'(z)\left(\frac{z}{f(z)}\right)^{1-\mu} = \left(\frac{g(z)}{z}\right)^{\alpha} h(z). \tag{4.2}$$

The collection of univalent functions that satisfies (4.2) is called Bazilevič functions of type (α, β) and is often denoted by $\mathcal{B}(\alpha, \beta)$ and $\mathcal{B} = \bigcup_{\alpha,\beta} \mathcal{B}(\alpha, \beta)$. Particular choices of the parameters α, β, and the functions $g(z)$, $h(z)$ yield convex, starlike, close-to-convex, and spiral like functions, see the books by Duren [6] and Goodman [12]. $\mathcal{B}$, the class of Bazilevič functions is the largest known subclass of $\mathcal{S}$.

In 1971, Sheil-Small [63] gave a geometric proof of Bazilevič theorem and presented an intrinsic characterization of this result along the lines of Kaplan's characterization of the close-to-convex functions [16].

Also in [20], it is shown that $\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{S}p \subset \mathcal{K} \subset \mathcal{B} \subset \mathcal{S}$.

For the special case $g(z) = z$, $\beta = 0$, and $\alpha = \mu \in \mathbb{R}$, the class $\mathcal{B}$ can be written in general as

$$\mathcal{U}_h(\mu) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left(\frac{z}{f(z)}\right)^{1-\mu} f'(z) \prec h(z), \quad z \in \Delta \right\},$$

where $\prec$ stands for subordination. We say for $f,\ g \in \mathcal{A}$, f is subordinate to g ($f \prec g$), if there is a Schwarz' function $w(z)$ (i.e., $|w(z)| \leq |z|$, $z \in \Delta$), such that $f(z) = g(w(z))$. In another way we can write for $f, g \in \mathcal{S}$, $f \prec g$, if $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$.

In the existing literature, $\mathcal{U}_h(\mu)$ has been studied extensively for different choices of $h(z)$. The followings are some of the subclasses of $\mathcal{B}$ so also $\mathcal{S}$, which are derived from $\mathcal{U}_h(\mu)$.

1. For the convex function $h(z) = \frac{1+z}{1-z}$, $\alpha = -\mu$, and $\mu < 0$, the class $\mathcal{U}_h(\mu)$ becomes

$$\mathcal{B}(\mu) = \left\{ f \in \mathcal{A} :\ \mathrm{Re}\left(\left(\frac{z}{f(z)}\right)^{1+\mu} f'(z)\right) > 0,\ \ z \in \Delta \right\},$$

which has been introduced in [19].

2. For $0 \leq \lambda < 1$, $\mu < 0$, and for the convex function $h(z) = \frac{1+(1-2\lambda)z}{1-z}$, the class $\mathcal{U}_h(\mu)$ becomes

$$\mathcal{B}_1(\lambda, \mu) = \left\{ f \in \mathcal{A} :\ \mathrm{Re}\left(\left(\frac{z}{f(z)}\right)^{1+\mu} f'(z)\right) > \lambda,\ \ z \in \Delta \right\},$$

which was introduced in [28].

3. For $0 < \lambda < 1$, $\mu < 0$, and for the function $h(z) = 1 + \lambda z$, the class $\mathcal{U}_h(\mu)$ becomes

$$\mathcal{B}_1(\lambda, \mu) = \left\{ f \in \mathcal{A} : \left|\left(\frac{z}{f(z)}\right)^{1+\mu} f'(z) - 1\right| < \lambda,\ \ z \in \Delta \right\},$$

which has been introduced by Ponnusamy in [46].

In 1983, Sheil-Small [64], obtained that if f satisfies the condition

$$\left|\arg\left(e^{i\gamma}\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z)\right)\right| \leq \frac{(1-2\mu)\pi}{2}, \quad z \in \Delta, \tag{4.3}$$

for suitable $\gamma \in \mathbb{R}$, then $f \in \mathcal{S}$ for $\mu \leq 1/2$ and for $\mu = 1/2$, f is spiral like. Thus, the above result showed that the range of μ can be extended from $\mu < 0$ to $\mu \leq 1/2$. So, for the Eq. (4.3), natural question aroused that, how to determine the region Ω of the complex plane such that

$$\left(\frac{z}{f(z)}\right)^{\mu+1} f'(z) \in \Omega, \quad z \in \Delta, \tag{4.4}$$

implies that f is univalent. In particular, the problem for $\Omega = \{w \in \mathbb{C} : |w-1| < \lambda\}$ and determining the condition on λ, so that f satisfies the condition (4.4), is starlike, convex, if $0 < \mu \leq 1$. Thus, one can still get univalent functions f having a clear analytical description without being in $\mathcal{B}$, which has been considered to be the largest known subclass of $\mathcal{S}$. So in this light, in 1999, Obradovic [31] introduced the following class for $\mu > 0$

$$\mathcal{U}(\lambda, \mu) = \left\{ f \in \mathcal{A} : \left|\left(\frac{z}{f(z)}\right)^{1+\mu} f'(z) - 1\right| < \lambda, \ z \in \Delta \right\}$$

For $\mu = 1$ and $\lambda = 1$, the class

$$\mathcal{U}(1,1) := \mathcal{U} = \left\{ f \in \mathcal{A} : \left|\left(\frac{z}{f(z)}\right)^{2} f'(z) - 1\right| < 1, \ z \in \Delta \right\}$$

was introduced in 1972 by Ozaki and Nunokowa in [43]. In [28], Obradovic established the following result:

Theorem 1 *If $f \in \mathcal{U}$, then $\frac{z}{f(z)} \prec (1+z)^2$ for $z \in \Delta$.*

This relation is valid for the starlike functions also. So it is possible to suppose that $\mathcal{U} \subset \mathcal{S}^*$. But Koebe function $z/(1-z)^2$ belongs to $\mathcal{U}$ and does not belong to $\mathcal{S}^*(\alpha)$, for $\alpha > 0$. Similarly, the bounded analytic function $f(z) = z + \frac{z^2}{2} \in \mathcal{U}$ but does not belong to $\mathcal{S}^*(\alpha)$, for $\alpha > 0$. Thus $\mathcal{U} \not\subset \mathcal{S}^*(\alpha)$, for any positive α. This inspired many mathematicians to define subclasses which will be contained in $\mathcal{S}^*(\alpha)$, for $\alpha > 0$. So in an attempt to solve this problem one subclass $\mathcal{U}(\lambda)$ is introduced by Obradovič in [29]. For $0 < \lambda \leq 1$ and $\mu = 1$, the class

$$\mathcal{U}(\lambda, 1) := \mathcal{U}(\lambda) = \left\{ f \in \mathcal{A} : \left|\left(\frac{z}{f(z)}\right)^{2} f'(z) - 1\right| < \lambda, \ z \in \Delta \right\}$$

was introduced in [28].

4.2 Characterization of the Classes $\mathcal{U}$ and $\mathcal{U}(\lambda)$

In 1972, Ozaki and Nunokowa [43] introduced the class $\mathcal{U}$ which is a special case of the class $\mathcal{U}(\lambda, \mu)$ for $\mu = 1$ and $\lambda = 1$, and the authors have proved that $\mathcal{U} \subset \mathcal{S}$. This is the origin of the problems involving the the class $\mathcal{U}(\lambda, \mu)$. In 1998, Obradovic [29] introduced the class $\mathcal{U}(\lambda)$, which is a special case of the class $\mathcal{U}(\lambda, \mu)$ for $\mu = 1$, and proved that $\mathcal{U}(\lambda) \subset \mathcal{S}$ for $0 < \lambda \leq 1$[1, 43]. Also observed the following remarks.

Remark 1 For $\lambda > 1, \mathcal{U}(\lambda)$ need not be univalent.

The class $\mathcal{U}(\lambda)$ is an interesting class as it contains Koebe function, which provides solution to many extremal function.

Remark 2 For the function

$$f(z) = \frac{z}{1 + \frac{1}{2}iz + \frac{1}{2}\lambda e^{i\beta}z^3},$$

where $0.87\cdots = \dfrac{\sqrt{10} - \sqrt{2}}{2} < \lambda \leq 1$

and

$$\arcsin \frac{2 - \lambda^2}{\sqrt{2}\lambda} - \frac{\pi}{4} < \beta < \frac{3\pi}{4} - \arcsin \frac{2 - \lambda^2}{\sqrt{2}\lambda}$$

we have $f \in \mathcal{U}(\lambda)$ and $f \notin \mathcal{S}^*(\beta)$ [28]. Also in [26], it was shown that

$$\mathcal{U}(\lambda) \not\subset \mathcal{S}^*, \quad 0 < \lambda < 1.$$

More information on the structure of the functions of the class $\mathcal{U}$ is given in the following results.

Theorem 2 [39] *If $f \in \mathcal{U}$, then*

$$\frac{z}{f(z)} = 1 - \frac{f''(0)}{2}z - z\int_0^z \frac{w(t)}{t^2}\,dt, \quad z \in \Delta,$$

where $w \in \mathcal{H}$ and $|w(z)| < 1$, $z \in \Delta$, $w(0) = w'(0) = 0$.

Theorem 3 *If $f \in \mathcal{U}$, then*

(i) $\left|\dfrac{z}{f(z)} - 1\right| \leq |z|\left(\dfrac{|f''(0)|}{2} + |z|\right), \quad z \in \Delta$

(ii) $\operatorname{Re}\left(\dfrac{z}{f(z)}\right) > 0$ *for all* $|z| < \frac{1}{4}(\sqrt{16 + |f''(0)|^2} - |f''(0)|)$

This result is sharp as the function

$$f(z) = \frac{z}{1 - az - z^2} \quad (0 \leq a \leq 2)$$

shows for $z = r > 0$.

4.2.1 Starlike-ness Conditions for the Class $\mathcal{U}$ and $\mathcal{U}(\lambda)$

Since $\mathcal{U}(\lambda) \not\subset \mathcal{S}^*$ for $0 < \lambda < 1$, $z \in \Delta$, so it is of interest to find a disc $|z| < r$, where we can have star-likeness, convexity, or other properties. In that sense, we cite the next few results.

Theorem 4 [38]. *Let $f(z) = z + a_2 z^2 + \cdots \in \mathcal{U}$ with $0 < |a_2| = a \le 2$, then*

(a) $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$, $|z| < r_1(a)$, *where* $r_1(a) \in (0,1)$, *is the positive root of the equation*

$$2r^4 + 2ar^3 + a^2r^2 - 1 = 0.$$

(b) $\left|\frac{zf'(z)}{f(z)} - \frac{1}{2\beta}\right| < \frac{1}{2\beta}$, $|z| < r$, *where β and r given by*

$$\beta = \begin{cases} \dfrac{1 - r(a+r)}{1+r^2}, & 0 < r \le \dfrac{2}{\sqrt{a^2+a+8}} = r_2(a) \\[2ex] \dfrac{1 - [(r(a+r))^2 + r^4]}{2(1-r^4)}, & r_2(a) < r < r_1(a) \end{cases}$$

where $r_1(a)$ as in (a).

(c) $\left|\frac{f(z)}{zf'(z)} - 1\right| < r \le 1$, $|z| < r$, $0 < r \le \dfrac{2}{\sqrt{a^2+a+12}}$.

Theorem 5 *If $f = z + a_2z^2 + \cdots \in \mathcal{U}$, then f is convex at least in the disc $|z| < r$, where r is the smallest positive root of the equation*

$$7r^4 + 5|a_2|r^3 - 8r^2 - 3|a_2|r + 1 = 0.$$

In the next theorem the author [30] *gives a sufficient condition for a function $f \in \mathcal{A}$ to be in the class $\mathcal{U}$.*

Theorem 6 *$f \in \mathcal{A}$ and let*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \frac{1}{2}\frac{zf''(z)}{f'(z)}\right\} > \frac{3}{4}, \quad z \in \Delta.$$

Then $f \in \mathcal{U}$.

As it is mentioned in Remarks 1 and 2 that $\mathcal{U} \not\subset \mathcal{S}^*(\alpha)$, $\alpha > 0$, and $\mathcal{U}(\lambda) \not\subset \mathcal{S}^*$, $\lambda < 1$, so it is of interest to find suitable conditions on $\lambda_0 < 1$ such that for $0 < \lambda < \lambda_0$, $\mathcal{U}(\lambda)$ is included in $\mathcal{S}^*$, $\mathcal{S}^*(\alpha)$, $\mathcal{S}_\alpha$, $\mathcal{S}^*_\alpha$, $\mathcal{C}$, $\mathcal{R}_\alpha$ or some other well-known subclasses of $\mathcal{S}$. Here

$$\mathcal{S}_\alpha = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z} \right)^\alpha, \quad z \in \Delta \right\}$$

is the class of strongly starlike functions of order α, $(0 < \alpha \leq 1)$.

$$\mathcal{S}_\alpha^* = \left\{ f \in \mathcal{S}^*(\alpha) : \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \alpha, \quad z \in \Delta \right\}$$

and

$$\mathcal{R}_\alpha = \left\{ f \in \mathcal{A} : \operatorname{Re} f'(z) > \alpha \quad z \in \Delta \right\}.$$

Next, we cite the following results due to Obradovič et al. [40].

Theorem 7 *Let $f \in \mathcal{U}(\lambda)$ and $\gamma \in [0, 1]$. Define*

$$\lambda_\gamma^* = \frac{-|f''(0)| \cos(\pi\gamma/4) + \sin(\pi\gamma/4)\sqrt{16\cos^2(\pi\gamma/4) - |f''(0)|^2}}{2\cos(\pi\gamma/4)}$$

and $\lambda_\gamma^{\mathcal{R}}$ is given by the inequality

$$\sin\frac{\pi\gamma}{2}\sqrt{4-\lambda^2} \geq \left(|f''(0)| + \lambda\right)\sqrt{4 - (|f''(0)| + \lambda)^2 + \lambda\cos(\pi\gamma/2)}.$$

Then

(i) $f \in \mathcal{U}(\lambda) \Rightarrow f \in \mathcal{S}_\gamma$ *for* $0 < \lambda \leq \lambda_\gamma^*/2$
(ii) $f \in \mathcal{U}(\lambda) \Rightarrow f \in \mathcal{R}_\gamma$ *for* $0 < \lambda \leq \lambda_\gamma^{\mathcal{R}}/2$.

Theorem 8 *Let $f(z) = z + \sum_{n=2}^\infty a_n z^n$,*

$$\lambda^* = \frac{-|a_2| + \sqrt{2 - |a_2|^2}}{2} \text{ and } \lambda^{\mathcal{R}} = \frac{\sqrt{8|a_2| + 9} - (4|a_2| + 1)}{4}.$$

Then

(a) $f \in \mathcal{U}(\lambda) \Rightarrow f \in \mathcal{S}^*$ *for* $0 < \lambda \leq \lambda^*$
(b) $f \in \mathcal{U}(\lambda) \Rightarrow f \in \mathcal{R}_1$ *for* $0 < \lambda \leq \lambda^{\mathcal{R}}$.

Theorem 9 *Let $f \in \mathcal{U}(\lambda)$, λ_γ^* and $\lambda_\gamma^{\mathcal{R}}$ be in Theorem 7. Then*

(i) *For $\lambda_\gamma^*/2 \leq \lambda \leq 1$, $f(z)$ is strongly starlike in $|z| < r_{\lambda,\gamma}^*$, where, $r_{\lambda,\gamma}^*$ is the positive root of the equation $E_{\lambda,\gamma}(r) = 0$, and*

$$E_{\lambda,\gamma}(r) = 2\lambda^2 r^4[(1 + \cos(\pi\gamma/2)) + 2\lambda|f''(0)|r^3(1 + \cos(\pi\gamma/2)) + |f''(0)|^2 r^2 - 4\sin^2(\pi\gamma/2)]$$

(ii) $\lambda_\gamma^{\mathcal{R}}/2 \le \lambda \le 1$, *$f \in \mathcal{R}_\gamma$ in $|z| < r_{\lambda,\gamma}^{\mathcal{R}}$, where $\gamma \in (0,1]$ and $0 < r_{\lambda,\gamma}^{\mathcal{R}} < 1$ are related by*

$$2 - (|f''(0)|r + \lambda r^2)^2 \ge \lambda r^2 \sin(\pi\gamma/2) + \sqrt{4 - \lambda^2 r^4} \cos(\pi\gamma/2).$$

Example 1 Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $\lambda \in (0,1]$. Then

(i) $0 < \lambda \le \dfrac{1 - 4|a_2|}{7}$, we have

$$f \in \mathcal{U}(\lambda) \Rightarrow \frac{zf'(z)}{f(z)} \prec \frac{2(1-z)}{2-z}, \quad z \in \Delta.$$

(ii) $\dfrac{1 - 4|a_2|}{7} < \lambda \le 1$, we have

$$f(z) \in \mathcal{U}(\lambda) \Rightarrow \frac{rzf'(rz)}{f(rz)} \prec \frac{2(1-rz)}{2-rz},$$

where $r = \dfrac{\sqrt{4|a_2|^2 + 7\lambda} - 2|a_2|}{7\lambda}$.

Theorem 10 *Let $f \in \mathcal{U}(\lambda)$ with $f''(0) = 0$ and $\gamma \in (0,1]$. Then we have the following:*

(i) *$f \in \mathcal{S}_\gamma$ for $0 < \lambda \le \sin(\pi\gamma/4)$. In particular, $f \in \mathcal{S}^*$ whenever $0 < \lambda \le 1/\sqrt{2}$.*
(ii) *$f \in \mathcal{S}_\gamma$ in $|z| < \sqrt{\frac{\sin(\pi\gamma/4)}{\lambda}}$, if $\sin(\pi\gamma/4) < \lambda \le 1$. In particular, $f \in \mathcal{S}^*$ in $|z| < 1/\sqrt{\sqrt{2}\lambda}$ whenever $1/\sqrt{2} < \lambda \le 1$.*
(iii) *$f \in \mathcal{R}_\gamma$ for $0 < \lambda \le \sin(\pi\gamma/6)$. In particular, $\mathcal{U}(\lambda) \subset \mathcal{R}_1$ for $0 < \lambda \le 1/2$.*
(iv) *$f \in \mathcal{R}_\gamma$ in $|z| < \sqrt{\frac{\sin(\pi\gamma/6)}{\lambda}}$, if $\sin(\pi\gamma/6) < \lambda \le 1$. In particular, $f \in \mathcal{R}_1$ in $|z| < 1/\sqrt{2\lambda}$ whenever $1/2 < \lambda \le 1$.*

Example 2 Let $f \in \mathcal{U}(\lambda)$, with $a_2 = 0$. Then

- $\dfrac{zf'(z)}{f(z)} \prec 1 + z$ when $0 < \lambda \le \dfrac{1}{3}$. But $f \in \mathcal{S}^*$ whenever $0 < \lambda \le \dfrac{1}{\sqrt{2}}$.
- $\dfrac{zf'(z)}{f(z)} \prec \dfrac{2(1-z)}{2-z}$ whenever $0 < \lambda \le \dfrac{1}{7}$.
- $\mathcal{U}(\lambda) \subset \mathcal{R}_1$ for $0 < \lambda \le \dfrac{1}{2}$ [40].

4.3 Characterization of the Class $\mathcal{U}(\lambda, \mu)$

In 1998, Obradovic [29] introduced and studied the class $\mathcal{U}(\lambda, \mu)$ which generalizes the class $\mathcal{U}(\lambda)$. In this section we discuss the historic development of this class and its coefficient properties.

In 1995, Ponnusamy [45] showed that, if $0 < \lambda \le 2/\sqrt{5}$ and $h \in \mathcal{A}$ satisfies $|h'(z) - 1| < \lambda$ in Δ, then $h \in \mathcal{S}^*(\alpha)$, where $\alpha = (2 - \lambda\sqrt{5})/(2 + \lambda)$ by using the method of differential subordination.

Theorem 11 *Let $\lambda \in [0, 1]$ and $|h'(z) - 1| < \lambda$ in Δ, then $h \in \mathcal{S}^*(\beta)$, where*

$$\beta = \begin{cases} \dfrac{1-\lambda}{1+\lambda/2}, & 0 \le \lambda \le 2/3, \\[2ex] \dfrac{4-5\lambda^2}{2(4-\lambda^2)}, & 2/3 \le \lambda < 1. \end{cases}$$

Next we sate an important convolution result due to Ruschweh and Sheil-Small [58], which is very useful to evaluate sharp bounds.

Theorem 12 *Let $\Phi, \Psi \in \mathcal{C}$ and suppose $f \prec \Psi$. Then $f * \Phi \prec \Psi * \Phi$.*

By using Theorem 12, Ponnusamy and Singh [51] proved the following improved version of the Theorem11,

Theorem 13 *Let $0 < \lambda \le 2/\sqrt{5}$ and $|h'(z) - 1| < \lambda$ in Δ, then $h \in \mathcal{S}^*(\beta)$, where*

$$\beta = \begin{cases} \dfrac{1-\lambda}{1+\lambda/2}, & 0 \le \lambda \le 2/\sqrt{5}, \\[2ex] \dfrac{4-5\lambda^2}{2(4-\lambda^2)}, & 2/\sqrt{5} \le \lambda < 1. \end{cases}$$

With the help of the above theorem, Ponnusamy [47] gave a sufficient condition for a function in $\mathcal{U}(\lambda, \mu)$ to be in the class of bounded starlike functions.

Theorem 14 *If $\mu < 0$, $0 < \alpha \le 1$, $f \in \mathcal{A}$ and*

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} - 1 \right| < \frac{(1-\mu)\alpha}{1-\alpha\mu-2\mu}, \quad z \in \Delta,$$

then $f \in \mathcal{S}_\alpha$.

Also in [47], the author found a similar type of result which dealt with an integral operator:

Theorem 15 *Let* $\mu < 0$*,* $\text{Re}c > \mu$*, and* $0 < \lambda \le (|c-\mu+1|(1-\mu))/(|c-\mu|(1-3\mu))$*. If* $f \in \mathcal{U}(\lambda,\mu)$*, then* $F \in \mathcal{S}_\alpha$*, where*

$$\alpha = \frac{(1-2\mu)(\lambda(c-\mu)/(c-\mu+1))}{1-\mu(\lambda(c-\mu)/(c-\mu+1))}$$

and

$$F(z) = z\left[\frac{c-\mu}{z^{c-\mu}}\int_0^z \left(\frac{t}{f(t)}\right)^{\mu} t^{c-\mu-1\,dt}\right]^{-\frac{1}{\mu}}. \tag{4.5}$$

Later on, Obradović introduced and studied the generalized class $\mathcal{U}(\lambda,\mu)$ for the higher range of μ, that is, for $\mu > 0$ in [31]. In the next section, we state some results on the same class for the higher range of μ.

4.3.1 Starlikeness Condition for the Class $\mathcal{U}(\lambda,\mu)$ with Higher Range of μ

Obradovic [31] proved the following results for $0 < \mu < 1$.

Theorem 16 [31] *Let* $f \in \mathcal{U}(\lambda,\mu)$*,* $0 < \mu < 1$*. Then we have the representation*

$$\left(\frac{z}{f(z)}\right)^{\mu} = 1 - \mu\lambda\int_0^1 \frac{w(z)}{t^{\mu+1}}\,dt,$$

where $w \in \mathcal{H}$*,* $w(0) = 0, |w(z)| < 1$*, for* $z \in \Delta$*.*

Theorem 17 [31] *Let* $f \in \mathcal{U}(\lambda,\mu)$*,* $0 < \mu < 1$*, and*

$$0 < \lambda \le \min\left\{1, \frac{1-\mu}{\mu}\right\} = \begin{cases} 1, & 0 < \mu \le 1/2 \\ \dfrac{1-\mu}{\mu}, & 1/2 \le \mu < 1, \end{cases}$$

then $\text{Re}\{(\frac{z}{f(z)})^{\mu}\} > 0$*, for* $z \in \Delta$*.*

Theorem 18 [31] *Let* $f \in \mathcal{U}(\lambda,\mu)$*,* $0 < \mu < 1$*, and* $0 < \lambda \le \dfrac{1-\mu}{\sqrt{(1-\mu)^2+\mu^2}}$*, then* $\mathcal{U}(\lambda,\mu) \subset \mathcal{S}^*$*.*

By the help of Theorem 16, Obradovic derived the following sufficient condition for the function in $\mathcal{U}(\lambda,\mu)$ to be in the class of starlike functions.

Theorem 19 [31] *Let* $f \in \mathcal{U}(\lambda, \mu)$*,* $0 < \mu < 1$*, and* $0 < \lambda \le \frac{1-\mu}{1+\mu}$*, then* f *is a starlike function and*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1, \quad for\, z \in \Delta$$

.

Theorem 20 [31] *Let* $f \in \mathcal{U}(\lambda, \mu)$*,* $\frac{1}{2} \le \mu < 1$*. Then,* $\mathrm{Re}\,(f'(z)) > 0$*,* $z \in \Delta$*, for* $0 < \lambda \le \lambda_0$*, where* λ_0 *is the smallest positive root of the equation*

$$a^2\lambda^2(3 - 4a^2\lambda^2)^2 + \lambda^2 - 1 = 0, \quad a = \frac{\mu}{1-\mu}.$$

Theorem 21 [29] *Let* $f \in \mathcal{U}(\lambda, \mu)$*,* $0 < \mu < 1$ *and let for* $c > \mu$*,* $F(z)$ *is defined by* (4.5)*. Then*

(a) $F \in \mathcal{S}^*$ *for* $\dfrac{(c-\mu)\lambda}{1+c-\mu} \le \dfrac{1-\mu}{\sqrt{(1-\mu)^2+\mu^2}}$, $0 < \mu < 1$

(b) $F \in \mathcal{S}^*(\beta)$*, where*

$$\beta = \begin{cases} \dfrac{1-\lambda_1}{1+\lambda_2}, & 0 < \lambda_1 < 1 - \mu \\[2ex] \dfrac{1-(\lambda_1^2+\lambda_2^2)}{2(1-\lambda_2^2)}, & 1 - \mu \le \lambda_1 \le \dfrac{1-\mu}{\sqrt{(1-\mu)^2+\mu^2}} \end{cases}$$

and $\lambda_1 = \dfrac{(c-\mu)\lambda}{1+c-\mu}$, $\lambda_2 = \dfrac{\mu}{1-\mu}$, $0 < \mu < 1/2$.

Remark 3 $\mathcal{U}(\lambda, 1)$ is not included in $\mathcal{S}^*$ for $\lambda < 1$ but on the other hand $f \in \mathcal{U}_2(\lambda, 1)$ $(a_2 = 0)$ is seen to be in $\mathcal{S}^*$ whenever $0 < \lambda \le \dfrac{1}{\sqrt{2}}$ [40].

This remark opened the door for the functions of missing coefficients which will extend the range of μ beyond the unit interval. In 2005, Ponusamy and Sahoo [48] introduced the class

$$\mathcal{U}_n(\lambda, \mu) = \left\{ f \in \mathcal{A}_n : \left| f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} - 1 \right| < \lambda, \quad z \in \Delta \right\} \equiv \mathcal{A}_n \cap \mathcal{U}(\lambda, \mu),$$

where $0 < \mu < n$, for a fixed $n \in \mathbb{N}$ and $\mathcal{A}_n$, to be the class of the functions in $\mathcal{A}$ with first n missing coefficients. The case $\mu = n$, which does produce a slightly different implication, has been discussed in [49]. We next cite some results on the class $\mathcal{U}(\lambda, \mu)$ with higher range of μ.

Theorem 22 *Let* $\gamma \in (0, 1]$*,* $n \ge 1$*,* $\mu \in (0, n)$ *and*

$$\lambda_*(\gamma, \mu, n) = \frac{(n-\mu)\sin(\gamma\pi/2)}{\sqrt{(n-\mu)^2 + \mu^2 + 2\mu(n-\mu)\cos(\gamma\pi/2)}}.$$

If $f \in \mathcal{U}_n(\lambda, \mu)$*, then* $f \in \mathcal{S}_\gamma$ *for* $0 < \lambda \le \lambda_*(\gamma, \mu, n)$*.*

Theorem 23 *If $f \in \mathcal{U}_n(\lambda, \mu)$ and $0 < \lambda \leq \dfrac{n-\mu}{\sqrt{(n-\mu)^2+\mu^2}}$, then $f \in \mathcal{S}^*$.*

Theorem 24 *Let $\alpha \in [0,1)$, $n \geq 1$, and $\mu \in (0,n)$. If $f(z) \in \mathcal{U}_n(\lambda,\mu)$, then $f \in \mathcal{S}^*(\alpha)$ for $0 < \lambda \leq \lambda^*(\alpha,\mu,n)$, where*

$$\lambda^*(\alpha,\mu,n) = \begin{cases} \dfrac{(n-\mu)\sqrt{1-2\alpha}}{\sqrt{(n-\mu)^2+\mu^2(1-2\alpha)}} & \text{for } 0 \leq \alpha \leq \dfrac{\mu}{n+\mu} \\ \dfrac{(n-\mu)(1-\alpha)}{n-\mu+\mu\alpha} & \text{for } \dfrac{\mu}{n+\mu} < \alpha < 1. \end{cases}$$

For $\mu = n$, the authors derived the following results:

Theorem 25 *If $f \in \mathcal{U}_n(\lambda)$ and $b = |a_{n+1}| \leq 1/n$, then $f \in \mathcal{S}^*(\alpha)$ whenever $0 < \lambda \leq \lambda_0(\alpha)$, where*

$$\lambda_0(\alpha) = \begin{cases} \dfrac{\sqrt{(1-2\alpha)(1+n^2(1-2\alpha-b^2))}-n^2b(1-2\alpha)}{1+n^2(1-2\alpha)} & \text{if } 0 \leq \alpha < \alpha_0(n,b), \\ \dfrac{1-\alpha(1+nb)}{1+n\alpha} \text{if } \alpha_0(n,b) \leq \alpha < \dfrac{1}{1+nb} \end{cases}$$

with $\alpha_0(n,b) = \frac{n(b+1)}{n(b+2)+1}$.

4.3.2 Sharpness of the Results on the Class $\mathcal{U}(\lambda)$ and $\mathcal{U}(\lambda,\mu)$

In the recent years the classes $\mathcal{U}(\lambda)$ and $\mathcal{U}(\lambda,\mu)$ have been studied more extensively only for real μ. In the previous sections we stated some of important properties and results on these classes for real μ, which are not sharp. But more recently, Ponnusamy, Obradovic and Fournier [9, 11, 56, 68] have studied these classes for complex μ and achieved even sharper results.

We next state the following theorem due to Fournier and Ponnusamy [11], which gives conditions, under which $\mathcal{U}(\lambda,\mu)$ has geometric significance (such as starlikeness and spirallikeness).

Theorem 26 [11] *Let $\mu \in \mathbb{C}$ with* $\mathrm{Re}(\mu) < 1$. *Then*

1. *$\mathcal{U}(\lambda,\mu) \subset \mathcal{S}^*$, if and only if $0 \leq \lambda \leq \frac{|1-\mu|}{\sqrt{|1-\mu|^2+|\mu|^2}}$.*
2. *$\mathcal{U}(\lambda,\mu) \subset \mathcal{S}p$, if and only if $0 \leq \lambda \leq \min\left(1, \frac{|1-\mu|}{|\mu|}\right)$, where $\mathcal{S}p$ is the class of spiral-like functions* [6].

In our discussion we stated many interesting properties and results on the class $\mathcal{U}(1,\mu)$ for real μ [11]. In the following theorem, we state condition on $\mu \in \mathbb{C}$ under which functions in the class $\mathcal{U}(1,\mu)$ belongs to the class of spiral-like functions:

Theorem 27 [11] $\mathcal{U}(1,\mu) \subset \mathcal{S}p$, *if and only if* $\text{Re}(\mu) \leq 1/2$.

These theorems are proved with the help of some lemmas related to Blaschke product. So we first define the Blaschke product.

Definition 1 A finite Blaschke product is a function of the type

$$b(z) = e^{i\gamma} \prod_{j=1}^{n} \frac{z - a_j}{1 - \bar{a_j} z}, \quad \{a_j\}_{j=1}^{n} \subset \Delta,\ \gamma \in \mathbb{R}.$$

Next, we state an important result on Blaschke product which is very useful to prove sharpness of several result.

Theorem 28 [14, 56] *Given ϕ and ψ in $\mathbb{R}$, there exists a sequence $\{b_n\}$ of finite Blaschke products such that $b_n(1) = e^{i\phi}$, $b_n(0) = 0$, and $b_n(z) \to e^{i\psi} z$ in the sense of convergence in $\mathcal{H}$.*

Theorem 29 *There exists an infinite sequence w_n of infinite Balschke products with the following property: given a function $w \in \mathcal{H}$ with $w(\Delta) \subseteq \Delta$ and two sets of nodes $\{\phi_k\}_{k=1}^{m}$ and $\{\psi_k\}_{k=1}^{m}$ in $\mathbb{R}$, where the $\phi_k^{\prime}s$ are assumed to be pairwise distinct mod 2π there exists a subsequence $\{w_{n_j}\}$ of $\{w_n\}$ such that*

$$w_{n_j}(e^{i\phi_k}) = e^{i\psi_k}, \quad 1 \leq k \leq m,\ j = 1, 2, \ldots$$

and

$$\lim_{j \to \infty} w_{n_j} = w \ \ in\ \mathcal{H}.$$

Following two theorems are due to Ruscheweyh [15, 57], which are the main tools for the proof of the sharpness of many of the results on class $\mathcal{U}(\lambda, \mu)$.

Theorem 30 *Let $c \in \mathbb{C}$ with $\text{Re} c < 1$ and $F_c(z) = \sum_{n=1}^{\infty} \frac{1-c}{n-c} z^{n-1} \in \mathcal{H}$. Then*

$$\sup_{z \in \Delta} |f * F_c(z)| \leq \sup_{z \in \Delta} |f(z)|, \quad for\ any\ f \in \mathcal{H}.$$

Theorem 31 *Let $c \in \mathbb{C}$ with $\text{Re} c < j$ and $\theta \in \mathbb{R}$. Then the functional*

$$I(w) = \sum_{k=1}^{\infty} \frac{a_k(w)}{k - c} e^{ik\theta}, \ \ w(z) = \sum_{k=1}^{\infty} a_k(w) z^k \in \mathcal{B}_j$$

is well defined and continuous over $\mathcal{B}_j = \{w \in \mathcal{B} | |w(z)| \leq |z|^j, z \in \Delta\}$.

4.3.3 Coefficient Characterization and Convolution Properties of the Functions in the Classes $\mathcal{U}$, $\mathcal{U}(\lambda)$ and $\mathcal{U}(\lambda, \mu)$

For analytic functions f in Δ of the form $\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots$, a sufficient condition for f to be in the class $\mathcal{U}$ is that, [34]

$$\sum_{n=2}^{\infty}(n-1)|b_n| \leq 1 \tag{4.6}$$

and even more, necessary condition for f to be in $\mathcal{S}$ is that, [33]

$$\sum_{n=2}^{\infty}(n-1)|b_n|^2 \leq 1. \tag{4.7}$$

On the other hand, no such simple necessary condition for the functions f in $\mathcal{A}$ to be in $\mathcal{U}$ seems to be known in the literature except the Bieberbach estimate. The sufficient condition (4.6) is useful especially for rational function. The class of convex functions is included in the class of starlike functions. Ponnusamy and Obradovic in [36], have derived a simple analog of this concerning the class $\mathcal{U}$, that is, if $f \in \mathcal{A}$ satisfies the condition

$$\left|\left(\frac{z}{f(z)}\right)''\right| \leq 2, \quad z \in \Delta,$$

then $f \in \mathcal{U}$.

If $f, g \in \mathcal{S}$, then the function F defined by $\frac{z}{F(z)} = \frac{z}{f(z)} * \frac{z}{g(z)}$ in $\mathcal{U}$ whenever $\frac{z}{f(z)} * \frac{z}{g(z)} \neq 0$ is in Δ [33]. The analogous property does not exist for the convolution $f(z) * g(z)$. For example $k(z) * k(z)$ is not univalent in Δ, where $k(z)$ is the Koebe function.

This observation gives the idea about a manner which proceed with a new approach to the theory of univalent functions although there does not seem to be direct geometrical meaning. It is sometimes convenient to consider functions $f \in \mathcal{U}(\lambda, \mu)$ of the form $(\frac{z}{f(z)})^{\mu} = 1 + b_1 z + b_2 z^2 + \cdots$, because the condition

$$\sum_{n=2}^{\infty}(n-\mu)|b_n|^2 \leq 1. \tag{4.8}$$

is necessary for the function f to be in the class $\mathcal{S}$ (see Theorem 11, p.193, vol. 2 of [12]).

On the other hand, the sufficient condition for the function of the form $(\frac{z}{f(z)})^{\mu} = \phi(z) = 1 + b_1 z + b_2 z^2 + \cdots$ to be in $\mathcal{U}(\lambda, \mu)$ [50] is

$$\sum_{n=2}^{\infty}(n-\mu)|b_n| \leq \lambda\mu. \tag{4.9}$$

As a motivation the authors, Ponnusamy and Obradovic, obtained the above necessary and sufficient coefficient conditions and verified results by considering special functions like Gaussian hypergeometric function. The Gaussian hypergeometric function is defined by

$$ {}_2F_1(a,b,c;z) = F(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n, \quad z \in \Delta, $$

where a, b, c in general, are complex numbers such that $c \neq -m$, $m = 0, 1, 2, \ldots$, and $(a)_n$ denotes the Pochhamer symbol

$$ (a)_0 = 1,\ (a)_n = a(a+1)(a+2)\cdots(a+n-1) \quad \text{for } n \in \mathbb{N}. $$

In [34], the authors use the following lemma, which was needed to prove the Theorems 32 and 33.

Lemma 1 *For $a > -1$, $b > -1$ with $ab > 0$ and $c \geq (a+1)(b+1)$. Then*

$$ \sum_{n=2}^{\infty} (n-1) \frac{(a)_n(b)_n}{(c)_n(1)_n} \leq 1. $$

Theorem 32 *Suppose that $a, b > -1$, $ab > 0$, and $c \geq \max\{ab, (a+1)(b+1)\}$. Then $\frac{z}{F(a,b,c;z)}$ is in $\mathcal{U}$.*

Theorem 33 *Suppose that $a \in \mathbb{C} - \{0\}$ and $c \geq \max\{|a|^2, |a+1|^2\}$. Then the function $\frac{z}{F(a,\bar{a},c;z)}$ is in $\mathcal{U}$.*

Remark 4 The following observations are from the Theorem 32,

(1) $\frac{z}{F(1,1,c;z)}$ is in $\mathcal{U}$, if $c \geq 4$.

(2) $\frac{z}{F(-1/2,-1/2,c;z)}$ is in $\mathcal{U}$, if $c \geq 1/4$.

Theorem 34 [34] *Suppose $f \in \mathcal{S}$ has the form*

$$ \left(\frac{z}{f(z)} \right) = 1 + b_1 z + b_2 z^2 + \cdots \neq 0, \quad b_n \geq 0. $$

Suppose that $a, b, c > -2$ with $c \neq 0, -1, -2, \ldots$ and satisfy

$$ 0 \leq \frac{ab(a+1)(b+1)}{2c(c+1)} \leq 1, \quad c \geq \max\left\{ a+b-1, \frac{2(a+b-1)+ab}{3} \right\} $$

and that $(z/f(z)) * F(a,b,c;z) \neq 0$ for all $z \in \Delta$. Then the transformation

$$ H(z) = \frac{z}{(z/f(z)) * F(a,b,c;z)}, \quad z \in \Delta, $$

is in $\mathcal{U}$.

Theorem 35 [32] *Let $f \in \mathcal{U}(\lambda)$ and $c \in \mathbb{C}$ with $\mathrm{Re}\, c \geq 0 \neq c$, such that*

$$(z/f(z)) * F(1, c, c+1; z) \neq 0 \text{ for all } z \in \Delta,$$

and $G = G_f^c$ be the transformation defined by

$$G(z) = \frac{z}{(z/f(z)) * F(1, c, c+1; z)}, \quad z \in \Delta. \tag{4.10}$$

Let $A = |\frac{c}{c+1} \frac{f''(0)}{2}| \leq 1$. Then

(1) $G \in \mathcal{U}(\lambda|c|/|c+2|)$. The result is sharp especially when $|f''(0)/2| \leq 1 - \lambda$. In particular, $G \in \mathcal{U}$ whenever $0 < \lambda \leq |(c+2)/c|$.
(2) $G \in \mathcal{S}^*$ whenever $0 < \lambda \leq \frac{|c+2|}{2|c|}(\sqrt{2 - A^2} - A)$. In particular, if $\lambda = 1$, $f''(0) = 0$, and $|c - 2| \leq 2\sqrt{2}$ with $\mathrm{Re}\, c \geq 0$, then $G \in \mathcal{S}^*$.

Theorem 36 [32] *Let $f \in \mathcal{A}_n$ in $\mathcal{U}(\lambda)$ and $c \in \mathbb{C}$ with $\mathrm{Re}\, c \geq 0 \neq c$, such that $(z/f(z)) * F(1, c, c+1; z) \neq 0$ for all $z \in \Delta$, and $G = G_f^c$ be the transformation defined by (4.10). Then*

(1) *$G \in \mathcal{U}(\lambda|c|/|c+n|)$. In particular, $G \in \mathcal{U}$ whenever $0 < \lambda \leq |(c+n)/c|$.*

(2) *$G \in \mathcal{S}^*$ whenever $0 < \lambda \leq \frac{|c+n|(n-1)}{|c|\sqrt{(n-1)^2+1}}$.*

4.3.4 Univalence of the Product of the Functions in the Classes $\mathcal{U}$ and $\mathcal{U}(\lambda)$

In [37], the authors discussed the radius of univalence of a product of functions $F(z) = g(z)h(z)/z$, where $g, h \in \mathcal{U}(\lambda)$. The authors considered the problem: For $g \in \mathcal{F}_1 \subset \mathcal{S}$ and $h \in \mathcal{F}_2 \subset \mathcal{S}$, then function $F = \frac{g(z)h(z)}{z}$ is starlike or belongs to the class $\mathcal{U}(\lambda)$ in the disk $|z| < r$. In that context, the authors proved the following theorems in [37].

Theorem 37 *Let $g, h \in \mathcal{S}^*$. Then the function $F(z) = g(z)h(z)/z$ is starlike in the disk $|z| < 1/3$. The result is sharp.*

Theorem 38 *Let $g, h \in \mathcal{U}$. Then the function $F(z) = g(z)h(z)/z$ belongs to the class $\mathcal{U}$ in the disk $|z| < 1/3$. The result is sharp.*

Theorem 39 *Let $g, h \in \mathcal{U}_2$. Then the function $F(z) = g(z)h(z)/z$ belongs to the class $\mathcal{U}_2$ in the disk $|z| < \frac{1}{\sqrt{3}}$. The result is sharp.*

Theorem 40 *Let $g \in \mathcal{U}_2(\lambda_1)$ and $g \in \mathcal{U}_2(\lambda_2)$. Then the function $F(z) = g(z)h(z)/z$ belongs to the class $\mathcal{U}_2(\lambda_3)$ in the disk $|z| < r$, where*

$$r = \sqrt{\frac{2\lambda_3}{\lambda_1 + \lambda_2 + \sqrt{(\lambda_1 + \lambda_2)^2 + 12\lambda_1\lambda_2\lambda_3}}}.$$

Theorem 41 *Let $g, h \in \mathcal{S}$. Then the function $F(z) = g(z)h(z)/z$ belongs to the class $\mathcal{U}$ in the disk $|z| < r0$, where $r_0 \approx 0.30294$ is the smallest positive root of the equation*

$$6r^2+2(\sqrt{2}+4)r^3+\frac{2r^4\sqrt{3-2r^2}}{1-r^2}+4r^2\left(\frac{r^2(6r^2-1-4r^4)}{(1-r^2)^2}+\log\left(\frac{1}{1-r^2}\right)\right)^{\frac{1}{2}}+\frac{r^4(3-2r)}{(1-r)^2}-1=0$$

in the interval $(0, 1)$.

Theorem 42 *Let $g, h \in \mathcal{S}$, such that $g''(0) = h''(0) = 0$. Then the function $F(z) = g(z)h(z)/z$ belongs to the class $\mathcal{U}$ in the disk $|z| < r_0$, where $r_0 \approx 0.435895$ is the smallest positive root of the equation*

$$2r^2 + 2\sqrt{2}r^3 + \frac{2r^4\sqrt{3-2r^2}}{1-r^2} + \frac{r^4(3-2r)}{(1-r)^2} - 1 = 0$$

in the interval $(0, 1)$. *Moreover, F is starlike in the disk $|z| < r_0$.*

The sharpness above two theorems are still open.

4.4 Characterization of the Class $\mathcal{U}(\alpha, \lambda, \mu)$

In 1988, Ponnusamy [44] defined and studied a special class of Bazilevič functions as

$$\mathcal{U}_h(\alpha,\lambda,\mu) = \left\{ f \in \mathcal{A} : (1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu} + \alpha f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} \prec h(z),\ z \in \Delta \right\},$$

where $0 \le \alpha < 1$, $\mu < 0$ and $h(z)$ is convex function in Δ with $h(0) = 1$. For $\alpha = 0$ and $h(z) = \frac{1-z}{1+z}$ has been studied in [42, 65].

In 2007, Zhu [71], considered the generalized class $\mathcal{U}_n(\alpha, M, \mu)$ for $\mu < 0$

$$\mathcal{U}_n(\alpha, M,\mu) = \left\{ f \in \mathcal{A}_n : \left|(1-\alpha)\left(\frac{z}{f(z)}\right)^{\mu} + \alpha f'(z)\left(\frac{z}{f(z)}\right)^{\mu+1} - 1\right| < M, z \in \Delta \right\}.$$

Zhu proved the following result by using a subordination result which is given in [22]:

Theorem 43 [71] *Let $\alpha > 0$, $\mu < 0$ and let*

$$M_n(\alpha,\lambda,\mu) = \begin{cases} \dfrac{(n\alpha-\mu)(1-\lambda)}{n-\mu(1-\lambda)} & \text{if } \alpha \ge \alpha_2, \\[2ex] \dfrac{(n\alpha-\mu)\sqrt{2\alpha(1-\lambda)-1}}{\sqrt{n^2\alpha^2+2[(1-\lambda)\mu^2-n\mu]\alpha}} & \text{if } \alpha_1 \le \alpha < \alpha_2 \\[2ex] \dfrac{\alpha(n\alpha-\mu)(1-\lambda)}{(n-\mu\lambda+\mu)\alpha-2\mu} & \text{if } 0 < \alpha \le \alpha_1, \end{cases}$$

where, $\alpha_2 = \frac{n-\mu(1-\lambda)}{n(1-\lambda)}$*, and*

$$\alpha_1 = \frac{\sqrt{9\mu^2 - 2n\mu + n^2 - (18\mu^2 - 2n\mu)\lambda + 9\mu^2\lambda^2} + 3\mu + n - 3\mu\lambda}{2n(1-\lambda)}.$$

If $p(z)$ *and* $q(z)$ *are analytic in* Δ *with* $p(z) = 1 + p_n z^n + \cdots$ *and* $q(z) = 1 + q_n z^n + \cdots$, *satisfy*

$$q(z) \prec 1 - \frac{\mu M z}{n\alpha - \mu},$$

then

$$q(z)[1 - \alpha + \alpha p(z)] \prec 1 + Mz,$$

where $0 < M \leq M_n(\alpha, \lambda, \mu)$*, then* $\operatorname{Re} p(z) > \lambda$ *for* $z \in \Delta$*.*

Using this theorem, the author also proved the following:

Theorem 44 [71] *Let* α, λ, μ, M, *and* $M_n(\alpha, \lambda, \mu)$ *be defined as in the above theorem. If* $f \in \mathcal{U}_n(\alpha, M, \mu)$*, then* $f \in \mathcal{S}_n^*(\lambda)$*.*

Theorem 45 [71] *Let* $\alpha > 0$ *and let*

$$M_n(\alpha) = \begin{cases} \dfrac{n\alpha + 1}{n+1} & \text{if } \alpha \geq \dfrac{n+1}{n} \\[2ex] \dfrac{(n\alpha - \mu)\sqrt{2\alpha - 1}}{\sqrt{n^2\alpha^2 + 2(n+1)\alpha}} & \text{if } \dfrac{\sqrt{9 + 2n + n^2} - 3 + n}{2n} \leq \alpha < \dfrac{n+1}{n} \\[2ex] \dfrac{\alpha(n\alpha + 1)}{(n-1)\alpha + 2} & \text{if } 0 < \alpha \leq \dfrac{\sqrt{9 + 2n + n^2} - 3 + n}{2n}. \end{cases}$$

If $f \in \mathcal{U}_n(\alpha, M, \mu)$*, where* $0 < M \leq M_n(\alpha)$*, then* $f \in \mathcal{S}_n^*(0)$*.*

For $n = 1$, the above theorem improves the result due to Mocanu in [23, 24].

Theorem 46 [71] *Let* $\alpha > 0$, $0 \leq \lambda < 1$ *and let*

$$M_n(\alpha, \lambda) = \begin{cases} \dfrac{(n\alpha + 1)(1-\lambda)}{n + 1 - \lambda} & \text{if } \alpha \geq \alpha_2, \\[2ex] \dfrac{(n\alpha + 1)\sqrt{2\alpha(1-\lambda) - 1}}{\sqrt{n^2\alpha^2 + 2(n + 1 - \lambda)\alpha}} & \text{if } \alpha_1 \leq \alpha < \alpha_2 \\[2ex] \dfrac{\alpha(n\alpha + 1)(1-\lambda)}{(n - 1 + \lambda +)\alpha + 2} & \text{if } 0 < \alpha \leq \alpha_1, \end{cases}$$

where, $\alpha_2 = \frac{n+1-\lambda}{n(1-\lambda)}$*, and*

$$\alpha_1 = \frac{\sqrt{9\mu^2 - 2n\mu + n^2 - (18 + 2n)\lambda + 9\lambda^2} - 3 + n + 3\lambda}{2n(1-\lambda)}.$$

If $f(z)$ satisfies $|f'(z)+\alpha z f''(z)-1| \le M$ for $z \in \Delta$, where $0 < M \le M_n(\alpha,\lambda)$, then $f(z) \in \mathcal{C}(\lambda)$.

For $n = 1$ the above theorem improves the result due to Mocanu, Mocanu and Fournier [10, 25].

For $\lambda > 0$, $\alpha > 0$, and $\mu > 0$, Sahoo and Singh [61] defined a new class $\mathcal{U}_n(\alpha,\lambda,\mu)$, of non-Bazilevič analytic functions.
For the special values of $\alpha = 1$, $n = 1$, $\mu = 1$, and $\lambda = 1$, studied in [28, 29, 40, 46, 48, 51].
The authors derived following conditions for which the class $\mathcal{U}_n(\alpha,\lambda,\mu)$ included well-known subclasses of $\mathcal{S}$.

Using the following Sahoo–Singh [59] have shown the Remark 5.

Lemma 2 [18] *Let $-1 \le B_1 \le B_2 < A_2 \le A_1 < 1$, then*

$$\frac{1+A_2 z}{1+B_2 z} \prec \frac{1+A_1 z}{1+B_1 z}, \quad z \in \Delta.$$

Remark 5 For $0 \le \lambda_2 < \lambda_1 < 1$, from Lemma 2, we have

$$\mathcal{U}(\alpha,\lambda_2,\mu) \subset \mathcal{U}(\alpha,\lambda_1,\mu).$$

The authors derived following results in [61]

Theorem 47 *Let $\gamma \in (0,1]$, $n \in \mathbb{N}$, $\alpha > 0$, $\mu \in (0, n\alpha)$, and*

$$\lambda_*(\alpha,\gamma,\mu,n) = \frac{\alpha(n\alpha-\mu)\sin\frac{\gamma\pi}{2}}{\sqrt{\alpha^2\mu^2+\{(n\alpha-\mu)+(1-\alpha)\mu\}^2+2\mu\alpha\{(n\alpha-\mu)+(1-\alpha)\mu\}\cos\frac{\gamma\pi}{2}}}. \tag{4.11}$$

If $f \in \mathcal{U}_n(\alpha,\lambda,\mu)$, then $f \in S_\gamma^n$ for $0 < \lambda \le \lambda_*(\alpha,\gamma,\mu,n)$; here $\lambda_*(\alpha,\gamma,\mu,n)$ is an increasing function of n and $\lambda_*(\alpha,\gamma,\mu,n) \to \alpha \sin\dfrac{\gamma\pi}{2}$ as $n \to \infty$. Here $S_\gamma^n \equiv S_\gamma \bigcap \mathcal{A}_n$.

Theorem 48 *Let $f \in \mathcal{U}_n(\alpha,\lambda,\mu)$ and $\lambda_*(\alpha,\gamma,\mu,n)$ be as in Theorem 47. Then, for $\lambda_*(\alpha,\gamma,\mu,n) < \lambda$, f is strongly starlike in $|z| < r = r(\alpha,\lambda,\gamma,\mu,n)$, where*

$$r = r(\alpha,\lambda,\gamma,\mu,n) = \left\{\frac{(n\alpha-\mu)\alpha\sin\frac{\gamma\pi}{2}}{\lambda\sqrt{\alpha^2\mu^2+\{(n\alpha-\mu)+|\alpha-1|\mu\}^2+2\mu\alpha\cos\frac{\gamma\pi}{2}\{(n\alpha-\mu)+|\alpha-1|\mu\}}}\right\}^{\frac{1}{n}}.$$

Theorem 49 *For $0 < \alpha \le \frac{1}{2}$ and $0 \le \beta < 1$ or $\alpha > \frac{1}{2}$ and $1-\frac{1}{2\alpha} \le \beta < 1$ or $\alpha > \frac{1}{2}$, $0 \le \beta < 1-\frac{1}{2\alpha}$ and $(n\alpha-\mu)|\alpha(1-\beta)-1| \ge \mu[2\alpha(1-\beta)-1]$, we define*

$$\lambda^*(\alpha,\beta,\mu,n) = \frac{\alpha(n\alpha-\mu)(1-\beta)}{(n\alpha-\mu)+\mu|1-\alpha(1-\beta)|}.$$

For $\alpha > \frac{1}{2}$, $0 \le \beta < 1-\frac{1}{2\alpha}$ and $(n\alpha-\mu)|\alpha(1-\beta)-1| < \mu[2\alpha(1-\beta)-1]$, we define

$$\lambda^*(\alpha,\beta,\mu,n) = \frac{(n\alpha-\mu)\sqrt{2\alpha(1-\beta)-1}}{\sqrt{(n\alpha-\mu)^2+\mu^2[2\alpha(1-\beta)-1]}}.$$

If $f \in \mathcal{U}_n(\alpha, \lambda, \mu)$*, then* $f \in \mathcal{S}_n^*(\beta)$ *for* $0 < \lambda \leq \lambda^*(\alpha, \beta, \mu, n)$.
Next, we consider the following integral transform $\mathrm{I}(f)$ of $f \in \mathcal{A}$ defined by

$$[\mathrm{I}(f)(z)] = F(z) = z\left[\frac{c+1-\mu}{z^{c+1-\mu}} \int_0^z \left(\frac{t}{f(t)}\right)^{\mu} t^{c-\mu} dt\right]^{1/\mu}, \qquad (c+1-\mu > 0). \tag{4.12}$$

This transform is similar to the Alexander transform when $c = \mu = 1$ and is similar to Bernardi transformation when $\mu = 1$ and $c > 0$.

Theorem 50 *Let* $f \in \mathcal{U}_n(\alpha, \lambda, \mu)$ *for* $\lambda > 0$, $n \geq 2$ *and* $\mu \in (0, n\alpha)$ *here* $\alpha \geq 0$. *For* $c + 1 - \mu > 0$ *and* $\beta < 1$, *let* $F(z)$ *defined by* (4.12). *Then* $F \in \mathcal{S}_\beta^*$ *whenever* c, λ, α *are related by*

$$0 < \lambda \leq \frac{(1-\beta)(n\alpha - \mu)(c+1+n-\mu)\{\alpha(c+1-\mu)+\mu\}}{\{\mu(1-\beta)+n\}\{\alpha(c+1)+\mu(1-\alpha)\}(c+1-\mu)}.$$

4.4.1 Fekete–Szegö Problem for the Class $\mathcal{U}(\alpha, \lambda, \mu)$

Until now, these classes have been studied with a view to find necessary conditions over μ, α, and λ so that, these classes included into the class of univalent functions or its well-known subclasses.

So far the most well-known Fekete–Szegö problem is open for these classes. Fekete and Szegö showed that for $f \in \mathcal{S}$ given by (4.1)

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & if \quad \mu \leq 0, \\ 1 + 2\exp\left(\frac{-2}{1-\mu}\right), & if \quad 0 \leq \mu < 1, \\ 4 - 3\mu, & if \quad \mu \geq 1. \end{cases}$$

As a result, many authors have also studied similar problems for some subclasses of $\mathcal{S}$ [4, 17]. In [8], sharp upper bound of Fekete–Szegö functional $|a_3 - \mu a_2^2|$ is obtained for all real α when $\beta = 0$ for (4.13). That result was extended by Darus [5] for a larger subclass satisfying

$$\mathrm{Re}\left\{\frac{zf'(z)}{f^{(1-\alpha)}(z)g^{\alpha}(z)}\right\} > \beta, \quad z \in \Delta \tag{4.13}$$

where $\alpha > 0, 0 \leq \beta < 1$.

Many authors studied Fekete–Szegö problem for subclasses of $\mathcal{B}(\lambda, \mu)$ [67, 69]. In this chapter we concentrate on the Fekete–Szegö problem for the class $\mathcal{U}(\alpha, \lambda, \mu)$.

By using the following lemma in [59], Sahoo–Singh established the Fekete–Szegö problem for the Class $\mathcal{U}(\alpha, \lambda, \mu)$

Lemma 3 *For* $w(z) = c_1 z + c_2 z^2 + \cdots$ *analytic in* Δ, *let* $|w(z)| < 1$ *in* Δ. *Then*

$$|c_1| \leq 1 \tag{4.14}$$

$$|c_2| \leq 1 - |c_1|^2 \tag{4.15}$$

and

$$|c_2 - sc_1^2| \leq \max\{1, |s|\} \tag{4.16}$$

The inequalities (4.14) *and* (4.15) *are in* (e.g., [27], p. 108) *and* (4.16) *is a trivial(e.g., [17]) consequence of the triangle inequality and the inequalities* (4.14) *and* (4.15). *Equality for* (4.16) *may be obtained with* $w(z) = z$ *when* $|s| \geq 1$ *and* $w(z) = z^2$ *when* $|s| < 1$.

Theorem 51 *Let* $f \in \mathcal{U}(\alpha, \lambda, \mu)$ *and* $\alpha > 0$, $0 < \lambda \leq 1$, $\mu > 0$. *Then*

$$|a_3 - \delta a_2^2| \leq \begin{cases} \dfrac{\lambda}{|2\alpha - \mu|}, & if \quad |\delta - k| \leq l, \\ \\ \dfrac{\lambda^2}{(\alpha - \mu)^2}|\delta - k|, & if \quad |\delta - k| \geq l, \end{cases} \tag{4.17}$$

where $k = \dfrac{\mu + 1}{2}$ *and* $l = \dfrac{(\alpha - \mu)^2}{\lambda|2\alpha - \mu|}$.

Corollary 1 *For* $\alpha = 1$, $f \in \mathcal{U}(\lambda, \mu)$ *and* $0 < \lambda \leq 1$, $\mu > 0$. *Then*

$$|a_3 - \delta a_2^2| \leq \begin{cases} \dfrac{\lambda}{|2 - \mu|}, & if \quad |\delta - k| \leq l, \\ \\ \dfrac{\lambda^2}{(1 - \mu)^2}|\delta - k|, & if \quad |\delta - k| \geq l, \end{cases} \tag{4.18}$$

where $k = \dfrac{\mu + 1}{2}$ *and* $l = \dfrac{(1 - \mu)^2}{\lambda|2 - \mu|}$.

4.5 Characterization of the Class $\mathcal{U}(\lambda, \mu)$ in $\mathbb{C}^n$

In [62], Nikolas Samaris dealt with analogous problems involving more generalized classes not only for the functions in complex plane $\mathbb{C}$ but also for the functions in $\mathbb{C}^n$.

To present these problems we first define the following notations and the definitions. For $\alpha = (\alpha_1, \alpha_2, \dots \alpha_k) \in \mathbb{C}^k$, we denote by D_α the operator

$$1 + \alpha_1 z \frac{d}{dz} + \alpha_2 z^2 \frac{d^2}{dz^2} + \cdots + \alpha_k z^k \frac{d^k}{dz^k}$$

For $\mu > 0,\ \alpha \in \mathbb{C}^k$, Nikolas introduced the classes $\mathcal{S}_n(\mu, \alpha, \lambda)$ and $\mathcal{P}_n(\mu, \alpha, \lambda)$ by

$$\mathcal{S}_n(\mu, \alpha, \lambda) = \left\{ f \in \mathcal{A}_n : \left| D_\alpha \left[f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} \right] - 1 \right| < \lambda,\ z \in \Delta \right\}$$

$$\mathcal{P}_n(\mu, \alpha, \lambda) = \left\{ f \in \mathcal{A}_n : \left| D_\alpha \left[\left(f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} \right)' \right] \right| < \lambda,\ z \in \Delta \right\}$$

From this we have

$$\mathcal{B}(\mu, \lambda) \equiv \mathcal{S}_1(\mu, 0, \lambda),\ \bar{\mathcal{R}}(\alpha, \lambda) \equiv \mathcal{S}_1(1, \alpha, \lambda),\ P_n(\lambda) \equiv \mathcal{P}_n(1, 0, \lambda).$$

If $\alpha \in \mathbb{C}^k$, we denote by p_α the polynomial $p_\alpha(x) \equiv 1 + \alpha_1 x + \cdots + \alpha_k x(x-1)\cdots(x-k+1)$. Here the author consider the polynomials $p_\alpha(x)$ has nonpositive real zeros or $\alpha = 0$.

If $t = (t_1, t_2, \dots, t_k) \in (0,1)^k$ and $\rho_1, \rho_2, \dots, \rho_k$ are the zeros of a polynomial p_α we will denote by t_α the number ${t_1}^{-1/\rho_1}.{t_2}^{-1/\rho_2} \cdots {t_k}^{-1/\rho_k}$.
If $p_\alpha \equiv 1$, we define $t_\alpha \equiv 1$. If $n = 0, 1, 2, \dots$ we denote by $\mathcal{W}_n$ the class of analytic functions in Δ for which $|w(z)| \le |z|^n$ in Δ.

Theorem 52 *Let $\mathcal{S}_n(\mu, \alpha, \lambda)$ be the class, such that $(\mu + n)p_\alpha(n) - \lambda\mu > 0$. Let also*

$$I_n(\mu, \alpha) = \int_{[0,1]^{k+1}} s_n(t_\alpha, t_{k+1}^{1/\mu}),$$

$s_n(t_1, t_2) = \sup\{|w(t_1 z) - w(t_1 t_2 z)|,\ w \in \mathcal{W}_n,\ z \in \Delta\}$ and

$$s_n^*(\mu, \alpha, \lambda) = \frac{\lambda I_n(\mu, \alpha) p_\alpha(n)(\mu + n)}{p_\alpha(n)(\mu + n) - \lambda\mu}.$$

(i) *If $s_n^*(\mu, \alpha, \lambda) \le r$, then $\mathcal{S}_n(\mu, \alpha, \lambda) \subset T_r$.*

(ii) *If $\lambda \le {I_n}^*(\mu, \alpha)$, then $\mathcal{S}_n(\mu, \alpha, \lambda) \subset T_1$ where*

$${I_n}^*(\mu, \alpha) = \left[I_n(\mu, \alpha) + \frac{\mu}{p_\alpha(n)(\mu + n)} \right]^{-1}.$$

(iii) *It holds that*

$$s_n^*(\mu, \alpha, \lambda) \le \frac{\lambda(2\mu + n)}{p_\alpha(n)(\mu + n) - \lambda\mu}, \quad I_n(\mu, \alpha) \ge \frac{p_\alpha(n)(\mu + n)}{3\mu + n}.$$

Remark 6 The relation of part(i) of the above theorem is equivalent to

$$\left|\frac{zf'(z)}{f(z)} - \gamma\right| \leq |1-\gamma| + s_n^*(\mu,\alpha,\lambda), \quad \gamma \in \mathbb{C},\ z \in \Delta.$$

If $s_n^*(\mu,\alpha,\lambda) < 1$ and $t \geq (1+s_n^*(\mu,\alpha,\lambda))/2$, then $\mathcal{S}_n(\mu,\alpha,\lambda) \subset (\mathcal{S}^*)_t$, where

$$(\mathcal{S}^*)_t = \left\{ f \in \mathcal{A}_1 : \left|\frac{zf'(z)}{f(z)} - t\right| \leq t, \quad z \in \Delta \right\}.$$

For particular values of α, μ, and γ, many authors have obtained nonsharp results which we discussed in previous sections, but here these results are sharp. The last inclusion is a new one, which in addition, is sharp. There are several other sharp results for this class which are analogous of the class of functions in $\mathbb{C}$ variable studied earlier.

Theorem 53 *Let $\mathcal{P}_n(\mu,\alpha,\lambda)$ be the class such that $n(n+\mu)p_\alpha(n-1) - \lambda\mu > 0$. Let also*

$$\mathcal{T}_n(\mu,\alpha) = \int_{[0,1]^{k+2}} s_n(t_\alpha t_{k+1}, t_0^{1/\mu}) t_\alpha^{-1} t_{k+1}^{-1},$$

$$s_n^{**}(\mu,\alpha,\lambda) = \frac{\lambda\mathcal{T}_n(\mu,\alpha,\lambda)}{1-\lambda\mu/(n(n+\mu)p_\alpha(n-1))},$$

and

$$\mathcal{T}_n^*(\mu,\alpha) = \left[\mathcal{T}_n(\mu,\alpha) + \frac{\mu}{n(n+\mu)p_\alpha(n-1)}\right]^{-1}.$$

(i) *If $s_n^{**}(\mu,\alpha,\lambda) \leq r$, then $\mathcal{P}_n(\mu,\alpha,\lambda) \subset T_r$.*

(ii) *If $\lambda \leq \mathcal{T}_n{}^*(\mu,\alpha)$, then $\mathcal{P}_n(\mu,\alpha,\lambda) \subset T_1$*

(iii) $$s^{**}{}_n(\mu,\alpha,\lambda) \leq \frac{\lambda(2\mu+n) - 2\lambda\mu I^{**}{}_{n-1}(\alpha)p_\alpha(n-1)}{n(n+\mu)p_\alpha(n-1)-\lambda\mu},$$
$$\mathcal{T}^*{}_n(\mu,\alpha) \geq \frac{n(n+\mu)p_\alpha(n-1)}{3\mu+n-2\mu I^{**}{}_{n-1}(\alpha)p_\alpha(n-1)}.$$

For particular value of α and μ the results are in [46, 52], but these results are more strong.

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Chapter 5
Uniqueness of Entire Functions Sharing Certain Values with Derivatives

Indrajit Lahiri

5.1 Preliminaries

Let f be a nonconstant meromorphic function in the open complex plane $\mathbb{C}$. We denote by $n(r,a;f)$ the number of a-points of f in $|z| \leq r$ for $a \in \mathbb{C} \cup \{\infty\}$, where an a-point is counted according to its multiplicity. We put

$$N(r,a;f) = \int_0^r \frac{n(t,a;f) - n(0,a;f)}{t} dt + n(0,a;f)\log r$$

and call it the integrated counting function of a-points of f. Also we define

$$m(r,f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta$$

and call it the proximity function of f, where $\log^+ x = \log x$ if $x > 1$ and $\log^+ x = 0$ if $0 \leq x \leq 1$.

For $a \in \mathbb{C}$, we put $m(r,a;f) = m(r, \frac{1}{f-a})$ and for $a = \infty$, we put $N(r,\infty;f) = N(r,f)$ and $m(r,\infty;f) = m(r,f)$.

The sum $T(r,f) = m(r,f) + N(r,f)$ is called Nevanlinna characteristic function of f . The number

$$\rho = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}$$

is called the order of f.

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We denote by $\overline{n}(r,a;f)$ the number of distinct a-points of f in $|z| \le r$, and $\overline{N}(r,a;f)$ is defined in terms of $\overline{n}(r,a;f)$ in the usual way for $a \in \mathbb{C} \cup \{\infty\}$.

We denote by $S(r, f)$ any quantity satisfying $\dfrac{S(r,f)}{T(r,f)} \to 0$ as $r \to \infty$, possibly outside a set of finite linear measure.

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$, we say that f and g share the value a counting multiplicities (CM) if f and g have the same a-points with the same multiplicities. If we do not consider the multiplicities then f and g are said to share the value a ignoring multiplicities (IM). The investigation of the relation between two entire or meromorphic functions sharing certain values is the main theme of the uniqueness theory. The theory was initiated by the two results of R. Nevanlinna [33] known as: the five value theorem and the four value theorem.

5.2 Entire Functions Sharing Two Values with Their Derivatives

The uniqueness problem of entire and meromorphic functions sharing values with their derivatives is a special case of the uniqueness theory. The research on this problem was started by L. A. Rubel and C. C. Yang [35] with the following result.

Theorem 1 *Let f be a nonconstant entire function. If f and f' share the distinct finite values a and b CM, then $f \equiv f'$.*

Considering $f = e^{e^z} \int_0^z e^{-e^t}(1 - e^t)dt$, we see that $f' - 1 = e^z(f - 1)$ and so the hypothesis that f and f' share two finite values is essential for Theorem 1.

In 1979, Mues and Steinmetz [32] improved Theorem 1 by considering IM shared values and proved the following result.

Theorem 2 *Let f be a nonconstant entire function, a and b be distinct finite values. If f and f' share the values a and b IM, then $f \equiv f'$.*

In 1992, Zheng and Wang [47] considered shared functions and improved Theorem 1 in the following manner.

Theorem 3 *Let f be a nonconstant entire function, $a = a(z)$ and $b = b(z)$ be distinct meromorphic functions satisfying $T(r,a) + T(r,b) = S(r, f)$. If $f - a$ and $f' - a$ share 0 CM, and $f - b$ and $f' - b$ share 0 CM, then $f \equiv f'$.*

In 2000, Qiu [34] replaced CM sharing by IM sharing in Theorem 3 and proved the following result.

Theorem 4 *Let f be a nonconstant entire function, $a = a(z)(\not\equiv \infty)$ and $b = b(z)(\not\equiv \infty)$ be distinct meromorphic functions satisfying $T(r,a) + T(r,b) = S(r, f)$. If $f - a$ and $f' - a$ share 0 IM and $f - b$ and $f' - b$ share 0 IM, then $f \equiv f'$.*

In 1990, Yang [39] considered the problem of uniqueness of an entire function when it shares two values with its k-th derivative. He proved the following results.

Theorem 5 *Let f be a nonconstant entire function, $k(\geq 2)$ be a positive integer and $a(\neq 0)$ be a finite value. Suppose that 0 is a Picard exceptional value of f and $f^{(k)}$, and that a is an IM shared value of f and $f^{(k)}$. Then $f \equiv f^{(k)}$.*

Theorem 6 *Let f be a nonconstant entire function, $k(\geq 2)$ be a positive integer, a and b be two distinct finite values. If f and $f^{(k)}$ share values a and b CM, then $f \equiv f^{(k)}$.*

Frank [12] in 1991 proposed the following conjecture:

Frank's Conjecture If an entire function f shares two finite values IM with its k-th derivative $(k \geq 1)$, then $f \equiv f^{(k)}$.

Li and Yang [25] settled this conjecture affirmatively in 2000 as a consequence of the following theorem.

Theorem 7 *Let f be a nonconstant entire function and a, b be two distinct complex numbers. Let $g = a_0 f + a_1 f^{(1)} + \cdots + a_k f^{(k)} (k \geq 1)$ and $\phi = \frac{f'(f-g)}{(f-a)(f-b)}$, where $a_0, a_1, \ldots, a_k (a_k \neq 0)$ are constants. If f and g share a, b IM, then ϕ must be a constant satisfying $a_0\phi + a_1\phi_2 + \cdots + a_k\phi^k = 0$.*

In 1994, Gu [13] extended Theorem 6 by considering a linear differential polynomial.

Theorem 8 *Let f be a nonconstant entire function, and let a and b be distinct finite values, $P(f) = f^{(n)} + a_1(z) f^{(n-1)} + \cdots + a_n(z) f(z)$, where $a_j(z)$ $(j = 1, 2, \ldots, n)$ are entire functions satisfying $T(r, a_j) = S(r, f)$ for $j = 1, 2, \ldots, n$. If f and $P(f)$ share finite values a and b CM, and $a + b \neq 0$ or $a_n(z) \not\equiv -1$, then $f \equiv P(f)$.*

Li and Yang [24] further generalised Theorem 6 and proved the following result.

Theorem 9 *Let f be a nonconstant entire function and $L(f) = b_{-1} + \sum_{j=0}^{n} b_j f^{(j)}$, where $b_n \not\equiv 0$ and $b_j (j = -1, 0, 1, \ldots, n)$ are meromorphic functions satisfying $T(r, b_j) = S(r, f)$ for $j = -1, 0, 1, \ldots, n$. Let a_1 and a_2 be two distinct finite values. If f and $L(f)$ share a_1 CM and a_2 IM, then $f \equiv L(f)$ or f and $L(f)$ have the following expressions*

$$f = a_2 + (a_1 - a_2)(1 - e^{\alpha})^2$$

and

$$L(f) = 2a_2 - a_1 + (a_1 - a_2)e^{\alpha},$$

where α is an entire function.

We now require the following definition.

Definition 1 *Let f and g be nonconstant meromorphic functions and a be a complex number. We denote by $\overline{N}_E(r, a; f, g)$ the reduced counting function of those common zeros of $f - a$ and $g - a$ having the same multiplicities. Also, we denote by $\overline{N}_0(r, a; f, g)$ the reduced counting function of the common zeros of $f - a$ and $g - a$.*

If $\overline{N}(r,a;f) = \overline{N}_0(r,a;f,g)+S(r,f)$ *and* $\overline{N}(r,a;g) = \overline{N}_0(r,a;f,g)+S(r,g)$, *we say that* f *and* g *share the value* a *IM*.*

If $\overline{N}(r,a;f) = \overline{N}_E(r,a;f,g)+S(r,f)$ *and* $\overline{N}(r,a;g) = \overline{N}_E(r,a;f,g)+S(r,g)$, *we say that* f *and* g *share the value* a *CM*.*

Yang and Li [42] extended Theorem 9 to meromorphic functions with few poles in the following manner.

Theorem 10 *Let* f *be a nonconstant meromorphic function satisfying* $\overline{N}(r,f) = S(r,f)$ *and* $L(f) = b_{-1} + \sum_{j=0}^{n} b_j f^{(j)}$, *where* $b_n \not\equiv 0$ *and* $b_j (j = -1, 0, 1, \ldots, n)$ *are meromorphic functions satisfying* $T(r,b_j) = S(r,f)$ *for* $j = -1, 0, 1, \ldots, n$. *Let* a_1 *and* a_2 *be two distinct meromorphic functions such that* $T(r,a_j) = S(r,f)$ *for* $j = 1, 2$. *If* f *and* $L(f)$ *share* a_1 *CM* and* a_2 *IM*, then* $f \equiv L(f)$ *or* f *and* $L(f)$ *have the following expressions*

$$f = a_2 + (a_1 - a_2)(1 - \alpha)^2$$

and

$$L(f) = 2a_2 - a_1 + (a_1 - a_2)\alpha,$$

where α *is a meromorphic function satisfying* $\overline{N}(r,\infty;\alpha) + \overline{N}(r,0;\alpha) = S(r,f)$.

In order to state the next theorem we need the following definition due to Mues [31].

Definition 2 *Let* f *be a nonconstant meromorphic function and* $L(f)$ *be a linear differential polynomial generated by* f. *We now define for* $a \in \mathbb{C}$, $\tau(a) = \tau(a;f,L(f)) = \liminf_{r\to\infty} \dfrac{\overline{N}_0\left(r,\frac{1}{f-a}\right)}{\overline{N}\left(r,\frac{1}{f-a}\right)}$ *if* $\overline{N}\left(r,\dfrac{1}{f-a}\right) \not\equiv 0$ *and* $\tau(a) = \tau(a;f,L(f)) = 1$ *if* $\overline{N}\left(r,\frac{1}{f-a}\right) \equiv 0$, where $\overline{N}_0\left(r,\dfrac{1}{f-a}\right)$ *denotes the reduced counting function of those* a*-points of* f *and* $L(f)$ *having the same multiplicities.*

Wang [36] improved Theorem 9 in the following manner.

Theorem 11 *Let* f *be a nonconstant entire function and* $L(f) = b_{-1} + \sum_{j=0}^{n} b_j f^{(j)}$, *where* $b_n \not\equiv 0$ *and* $b_j (j = -1, 0, 1, \ldots, n)$ *are meromorphic functions satisfying* $T(r,b_j) = S(r,f)$ *for* $j = -1, 0, 1, \ldots, n$. *If* f *and* $L(f)$ *share two complex numbers* a_1 *and* a_2 *IM and if* $\tau(a_1) > \dfrac{n+2}{n+3}$, *where* n *is the highest order of the derivative involved in* $L(f)$, *then either* $f \equiv L(f)$ *or* f *and* $L(f)$ *have the following expressions*

$$f = a_2 + (a_1 - a_2)(1 - e^{\alpha})^2$$

and

$$L(f) = 2a_2 - a_1 + (a_1 - a_2)e^{\alpha},$$

where α *is an entire function.*

In the same paper J. P. Wang conjectured that the condition $\tau(a_1) > \dfrac{n+2}{n+3}$ might be replaced by $\tau(a_1) > \dfrac{1}{2}$.

In response to this conjecture in 2005 Yang and Li [42] proved the following result.

Theorem 12 *Let* f *be a nonconstant meromorphic function satisfying* $\overline{N}(r, f) = S(r, f)$ *and* $g = L(f) = b_{-1} + \sum_{j=0}^{n} b_j f^{(j)}$, *where* $b_n \not\equiv 0$ *and* $b_j (j = -1, 0, 1, \ldots, n)$ *are meromorphic functions satisfying* $T(r, b_j) = S(r, f)$ *for* $j = -1, 0, 1, \ldots, n$. *Let* a_1 *and* a_2 *be two distinct finite values. If* $\max\{\tau(a_1), \tau(a_2)\} > \dfrac{1}{2}$, *then* f *and* g *assume one of the following cases:*

(i) $f \equiv g$;
(ii) $f = a_2 + (a_1 - a_2)(1 - h)^2$ *and* $g = 2a_2 - a_1 + (a_1 - a_2)h$, *where* h *is a meromorphic function satisfying* $\overline{N}(r, \infty; h) + \overline{N}(r, 0; h) = S(r, f)$;
(iii) $f = a_1 + (a_2 - a_1)(1 - h)^2$ *and* $g = 2a_1 - a_2 + (a_2 - a_1)h$, *where* h *is a meromorphic function satisfying* $\overline{N}(r, \infty; h) + \overline{N}(r, 0; h) = S(r, f)$;
(iv) there exists an integer $k \geq 3$ *such that* $k\alpha = \phi$, *where* $\alpha = \dfrac{f' - g'}{f - g} - \dfrac{g'}{g - a_1} - \dfrac{g'}{g - a_2}$ *and* $\phi = \dfrac{f'(f - g)}{(f - a_1)(f - a_2)}$.

If, further, $\max\{\tau(a_1), \tau(a_2)\} > \dfrac{2}{3}$, *then one of (i) – (iii) holds.*

When the linear differential polynomial $g = L(f)$ involves only the first derivative of f, then Yang and Li [42] obtained the following theorem.

Theorem 13 *Suppose that* f *is a nonconstant meromorphic function satisfying* $\overline{N}(r, f) = S(r, f)$ *and* $g = L(f) = b_{-1} + b_0 f + b_1 f'$, *where* $b_j (j = -1, 0, 1)$ *are meromorphic functions satisfying* $T(r, b_j) = S(r, f)$ *for* $j = -1, 0, 1$. *Let* a_1 *and* a_2 *be two distinct meromorphic functions such that* $T(r, a_j) = S(r, f)$ *for* $j = 1, 2$. *If* f *and* g *share* a_1 *and* a_2 *IM*, then one of the following cases holds:*

(i) $f \equiv g$;
(ii) $f = a_2 + (a_1 - a_2)(1 - h)^2$ *and* $g = 2a_2 - a_1 + (a_1 - a_2)h$;
(iii) $f = a_1 + (a_2 - a_1)(1 - h)^2$ *and* $g = 2a_1 - a_2 + (a_2 - a_1)h$;
(iv) $f = \dfrac{a_1 + a_2}{2} + \dfrac{(a_2 - a_1)\left(h + \frac{1}{h}\right)}{4}$ *and* $g = \dfrac{a_1 + a_2}{2} + \dfrac{(a_2 - a_1)h}{2}$,

where h *is a meromorphic function satisfying* $\overline{N}(r, 0; h) + \overline{N}(r, \infty; h) = S(r, f)$.

5.3 f, f' and f'' Sharing One Value

Jank et al. [16] considered the problem of uniqueness of entire and meromorphic functions sharing a value with two of its derivatives. Their results can be stated as follows.

Theorem 14 *Let f be a nonconstant entire function and $a \neq 0$ be a finite constant. If f, f' share the value a IM and $f''(z) = a$ whenever $f(z) = a$, then $f \equiv f'$.*

Theorem 15 *Let f be a nonconstant meromorphic function and $a \neq 0$ be a finite constant. If f, f' and f'' share the value a CM, then $f \equiv f'$.*

Extending Theorem 14, Chang and Fang [9] proved the following two results.

Theorem 16 *Let f be a nonconstant entire function and a, c be two nonzero constants. If $f'(z) = a$ whenever $f(z) = a$ and $f''(z) = c$ whenever $f'(z) = a$, then either $f(z) = A\exp\left(\frac{cz}{a}\right) + \frac{ac - a^2}{c}$ or $f(z) = A\exp\left(\frac{cz}{a}\right) + a$, where A is a nonzero constant.*

Theorem 17 *Let f be a nonconstant entire function. If $f(z) - z$ and $f'(z) - z$ share 0 IM and $f''(z) = z$ whenever $f'(z) = z$, then $f \equiv f'$.*

We now require the following notation.

Definition 3 *Let f be a meromorphic function in $\mathbb{C}$ and a meromorphic function $a = a(z)$ is called a small function of f if $T(r, a) = S(r, f)$. We now denote by $E(a; f)$ the set of distinct zeros of $f - a$, where a is a small function of f.*

Improving Theorem 17 following result is recently proved by Lahiri and Ghosh [20].

Theorem 18 *Let f be a nonconstant entire function and $a = \alpha z + \beta$, where $\alpha(\neq 0)$ and β are constants. If $E(a; f) \subset E(a; f')$ and $E(a; f') \subset E(a; f'')$, then either $f = Ae^z$ or $f = \alpha z + \beta + (\alpha z + \beta - 2\alpha)\exp\left\{\frac{\alpha z + \beta - 2\alpha}{\alpha}\right\}$.*

In 2004 Wang and Yi [38] also generalised Theorem 14 and proved the following two results.

Theorem 19 *Let f be a nonconstant entire function and $k(\geq 2)$ be a positive integer. If $f(z) - z$ and $f'(z) - z$ share 0 CM and if $f^{(k)}(z) = z$ whenever $f(z) = z$, then $f \equiv f'$.*

Theorem 20 *Let f be a nonconstant entire function and $k(\geq 2)$ be a positive integer. If f, f' and $f^{(k)}$ have the same fixed points with the same multiplicities, then $f \equiv f'$.*

In 1995 Zhong [48] considered the higher order derivative of an entire function and proved the following theorem.

Theorem 21 *Let f be a nonconstant entire function, and let n be a positive integer. If f and f' share a finite nonzero value a CM and if $f^{(n)}(z) = f^{(n+1)}(z) = a$ whenever $f(z) = a$, then $f \equiv f^{(n)}$.*

In this direction Li and Yang [26] also proved the following theorem.

Theorem 22 *Let f be a nonconstant entire function, a be a finite nonzero constant and let n be a positive integer. If f, $f^{(n)}$ and $f^{(n+1)}$ share the value a CM, then $f \equiv f'$.*

In the above two theorems we see that two consecutive derivatives are involved. However in 2003 Wang and Yi [37] proved the following two results, in the first of which they did not consider two consecutive derivatives of f.

Theorem 23 *Let f be a nonconstant entire function, a be a finite nonzero constant, and n, $m(>n)$ be positive integers. If f and f' share the value a CM and if $f^{(n)}(z) = f^{(m)}(z) = a$ whenever $f(z) = a$, then*

$$f = A\exp(\lambda z) + a - \frac{a}{\lambda},$$

where $A(\neq 0)$ and λ are constants satisfying $\lambda^{n-1} = \lambda^{m-1} = 1$.

Theorem 24 *Let f be a nonconstant entire function, a be a finite nonzero constant, and $n(\geq 2)$ be a positive integer. If f and f' share the value a CM and if $f^{(n)}(z) = a$ whenever $f(z) = a$, then*

$$f = A\exp(\lambda z) + a - \frac{a}{\lambda},$$

where $A(\neq 0)$ and λ are constants satisfying $\lambda^{n-1} = 1$.

Recently, Lu and Xu [30] used the theory of normal families to replace CM sharing by IM sharing in Theorem 24 but at the cost of an additional condition as we see in the following theorem.

Theorem 25 *Let f be a nonconstant entire function, a be a finite nonzero constant, and $n(\geq 2)$ be a positive integer. If f and f' share the value a IM and if $f^{(n)}(z) = a$ whenever $f(z) = a$ and if there exists $z_0 \in \mathbb{C}$ satisfying $f^{(n)}(z_0) = f'(z_0) = b$, where $b \neq a$ is a constant, then*

$$f = A\exp(\lambda z) + a - \frac{a}{\lambda},$$

where $A(\neq 0)$ and λ are constants satisfying $\lambda^{n-1} = 1$.

However, in the same year Chang and Fang [11] were able to replace the CM sharing by IM sharing in Theorem 24 without any additional hypothesis. Their result can be stated as follows.

Theorem 26 *Let f be a nonconstant entire function, a be a finite nonzero complex number and $k(\geq 2)$ be a positive integer. Suppose that $f'(z) = a$ whenever $f(z) = a$ and $f^{(k)}(z) = a$ whenever $f'(z) = a$. Then either $f = c\exp\{\lambda z\}$ or $f = c\exp\{\lambda z\} + \frac{a(\lambda-1)}{\lambda}$, where c, λ are nonzero constants with $\lambda^{k-1} = 1$.*

Let us now see another way of sharing values by an entire function and its derivative. In 1995, Zhong [48] introduced the following notion.

Definition 4 *Let f and g be two nonconstant meromorphic functions defined in $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$, we put $A = \{E(a; f)\backslash E(a; g)\} \cup \{E(a; g)\backslash E(a; f)\}$. For $B \subset \mathbb{C}$,* *we denote by* $N_B(r, a; f)$ *the counting function (counted with multiplicities) of those* a*-points of* f *which belong to the set* B.

The functions f *and* g *are said to share a value* a *IMN if* $N_A(r, a; f) = S(r, f)$ *and* $N_A(r, a; g) = S(r, g)$*. If, in addition, the common* a*-points of* f *and* g *have the same multiplicities, then* f *and* g *are said to share a value* a *CMN.*

Using this idea Zhong [48] proved the following uniqueness theorem.

Theorem 27 *Let* f *be a nonconstant entire function,* f, $f^{(n)} (n \geq 1)$ *share a value* $a(\neq 0, \infty)$ *IMN and* $f^{(1)}(z) = f^{(n+1)}(z) = a$ *when* $f(z) = a$*. If for any set* E *of finite linear measure*

$$\lim_{r\to\infty, r\notin E} \frac{N_A\left(r, a; f^{(n+1)}\right)}{N(r, a; f^{(n+1)})} \neq \frac{1}{2}, \tag{5.1}$$

then $f \equiv f^{(n)}$*, where* $A = E(a; f)\Delta E(a; f^{(n+1)})$.

Zhong left the possibility of removing the condition (5.1) as open. In 1997 Yang [40] indeed removed the condition (5.1) and proved the following result.

Theorem 28 *Let* f *be a nonconstant entire function. If* f *and* $f^{(n)} (n \geq 1)$ *share the value* $a(\neq 0, \infty)$ *IMN and* $f^{(1)}(z) = f^{(n+1)}(z) = a$ *when* $f = a$*, then* $f = ce^z$*, where* c *is a nonzero constant.*

Recently a result of Zhong (Theorem 21) is improved in the following manner by Lahiri and Ghosh [21].

Theorem 29 *Let* f *be a nonconstant entire function and* a*,* b *be two nonzero finite constants. Suppose further that* $A = E(a; f)\backslash E(a; f^{(1)})$ *and* $B = E(a; f^{(1)})\backslash \{E(a; f^{(n)}) \cap E(b; f^{(n+1)})\}$ *for* $n \geq 1$*. If each common zero of* $f - a$ *and* $f^{(1)} - a$ *has the same multiplicity and* $N_A(r, a; f) + N_B(r, a; f^{(1)}) = S(r, f)$*, then* $f = \lambda \exp\left\{\frac{bz}{a}\right\} + \frac{ab-b^2}{b}$ *or* $f = \lambda \exp\left\{\frac{bz}{a}\right\} + a$*, where* $\lambda(\neq 0)$ *is a constant.*

The theorem of L. Z. Yang (Theorem 28) is also improved in the following way [21].

Theorem 30 *Let* f *be a nonconstant entire function and* a*,* b *be two nonzero finite constants. Suppose that* $A = E(a; f)\backslash E(a; f^{(n)})$ *and* $B = E(a; f^{(n)})\backslash \{E(a; f^{(1)}) \cap E(b; f^{(n+1)})\}$ *for* $n \geq 1$*. If* $N_A(r, a; f) + N_B(r, a; f^{(n)}) = S(r, f)$*, then either* $f = \lambda \exp\{z\}$ *or* $f = \lambda \exp\{z\} + a$*, where* $\lambda(\neq 0)$ *is a constant.*

Remark 1 Theorem 29 will look smarter if one can get rid of the hypothesis: each common zero of $f - a$ *and* $f^{(1)} - a$ *has the same multiplicity.*

If $A \cup B = \phi$*, then by Theorems 16 and 26 one can remove this hypothesis. Again if* $A \cup B \neq \phi$ *and* $n = 1$*, then also this hypothesis is needless for Theorem 29 as is evident from its proof. So if* $A \cup B \neq \phi$ *and* $n \geq 2$*, then it is an interesting open problem to investigate the possibility of removing the said hypothesis from Theorem 29.*

Yang [41] gave a generalisation of a result of Jank-Mues-Volkmann (Theorem 14) in the following manner.

Theorem 31 *Let* f *be a nonconstant entire function. If* f *and* $f^{(n)}$ *share a finite value* $a(\neq 0)$ *IM and* $E(a; f) \subset E(a; f^{(1)}) \cap E(a; f^{(n+1)})$*, then* $f \equiv f^{(n)}$.

In 1999, Li [23] further extended Theorem 31 by considering a linear differential polynomial instead of the derivative. The result of P. Li may be stated as follows.

Theorem 32 *Let f be a nonconstant entire function and $L = a_1 f^{(1)} + a_2 f^{(2)} + \cdots + a_n f^{(n)}$, where $a_1, a_2, \ldots a_n (\neq 0)$ are constants. If f and L share a finite nonzero value a IM, $E(a; f) \subset E(a; f^{(1)}) \cap E(a; L^{(1)})$ and $\sum_{k=1}^{n} 2^k a_k \neq 0$ or $\sum_{k=1}^{n} a_k \neq -1$, then $f \equiv f^{(1)} \equiv L$.*

Considering the following example P. Li showed that the condition $\sum_{k=1}^{n} 2^k a_k \neq 0$ or $\sum_{k=1}^{n} a_k \neq -1$ is essential.

Example 1 Let $a_1, a_2, \ldots, a_n$ be constants satisfying $\sum_{k=1}^{n} 2^k a_k = 0$ and $\sum_{k=1}^{n} a_k = -1$. We put $f = a + \frac{e^{2z}}{a} - e^z$, where $a(\neq 0, \infty)$ is a constant. Then $f^{(k)} = \frac{2^k e^{2z}}{a} - e^z$ and $L(z) = e^z$. Hence f, L share the value a CM and $E(a; f) \subset E(a; f^{(1)}) \cap E(a; L^{(1)})$ but $f \not\equiv f^{(1)}$.

Seeing the result and example of Li, Lahiri et al. [22] raised the following questions: *Is it possible to further relax the nature of value sharing in Theorem 32? Is the example of P. Li only of its kind?*

In order to answer this question the following two results are proved in [22].

Theorem 33 *Let f be a nonconstant entire function in $\mathbb{C}$, a be a finite nonzero complex number and $L = a_1 f^{(1)} + a_2 f^{(2)} + \cdots + a_n f^{(n)}$, where $a_1, a_2, \ldots a_n (\neq 0)$ are constants.*

Further suppose that $E_{1)}(a; f) \subset E(a; f^{(1)})$ and $N_A(r, a; f) + N_B(r, a; L) = S(r, f)$, where $A = E(a; f) \setminus E(a; L)$, $B = E(a; L) \setminus \{E(a; f^{(1)}) \cap E(a; L^{(1)})\}$ and $E_{1)}(a; f)$ are the set of simple a-points of f. Then one of the following cases holds:

- *(i) $f = a + \alpha e^z$ and $L = \alpha e^z$, where α is a nonzero constant;*
- *(ii) $f = L = \alpha e^z$, where α is a nonzero constant;*
- *(iii) $f = a + \frac{\alpha^2}{a} e^{2z} - \alpha e^z$ and $L = \alpha e^z$, where $\sum_{k=1}^{n} 2^k a_k = 0$, $\sum_{k=1}^{n} a_k = -1$ and α is a nonzero constant.*

Theorem 34 *Let f be a nonconstant entire function in $\mathbb{C}$, a be a finite nonzero complex number and $L = a_1 f^{(1)} + a_2 f^{(2)} + \cdots + a_n f^{(n)}$, where $a_1, a_2, \ldots a_n (\neq 0)$ are constants.*

Further, let $N_A(r, a; f) + N_B(r, a; L) = S(r, f)$, where $A = E(a; f) \setminus E(a; L)$, $B = E(a; L) \setminus \{E(a; f^{(1)}) \cap E(a; L^{(1)})\}$. If $f \not\equiv L$, then one of the following cases holds:

(i) $f = a + \alpha e^z$ *and* $L = \alpha e^z$*, where* α *is a nonzero constant;*

(ii) $f = a + \dfrac{\alpha^2}{a} e^{2z} - \alpha e^z$ *and* $L = \alpha e^z$*, where* $\sum_{k=1}^{n} 2^k a_k = 0$*,* $\sum_{k=1}^{n} a_k = -1$ *and* α *is a nonzero constant.*

In connection to a result of Jank-Mues-Volkmann (Theorem 15) following question was asked [14, 15]: *Let* f *be a nonconstant meromorphic function, a be a nonzero finite complex number and k, m be two distinct positive integers. Suppose that* f*,* $f^{(k)}$ *and* $f^{(m)}$ *share a CM. Can we get* $f \equiv f^{(k)}$*?*

Following example shows that the answer to the above question is, in general, negative [40].

Example 2 Let k, m be positive integers satisfying $m > k + 1$, b be a nonzero constant such that $b^k = b^m \neq 1$ and $a = b^k$. Put $f = e^{bz} + a - 1$. Then f, $f^{(k)}$ and $f^{(m)}$ share a CM but $f \not\equiv f^{(k)}$. However, we see that $f^{(k)} \equiv f^{(m)}$.

In view of this example Chang and Fang [10] asked: *Let* f *be a nonconstant meromorphic function, a be a nonzero finite complex number and k, m be two distinct positive integers. Suppose that* f*,* $f^{(k)}$ *and* $f^{(m)}$ *share a CM. Can we get* $f^{(k)} \equiv f^{(m)}$*?*

To answer this question affirmatively Chang and Fang [10] proved the following results.

Theorem 35 *Let* f *be a nonconstant entire function, a be a finite complex number, k and m be two distinct positive integers, and* (k, m) *be the greatest common divisor of k and m. If* f*,* $f^{(k)}$ *and* $f^{(m)}$ *share a CM, then*

$$f(z) = \left(1 - \frac{1}{c}\right) a + \sum_{j=1}^{q} C_j e^{\lambda_j z},$$

where q is a positive integer with $q \leq (k, m)$*, c and* $C_j, 1 \leq j \leq q$*, are nonzero constants, and* $\lambda_j, 1 \leq j \leq q$*, are distinct nonzero constants satisfying* $(\lambda_j)^k = (\lambda_j)^m = c$ *for* $a \neq 0$ *and* $(\lambda_j)^k = c$*,* $(\lambda_j)^m = d$ *for* $a = 0$*, where d is a nonzero constant.*

Theorem 36 *Let* f *be a nonconstant meromorphic function, a be a nonzero finite complex number and k, m be two distinct positive integers. Suppose that* f*,* $f^{(k)}$ *and* $f^{(m)}$ *share a CM. Then* $f^{(k)} \equiv f^{(m)}$*.*

5.4 Entire and Meromorphic Functions Sharing a Single Value with Their Derivatives

We start this section with following definition.

Definition 5 Let f be a nonconstant meromorphic function, the hyper-order of f, denoted by $\rho_2(f)$, is defined by

$$\rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

Brück [8] proved the following results.

Theorem 37 *Let f be a nonconstant entire function satisfying $\rho_2(f) < \infty$, where $\rho_2(f)$ is not a positive integer. If f and f' share the value 0 CM, then $f \equiv cf'$ for some nonzero constant c.*

Theorem 38 *Let f be a nonconstant entire function. If f and f' share the value 1 CM and if $N(r,0;f') = S(r,f)$, then $f-1 \equiv c(f'-1)$ for some constant $c(\neq 0)$.*

Zhang [44] extended Theorem 38 to meromorphic functions and proved the following results.

Theorem 39 *Let f be a nonconstant meromorphic function. If f and f' share 1 CM and if*

$$\overline{N}(r,\infty;f) + N(r,0;f') < \{\lambda + o(1)\}T(r,f')$$

for some constant $\lambda \in \left(0,\frac{1}{2}\right)$, then $\frac{f'-1}{f-1}$ is a nonzero constant.

Theorem 40 *Let f be a nonconstant meromorphic function. If f and $f^{(k)}$ share 1 CM and if*

$$2\overline{N}(r,\infty;f) + \overline{N}(r,0;f') + N(r,0;f^{(k)}) < \{\lambda + o(1)\}\, T\left(r,f^{(k)}\right)$$

for some constant $\lambda \in (0,1)$, then $\frac{f^{(k)}-1}{f-1}$ is a nonzero constant.

A meromorphic function a is called small with respect to a meromorphic function f if $T(r,a) = S(r,f)$. In 2003, Yu [43] considered the uniqueness problem of an entire or a meromorphic function when it shares one small function with its derivative. K. W. Yu proved the following two results.

Theorem 41 *Let f be a nonconstant entire function and $a(\not\equiv 0,\infty)$ be a small function of f. If $f-a$ and $f^{(k)}-a$ share the value 0 CM and $\delta(0;f) > \frac{3}{4}$, then $f \equiv f^{(k)}$, where k is a positive integer.*

Theorem 42 *Let f be a nonconstant non-entire meromorphic function and $a(\not\equiv 0,\infty)$ be a small function of f. If*

(i) f and a have no common pole,
(ii) $f-a$ and $f^{(k)}-a$ share the value 0 CM,
(iii) $4\delta(0;f) + 2(8+k)\Theta(\infty;f) > 19+2k$,

then $f \equiv f^{(k)}$, where k is a positive integer. Further if k is an odd integer, then the hypothesis (i) can be dropped.

In his paper K. W. Yu asked the following open questions:

1. Can CM shared value be replaced by an IM shared value?
2. Can the condition $\delta(0; f) > \frac{3}{4}$ of Theorem 41 be further relaxed?
3. Can the condition (iii) of Theorem 42 be further relaxed?
4. Can, in general, the condition (i) of Theorem 42 be dropped?

On these questions a considerable amount of work has been done. In connection to the second question of K. W. Yu, in 2004 Liu and Gu [27] proved the following result.

Theorem 43 *Let $k \geq 1$ and f be a nonconstant meromorphic function, $a(\not\equiv 0, \infty)$ be a small function of f. If $f - a$ and $f^{(k)} - a$ share the value 0 CM, $f^{(k)}$ and a do not have any common pole of same multiplicity and $2\delta(0; f) + 4\Theta(\infty; f) > 5$, then $f \equiv f^{(k)}$.*

If f is entire, then Theorem 43 reduces to the following.

Theorem 44 *Let $k \geq 1$ and f be a nonconstant entire function, $a(\not\equiv 0, \infty)$ be a small function of f. If $f - a$ and $f^{(k)} - a$ share the value 0 CM and $\delta(0; f) > \frac{1}{2}$, then $f \equiv f^{(k)}$.*

In order to state the next results we require the following definition [17, 18].

Definition 6 *Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.*

The definition implies that if f, g share a value a with weight k then z_o is a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\leq k)$ and z_o is a zero of $f - a$ with multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly, if f, g share (a, k) then f, g share (a, p) for all integers p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Following definition is well-known.

Definition 7 *We denote by $\delta_p(a; f)$ the quantity*

$$\delta_p(a; f) = 1 - \limsup_{r \to \infty} \frac{N_p(r, a; f)}{T(r, f)},$$

where p is a positive integer and $N_p(r, a; f)$ denotes the counting function of a-points of f, an a-point of multiplicity m being counted m times if $m \leq p$ and p times if $m > p$.

In 2004 Lahiri and Sarkar [19] used the above idea to prove the following two results in response to the first question of K. W. Yu for a shared value and also to prove a result for a shared small function.

Theorem 45 *Let f be a nonconstant meromorphic function and k be a positive integer. If f, $f^{(k)}$ share $(1,2)$ and*

$$2\overline{N}(r,\infty;f)+N_2(r,0;f^{(k)})+N_2(r,0;f')<\{\lambda+o(1)\}T(r,f^{(k)})$$

for $r\in I$, where $0<\lambda<1$ and I is a set of infinite linear measure, then $\dfrac{f^{(k)}-1}{f-1}$ is a nonzero constant.

Theorem 46 *Let f be a nonconstant meromorphic function and k be a positive integer. If f, $f^{(k)}$ share $(1,1)$ and*

$$2\overline{N}(r,\infty;f)+N_2(r,0;f^{(k)})+2\overline{N}(r,0;f')<\{\lambda+o(1)\}T(r,f^{(k)})$$

for $r\in I$, where $0<\lambda<1$ and I is a set of infinite linear measure, then $\dfrac{f^{(k)}-1}{f-1}$ is a nonzero constant.

Theorem 47 *Let f be a nonconstant meromorphic function and k be a positive integer. Let $a(\neq 0,\infty)$ be a small function of f. If*

(i) a has no zero (pole) which is also a zero (pole) of f or $f^{(k)}$ with the same multiplicity,
(ii) $f-a$ and $f^{(k)}-a$ share $(0,2)$,
(iii) $2\delta_{2+k}(0;f)+(4+k)\Theta(\infty;f)>5+k$,

then $f\equiv f^{(k)}$.

Liu and Yang [29] considered an IM shared value by a meromorphic function and its derivative. Their results can be stated as follows:

Theorem 48 *Let f be a nonconstant meromorphic function. If f and f' share the value 1 IM and if*

$$\overline{N}(r,\infty;f)+\overline{N}(r,0;f')<\{\lambda+o(1)\}T(r,f'),$$

where $0<\lambda<\dfrac{1}{4}$, then $\dfrac{f'-1}{f-1}$ is a nonzero constant.

Theorem 49 *Let f be a nonconstant meromorphic function, k be a positive integer. If f and $f^{(k)}$ share the value 1 IM and if*

$$(3k+6)\overline{N}(r,\infty;f)+5\,N(r,0;f)<\{\lambda+o(1)\}T(r,f^{(k)}),$$

where $0<\lambda<1$, then $\dfrac{f^{(k)}-1}{f-1}$ is a nonzero constant.

Considering a small function Zhang [45] proved the following theorems which improve the results of Lahiri and Sarkar [19].

Theorem 50 *Let f be a nonconstant meromorphic function and $k(\geq 1)$, $l(\geq 0)$ be integers. Also let for a small function $a(\not\equiv 0,\infty)$ of f, $f-a$ and $f^{(k)}-a$ share $(0,l)$. If one of the following holds, then $\dfrac{f^{(k)}-a}{f-a}$ is a nonzero constant:*

(i) $l \geq 2$ *and* $2\overline{N}(r,\infty;f) + N_2(r,0;f^{(k)}) + N_2(r,0;(f/a)') < \{\lambda + o(1)\} T(r,f^{(k)})$,
(ii) $l = 1$ *and* $2\overline{N}(r,\infty;f) + N_2(r,0;f^{(k)}) + 2\overline{N}(r,0;(f/a)') < \{\lambda + o(1)\} T(r,f^{(k)})$,
(iii) $l = 0$ *and* $4\overline{N}(r,\infty;f) + 3N_2(r,0;f^{(k)}) + 2\overline{N}(r,0;(f/a)') < \{\lambda + o(1)\} T(r,f^{(k)})$,

for $r \in I$, *where* $0 < \lambda < 1$ *and* I *is a set of infinite linear measure.*

Theorem 51 *Let* f *be a nonconstant meromorphic function and* $k(\geq 1)$, $l(\geq 0)$ *be integers. Also let for a small function* $a(\not\equiv 0,\infty)$ *of* f, $f-a$ *and* $f^{(k)}-a$ *share* $(0,l)$. *If one of the following holds, then* $f \equiv f^{(k)}$:

(i) $l \geq 2$ *and* $(3+k)\Theta(\infty;f) + 2\delta_{2+k}(0;f) > k+4$,
(ii) $l = 1$ *and* $(4+k)\Theta(\infty;f) + 3\delta_{2+k}(0;f) > k+6$,
(iii) $l = 0$ *and* $(6+2k)\Theta(\infty;f) + 5\delta_{2+k}(0;f) > 2k+10$.

In 2007 two improvements were made over the results of Q. C. Zhang (Theorems 50 and 51) by J. L. Zhang, L. Z. Yang and by A. Banerjee. First, we state the result of J. L. Zhang and L. Z. Yang [46].

Theorem 52 *Let* f *be a nonconstant meromorphic function,* $k(\geq 1)$ *and* $l(\geq 0)$ *be two integers. Let* $L(f) = f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_0 f$, *where* $a_0, a_1, \ldots, a_{k-1}$ *are small functions of* f. *If for a small function* $a(\not\equiv 0,\infty)$, $f-a$ *and* $L(f)-a$ *share* $(0,l)$, *then* $f = L(f)$ *provided one of the following holds:*

(i) $l \geq 2$ *and* $\delta_{2+k}(0;f) + \delta_2(0;f) + 3\Theta(\infty;f) + \delta(a;f) > 4$,
(ii) $l = 1$ *and* $\delta_{2+k}(0;f) + \delta_2(0;f) + \frac{1}{2}\delta_{1+k}(0;f) + \frac{k+7}{2}\Theta(\infty;f) + \delta(a;f) > \frac{k}{2}+5$,
(iii) $l = 0$ *and* $\delta_{2+k}(0;f) + 2\delta_{1+k}(0;f) + \delta_2(0;f) + \Theta(0;f) + (6+2k)\Theta(\infty;f) + \delta(a;f) > 2k+10$.

We now require the following definition.

Definition 8 *We denote by* $N(r,a;f \mid\geq k)$ $(\overline{N}(r,a;f \mid\geq k))$ *the counting function (reduced counting function) of those* a*-points of* f *whose multiplicities are not less than* k.

In a similar manner we define $N(r,a;f \mid\leq k)$ *and* $\overline{N}(r,a;f \mid\leq k)$.

Banerjee [7] proved the following two results in the same direction.

Theorem 53 *Let* f *be a nonconstant meromorphic function and* $k(\geq 1)$, $l(\geq 0)$ *be integers. If* $f-a$ *and* $f^k - a$ *share* $(0,l)$ *for some nonconstant small function* a *of* f, *then* $\dfrac{f^{(k)}-a}{f-a}$ *is a nonzero constant, provided one of the following holds:*

(i) $l \geq 2$ *and* $2\overline{N}(r,\infty;f) + N_2(r,0;f^{(k)}) + N_2(r,0;(f/a)') - N(r,0;f/a \mid\geq 3) < \{\lambda + o(1)\}T(r,f^{(k)})$,
(ii) $l = 1$ *and* $2\overline{N}(r,\infty;f) + N_2(r,0;f^{(k)}) + 2\overline{N}(r,0;(f/a)') - \overline{N}(r,0;f/a \mid\geq 2) < \{\lambda + o(1)\}T(r,f^{(k)})$,

(iii) $l = 0$ *and* $4\overline{N}(r,\infty; f) + 2N_2(r,0; f^{(k)}) + N(r,0; f^{(k)} \mid\leq 1) + 2\overline{N}(r,0; (f/a)')$ $-\overline{N}(r,0; f/a \mid\geq 2) < \{\lambda + o(1)\}T(r, f^{(k)})$,

for $r \in I$, *where* $0 < \lambda < 1$ *and* I *is a set of infinite linear measure.*

Theorem 54 *Let* f *be a nonconstant meromorphic function and* $k(\geq 1)$, $l(\geq 0)$ *be integers and* a *be a nonconstant small function of* f. *Suppose that* $f - a$ *and* $f^{(k)} - a$ *share* $(0,l)$. *Then* $f \equiv f^{(k)}$ *provided one of the following holds:*

(i) $l = 1$ *and* $\left(\frac{7}{2} + k\right)\Theta(\infty; f) + \frac{3}{2}\delta_2(0; f) + \delta_{2+k}(0; f) > 5 + k$,
(ii) $l = 0$ *and* $(6+2k)\Theta(\infty; f) + 2\Theta(0; f) + \delta_2(0; f) + \delta_{1+k}(0; f) + \delta_{2+k}(0; f) > 2k + 10$.

In 2006, Lin and Lin [28] introduced the notion of weakly weighted sharing of values to investigate the questions of K. W. Yu. In the following definition we explain this notion.

Definition 9 *We denote by* $\overline{N}^{E}_{k)}(r,a; f,g)$ *the reduced counting function of those common* a*-points of* f *and* g *whose corresponding multiplicities are equal, both of their multiplicities are not greater than* k.

Also we denote by $\overline{N}^{0}_{(k}(r,a; f,g)$ *the reduced counting function of those* a *-points of* f *which are* a*-points of* g, *both of their multiplicities are not less than* k.

If for some k *(a positive integer or* ∞*) and* $a \in \mathbb{C}\cup\{\infty\}$ *we get* $\overline{N}(r,a; f \mid\leq k) = \overline{N}^{E}_{k)}(r,a; f,g) + S(r,f)$, $\overline{N}(r,a; g \mid\leq k) = \overline{N}^{E}_{k)}(r,a; f,g) + S(r,g)$, $\overline{N}(r,a; f \mid\geq 1+k) = \overline{N}^{0}_{(1+k}(r,a; f,g) + S(r,f)$, $\overline{N}(r,a; g \mid\geq 1+k) = \overline{N}^{0}_{(1+k}(r,a; f,g) + S(r,g)$ *and for* $k = 0$ *we get* $\overline{N}(r,a; f) = \overline{N}_0(r,a; f,g) + S(r,f)$, $\overline{N}(r,a; g) = \overline{N}_0(r,a; f,g) + S(r,g)$, *then we say that* f *and* g *weakly share* a *with weight* k. *We write* f, g *share '*(a,k)*' to mean that* f *and* g *weakly share* a *with weight* k.

Obviously, if f *and* g *share '*(a,k)*', then* f *and* g *share '*(a,p)*' for any integer* $p(0 \leq p < k)$. *Also we note that* f *and* g *share IM* or CM* if and only if* f *and* g *share '*$(a,0)$*' or '*(a,∞)*' respectively.*

We now state the results of Lin and Lin [28].

Theorem 55 *Let* $k \geq 1$ *and* $2 \leq m \leq \infty$ *and* f *be a nonconstant meromorphic function. If* $f - a$ *and* $f^{(k)} - a$ *share "*$(0,m)$*" for some small function* $a(\not\equiv 0,\infty)$ *of* f *and* $2\delta_{2+k}(0; f) + 4\Theta(\infty; f) > 5$, *then* $f \equiv f^{(k)}$.

Theorem 56 *Let* $k \geq 1$ *and* f *be a nonconstant meromorphic function. If* $f - a$ *and* $f^{(k)} - a$ *share* "(0, 1)" *for some small function* $a(\not\equiv 0,\infty)$ *of* f *and* $\frac{5}{2}\delta_{2+k}(0; f) + \frac{k+9}{2}\Theta(\infty; f) > \frac{k}{2} + 6$, *then* $f \equiv f^{(k)}$.

Theorem 57 *Let* $k \geq 1$ *and* f *be a nonconstant meromorphic function. If* $f - a$ *and* $f^{(k)} - a$ *share* '(0, 0)' *for some small function* $a(\not\equiv 0,\infty)$ *of* f *and* $5\delta_{2+k}(0; f) + (2k+7)\Theta(\infty; f) > 2k + 11$, *then* $f \equiv f^{(k)}$.

A. H. H. Al-Klahaldi worked a lot on Brück's result (Theorem 38) and so it needs separate attention. Considering the sharing of small functions by an entire function and its derivative, he [1] exhibited by the following example that it is not possible to replace straightway the shared value by a shared function in R. Brück's theorem (Theorem 38).

Example 3 *Let* $f = \exp e^z$ *and* $a = \frac{e^z}{e^z-1}$. *It is easy to see that* $f - a$ *and* $f' - a$ *share* 0 *CM and* $N(r, 0; f') \equiv 0$ *but* $f - a \not\equiv c(f' - a)$ *for any nonzero constant* c.

Instead Al-Khaladi [1] proved the following theorem.

Theorem 58 *Let* f *be a nonconstant entire function satisfying* $N(r, 0; f') = S(r, f)$ *and let* $a = a(z)(\not\equiv 0, \infty)$ *be a meromorphic small function of* f. *If* $f - a$ *and* $f' - a$ *share* 0 *CM, then* $f - a = (1 - \frac{k}{a})(f' - a)$, *where* $1 - \frac{k}{a} = e^{\beta}$, k *is a constant and* β *is an entire function.*

In 2005 Al-Khaladi [2] proved that Brück's result (Theorem 38) is also valid for meromorphic functions. He also verified by the following examples that the hypotheses of Brück's result (Theorem 38) are essential.

Example 4 *Let* $f = 1 + \tan z$. *Then* $f' - 1 = (f - 1)^2$ *so that* f *and* f' *share* 1 *IM and* $N(r, 0; f') \equiv 0$. *However, the conclusion of Brück's result (Theorem 38) does not hold.*

Example 5 *Let* $f = \frac{ze^z}{1+e^z}$. *Then* f *and* f' *share the value* 1 *CM and* $N(r, 0; f') \neq S(r, f)$. *Clearly* $f' - 1 \neq c(f - 1)$ *for any nonzero constant* c.

Al-Khaladi [3] extended Theorem 58 to higher order derivative and proved the following theorem.

Theorem 59 *Let* f *be a nonconstant entire function satisfying* $\overline{N}(r, 0; f^{(k)}) = S(r, f)$ *and let* $a = a(z)(\not\equiv 0, \infty)$ *be a meromorphic small function of* f. *If* $f - a$ *and* $f^{(k)} - a$ *share* 0 *CM, then* $f - a = \left(1 - \frac{P_{k-1}}{a}\right)\left(f^{(k)} - a\right)$, *where* $1 - \frac{P_{k-1}}{a} = e^{\beta}$, P_{k-1} *is a polynomial of degree at most* $k - 1$ *and* β *is an entire function.*

Al-Khaladi [4] also considered the situation when a meromorphic function share a nonzero value ignoring multiplicities with its first derivative and proved the following result.

Theorem 60 *Let* f *be a nonconstant meromorphic function. If* f *and* f' *share a value* $a(\neq 0, \infty)$ *IM and if* $\overline{N}(r, 0; f) + \overline{N}(r, 0; f') = S(r, f)$, *then either* $f \equiv f'$ *or* $f = \frac{2a}{1-Ae^{-2z}}$, *where* A *is a nonzero constant.*

In 2010 Al-Khaladi [5] improved Theorem 59 in the following manner.

Theorem 61 *Let* f *be a nonconstant meromorphic function and* $a = a(z)(\not\equiv 0, \infty)$ *be a meromorphic small function of* f. *If* $f - a$ *and* $f^{(k)} - a$ *share* 0 *CM and if*

$$\overline{N}(r, \infty; f) + \overline{N}\left(r, 0; f^{(k)}\right) < \lambda T\left(r, f^{(k)}\right) + S\left(r, f^{(k)}\right)$$

for some real constant $\lambda \in \left(0, \frac{1}{k+1}\right)$, *then* $f - a = \left(1 - \frac{P_{k_1}}{a}\right)\left(f^{(k)} - a\right)$, *where* P_{k-1} *is a polynomial of degree at most* $k - 1$ *and* $1 - \frac{P_{k-1}}{a} \not\equiv 0$.

Al-Khaladi [6] further extended Theorem 59 to meromorphic functions and proved the following theorem.

Theorem 62 *Let f be a nonconstant meromorphic function satisfying $\overline{N}(r, 0; f^{(k)}) = S(r, f)$ and let $a = a(z)(\not\equiv 0, \infty)$ be a meromorphic small function of f. If $f - a$ and $f^{(k)} - a$ share 0 CM, then $f - a = \left(1 - \frac{P_{k-1}}{a}\right)\left(f^{(k)} - a\right)$, where $1 - \frac{P_{k-1}}{a} \not\equiv 0$ and P_{k-1} is a polynomial of degree at most $k - 1$.*

In this short survey, it is not possible to cover a reasonably wide area of the literature. There are many results which we cannot mention, though many of those deserve so for their own values and intricacies. We just tried to give an overall idea of the flow of research on the topic and our efforts will be meaningful only if it can create some interest in the reader for the topic.

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Chapter 6
Differential Superordinations and Sandwich-Type Results

Teodor Bulboacă, Nak Eun Cho and Pranay Goswami

6.1 The General Theory of Differential Superordinations

Let $\Omega \subset \mathbb{C}$, let p be analytic in the unit disc $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$, and let $\psi(r, s, t; z) : \mathbb{C}^3 \times \mathrm{U} \to \mathbb{C}$. In a series of articles, S. S. Miller, P. T. Mocanu and many others have determined properties of functions ψ that satisfy the differential subordination (i.e. the differential inclusion)

$$\{\psi(p(z), zp'(z), z^2\ p''(z); z) : z \in \mathrm{U}\} \subset \Omega.$$

Reversely, let us consider the dual problem of determining properties of functions ψ that satisfy the differential superordination

$$\Omega \subset \{\psi(p(z), zp'(z), z^2 p''(z); z) : z \in \mathrm{U}\}.$$

Since many of these kind of results can be expressed in terms of subordination and superordination, we will give the required definitions, and note that these results have been first presented in [21].

Definition 1 Let $f,\ F \in H(\mathrm{U})$, where $H(\mathrm{U})$ denotes the set of all analytic functions in U. The function f is said to be *subordinate* to F, or F is said to be *superordinate* to f, if there exists a function w analytic in U, with $w(0) = 0$ and

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$|w(z)| < 1$, and such that $f(z) = F(w(z))$, for all $z \in \mathrm{U}$. In such a case we write $f(z) \prec F(z)$.

It is well-known that, if F is univalent in U, then

$$f(z) \prec F(z) \Leftrightarrow f(0) = F(0) \text{ and } f(\mathrm{U}) \subset F(\mathrm{U}).$$

Let $\Omega,\ \Delta \subset \mathbb{C}$, let p be analytic in the unit disc U, and let $\varphi(r, s, t; z) : \mathbb{C}^3 \times \mathrm{U} \to \mathbb{C}$. In [21], authors determined sufficient conditions on Ω, Δ and φ for which the following implication holds:

$$\Omega \subset \{\varphi(p(z), zp'(z), z^2 p''(z); z) : z \in \mathrm{U}\} \Rightarrow \Delta \subset p(\mathrm{U}). \qquad (6.1)$$

If either Ω or U in (6.1) is a simply connected domain, then it is possible to rephrase (6.1) in terms of superordination. If p is univalent in U, and if U is a simply connected domain with $\Delta \neq \mathbb{C}$, then there exists a conformal mapping q of U onto U, such that $q(0) = p(0)$. Thus, (6.1) can be rewritten as

$$\Omega \subset \{\varphi(p(z), zp'(z), z^2 p''(z); z) : z \in \mathrm{U}\} \Rightarrow q(z) \prec p(z). \qquad (6.2)$$

If Ω is also a simply connected domain with $\Omega \neq \mathbb{C}$, then there exists a conformal mapping h of U onto Ω, such that $h(0) = \varphi(p(0), 0, 0; 0)$. If in addition, the function $\varphi(p(z), zp'(z), z^2\ p''(z); z)$ is univalent in U, then (6.2) can be rewritten as

$$h(z) \prec \varphi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) \prec p(z). \qquad (6.3)$$

Definition 2 Let $\varphi : \mathbb{C}^3 \times \mathrm{U} \to \mathbb{C}$ and let h be analytic in U. If p and $\varphi(p(z), zp'(z), z^2\ p''(z); z)$ are univalent in U and satisfy the (*second-order*) *differential superordination*

$$h(z) \prec \varphi(p(z), zp'(z), z^2 p''(z); z), \qquad (6.4)$$

then p is called a *solution* of the differential superordination. An analytic function q is called a *subordinant of the solutions of the differential superordination*, or more simply a *subordinant* if $q \prec p$ for all p satisfying (6.4). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (6.4) is said to be the *best subordinant*.

Note that the best subordinant is unique up to a rotation of U.

In the special case when the set inclusions of (6.1) can be replaced by the superordinations of (6.3), there are three distinct cases to consider in analysing this implication:

Problem 1 Given analytic functions h and q, find a class of admissible functions $\Phi[h, q]$ such that (6.3) holds.

Problem 2 Given the differential superordination in (6.3), find a subordinant q. Moreover, find the best subordinant.

Problem 3 Given φ and subordinant q, find the largest class of analytic functions h such that (6.3) holds.

Now, we will introduce a class of univalent functions, with nice boundary properties:

Definition 3 Let us denote by $\mathcal{Q}$, the set of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \left\{ \zeta \in \partial \mathrm{U} : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial \mathrm{U} \setminus E(f)$.

The subclass of $\mathcal{Q}$ for which $f(0) = a$ is denoted by $\mathcal{Q}(a)$.

The following well-known lemma [20, p. 24] plays a crucial role in determining subordinants of differential superordinations; for the number $0 < r < 1$, we will use the notation $\mathrm{U}_r = \{z \in \mathbb{C} : |z| < r\}$.

Lemma 1 *Let $p \in \mathcal{Q}(a)$, and let $q \in H[a,n]$, where*

$$H[a,n] = \left\{ f \in H(\mathrm{U}) : \ f(z) = a + a_n z^n + \ldots \right\},$$

with $q(z) \not\equiv a$ and $n \geq 1$. If q is not subordinate to p, then there exists points $z_0 = r_0 e^{i\theta_0} \in \mathrm{U}$ and $\zeta_0 \in \partial \mathrm{U} \setminus E(p)$, and an $m \geq n \geq 1$ for which $q(\mathrm{U}_{r_0}) \subset p(\mathrm{U})$,

(i) $q(z_0) = p(\zeta_0)$
(ii) $z_0 q'(z_0) = m \zeta_0 p'(\zeta_0)$
(iii) $\mathrm{Re}\, \frac{z_0 q''(z_0)}{q'(z_0)} + 1 \geq m\, \mathrm{Re} \left[\frac{\zeta_0 p''(\zeta_0)}{p'(\zeta_0)} + 1 \right]$

The class of admissible functions referred to in the introduction is defined as follows:

Definition 4 Let Ω be a set in $\mathbb{C}$ and $q \in H[a,n]$. The *class of admissible functions* $\Phi_n[\Omega, q]$, consists of those functions $\varphi : \mathbb{C}^3 \times \overline{\mathrm{U}} \to \mathbb{C}$ that satisfy the *admissibility condition*:

$$\varphi(r, s, t; \zeta) \in \Omega, \tag{6.5}$$

whenever

$$r = q(z),\ s = \frac{zq'(z)}{m},\ \mathrm{Re}\, \frac{t}{s} + 1 \leq \frac{1}{m} \mathrm{Re} \left[\frac{zq''(z)}{q'(z)} + 1 \right],$$

where $\zeta \in \partial \mathrm{U}$, $z \in \mathrm{U}$ and $m \geq n \geq 1$.

Remark 1 We will use the next simplified notations:

1. When $n = 1$ we write $\Phi_1[\Omega, q]$ as $\Phi[\Omega, q]$;
2. In the special case when h is an analytic mapping of U onto $\Omega \neq \mathbb{C}$, we denote the class $\Phi_n[h(\mathrm{U}), q]$ by $\Phi_n[h, q]$;
3. If $\varphi : \mathbb{C}^2 \times \overline{\mathrm{U}} \to \mathbb{C}$, then the admissibility condition (6.5) reduces to

$$\varphi\left(q(z), \frac{zq'(z)}{m}; \zeta \right) \in \Omega, \tag{6.6}$$

where $z \in \mathrm{U}$, $\zeta \in \partial\mathrm{U}$ and $m \geq n \geq 1$.

The next theorem is a key result in the theory of first and second order differential superordinations, and its proof follows immediately from [20, Lemma 2.2d.]:

Theorem 1 [21] *Let $\Omega \subset \mathbb{C}$, let $q \in H[a,n]$ and let $\varphi \in \Phi_n[\Omega, q]$. If $p \in \mathcal{Q}(a)$ and $\varphi(p(z), zp'(z), z^2\, p''(z); z)$ is univalent in* U*, then*

$$\Omega \subset \{\varphi(p(z), zp'(z), z^2 p''(z); z) : z \in \mathrm{U}\} \tag{6.7}$$

implies $q(z) \prec p(z)$.

For the special case when h is analytic on U and $h(\mathrm{U}) = \Omega \neq \mathbb{C}$, the class $\Phi_n[h(\mathrm{U}), q]$ is written as $\Phi_n[h, q]$, while the following result is an immediate consequence of Theorem 1:

Theorem 2 [21] *Let $q \in H[a,n]$, let h be analytic and let $\varphi \in \Phi_n[h, q]$. If $p \in \mathcal{Q}(a)$ and $\varphi(p(z), zp'(z), z^2\, p''(z); z)$ is univalent in* U*, then*

$$h(z) \prec \varphi(p(z), zp'(z), z^2 p''(z); z), \tag{6.8}$$

implies $q(z) \prec p(z)$.

The above Theorems 1 and 2 are very useful to obtain subordinants of a differential superordination of the form (6.7) or (6.8), by checking that the function φ is an admissible function, which requires that φ satisfies condition (6.5), that is a simple algebraic condition.

Otherwise, the subordinants of various differential superordinations are difficult to be obtained directly. For checking that the function φ is an admissible function, we can use three different techniques:

(i) an elementary technique deals with those cases in which the equation of the boundary of Ω is known;
(ii) a second technique concerns those cases for which the geometry of the domain Ω is of particular form (convex, starlike, . . .);
(iii) the third technique uses a more sophisticated method that employs subordination chains.

The next theorem, that is an immediate consequence of Theorem 2, proves the existence of the best subordinant of (6.8) for certain φ, and also gives us a method for finding the best subordinant:

Theorem 3 [21] *Let h be analytic in* U *and let $\varphi : \mathbb{C}^3 \times \mathrm{U} \to \mathbb{C}$. Suppose that the differential equation*

$$\varphi(q(z), zq'(z), z^2 q''(z); z) = h(z) \tag{6.9}$$

has a solution $q \in \mathcal{Q}(a)$. If $\varphi \in \Phi[h, q]$, $p \in \mathcal{Q}(a)$ and $\varphi(p(z), zp'(z), z^2 p''(z); z)$ is univalent in U*, then*

$$h(z) \prec \varphi(p(z), zp'(z), z^2 p''(z); z) \tag{6.10}$$

implies $q(z) \prec p(z)$ *and* q *is the best subordinant.*

This theorem shows us that the problem of finding the best subordinant of (6.10) reduces to showing that differential equation (6.9) has a univalent solution, and checking further that $\varphi \in \Phi[h, q]$.

6.2 First-Order Differential Superordinations

For the case of first-order differential subordinations, the Theorems 1, 2 and 3 can be simplified as follows, and these results are immediately obtained by using these theorems and admissibility condition (6.6):

Theorem 4 [21] *Let* $\Omega \subset \mathbb{C}$, $q \in H[a,n]$, $\varphi : \mathbb{C}^2 \times \overline{\mathrm{U}} \to \mathbb{C}$, *and suppose that*

$$\varphi(q(z), tzq'(z); \zeta) \in \Omega,$$

for $z \in \mathrm{U}$, $\zeta \in \partial\mathrm{U}$ *and* $0 < t \leq \dfrac{1}{n} \leq 1$. *If* $p \in \mathcal{Q}(a)$ *and* $\varphi(p(z), zp'(z); z)$ *is univalent in* U, *then*

$$\Omega \subset \{\varphi(p(z), zp'(z); z) : z \in \mathrm{U}\} \Rightarrow q(z) \prec p(z).$$

Theorem 5 [21] *Let* h *be analytic in* U, $q \in H[a,n]$, $\varphi : \mathbb{C}^2 \times \overline{\mathrm{U}} \to \mathbb{C}$, *and suppose that*

$$\varphi(q(z), tzq'(z); \zeta) \in h(\mathrm{U}),$$

for $z \in \mathrm{U}$, $\zeta \in \partial\mathrm{U}$ *and* $0 < t \leq \dfrac{1}{n} \leq 1$. *If* $p \in \mathcal{Q}(a)$ *and* $\varphi(p(z), zp'(z); z)$ *is univalent in* U, *then*

$$h(z) \prec \varphi(p(z), zp'(z); z) \Rightarrow q(z) \prec p(z).$$

Furthermore, if $\varphi(q(z), zq'(z); z) = h(z)$ *has a univalent solution* $q \in \mathcal{Q}(a)$, *then* q *is the best subordinant.*

Hallenbeck and Ruscheweyh [16], [20, p. 71] considered the differential subordination

$$p(z) + \frac{zp'(z)}{\gamma} \prec h_2(z), \tag{6.11}$$

where h_2 is convex in U, $h_2(0) = a$, $\gamma \neq 0$ and $\operatorname{Re}\gamma \geq 0$. They showed that if $p \in H[a, 1]$ satisfies (6.11), then

$$p(z) \prec q_2(z) \prec h_2(z),$$

where

$$q_2(z) = \frac{\gamma}{z^{\gamma}} \int_0^z h_2(t) t^{\gamma - 1}\, \mathrm{d}t,$$

and the function q_2 is a convex function and is the best dominant of (6.11).

The next theorem is an analogous result for the corresponding differential superordination:

Theorem 6 [21] *Let* h_1 *be convex in* U, *with* $h_1(0) = a$, $\gamma \neq 0$ *with* $\operatorname{Re}\gamma \geq 0$ *and* $p \in H[a,1] \cap \mathcal{Q}$. *If* $\dfrac{p(z) + zp'(z)}{\gamma}$ *is univalent in* U,

$$h_1(z) \prec p(z) + \frac{zp'(z)}{\gamma}$$

and

$$q_1(z) = \frac{\gamma}{z^\gamma} \int_0^z h_1(t) t^{\gamma-1}\, \mathrm{d}t,$$

then $q_1(z) \prec p(z)$. *The function* q_1 *is convex and is the best subordinant.*

Combining the last theorem with the above mentioned Hallenbeck and Ruscheweyh result, we obtain the following differential *sandwich-type theorem*:

Corollary 1 [21] *Let* h_1 *and* h_2 *be convex in* U, *with* $h_1(0) = h_2(0) = a$. *Let* $\gamma \neq 0$, *with* $\operatorname{Re}\gamma \geq 0$, *and let the functions* q_i *be defined by*

$$q_i(z) = \frac{\gamma}{z^\gamma} \int_0^z h_i(t) t^{\gamma-1}\, \mathrm{d}\, t,$$

for $i = 1, 2$. *If* $p \in H[a,1] \cap \mathcal{Q}$ *and* $\dfrac{p(z) + zp'(z)}{\gamma}$ *is univalent, then*

$$h_1(z) \prec p(z) + \frac{zp'(z)}{\gamma} \prec h_2(z) \Rightarrow q_1(z) \prec p(z) \prec q_2(z). \tag{6.12}$$

The functions q_1 *and* q_2 *are convex and they are respectively the best subordinant and best dominant.*

If we denote $f(z) = \dfrac{p(z) + zp'(z)}{\gamma}$, then (6.12) can be expressed as the following *sandwich-type theorem* involving subordination-preserving integral operators:

Corollary 2 [21] *Let* h_1 *and* h_2 *be convex in* U *and* f *be univalent in* U, *with* $h_1(0) = h_2(0) = f(0)$. *Let* $\gamma \neq 0$ *with* $\operatorname{Re}\gamma \geq 0$. *If*

$$h_1(z) \prec f(z) \prec h_2(z),$$

then

$$\frac{\gamma}{z^\gamma} \int_0^z h_1(t) t^{\gamma-1}\, \mathrm{d}t \prec \frac{\gamma}{z^\gamma} \int_0^z f(t) t^{\gamma-1}\, \mathrm{d}t \prec \frac{\gamma}{z^\gamma} \int_0^z h_2(t) t^{\gamma-1}\, \mathrm{d}t,$$

when the middle integral is univalent.

Definition 5 [25, p. 157] A function $L(z;t)$, with $z \in$ U and $t \geq 0$, is a *subordination (or a Loewner) chain* if $L(\cdot;t)$ is analytic and univalent in U for all $t \geq 0$, $L(z;\cdot)$ is continuously differentiable on $[0, +\infty)$ for all $z \in$ U, and $L(z;s) \prec L(z;t)$, when $0 \leq s \leq t$.

The following lemma provides a sufficient condition for $L(z,t)$ to be a subordination chain and it was used in many of these proofs:

Lemma 2 [25, p. 159] *The function* $L(z;t) = a_1(t)z + a_2(t)z^2 + \ldots$, *with* $a_1(t) \neq 0$ *for* $t \geq 0$, *and* $\lim_{t\to\infty} |a_1(t)| = \infty$, *is a subordination chain if and only if there exist constants* $r \in (0,1]$ *and* $M > 0$ *such that:*

(i) $L(z;t)$ *is analytic in* $|z| < r$ *for each* $t \geq 0$, *locally absolutely continuous in* $[0,\infty)$ *for each* $|z| < r$, *and satisfies*

$$|L(z;t)| \leq M|a_1(t)|, \text{ for } |z| < r \text{ and } t \geq 0;$$

(ii) there exists a function $p(z,t)$ *analytic in* U *for all* $t \in [0,\infty)$ *and measurable in* $[0,\infty)$ *for each* $z \in$ U, *such that* $\operatorname{Re} p(z,t) > 0$ *for* $z \in$ U, $t \in [0,\infty)$, *and*

$$\frac{\partial L(z;t)}{\partial t} = z\frac{\partial L(z;t)}{\partial z}p(z,t), \text{ for } |z| < r \text{ and for almost all } t \in [0,\infty).$$

The following result allows to obtain subordinants of a differential superordination by applying the theory of subordination chains:

Theorem 7 [21, Theorem 7] *Let* $q \in H[a,1]$, *let* $\varphi : \mathbb{C}^2 \to \mathbb{C}$ *and set* $\varphi(q(z), zq'(z)) \equiv h(z)$. *If* $L(z;t) = \varphi(q(z), tzq'(z))$ *is a subordination chain, and* $p \in H[a,1] \cap \mathcal{Q}$, *then*

$$h(z) \prec \varphi(p(z), zp'(z)) \Rightarrow q(z) \prec p(z).$$

Furthermore, if $\varphi(q(z), zq'(z)) = h(z)$ *has a univalent solution* $q \in \mathcal{Q}$, *then* q *is the best subordinant.*

An application of this above result could be done by considering again the differential superordination

$$h(z) \prec p(z) + \frac{zp'(z)}{\gamma}, \tag{6.13}$$

and the corresponding differential equation

$$q(z) + \frac{zq'(z)}{\gamma} = h(z). \tag{6.14}$$

While in Theorem 6 it was assumed that the function h of (6.14) was convex (which implied that solution q was also convex), now the assumption is a weaker one by assuming that q is convex and that h is defined by (6.14).

In this new case, it is easy to see that h is a univalent (more precisely, a close-to-convex) function, and the proof is based on the subordination chain argument as given in Theorem 7:

Theorem 8 [21] *Let q be convex in* U *and let h be defined by* (6.14), with $\operatorname{Re}\gamma > 0$. If $p \in H[a,1] \cap \mathcal{Q}$, $\dfrac{p(z) + zp'(z)}{\gamma}$ *is univalent in* U, *and* (6.13) *is satisfied, then* $q(z) \prec p(z)$, *where*

$$q(z) = \frac{\gamma}{z^{\gamma}} \int_0^z h(t)t^{\gamma-1}\,\mathrm{d}t.$$

The function q is the best subordinant.

This last theorem is an example of a solution of Problem 3 referred to in the Sect. 6.1:

Theorem 9 [21] *Let h be starlike in* U*, with $h(0) = 0$. If $p \in H[0,1] \cap \mathcal{Q}$ and $zp'(z)$ is univalent in* U*, then*

$$h(z) \prec zp'(z) \Rightarrow q(z) \prec p(z),$$

where

$$q(z) = \int_0^z h(t)t^{-1}\,\mathrm{d}t.$$

The function q is convex and is the best subordinant.

There exists a corresponding result of Theorem 9 for differential subordinations of the form $zp'(z) \prec h(z)$ due to Suffridge [26], [20, p. 76], and combining that result with Theorem 9 we obtain the following *sandwich-type result*:

Corollary 3 [21] *Let h_1 and h_2 be starlike in* U*, with $h_1(0) = h_2(0) = 0$, and let the functions q_i be defined by*

$$q_i(z) = \int_0^z h_i(t)t^{-1}\,\mathrm{d}t,$$

for $i = 1, 2$. If $p \in H[0,1] \cap \mathcal{Q}$ and $zp'(z)$ is univalent in U*, then*

$$h_1(z) \prec zp'(z) \prec h_2(z) \Rightarrow q_1(z) \prec p(z) \prec q_2(z).$$

The functions q_1 and q_2 are convex and they are respectively the best subordinant and best dominant.

Setting $f(z) = zp'(z)$, then the above corollary can be expressed as the following *sandwich-type theorem* involving subordination-preserving integral operators:

Corollary 4 [21] *Let h_1 and h_2 be starlike in* U *and f be univalent in* U*, with $h_1(0) = h_2(0) = f(0) = 0$. If*

$$h_1(z) \prec f(z) \prec h_2(z),$$

then

$$\int_0^z h_1(t)t^{-1}\,\mathrm{d}t \prec \int_0^z f(t)t^{-1}\,\mathrm{d}t \prec \int_0^z h_2(t)t^{-1}\,\mathrm{d}t$$

when the middle integral is univalent.

6.3 Classes of First-Order Differential Superordinations

Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$ be an analytic function in a domain $\Delta \subset \mathbb{C}^2$, let $p \in H(\mathrm{U})$ such that $\varphi(p(z), zp'(z))$ is univalent in U and suppose that p satisfies the first-order differential superordination

$$h(z) \prec \varphi(p(z), zp'(z)).$$

In the case when

$$\varphi(p(z), zp'(z)) = \alpha(p(z)) + \beta(p(z))\,\gamma(zp'(z))$$

it is possible to determine conditions on h, α, β and γ so that the above subordination implies $q(z) \prec p(z)$, where q is the *largest* function so that $q(z) \prec p(z)$ for all p functions satisfying the first-order differential superordination, i.e. the *best subordinant* q.

Theorem 10 [5] *Let q be a convex (univalent) function in the unit disc* U*, let α, $\beta \in H(D)$, where $D \supset q(\mathrm{U})$ is a domain and let $\gamma \in H(\mathbb{C})$. Suppose that*

$$\mathrm{Re}\,\frac{\alpha'(q(z)) + \beta'(q(z))\,\gamma(tzq'(z))}{\beta(q(z))\,\gamma'(tzq'(z))} > 0,\ z \in \mathrm{U}\ \textit{and}\ \ t \geq 0.$$

If $p \in H[q(0), 1] \cap \mathcal{Q}$, with $p(\mathrm{U}) \subset D$, and $\alpha(p(z)) + \beta(p(z))\,\gamma(zp'(z))$ is univalent in U*, then*

$$\alpha(q(z)) + \beta(q(z))\,\gamma(zq'(z)) = h(z) \prec \alpha(p(z)) + \beta(p(z))\,\gamma(zp'(z)) \Rightarrow q(z) \prec p(z),$$

and q is the best subordinant.

For the particular case when $\gamma(w) = w$, using a similar proof as in Theorem 10 we obtain:

Corollary 5 [5] *Let q be a univalent function in the unit disc* U *and let α, $\beta \in H(D)$, where $D \supset q(\mathrm{U})$ is a domain. Suppose that*

(i) $\mathrm{Re}\,\dfrac{\alpha'(q(z))}{\beta(q(z))} > 0,\ z \in \mathrm{U}$
(ii) $Q(z) = zq'(z)\beta(q(z))$ *is a starlike (univalent) function in* U

If $p \in H[q(0), 1] \cap \mathcal{Q}$, with $p(\mathrm{U}) \subset D$, and $\alpha(p(z)) + zp'(z)\,\beta(p(z))$ is univalent in U*, then*

$$\alpha(q(z)) + zq'(z)\,\beta(q(z)) \prec \alpha(p(z)) + zp'(z)\beta(p(z))\ \Rightarrow\ q(z) \prec p(z),$$

and q is the best subordinant.

For the case $\beta(w) = 1$, using the fact that the function $Q(z) = zq'(z)$ is starlike (univalent) in U if and only if q is convex (univalent) in U, Corollary 5 becomes:

Corollary 6 [5] *Let q be a convex (univalent) function in the unit disc* U *and let* $\alpha \in H(D)$*, where* $D \supset q(\mathrm{U})$ *is a domain. Suppose that*

$$\mathrm{Re}\,\alpha'(q(z)) > 0,\ z \in \mathrm{U}.$$

If $p \in H[q(0), 1] \cap \mathcal{Q}$*, with* $p(\mathrm{U}) \subset D$*, and* $\alpha(p(z)) + zp'(z)$ *is univalent in* U*, then*

$$\alpha(q(z)) + zq'(z) \prec \alpha(p(z)) + zp'(z) \ \Rightarrow\ q(z) \prec p(z),$$

and q is the best subordinant.

Next we will give some particular cases of the above results obtained for appropriate choices of the q, α and β functions.

Taking $\alpha(w) = w$ and $\beta(w) = 1/\gamma$, $\mathrm{Re}\,\gamma > 0$, in Corollary 5, condition (i) holds if $\mathrm{Re}\,\gamma > 0$ and (ii) holds if and only if q is a convex (univalent) function in U, hence we obtain:

Example 1 [5, 21, Theorem 8], Let q be a convex (univalent) function in the unit disc U and let $\gamma \in \mathbb{C}$, with $\mathrm{Re}\,\gamma > 0$. If $p \in H[q(0), 1] \cap \mathcal{Q}$ and $p(z) + \dfrac{zp'(z)}{\gamma}$ is univalent in U, then

$$q(z) + \frac{zq'(z)}{\gamma} \prec p(z) + \frac{zp'(z)}{\gamma} \ \Rightarrow\ q(z) \prec p(z),$$

and q is the best subordinant.

Considering in Corollary 6 the special case $\alpha(w) = e^w$, the assumption becomes

$$\mathrm{Re}\,\alpha'(q(z)) = e^{\mathrm{Re}\,q(z)} \cos(\mathrm{Im}\,q(z)) > 0,\ z \in \mathrm{U},$$

and we obtain:

Example 2 Let q be a convex (univalent) function in the unit disc U and suppose that

$$|\,\mathrm{Im}\,q(z)| < \frac{\pi}{2},\ z \in \mathrm{U}.$$

If $p \in H[q(0), 1] \cap \mathcal{Q}$ and $e^{p(z)} + zp'(z)$ is univalent in U, then

$$e^{q(z)} + zq'(z) \prec e^{p(z)} + zp'(z) \ \Rightarrow\ q(z) \prec p(z),$$

and q is the best subordinant.

Remark 2 Taking $q(z) = \lambda z$, $|\lambda| \le \pi/2$ in Example 2 we have the next result:

If $p \in H[0, 1] \cap \mathcal{Q}$ such that $e^{p(z)} + zp'(z)$ is univalent in U and $|\lambda| \le \pi/2$, then

$$e^{\lambda z} + \lambda z \prec e^{p(z)} + zp'(z) \ \Rightarrow\ \lambda z \prec p(z),$$

and λz is the best subordinant.

If we consider in Corollary 6 the case $\alpha(w) = \dfrac{w^2}{2} - \beta w$, then we may easily obtain the next result:

Example 3 Let q be a convex (univalent) function in the unit disc U and suppose that

$$\operatorname{Re} q(z) > \beta,\ z \in \mathrm{U}.$$

If $p \in H[q(0), 1] \cap \mathcal{Q}$ and $\dfrac{p^2(z)}{2} - \beta p(z) + zp'(z)$ is univalent in U, then

$$\frac{q^2(z)}{2} - \beta q(z) + zq'(z) \prec \frac{p^2(z)}{2} - \beta p(z) + zp'(z) \ \Rightarrow\ q(z) \prec p(z),$$

and q is the best subordinant.

6.4 Briot–Bouquet Differential Superordinations

Let $\beta, \gamma \in \mathbb{C}$, let $\Omega_2, \Delta_2 \subset \mathbb{C}$, and let $p \in H(\mathrm{U})$. In a series of articles, S. S. Miller and P. T. Mocanu and many other authors [20, pp. 80–119] have determined conditions such that

$$\left\{ p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} : z \in \mathrm{U} \right\} \subset \Omega_2 \Rightarrow p(\mathrm{U}) \subset \Delta_2.$$

The above differential operator is known as the *Briot–Bouquet differential operator*, and the main investigation in this subject is to find the smallest set $\Delta_2 \subset \mathbb{C}$ for which the above implication holds. We emphasise that this particular differential implication has many applications in univalent function theory.

Now we will discuss the dual problem of determining conditions so that

$$\Omega_1 \subset \left\{ p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} : z \in \mathrm{U} \right\} \Rightarrow \Delta_1 \subset p(\mathrm{U}),$$

and, in particular, we are interested in determining the largest set $\Delta_1 \subset \mathbb{C}$ for which this implication holds.

If the sets $\Omega_1, \Omega_2, \Delta_1, \Delta_2 \subset \mathbb{C}$ are simply connected domains not equal to $\mathbb{C}$, then it is possible to rephrase the above expressions in terms of subordination and superordination in the forms

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h_2(z) \Rightarrow p(z) \prec q_2(z) \tag{6.15}$$

and

$$h_1(z) \prec p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \Rightarrow q_1(z) \prec p(z). \tag{6.16}$$

We mention that the left-hand side of (6.15) is called a *Briot–Bouquet differential subordination*, and the function q_2 is called a *dominant* of the differential subordination. The *best dominant*, which provides a sharp result, is the dominant that is subordinate to all other dominants. Many results and applications of these topics can be found in [20, pp. 80–119].

In the Sect. 6.1 the dual concept of differential superordination was presented as introduced in [21]. In the light of those results the left side of (6.16) is called a *Briot–Bouquet differential superordination*, and the function q_1 is called a *subordinant* of the differential superordination. Also, the *best subordinant*, that provides a sharp result is the subordinant that is superordinate to all other subordinants.

Now, we will present the results of [22], where the authors combined (6.15) and (6.16) to obtain conditions so that the *Briot–Bouquet sandwich-type result*

$$h_1(z) \prec p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h_2(z),$$

implies that $q_1(z) \prec p(z) \prec q_2(z)$.

Theorem 11 [22] *Let h be convex in* U *with $h(0) = a$, and let $\Theta, \Phi \in H(D)$ where $D \subset \mathbb{C}$ is a domain. Let $p \in H[a, 1] \cap \mathcal{Q}$ and suppose that the function $\Theta(p(z)) + zp'(z)\Phi(p(z))$ is univalent in* U. *If the differential equation*

$$\Theta(q(z)) + zq'(z)\Phi(q(z)) = h(z)$$

has a univalent solution q that satisfies $q(0) = a$, $q(\mathrm{U}) \subset D$, and

$$\Theta(q(z)) \prec h(z),$$

then

$$h(z) \prec \Theta(p(z)) + zp'(z)\Phi(p(z)) \Rightarrow q(z) \prec p(z).$$

The function q is the best subordinant.

In the special case when $\Theta(w) = w$ and $\Phi(w) = \dfrac{1}{\beta w + \gamma}$ we obtain the following result for the Briot–Bouquet differential superordination:

Corollary 7 [22] *Let $\beta, \gamma \in \mathbb{C}$, and let h be convex in* U *with $h(0) = a$. Suppose that the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \tag{6.17}$$

has a univalent solution q that satisfies $q(0) = a$, and $q(z) \prec h(z)$. If $p \in H[a, 1] \cap \mathcal{Q}$ and $p(z) + \dfrac{zp'(z)}{\beta p(z) + \gamma}$ is univalent in U, *then*

$$h(z) \prec p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \Rightarrow q(z) \prec p(z).$$

The function q is the best subordinant.

Some conditions and examples for which the Briot–Bouquet differential Equation (6.17) has univalent solutions may be found in [19] and [20, p. 91].

There is a complete analogue of Theorem 11 for differential subordinations, which is given in [18, p. 189] and [20, p. 125]. We can combine that result with Theorem 11 and obtain the following *sandwich-type theorem*:

Theorem 12 [22] *Let h_1 and h_2 be convex in* U *with $h_1(0) = h_2(0) = a$, and let $\Theta, \Phi \in H(D)$, where $D \subset \mathbb{C}$ is a domain. Let $p \in H[a,1] \cap \mathcal{Q}$ and suppose that $\Theta(p(z)) + zp'(z)\Phi(p(z))$ is univalent in* U. *If the differential equations*

$$\Theta(q(z)) + zq'(z)\Phi(q(z)) = h_k(z)$$

have the univalent solutions q_k that satisfy $q_k(0) = a$, $q_k(\mathrm{U}) \subset D$, and

$$\Theta(q_k(z)) \prec h(z),$$

for $k = 1, 2$, then

$$h_1(z) \prec \Theta(p(z)) + zp'(z)\Phi(p(z)) \prec h_2(z) \Rightarrow q_1(z) \prec p(z) \prec q_2(z).$$

The functions q_1 and q_2 are the best subordinant and the best dominant, respectively.

In the special case when $\Theta(w) = w$ and $\Phi(w) = \dfrac{1}{\beta w + \gamma}$ we obtain the following result for the Briot–Bouquet *sandwich-type result*:

Corollary 8 [22] *Let $\beta, \gamma \in \mathbb{C}$, and let h_k be convex in* U *with $h_k(0) = a$, for $k = 1, 2$. Suppose that the differential equations*

$$q_k(z) + \frac{zq_k'(z)}{\beta q_k(z) + \gamma} = h_k(z) \tag{6.18}$$

have the univalent solutions q_k that satisfy $q_k(0) = a$, and $q_k(z) \prec h(z)$, for $k = 1, 2$. If $p \in H[a,1] \cap \mathcal{Q}$ and $p(z) + \dfrac{zp'(z)}{\beta p(z) + \gamma}$ is univalent in U, *then*

$$h_1(z) \prec p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h_2(z) \Rightarrow q_1(z) \prec p(z) \prec q_2(z).$$

The functions q_1 and q_2 are the best subordinant and the best dominant, respectively.

If $\beta = 0$ and $\gamma \neq 0$ with $\operatorname{Re} \gamma \geq 0$, then (6.18) has univalent (convex) solutions given by

$$q_k(z) = \frac{\gamma}{z^\gamma} \int_0^z h_k(t) t^{\gamma - 1}\, \mathrm{d}t,$$

for $k = 1, 2$, and in this case we obtain the *sandwich-type result* of Corollary 1.

Theorem 11 dealt with finding a subordinant, or the best subordinant, for a differential superordination for a given h function. Handling the problem from a different direction, i.e. first select the subordinant q and then find the appropriate h corresponding to this q, we have the following result:

Theorem 13 [22] *Let $\Theta, \Phi \in H(D)$, where $D \subset \mathbb{C}$ is a domain, and let q be univalent in* U *with $q(0) = a$ and $q(\mathrm{U}) \subset D$. Set $Q(z) = zq'(z)\Phi(q(z))$, $h(z) = \Theta(q(z)) + Q(z)$ and suppose that*

(i) $\operatorname{Re}\dfrac{\Theta'(q(z))}{\Phi(q(z))} > 0,\ z \in \mathrm{U}$ *and*
(ii) Q *is starlike in* U

If $p \in H[a,1] \cap \mathcal{Q}$ *and* $\Theta(p(z)) + zp'(z)\Phi(p(z))$ *is univalent in* U*, then*

$$h(z) \prec \Theta(p(z)) + zp'(z)\Phi(p(z)) \Rightarrow q(z) \prec p(z),$$

and q is the best subordinant.

In the special case when $\Theta(w) = w$ and $\Phi(w) = \dfrac{1}{\beta w + \gamma}$, Theorem 13 simplifies to the following result for the Briot–Bouquet differential superordinations:

Corollary 9 [22] *Let* $\beta, \gamma \in \mathbb{C}$*, and let q be univalent in* U *with* $q(0) = a$*. Set*

$$h(z) = q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \tag{6.19}$$

and suppose that

$$(i)\operatorname{Re}\left[\beta q(z) + \gamma\right] > 0,\ z \in \mathrm{U}, \tag{6.20}$$

and

$$(ii)\frac{zq'(z)}{\beta q(z) + \gamma}\ \textit{is starlike in}\ \mathrm{U}. \tag{6.21}$$

If $p \in H[a,1] \cap \mathcal{Q}$ *and* $p(z) + \dfrac{zp'(z)}{\beta p(z) + \gamma}$ *is univalent in* U*, then*

$$h(z) \prec p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \Rightarrow q(z) \prec p(z)$$

and q is the best subordinant.

Several previous results of the authors of [22] enable us to replace the conditions that q be univalent and that (6.20) be satisfied in the above result with weaker conditions. In [20, pp. 86–91], it was shown that if R_c represents the *open door* function defined in [20, Definition 2.5a], and

$$\beta h(z) + \gamma \prec R_{\beta a+\gamma}(z),$$

then the differential equation (6.19) has an analytic solution q that satisfies the condition (6.20). Moreover, the condition (6.21) implies that this solution q is univalent. Combining these results with Corollary 9 we obtain the following improved result:

Corollary 10 [22] *Let* $h \in H(\mathrm{U})$ *with* $h(0) = a$*, let the numbers* $\beta, \gamma \in \mathbb{C}$ *with* $\operatorname{Re}[\beta a + \gamma] > 0$*, and suppose that*

$$(i)\ \ \beta h(z) + \gamma \prec R_{\beta+\gamma}(z)$$

Let q be the analytic solution of the Briot–Bouquet differential equation

$$h(z) = q(z) + \frac{zq'(z)}{\beta q(z) + \gamma}$$

and suppose that

$$(ii)\frac{zq'(z)}{\beta q(z) + \gamma} \text{ is starlike in U.}$$

If $p \in H[a, 1] \cap \mathcal{Q}$ *and* $p(z) + \dfrac{zp'(z)}{\beta p(z) + \gamma}$ *is univalent in* U*, then*

$$h(z) \prec p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \Rightarrow q(z) \prec p(z)$$

and q is the best subordinant.

Since there is a complete analogue of Theorem 13 for differential subordinations, which is given in [18, p. 190] and [20, p. 132], combining that result with Theorem 13 we get the following *sandwich-type theorem*:

Theorem 14 [22] *Let* $\Theta, \Phi \in H(D)$*, where* $D \subset \mathbb{C}$ *is a domain, and let* q_1 *and* q_2 *be univalent in* U *with* $q_k(0) = a$ *and* $q_k(\mathrm{U}) \subset D$*, for* $k = 1, 2$*. Set* $Q_k(z) = zq_k'(z)\Phi(q_k(z))$*,* $h_k(z) = \Theta(q_k(z)) + Q_k(z)$ *and suppose that*

$$(i) \ \operatorname{Re} \frac{\Theta'(q_k(z))}{\Phi(q_k(z))} > 0, \ z \in \mathrm{U},$$

and

$$(ii) \ Q_k \text{ is starlike in U.}$$

If $p \in H[a, 1] \cap \mathcal{Q}$ *and* $\Theta(p(z)) + zp'(z)\Phi(p(z))$ *is univalent in* U*, then*

$$h_1(z) \prec \Theta(p(z)) + zp'(z)\Phi(p(z)) \prec h_2(z) \Rightarrow q_1(z) \prec p(z) \prec q_2(z).$$

The functions q_1 *and* q_2 *are the best subordinant and the best dominant, respectively.*

For the special case of the Briot–Bouquet differential operator this result becomes:

Corollary 11 [22] *For* $k = 1, 2$*, let* $h_k \in H(\mathrm{U})$ *with* $h_k(0) = a$*. Let* $\beta, \gamma \in \mathbb{C}$ *with* $\operatorname{Re}[\beta a + \gamma] > 0$*, and suppose that*

$$(i) \ \beta h_k(z) + \gamma \prec R_{\beta a + \gamma}(z).$$

Let q_k *be analytic solutions of the Briot–Bouquet differential equation*

$$h_k(z) = q_k(z) + \frac{zq_k'(z)}{\beta q_k(z) + \gamma}$$

for $k = 1, 2$ *and suppose that*

$$(ii) \ \frac{zq_k'(z)}{\beta q_k(z) + \gamma} \text{ is starlike in U.}$$

If $p \in H[a,1] \cap \mathcal{Q}$ *and* $p(z) + \dfrac{zp'(z)}{\beta p(z) + \gamma}$ *is univalent in* U, *then*

$$h_1(z) \prec p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h_2(z) \Rightarrow q_1(z) \prec p(z) \prec q_2(z).$$

The functions q_1 *and* q_2 *are the best subordinant and the best dominant, respectively.*

6.5 Generalized Briot–Bouquet Differential Subordinations and Superordinations

Let $\psi : \mathbb{C}^2 \to \mathbb{C}$ be analytic in a domain $D \subset \mathbb{C}$, let $h \in H(\mathrm{U})$ be univalent in U, and suppose that $p \in H(\mathrm{U})$ satisfies the first-order differential subordination

$$\psi(p(z), zp'(z)) \prec h(z). \tag{6.22}$$

In [18] the authors determined conditions on ψ and h so that (6.22) implies $p(z) \prec q(z)$, when

$$\psi(p(z), zp'(z)) = \Theta(p(z)) + zp'(z)\Phi(p(z)).$$

This result had been generalised in [1] and [2] for the cases

$$\psi(p(z), zp'(z)) = \alpha(p(z)) + \beta(p(z))\gamma(zp'(z)) \tag{6.23}$$

and

$$\psi(p(z), zp'(z)) = \alpha(zp'(z)) + \beta(zp'(z))\gamma(p(z)), \tag{6.24}$$

respectively.

Now, we will show an extension of the results from [1] and [2], and moreover we will determine sufficient conditions on ψ so that

$$h(z) \prec \psi(p(z), zp'(z)) \Rightarrow q(z) \prec p(z),$$

where ψ is given by (6.23) and (6.24). Combining those results we will obtain the *sandwich-type theorems* and we will give some particular cases of those main results obtained for the appropriate choice of α, β, γ and of the subordinants and the dominants.

Note that the first type of differential subordinations represents a generalisation of the *Briot–Bouquet* differential subordination, i.e.

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z),$$

obtained from (6.23) for $\alpha(w) = w$, $\beta(w) = \dfrac{1}{\beta w + \gamma}$ and $\gamma(w) = w$.

Theorem 15 [8] *Let q be univalent in* U*, let $\alpha, \beta \in H(D)$, where $D \subset \mathbb{C}$ is a domain so that $D \supset q(\mathrm{U})$, and let $\gamma \in H(\mathbb{C})$. Suppose that*

$$\operatorname{Re}\left\{\frac{\alpha'(q(z)) + \beta'(q(z))\gamma((1+t)zq'(z))}{\beta(q(z))\gamma'((1+t)zq'(z))} + (1+t)\left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0,$$

$$z \in \mathrm{U},\ t \geq 0,$$

$$Q(z) = zq'(z)\beta(q(z))\gamma'(zq'(z)) \quad \textit{is starlike in } \mathrm{U}.$$

If $p \in H(\mathrm{U})$, with $p(0) = q(0)$ and $p(\mathrm{U}) \subset D$, then

$$\alpha(p(z)) + \beta(p(z))\gamma(zp'(z)) \prec \alpha(q(z)) + \beta(q(z))\gamma(zq'(z)) = h(z)$$

implies $p(z) \prec q(z)$, and q is the best dominant.

Note that this result improves Theorem 1 from [1], where we assumed in addition, that q is a convex function in U. For the particular case $\gamma(w) = w, w \in \mathbb{C}$, the previous theorem reduces to Theorem 3 from [18].

The dual result of Theorem 15 for differential supeordinations is the following one:

Theorem 16 [8] *Let q be univalent in* U*, let $\alpha, \beta \in H(D)$, where $D \subset \mathbb{C}$ is a domain so that $D \supset q(\mathrm{U})$, and let $\gamma \in H(\mathbb{C})$. Suppose that*

$$\operatorname{Re}\left\{\frac{\alpha'(q(z)) + \beta'(q(z))\gamma(tzq'(z))}{\beta(q(z))\gamma'(tzq'(z))} + t\left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0,\ z \in \mathrm{U},\ t \geq 0.$$

If $p \in H[q(0), 1] \cap \mathcal{Q}$, with $p(\mathrm{U}) \subset D$ and $\alpha(p(z)) + \beta(p(z))\gamma(zp'(z))$ is univalent in U*, then*

$$\alpha(q(z)) + \beta(q(z))\gamma(zq'(z)) = h(z) \prec \alpha(p(z)) + \beta(p(z))\gamma(zp'(z))$$

implies $q(z) \prec p(z)$, and q is the best subordinant.

Note that this result improves Theorem 3.1 from [5], where we assumed in addition, that q is a convex function in U.

Combining the last theorem with Theorem 15, we obtain the following differential *sandwich-type theorem*:

Theorem 17 [8] *Let q_1, q_2 be univalent functions in* U*, with $q_1(0) = q_2(0)$. Let $\alpha, \beta \in H(D)$, where $D \subset \mathbb{C}$ is a domain so that $D \supset q_1(\mathrm{U}) \cup q_2(\mathrm{U})$, and let $\gamma \in H(\mathbb{C})$. If we denote by*

$$L[q_k](z;t) = \frac{\alpha'(q_k(z)) + \beta'(q_k(z))\gamma(tzq_k'(z))}{\beta(q_k(z))\gamma'(tzq_k'(z))} + t\left(1 + \frac{zq_k''(z)}{q_k'(z)}\right),\ k = 1, 2,$$

suppose that

$$\operatorname{Re} L[q_1](z;t) > 0,\ z \in \mathrm{U},\ t \geq 0,$$
$$\operatorname{Re} L[q_2](z;1+t) > 0,\ z \in \mathrm{U},\ t \geq 0,$$

$$Q(z) = zq_2'(z)\beta(q_2(z))\gamma'(zq_2'(z)) \quad \textit{is starlike in } \mathrm{U}.$$

If $p \in H[q_1(0), 1] \cap \mathcal{Q}$*, with* $p(\mathrm{U}) \subset D$ *and* $\alpha(p(z)) + \beta(p(z))\gamma(zp'(z))$ *is univalent in* U*, then*

$$\alpha(q_1(z)) + \beta(q_1(z))\gamma(zq_1'(z)) \prec \alpha(p(z)) + \beta(p(z))\gamma(zp'(z)) \prec$$
$$\prec \alpha(q_2(z)) + \beta(q_2(z))\gamma(zq_2'(z))$$

implies $q_1(z) \prec p(z) \prec q_2(z)$*. Moreover, the functions* q_1 *and* q_2 *are the best subordinant and the best dominant, respectively.*

An interesting particular case of the above theorem may be obtained for $\gamma(w) = w$, $w \in \mathbb{C}$, presented in the next corollary:

Corollary 12 [8] *Let* q_1, q_2 *be univalent functions in* U*, with* $q_1(0) = q_2(0)$*. Let* $\alpha, \beta \in H(D)$*, where* $D \subset \mathbb{C}$ *is a domain so that* $D \supset q_1(\mathrm{U}) \cup q_2(\mathrm{U})$*, and let* $\gamma \in H(\mathbb{C})$*. If we denote by*

$$Q_k(z) = zq_k'(z)\beta(q_k(z)),\ k = 1, 2,$$

suppose that

$$Q_1, Q_2 \quad \textit{are starlike in } \mathrm{U},$$
$$\mathrm{Re}\left[\frac{\alpha'(q_2(z))}{\beta(q_2(z))} + \frac{zq_2'(z)}{Q_2(z)}\right] > 0,\ z \in \mathrm{U},$$
$$\mathrm{Re}\,\frac{\alpha'(q_1(z))}{\beta(q_1(z))} > 0,\ z \in \mathrm{U}.$$

If $p \in H[q_1(0), 1] \cap \mathcal{Q}$*, with* $p(\mathrm{U}) \subset D$ *and* $\alpha(p(z)) + zp'(z)\beta(p(z))$ *is univalent in* U*, then*

$$\alpha(q_1(z)) + zq_1'(z)\beta(q_1(z)) \prec \alpha(p(z)) + zp'(z)\beta(p(z)) \prec$$
$$\prec \alpha(q_2(z)) + zq_2'(z)\beta(q_2(z))$$

implies $q_1(z) \prec p(z) \prec q_2(z)$*. Moreover, the functions* q_1 *and* q_2 *are the best subordinant and the best dominant, respectively..*

Considering in this corollary the particular case $\alpha(w) = w$ and $\beta(w) = \dfrac{1}{1 + \lambda w}$, $w \in \mathbb{C} \setminus \left\{-\dfrac{1}{\lambda}\right\}$, if $\lambda \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, we easily deduce the next *sandwich-type theorem* related to the *Briot–Bouquet* differential subordination and superordination:

Corollary 13 [8] *Let* q_1, q_2 *be univalent functions in* U*, with* $q_1(0) = q_2(0)$ *and* $q_1(\mathrm{U}) \cup q_2(\mathrm{U}) \subset \mathbb{C} \setminus \left\{-\dfrac{1}{\lambda}\right\}$*, if* $\lambda \in \mathbb{C}^*$*. If we denote by*

$$Q_k(z) = \frac{zq_k'(z)}{1 + \lambda q_k(z)},\ k = 1, 2,$$

suppose that

$$Q_1, Q_2 \quad \textit{are starlike in } \mathrm{U},$$

$$\mathrm{Re}\left[1 + \lambda q_2(z) + \frac{zq_2'(z)}{Q_2(z)}\right] > 0,\ z \in \mathrm{U},$$
$$\mathrm{Re}\,[1 + \lambda q_1(z)] > 0,\ z \in \mathrm{U}.$$

If $p \in H[q_1(0), 1] \cap \mathcal{Q}$, *with* $p(\mathrm{U}) \subset \mathbb{C} \setminus \left\{-\frac{1}{\lambda}\right\}$, *if* $\lambda \in \mathbb{C}^*$, *and* $p(z) + \dfrac{zp'(z)}{1 + \lambda p(z)}$ *is univalent in* U, *then*

$$q_1(z) + \frac{zq_1'(z)}{1 + \lambda q_1(z)} \prec p(z) + \frac{zp'(z)}{1 + \lambda p(z)} \prec q_2(z) + \frac{zq_2'(z)}{1 + \lambda q_2(z)}$$

implies $q_1(z) \prec p(z) \prec q_2(z)$. *Moreover, the functions* q_1 *and* q_2 *are the best subordinant and the best dominant, respectively.*

The next three results deal with similar kinds of subordination and superordination theorems, for the case when ψ and φ, respectively will have the form $\alpha(zp'(z)) + \beta(zp'(z))\gamma(p(z))$.

Theorem 18 [8] *Let* q *be univalent in* U, *let* $\alpha, \beta \in H(\mathbb{C})$, *and let* $\gamma \in H(D)$, *where* $D \subset \mathbb{C}$ *is a domain so that* $D \supset q(\mathrm{U})$. *Suppose that*

$$\mathrm{Re}\left\{\frac{\beta((1+t)zq'(z))\gamma'(q(z))}{\alpha'((1+t)zq'(z)) + \beta'((1+t)zq'(z))\gamma(q(z))} + (1+t)\left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0,\ z \in \mathrm{U},\ t \geq 0,$$

$$Q(z) = zq'(z)[\alpha'(zq'(z)) + \beta'(zq'(z))\gamma(q(z))] \text{ is starlike in } \mathrm{U}.$$

If $p \in H(\mathrm{U})$, *with* $p(0) = q(0)$ *and* $p(\mathrm{U}) \subset D$ *then*

$$\alpha(zp'(z)) + \beta(zp'(z))\gamma(p(z)) \prec \alpha(zq'(z)) + \beta(zq'(z))\gamma(q(z)) = h(z)$$

implies $p(z) \prec q(z)$, *and* q *is the best dominant.*

This result improves Theorem 1 from [2], where it was presumed the strong assumption that q is a convex function in U.

The dual result of Theorem 18 for differential supeordinations is the following one:

Theorem 19 [8] *Let* q *be univalent in* U, *let* $\alpha, \beta \in H(\mathbb{C})$, *and let* $\gamma \in H(D)$, *where* $D \subset \mathbb{C}$ *is a domain so that* $D \supset q(\mathrm{U})$. *Suppose that*

$$\mathrm{Re}\left\{\frac{\beta(tzq'(z))\gamma'(q(z))}{\alpha'(tzq'(z)) + \beta'(tzq'(z))\gamma(q(z))} + t\left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0,\ z \in \mathrm{U},\ t \geq 0.$$

If $p \in H[q(0), 1] \cap \mathcal{Q}$, *with* $p(\mathrm{U}) \subset D$, *and* $\alpha(zp'(z)) + \beta(zp'(z))\gamma(p(z))$ *is univalent in* U, *then*

$$\alpha(zq'(z)) + \beta(zq'(z))\gamma(q(z)) = h(z) \prec \alpha(zp'(z)) + \beta(zp'(z))\gamma(p(z))$$

implies $q(z) \prec p(z)$, *and* q *is the best subordinant.*

Combining Theorem 18 with Theorem 19 we obtain the next differential *sandwich-type theorem*:

Theorem 20 [8] *Let q_1, q_2 be univalent functions in* U*, with $q_1(0) = q_2(0)$, let $\alpha, \beta \in H(\mathbb{C})$ and let $\gamma \in H(D)$, where $D \subset \mathbb{C}$ is a domain so that $D \supset q_1(\mathrm{U}) \cup q_2(\mathrm{U})$. If we denote by*

$$\Lambda[q_k](z;t) = \frac{\beta(tzq_k'(z))\gamma'(q_k(z))}{\alpha'(tzq_k'(z)) + \beta'(tzq_k'(z))\gamma(q_k(z))} + t\left(1 + \frac{zq_k''(z)}{q_k'(z)}\right),\ k = 1,2,$$

suppose that

$$\operatorname{Re} \Lambda[q_1](z;t) > 0,\ z \in \mathrm{U},\ t \geq 0,$$
$$\operatorname{Re} \Lambda[q_2](z;1+t) > 0,\ z \in \mathrm{U},\ t \geq 0,$$
$$Q(z) = zq_2'(z)[\alpha'(zq_2'(z)) + \beta'(zq_2'(z))\gamma(q_2(z))] \text{ is starlike in } \mathrm{U}.$$

If $p \in H[q_1(0), 1] \cap \mathcal{Q}$, with $p(\mathrm{U}) \subset D$ and $\alpha(zp'(z)) + \beta(zp'(z))\gamma(p(z))$ is univalent in U*, then*

$$\alpha(zq_1'(z)) + \beta(zq_1'(z))\gamma(q_1(z)) \prec \alpha(zp'(z)) + \beta(zp'(z))\gamma(p(z)) \prec$$
$$\prec \alpha(zq_2'(z)) + \beta(zq_2'(z))\gamma(q_2(z))$$

implies $q_1(z) \prec p(z) \prec q_2(z)$. Moreover, the functions q_1 and q_2 are the best subordinant and the best dominant, respectively.

Taking in this last theorem $\alpha(w) = w$ and $\beta(w) = 1$, $w \in \mathbb{C}$, then we obtain the next corollary:

Corollary 14 *Let q_1, q_2 be convex functions in* U*, with $q_1(0) = q_2(0)$ and let $\gamma \in H(D)$, where $D \subset \mathbb{C}$ is a domain so that $D \supset q_1(\mathrm{U}) \cup q_2(\mathrm{U})$. Suppose that*

$$\operatorname{Re} \gamma'(q_1(z)) > 0,\ z \in \mathrm{U},$$
$$\operatorname{Re}\left[\gamma'(q_2(z)) + 1 + \frac{zq_2''(z)}{q_2'(z)}\right] > 0,\ z \in \mathrm{U}.$$

If $p \in H[q_1(0), 1] \cap \mathcal{Q}$, with $p(\mathrm{U}) \subset D$ and $zp'(z) + \gamma(p(z))$ is univalent in U*, then*

$$zq_1'(z) + \gamma(q_1(z)) \prec zp'(z) + \gamma(p(z)) \prec zq_2'(z) + \gamma(q_2(z))$$

implies $q_1(z) \prec p(z) \prec q_2(z)$. Moreover, the functions q_1 and q_2 are the best subordinant and the best dominant, respectively.

Next we will present some particular cases of the main results, obtained for convenient choices of the subordinants and of the dominants, i.e. $q_k(z) = r_k z$, $k = 1,2$, where $0 \leq r_1 < r_2$.

Considering $\beta(w) = 1$ and $\gamma(w) = w$, $w \in \mathbb{C}$, in Corollary 12, for the above mentioned subordinants and dominants we have the next result.

Example 4 Let $0 \le r_1 < r_2$ and let $\alpha \in H(D)$, where $D \subset \mathbb{C}$ is a domain so that $D \supset \{w \in \mathbb{C} : |w| < r_2\}$. Suppose that

$$\begin{aligned} &\operatorname{Re}\alpha'(r_1 z) > 0,\ z \in \mathrm{U},\\ &\operatorname{Re}\alpha'(r_2 z) > -1,\ z \in \mathrm{U}. \end{aligned}$$

If $p \in H[0,1] \cap \mathcal{Q}$, with $p(\mathrm{U}) \subset D$ and $\alpha(p(z)) + zp'(z)$ is univalent in U, then

$$\alpha(r_1 z) + r_1 z \prec \alpha(p(z)) + zp'(z) \prec \alpha(r_2 z) + r_2 z$$

implies $r_1 z \prec p(z) \prec r_2 z$. Moreover, the functions $r_1 z$ and $r_2 z$ are the best subordinant and the best dominant, respectively .

If we consider the functions $q_k(z) = r_k z$, $k = 1, 2$, where $0 \le r_1 < r_2$, in Corollary 13, we have the next example:

Example 5 Let $0 \le r_1 < r_2$ and let $\lambda \in \mathbb{C}$ with $|\lambda| \le \dfrac{1}{r_2}$. Suppose that $p \in H[0,1] \cap \mathcal{Q}$, with $p(\mathrm{U}) \subset \mathbb{C} \setminus \left\{-\dfrac{1}{\lambda}\right\}$, if $\lambda \in \mathbb{C}^*$, and $p(z) + \dfrac{zp'(z)}{1+\lambda p(z)}$ is univalent in U. Then

$$r_1 z + \frac{r_1 z}{1+\lambda r_1 z} \prec p(z) + \frac{zp'(z)}{1+\lambda p(z)} \prec r_2 z + \frac{r_2 z}{1+\lambda r_2 z}$$

implies $r_1 z \prec p(z) \prec r_2 z$, and the functions $r_1 z$ and $r_2 z$ are the best subordinant and the best dominant, respectively.

6.6 Sandwich-type Theorems for a Class of Integral Operators: the $\mathrm{I}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}$ Operator

For the functions $\phi, \varphi \in \mathcal{D}$, where

$$\mathcal{D} = \{\varphi \in H(\mathrm{U}) :\ \varphi(0) = 1,\ \varphi(z) \ne 0,\ z \in \mathrm{U}\},$$

we consider the integral operator $\mathrm{I}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta} : \mathcal{A}_{\varphi;\alpha,\delta} \to H(\mathrm{U})$ by

$$\mathrm{I}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}(f)(z) = \left[\frac{\beta+\gamma}{z^{\gamma}\phi(z)} \int_0^z t^{\delta-1} f^{\alpha}(t)\varphi(t)\,\mathrm{d}t\right]^{1/\beta}, \tag{6.25}$$

where the complex parameters α, β, γ and δ are suitably chosen and all the powers in (6.25) are principal ones. The subset $\mathcal{A}_{\varphi;\alpha,\delta} \subset H(\mathrm{U})$ was determined in [20] as follows:

Lemma 3 [20] *Let* $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ *with* $\beta \ne 0$, $\alpha + \delta = \beta + \gamma$, $\operatorname{Re}(\alpha+\delta) > 0$ *and* $\phi, \varphi \in \mathcal{D}$. *If* $f \in \mathcal{A}_{\varphi;\alpha,\delta}$, *where*

$$\mathcal{A}_{\varphi;\alpha,\delta} = \left\{ f \in A : \alpha\frac{zf'(z)}{f(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta \prec R_{\alpha+\delta}(z) \right\}, \tag{6.26}$$

$$A = \{f \in H(\mathrm{U} : f(0) = f'(0) - 1 = 0\},$$

and $R_{\alpha+\delta}$ *is the open door function* [20, Definition 2.5a]*, then*

$$\mathrm{I}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}(f) \in A, \quad \frac{\mathrm{I}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}(f)(z)}{z} \neq 0,\ z \in \mathrm{U},$$

and

$$\mathrm{Re}\left\{\beta\frac{z\left(\mathrm{I}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}(f)(z)\right)'}{\mathrm{I}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}(f)(z)} + \frac{z\phi'(z)}{\phi(z)} + \gamma\right\} > 0,\ z \in \mathrm{U},$$

where $\mathrm{I}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}$ *is the integral operator defined by (6.25).*

A general subordination property involving the integral operator $\mathrm{I}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}$ defined by (6.25) is contained in Theorem 21 below:

Theorem 21 [13] *Let* $f, g \in \mathcal{A}_{\varphi;\alpha,\delta}$*, where* $\mathcal{A}_{\varphi;\alpha,\delta}$ *is defined by* (6.26)*. Suppose also that*

$$\mathrm{Re}\left[1 + \frac{z\nu''(z)}{\nu'(z)}\right] > -\rho,\ z \in \mathrm{U},$$

where $\nu(z) = z\left[\frac{g(z)}{z}\right]^{\alpha}\varphi(z)$ *and*

$$\rho = \frac{1 + |\beta+\gamma-1|^2 - |1-(\beta+\gamma-1)^2|}{4\,\mathrm{Re}\,(\beta+\gamma-1)},\ \textit{with}\ \mathrm{Re}\,(\beta+\gamma-1) > 0. \quad (6.27)$$

Then, the following subordination relation

$$z\left[\frac{f(z)}{z}\right]^{\alpha}\varphi(z) \prec z\left[\frac{g(z)}{z}\right]^{\alpha}\varphi(z)$$

implies that

$$z\left[\frac{\mathrm{I}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}(f)(z)}{z}\right]^{\beta}\phi(z) \prec z\left[\frac{\mathrm{I}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}(g)(z)}{z}\right]^{\beta}\phi(z),$$

where $\mathrm{I}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}$ is *the integral operator defined by* (6.25)*. Moreover, the function* $z\left[\frac{\mathrm{I}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}(g)(z)}{z}\right]^{\beta}\phi(z)$ *is the best dominant.*

Remark 3 If we choose the complex parameters α, β, γ and δ with $\phi(z) = \varphi(z) = 1$, $\alpha = \beta$, $\gamma = \delta$ and $1 < \beta + \gamma \leq 2$ in Theorem 21, then we have the results obtained in [4] and [6].

Next, we present a solution to a dual problem of Theorem 21, in the sense that the subordinations are replaced by superordinations:

Theorem 22 [13] *Let* $f, g \in \mathcal{A}_{\varphi;\alpha,\delta}$, *where* $\mathcal{A}_{\varphi;\alpha,\delta}$ *is defined by* (6.26). Suppose also that

$$\mathrm{Re}\left[1 + \frac{z\nu''(z)}{\nu'(z)}\right] > -\rho,\ z \in \mathrm{U},$$

where $\nu(z) = z\left[\frac{g(z)}{z}\right]^{\alpha}\varphi(z)$ and ρ is given by (6.27). If the function $z\left[\frac{f(z)}{z}\right]^{\alpha}\varphi(z)$ is univalent in U and $z\left[\frac{\mathrm{I}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}(f)(z)}{z}\right]^{\beta}\phi(z) \in \mathcal{Q}$, where $\mathrm{I}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}$ is the integral operator defined by (6.25), then the following superordination relation

$$z\left[\frac{g(z)}{z}\right]^{\alpha}\varphi(z) \prec z\left[\frac{f(z)}{z}\right]^{\alpha}\varphi(z)$$

implies that

$$z\left[\frac{\mathrm{I}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}(g)(z)}{z}\right]^{\beta}\phi(z) \prec z\left[\frac{\mathrm{I}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}(f)(z)}{z}\right]^{\beta}\phi(z).$$

Moreover, the function $z\left[\frac{\mathrm{I}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}(g)(z)}{z}\right]^{\beta}\phi(z)$ is the best subordinant.

Remark 4 If we take the complex parameters α, β, γ and δ in Theorem 22 as in Lemma 3, then we also have the results obtained in [7].

If we combine the above two theorems, we obtain the following *sandwich-type theorem*:

Theorem 23 [13] *Let* $f, g_k \in \mathcal{A}_{\varphi;\alpha,\delta}$, $k = 1, 2$, *where* $\mathcal{A}_{\varphi;\alpha,\delta}$ *is defined by* (6.26). Suppose also that

$$\mathrm{Re}\left[1 + \frac{z\nu_k''(z)}{\nu_k'(z)}\right] > -\rho,\ z \in \mathrm{U},$$

where $\nu_k(z) = z\left[\frac{g_k(z)}{z}\right]^{\alpha}\varphi(z)$, $k = 1, 2$, and ρ is given by (6.27). If the function $z\left[\frac{f(z)}{z}\right]^{\alpha}\varphi(z)$ is univalent in U and $z\left[\frac{\mathrm{I}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}(f)(z)}{z}\right]^{\beta}\phi(z) \in \mathcal{Q}$, where $\mathrm{I}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}$ is the integral operator defined by (6.25), then the following subordination relations

$$z\left[\frac{g_1(z)}{z}\right]^{\alpha}\varphi(z) \prec z\left[\frac{f(z)}{z}\right]^{\alpha}\varphi(z) \prec z\left[\frac{g_2(z)}{z}\right]^{\alpha}\varphi(z)$$

imply that

$$z\left[\frac{\mathrm{I}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}(g_1)(z)}{z}\right]^{\beta}\phi(z) \prec z\left[\frac{\mathrm{I}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}(f)(z)}{z}\right]^{\beta}\phi(z) \prec z\left[\frac{\mathrm{I}^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}(g_2)(z)}{z}\right]^{\beta}\phi(z).$$

Moreover, the functions $z\left[\frac{\mathrm{I}_{\alpha,\beta,\gamma,\delta}^{\phi,\varphi}(g_1)(z)}{z}\right]^{\beta}\phi(z)$ and $z\left[\frac{\mathrm{I}_{\alpha,\beta,\gamma,\delta}^{\phi,\varphi}(g_2)(z)}{z}\right]^{\beta}\phi(z)$ are the best subordinant and the best dominant, respectively.

Remark 5 1. If we choose the complex parameters α, β, γ and δ with $1 < \beta+\gamma \leq 2$ in Theorem 23, then we have the results obtained in [11].

2. We also note that Theorem 23 is an extension of the results obtained by Bulboacă in [7], and Cho and Bulboacă in [12].

6.7 Sandwich-type Theorems for a Class of Integral Operators: the $\mathrm{A}_{h,\beta}$ Operator

For a function $h \in \mathbf{A}$, where

$$\mathbf{A} = \left\{h \in A : h(z) \neq 0,\ z \in \dot{\mathrm{U}} = \mathrm{U} \setminus \{0\}, h'(z) \neq 0,\ z \in \mathrm{U}\right\},$$

we define the integral operator $\mathrm{A}_{h,\beta} : \mathcal{K}_{h;\beta} \to H(\mathrm{U})$ by

$$\mathrm{A}_{h,\beta}(f)(z) = \left[\beta \int_0^z f^{\beta}(t)h^{-1}(t)h'(t)\,\mathrm{d}t\right]^{1/\beta},$$

where $\beta \in \mathbb{C}^*$, and all powers are the principal ones. The subset $\mathcal{K}_{h;\beta} \subset H(\mathrm{U})$ was determined in [10], as follows:

Lemma 4 [10] *Let* $\beta \in \mathbb{C}$ *with* $\operatorname{Re}\beta > 0$, *let* $h \in \mathbf{A}$ *and denote by*

$$J(\gamma, h)(z) = (\gamma - 1)\frac{zh'(z)}{h(z)} + 1 + \frac{zh''(z)}{h'(z)}.$$

If R_β *represents the open door function, and if*

$$\widetilde{\mathcal{K}}_{h;\beta} = \left\{f \in A : \beta\frac{zf'(z)}{f(z)} + J(0,h)(z) \prec R_\beta(z)\right\},\ \textit{for}\ \beta \neq 1,$$
$$\widetilde{\mathcal{K}}_{h;1} = \{f \in H(\mathrm{U}) : f(0) = 0\},\ \textit{for}\ \beta = 1,$$

then the integral operator $\mathrm{A}_{h,\beta}$ *is well-defined on* $\widetilde{\mathcal{K}}_{h;\beta}$.

Lemma 5 [10] *Let* $\beta \in \mathbb{C}$ *with* $\operatorname{Re}\beta > 0$, *and let* $h \in \mathbf{A}$. *If*

$$\mathcal{K}_{h;\beta} = \widetilde{\mathcal{K}}_{h;\beta},\ \textit{for}\ \beta \neq 1,$$
$$\mathcal{K}_{h;1} = \left\{f \in \widetilde{\mathcal{K}}_{h;1} : f'(0) \neq 0\right\},\ \textit{for}\ \beta = 1,$$

then the integral operator $\mathrm{A}_{h,\beta}$ *is well-defined on* $\mathcal{K}_{h;\beta}$ *and satisfies the following conditions:*

$$F = \mathrm{A}_{h;\beta}[f] \in A,\ \frac{F(z)}{z} \neq 0,\ z \in \mathrm{U},\ \operatorname{Re}\left[\beta\frac{zF'(z)}{F(z)}\right] > 0,\ z \in \mathrm{U},\ \textit{for}\ \beta \neq 1,$$

and

$$F(z) = \mathrm{A}_{h,\beta}[f](z) = f'(0)z + \dots, \ z \in \mathrm{U}, \quad \textit{for } \beta = 1.$$

In [3] the author determined conditions on the h and g functions and on the parameter β, such that

$$\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g(z) \Rightarrow \mathrm{A}_{h,\beta}(f)(z) \prec \mathrm{A}_{h,\beta}(g)(z). \tag{6.28}$$

Now we will show an improvement of the above result, then we will study the reverse problem to determine simple sufficient conditions on h, g and β, such that

$$\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z) \Rightarrow \mathrm{A}_{h,\beta}(g)(z) \prec \mathrm{A}_{h,\beta}(f)(z), \tag{6.29}$$

and under our assumptions this result is sharp.

Combining these results we obtained a so-called *sandwich-type theorem*, and we gave some interesting particular results obtained for convenient choices of the h function.

The next result deals with the subordination of the form (6.28) and gives us an extension of Theorem 1 of [3]:

Theorem 24 [10] *Let $\beta > 0$ and let $h \in \mathbf{A}$. Let $f, g \in \mathcal{K}_{h;\beta}$ and suppose that*

$$\operatorname{Re}\frac{zg'(z)}{g(z)} > -\frac{1}{\beta}\operatorname{Re} J(0,h)(z), \ z \in \mathrm{U},$$

Then,

$$\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g(z) \Rightarrow \mathrm{A}_{h,\beta}(f)(z) \prec \mathrm{A}_{h,\beta}(g)(z),$$

and the function $\mathrm{A}_{h,\beta}(g)$ is the best dominant of the subordination.

The next theorem represents a dual result of Theorem 24, in the sense that the subordinations are replaced by superordinations.

Theorem 25 [10] *Let $\beta > 0$ and let $h \in \mathbf{A}$. Let $g \in \mathcal{K}_{h;\beta}$ and suppose that*

$$\operatorname{Re}\frac{zg'(z)}{g(z)} > -\frac{1}{\beta}\operatorname{Re} J(0,h)(z), \ z \in \mathrm{U}.$$

Let $f \in \mathcal{Q} \cap \mathcal{K}_{h;\beta}$ such that $\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z)$ and $\mathrm{A}_{h,\beta}(f)(z)$ are univalent functions in U.

Then,

$$\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z) \Rightarrow \mathrm{A}_{h,\beta}(g)(z) \prec \mathrm{A}_{h,\beta}(f)(z),$$

and the function $\mathrm{A}_{h,\beta}(g)$ is the best subordinant of the superordination.

If we combine these two results we obtain the following *sandwich-type theorem*:

Theorem 26 [10] *Let $\beta > 0$ and let $h \in \mathbf{A}$. Let $g_1, g_2 \in \mathcal{K}_{h;\beta}$ and suppose that the next two conditions are satisfied*

$$\operatorname{Re}\frac{zg_k'(z)}{g_k(z)} > -\frac{1}{\beta}\operatorname{Re}J(0,h)(z),\ z \in \mathrm{U}, \quad \textit{for} \quad k = 1, 2. \tag{6.30}$$

Let $f \in \mathcal{Q} \cap \mathcal{K}_{h;\beta}$ such that $\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z)$ and $\mathrm{A}_{h,\beta}(f)(z)$ are univalent functions in U.

Then,

$$\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g_1(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g_2(z)$$

implies

$$\mathrm{A}_{h,\beta}(g_1)(z) \prec \mathrm{A}_{h,\beta}(f)(z) \prec \mathrm{A}_{h,\beta}(g_2)(z).$$

Moreover, the functions $\mathrm{A}_{h,\beta}(g_1)$ and $\mathrm{A}_{h,\beta}(g_2)$ are the best subordinant and the best dominant, respectively .

Since in the assumption of the above theorem we need to suppose that the functions $\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z)$ and $\mathrm{A}_{h,\beta}(f)(z)$ are univalent in U, the next similar result will give us, in addition, sufficient conditions that imply the univalence of these functions:

Corollary 15 [10] *Let $\beta > 0$ and let $h \in \mathbf{A}$. Let $g_1, g_2 \in \mathcal{K}_{h;\beta}$ and suppose that the conditions* (6.30) are satisfied.

Let $f \in \mathcal{Q} \cap \mathcal{K}_{h;\beta}$ such that

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > -\frac{1}{\beta}\operatorname{Re}J(0,h)(z),\ z \in \mathrm{U}. \tag{6.31}$$

Then,

$$\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g_1(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g_2(z)$$

implies

$$\mathrm{A}_{h,\beta}(g_1)(z) \prec \mathrm{A}_{h,\beta}(f)(z) \prec \mathrm{A}_{h,\beta}(g_2)(z).$$

Moreover, the functions $\mathrm{A}_{h,\beta}(g_1)$ and $\mathrm{A}_{h,\beta}(g_2)$ are the best subordinant and the best dominant, respectively.

Next we will discuss some particular cases of Theorem 26 obtained for appropriate choices of the h function.

1°. For the special case $h(z) = z\exp(\lambda z)$, $|\lambda| < 1$, we obtain the next example:

Example 6 Let $\beta > 0$ and $g_1, g_2 \in \mathcal{K}_{z\exp(\lambda z);\beta}$, where $|\lambda| < 1$. Suppose that the next two conditions are satisfied

$$\operatorname{Re}\frac{zg_k'(z)}{g_k(z)} > \frac{1}{\beta}\frac{|\lambda|}{1-|\lambda|},\ z \in \mathrm{U}, \quad \text{for} \quad k = 1, 2.$$

Let $f \in \mathcal{Q} \cap \mathcal{K}_{z\exp(\lambda z);\beta}$ such that $(1+\lambda z)^{1/\beta} f(z)$ and $\left[\beta \int_0^z f^\beta(t)\frac{1+\lambda t}{t}\,\mathrm{d}t\right]^{1/\beta}$ are univalent functions in U.

Then,

$$(1+\lambda z)^{1/\beta} g_1(z) \prec (1+\lambda z)^{1/\beta} f(z) \prec (1+\lambda z)^{1/\beta} g_2(z)$$

implies

$$\left[\beta \int_0^z g_1^\beta(t)\frac{1+\lambda t}{t}\,\mathrm{d}t\right]^{1/\beta} \prec \left[\beta \int_0^z f^\beta(t)\frac{1+\lambda t}{t}\,\mathrm{d}t\right]^{1/\beta} \prec \left[\beta \int_0^z g_2^\beta(t)\frac{1+\lambda t}{t}\,\mathrm{d}t\right]^{1/\beta}.$$

Moreover, the functions $\left[\beta \int_0^z g_1^\beta(t)\frac{1+\lambda t}{t}\,\mathrm{d}t\right]^{1/\beta}$ and $\left[\beta \int_0^z g_2^\beta(t)\frac{1+\lambda t}{t}\,\mathrm{d}t\right]^{1/\beta}$ are the best subordinant and the best dominant, respectively.

Remark 6 1. According to Corollary 15, if $f \in \mathcal{Q} \cap \mathcal{K}_{z\exp(\lambda z);\beta}$ satisfies the condition

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \frac{1}{\beta}\frac{|\lambda|}{1-|\lambda|},\ z \in \mathrm{U},$$

then it is not necessary to assume that $(1+\lambda z)^{1/\beta} f(z)$ and $\left[\beta \int_0^z f^\beta(t)\frac{1+\lambda t}{t}\,\mathrm{d}t\right]^{1/\beta}$ are univalent functions in U.

2. For the special case $\beta = 1$ and $\lambda = 0$, the right-hand side of the Example 6 represents a generalisation of a result due to Suffridge [26]. In addition, the left-hand side generalises Theorem 9 from [21].

2°. For the special case $h(z) = \dfrac{z}{1+\lambda z}$, $|\lambda| \le 1$, from Theorem 26 we have:

Example 7 Let $\beta > 0$ and $g_1, g_2 \in \mathcal{K}_{\frac{z}{1+\lambda z};\beta}$, where $|\lambda| \le 1$. Suppose that the next two conditions are satisfied

$$\operatorname{Re}\frac{zg_k'(z)}{g_k(z)} > \frac{1}{\beta}\frac{|\lambda|}{1+|\lambda|},\ z \in \mathrm{U}, \quad \text{for} \quad k = 1, 2.$$

Let $f \in \mathcal{Q} \cap \mathcal{K}_{\frac{z}{1+\lambda z};\beta}$ such that $\dfrac{f(z)}{(1+\lambda z)^{1/\beta}}$ and $\left[\beta \int_0^z \dfrac{f^\beta(t)}{t(1+\lambda t)}\,\mathrm{d}t\right]^{1/\beta}$ are univalent functions in U.

Then,

$$\frac{g_1(z)}{(1+\lambda z)^{1/\beta}} \prec \frac{f(z)}{(1+\lambda z)^{1/\beta}} \prec \frac{g_2(z)}{(1+\lambda z)^{1/\beta}}$$

implies

$$\left[\beta\int_0^z \frac{g_1^{\beta}(t)}{t(1+\lambda t)}\,\mathrm{d}t\right]^{1/\beta} \prec \left[\beta\int_0^z \frac{f^{\beta}(t)}{t(1+\lambda t)}\,\mathrm{d}t\right]^{1/\beta} \prec \left[\beta\int_0^z \frac{g_2^{\beta}(t)}{t(1+\lambda t)}\,\mathrm{d}t\right]^{1/\beta}.$$

Moreover, the functions $\left[\beta\int_0^z \frac{g_1^{\beta}(t)}{t(1+\lambda t)}\,\mathrm{d}t\right]^{1/\beta}$ and $\left[\beta\int_0^z \frac{g_2^{\beta}(t)}{t(1+\lambda t)}\,\mathrm{d}t\right]^{1/\beta}$ are the best subordinant and the best dominant, respectively.

Remark 7 1. From the Corollary 15, we deduce that, if $f \in \mathcal{Q} \cap \mathcal{K}_{\frac{z}{1+\lambda z};\beta}$ satisfies the condition

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \frac{1}{\beta}\frac{|\lambda|}{1+|\lambda|},\ z \in \mathrm{U},$$

then it is not necessary to assume that $\frac{f(z)}{(1+\lambda z)^{1/\beta}}$ and $\left[\beta\int_0^z \frac{f^{\beta}(t)}{t(1+\lambda t)}\,\mathrm{d}t\right]^{1/\beta}$ are univalent functions in U.

2. For the special case $\beta = 1$ and $\lambda = 0$, the right-hand side of this Example generalises a result due to Suffridge [26], and the left-hand side generalises Theorem 9 from [21].

3°. For the special case $h(z) = z\exp\int_0^z \frac{e^{\lambda t}-1}{t}\,\mathrm{d}t$, $\lambda \in \mathbb{C}$, Theorem 26 reduces to the next example:

Example 8 Let $\beta > 0$ and $g_1, g_2 \in \mathcal{K}_{h;\beta}$, where

$$h(z) = z\exp\int_0^z \frac{e^{\lambda t}-1}{t}\,\mathrm{d}\,t$$

and $\lambda \in \mathbb{C}$. Suppose that the next two conditions are satisfied

$$\operatorname{Re}\frac{zg_k'(z)}{g_k(z)} > \frac{|\lambda|}{\beta},\ z \in \mathrm{U}, \quad \text{for} \quad k = 1, 2.$$

Let $f \in \mathcal{Q} \cap \mathcal{K}_{h;\beta}$ such that $f(z)\exp(\lambda z/\beta)$ and $\left[\beta\int_0^z f^{\beta}(t)\frac{\exp(\lambda t)}{t}\,\mathrm{d}t\right]^{1/\beta}$ are univalent functions in U.

Then,

$$g_1(z)\exp\left(\frac{\lambda z}{\beta}\right) \prec f(z)\left(\frac{\lambda z}{\beta}\right) \prec g_2(z)\exp\left(\frac{\lambda z}{\beta}\right)$$

implies

$$\left[\beta\int_0^z g_1^{\beta}(t)\frac{\exp(\lambda t)}{t}\,\mathrm{d}t\right]^{1/\beta} \prec \left[\beta\int_0^z f^{\beta}(t)\frac{\exp(\lambda t)}{t}\,\mathrm{d}t\right]^{1/\beta} \prec \left[\beta\int_0^z g_2^{\beta}(t)\frac{\exp(\lambda t)}{t}\,\mathrm{d}t\right]^{1/\beta}.$$

Moreover, the functions $\left[\beta\int_0^z g_1^{\beta}(t)\frac{\exp(\lambda t)}{t}\,\mathrm{d}t\right]^{1/\beta}$ and $\left[\beta\int_0^z g_2^{\beta}(t)\frac{\exp(\lambda t)}{t}\,\mathrm{d}t\right]^{1/\beta}$ are the best subordinant and the best dominant, respectively.

Remark 8 1. As in the previous remarks, from Corollary 15 we obtain that if $f \in \mathcal{Q}\cap\mathcal{K}_{h;\beta}$, where $h(z) = z\exp\int_0^z \frac{e^{\lambda t}-1}{t}\,\mathrm{d}t$, satisfies the condition

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \frac{|\lambda|}{\beta},\ z \in \mathrm{U},$$

then it is not necessary to assume that $f(z)\exp(\lambda z/\beta)$ and $\left[\beta\int_0^z f^{\beta}(t)\frac{\exp(\lambda t)}{t}\,\mathrm{d}t\right]^{1/\beta}$ are univalent functions in U.

2. For the special case $\beta = 1$ and $\lambda = 0$, the right-hand side of the Example 8 extends a result of T. J. Suffridge [26]. In addition, the left-hand side is an extension of Theorem 9 from [21].

For $\alpha \in \mathbb{R}$ and $\theta < 1$, a function $f \in H(\mathrm{U})$ with $f(0) = 0$ and $f'(0) \neq 0$ is called to be an α-*convex* (not necessarily normalized) *function of order* θ, if

$$\operatorname{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf''(z)}{f'(z)}+1\right)\right] > \theta,\ z \in \mathrm{U},$$

and we denote this class by $M_{\alpha}(\theta)$. For $\theta = 0$ we have $M_{\alpha} \equiv M_{\alpha}(0)$, where M_{α} represents the class of α-*convex* (not necessarily normalized) *functions* introduced in [24]. Note that all α-convex functions are univalent and starlike [23], that is $M_{\alpha} \subset M_0$.

The next two results deal with the subordination of the form (6.28) and give us extensions of Theorem 1 of [3]:

Theorem 27 [9] *Let* $\alpha, \beta, \theta \in \mathbb{R}$, *with* $\beta > 0$, $\alpha\beta \geq 1$ *and* $0 \leq \theta < 1$. *Let* $f, g \in \mathcal{K}_{h;\beta}$, *where* $h \in \mathbf{A}$, *and suppose that*

$$\begin{aligned} &g \in M_{\alpha}(\theta),\\ &\operatorname{Re} J(0,h)(z) > -\frac{\theta}{\alpha},\ z \in \mathrm{U}. \end{aligned} \tag{6.32}$$

Then,

$$\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g(z) \Rightarrow \mathrm{A}_{h,\beta}(f)(z) \prec \mathrm{A}_{h,\beta}(g)(z),$$

and the function $\mathrm{A}_{h,\beta}(f)(g)$ *is the best dominant of the subordination.*

The next theorem is an improvement of the above, that means the conclusion holds if we make stronger assumption under the parameter θ, and the condition (6.32) is replaced by a weaker one:

Theorem 28 [9] Let $\alpha, \beta, \theta \in \mathbb{R}$, *with* $\beta > 0$, $\alpha\beta \geq 1$ *and*

$$\max\left\{\frac{\alpha(\beta-1)}{2}; 0\right\} \leq \theta < 1.$$

Let $f, g \in \mathcal{K}_{h;\beta}$, *where* $h \in \mathbf{A}$, *and suppose that*

$$g \in M_\alpha(\theta),$$

$$\operatorname{Re} J(0, h)(z) > -\frac{\beta}{{}_2F_1\left(1, 2\left(\beta - \frac{\theta}{\alpha}\right), \beta + 1; 1/2\right)},\ z \in \mathrm{U}.$$

Then,

$$\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g(z) \Rightarrow \mathrm{A}_{h,\beta}(f)(z) \prec \mathrm{A}_{h,\beta}(g)(z),$$

and the function $\mathrm{A}_{h,\beta}(g)$ *is the best dominant of the subordination.*

The next result represents a dual result of Theorem 27, in the sense that the subordinations are replaced by superordinations, and it deals with the superordination of the form (6.29):

Theorem 29 [9] *Let* $\alpha, \beta, \theta \in \mathbb{R}$, *with* $\beta > 0$, $\alpha\beta \geq 1$ *and* $0 \leq \theta < 1$. *Let* $g \in \mathcal{K}_{h;\beta}$, *where* $h \in \mathbf{A}$, *and suppose that*

$$g \in M_\alpha(\theta),$$

$$\operatorname{Re} J(0, h)(z) > -\frac{\theta}{\alpha},\ z \in \mathrm{U}.$$

Let $f \in \mathcal{Q} \cap \mathcal{K}_{h;\beta}$ *such that* $\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z)$ *and* $\mathrm{A}_{h,\beta}(f)(z)$ *are univalent functions in* U.

Then,

$$\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z) \Rightarrow \mathrm{A}_{h,\beta}(g)(z) \prec \mathrm{A}_{h,\beta}(f)(z),$$

and the function $\mathrm{A}_{h,\beta}(g)$ *is the best subordinant of the superordination.*

The next theorem represents a similar dual result of Theorem 28, where the subordinations are replaced by superordinations. It also deals with the superordination of the form (6.29):

Theorem 30 [9] *Let* $\alpha, \beta, \theta \in \mathbb{R}$, *with* $\beta > 0$, $\alpha\beta \geq 1$ *and*

$$\max\left\{\frac{\alpha(\beta-1)}{2}; 0\right\} \leq \theta < 1.$$

Let $g \in \mathcal{K}_{h;\beta}$, *where* $h \in \mathbf{A}$, *and suppose that*

$$g \in M_\alpha(\theta),$$

$$\operatorname{Re} J(0,h)(z) > -\frac{\beta}{{}_2F_1\left(1, 2\left(\beta - \frac{\theta}{\alpha}\right), \beta+1; 1/2\right)}, \ z \in \mathrm{U}.$$

Let $f \in \mathcal{Q} \cap \mathcal{K}_{h;\beta}$ *such that* $\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z)$ *and* $\mathrm{A}_{h,\beta}(f)(z)$ *are univalent functions in* U.

Then,

$$\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z) \Rightarrow \mathrm{A}_{h,\beta}(g)(z) \prec \mathrm{A}_{h,\beta}(f)(z),$$

and the function $\mathrm{A}_{h,\beta}(g)$ *is the best subordinant of the superordination.*

If we combine Theorem 27 with Theorem 29, and respectively Theorem 28 with Theorem 30, we obtain the next two corollaries, that represent the *sandwich-type theorems*:

Corollary 16 [9] *Let* $\alpha, \beta, \theta \in \mathbb{R}$, *with* $\beta > 0$, $\alpha\beta \geq 1$ *and* $0 \leq \theta < 1$. *Let* $g_1, g_2 \in \mathcal{K}_{h;\beta}$, *where* $h \in \mathbf{A}$, *and suppose that the next two conditions are satisfied:*

$$g_1, g_2 \in M_\alpha(\theta), \tag{6.33}$$

$$\operatorname{Re} J(0,h)(z) > -\frac{\theta}{\alpha}, \ z \in \mathrm{U}. \tag{6.34}$$

Let $f \in \mathcal{Q} \cap \mathcal{K}_{h;\beta}$ *such that* $\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z)$ *and* $\mathrm{A}_{h,\beta}(f)(z)$ *are univalent functions in* U.

Then,

$$\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g_1(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g_2(z)$$

implies

$$\mathrm{A}_{h,\beta}(g_1)(z) \prec \mathrm{A}_{h,\beta}(f)(z) \prec \mathrm{A}_{h,\beta}(g_2)(z).$$

Moreover, the functions $\mathrm{A}_{h,\beta}(f)(g_1)$ and $\mathrm{A}_{h,\beta}(f)(g_2)$ are the best subordinant and the best dominant, respectively.

Corollary 17 [9] *Let* $\alpha, \beta, \theta \in \mathbb{R}$, *with* $\beta > 0$, $\alpha\beta \geq 1$ *and*

$$\max\left\{\frac{\alpha(\beta-1)}{2}; 0\right\} \leq \theta < 1.$$

Let $g_1, g_2 \in \mathcal{K}_{h;\beta}$, *where* $h \in \mathbf{A}$, *and suppose that the next two conditions are satisfied:*

$$g_1, g_2 \in M_\alpha(\theta), \tag{6.35}$$

$$\operatorname{Re} J(0,h)(z) > -\frac{\beta}{{}_2F_1\left(1, 2\left(\beta - \frac{\theta}{\alpha}\right), \beta+1; 1/2\right)}, \ z \in \mathrm{U}. \tag{6.36}$$

Let $f \in \mathcal{Q} \cap \mathcal{K}_{h;\beta}$ *such that* $\left[\dfrac{zh'(z)}{h(z)}\right]^{1/\beta} f(z)$ *and* $\mathrm{A}_{h,\beta}(f)(z)$ *are univalent functions in* U.

Then,

$$\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g_1(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g_2(z)$$

implies

$$\mathrm{A}_{h,\beta}(g_1)(z) \prec \mathrm{A}_{h,\beta}(f)(z) \prec \mathrm{A}_{h,\beta}(g_2)(z).$$

Moreover, the functions $\mathrm{A}_{h,\beta}(g_1)$ *and* $\mathrm{A}_{h,\beta}(g_2)$ *are the best subordinant and the best dominant, respectively.*

In the assumption of the above two corollaries we need to suppose that the functions $\left[\dfrac{zh'(z)}{h(z)}\right]^{1/\beta} f(z)$ and $\mathrm{A}_{h,\beta}(f)(z)$ are univalent in U. The next similar two results will give us sufficient conditions that imply, in addition, the univalence of these functions:

Corollary 18 [9] *Let* $\alpha, \beta, \theta \in \mathbb{R}$, *with* $\beta > 0$, $\alpha\beta \geq 1$ *and* $0 \leq \theta < 1$. *Let* $g_1, g_2 \in \mathcal{K}_{h;\beta}$, *where* $h \in \mathbf{A}$, *and suppose that the conditions* (6.33) and (6.34) are satisfied. Let $f \in \mathcal{Q} \cap \mathcal{K}_{h;\beta}$ such that $f \in M_\alpha(\theta)$.

Then,

$$\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g_1(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g_2(z)$$

implies

$$\mathrm{A}_{h,\beta}(g_1)(z) \prec \mathrm{A}_{h,\beta}(f)(z) \prec \mathrm{A}_{h,\beta}(g_2)(z).$$

Moreover, the functions $\mathrm{A}_{h,\beta}(g_1)$ and $\mathrm{A}_{h,\beta}(g_2)$ are the best subordinant and the best dominant, respectively.

Corollary 19 [9] *Let* $\alpha, \beta, \theta \in \mathbb{R}$, *with* $\beta > 0$, $\alpha\beta \geq 1$ *and*

$$\max\left\{\frac{\alpha(\beta-1)}{2}; 0\right\} \leq \theta < 1.$$

Let $g_1, g_2 \in \mathcal{K}_{h;\beta}$, *where* $h \in \mathbf{A}$, *and suppose that the conditions* (6.35) and (6.36) are satisfied. Let $f \in \mathcal{Q} \cap \mathcal{K}_{h;\beta}$ such that $f \in M_\alpha(\theta)$.

Then,

$$\left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g_1(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} f(z) \prec \left[\frac{zh'(z)}{h(z)}\right]^{1/\beta} g_2(z)$$

implies

$$\mathrm{A}_{h,\beta}(g_1)(z) \prec \mathrm{A}_{h,\beta}(f)(z) \prec \mathrm{A}_{h,\beta}(g_2)(z).$$

Moreover, the functions $\mathrm{A}_{h,\beta}(g_1)$ and $\mathrm{A}_{h,\beta}(g_2)$ are the best subordinant and the best dominant, respectively.

6.8 Sandwich-Type Theorems for a Class of Integral Operators: the $\mathcal{I}_\lambda^\alpha$ Operator

In a recent paper, Li and Srivastava [14] introduced and studied the function class Φ defined by

$$\Phi = \left\{ \lambda : [0,1] \to \mathbb{R} : \lambda(t) \geq 0 \text{ and } \int_0^1 \lambda(t)\,\mathrm{d}t = 1 \right\}.$$

Fournier and Ruscheweyh [14] (see also [17]) have considered an integral operator which involves a non-negative function

$$\lambda_\alpha : [0,1] \to \mathbb{R} \quad \text{such that} \quad \int_0^1 \lambda_\alpha(t)\,\mathrm{d}t = 1.$$

Many applications of the real valued function λ_α depends also upon a suitable parameter α. Thus, in [15] the authors considered the Fournier–Ruscheweyh integral operator in the following modified form (see also [17, Eq. (2.2), p. 131])

$$\mathcal{I}_\lambda^\alpha f(z) = \int_0^1 \lambda_\alpha(t) \frac{f(tz)}{t}\,\mathrm{d}t, \quad f \in A,$$

where the real-valued functions λ_α and $\lambda_{\alpha-1}$ satisfy the following conditions:

(i) for a suitable parameter α,

$$\lambda_{\alpha-1} \in \Phi,\ \lambda_\alpha \in \Phi \text{ and } \lambda_\alpha(1) = 0; \tag{6.37}$$

(ii) there exists a constant $c \in (-1, 2]$, such that

$$c\lambda_\alpha(t) - t\lambda_\alpha'(t) = (c+1)\lambda_{\alpha-1}(t),\ 0 < t < 1. \tag{6.38}$$

Remark that, for $\mathcal{I}_\lambda^\alpha$ operator, under the conditions (6.37) and (6.38), we have

$$z\left(\mathcal{I}_\lambda^\alpha f(z)\right)' = -c\,\mathcal{I}_\lambda^\alpha f(z) + (c+1)\,\mathcal{I}_\lambda^{\alpha-1} f(z).$$

Moreover, we assume in this section that all powers are the principal ones.

Theorem 31 [15] *Let q be univalent in the unit disc* U, *let $\beta \in \mathbb{C}^*$ and $0 < \mu < 1$. Suppose that the function q satisfies*

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\operatorname{Re}\frac{\mu}{\beta}\right\},\ z \in \mathrm{U}, \tag{6.39}$$

and the conditions (6.37) and (6.38) hold. If $f \in A$ satisfies the subordination

$$(1+\beta+\beta c)\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu - \beta(1+c)\left(\frac{\mathcal{I}_\lambda^{\alpha-1} f(z)}{\mathcal{I}_\lambda^\alpha f(z)}\right)\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu \prec q(z) + \frac{\beta}{\mu} zq'(z),$$

then

$$\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^\mu \prec q(z),$$

and q is the best dominant.

Some interesting special cases of Theorem 31, obtained for the dominant $q(z) = \dfrac{1+Az}{1+Bz}$, with $-1 \le A < B \le 1$, and $q(z) = \left(\dfrac{1+z}{1-z}\right)^{\rho}$, with $0 < \rho \le 1$, may be found in [15].

Theorem 32 [15] *Let q be univalent in the unit disc* U*, such that that the function* $\dfrac{zq'(z)}{q(z)}$ *is starlike univalent in* U *and q satisfies*

$$\operatorname{Re}\left(1 + \frac{\rho}{\beta}q(z) + \frac{2\delta}{\beta}q^2(z) - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right) > 0,\ z \in \mathrm{U}, \tag{6.40}$$

where, $\delta, \rho, \in \mathbb{C}$, $\beta \in \mathbb{C}^*$.

Suppose also that the conditions (6.37) and (6.38) are satisfied and for a function $f \in A$ let denote

$$\Psi_{\lambda,\mu,\rho,\eta,\alpha,\beta,\gamma,\delta}(f)(z) = \gamma + \rho\left(\frac{z}{(1-\eta)\mathcal{I}_\lambda^\alpha f(z) + \eta\mathcal{I}_\lambda^{\alpha+1} f(z)}\right)^{\mu} + \tag{6.41}$$

$$\delta\left(\frac{z}{(1-\eta)\mathcal{I}_\lambda^\alpha f(z) + \eta\mathcal{I}_\lambda^{\alpha+1} f(z)}\right)^{2\mu} + \beta\mu(c+1)\left[1 - \frac{(1-\eta)\mathcal{I}_\lambda^{\alpha-1} f(z) + \eta\mathcal{I}_\lambda^{\alpha} f(z)}{(1-\eta)\mathcal{I}_\lambda^\alpha f(z) + \eta\mathcal{I}_\lambda^{\alpha+1} f(z)}\right],$$

where $0 \le \eta \le 1$, $0 < \mu < 1$.

If

$$\Psi_{\lambda,\mu,\rho,\eta,\alpha,\beta,\gamma,\delta}(f)(z) \prec \gamma + \rho q(z) + \delta q^2(z) + \beta\frac{zq'(z)}{q(z)},$$

then

$$\left(\frac{z}{(1-\eta)\mathcal{I}_\lambda^\alpha f(z) + \eta\mathcal{I}_\lambda^{\alpha+1} f(z)}\right)^{\mu} \prec q(z),$$

and q is the best dominant.

Upon setting $q(z) = e^{Az}$ and $\eta = 1$ in Theorem 32, the authors obtained a simple special case given in [15].

Theorem 33 [15] *Let q be a convex univalent function in the unit disc* U*, let $\beta \in \mathbb{C}^*$ with* $\operatorname{Re}\beta > 0$*, and* $0 < \mu < 1$*. Moreover, suppose that the conditions (6.37) and (6.38) are satisfied. For $f \in A$ suppose that*

$$\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^{\mu} \in H[q(0), 1] \cap \mathcal{Q}, \tag{6.42}$$

and

$$(1+\beta+\beta c)\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^{\mu} - \beta(1+c)\left(\frac{\mathcal{I}_\lambda^{\alpha-1} f(z)}{\mathcal{I}_\lambda^\alpha f(z)}\right)\left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)}\right)^{\mu} \text{ is univalent in U.} \tag{6.43}$$

Then,

$$q(z)+\frac{\beta}{\mu}zq'(z)\prec(1+\beta+\beta c)\left(\frac{z}{\mathcal{I}_{\lambda}^{\alpha}f(z)}\right)^{\mu}-\beta(1+c)\left(\frac{\mathcal{I}_{\lambda}^{\alpha-1}f(z)}{\mathcal{I}_{\lambda}^{\alpha}f(z)}\right)\left(\frac{z}{\mathcal{I}_{\lambda}^{\alpha}f(z)}\right)^{\mu}$$

implies that

$$q(z)\prec\left(\frac{z}{\mathcal{I}_{\lambda}^{\alpha}f(z)}\right)^{\mu},$$

and q is the best subordinant.

Taking $q(z)=\dfrac{1+Az}{1+Bz}$, $-1\le A<B\le 1$ in Theorem 33, we easily get the next result:

Corollary 20 [15] *Let $-1\le A<B\le 1$, let $\beta\in\mathbb{C}$ with* $\operatorname{Re}\beta>0$, *and* $0<\mu<1$. *Also, suppose that the conditions* (6.37) *and* (6.38) *hold. For $f\in A$ suppose that*

$$\left(\frac{z}{\mathcal{I}_{\lambda}^{\alpha}f(z)}\right)^{\mu}\in H[q(0),1]\cap\mathcal{Q}$$

and

$$(1+\beta+\beta c)\left(\frac{z}{\mathcal{I}_{\lambda}^{\alpha}f(z)}\right)^{\mu}-\beta(1+c)\left(\frac{\mathcal{I}_{\lambda}^{\alpha-1}f(z)}{\mathcal{I}_{\lambda}^{\alpha}f(z)}\right)\left(\frac{z}{\mathcal{I}_{\lambda}^{\alpha}f(z)}\right)^{\mu}$$

is univalent in U.

Then,

$$\frac{\beta(A-B)z}{\alpha(1+Bz)^{2}}+\frac{1+Az}{1+Bz}\prec(1+\beta+\beta c)\left(\frac{z}{\mathcal{I}_{\lambda}^{\alpha}f(z)}\right)^{\mu}-\beta(1+c)\left(\frac{\mathcal{I}_{\lambda}^{\alpha-1}f(z)}{\mathcal{I}_{\lambda}^{\alpha}f(z)}\right)\left(\frac{z}{\mathcal{I}_{\lambda}^{\alpha}f(z)}\right)^{\mu}$$

implies that

$$\frac{1+Az}{1+Bz}\prec\left(\frac{z}{\mathcal{I}_{\lambda}^{\alpha}f(z)}\right)^{\mu},$$

and the function $\dfrac{1+Az}{1+Bz}$ is the best subordinant.

Theorem 34 [15] *Let q be convex univalent in the unit disc* U *and* $0<\mu<1$. *Suppose that $\dfrac{zq'(z)}{q(z)}$ is starlike univalent in* U *and q satisfies*

$$\operatorname{Re}\left(1+\frac{\rho}{\beta}q(z)+\frac{2\delta}{\beta}q^{2}(z)\right)>0,\ z\in\mathrm{U},\tag{6.44}$$

where $\delta,\rho\in\mathbb{C}$, $\beta\in\mathbb{C}^{*}$, and the conditions (6.37) *and* (6.38) *hold. For $f\in A$, assume that the function*

$$\left(\frac{z}{(1-\eta)\mathcal{I}_{\lambda}^{\alpha}f(z)+\eta\mathcal{I}_{\lambda}^{\alpha+1}f(z)}\right)^{\mu}\in H[q(0),1]\cap\mathcal{Q},\tag{6.45}$$

and the function $\Psi_{\lambda,\mu,\rho,\eta,\alpha,\beta,\gamma,\delta}(f)(z)$ *defined by* (6.41) *is univalent in the unit disc* U.

Then,

$$\gamma + \rho q(z) + \delta(q(z))^2 + \beta \frac{zq'(z)}{q(z)} \prec \Psi_{\lambda,\mu,\rho,\eta,\alpha,\beta,\gamma,\delta}(f)(z)$$

implies that

$$q(z) \prec \left(\frac{z}{(1-\eta)\mathcal{I}_\lambda^\alpha f(z) + \eta \mathcal{I}_\lambda^{\alpha+1} f(z)} \right)^\mu ,$$

and q *is the best subordinant.*

Combining Theorem 31 with Theorem 33, and Theorem 32 with Theorem 34, we obtain the following *sandwich-type results*, respectively:

Theorem 35 [15] *Let* q_1 *be convex univalent and let* q_2 *be univalent functions in the unit disc* U*, let* $\beta \in \mathbb{C}^*$*, with* $\operatorname{Re} \beta > 0$*, and* $0 < \mu < 1$*. Suppose that* q_2 *satisfies the condition* (6.39) *for* $q = q_2$ *and the conditions* (6.37) *and* (6.38) *are satisfied. For* $f \in A$ *assume that the assumptions* (6.42) *for* $q = q_1$ *and* (6.43) *hold.*

Then,

$$q_1(z) + \frac{\beta}{\mu} z q_1'(z) \prec (1 + \beta + \beta c) \left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)} \right)^\mu - \beta(1+c)$$
$$\left(\frac{\mathcal{I}_\lambda^{\alpha-1} f(z)}{\mathcal{I}_\lambda^\alpha f(z)} \right) \left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)} \right)^\mu \prec q_2(z) + \frac{\beta}{\mu} z q_2'(z)$$

implies that

$$q_1(z) \prec \left(\frac{z}{\mathcal{I}_\lambda^\alpha f(z)} \right)^\mu \prec q_2(z),$$

and q_1 *and* q_2 *are the best subordinant and the best dominant, respectively.*

Theorem 36 [15] *Let* q_1 *be convex univalent in the unit disc* U *and* $0 < \mu < 1$*. Suppose that* $\dfrac{zq_1'(z)}{q_1(z)}$ *is starlike univalent in* U *and* q_1 *satisfies the condition* (6.44) *for* $q = q_1$*, where* $\delta, \rho \in \mathbb{C}$*,* $\beta \in \mathbb{C}^*$*. Let* q_2 *be univalent in the unit disc* U*, such that that the function* $\dfrac{zq_2'(z)}{q_2(z)}$ *is starlike univalent in* U *and* q_2 *satisfies the condition* (6.40) *for* $q = q_2$*, and the conditions* (6.37) *and* (6.38) *hold.*

For a function $f \in A$ *suppose that the function* $\Psi_{\lambda,\mu,\rho,\eta,\alpha,\beta,\gamma,\delta}(f)$ *defined by* (6.41) *is univalent in the unit disc* U*, and the condition* (6.45) *holds for* $q = q_1$ *and* $0 \leq \eta \leq 1$.

Then,

$$\gamma + \rho q_1(z) + \delta(q_1(z))^2 + \beta \frac{z{q_1}'(z)}{q_1(z)} \prec \Psi_{\lambda,\mu,\rho,\eta,\alpha,\beta,\gamma,\delta}(f)(z)$$
$$\prec \gamma + \rho q_2(z) + \delta(q_2(z))^2 + \beta \frac{z{q_2}'(z)}{q_2(z)}$$

implies that

$$q_1(z) \prec \left(\frac{z}{(1-\eta)\mathcal{I}_\lambda^\alpha f(z) + \eta \mathcal{I}_\lambda^{\alpha+1} f(z)}\right)^\mu \prec q_2(z),$$

and q_1 and q_2 are the best subordinant and the best dominant, respectively.

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Chapter 7
Starlikeness and Convexity of Certain Integral Transforms by using Duality Technique

Satwanti Devi and A. Swaminathan

7.1 Introduction

Define $\mathcal{A}$ as the class of all analytic functions f in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ which are normalized by the condition $f(0) = 0 = f'(0) - 1$. Hence, the Taylor series representation of the functions $f(z) \in \mathcal{A}$ is of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in \mathbb{D}.$$

The univalent class of functions denoted by $\mathcal{S}$ is the subclass of $\mathcal{A}$, containing all the one–one functions in $\mathbb{D}$. Few important subclasses of the class $\mathcal{S}$ are defined below:

A domain Ω is starlike with respect to a point $a \in \Omega$, if the line segment joining a point a to any point in Ω also lies entirely in Ω. A domain is starlike if it is starlike with respect to the origin. Now we define the subclass $\mathcal{S}^* \subset \mathcal{S}$ which contains all the starlike functions. The function f is said to be in the subclass of starlike functions $\mathcal{S}^*$, if it maps the domain $\mathbb{D}$ conformally onto the region which bounds a starlike domain with respect to the origin. The analytic description of the function $f(z) \in \mathcal{S}^*$ is given by

$$\mathcal{S}^* := \left\{ f \in \mathcal{A} : \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \qquad z \in \mathbb{D} \right\}.$$

If the domain Ω is starlike with respect to each of its points, then the domain Ω is said to be convex, i.e., when the line segment joining any two points in Ω also lies in Ω entirely. Now we define the subclass $\mathcal{C} \subset \mathcal{S}$ which contains all the convex

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functions. A function f is said to be in the subclass of convex functions $\mathcal{C}$, if it maps $\mathbb{D}$ conformally onto the region which bounds the convex domain. The analytic description of the function $f(z) \in \mathcal{C}$ is given by

$$\mathcal{C} := \left\{ f \in \mathcal{A} : \text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \qquad z \in \mathbb{D} \right\}.$$

Another important subclass of $\mathcal{S}$ is the class of close-to-convex functions $\mathcal{K}$ introduced by W. Kaplan [18]. A function $f(z) \in \mathcal{K}$ is said to be in the class of close-to-convex functions with respect to the starlike function g, if the class consists of linearly accessible functions, i.e., the function f satisfies the geometric property that the complement of the image of open unit disk $\mathbb{D}$ under f is the union of closed half-lines such that the corresponding open half-lines are nonintersecting. The analytic characterization of the function $f(z) \in \mathcal{K}$ is given by

$$\mathcal{K} := \left\{ f \in \mathcal{A} : \text{Re}\left(e^{i\theta} \frac{zf'(z)}{g(z)}\right) > 0, \qquad \theta \in \mathbb{R}, \;\; g \in \mathcal{S}^* \text{ and } z \in \mathbb{D} \right\}.$$

Note that $\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}$.

In the sequel, we also use the following generalization of the classes $\mathcal{S}^*$ and $\mathcal{C}$, given respectively as follows:

$$\mathcal{S}^*(\xi) := \left\{ f \in \mathcal{A} \; : \; \text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \xi, \; 0 \le \xi < 1, z \in \mathbb{D} \right\}$$

and

$$\mathcal{C}(\xi) := \left\{ f \in \mathcal{A} \; : \; \text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \xi, \; 0 \le \xi < 1, z \in \mathbb{D} \right\}.$$

Note that $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ and $\mathcal{C}(0) \equiv \mathcal{C}$. Further, it is interesting to observe that for $0 \le \xi < 1, zf' \in \mathcal{S}^*(\xi) \Longleftrightarrow f \in \mathcal{C}(\xi)$.

For $0 \le \sigma \le 1$ and $0 \le \xi < 1$, the function $f \in \mathcal{A}$ is said to be in Pascu class of σ-convex functions of order ξ [25] denoted by $M(\sigma, \xi)$, if

$$\sigma z f'(z) + (1 - \sigma) f(z) \in \mathcal{S}^*(\xi).$$

Note that $M(0, \xi) \equiv \mathcal{S}^*(\xi)$ and $M(1, \xi) \equiv \mathcal{C}(\xi)$ which means that the Pascu class unifies the class of convex and starlike functions, i.e., it provides the smooth passage between the class of convex and starlike functions. It contains some nonunivalent functions also. Analytically, this class is defined as

$$f(z) \in M(\sigma, \xi) \Longleftrightarrow \text{Re}\left(\frac{\sigma z(zf'(z))' + (1-\sigma)zf'(z)}{\sigma z f'(z) + (1-\sigma) f(z)}\right) > \xi.$$

We usually set $M(\sigma) := M(\sigma, 0)$.

7.2 Motivation for the Main Problem and Its Consequences

For each normalized and univalent function $f(z) \in \mathcal{S}$, Bieberbach [12] gave the Bieberbach's theorem which established the estimation for second coefficient, $|a_2| \le 2$. The equality holds if the function f is the rotation of Koebe function. The consequence of Bieberbach's theorem is the Koebe distortion theorem [14, p. 32] which provides the existence of positive sharp upper and lower bounds of $|f'(z)|$ that represent the infinitesimal magnification factor of image curve under the mapping f. For the function $f \in \mathcal{S}$, the Koebe distortion theorem leads to the following

$$\frac{1-|z|}{(1+|z|)^3} \le |f'(z)| \le \frac{1+|z|}{(1-|z|)^3}, \qquad z \in \mathbb{D}.$$

Equality holds only when $f(z)$ is the Koebe function or one of its rotations. This result is useful to obtain the growth theorem [14, p. 33] which states that for the function $f \in \mathcal{S}$,

$$\frac{1}{(1+|z|)^2} \le \left|\frac{f(z)}{z}\right| \le \frac{1}{(1-|z|)^2}, \qquad z \in \mathbb{D}.$$

Equality holds only if $f(z)$ is the Koebe function or one of its rotations.

Since $\operatorname{Re} \dfrac{f(z)}{z} \le \left|\dfrac{f(z)}{z}\right|$, it is interesting to determine the conditions under which the following inequality

$$\frac{1}{(1+|z|)^2} \le \operatorname{Re}\left(\frac{f(z)}{z}\right),$$

holds true.

For the function $f(z) \in \mathcal{S}$, H. Grunsky [16] has obtained the range $0 < |z| \le (e-1)/(e+1) \simeq .462$, for which the above inequality holds. For this purpose, R. Fournier and S. Ruscheweyh [15] considered the real-valued function Λ, which is defined on $[0,1]$ and positive on (0,1). For $f(z) \in \mathcal{S}$, the function L_Λ is defined as

$$L_\Lambda(f)(z) = \inf_{z\in\mathbb{D}} \int_0^1 \Lambda(t) \left(\operatorname{Re}\frac{f(tz)}{tz} - \frac{1}{(1+t)^2}\right) dt.$$

If we consider $f(z)$ as the Koebe function, then $L_\Lambda(f)(z) \le 0$ for all acceptable weight functions Λ.

This concept leads to investigating various subclasses of $\mathcal{S}$ for which results of this type hold true and the objective of this chapter is to outline all related works and their consequences in this direction available in the literature. Hence for important subclasses of $\mathcal{S}$, corresponding weight functions are obtained for which $L_\Lambda(\mathcal{S}) = 0$.

Initially R. Fournier and S. Ruscheweyh [15] obtained the condition on weight function $\Lambda(t)$ and gave the following result.

Theorem 1 [15] *Let the function $\Lambda(t)$ be integrable on* $[0, 1]$ *and positive on* $(0, 1)$. *If*

$$\frac{\Lambda(t)}{1-t^2}, \qquad t \in (0,1) \tag{7.1}$$

is decreasing, then $L_\Lambda(\mathcal{K}) = 0$.

The following function

$$\Lambda_c(t) = \begin{cases} \dfrac{(1-t^c)}{c}, & -1 < c \le 2, c \neq 0; \\ \\ log(1/t), & c = 0, \end{cases} \tag{7.2}$$

was considered, since it satisfies the condition (7.1). For larger values of $c > 7$, the corresponding function $L_{\Lambda_c}(\mathcal{K}) < 0$.

For the function $f \in \mathcal{A}$, R. Fournier and S. Ruscheweyh [15] introduced the class $P(\beta)$ which satisfies the property that

$$P(\beta) = \left\{ f \in \mathcal{A} \,:\, \mathrm{Re}\, e^{i\phi}(f'(z) - \beta) > 0,\ \beta < 1,\ z \in \mathbb{D} \right\}. \tag{7.3}$$

The integral operator $V_\lambda(f)(z)$ was considered which is defined as

$$V_\lambda(f)(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t}\, dt, \tag{7.4}$$

where $\lambda(t)$ is the real-valued nonnegative function of $t \in [0, 1]$ with the property that $\int_0^1 \lambda(t) dt = 1$.

Theorem 2 [15] *Let* $f \in P(\beta)$. *If* $\beta < 1$ *satisfies*

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t) \frac{(1-t)}{(1+t)}\, dt, \tag{7.5}$$

then, $V_\lambda(f)(z) \in \mathcal{S}$. *Further, if* β *and* λ *satisfy* (7.5) *and in addition* $t\,\Lambda(t) = t \int_t^1 \frac{\lambda(s)}{s} ds \to 0$, *as* $t \to 0$, *then* $V_\lambda(P(\beta)) \in \mathcal{S}^* \iff L_\Lambda(\mathcal{K}) = 0$.

These two results lead to various generalizations. In one direction, these results are proved for the class $\mathcal{S}^*(\xi)$ and are further extended for $\mathcal{C}$ and $\mathcal{C}(\xi)$, $0 \le \xi < 1$. In another direction, the class $P(\beta)$ given in (7.3) has been generalized to study the result for more general classes. All these results are based on Theorems 1 and 2, wherein the condition (7.1) is modified as per requirement. The proofs also follow in a somewhat similar fashion as the proofs of these theorems. Hence, the proofs of Theorems 1 and 2 are important. The best way to understand the proofs given in [15] is as follows: (1) Prove the first part of Theorem 2, i.e., $V_\lambda(P(\beta)) \subset \mathcal{S}$, (2) Prove the second part of Theorem 2, namely the condition for which $L_\Lambda(\mathcal{K}) = 0$. Using this condition, it is easy to understand the proof of Theorem 1.

The results were further modified by S. Ponnusamy and F. Ronning [27] using duality method for convolution and they obtained the condition on the weight function $\Lambda(t)$ that, when

$$\frac{\Lambda(t)}{(1+t)(1-t)^{1+2\xi}}, \qquad t \in (0,1)$$

is decreasing, then $L_\Lambda(\mathcal{C}(\xi)) = 0$ for $0 \leq \xi \leq 1/2$.

Further the relation was derived between β and λ so that the integral operator V_λ maps the function $f(z) \in P(\beta)$ into $\mathcal{S}^*(\xi)$ for $0 \leq \xi < 1$. With the condition

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t)\left(\frac{((1+\xi)-(1-\xi)t)}{(1-\xi)(1+t)} - \frac{2\xi \log(1+t)}{(1-\xi)t}\right) dt,$$

$V_\Lambda(f)(z) \in \mathcal{S}^*(\xi) \iff L_\Lambda(\mathcal{C}(\xi)) \geq 0$. For particular value of $\xi = 0$, the result reduces to the result given by R. Fournier and S. Ruscheweyh [15].

Likewise, the relation between β and λ was obtained by R. M. Ali and V. Singh [2] so that the integral operator V_λ maps the function $f(z) \in P(\beta)$ into $\mathcal{C}$ and defines the functional $M_\Lambda(f)(z)$ as

$$M_\Lambda(f)(z) = \inf_{z\in\mathbb{D}} \int_0^1 \Lambda(t)\left(\operatorname{Re} f'(tz) - \frac{1-t}{(1+t)^3}\right) dt.$$

R. M. Ali and V. Singh [2] gave the condition on β and λ for which the following result holds good.

Theorem 3 [2] *If* $\beta < 1$ *satisfies*

$$\frac{\beta - \frac{1}{2}}{1-\beta} = -\int_0^1 \lambda(t)\frac{1}{(1+t)^2} dt,$$

and $t\,\Lambda(t) = t\int_t^1 \frac{\lambda(s)}{s} ds \to 0$*, as* $t \to 0$*, then* $V_\lambda(P(\beta)) \in \mathcal{C} \iff M_\Lambda(\mathcal{K}) = 0$.

It is difficult to obtain the condition on $M_\Lambda(\mathcal{K})$, hence the corresponding sufficient result was given.

Theorem 4 [2] *If* $t\,\Lambda'(t)/(1-t^2)$ *is decreasing for* $t \in (0,1)$*, then* $M_\Lambda(\mathcal{K}) = 0$.

R. M. Ali and V. Singh [2] introduced the generalized integral operator $\mathcal{V}_\lambda(f)(z)$ which is the convex combination of z and $V_\lambda(f)(z)$. They defined

$$\mathcal{V}_\lambda(f)(z) := \rho z + (1-\rho)V_\lambda(f)(z) = z\int_0^1 \lambda(t)\frac{1-\rho t z}{1-tz}dt * f(z) \qquad \rho < 1.$$

For the function $f(z) \in P(\beta)$, R. M. Ali and V. Singh [2] derived the relation between β and λ so that the new integral operator $\mathcal{V}_\lambda(f)(z)$ belongs to the subclasses of univalent function class $\mathcal{S}^*$ and $\mathcal{C}$.

Theorem 5 [2] *If $\beta < 1$ satisfies the condition*

$$\frac{1}{2(1-\rho)(1-\beta)} = \int_0^1 \lambda(t)\frac{t}{1+t}dt, \tag{7.6}$$

then $\mathcal{V}_\lambda(f)(z) \subset \mathcal{S}$. Further, $\mathcal{V}_\lambda(f)(z) \subset \mathcal{S}^* \iff L_\Lambda(\mathcal{C}) = 0$, if $\beta < 1$ satisfies (7.6) and $\Lambda(t) = \int_t^1 \frac{\lambda(s)}{s}ds$ satisfies the condition that $t\Lambda(t) \to 0$ as $t \to 0^+$.

To obtain the convexity result for $\mathcal{V}_\lambda(P(\beta))$, R. M. Ali and V. Singh [2] gave the following condition on $\beta < 1$ as

$$\frac{1}{2(1-\rho)(1-\beta)} = \int_0^1 \lambda(t)\frac{t(t+2)}{(1+t)^2}dt,$$

then $\mathcal{V}_\lambda(f)(z) \subset \mathcal{C} \iff M_\Lambda(\mathcal{C}) = 0$, provided $\Lambda(t) = \int_t^1 \frac{\lambda(s)}{s}ds$ satisfies the condition that $t\Lambda(t) \to 0$ as $t \to 0^+$.

Further extensions of subclass $P(\beta)$ of the class $\mathcal{A}$ are the classes $P_\gamma(\beta)$, $R_\gamma(\beta)$, and $\mathcal{W}_\beta(\alpha,\gamma)$, which are defined below and are used to generalize the result given above.

The class $P_\gamma(\beta)$ introduced by Y. C. Kim and F. Ronning in [19] is the linear combination of two functionals that modifies the class $P(\beta)$. For $\beta < 1$ and $0 \leq \gamma < 1$, $P_\gamma(\beta)$ denotes the normalized class of analytic function defined by

$$P_\gamma(\beta) = \left\{ f\in\mathcal{A} : \exists\theta \in \mathbb{R} \mid \text{Re}\left(e^{i\theta}\left((1-\gamma)\frac{f(z)}{z} + \gamma f'(z) - \beta\right)\right) > 0, \quad z \in \mathbb{D}\right\},$$

for some $\theta \in \mathbb{R}$. Note that $P_1(\beta) \equiv P(\beta)$.

The class $R_\gamma(\beta)$ introduced by S. Ponnusamy and F. Ronning in [28] is the set of all normalized and analytic functions defined by

$$R_\gamma(\beta) = \left\{ f \in \mathcal{A} : \exists\theta \in \mathbb{R} \mid \text{Re}\left(e^{i\theta}(f'(z) + \gamma z f''(z) - \beta)\right) > 0, \quad z \in \mathbb{D}\right\},$$

for $\gamma \geq 0$ and $\beta < 1$. Note that $f(z) \in R_\gamma(\beta) \iff zf'(z) \in P_\gamma(\beta)$. Hence, the class $R_\gamma(\beta)$ is closely related to the class $P_\gamma(\beta)$. Another simple observation about the relationship between the class $P(\beta)$ and $R_\gamma(\beta)$ is that $R_0(\beta) \equiv P(\beta)$.

For $\alpha \geq 0$, $\gamma \geq 0$ and $\beta < 1$, R. M. Ali et al. [3] defined the class $\mathcal{W}_\beta(\alpha,\gamma)$ as

$$\begin{aligned}\mathcal{W}_\beta(\alpha,\gamma) = \{ f \in \mathcal{A} : \exists\phi \in \mathbb{R} \mid \\ \text{Re}\, e^{i\phi}\left((1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) + \gamma z f''(z) - \beta\right) \\ > 0, z \in \mathbb{D}\}.\end{aligned}$$

It unifies both the classes $P_\gamma(\beta)$ and $R_\gamma(\beta)$. Note that $P(\beta) \equiv \mathcal{W}_\beta(1,0)$, $P_\alpha(\beta) \equiv \mathcal{W}_\beta(\alpha,0)$ and $R_\gamma(\beta) \equiv \mathcal{W}_\beta(1+2\gamma,\gamma)$.

7.3 Application for the Class $P(\beta)$

Results given in Sect. 7.2 are more interesting whenever particular values of $\lambda(t)$ are chosen for the operator given by (7.4). Various values of $\lambda(t)$ lead to different interesting consequences. One such application is related to (7.2) that provides Bernardi operator, which will be defined later. For further applications, we need the following preliminaries:

Consider the analytic functions $f_i(z)$, for $i = 1, 2$ represented in the series form as

$$f_i(z) = z + \sum_{n=2}^{\infty} a_{n,i} z^n \quad (a_{n,i} \geq 0),$$

then the Hadamard product (or convolution) $f_1 * f_2$ is given by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n.$$

For the complex numbers a, b and c, the Gaussian hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by

$$F(a, b; c; z) \equiv {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \qquad z \in \mathbb{D}.$$

where $c \neq 0, -1, -2, \ldots$ and the Pochhammer symbol, $(\chi)_n$ is expressed as

$$(\chi)_n = \begin{cases} 1 & n = 0, \\ \chi(\chi + 1) \cdots (\chi + n - 1) & n \in \mathbb{N}. \end{cases}$$

The integral operator $V_\lambda(f)(z)$ given in (7.4) under special cases of $\lambda(t)$ reduces to various well-known operators that were studied in detail by many authors. For further details, see [9, 10, 17, 20, 21].

(1) Consider

$$\lambda(t) = \frac{(1 + c)^\delta}{\Gamma(\delta)} t^c \left(log \frac{1}{t} \right)^{\delta - 1}, \quad \text{for} \quad \delta \geq 0 \text{ and } c > -1.$$

Then the integral operator defined corresponding to the weight function $\lambda(t)$ given above is denoted by $V_\lambda(f(z)) := F_{c,\delta}(f(z))$ which is known as the Komatu operator [20]. The particular cases are:

(i) Let $\delta = 1$, so that

$$\lambda(t) = (1 + c)t^c, \quad c > -1.$$

Then the Komatu operator reduces to the Bernardi operator ($\mathcal{B}_c(f)(z)$) [9]. Note that this $\lambda(t)$ is related to $\Lambda_c(t)$ given in (7.2) by $\Lambda_c(t) = \int_t^1 \frac{\lambda(s)}{s} ds$.

(ii) When $c = 0$ and $\delta = 1$, the Komatu operator reduces to Alexander or Biernacki operator [1] which is denoted as

$$\mathcal{A}(f)(z) = \int_0^z \frac{f(t)}{t} dt$$

that maps the function belonging to $\mathcal{S}^*$ onto $\mathcal{C}$, which means

$$f(z) \in \mathcal{S}^*(\xi) \Longleftrightarrow \mathcal{A}(f)(z) \in \mathcal{C}(\xi).$$

(iii) When $c = 1$ and $\delta = 1$, the Komatu operator reduces to the Libera operator [21] which is denoted by

$$F(f)(z) = \frac{2}{z} \int_0^z f(t)dt.$$

(2) Consider

$$\lambda(t) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)} t^{b-1}(1-t)^{c-a-b} \, {}_2F_1(c-a, 1-a; c-a-b+1; 1-t),$$

for which the integral operator $V_\lambda(f(z))$ is known as Hohlov operator [17] denoted by $H_{a,b,c}(f)(z)$. We note that the Hohlov operator can be written in terms of convolution as [19]

$$V_\lambda(f)(z) := H_{a,b,c}(f)(z) = z\,{}_2F_1(a, b; c; z) * f(z).$$

(i) The case $a = 1$ gives

$$H_{1,b,c}(f)(z) := \mathcal{L}(b, c)(f)(z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-2}(1-t)^{c-b-1} f(tz)dt$$

where $\mathcal{L}(b, c)(f)(z)$ is known as Carlson–Shaffer operator [10].

(ii) For the case $a = 1$, $b = \vartheta + 1$, and $c = \vartheta + 2$, Hohlov operator reduces to Bernardi operator, i.e., $H_{1,\vartheta+1,\vartheta+2}(f)(z) = \mathcal{B}_\vartheta(f)(z)$ for $Re\,\vartheta > -1$.

(3) For the two complex numbers, $a, b > -1$. Consider

$$\lambda(t) = \begin{cases} (a+1)(b+1)\frac{t^a(1-t^{b-a})}{b-a}, & b \neq a, \\ (a+1)^2 t^a \log(1/t), & b = a. \end{cases}$$

Then the corresponding integral operator $V_\lambda(f)(z) = G_f(a, b, z)$ was introduced and studied by S. Ponnusamy [26].

Note that even though for particular values the Hohlov and Komatu operators reduce to the Carlson–Shaffer operator, these two operators are different and have different consequences. We summarize the applications with the following outline:

For application purpose, R. Fournier and S. Ruscheweyh [15] considered only the Bernardi integral operator $\mathcal{B}_c(f)(z)$. If $\beta < 1$ satisfies the condition

$$\frac{\beta}{1-\beta} = -(1+c)\int_0^1 t^c \frac{(1-t)}{(1+t)}, \qquad (-1 < c \leq 2),$$

then for $f(z) \in P(\beta)$, $\mathcal{B}_c(f)(z) \in \mathcal{S}^*$ and the result is sharp.

For $\beta = -0.262\ldots$, M. Nunokawa and D. K. Thomas [22] have shown that the function $f(z) \in P(\beta)$ implies $\mathcal{B}_0(f)(z)$ is starlike and if $\beta = -0.0175\ldots$ then $\mathcal{B}_1(f)(z)$ is starlike. These conditions are weak when compared to the result of R. Fournier and S. Ruscheweyh [15] which provides that for $\beta = -0.629\ldots$, $\mathcal{B}_0(P(\beta))$ is starlike and for $\beta = -0.294\ldots$, $\mathcal{B}_1(P(\beta))$ is starlike.

R. M. Ali and V. Singh [2] obtained various ranges of inclusion for the Bernardi operator that are given by $\mathcal{B}_0(P(0)) \in \mathcal{S}^*(0.1174\ldots)$, $\mathcal{B}_1(P(0)) \in \mathcal{S}^*(0.05685\ldots)$, and $\mathcal{B}_2(P(0)) \in \mathcal{S}^*(0.0351\ldots)$.

S. Ponnusamy and F. Ronning [27] considered the operator $G_f(a, b, z)$ for the function $f(z) \in P(\beta)$. If the relation between $\beta < 1$ and $\lambda(t)$ is given by

$$\frac{\beta}{1-\beta} = \int_0^1 \lambda(t)\frac{1-t}{1+t}dt, \qquad a \in (-1, 2] \text{ and } a \leq b,$$

then $G_f(a, b, z) \in \mathcal{S}^*$.

R. M. Ali and V. Singh [2] considered the operator V_λ, where $\lambda(t) = (1-a)(3-a)t^{-a}(1-t^2)/2$ and derived the condition on $\beta < 1$ under which

$$\frac{\beta - 1/2}{1-\beta} = -\frac{(1-a)(3-a)}{2}\int_0^1 t^{-a}\frac{1-t}{1+t}dt,$$

then $V_\lambda(P(\beta)) \in \mathcal{C}$ for $0 \leq a < 1$.

R. M. Ali and V. Singh [2] proved that $H_{1,a,a+b}(P(\beta)) \in \mathcal{C}$, if $\beta < 1$ satisfies

$$\frac{\beta - 1/2}{1-\beta} = -\,{}_2F_1(2, a; a+b; -1) \qquad 0 < a < 1,\ b > 2,$$

and the estimate is sharp.

Since duality technique for convolution gives better bounds, it became an adequate tool while dealing with such type of integral transforms.

7.4 The Class $\mathcal{W}_\beta(\alpha, \gamma)$

The class $\mathcal{W}_\beta(\alpha, \gamma)$ defined in Sect. 7.2 unifies all the other classes existing in the literature. Hence, we provide results only for the class $\mathcal{W}_\beta(\alpha, \gamma)$ and notify particular cases wherever necessary. R. M. Ali et al. [3] discussed the starlikeness of the integral operator $V_\lambda(\mathcal{W}_\beta(\alpha,\gamma)$. To discuss the main result, the following notations are useful:

Let $\mu,\ \nu \geq 0$ satisfy the relation, $\mu+\nu=\alpha-\gamma$ and $\mu\nu=\gamma$. Further, consider the positive and integrable functions $\Pi_{\mu,\nu}(t)$ and $\Lambda_\nu(t)$ that are positive on (0, 1) and integrable on [0, 1], and are defined as

$$\Pi_{\mu,\nu}(t)=\begin{cases}\displaystyle\int_t^1 \Lambda_\nu(s)s^{1/\nu-1/\mu-1}ds, & \gamma>0\\ \Lambda_\alpha(t), & \gamma=0,\mu=0,\nu=\alpha>0,\end{cases} \tag{7.7}$$

where

$$\Lambda_\nu(t)=\int_t^1 \frac{\lambda(s)}{s^{1/\nu}}ds. \tag{7.8}$$

The functional $M_{\Pi_{\mu,\nu}}(h_{\xi,\sigma,z}(t))$ is defined as

$$M_{\Pi_{\mu,\nu}}(h_{\xi,\sigma,z}(t))=\begin{cases}\mathrm{Re}\displaystyle\int_0^1 \Pi_{\nu,\mu}(t)t^{1/\mu-1}h_{\xi,\sigma,z}(t)dt, & \gamma>0,\\ \mathrm{Re}\displaystyle\int_0^1 \Pi_{0,\alpha}(t)t^{1/\alpha-1}h_{\xi,\sigma,z}(t)dt, & \gamma=0,\end{cases} \tag{7.9}$$

where

$$h_{\xi,\sigma,z}(t)=(1-\sigma)\left(\frac{h_\xi(tz)}{tz}-\frac{1-\xi(1+t)}{(1-\xi)(1+t)^2}\right)+\sigma\left(h'_\xi(tz)-\frac{1-\xi-(1+\xi)t}{(1-\xi)(1+t)^3}\right).$$

Consider $g_\xi(t)$ to be the solution of initial value problem

$$\frac{d}{dt}t^{1/\nu}(1+g_\xi(t))=\begin{cases}\dfrac{2}{\mu\nu}t^{1/\nu-1}\displaystyle\int_0^1 s^{1/\mu-1}\frac{(1-\xi(1+st))}{(1-\xi)(1+st)^2}ds, & \gamma>0\\ \dfrac{2(1-\xi(1+t))}{\alpha(1-\xi)(1+t)^2}t^{1/\alpha-1}, & \gamma=0,\alpha>0,\end{cases} \tag{7.10}$$

satisfying $g_\xi(0)=1$.

For $\sigma\in[0,1]$, define $\beta<1$ by

$$\frac{\beta}{1-\beta}=-\int_0^1 \lambda(t)[(1-\sigma)g_\xi(t)+\sigma(2q_\xi(t)-1)]dt. \tag{7.11}$$

Here, $q_\xi(t)$ is the solution of the initial value problem

$$\frac{d}{dt}t^{1/\nu}q_\xi(t)=\begin{cases}\dfrac{1}{\mu\nu}t^{1/\nu-1}\displaystyle\int_0^1 s^{1/\mu-1}\frac{(1-\xi-(1+\xi)st)}{(1-\xi)(1+st)^3}ds, & \gamma>0\\ t^{1/\alpha-1}\dfrac{(1-\xi-(1+\xi)t)}{\alpha(1-\xi)(1+t)^3}, & \gamma=0,\alpha>0.\end{cases} \tag{7.12}$$

satisfying $q_\xi(0)=1$.

Recently, S. Verma et al. [32] (see also [23] where similar results are obtained) considered the function $f(z)$ belonging to the class of analytic function $\mathcal{W}_\beta(\alpha,\gamma)$ and discussed the conditions between the constraint β and the function $\lambda(t)$ under which the integral transform $V_\lambda(f)(z)$ belongs to $\mathcal{S}^*(\xi)$. The following results were formulated.

Theorem 6 [23, 32] *Consider $\mu,\nu \geq 0$ and β satisfy (7.11), for the case $\sigma = 0$. Assume that $t^{1/\nu}\Lambda_\nu(t) \to 0$ and $t^{1/\nu}\Pi_{\mu,\nu}(t) \to 0$ as $t \to 0^+$. Then*

$$V_\lambda(\mathcal{W}_\beta(\alpha,\gamma)) \in \mathcal{S}^*(\xi) \Longleftrightarrow M_{\Pi_{\mu,\nu}}(h_{\xi,0,z}(t)) \geq 0$$

for $\xi \in [0, 1/2]$.

Particular cases for Theorem 6 are discussed as under:

Remark 1

1. For $\xi = 0$, Theorem 6 yields [3, Theorem 3.1].
2. For the case $\alpha = 1+2\gamma$, the class $\mathcal{W}_\beta(1+2\gamma,\gamma)$ reduces to the class $R_\gamma(\beta)$ and Theorem 6 coincides with [7, Theorem 2.1].
3. For the class $\mathcal{W}_\beta(1+2\gamma,\gamma) := R_\gamma(\beta)$ and the case $\xi = 0$, Theorem 6 gives similar result obtained in [28, Theorem 2.2].
4. For the case $\gamma = 0$, the class $\mathcal{W}_\beta(\alpha, 0)$ reduces to the class $P_\alpha(\beta)$ and Theorem 6 reduces to [5, Theorem 1.2].
5. For the class $\mathcal{W}_\beta(\alpha, 0) := P_\alpha(\beta)$ and the case $\xi = 0$, Theorem 6 gives the result obtained by [19, Theorem 2.1].

It is difficult to verify the condition on $M_{\Pi_{\mu,\nu}}(h_{\xi,0,z}(t))$. Hence, Theorem 6 cannot be used directly. From an application point of view, the following sufficient condition has been derived in [23, 32].

Theorem 7 [23, 32] *Let $\mu \geq 1$, $\xi \in [0, 1/2]$ and $\dfrac{\Pi_{\mu,\nu}(t)}{(1+t)(1-t)^{1+2\xi}}$ be decreasing on $(0, 1)$. Then $V_\lambda(\mathcal{W}_\beta(\alpha,\gamma)) \in \mathcal{S}^*(\xi)$ where β satisfies (7.11) and $\sigma = 0$.*

Considering particular values for ξ, α, and γ, sufficient condition given in Theorem 7 is similar to [3, Theorem 4.1], [7 ,Theorem 2.2], and [28, Theorem 2.3]. This condition cannot be reduced to the result obtained by Y. C. Kim and F. Ronning [19] for the case, $\gamma = 0$ and $\xi = 0$, since the condition in [19] contains the function $(log(1/t))^{1+2\xi}$ instead of $(1+t)(1-t)^{1+2\xi}$ in the denominator part. Both the functions are decreasing and tend to 0 as $t \to 1$. Hence to obtain the sufficient conditions corresponding to Theorem 6, for the case $\gamma = 0$ and $\xi = 0$, the following result by Y. C. Kim and F. Ronning [19] is useful.

Theorem 8 [19] *If $\dfrac{\Lambda_\alpha(t)}{\log(1/t)}$ is decreasing on $(0, 1)$, then $M_{\Pi_{\mu,\nu}}(h_{0,0,z}(t)) \geq 0$ for $\alpha \in [1/2, 1]$ and $\gamma = 0$.*

This result is further generalized by R. Balasubramanian et al. [5] to obtain the sufficient condition so that the integral transform of the functions belonging to the class $\mathcal{W}_\beta(\alpha, 0)$ is a member of $\mathcal{S}^*(\xi)$.

Theorem 9 [5] *If* $\dfrac{\Lambda_\alpha(t)}{(\log(1/t))^{1+2\xi}}$ *is decreasing on* $(0,1)$*, then* $M_{\Pi_{\mu,\nu}}(h_{\xi,0,z}(t)) \geq 0$ *for* $\alpha \in [1/2, 1]$*,* $\xi \in [0, 1/2]$*, and* $\gamma = 0$*.*

The sufficient conditions given in Theorems 8 and 9 can be applied only when $\alpha \in [1/2, 1]$. Hence the results obtained are weak as compared to the Theorem 7.

Alternative result corresponding to Theorem 7 is given below.

Theorem 10 [23, 32] *Let* Λ_ν *and* $\Pi_{\mu,\nu}$ *be defined in* (7.8) *and* (7.7)*, respectively. Consider the case* $\sigma = 0$ *and* $\beta < 1$ *is given by* (7.11)*. If* $\lambda(t)$ *satisfies the condition*

$$\frac{t\lambda'(t)}{\lambda(t)} \leq \begin{cases} 1 + \dfrac{1}{\mu}, & \dfrac{\nu}{(1-\nu)} \geq \mu \geq 1 \ (\gamma > 0), \\ 3 - \frac{1}{\alpha}, & \gamma = 0, \ \alpha \in (0, 1/3], \end{cases}$$

then $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \in \mathcal{S}^*(\xi)$ *for* $\xi \in (0, 1/2]$*.*

Following comparisons are made on substituting certain values of ξ, α, and γ in Theorem 10:

Remark 2

1. The condition given in [3, Theorem 4.2] is better than that obtained from Theorem 10, for $\xi = 0$. This is due to the fact that the domain for α is larger for the case $\gamma = 0$.
2. Theorem 10 reduces to the result given in [7, Theorem 3.1] for the case $\alpha = 1+2\gamma$. The condition given in [7] is weak since the result does not hold for $0 \leq \gamma < 1/2$.
3. For the case $\alpha = 1+2\gamma$ and $\xi = 0$, the condition on $\lambda(t)$ of Theorem 10 coincides with [28, Theorem 3.1].

Using Theorem 10, a number of applications for various well-known integral operators are given.

Theorem 11 [23, 32] *Let* $\beta < 1$ *satisfy* (7.11) *with* $\sigma = 0$ *and* $\lambda(t) = (c+1)t^c$*,* $c > -1$*. Then* $\mathcal{B}_c(\mathcal{W}_\beta(\alpha, \gamma)) \in \mathcal{S}^*(\xi)$ *for* $\xi \in (0, 1/2]$*, if*

$$c \leq \begin{cases} 1 + \dfrac{1}{\mu}, & \dfrac{\nu}{(1-\nu)} \geq \mu \geq 1 \ (\gamma > 0), \\ 3 - \frac{1}{\alpha}, & \gamma = 0, \ \alpha \in (1/4, 1/3]. \end{cases}$$

Particular cases are discussed for Bernardi operator.

Remark 3

1. The condition obtained by [3, Theorem 5.1] is better than that obtained from Theorem 11, for $\xi = 0$. This is due to the fact that the Bernardi integral operator is starlike if $c \leq 3 - 1/\alpha$, for the case $\gamma = 0$ and $\alpha \in (0, 1/3] \cup [1, \infty)$.
2. The result given in [7, Theorem 3.1] is weaker than Theorem 11, for $\alpha = 1+2\gamma$. Further, the condition coincides for the case $\gamma \geq 1/2$.
3. For $\alpha = 1 + 2\gamma$ and $\xi = 0$, Theorem 11 coincides with [28, Corollary 3.2].

The case $c = 0$ gives rise to the following application:

Corollary 1 [3] *Let $\beta < 1$ satisfy (7.11) with $\sigma = 0$, $\xi = 0$ and $\lambda(t) = 1$. Then the function $F(z) \in \mathcal{A}$ satisfying*

$$\mathrm{Re}(f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z)) > \beta$$

belongs to $\mathcal{S}^*$, for $\alpha \geq \gamma > 0$ or $\gamma = 0$ and $\alpha \geq 1/3$.

Following results are immediate:

Example 1 Consider the function $f \in \mathcal{A}$ satisfying the condition $\mathrm{Re}(f'(z) + zf''(z)) > (1 - 2\ln 2)/2(1 - \ln 2)$. Then $f(z)$ is starlike.

Example 2 Consider the function $f \in \mathcal{A}$ satisfying the condition $\mathrm{Re}(f'(z) + 3zf''(z) + z^2 f'''(z)) > (6 - \pi^2)/(12 - \pi^2)$. Then $f(z)$ is starlike.

Theorem 12 [23, 32] *Consider*

$$\lambda(t) = \frac{(1+c)^\delta}{\Gamma(\delta)} t^c \left(log \frac{1}{t}\right)^{\delta - 1}, \quad \text{for} \quad \delta \geq 1 \text{ and } c > -1,$$

and $\beta < 1$ satisfy (7.11) for the case $\sigma = 0$. Then $F_{c,\delta}(\mathcal{W}_\beta(\alpha, \gamma)) \in \mathcal{S}^*(\xi)$ for $\xi \in (0, 1/2]$, if

$$c \leq \begin{cases} 1 + \dfrac{1}{\mu}, & \dfrac{\nu}{(1-\nu)} \geq \mu \geq 1 \ (\gamma > 0), \\ 3 - \frac{1}{\alpha}, & \gamma = 0, \ \alpha \in (1/4, 1/3]. \end{cases}$$

Remark 4

1. The result obtained by [3 ,Theorem 5.4] is better than the result corresponding to Theorem 12, for $\xi = 0$. This is due to the fact that the condition on c, when $\gamma = 0$, exists in the bigger domain for $\alpha \in (0, 1/3] \cup [1, \infty)$.
2. For $\alpha = 1 + 2\gamma$, Theorem 12 is better than the result in [7, Theorem 4.2] and reduces when $\gamma \geq 1/2$.
3. Since the bound for c is larger in [5 ,Theorem 2.1], Theorem 12, for $\gamma = 0$, is weaker.
4. For $\gamma = 0$ and $\xi = 0$, Theorem 12 has smaller bound for c than [19, Theorem 2.3].

Theorem 13 [23, 32] *Consider*

$$\lambda(t) = K t^{b-1} (1 - t)^{c-a-b} \Psi(1 - t) \quad \text{for} \quad a, b, c > 0$$

and $\beta < 1$ satisfy (7.11) for the case $\sigma = 0$. Then $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \in \mathcal{S}^*(\xi)$ for $\xi \in [0, 1/2]$, if

$$c < (a + b) \text{ and } 0 < b \leq 1,$$

or

$$c \geq (a+b) \quad \text{and} \quad b \leq \begin{cases} 2+\dfrac{1}{\mu}, & \dfrac{\nu}{(1-\nu)} \geq \mu \geq 1 \ (\gamma > 0), \\ 4-\frac{1}{\alpha}, & \gamma = 0, \ \alpha \in (1/4, 1/3]. \end{cases}$$

Following particular cases are of interest and available in the literature.

Remark 5

1. The result [3, Theorem 5.5] is better than the result of Theorem 13, for $\xi = 0$. This is due to the fact that the condition on b, when $\gamma = 0$, is defined on the bigger domain for $\alpha \in (0, 1/3] \cup [1, \infty)$.
2. Result given in [7, Theorem 4.3] holds for $\gamma \geq 1/2$ whereas Theorem 13, for $\alpha = 1 + 2\gamma$, is true for $\gamma > 0$. Hence the latter result is better.
3. For $\gamma = 0$ in Theorem 13, the parameters have the range $a > 0$, $c \geq a + b$, and $b \leq 4-1/\alpha$, whereas the range of a, b, and c in [5, Theorem 2.2] lies in the interval $a \in (0, 1]$ and $b+2\xi \leq c-a \leq (c-a-b+1)/\alpha(a(c-a-b+1)+2\xi(1-a))$ for $\alpha \in [1/2, 1]$.
4. Substituting $\gamma = 0$ and $\xi = 0$ in Theorem 13, the a, b, and α are different when compared to [19, Theorem 2.4], defined in the range $0 < a \leq 1$, $0 < b \leq 1/\alpha$ and $\alpha \in [1/2, 1]$.
5. The bound for b is $0 < b \leq (c-a) \leq 1/a\alpha$ in [11, Theorem 2], which is larger than the bounds in Theorem 13, for the case $\gamma = 0$ and $\xi = 0$.

For particular values of a, b and c, J. H. Choi et al. [11] gave the following result:
If $\beta < 1$ satisfies

$$\beta = 1 - \frac{1}{2(1 - {}_2F_1(1, 1/\alpha; 1 + 1/\alpha; -1))}, \qquad (1/2 \leq \alpha \leq 1),$$

then $H_{1,1,2}(\mathcal{W}_\beta(\alpha, 0)) \in \mathcal{S}^*$.

Theorem 14 [23, 32] *Let $a, b > -1$, $\lambda(t)$ be given by*

$$\lambda(t) = \begin{cases} (a+1)(b+1)\frac{t^a(1-t^{b-a})}{b-a}, & b \neq a, \\ (a+1)^2 t^a \log(1/t), & b = a, \end{cases}$$

and $\beta < 1$ satisfy (7.11) for the case $\sigma = 0$. Then for the function $f(z) \in \mathcal{W}_\beta(\alpha, \gamma)$, the operator $G_f(a, b, z) \in \mathcal{S}^*(\xi)$ for $\xi \in [0, 1/2]$, if

$$a \leq \begin{cases} 1+\dfrac{1}{\mu}, & \dfrac{\nu}{(1-\nu)} \geq \mu \geq 1 \ (\gamma > 0), \\ 3-\frac{1}{\alpha}, & \gamma = 0, \ \alpha \in (1/4, 1/3]. \end{cases}$$

The particular cases of the above result are of interest.

Remark 6

1. Theorem 14, when $\xi = 0$, leads to the result similar to [3, Theorem 5.3]. But the condition on "a," when $\gamma = 0$ is true for the larger domain as $\alpha \in (0, 1/3] \cup [1, \infty)$ in [3] which leads to the better result.

2. For $\alpha = 1 + 2\gamma$, Theorem 14 gives similar result as in [7, Theorem 4.1].
3. [5, Theorem 2.4] gives better bounds for "a" than that obtained in the Theorem 14 for the case $\gamma = 0$.

R. M. Ali et al. [4] gave the following results corresponding to the integral operator $\mathcal{V}_\lambda(f)(z)$.

Theorem 15 [4] *Consider $\mu, \nu \geq 0$ and $\beta < 1$ satisfy*

$$\frac{1}{2(1-\rho)(1-\beta)} = -\int_0^1 \lambda(t)\frac{g_0(t)-1}{2}dt \qquad \rho < 1,$$

where $g_0(t)$ is the solution of initial value problem given in (7.10). Assume that $t^{1/\nu}\Lambda_\nu(t) \to 0$ and $t^{1/\nu}\Pi_{\mu,\nu}(t) \to 0$ as $t \to 0^+$. Then

$$\mathcal{V}_\lambda(\mathcal{W}_\beta(\alpha,\gamma)) \in \mathcal{S}^* \iff M_{\Pi_{\mu,\nu}}(h_{0,0,z}(t)) \geq 0.$$

For $\gamma = 0$, Theorem 15 gives [6, Theorem 2.4], for $\xi = 0$.

R. M. Ali et al. [4] discussed the corresponding convexity results for the integral operator $V_\lambda(\mathcal{W}_\beta(\alpha,\gamma)$. Following notations are used for further discussion. The integrable functions $\Pi_{\mu,\nu}$ and Λ_ν are defined in (7.7) and (7.8). The functional $M_{\Pi_{\mu,\nu}}(h_{\xi,\sigma,z}(t))$ is given in (7.9).

Recently R. Omar et al. [24] obtained the condition between β and $\lambda(t)$, so that the integral transform V_λ maps the function $f(z) \in \mathcal{W}_\beta(\alpha,\gamma)$ into the convex class of functions of order ξ. The following results are formulated:

Theorem 16 [24] *Consider $\mu, \nu \geq 0$, and $\beta < 1$ satisfy (7.11) for the case $\sigma = 1$. Assume that $t^{1/\nu}\Lambda_\nu(t) \to 0$ and $t^{1/\nu}\Pi_{\mu,\nu}(t) \to 0$ as $t \to 0^+$. Then*

$$V_\lambda(\mathcal{W}_\beta(\alpha,\gamma)) \in \mathcal{C}(\xi) \iff M_{\Pi_{\mu,\nu}}(h_{\xi,1,z}(t)) \geq 0$$

for $\xi \in [0, 1/2]$.

Note that these results are also available in the work of S. Verma et al. [31]. Further, particular cases of Theorem 16 are given in the literature.

Remark 7

1. For $\xi = 0$, Theorem 16 reduces to [4, Theorem 3.1].
2. For $\gamma = 0$, Theorem 16 gives similar result as in [6, Theorem 2.3].

To verify the condition $M_{\Pi_{\mu,\nu}}(h_{\xi,1,z}(t)) \geq 0$, Theorem 16 cannot be used directly. Hence for application purposes, the following sufficient condition is useful:

Theorem 17 [24] *Let $\mu \geq 1$ and $\dfrac{-t\Pi'_{\mu,\nu}(t) + (1-1/\mu)\Pi_{\mu,\nu}(t)}{(1+t)(1-t)^{1+2\xi}}$, $\xi \in [0, 1/2]$ be decreasing on $(0, 1)$. Then $V_\lambda(\mathcal{W}_\beta(\alpha,\gamma)) \in \mathcal{C}(\xi)$ where $\beta < 1$ satisfies (7.11) for the case $\sigma = 1$.*

For $\xi = 0$, Theorem 17 is similar to [4, Theorem 4.2]. When $\gamma = 0$, Theorem 17 cannot be reduced to the result of R. Balasubramanian et al. [6, Theorem 2.3]. This is due to the fact that the denominator part of [6, Theorem 2.3] contains $(\log(1/t))^{1+2\xi}$

instead of $(1+t)(1-t)^{1+2\xi}$. Therefore the sufficient condition corresponding to Theorem 17 for the case $\gamma = 0$ is given as under:

Theorem 18 [6] *Let* $\xi \in [0, 1/2]$, $\dfrac{(1-1/\alpha)\Lambda_\alpha(t) - t\Lambda'_\alpha(t)}{[\log(1/t)]^{1+2\xi}}$ *be decreasing on* $(0, 1)$. *Then* $M_{\Pi_{\mu,\nu}}(h_{\xi,1,z}(t)) \geq 0$ *for* $\alpha \in [1/2, 1]$ *and* $\gamma = 0$.

Alternative criteria of convexity corresponding to Theorem 18 as in [24] are given below.

Theorem 19 [24] *Let* $\beta < 1$ *be given by* (7.11) *for the case* $\sigma = 1$. *Further suppose that* $\lambda(t)$ *satisfies the inequality*

$$\frac{t\lambda'(t)}{\lambda(t)} \leq 2 + \frac{1}{\mu} - \frac{1}{\nu}, \qquad \nu \geq \mu \geq 1.$$

Then, $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \in \mathcal{C}(\xi)$ *for* $\xi \in [0, 1/2]$.

Remark 8

1. For $\xi = 0$, Theorem 19 gives the result as in [4, Theorem 4.3].
2. For $\alpha = 1 + 2\gamma$ and $\xi = 0$, Theorem 19 reduces to [4, Corollary 4.4].

Using Theorem 19, applications for different integral operators discussed in Sect. 7.3 can be obtained.

Theorem 20 [31] *Let* $\lambda(t) = (c+1)t^c$, $c > -1$, *and* $\beta < 1$ *satisfy* (7.11) *for the case* $\sigma = 1$. *Then* $\mathcal{B}_c(\mathcal{W}_\beta(\alpha, \gamma)) \in \mathcal{C}(\xi)$ *for* $0 < \gamma \leq \alpha \leq 1 + 2\gamma$ *and* $\xi \in (0, 1/2]$, *if*

$$c \leq 2 + 1/\mu - 1/\nu, \qquad \nu \geq \mu \geq 1.$$

Remark 9

1. For $\xi = 0$, Theorem 20 reduces to the result as in [4, Theorem 5.1].
2. For $\alpha = 1 + 2\gamma$ and $\xi = 0$, Theorem 20 reduces to [4, Corollary 5.2].

For the case $c = 0$, Theorem 20 gives the following application:

Corollary 2 [4] *Let* $\beta < 1$ *satisfy* (7.11), *where* $\sigma = 1$, $\xi = 0$, *and* $\lambda(t) = 1$. *Then the function* $F(z) \in \mathcal{A}$ *satisfying*

$$\mathrm{Re}(F'(z) + \alpha z F''(z) + \gamma z^2 F'''(z)) > \beta$$

belongs to $\mathcal{C}$, *for* $0 < \gamma \leq \alpha \leq (1 + 2\gamma)$.

Following result is immediate.

Example 3 Consider the function $f \in \mathcal{A}$ satisfying the condition $\mathrm{Re}(f'(z) + zf''(z)) > (1 - 2\ln 2)/(2(1 - \ln 2))$. Then $f(z)$ is convex.

Theorem 21 [31] *Consider*

$$\lambda(t) = \frac{(1+c)^\delta}{\Gamma(\delta)} t^c \left(log\frac{1}{t}\right)^{\delta-1}, \quad \text{for} \quad c > \delta - 2 > -1,$$

and $\beta < 1$ *be defined by* (7.11), *for the case* $\sigma = 1$. *Then* $F_{c,\delta}(\mathcal{W}_\beta(\alpha,\gamma)) \in C(\xi)$ *for* $0 < \gamma \le \alpha \le 1 + 2\gamma$ *and* $\xi \in (0, 1/2]$, *if*

$$c \le 2 + \frac{1}{\mu} - \frac{1}{\nu} \qquad \nu \ge \mu \ge 1.$$

Following particular cases are available in the literature.

Remark 10

1. The result obtained by [4, Theorem 5.6] coincides with the result obtained by Theorem 21, for the case $\xi = 0$.
2. The case $\gamma = 0$ implies $\mu = 0$ and $\nu = \alpha$. Hence, Theorem 21 cannot be used for the case $\gamma = 0$ and the comparison cannot be done with [6, Theorem 3.3].

The result for Komatu operator given in Theorem 21 cannot be used for $\gamma = 0$. So the following result is given in [6, Theorem 3.3].

Theorem 22 [6] *If* $\xi \in [0, 1/2]$ *and* $\beta < 1$ *satisfies* (7.11) *for* $\sigma = 1$, *then* $F_{c,\delta}(\mathcal{W}_\beta(\alpha, 0)) \in \mathcal{C}(\xi)$, *for* $\alpha \in [1/2, 1]$, $0 < c+1 \le 1/(2\alpha) - \xi$, *and* $\delta \ge 2(1+\xi)$.

The Hohlov operator was not considered either by R. Omar et al. [24] or S. Verma et al. [31] for convexity case. Hence, we mention the result given by R. M. Ali et al. [4].

Theorem 23 [4] *Consider*

$$\lambda(t) = Kt^{b-1}(1-t)^{c-a-b}\Psi(1-t) \quad \text{for} \quad a,\, b,\, c > 0$$

and $\beta < 1$ *be defined by* (7.11) *for the case* $\sigma = 1$. *Then* $V_\lambda(\mathcal{W}_\beta(\alpha,\gamma)) \in \mathcal{C}$ *if*

$$c \ge (a+b+1) \text{ and } 0 < b \le 1.$$

Remark 11

1. For $\gamma = 0$, Theorem 23 gives better results than that obtained in [6, Theorem 3.2], for $\xi = 0$. This is due to the fact that in [6, Theorem 3.2], the bounds of $0 < a \le 1$ and $0 < b \le 1/(2\alpha)$, for $\alpha \in [1/2, 1]$ are smaller as compared to the bounds of Theorem 23.
2. The conditions in [11, Theorem 1] are $0 < a \le 1$, $b \le (c-a-2)/(2\alpha(c-a-1)-1)$, $c > a+2$, and $\alpha \in [1/2, 1]$ which are different from the conditions in Theorem 23, for $\gamma = 0$.

If $\beta < 1$ satisfies

$$\beta = 1 - \frac{1}{2(1 - {}_4F_3(2, 2, b, 1/\alpha; 1, c, 1+1/\alpha; -1))},$$

for $0 < b \le (c-3)/(2\alpha c - 4\alpha - 1)$, $c > 3$, and $\alpha \in [1/2, 1]$, then J. H. Choi et al. [11] proved that the function $f(z) \in \mathcal{W}_\beta(\alpha, 0)$ implies $\mathcal{L}(b, c)(f)(z) \in \mathcal{C}$.

Theorem 24 [31] *Let* $a, b > -1$, $\lambda(t)$ *be given by*

$$\lambda(t) = \begin{cases} (a+1)(b+1)\frac{t^a(1-t^{b-a})}{b-a}, & b \ne a, \\ (a+1)^2 t^a \log(1/t), & b = a, \end{cases}$$

and $\beta < 1$ *satisfy* (7.11) *for the case* $\sigma = 1$. *Then for the function* $f(z) \in \mathcal{W}_\beta(\alpha, \gamma)$, *the operator* $G_f(a, b, z) \in \mathcal{C}(\xi)$ *for* $0 < \gamma \leq \alpha \leq 1 + 2\gamma$ *and* $\xi \in [0, 1/2]$, *if*

$$a \leq 2 + \frac{1}{\mu} - \frac{1}{\nu}, \text{ for } \nu \geq \mu \geq 1.$$

Theorem 24 for $\xi = 0$ gives better bounds for "a" than the result given in [4, Theorem 5.7].

For $\gamma = 0$, the result is not valid. In this context, following result is given in [6, Theorem 3.4].

Theorem 25 *Let* $a, b > -1$, $\lambda(t)$ *be defined as in Theorem* 24 *and* $\beta < 1$ *satisfy* (7.11) *for the case* $\sigma = 1$. *Then for the function* $f(z) \in \mathcal{W}_\beta(\alpha, 0)$, $\alpha \in [1/2, 1]$, *the operator* $G_f(a, b, z) \in \mathcal{C}$, *if* $a \leq 0$ *and* b, *satisfies* $b = a$ *or* $b \leq 1/\alpha - 1$.

For the generalized integral operator $\mathcal{V}_\lambda(f)(z)$, R. M. Ali et al. [4] gave the following result.

Theorem 26 [4] *Consider* $\mu, \nu \geq 0$ *and* $\beta < 1$ *satisfy*

$$\frac{1}{2(1-\rho)(1-\beta)} = -\int_0^1 \lambda(t)(q_0(t) - 1)dt \qquad \rho < 1,$$

where $q_0(t)$ *is the solution of initial value problem given in* (7.12). *Assume further that* $t^{1/\nu}\Lambda_\nu(t) \to 0$ *and* $t^{1/\nu}\Pi_{\mu,\nu}(t) \to 0$ *as* $t \to 0^+$. *Then*

$$\mathcal{V}_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \in \mathcal{C} \Longleftrightarrow M_{\Pi_{\mu,\nu}}(h_{0,1,z}(t)) \geq 0.$$

For $\gamma = 0$, Theorem 26 reduces to [6, Theorem 2.5], putting $\xi = 0$.

Until now, the conditions under which the integral operator V_λ maps the function belonging to the subclasses of analytic function class $\mathcal{A}$ to the subclass of univalent function class $\mathcal{S}$ are discussed. We will now discuss the relation between the constant β and the function $\lambda(t)$ and obtain the condition so that the integral operator maps the function belonging to the subclass of normalized and analytic function class to another subclass of normalized and analytic function class.

Recently S. Verma et al. [30] obtained sharp values for β so that $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \in \mathcal{W}_\omega(1, 0))$ and $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \in \mathcal{W}_\omega(\alpha, \gamma))$, for $\omega < 1$ and gave the following results:

Theorem 27 *Consider* $\mu, \nu \geq 0$ *and* $\beta < 1$ *satisfying*
$\beta =$

$$1 - \frac{1-\omega}{2}\left[1 - \frac{1}{\nu}\int_0^1 \lambda(t)\int_0^1 \frac{ds}{1 + ts^\mu}dt - \left(1 - \frac{1}{\nu}\right)\int_0^1 \lambda(t)\int_0^1\int_0^1 \frac{d\eta d\zeta}{1 + t\eta^\nu\zeta^\mu}dt\right]^{-1}, \quad \text{for} \quad \gamma \neq 0,$$

$$1 - \frac{1-\omega}{2}\left[1 - \frac{1}{\alpha}\int_0^1 \frac{\lambda(t)}{1+t}dt - \left(1 - \frac{1}{\alpha}\right)\int_0^1 \lambda(t)\int_0^1 \frac{d\eta}{1 + t\eta^\alpha}dt\right]^{-1}, \quad \gamma = 0.$$

Then, $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \in \mathcal{W}_\omega(1, 0))$.

Remark 12

1. For $\alpha = 1$ and $\gamma = 0$, Theorem 27 reduces to the result in [15 ,Theorem 2].
2. For $\gamma = 0$, Theorem 27 is similar to [8, Theorem 1.5].

Theorem 28 *Consider* $\alpha, \gamma \geq 0$*, and* $\beta < 1$ *satisfying*

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t) \frac{1 - \dfrac{(1+\omega)}{(1-\omega)}t}{(1+t)} dt.$$

Then, $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \in \mathcal{W}_\omega(\alpha, \gamma))$.

When $\gamma = 0$, Theorem 28 reduces to the result in [19, Theorem 2.6].

Put $\delta = 1$ and $c = (1-\alpha)/\alpha$ in $\lambda(t) = \dfrac{(1+c)}{\Gamma(\delta)} t^c (\log(1/t))^{\delta - 1}$. Then using Theorem 28 for the case $\gamma = 0$, Y. C. Kim and F. Ronning [19] proved that $f(z) \in \mathcal{W}_\beta(\alpha, 0)$ implies $\operatorname{Re} e^{i\phi}(f(z)/z - \omega) > 0$.

7.5 The Pascu Class

The results corresponding to the class $\mathcal{W}_\beta(\alpha, \gamma)$, given in [3] and [4], are unified in [13]. For the function $f(z) \in \mathcal{W}_\beta(\alpha, \gamma)$, S. Devi and A. Swaminathan [13] obtained the conditions between β and λ so that $V_\lambda(f(z)) \in M(\sigma, 0)$. The following results are formulated:

Theorem 29 [13] *Consider* $\mu, \nu \geq 0$ *and* $\beta < 1$ *satisfying* (7.11) *for* $\xi = 0$*. Further assume that* $t^{1/\nu}\Lambda_\nu(t) \to 0$ *and* $t^{1/\nu}\Pi_{\mu,\nu}(t) \to 0$ *as* $t \to 0^+$*. Then*

$$V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \in M(\sigma, 0) \Longleftrightarrow M_{\Pi_{\mu,\nu}}(h_{0,\sigma,z}(t)) \geq 0.$$

For $\gamma = 0$, Theorem 29 reduces to [29, Theorem 2.1]. For application purposes, the following sufficient condition is given:

Theorem 30 [13] *Let* $\sigma \in [0, 1]$*,* $\mu \geq 1$*, and* $\beta < 1$ *satisfy* (7.11) *for* $\xi = 0$*. If*

$$\frac{\sigma t^{1/\sigma - 1/\mu + 1} d(t^{1/\mu - 1/\sigma} \Pi_{\mu,\nu}(t))}{1 - t^2}$$

is increasing on $(0, 1)$*, then* $V_\lambda(\mathcal{W}_\beta(\alpha, \gamma)) \in M(\sigma, 0)$.

This result cannot be reduced to the result of K. Raghavendar and A. Swaminathan [29]. This is because the sufficient condition in [29, Theorem 2.1], for $\xi = 0$, contains $(\log(1/t))$ in the denominator whereas in Theorem 30, contains the term $(1 - t^2)$. Hence the sufficient condition corresponding to Theorem 30 is given below:

Theorem 31 [29] *Let* $\xi \in [0, 1/2]$*,* $\alpha \in [1/2, 1]$*, and* $\beta < 1$ *satisfy* (7.11) *for* $\gamma = 0$*. If*

$$\frac{\sigma t^{1/\sigma - 1/\alpha + 1} d(t^{1/\alpha - 1/\sigma} \Lambda_\alpha(t))}{(\log(1/t))^{1+2\xi}}$$

is increasing on $(0,1)$*, then* $V_\lambda(\mathcal{W}_\beta(\alpha,0)) \in M(\sigma,\xi)$.

Following alternative condition corresponding to Theorem 30 is used:

Theorem 32 [13] *If* β *is given by* (7.11) *for the case* $\xi = 0$ *and* $\lambda(t)$ *satisfy the condition*

$$(1-\sigma)\left[\left(1+\frac{1}{\mu}\right)\lambda(t) - t\lambda'(t)\right] + \sigma\left[t^2\lambda''(t) - \frac{1}{\mu}t\lambda'(t)\right] \geq 0.$$

whenever

$$\sigma\frac{\lambda'(1)}{\lambda(1)} \leq 1 + \sigma\left(1+\frac{1}{\mu}-\frac{1}{\nu}\right),$$

then $V_\lambda(\mathcal{W}_\beta(\alpha,\gamma)) \in M(\sigma,0)$.

The case $\gamma = 0$ implies that $\mu = 0$ and $\nu = \alpha$. Hence the above result cannot be used for $\gamma = 0$. Using Theorem 32, a number of applications are discussed.

Theorem 33 [13] *Let* $\lambda(t) = (1+c)t^c$, $c > -1$ *and* $\beta < 1$ *satisfy* (7.11) *for the case* $\xi = 0$. *If*

$$c \leq min\left[(1 + 1/\mu - 1/\nu), (1 + 1/\mu - \xi)/(1 + 2\xi)\right],$$

then $\mathcal{B}_c(\mathcal{W}_\beta(\alpha,\gamma)) \in M(\sigma,0)$*, for* $\sigma \in [0,1]$.

The above result is not valid for $\gamma = 0$. Hence, [13, Theorem 3.2] gives the following result for above theorem. If $c > 1 + 1/\alpha$ and $\sigma \geq \alpha$, then $\mathcal{B}_c(\mathcal{W}_\beta(\alpha,0)) \in M(\sigma,0)$, for $\mu \geq 1$ and $\sigma \in (0,1)$.

This condition of above result for $\gamma = 0$ cannot be compared due to the different bounds for c and α. The case $\gamma = 0$ [29, Theorem 3.1] gave the following result:

Theorem 34 [29] *Let* $\lambda(t) = (1+c)t^c$, $c > -1$*, and* $\beta < 1$ *satisfy* (7.11). *If*

$$c \leq \min\{(1/\sigma - 1), (1/\alpha - 1)\},$$

then $\mathcal{B}_c(\mathcal{W}_\beta(\alpha,0)) \in M(\sigma,\xi)$*, for* $\alpha \in [1/2,1]$*,* $\gamma = 0$*,* $\xi \in [0,1/2]$*, and* $\sigma \in (0,1)$.

The case $c = 0$ leads to the following application.

Corollary 3 [13] *Let* $\beta < 1$ *satisfy* (7.11) *for* $\xi = 0$ *and* $\lambda(t) = 1$. *Then the function* $F(z) \in \mathcal{A}$ *satisfying*

$$\mathrm{Re}(f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z)) > \beta$$

belongs to $M(\sigma,0)$.

Following results are immediate.

Example 4 Consider the function $f \in \mathcal{A}$ satisfying the condition $\mathrm{Re}(f'(z) + 3zf''(z) + z^2 f'''(z)) > (2(1-(1-\sigma)\pi^2/12 - \sigma\log 2))^{-1}$. Then $f(z) \in M(\sigma,0)$.

Theorem 35 [13] *Consider*

$$\lambda(t) = \frac{(1+c)^\delta}{\Gamma(\delta)}t^c\left(log\frac{1}{t}\right)^{\delta-1}, \quad \text{for} \quad c < 0 \ \text{ and } \ \delta > 2.$$

If $\beta < 1$ *satisfies* (7.11) *for* $\xi = 0$*, then* $F_{c,\delta}(\mathcal{W}_\beta(\alpha,\gamma)) \in M(\sigma,0)$ *for* $\mu \geq 1$.

For $\gamma = 0$, Theorem 35 is better and coincides with the result given in [29, Theorem 3.4], whenever $\xi = 0$ and $\alpha \in [1/2, 1]$.

Theorem 36 [13] *Consider*

$$\lambda(t) = Kt^{b-1}(1-t)^{c-a-b}\Psi(1-t) \quad \text{for} \quad a,\, b,\, c > 0$$

and $\beta < 1$ *satisfy* (7.11) *for the case* $\xi = 0$*. Then* $V_\lambda(\mathcal{W}_\beta(\alpha,\gamma)) \in M(\sigma,0)$ *for* $\mu \geq 1$ *if*

$$b < \min\{1, (2+1/\mu), (c-a-1)\}.$$

For $\gamma = 0$, Theorem 36 coincides with the result given in [29, Theorem 3.2] for $\xi = 0$.

Theorem 37 [13] *Let* $a, b > -1$ *and* $\lambda(t)$ *be given by*

$$\lambda(t) = \begin{cases} (a+1)(b+1)\frac{t^a(1-t^{b-a})}{b-a}, & b \neq a, \\ (a+1)^2 t^a \log(1/t), & b = a. \end{cases}$$

Consider $\beta < 1$ *satisfy* (7.11) *for the case* $\xi = 0$*. If* a*,* b*, and* μ *satisfy any one of the following conditions*

1. $b > a, \quad -1 < a < 0$ *and* $\quad b + a - 1 < 1/\mu < b - 1,$
2. $b < a, \quad -1 < b < 0$ *and* $\quad b + a - 1 < 1/\mu < a - 1,$
3. $b = a < 0,$

then $f(z) \in \mathcal{W}_\beta(\alpha,\gamma)$ *implies* $G_f(a,b;z) \in M(\sigma,0)$*, for* $\mu \geq 1$ *and* $\sigma \in [0,1]$.

For the generalized integral operator $\mathcal{V}_\lambda(f)(z)$, following result is given in [13, Theorem 4.1].

Theorem 38 [13] *Let* $\mu, \nu > 0$ *and* $\beta < 1$ *satisfy*

$$\frac{1}{2(1-\rho)(1-\beta)} = -\int_0^1 \lambda(t)\left((1-\sigma)\left(\frac{g_0(t)-1}{2}\right) + \sigma(q_0(t)-1)\right)dt \quad \rho < 1,$$

where $g_0(t)$ *and* $q_0(t)$ *are the solutions of the initial value problem given in* (7.10) *and* (7.12)*, respectively. Assume further that* $t^{1/\nu}\Lambda_\nu(t) \to 0$ *and* $t^{1/\nu}\Pi_{\mu,\nu}(t) \to 0$ *as* $t \to 0^+$*. Then*

$$\mathcal{V}_\lambda(\mathcal{W}_\beta(\alpha,\gamma)) \in M(\sigma,0) \Longleftrightarrow M_{\Pi_{\mu,\nu}}(h_{0,\sigma,z}(t)) \geq 0.$$

7.6 Conclusion and Further Problems

The duality technique of convolutions is useful for an integral operator to carry a function from a class of analytic functions into the class of univalent functions and to the class of analytic functions as well. It is a strong technique because it gives sharp result. The sufficient condition used to evaluate the results has to be modified to obtain better results. Hence the following open problems will be of interest for further research.

Problem 1 To generalize the class $\mathcal{W}_\beta(\alpha, \gamma)$, so that the results can be extended for the new class.

Problem 2 To study the results for various other subclasses of $\mathcal{S}$ different from Pascu class.

Problem 3 To find other cases of $\lambda(t)$, so that the new applications by means of different integral operator are obtained.

Problem 4 To analyze the ranges of the parameters of $\lambda(t)$ that were not covered by the results existing in the literature.

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Chapter 8
Eneström–Kakeya Theorem and Some of Its Generalizations

Robert B. Gardner and N. K. Govil

8.1 Introduction and History

The study of the zeros of polynomials has a very rich history. In addition to having numerous applications, this study has been the inspiration for much theoretical research (including being the initial motivation for modern algebra). Algebraic and analytic methods for finding zeros of a polynomial, in general, can be quite complicated, so it is desirable to put some restrictions on polynomials. Historically speaking, the subject dates from about the time when the geometric representation of the complex numbers was introduced into mathematics, and the first contributors to the subject were Gauss and Cauchy. Gauss, as part of his 1816 explorations of the fundamental theorem of algebra, proved (see, for example, [26]):

Theorem 1 *If $p(z) = z^n + a_1 z^{n-1} + \cdot + a_n$ is a polynomial of degree n with real coefficients, then all the zeros of p lie in*

$$|z| \leq R = \max_{1 \leq k \leq n} \{(n\sqrt{2}|a_k|)^{1/k}\}.$$

In the case of arbitrary real or complex a_j, he [26] showed in 1849 that R may be taken as the positive root of the equation

$$z^n - \sqrt{2}(|a_1|z^{n-1} + \cdots + |a_n|) = 0.$$

Cauchy [13, 51] improved the result of Gauss in Theorem 1, and proved:

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Theorem 2 *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients, then all the zeros of p lie in* $|z| \le 1 + \max_{0 \le j \le n-1} \left| \frac{a_j}{a_n} \right|$.

Notice that neither Theorem 1 nor Theorem 2 put any restrictions on the coefficients of p (beyond the restriction that they either lie in $\mathbb{R}$ or $\mathbb{C}$, respectively). See [1, 2, 3, 19] for several related results which apply to all polynomials with complex coefficients.

In this survey, we explore the Eneström–Kakeya theorem and its generalizations. By this, we mean that we explore results which give the location of zeros of a polynomial in terms of their moduli based on hypotheses imposed on the coefficients of the polynomial. We give a mostly chronological presentation. The well-known Eneström–Kakeya theorem is most commonly stated as follows:

Theorem 3 *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying $0 \le a_0 \le a_1 \le \cdots \le a_n$, then all the zeros of p lie in $|z| \le 1$.*

Proof Define f by the equation

$$\begin{aligned} p(z)(1-z) &= a_0 + (a_1 - a_0)z + (a_2 - a_1)z^2 + \cdots + (a_n - a_{n-1})z^n - a_n z^{n+1} \\ &= f(z) - a_n z^{n+1}. \end{aligned}$$

Then for $|z| = 1$, we have

$$\begin{aligned} |f(z)| &\le |a_0| + |a_1 - a_0| + |a_2 - a_1| + \cdots + |a_n - a_{n-1}| \\ &= a_0 + (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_n - a_{n-1}) \\ &= a_n. \end{aligned}$$

Notice that the function $z^n f(1/z) = \sum_{j=0}^{n} (a_j - a_{j-1}) z^{n-j}$, $a_{-1} = 0$ has the same bound on $|z| = 1$ as f. Namely, $|z^n f(1/z)| \le a_n$ for $|z| = 1$. Since $z^n f(1/z)$ is analytic in $|z| \le 1$, we have $|z^n f(1/z)| \le a_n$ for $|z| \le 1$ by the maximum modulus theorem. Hence, $|f(1/z)| \le a_n/|z|^n$ for $|z| \le 1$. Replacing z with $1/z$, we see that $|f(z)| \le a_n |z|^n$ for $|z| \ge 1$, and making use of this we get,

$$\begin{aligned} |(1-z)p(z)| &= |f(z) - a_n z^{n+1}| \\ &\ge a_n |z|^{n+1} - |f(z)| \\ &\ge a_n |z|^{n+1} - a_n |z|^n \\ &= a_n |z|^n (|z| - 1). \end{aligned}$$

So if $|z| > 1$ then $(1-z)p(z) \ne 0$. Therefore, all the zeros of p lie in $|z| \le 1$. □

The proof given here is modeled on a proof of a generalization of the Eneström–Kakeya theorem given by Joyal, Labelle, and Rahman [46]. The original statement of the result is slightly different and has a complicated history.

It seems that G. Eneström was the first to get a result of this nature when he was studying a problem in the theory of pension funds. He published his work in Swedish in 1893 in the journal *Öfversigt af Vetenskaps-Akademiens Förhandlingar* [22]. He mentioned his result again in publications of 1893–1894 and 1895. In 1912, S. Kakeya [47] published a paper (in English) in the *Tôhoku Mathematical Journal* which contained the more general result:

Theorem 4 *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real and positive coefficients, then all the zeros of p lie in the annulus $R_1 \leq |z| \leq R_2$ where $R_1 = \min_{0\leq j\leq n-1} a_j/a_{j+1}$ and $R_2 = \max_{0\leq j\leq n-1} a_j/a_{j+1}$.*

In the final few lines of Kakeya's paper, he mentioned that the monotonicity assumption of Theorem 3 implies that all zeros of p lie in $|z| \leq 1$. The paper gave no references and Kakeya seems to have been unaware of Eneström's earlier work. Kakeya's paper received a bit of attention and was mentioned in at least three other papers in the *Tôhoku Mathematical Journal* during 1912 and 1913; one is in German [42] and two are in English [40, 41]. The two papers in English are by T. Hayashi. At some point, Hayashi must have learned of Eneström's earlier result. Hayashi encouraged Eneström to publish his own results in the *Tôhoku Mathematical Journal* and in 1920, Eneström [23] published in French: "Remarque sur un théorème relatif aux racines de l'equation $a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0 = 0$ où tous les coefficientes a sont réels et positifs" ("Remark on a Theorem on the Roots of the Equation $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ where all Coefficients are Real and Positive") . In this work [23], Eneström presented a "verbatim" (*textuellement*) translation of his original 1893 paper. We can now see that Eneström was the first to publish a proof of Theorem 3 in 1893 and that Kakeya independently proved the result in 1912. This could therefore be a reason to refer to Theorem 3 as the "Eneström–Kakeya theorem." Since Eneström's argument is so historically important, we present a complete English translation of this paper of Eneström [23] in the appendix of this chapter.

8.2 Generalizations of Eneström–Kakeya Theorem During the 1960s

The Eneström–Kakeya theorem gives an upper bound on the modulus of the zeros of polynomials in a certain class (namely, those polynomials with real, nonnegative, monotone increasing coefficients). We can easily obtain a zero-free region for a related class of polynomials in the sense that we can get a lower bound on the modulus of the zeros. By applying Theorem 3 to $z^n p(1/z)$ where p has real, nonnegative, monotone decreasing coefficients, we get the following:

Theorem 5 *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying $a_0 \geq a_1 \geq \cdots \geq a_n \geq 0$, then all the zeros of p lie in $|z| \geq 1$.*

In 1963, Cargo and Shisha [12] introduced the "backward-difference operator" on the coefficients of polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$ by defining $\nabla a_j = a_j - a_{j-1}$ (when speaking of ∇a_0 or ∇a_{n+1}, we will assume $a_{-1} = a_{n+1} = 0$). More generally, they also defined "fractional order differences" for any complex α as

$$\nabla^{\alpha} a_n = \sum_{m=0}^{k} (-1)^m \begin{pmatrix} \alpha \\ m \end{pmatrix} a_{k-m}.$$

See Cargo and Shisha [12] (also [52, 54])

Theorem 6 *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real, nonnegative coefficients satisfying $\nabla^{\alpha} a_j \leq 0$ for $j = 1, 2, \ldots, n$ and $0 < \alpha \leq 1$, then all the zeros of p lie in $|z| \geq 1$.*

Cargo and Shisha showed that Theorem 6 reduces to Theorem 5 when $\alpha = 1$. They also gave specific polynomials to which Theorem 6 applies, but Theorem 5 does not, thus showing that the hypotheses are weaker in their result, even though the conclusion is the same as that of Theorem 5.

The generalization of Eneström–Kakeya theorem for functions of several variables was given by Mond and Shisha [53].

In 1967, Joyal, Labelle, and Rahman [46] published a result which might be considered the foundation of the studies which we are currently surveying. The Eneström–Kakeya theorem, as stated in Theorem 3, deals with polynomials with non-negative coefficients which form a monotone sequence. Joyal, Labelle, and Rahman generalized Theorem 3 by dropping the condition of nonnegativity and maintaining the condition of monotonicity. Namely, they proved:

Theorem 7 *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying $a_0 \leq a_1 \leq \cdots \leq a_n$, then all the zeros of p lie in $|z| \leq (a_n - a_0 + |a_0|)/|a_n|$.*

Of course, when $a_0 \geq 0$ then Theorem 7 reduces to Theorem 3. The Joyal–Labelle–Rahman result, like the original Eneström–Kakeya theorem, is only applicable to polynomials with real coefficients. In 1968, Govil and Rahman [30] presented a result that is applicable to polynomials with complex coefficients:

Theorem 8 *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients satisfying $|\arg a_j - \beta| \leq \alpha \leq \pi/2$ for some α and β and for $j = 0, 1, 2, \ldots, n$ and $|a_0| \leq |a_1| \leq \cdots \leq |a_n|$, then all the zeros of p lie in $|z| \leq \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|} \sum_{j=0}^{n-1} |a_j|$.*

With $\alpha = \beta = 0$, Theorem 8 reduces to Theorem 3. In the same paper, Govil and Rahman gave a result for polynomials with complex coefficients but impose a nonnegativity and monotonicity condition on the real or imaginary parts of the coefficients of the polynomial:

Theorem 9 *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients where $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$ for $j = 0, 1, 2, \ldots, n$, satisfying $0 \leq \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_n$, $\alpha_n \neq 0$, then all the zeros of p lie in $|z| \leq 1 + \frac{2}{\alpha_n} \sum_{j=0}^{n} |\beta_j|$.*

With each $\beta_k = 0$, Theorem 9 reduces to Theorem 3.

8.3 Generalizations of Eneström–Kakeya Theorem During the 1970s and 1980s

In 1973, Govil and Jain [28] refined Theorem 8 by giving a zero-free region about the origin and thus restricting the location of the zeros to an annulus:

Theorem 10 *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients satisfying $|\arg a_j - \beta| \leq \alpha \leq \pi/2$ for some α and β and for $j = 0, 1, 2, \ldots, n$ and $0 \neq |a_0| \leq |a_1| \leq \cdots \leq |a_n|$, then all the zeros of p lie in*

$$\frac{1}{R^{n-1}[2R(|a_n|/|a_0|) - (\cos\alpha + \sin\alpha)]} \leq |z| \leq R$$

where $R = \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|} \sum_{j=0}^{n-1} |a_j|$.

In the same paper, Govil and Jain similarly refined Theorem 9 by giving a zero-free annular region and improving the outer radius when the real or imaginary part of the coefficients satisfy monotonicity condition:

Theorem 11 *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients where $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$ for $j = 0, 1, 2, \ldots, n$, satisfying $0 \leq \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n$, $\alpha_n \neq 0$, then all the zeros of p lie in*

$$\frac{|a_0|}{R^{n-1}[2R\alpha_n + R|\beta_n| - (\alpha_0 + |\beta_0|)]} \leq |z| \leq R$$

where $R = 1 + \frac{1}{\alpha_n}\left(2\sum_{j=0}^{n-1} |\beta_j| + |\beta_n|\right)$.

In a "sequel" paper Govil and Jain [29] further refined Theorems 10 and 11. The refinement was accomplished by using a more sophisticated technique of proof to improve the inner and outer radii of the annulus containing the zeros of the polynomial. The refinements are, respectively:

Theorem 12 *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients satisfying $|\arg a_j - \beta| \leq \alpha \leq \pi/2$ for some α and β and for $j = 0, 1, 2, \ldots, n$ and $|a_0| \leq |a_1| \leq \cdots \leq |a_n|$, then all the zeros of p lie in*

$$\frac{1}{2M_2^2}[-R^2|b|(M_2 - |a_0|) + \{4|a_0|R^2M_2^3 + R^4|b|^2(M_2 - |a_0|)^2\}^{1/2}] \leq |z| \leq R$$

where

$$R = \frac{c}{2}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right) + \left\{\frac{c^2}{4}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right)^2 + \frac{M_1}{|a_n|}\right\}^{1/2}$$

and $M_1 = |a_n|r$, $M_2 = |a_n|R^n\left[r + R - \frac{|a_0|}{|a_n|}(\cos\alpha + \sin\alpha)\right]$, $c = |a_n - a_{n-1}|$, $b = a_1 - a_0$, and $r = \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|}\sum_{j=0}^{n-1}|a_j|$.

Theorem 13 *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients where $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$ for $j = 0, 1, 2, \ldots, n$, satisfying $0 \leq \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n$, $\alpha_n \neq 0$, then all the zeros of p lie in*

$$\frac{1}{2M_4^2}\left[-R^2|b|(M_4 - |a_0|) + \{4|a_0|R^2M_4^3 + R^4|b|^2(M_4 - |a_0|)^2\}^{1/2}\right] \leq |z| \leq R$$

where

$$R = \frac{c}{2}\left(\frac{1}{\alpha_n} - \frac{1}{M_3}\right) + \left\{\frac{c^2}{4}\left(\frac{1}{\alpha_n} - \frac{1}{M_3}\right)^2 + \frac{M_3}{\alpha_n}\right\}^{1/2}$$

and $M_3 = \alpha_n r$, $M_4 = R^n\left[(\alpha_n + |\beta_n|)R + \alpha_n r - (\alpha_0 + |\beta_0|)\right]$, $c = |a_n - a_{n-1}|$, $b = a_1 - a_0$, *and* $r = 1 + \frac{1}{\alpha_n}(2\sum_{j=0}^{n-1}|\beta_j| + |\beta_n|)$.

In 1984, Dewan and Govil [21] considered polynomials with real monotone coefficients and obtained:

Theorem 14 *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree n with real coefficients satisfying* $a_0 \le a_1 \le \cdots \le a_n$, *then all the zeros of p lie in*

$$\frac{1}{2M_2^2}\left[-R^2 b(M_2 - |a_0|) + \{4|a_0|R^2 M_2^3 + R^4 b^2 (M_2 - |a_0|)^2\}^{1/2}\right] \le |z| \le R$$

where

$$R = \frac{c}{2}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right) + \left\{\frac{c^2}{4}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right)^2 + \frac{M_1}{|a_n|}\right\}^{1/2}$$

and $M_1 = a_n - a_0 + |a_0|$, $M_2 = R^n(|a_n|R + a_n - a_0)$, $c = a_n - a_{n-1}$, *and* $b = a_1 - a_0$.

Dewan and Govil also showed that $R \le \frac{a_n - a_0 + |a_0|}{|a_n|}$ and that the inner radius of the zero containing region is less than 1, indicating that this result is an improvement of the result of Joyal, Labelle, and Rahman (Theorem 7); hence it is also an improvement of the Eneström–Kakeya theorem. They also gave specific examples of polynomials for which their result gives sharper bound than obtainable from Theorem 7 of Joyal, Labelle, and Rahman.

In 1980, Aziz and Mohammad [6] introduced a condition on the coefficients to produce the following generalization of Theorem 3:

Theorem 15 *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree n with real, positive coefficients. If* $t_1 > t_2 \ge 0$ *can be found such that*

$$a_j t_1 t_2 + a_{j-1}(t_1 - t_2) - a_{j-2} \ge 0, \quad \text{for } j = 1, 2, \ldots, n+1$$

where we take $a_{-1} = a_{n+1} = 0$, *then all zeros of p lie in* $|z| \le t_1$.

With $t_1 = 1$ and $t_2 = 0$, Theorem 15 implies the Eneström–Kakeya theorem. In the same paper, Aziz and Mohammad [6] introduced an interesting and general condition on the coefficients of a power series representation $\sum_{j=0}^{\infty} a_j z^j$ of an analytic function in order to restrict the location of the zeros. The condition is that $|\arg a_j - \beta| \le \alpha \le \pi/2$ for some α and β and for $j = 0, 1, 2, \ldots$ and $|a_0| \le t|a_1| \le \cdots \le t^{k-1}|a_{k-1}| \le t^k|a_k| \ge t^{k+1}|a_{k+1}| \ge \cdots$ for some $t > 0$ and some $k = 0, 1, \ldots$. Aziz and Mohammad [7] imposed similar conditions on the coefficients of polynomials and proved the following three theorems.

Theorem 16 *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients satisfying*

$$|a_0| \leq t|a_1| \leq \cdots \leq t^k|a_k| \geq t^{k+1}|a_{k+1}| \geq \cdots \geq t^n|a_n|$$

for some $k = 0, 1, \ldots, n$ and some $t > 0$, then all zeros of p lie in

$$|z| \leq t\left(\frac{2t^k|a_k|}{t^n|a_n|} - 1\right) + 2\sum_{j=0}^{n} \frac{|a_j - |a_j||}{t^{n-j-1}|a_n|}.$$

Theorem 17 *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients where $Re\, a_j = \alpha_j$ and $Im\, a_j = \beta_j$ for $j = 0, 1, 2, \ldots, n$, satisfying*

$$0 \leq \alpha_0 \leq t\alpha_1 \leq \cdots \leq t^k\alpha_k \geq t^{k+1}\alpha_{k+1} \geq \cdots \geq t^n\alpha_n > 0$$

for some $k = 0, 1, \ldots, n$ and some $t > 0$, then all zeros of p lie in

$$|z| \leq t\left(\frac{2t^k\alpha_k}{t^n\alpha_n} - 1\right) + \frac{2}{\alpha_n}\sum_{j=0}^{n} \frac{|\beta_j|}{t^{n-j-1}}.$$

Theorem 18 *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients where $Re\, a_j = \alpha_j$ and $Im\, a_j = \beta_j$ for $j = 0, 1, 2, \ldots, n$, satisfying*

$$0 \leq \alpha_0 \leq t\alpha_1 \leq \cdots \leq t^k\alpha_k \geq t^{k+1}\alpha_{k+1} \geq \cdots \geq t^n\alpha_n > 0$$

and

$$0 \leq \beta_0 \leq t\beta_1 \leq \cdots \leq t^r\beta_r \geq t^{r+1}\alpha_{r+1} \geq \cdots \geq t^n\beta_n \geq 0$$

for some $k = 0, 1, \ldots, n$, some $r = 0, 1, \ldots, n$, and some $t > 0$, then all zeros of p lie in

$$|z| \leq \frac{t}{|a_n|}\{2(t^{k-n}\alpha_k + t^{r-n}\beta_r) - (\alpha_n + \beta_n)\}.$$

Notice that each of the three previous results imply Theorem 3 for the appropriate choices of t, k, and β_j.

8.4 Generalizations of Eneström–Kakeya Theorem During the 1990s

In the style of Aziz and Mohammad [7], Dewan and Bidkham [20] dropped the nonnegativity condition of Theorem 17 and proved for polynomials with real coefficients:

Theorem 19 *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying*

$$a_0 \le ta_1 \le \cdots \le t^k a_k \ge t^{k+1} a_{k+1} \ge \cdots \ge t^n a_n$$

for some $k = 0, 1, \dots, n$ and some $t > 0$, then all zeros of p lie in

$$|z| \le t\left(\frac{2t^k a_k}{t^n |a_n|} - \frac{a_n}{|a_n|}\right) + \frac{1}{t^{n-1}|a_n|}(|a_0| - a_0).$$

With $a_0 > 0$ and $a_n > 0$ in Theorem 19, we see that the zeros of p lie in

$$|z| \le t\left(\frac{2t^k a_k}{t^n a_n} - 1\right).$$

The above result also follows from Theorem 17 if we take each $\beta_j = 0$, and in this sense Dewan and Bidkham's result overlaps with that of Aziz and Mohammad [7].

Related to the hypotheses of Theorem 19, Gardner and Govil [24] proved the following in 1994 which was inspired by a result by Aziz and Mohammad [6] for analytic functions:

Theorem 20 *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients where $Re\, a_j = \alpha_j$ and $Im\, a_j = \beta_j$ for $j = 0, 1, 2, \dots, n$, satisfying*

$$\alpha_0 \le t\alpha_1 \le \cdots \le t^k \alpha_k \ge t^{k+1}\alpha_{k+1} \ge \cdots \ge t^n \alpha_n$$

and

$$\beta_0 \le t\beta_1 \le \cdots \le t^r \beta_r \ge t^{r+1}\beta_{r+1} \ge \cdots \ge t^n \beta_n$$

for some $k = 0, 1, \dots, n$, some $r = 0, 1, \dots, n$, and some $t > 0$, then all zeros of p lie in $R_1 \le |z| \le R_2$, where

$$R_1 = \min\left\{(t|a_0|/(2(t^k\alpha_k + t^r\beta_r) - (\alpha_0 + \beta_0) - t^n(\alpha_n + \beta_n - |a_n|)), t\right\}$$

and

$$\begin{aligned} R_2 = \max\Bigg\{\Bigg[&|a_0|t^{n+1} - t^{n-1}(\alpha_0 + \beta_0) - t(\alpha_n + \beta_n) + (t^2 + 1)(t^{n-k-1}\alpha_k \\ &+ t^{n-r-1}\beta_r) + (t^2 - 1)\left(\sum_{j=1}^{k-1} t^{n-j-1}\alpha_j + \sum_{j=1}^{r-1} t^{n-j-1}\beta_j\right) \\ &+ (1 - t^2)\left(\sum_{j=k+1}^{n-1} t^{n-j-1}\alpha_j + \sum_{j=r+1}^{n-1} t^{n-j-1}\beta_j\right)\Bigg] \Big/ |a_n|, \frac{1}{t}\Bigg\}. \end{aligned}$$

The flexibility of Theorem 20 is revealed by considering the corollaries which result by letting $t = 1$, and $k, r \in \{0, n\}$. For example, with $t = 1$, $k = n$, and $r = n$, it implies the following, which is clearly a generalization and refinement of the result of Joyal, Labelle, and Rahman (Theorem 7):

Corollary 1 *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients where $Re\, a_j = \alpha_j$ and $Im\, a_j = \beta_j$ for $j = 0, 1, 2, \ldots, n$, satisfying*

$$\alpha_0 \le \alpha_1 \le \cdots \le \alpha_n \text{ and } \beta_0 \le \beta_1 \le \cdots \le \beta_n$$

for some $t > 0$, then all zeros of p lie in

$$\frac{|a_0|}{|a_n| - (\alpha_0 + \beta_0) + (\alpha_n + \beta_n)} \le |z| \le \frac{|a_0| - (\alpha_0 + \beta_0) + (\alpha_n + \beta_n)}{|a_n|}.$$

By making suitable choice of t and k and using appropriate transformations Gardner and Govil [24] also obtained several results analogous to the above corollary when, for example, real parts of the coefficients is monotonically decreasing and imaginary parts monotonically decreasing, or real parts of the coefficients monotonic increasing and imaginary parts monotonic decreasing.

In order to apply the above Theorem 20 of Gardner and Govil [24], both the real and imaginary parts of the coefficients have to be monotonic, but if this does not happen and instead only the real or imaginary parts of the coefficients satisfy this condition then the Theorem 20 is not applicable. In this regard, Gardner and Govil [25] proved a result related to Theorem 20, but with hypotheses restricted to just the the real parts or imaginary parts of the coefficients. To be more precise their result is the following:

Theorem 21 *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients where $Re\ a_j = \alpha_j$ and $Im\, a_j = \beta_j$ for $j = 0, 1, 2, \ldots, n$, satisfying*

$$\alpha_0 \le t\alpha_1 \le \cdots \le t^k \alpha_k \ge t^{k+1}\alpha_{k+1} \ge \cdots \ge t^n \alpha_n$$

for some $k = 0, 1, \ldots, n$, and some $t > 0$, then all zeros of p lie in $R_1 \le |z| \le R_2$, where

$$R_1 = t|a_0| \Bigg/ \left(2(t^k \alpha_k - \alpha_0 - t^n \alpha_n + t^n |a_n| + |\beta_0| + |\beta_n| t^n + 2\sum_{j=1}^{n-1} |\beta_j| t^j \right)$$

and

$$R_2 = \max\left\{ \left(|a_0| t^{n+1} + (t^2+1) t^{n-k-1} \alpha_k - t^{n-1}\alpha_0 - t\alpha_n + (t^2 - 1)\sum_{j=1}^{k-1} t^{n-j-1}\alpha_j \right.\right.$$
$$\left.\left. + (1 - t^2) \sum_{j=k+1}^{n-1} t^{n-j-1}\alpha_j + \sum_{j=1}^{n} (|\beta_{j-1}| + t|\beta_j|) t^{n-j} \right) \Bigg/ |a_n|, \frac{1}{t} \right\}.$$

By using suitable transformations, Gardner and Govil [25] also obtained results analogous to the above Theorem 21 when the condition is satisfied by imaginary parts of the coefficients.

In the same paper that contained Theorem 19, Dewan and Bidkham [20] also generalized Theorem 14 due to Dewan and Govil, and proved the following:

Theorem 22 *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying*

$$a_0 \leq a_1 \leq \cdots \leq a_k \geq a_{k+1} \geq \cdots \geq a_n$$

for some $k = 0, 1, \ldots, n$, then all zeros of p lie in

$$\frac{1}{2M_2^2}\left[-R^2 b(M_2 - |a_0|) + \{4|a_0|R^2 M_2^3 + R^4 b^2 (M_2 - |a_0|)^2\}^{1/2}\right] \leq |z| \leq R$$

where

$$R = \frac{c}{2}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right) + \left\{\frac{c^2}{4}\left(\frac{1}{|a_n|} - \frac{1}{M_1}\right)^2 + \frac{M_1}{|a_n|}\right\}^{1/2}$$

and $M_1 = -a_n + 2a_k - a_0 + |a_0|$, $M_2 = R^n(|a_n|R + 2a_k - a_n - a_0)$, $c = |a_n - a_{n-1}|$, and $b = a_1 - a_0$.

In 1998, Aziz and Shah [8] introduced a very general condition on the coefficients of a polynomial. Though the condition is complicated, it allowed them to conclude several of the previous results mentioned above. They proved:

Theorem 23 *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. Suppose for some $t > 0$ we have*

$$\max_{|z|=r} |ta_0 z^{n+1} + (ta_1 - a_0)z^n + \cdots + (ta_n - a_{n-1})z| \leq M_1$$

and

$$\max_{|z|=r} | - a_n z^{n+1} + (ta_n - a_{n-1})z^n + \cdots + (ta_1 - a_0)z| \leq M_2,$$

where r is any positive real number. Then all zeros of p lie in

$$\frac{1}{2M_2^2}\left[-r^2 b(M_2 - t|a_0|) + \{4t|a_0|r^2 M_2^3 + r^4 b^2 (M_2 - t|a_0|)^2\}^{1/2}\right] \leq |z| \leq R$$

where

$$R = 2M_1^2[-c(M_1 - |a_n|)r^2 + \{4|a_n|r^2 M_1^3 + r^4 c^2 (M_1 - |a_n|)^2\}^{1/2}]^{-1},$$
$$c = |ta_n - a_{n-1}|, \quad \text{and} \quad b = |ta_1 - a_0|.$$

Aziz and Shah proved that Theorem 23 implies Theorem 12 due to Govil and Jain and stated that a similar argument shows that Theorem 23 implies Dewan and Bidkham's Theorem 22, as well as Govil and Jain's Theorem 13. In the same paper, Aziz and Shah also gave the following result with similar type of hypotheses which implies Theorem 19 due to Dewan and Bidkham:

Theorem 24 *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with complex coefficients. Suppose for some* $t > 0$ *we have*

$$\max_{|z|=r} |ta_0 z^n + (ta_1 - a_0)z^{n-1} + \cdots + (ta_n - a_{n-1})| \le M$$

where r is any positive real number. Then all zeros of p lie in

$$|z| \le \max\left\{\frac{M}{|a_n|}, \frac{1}{r}\right\}.$$

Aziz and Zargar [9] relaxed the monotonicity condition of Joyal, Labelle, and Rahman [46] and obtained a result related to the Eneström–Kakeya theorem. Here,the disk obtained is not necessarily centered at the origin. This result involves a modification of the monotonicity condition by introducing a parameter λ, in the sense that the first $(n-1)$ coefficients of the polynomial satisfy the monotonicity condition while the last coefficient a_n does not follow this pattern, and is free. This λ condition will appear often in research on the Eneström–Kakeya theorem in the new millennium:

Theorem 25 *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with real coefficients satisfying* $a_0 \le a_1 \le \cdots \le a_{n-1} \le \lambda a_n$. *Then all the zeros of p lie in* $|z + (\lambda - 1)| \le (\lambda a_n - a_0 + |a_0|)/|a_n|$.

Of course, with $\lambda = 1$, Theorem 25 reduces to Joyal, Labelle, and Rahman's Theorem 7. With $\lambda = a_{n-1}/a_n$, we see from Theorem 25 that the zeros of a polynomial with monotone coefficients $a_0 \le a_1 \le \cdots \le a_{n-1}$ has all its zeros in $|z + (a_{n-1}/a_n - 1)| \le (a_{n-1} - a_0 + |a_0|)/|a_n|$. Later, Aziz and Zargar [9] generalized their own Theorem 25 and proved:

Theorem 26 *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree n with real coefficients satisfying*

$$a_0 \le ta_1 \le \cdots \le t^k a_k \ge t^{k+1} a_{k+1} \ge \cdots \ge t^n a_n$$

for some $k = 0, 1, \ldots, n-1$ *and some* $t > 0$, *then all zeros of p lie in*

$$\left| z + \left(\frac{a_{n-1}}{a_n} - t\right)\right| \le t\left(\frac{2t^k a_k}{t^n |a_n|} - \frac{a_{n-1}}{t|a_n|}\right) + \frac{1}{t^{n-1}|a_n|}(|a_0| - a_0).$$

In the same paper, Aziz and Zargar [9] proved a result related to Theorem 26, but with a hypothesis concerning the even-indexed and odd-indexed coefficients separately. Their result is:

Theorem 27 *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with real coefficients satisfying*

$$0 < a_0 \le t^2 a_2 \le t^4 a_4 \le \cdots \le t^{2\lfloor n/2 \rfloor} a_{2\lfloor n/2 \rfloor}$$

and

$$0 < a_1 \le t^2 a_3 \le t^4 a_5 \le \cdots \le t^{2\lfloor n/2 \rfloor} a_{2\lfloor (n+1)/2 \rfloor - 1}.$$

Then all of the zeros of p lie in

$$\left| z - \frac{a_{n-1}}{a_n} \right| \le t + \frac{a_{n-1}}{a_n}.$$

8.5 Generalizations in the New Millennium

Cao and Gardner [11] extended the even- and odd-indexed coefficient condition of Aziz and Zargar's Theorem 27 to polynomials with complex coefficients to prove the following:

Theorem 28 *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients where $Re\, a_j = \alpha_j$ and $Im\, a_j = \beta_j$ for $j = 0, 1, 2, \ldots, n$, satisfying*

$$\alpha_0 \le t^2\alpha_2 \le t^4\alpha_4 \le \cdots \le t^{2k}\alpha_{2k} \ge t^{2k+2}\alpha_{2k+2} \ge \cdots \ge t^{2\lfloor n/2\rfloor}\alpha_{2\lfloor n/2\rfloor},$$

$$\alpha_1 \le t^2\alpha_3 \le t^4\alpha_5 \le \cdots \le t^{2\ell-2}\alpha_{2\ell-1} \ge t^{2\ell}\alpha_{2\ell+1} \ge \cdots \ge t^{2\lfloor n/2\rfloor}\alpha_{2\lfloor (n+1)/2\rfloor-1},$$

$$\beta_0 \le t^2\beta_2 \le t^4\beta_4 \le \cdots \le t^{2s}\beta_{2s} \ge t^{2s+2}\beta_{2s+2} \ge \cdots \ge t^{2\lfloor n/2\rfloor}\beta_{2\lfloor n/2\rfloor},$$

and

$$\beta_1 \le t^2\beta_3 \le t^4\beta_5 \le \cdots \le t^{2q-2}\beta_{2q-1} \ge t^{2q}\beta_{2q+1} \ge \cdots \ge t^{2\lfloor n/2\rfloor}\beta_{2\lfloor (n+1)/2\rfloor-1}$$

for some k, ℓ, s, q in $\{0, 1, \ldots, \lfloor n/2\rfloor\}$. Then all the zeros of p lie in $R_1 \le |z| \le R_2$ where $R_1 = \min\left\{\frac{t|a_0|}{M_1}, t\right\}$, $R_2 = \max\left\{\frac{M_2}{|a_n|}, \frac{1}{t}\right\}$ and

$$\begin{aligned}
M_1 = & -(\alpha_0+\beta_0) + (|\alpha_1|+|\beta_1|)t - (\alpha_1+\beta_1)t \\
& + 2[\alpha_{2k}t^{2k} + 2_{2\ell-1}t^{2\ell-1} + \beta_{2s}t^{2s} + \beta_{2q-1}t^{2q-1}] - (\alpha_{n-1}+\beta_{n-1})t^{n-1} \\
& - (\alpha_n+\beta_n)t^n + (|\alpha_{n-1}|+|\beta_{n-1}|)t^{n-1} + (|\alpha_n|+|\beta_n|)t^n \\
M_2 = & \, t^{n+3}(|a_0| - \alpha_0 - \beta_0) + (|\alpha_1| - \alpha_1 - \beta_1)t^{n+2} \\
& + (t^4+1)(\alpha_{2k}t^{n-1-2k} + \alpha_{2\ell-1}t^{n-2\ell} \\
& + \beta_{2s}t^{n-1-2s} + \beta_{2q-1}t^{n-2q}) - (\alpha_{n-1}+\beta_{n-1}) + |a_{n-1}| - (\alpha_n+\beta_n)t^{-1} \\
& + (t^4-1)\left(\sum_{j=0,\, j\ even}^{2k-2} \alpha_j t^{n-1-j} + \sum_{j=1,\, j\ odd}^{2\ell-3} \alpha_j t^{n-1-j} + \sum_{j=0,\, j\ even}^{2s-2} \beta_j t^{n-1-j}\right. \\
& + \sum_{j=1,\, j\ odd}^{2q-3} \beta_j t^{n-1-j} - \sum_{j=2k+2,\, j\ even}^{2\lfloor n/2\rfloor} \alpha_j t^{n-1-j} - \sum_{j=2\ell+1,\, j\ odd}^{2\lfloor (n+1)/2\rfloor-1} \alpha_j t^{n-1-j} \\
& \left. - \sum_{j=2s+2,\, j\ even}^{2\lfloor n/2\rfloor} \beta_j t^{n-1-j} - \sum_{j=2q+1,\, j\ odd}^{2\lfloor (n+1)/2\rfloor-1} \beta_j t^{n-1-j}\right).
\end{aligned}$$

The flexible hypotheses of Theorem 28 allow to obtain a large number of corollaries. For example, monotonicity conditions can be imposed on the even and odd indexed coefficients to prove the following result for polynomials with real coefficients:

Corollary 2 *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with real coefficients satisfying*

$$a_0 \leq a_2 \leq a_4 \leq \cdots a_{2\lfloor n/2 \rfloor},$$

and

$$a_1 \geq a_3 \geq a_5 \geq \cdots a_{2\lfloor (n+1)/2 \rfloor -1}.$$

Then all the zeros of p lie in $R_1 \leq |z| \leq R_2$ where $R_1 = \min\left\{\frac{|a_0|}{M_1}, 1\right\}$ and $R_2 = \max\left\{\frac{M_2}{|a_n|}, 1\right\}$ for $M_1 = -a_0 + |a_1| + a_1 + 2a_{2\lfloor n/2 \rfloor} + |a_{n-1}| - a_{n-1} + |a_n| - a_n$ and $M_2 = |a_0| - a_0 + |a_1| + a_1 + 2a_{2\lfloor n/2 \rfloor} + |a_{n-1}| - a_{n-1} - a_n$.

Cao and Gardner gave specific examples of polynomials showing that these results sometimes give improvements over previous results. In addition, they [11] addressed a similar condition on the moduli of the even and odd indexed coefficients for polynomials with complex coefficients:

Theorem 29 *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients satisfying $|\arg a_j - \beta| \leq \alpha \leq \pi/2$ for some α and β and for $j = 0, 1, 2, \ldots, n$ and*

$$|a_0| \leq t^2|a_2| \leq t^4|a_4| \leq \cdots \leq t^{2k}|a_{2k}| \geq t^{2k+2}|a_{2k+2}| \geq \cdots \geq t^{2\lfloor n/2 \rfloor}|a_{2\lfloor n/2 \rfloor}|,$$
$$|a_1| \leq t^2|a_3| \leq t^4|a_5| \leq \cdots \leq t^{2\ell-2}|a_{2\ell-1}| \geq t^{2\ell}|a_{2\ell+1}| \geq$$
$$\cdots \geq t^{2\lfloor n/2 \rfloor}|a_{2\lfloor (n+1)/2 \rfloor -1}|$$

for some $k = 0, 1, \ldots, \lfloor n/2 \rfloor$ and $\ell = 0, 1, \ldots, \lfloor n/2 \rfloor$. Then all the zeros of p lie in $R_1 \leq |z| \leq R_2$ where $R_1 = \min\left\{\frac{t|a_0|}{M_1}, t\right\}$, $R_2 = \max\left\{\frac{M_2}{|a_n|}, \frac{1}{t}\right\}$,

$$M_1 = |a_1|t + |a_{n-1}|t^{n-1} + |a_n|t^n + \cos\alpha[-|a_0| - |a_1|t + 2|a_{2k}|t^{2k} + 2|a_{2\ell-1}|t^{2\ell-1}$$
$$-|a_{n-1}|t^{n-1} - |a_n|t^n] + \sin\alpha\left[2\sum_{j=0}^{n-2}|a_j|t^j + |a_0| + |a_1|t + |a_{n-1}|t^{n-1} + |a_n|t^n\right]$$

and

$$M_2 = |a_0|t^{n+3} + |a_1|t^{n+2} + |a_{n-1}| + \cos\alpha\left\{(t^4 - 1)\left(\sum_{j=0,\, j\ even}^{2k-2}|a_j|t^{n-1-j}\right.\right.$$
$$\left. + \sum_{j=1,\, j\ odd}^{2\ell-3}|a_j|t^{n-1-j} - \sum_{j=2k+2,\, j\ even}^{2\lfloor n/2 \rfloor}|a_j|t^{n-1-j} - \sum_{j=2\ell+1,\, j\ odd}^{2\lfloor (n+1)/2 \rfloor -1}|a_j|t^{n-1-j}\right)$$
$$+ (t^4 + 1)(|a_{2k}|t^{n-1-2k} + |a_{2\ell-1}|t^{n-2\ell}) - |a_0|t^{n+3}$$
$$\left. - |a_1|t^{n+2} - |a_{n-1}| - |a_n|t^{-1}\right\}$$
$$+ \sin\alpha\left\{(t^4 + 1)\sum_{j=2}^{n-2}|a_j|t^{n-1-j} + |a_0|t^{n-1} + |a_1|t^{n-2} + |a_{n-1}|t^4 + |a_n|t^3\right\}.$$

The hypotheses of Theorem 29, as well as several of the other results above, involve a reversal of an inequality condition on the coefficients of a polynomial. In 2005, Chattopadhyay, Das, Jain, and Konwar [14] took this idea of a reversal of the inequality to its logical conclusion, and introduced hypotheses concerning an arbitrary number of reversals in an inequality on the coefficients. As an example, they [14] proved:

Theorem 30 *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients where $Re\, a_j = \alpha_j$ and $Im\, a_j = \beta_j$ for $j = 0, 1, 2, \dots, n$, satisfying for some $t > 0$*

$$\alpha_0 \leq t\alpha_1 \leq \cdots \leq t^{k_1}\alpha_{k_1} \geq t^{k_1+1}\alpha_{k_1+1} \geq \cdots \geq t^{k_2}\alpha_{k_2} \leq t^{k_2+1}\alpha_{k_2+1} \leq \cdots t^n\alpha_n$$

and

$$\beta_0 \leq t\beta_1 \leq \cdots \leq t^{r_1}\beta_{r_1} \geq t^{r_1+1}\beta_{r_1+1} \geq \cdots \geq t^{r_2}\beta_{r_2} \leq t^{r_2+1}\beta_{r_2+1} \leq \cdots t^n\beta_n,$$

where the inequalities involving the real parts reverse at each of the indices $k_1, k_2, \dots, k_p$, and the inequalities involving the imaginary parts reverse at each of the indices $r_1, r_2, \dots, r_q$. Then all zeros of p lie in $R_1 \leq |z| \leq R_2$, where

$$R_1 = \min\left\{\frac{t|a_0|}{M_1}, t\right\}, R_2 = \max\left\{\frac{M_2}{|a_n|}, \frac{1}{t}\right\},$$

$$M_1 = -\left(\alpha_0 + (-1)^{p+1}\alpha_n t^n + \sum_{j=1}^{p}(-1)^j\alpha_{k_j}t^{k_j}\right) - \left(\beta_0 + (-1)^{q+1}\beta_n t^n + \sum_{j=1}^{q}(-1)^j\beta_{r_j}t^{r_j}\right) + |a_n|t^n,$$

and

$$\begin{aligned} M_2 = \Bigg[& -\alpha_0 t^{n-1} + (-1)^{p+1}\alpha_n t + (t^2+1)\sum_{j=1}^{p}(-1)^j\alpha_{k_j}t^{n-k_j-1} \\ & + (t^2-1)\sum_{j=0}^{p}\left\{(-1)^{j+1}\sum_{m=k_j+1}^{k_{j+1}-1}\alpha_m t^{n-m-1}\right\}\Bigg] \\ & - \Bigg[\beta_0 t^{n-1} + (-1)^{q+1}\beta_n t + (t^2+1)\sum_{j=1}^{q}(-1)^j\beta_{r_j}t^{n-r_j-1} \\ & + (t^2-1)\sum_{j=0}^{q}\left\{(-1)^{j+1}\sum_{m=r_j+1}^{r_{j+1}-1}\beta_m t^{n-m-1}\right\}\Bigg] + |a_0|t^{n+1}, \end{aligned}$$

where we take $k_0 = r_0 = 0$ and $k_{p+1} = r_{q+1} = n$.

With $k_1 = k, r_1 = r$, and the number of reversals $p = q = 1$, Theorem 30 reduces to Gardner and Govil's Theorem 20. In the same paper, Chattopadhyay, Das, Jain, and Konwar also gave a result which hypothesizes a number of reversals in an inequality concerning the moduli of the coefficients, thus giving a generalization of Theorem 16 due to Aziz and Mohammad.

In 2007, Shah and Liman [56] extended Aziz and Zargar's idea from Theorem 25 to complex polynomials by hypothesizing a condition on the moduli of the polynomial. Their result is as follows:

Theorem 31 *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients where $Re\, a_j = \alpha_j$ and $Im\, a_j = \beta_j$ where $|arg\, a_j - \beta| \leq \alpha \leq \pi/2$ for some α and β for $j = 0, 1, 2, \ldots, n$. If for some $\lambda \geq 1$ we have*

$$|a_0| \leq |a_1| \leq \ldots |a_{n-1}| \leq \lambda |a_n|$$

then all the zeros of p lie in

$$|z + (\lambda - 1)| \leq \left\{ (\lambda|a_n| - |a_0|)(\sin\alpha + \cos\alpha) + |a_0| + 2\sin\alpha \sum_{j=0}^{n-1} |a_j| \right\} \Big/ |a_n|.$$

In the same paper, Shah and Liman produced similar results by imposing the "λ condition" on the real parts and by combining this with a reversal in the monotonicity condition.

In 2009, in a paper dealing mostly with the number of zeros in a region, Jain [45] produced a corollary involving a fairly simple monotonicity condition very similar to the original Eneström–Kakeya theorem, but combined with an additional hypothesis on coefficients a_0, a_{n-1}, and a_n:

Theorem 32 *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying $0 < a_0 \leq a_1 \leq \cdots \leq a_{n-1} < a_n$ and such that*

$$(n+1)^n a_n^{n-1} \{(n+1)a_0 a_n + n(a_n - a_{n-1})(a_{n-1} - a_0)\} < n^n (a_n - a_{n-1})^{n+1},$$

then all the zeros of p lie in

$$|z| \leq \frac{n}{n+1} \frac{a_n - a_{n-1}}{a_n} < 1.$$

Jain [45] also showed by example that for some polynomials satisfying both the hypotheses of the Eneström–Kakeya theorem and the hypotheses of his Theorem 32, the location of the zeros can be more finely constrained by his result than by the Eenström–Kakeya theorem (which will, of course, restrict the zeros to $|z| \leq 1$).

Choo [16] generalized Theorem 29 by introducing another parameter in each of the monotonicity-type hypotheses on the coefficients. In addition, he gave a simpler expression for the upper bound on the zero containing region:

Theorem 33 *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients satisfying $|\arg a_j - \beta| \leq \alpha \leq \pi/2$ for some α and β and for $j = 0, 1, 2, \ldots, n$ and*

$$|a_0| \leq t^2|a_2| \leq t^4|a_4| \leq \cdots \leq t^{2k}|a_{2k}| \geq t^{2k+2}|a_{2k+2}| \geq \cdots \geq \lambda_1 t^{2\lfloor n/2 \rfloor}|a_{2\lfloor n/2 \rfloor}|,$$

$$|a_1| \leq t^2|a_3| \leq t^4|a_5| \leq \cdots \leq t^{2\ell-2}|a_{2\ell-1}| \geq t^{2\ell}|a_{2\ell+1}| \geq$$

$$\cdots \geq \lambda_2 t^{2\lfloor n/2 \rfloor}|a_{2\lfloor (n+1)/2 \rfloor - 1}|$$

for some $k = 0, 1, \ldots, \lfloor n/2 \rfloor$, $\ell = 0, 1, \ldots, \lfloor n/2 \rfloor$, $\lambda_1 > 0$, and $\lambda_2 > 0$. Then all the zeros of p lie in $R_1 \leq |z| \leq R_2$ where $R_1 = \min\left\{\frac{t|a_0|}{M_1}, t\right\}$, $R_2 = \max\left\{\frac{M_2}{t^{n-1}|a_n|}, \frac{1}{t}\right\}$,

$$M_1 = |a_1|t + |a_{n-1}|t^{n-1} + |a_n|t^n + |(\lambda^{**} - 1)a_{n-1}|t^{n-1} + |(\lambda^* - 1)a_n|t^n$$

$$+ \cos\alpha[-|a_0| - |a_1|t + 2|a_{2k}|t^{2k} + 2|a_{2\ell-1}|t^{2\ell-1} - \lambda^{**}|a_{n-1}|t^{n-1} - \lambda^*|a_n|t^n]$$

$$+ \sin\alpha\left[2\sum_{j=0}^{n-2}|a_j|t^j + |a_0| + |a_1|t + \lambda^{**}|a_{n-1}|t^{n-1} + \lambda^*|a_n|t^n\right]$$

and

$$M_2 = |a_0| + |a_1|t + |a_{n-1}|t^{n-1} + |(\lambda^{**} - 1)a_{n-1}|t^{n-1} + |(\lambda^* - 1)a_n|t^n$$

$$+ \cos\alpha\left[2|a_{2k}|t^{2k} + 2|a_{2\ell-1}|t^{2\ell-1} - \lambda^*|a_n|t^n - \lambda^{**}|a_{n-1}|t^{n-1} - |a_1|t - |a_0|\right]$$

$$+ \sin\alpha\left\{\lambda^*|a_n|t^n + \lambda^{**}|a_{n-1}|t^{n-1} + |a_1|t + |a_0| + 2\sum_{j=2}^{n-2}|a_j|t^j\right\}.$$

For n even we have $\lambda^* = \lambda_1$ and $\lambda^{**} = \lambda_2$, but for n odd we have $\lambda^* = \lambda_2$ and $\lambda^{**} = \lambda_1$.

In the same paper, Choo [16] gave a similar generalization of Theorem 28 due to Cao and Gardener.

In 2010, Singh and Shah [57] combined the hypotheses of Aziz and Mohammad's Theorem 15 (but applied to complex coefficients, as opposed to real coefficients) with the hypotheses of Aziz and Zargar's Theorem 25 to get the following:

Theorem 34 *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients where $Re\, a_j = \alpha_j$ and $Im\, a_j = \beta_j$ for $j = 0, 1, 2, \ldots, n$. If $t_1 > t_2 \geq 0$, can be found such that we have*

$$\alpha_j t_1 t_2 + \alpha_{j-1}(t_1 - t_2) - \alpha_{j-2} \geq 0, \quad \textit{for } j = 2, 3, \ldots, n,$$

$$\beta_j t_1 t_2 + \beta_{j-1}(t_1 - t_2) - \beta_{j-2} \geq 0, \quad \textit{for } j = 2, 3, \ldots, n,$$

where we take $\alpha_{n+1} = \beta_{n+1} = 0$, and for some $\lambda \geq 1$,

$$\lambda\alpha_n(t_1 - t_2) - \alpha_{n-1} \geq 0 \textit{ and } \lambda\beta_n(t_1 - t_2) - \beta_{n-1} \geq 0,$$

then all zeros of p lie in $|z + (\lambda - 1)(t_1 - t_2)| \le R$ *where*

$$R = \frac{1}{|\alpha_n|}\left\{\lambda(\alpha_n + \beta_n)(t_1 - t_2) + (\alpha_n + \beta_n)t_2 - (\alpha_1 + \beta_1)\frac{t_2}{t_1^{n-1}} - (\alpha_0 + \beta_0)\frac{1}{t_1^{n-1}}\right.$$
$$\left. + (|\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)| + |\beta_1 t_1 t_2 + \beta_0(t_1 - t_2)|)\frac{1}{t_1^n} + (|\alpha_0| + |\beta_0|)\frac{t_2}{t_1^n}\right\}.$$

With all the coefficients real and positive, and $\lambda = 1$, the Theorem 34 reduces to Theorem 15 due to Aziz and Mohammad. With $t_1 = 1$, $t_2 = 0$, and $\lambda = 1$, Theorem 34 reduces to Theorem 25 of Aziz and Zargar. In the same paper, Singh and Shah [57] modified the hypotheses

$$\lambda\alpha_n(t_1 - t_2) - \alpha_{n-1} \ge 0 \quad \text{and} \quad \lambda\beta_n(t_1 - t_2) - \beta_{n-1} \ge 0,$$

to

$$\lambda_1\alpha_n(t_1 - t_2) - \alpha_{n-1} \ge 0 \quad \text{and} \quad \lambda_2\beta_n(t_1 - t_2) - \beta_{n-1} \ge 0$$

where $\lambda_1 \ge 1$ and $\lambda_2 \ge 1$, and proved a result concerning the location of zeros in a disk (not necessarily centered at origin) which includes many of the other results mentioned above. In a related result, but concerning zeros in a disk centered at origin, Singh and Shah [58] in 2011 presented the following:

Theorem 35 *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with complex coefficients where* $\mathrm{Re}\, a_j = \alpha_j$ *and* $\mathrm{Im}\, a_j = \beta_j$ *for* $j = 0, 1, 2, \ldots, n$. *If* $t_1 > t_2 \ge 0$ *can be found such that for* $j = 1, 2, \ldots, n+1$ *we have*

$$\alpha_j t_1 t_2 + \alpha_{j-1}(t_1 - t_2) - \alpha_{j-2} \ge 0$$

and

$$\beta_j t_1 t_2 + \beta_{j-1}(t_1 - t_2) - \beta_{j-2} \ge 0,$$

where we take $\alpha_{-1} = \alpha_{n+1} = \beta_{-1} = \beta_{n+1} = 0$. *Then all zeros of p lie in* $|z| \le (|\alpha_n + \beta_n + M|t_1/|a_n|$ *where*

$$M = -\alpha_1\frac{t_2}{t_1^n} - \frac{\alpha_0}{t_1^n} + |\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)|\frac{1}{t_1^{n+1}} + |\alpha_0 t_1 t_2|\frac{1}{t_1^{n+2}}$$
$$-\beta_1\frac{t_2}{t_1^n} - \frac{\beta_0}{t_1^n} + |\beta_1 t_1 t_2 + \beta_0(t_1 - t_2)|\frac{1}{t_1^{n+1}} + |\beta_0 t_1 t_2|\frac{1}{t_1^{n+2}}.$$

Again, this result implies Aziz and Mohammad's Theorem 15.

Using the same hypotheses as Theorem 35, Singh and Shah [59] proved another result concerning the location of zeros, but this time obtained an annulus region containing all the zeros:

Theorem 36 *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with complex coefficients where* $\mathrm{Re}\, a_j = \alpha_j$ *and* $\mathrm{Im}\, a_j = \beta_j$ *for* $j = 0, 1, 2, \ldots, n$. *If* $t_1 \ge t_2$, $t_1 \ne 0$, *can be found such that for* $j = 1, 2, \ldots, n+1$ *we have*

$$\alpha_j t_1 t_2 + \alpha_{j-1}(t_1 - t_2) - \alpha_{j-2} \ge 0$$

and

$$\beta_j t_1 t_2 + \beta_{j-1}(t_1 - t_2) - \beta_{j-2} \geq 0,$$

where we take $\alpha_{-1} = \alpha_{n+1} = \beta_{-1} = \beta_{n+1} = 0$. *Then all zeros of p lie in* $\min\{R_2, 1/t_1\} \leq |z| \leq \max\{R_1, t_1\}$. *Here,*

$$R_1 = \Big\{-(|a_n| - K_1)|a_n(t_1 - t_2) - a_{n-1}| + \big[(|a_n| - K_1)^2 |a_n(t_1 - t_2) - a_{n-1}|^2 + 4K_1^3 t_1^2 |a_n|\big]^{1/2}\Big\}/(2K_1|a_n|),$$

$$R_2 = \Big\{-(|a_0|t_1t_2 - K_2)|a_1t_1t_2 + a_0(t_1 - t_2)|t_1^2 + \big[(|a_0|t_1t_2 - K_2)^2 |a_1t_1t_2 + a_0(t_1 - t_2)|^2 t_1^4 + 4K_2^3|a_0|t_1^3t_2\big]^{1/2}\Big\}/(2K_2^2),$$

$K_1 = (\alpha_n + \beta_n) + (|\alpha_0| - \alpha_0)t_2/t_1^{n+1} + (|\beta_0| - \beta_0)t_2/t_1^{n+1}$, *and* $K_2 = (\alpha_n + \beta_n)t_1^{n+2} + (|\alpha_n| + |\beta_n|)t_1^{n+2} - (\alpha_0 + \beta_0)t_1t_2$.

When each $\beta_j = 0$, Theorem 36 reduces to Theorem 15. With all coefficients real and positive, and $t_1 = 1$ and $t_2 = 0$, Theorem 36 implies the following clean refinement of Theorem 3:

Corollary 3 *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with real coefficients satisfying* $0 \leq a_0 \leq a_1 \leq \cdots \leq a_n$, *then all the zeros of p lie in* $\dfrac{a_0}{2a_n} \leq |z| \leq 1$.

In the same paper, Singh and Shah [59] introduced a reversal in the inequality imposed on the coefficients at a particular point and proved:

Theorem 37 *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with complex coefficients where* $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $j = 0, 1, 2, \ldots, n$. *If* $t_1 > t_2 \geq 0$ *can be found such that*

$$\alpha_j t_1 t_2 + \alpha_{j-1}(t_1 - t_2) - \alpha_{j-2} \geq 0, \text{ for } j = 2, 3, \ldots, r+1,$$

$$\alpha_j t_1 t_2 + \alpha_{j-1}(t_1 - t_2) - \alpha_{j-2} \leq 0, \text{ for } j = r+2, r+3, \ldots, n+1,$$

$$\beta_j t_1 t_2 + \beta_{j-1}(t_1 - t_2) - \beta_{j-2} \geq 0, \text{ for } j = 2, 3, \ldots, r+1,$$

$$\beta_j t_1 t_2 + \beta_{j-1}(t_1 - t_2) - \beta_{j-2} \leq 0, \text{ for } j = r+2, r+3, \ldots, n+1,$$

for some λ *with* $1 \leq r \leq n$, *where we take* $\alpha_{n+1} = \beta_{n+1} = 0$, *then all zeros of p lie in*

$$|z| \leq \frac{t_1}{|a_n|}\left\{\left(\frac{2\alpha_r}{t_1^{n-r}} - \alpha_n\right) + \frac{1}{t_1^n}(|\alpha_0| - \alpha_0)\right\} + \frac{t_2}{|a_n|}\left\{\frac{2\alpha_{r+1}}{t_1^{n-r-1}} + \frac{1}{t_1^n}(|\alpha_0| - \alpha_0)\right\}$$
$$+ \frac{t_1}{|a_n|}\left\{\left(\frac{2\beta_r}{t_1^{n-r}} - \beta_n\right) + \frac{1}{t_1^n}(|\beta_0| - \beta_0)\right\} + \frac{t_2}{|a_n|}\left\{\frac{2\beta_{r+1}}{t_1^{n-r-1}} + \frac{1}{t_1^n}(|\beta_0| - \beta_0)\right\}.$$

Singh and Shah remarked that Theorem 37 reduces to Dewan and Bidkham's Theorem 19 when each coefficient is real and $t_2 = 0$, and further reduces to Theorem 3 when $r = n$ and $a_0 \geq 0$. In the same paper, Singh and Shah [59] also presented a related generalization of Theorem 26 due to Aziz and Zargar.

In 2013, Singh and Shah [60] gave another result related to Theorem 36:

Theorem 38 *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients where $Re\, a_j = \alpha_j$ and $Im\, a_j = \beta_j$ for $j = 0, 1, 2, \dots, n$. If $t_1 > t_2 \geq 0$, can be found such that for $j = 2, 3, \dots, n$ we have*

$$\alpha_j t_1 t_2 + \alpha_{j-1}(t_1 - t_2) - \alpha_{j-2} \geq 0,$$

$$\beta_j t_1 t_2 + \beta_{j-1}(t_1 - t_2) - \beta_{j-2} \geq 0,$$

and for some real λ_1 and λ_2 we have

$$(\alpha_n + \lambda_1)(t_1 - t_2) - \alpha_{n-1} \geq 0, \textit{and}$$

$$(\beta_n + \lambda_2)(t_1 - t_2) - \beta_{n-1} \geq 0,$$

then all zeros of p lie in

$$\left| z + \frac{(\lambda_1 + i\lambda_2)(t_1 - t_2)}{a_n} \right| \leq R,$$

where

$$R = \{[(\alpha_n + \lambda_1) + (\beta_n + \lambda_2)](t_1 - t_2) + (\alpha_n + \beta_n)t_2 - (\alpha_1 + \beta_1)t_2/t_1^{n-1} - (\alpha_0 + \beta_0)/t_1^{n-1}$$
$$+ (|\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)| + |\beta_1 t_1 t_2 + \beta_0(t_1 - t_2)|)/t_1^n + (|\alpha_0| + |\beta_0|)t_2/t_1^n\}/|a_n|.$$

With $t_2 = 0$ in Theorem 38, one easily gets as a corollary the following:

Corollary 4 *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients where $Re\, a_j = \alpha_j$ and $Im\, a_j = \beta_j$ for $j = 0, 1, 2, \dots, n$. If $t > 0$, can be found such that*

$$t^n(\alpha_n + \lambda_1) \geq t^{n-1}\alpha_{n-1} \geq t^{n-2}\alpha_{n-1} \geq \cdots \geq t\alpha_1 \geq \alpha_0$$

and

$$t^n(\beta_n + \lambda_2) \geq t^{n-1}\beta_{n-1} \geq t^{n-2}\beta_{n-1} \geq \cdots \geq t\beta_1 \geq \beta_0$$

for some real λ_1 and λ_2, then all zeros of p lie in

$$\left| z + \frac{(\lambda_1 + i\lambda_2)t}{a_n} \right| \leq R$$

where

$$R = t\{(\alpha_n + \lambda_1) + (\beta_n + \lambda_2) - [\alpha_0 + \beta_0 - |\alpha_0| - |\beta_0|]/t^n\}/|a_n|.$$

Among the many results listed above which overlap with Corollary 4 is included Joyal, Labelle, and Rahman's Theorem 7, which follows from the corollary when $\lambda_1 = \lambda_2 = 0$ and $t = 1$.

Many of the results above, such as Corollary 4, involve the parameter $t > 0$ in such a way that coefficient a_j (or possibly its real part, imaginary part, or modulus) is multiplied by t^j and then involved in some type of monotonicity condition. Such a result often will follow from a simpler result which does not involve parameter t by applying the simpler result to polynomial $p(tz)$. Recently, Gulzar, Liman, and Shah [39] introduced a condition on the coefficients which is somewhat more subtle and does not yield a result which will easily follow from a simpler theorem. They required the parameter t to follow a pattern similar to that given in Corollary 4, but only for *some* of the coefficients. They prove:

Theorem 39 *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with real coefficients satisfying*

$$a_0 \le a_1 \le \cdots \le a_{k-1} \le t a_k \le t^2 a_{k+1} \le \cdots \le t^{n-k} a_{n-1} \le t^{n-k+1} a_n$$

for some $t > 0$ and $1 \le k \le n$. Then all the zeros of p lie in

$$|z + (t-1)| \le \frac{a_n - a_0 + |a_0| + (t-1)\left\{\sum_{j=k}^{n} (a_j + |a_j|) - |a_n|\right\}}{|a_n|}.$$

With $k = n$ and $t = \lambda$, Theorem 39 implies Aziz and Zargar's Theorem 25. With $k = 1$, the hypotheses of Theorem 39 is similar to several of the results above (though there is no resulting reversal in the monotonicity hypothesis).

Choo and Choi [18] gave an interesting result in 2011 related to the hypothesis of monotonicity of the coefficients in the Eneström–Kakeya theorem. They allowed one coefficient, say a_k, to violate the monotonicity condition and then constrained the deviation of a_k from a_{k-1} and a_{k+1} such that the zeros of the polynomial would still lie in $|z| \le 1$:

Theorem 40 *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying $a_{r_i} = a_{r_i+1} = \cdots = a_{r_{i+1}-1}$ for $i = 0, 1, 2, \ldots, m$, where $r_0 = 0 < r_1 < r_2 < \cdots < r_m < r_{m+1} = n+1$. Suppose that for some $0 \le k \le m-1$,*

$$a_{r_m} > a_{r_{m-1}} > \cdots > a_{r_{k+1}} > a_{r_{k-1}} > a_{r_{k-2}} > \cdots > a_{r_n} > 0$$

and let

$$\rho = \max\left\{\frac{a_{r_{m-1}}}{a_{r_m}}, \frac{a_{r_{m-2}}}{a_{r_{m-1}}}, \ldots, \frac{a_{r_{k+1}}}{a_{r_{k+2}}}, \frac{a_{r_{k-1}}}{a_{r_{k+1}}}, \frac{a_{r_{k-2}}}{a_{r_{k-1}}}, \ldots, \frac{a_{r_0}}{a_{r_1}}\right\}.$$

If $\rho < 1$, then p has all its zeros in the disk $|z| \le 1$ provided $a_{r_{k-1}} - \epsilon_1 \le a_{r_k} \le a_{r_{k+1}} + \epsilon_2$ where

$$\epsilon_1 = \frac{(1-\rho)R_1}{1+\rho+(1-\rho)(r_{k+1}-r_k-1)} \text{ and } \epsilon_2 = \frac{(1-\rho)R_2}{1+\rho+(1-\rho)(r_{k+1}-r_k-1)},$$

$$R_1 = a_{r_m} + a_{r_{m-1}} + \cdots + a_{r_{k+1}} + a_{r_{k-2}} + \cdots + a_{r_1} + a_0$$
$$- \{(n-r_m)a_{r_m} + (r_m - r_{m-1} - 1)a_{r_{m-1}} + \cdots + (r_{k+2} - r_{k+1} - 1)a_{r_{k+1}}$$

$$+ (r_{k+1} - r_{k-1} - 2)a_{r_{k-1}} + (r_{k-1} - r_{k-2} - 1)a_{r_{k-2}} + \cdots + (r_2 - r_1 - 1)a_{r_1} + (r_1 - 1)a_0\},$$

and

$$R_2 = a_{r_m} + a_{r_{m-1}} + \cdots + a_{r_{k+2}} + a_{r_{k-1}} + \cdots + a_{r_1} + a_0$$
$$-\{(n - r_m)a_{r_m} + (r_m - r_{m-1} - 1)a_{r_{m-1}} + \cdots + (r_{k+2} - r_k - 2)a_{r_{k+1}}$$
$$+(r_k - r_{k-1} - 1)a_{r_{k-1}} + (r_{k-1} - r_{k-2} - 1)a_{r_{k-2}} + \cdots + (r_2 - r_1 - 1)a_{r_1} + (r_1 - 1)a_0\}.$$

Choo and Choi gave examples of polynomials illustrating their result. In particular, they gave $P(z) = 3.6z^6 + 5z^5 + 4z^4 + 3.2z^3 + 2.5z^2 + 2z + 1.5$ as an example of a polynomial which violates the monotonicity condition of Eneström–Kakeya, but which still has its zeros in $|z| \leq 1$. Coefficient a_6 violates the monotonicity condition; the authors computed $\epsilon_1 = 1.4667$ and observed that $a_6 \geq a_5 - \epsilon_1$, thus indicating that the hypotheses of their theorem are satisfied. In the same issue of the same journal, Choo and Choi [17] introduced the following generalization of the Eneström–Kakeya theorem:

Theorem 41 *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying*

$$a_0 \leq a_1 \leq \cdots \leq a_{n-k-1} \leq \lambda a_{n-k} \leq a_{n-k+1} \leq \cdots \leq a_n$$

for some real λ, then the zeros of p lie in $|z| \geq R$ where

$$R = \frac{|a_0|}{|a_n| + a_n + |(\lambda - 1)a_{n-k}| + (\lambda - 1)a_{n-k} - a_0} \quad \text{if } a_{n-k-1} \geq a_{n-k},$$

and

$$R = \frac{|a_0|}{|a_n| + a_n + |(\lambda - 1)a_{n-k}| + (1 - \lambda)a_{n-k} - a_0} \quad \text{if } a_{n-k} \geq a_{n-k+1}.$$

In the same paper, Choo and Choi gave a similar result by hypothesizing that the real parts of p satisfy the conditions of Theorem 41 and that the imaginary parts are monotone increasing. They also gave the following result which has a hypothesis concerning the moduli of the coefficients of p:

Theorem 42 *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients where $\mathrm{Re}\, a_j = \alpha_j$ and $\mathrm{Im}\, a_j = \beta_j$ where $|\arg a_j - \beta| \leq \alpha \leq \pi/2$ for some α and β for $j = 0, 1, 2, \ldots, n$. If*

$$|a_0| \leq |a_1| \leq \cdots \leq |a_{n-k-1}| \leq \lambda |a_{n-k}| \leq |a_{n-k+1}| \leq \cdots \leq |a_n|$$

for some $\lambda > 0$, then the zeros of p lie in $|z| \geq R$ where

$$R = \frac{|a_0|}{(|a_n| + (\lambda - 1)|a_{n-k}|)(\cos\alpha + \sin\alpha) - |a_0|(\cos\alpha - \sin\alpha) + 2\sin\alpha \sum_{j=1}^{n-1} |a_j|}$$

if $|a_{n-k-1}| \geq |a_{n-k}|$, and

$$R = \frac{|a_0|}{(|a_n| + (1 - \lambda)|a_{n-k}|)(\cos\alpha + \sin\alpha) - |a_0|(\cos\alpha - \sin\alpha) + 2\sin\alpha \sum_{j=1}^{n-1} |a_j|}$$

if $|a_{n-k}| \geq |a_{n-k+1}|$.

In 2012, Aziz and Zargar [10] modified the hypotheses of their own 1996 result, Theorem 25, and proved the following three theorems.

Theorem 43 *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree n with real coefficients such that for some* $\lambda \geq 1$ *and* $0 < \rho \leq 1$ *we have*

$$0 \leq \rho a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_{n-1} \leq \lambda a_n,$$

then all the zeros of p lie in $|z + \lambda - 1| \leq \lambda + 2a_0(1 - \rho)/a_n$.

Theorem 44 *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree n with real coefficients such that for some* $0 < \rho \leq 1$ *and some* $0 \leq k \leq n$ *we have*

$$\rho a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_k \geq a_{k+1} \geq \cdots \geq a_n,$$

then all the zeros of p lie in

$$\left| z + \frac{a_{n-1}}{a_n} - 1 \right| \leq \frac{1}{|a_n|}\{2a_k - a_{n-1} + (2 - \rho)|a_0| - \rho a_0\}.$$

Theorem 45 *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree n with real coefficients such that for some* $0 < \rho \leq 1$ *and some* $0 \leq k \leq n$ *we have*

$$\rho a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_k \geq a_{k+1} \geq \cdots \geq \lambda a_n,$$

then all the zeros of p lie in

$$|z| \leq \frac{2a_k - a_n + (2 - \rho)|a_0| + \rho a_0}{|a_n|}.$$

Aziz and Zargar [10] also showed that each of these implies Theorem 7 of Joyal, Labelle, and Rahman, and hence it is a generalization of the Eneström–Kakeya theorem.

Recently, Gulzar [33] (see also [31]) proved:

Theorem 46 *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with complex coefficients where* $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *for* $j = 0, 1, 2, \ldots, n$*, satisfying*

$$\rho\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_{n-1} \leq \sigma + \alpha_n$$

for some $\sigma \geq 0$ *and* $0 < \rho \leq 1$*, then the zeros of p lie in*

$$\left| z + \frac{\sigma}{\alpha_n} \right| \leq \frac{\sigma + \alpha_n - \rho(|\alpha_0| + \alpha_0) + 2|\alpha_0| + 2\sum_{j=0}^{n} |\beta_j|}{|\alpha_n|}.$$

Under the same hypotheses, Gulzar [38] gave an inner radius for a zero-free region for p as given in Theorem 46 by showing that p has no zeros in $|z| \leq |a_0|/\{2\sigma + a_n + |a_n| - \rho(a_0 + |a_0|) + |a_0|\}$. With similar monotonicity, hypotheses concerning the coefficients, but with the added factor σ as in Theorem 46 and with $\rho = 1$, Liman, Shah, and Ahmad [49] gave additional related results in a 2013 publication.

In a result related to Choo and Choi's Theorems 41 and 46, Gulzar [32] proved:

Theorem 47 *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients where $\mathrm{Re}\, a_j = \alpha_j$ and $\mathrm{Im}\, a_j = \beta_j$ for $j = 0, 1, 2, \ldots, n$, satisfying*

$$\alpha_0 \le \alpha_1 \le \cdots \le \alpha_{n-k-1} \le \lambda\alpha_{n-k} \le \alpha_{n-k+1} \le \cdots \le \alpha_{n-1} \le \sigma + \alpha_n$$

for some $\sigma \ge 0$ and real λ, where $\alpha_{n-k} \ne 0$, then the zeros of p lie in

$$\left| z + \frac{\sigma}{a_n} \right| \le \frac{\sigma + \alpha_n + (\lambda - 1)\alpha_{n-k} + |\lambda - 1||\alpha_{n-k}| + |\alpha_0| - \alpha_0 + 2\sum_{j=0}^{n} |\beta_j|}{|a_n|}$$

if $\alpha_{n-k-1} > \alpha_{n-k}$, and the zeros lie in

$$\left| z + \frac{\sigma}{a_n} \right| \le \frac{\sigma + \alpha_n + (1 - \lambda)\alpha_{n-k} + |\lambda - 1||\alpha_{n-k}| + |\alpha_0| - \alpha_0 + 2\sum_{j=0}^{n} |\beta_j|}{|a_n|}$$

if $\alpha_{n-k} > \alpha_{n-k+1}$.

With $\sigma = 0$ and each $\beta_j = 0$ in Gulzar's Theorem 47, one can produce an annulus (centered at origin) containing all the zeros of the polynomial where the inner radius is given by Choo and Choi's Theorem 41.

In [35], Gulzar combines the hypotheses of his own Theorems 46 and 47 (with parameters ρ, σ, and λ) to present three generalizations of the Eneström–Kakeya theorem (with hypotheses on (1) the real part, (2) the imaginary part, and (3) the modulus of the coefficients).

8.6 Related Results

In this survey, we have have tried to present results that put a restriction on the modulus of the zeros of a polynomial *explicitly* in terms of the coefficients of the polynomial, as the original Eneström–Kakeya theorem does. There are other results related to the Eneström–Kakeya theorem which we have not yet been able to mention due to restrictions in the length of this chapter, but will now describe them briefly.

We say "explicitly" in the previous paragraph, because there are a number of results which restrict the modulus of the zeros of a polynomial, but the restrictions are given indirectly in the sense of being given by a root of a polynomial itself. This type of result was first given by Cauchy [13], who proved the following:

Theorem 48 *Let $p(z) = z^n + \sum_{j=0}^{n-1} a_j z^j$, be a complex polynomial. Then all the zeros of $p(z)$ lie in the disk*

$$\{z : |z| < \eta\} \subset \{z : |z| < 1 + A\}, \tag{8.1}$$

where

$$A = \max_{0 \le j \le n-1} |a_j|,$$

and η *is the unique positive root of the real-coefficient polynomial*

$$Q(x) = x^n - |a_{n-1}|x^{n-1} - |a_{n-2}|x^{n-2} - \cdots - |a_1|x - |a_0|. \tag{8.2}$$

Govil and Rahman [30] also gave this type of result, and the same is stated as follows:

Theorem 49 *Let* $p(z) = \sum_{j=0}^{n} a_j z^j \not\equiv 0$ *be a polynomial of degree n with complex coefficients such that for some* $a > 0$*, we have* $a^n|a_0| \le a^{n-1}|a_1| \le \cdots \le a|a_{n-1}| \le |a_n|$*. Then all the zeros of p lie in* $|z| \le \left(\frac{1}{a}\right) M$ *where M is the greatest positive root of the trinomial equation* $x^{n+1} - 2x^n + 1 = 0$.

Related results concerning the location of the zeros of a polynomial have also been presented by Aziz and Mohammad [7], Sun and Hsieh [61], Affane-Aji, Agarwal, and Govil [2], Affane-Aji, Biaz and Govil [3], Choo [15], Choo and Choi [17], Dalal and Govil [19], Gulzar [34, 36], and Gilani [27].

The hypotheses of the following result, due to Jain [43] in 1988, are very much in the spirit of the Eneström–Kakeya theorem, although the conclusion involves the size of the real part of the zeros instead of the modulus:

Theorem 50 *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with complex coefficients where* $Re\, a_j = \alpha_j$ *and* $Im\, a_j = \beta_j$ *where* $|\arg a_j - \beta| \le \alpha \le \pi/2$ *for some* α *and* β*, and* $j = 0, 1, 2, \ldots, n$*. If* $0 < |a_0| \le |a_1| \le \cdots \le |a_{n-1}| \le |a_n| = 1$*. Then all the zeros of p lie in the vertical strip* $\{z : -\max\{1, \delta_2\} \le Re(z) \le \delta_2\}$ *where* $\delta_1 = [(1-\alpha_1) + \{(1-\alpha_2)^2 + 4M\}^{1/2}]/2$, $\delta_2 = [-(1-\alpha_1) + \{(1-\alpha_2)^2 + 4M\}^{1/2}]/2$, *and* $M = (|a_1| - |a_n|)(\cos\alpha + \sin\alpha) + 2\sin\alpha(\sum_{j=2}^{n} |a_j|) + |a_n|$.

In the same paper, Jain gave a result by putting the monotonicity hypothesis on the real parts of the coefficients. He also presented the corresponding result which put restrictions on the imaginary parts of the zeros of the polynomial p. Also, Jain [44] in 1993 gave a result which restricts the real part of the zeros, but with no condition on the coefficients (and hence not really related to the Eneström–Kakeya theorem). It seems that this type of approach to the restriction of the zeros has been relatively little studied.

The techniques used in proving many of the theorems above can also be used to establish a bound on the moduli of the zeros of an analytic function which has a related monotonicity-type condition on the coefficients of its series representation. For example, Govil and Rahman [30] included the following result in their 1968 paper which was primarily devoted to polynomials:

Theorem 51 *Let* $f(z) = \sum_{j=0}^{\infty} a_j z^j$ *be an analytic function in* $|z| \le 1$*. Suppose* $|\arg a_j - \beta| \le \alpha \le \pi/2$ *for some* α *and* β *for* $j = 0, 1, 2, \ldots$ *and* $|a_0| \ge |a_1| \ge |a_2| \ge \cdots$*. Then the zeros of f lie in* $|z| \ge \{\cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_0|} \sum_{j=1}^{\infty} |a_j|\}^{-1}$.

Also closely related to the results of this survey is the following which is due to Aziz and Mohammad [6] and appeared in 1980.

Theorem 52 *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be an analytic function in $|z| \leq t$. Suppose $0 < a_0 \leq ta_1 \leq t^2 a_2 \leq \cdots$. Then all the zeros of f lie in $|z| \geq t$.*

Related results have been presented by Krishnaiah [48], Aziz and Shah [8], Lin, Huang, Cao, and Gardner [50], Shah and Liman [56], Choo [16], and Gulzar [37].

A natural question is: "Is the Eneström–Kakeya theorem sharp?" In other words, is there a polynomial p satisfying the hypotheses of the the Eneström–Kakeya theorem for which there is a zero of modulus 1 (thus indicating that the given bound cannot be improved)? For $p_n(z) = 1 + z + z^2 + \cdots + z^n$, the zeros of p_n are the $(n+1)$th roots of unity, $\cos\theta + i\sin\theta$ for $\theta = 2k\pi/(n+1)$ for $k = 1, 2, \ldots, n$. Therefore, the Eneström–Kakeya theorem (Theorem 3) is sharp. In fact, this example shows that the alternate version of the Eneström–Kakeya theorem (Theorem 4) is also sharp.

However it is possible to sharpen the Eneström–Kakeya theorem by taking away a part of the unit disk that does not contain the zeros of the polynomial, and this has been done, among others, by Govil and Rahman (see Theorem 5 in [30]), and by Rubinstein (see Corollary 1 in [55]).

In 1912–1913, Hurwitz [42] characterized polynomials for which the Eneström–Kakeya theorem is sharp. In 1979, Anderson, Saff, and Varga [4] gave a proof (and correction) of Hurwitz's result based on matrix methods. In a sense, their result states that a polynomial satisfying the hypotheses of the Eneström–Kakeya theorem has a zero of modulus 1 only if the polynomial has p_n as a factor for some n (this is an oversimplification of their result, but somewhat reflects the importance of this result). An interesting corollary to their main theorem deals with the version of the Eneström–Kakeya theorem as stated in Theorem 4:

Corollary 5 *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real and positive coefficients, then all the zeros of p lie in the annulus $R_1 \leq |z| \leq R_2$ where $R_1 = \min_{0 \leq j \leq n-1} a_j/a_{j+1}$ and $R_2 = \max_{0 \leq j \leq n-1} a_j/a_{j+1}$. If $R_1 < R_2$, then it is not possible for p to simultaneously have zeros on $|z| = R_1$ and on $|z| = R_2$.*

In a related result, Anderson, Saff, and Varga [5] in 1980 introduced a "generalized Eneström–Kakeya functional" and established a result concerning the location of zeros of polynomials and showed that their result is asymptotically sharp.

Appendix

Remark on a Theorem on the Roots of the Equation
$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$ Where
All Coefficients Are Real and Positive
by
G. Eneström, Stockholm, Sweden
Tôhoku Mathematical Journal, **18** (1920), 34–36
A translation of "Remarque sur un théorème relatif aux racines de l'equation
$a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0 = 0$ où tous les coefficientes a sont réels et
positifs" by G. Eneström. Translated by Robert Gardner.

In 1912 M. S. Kakeya demonstrated in a paper in this journal that the absolute value of each root of the equation above [in the title] is between the smallest and largest values of

$$\frac{a_{n-1}}{a_n}, \frac{a_{n-2}}{a_{n-1}}, \ldots, \frac{a_0}{a_1},$$

and therefore for [positive] $a_n > a_{n-1} > \cdots > a_0$, the absolute value of each root is less than 1.

This theorem has already been proposed and demonstrated by me in 1893 in a footnote to a problem on pension funds.[1] This problem leads us to the equation

$$k^{s-1} + a_1 k^{s-2} + \cdots + a_{s-2}k + a_{s-1} = 0 \tag{A}$$

where all of these coefficients are real and positive and for which

$$1 > a_1 \geq a_2 \geq \cdots \geq a_{s-1}.$$

The reference cited in the footnote is written in Swedish and at the request of Mr. Hayashi I now translate verbatim the part about the roots of this equation.

Define α_1 as the smallest of the quantities

$$a_1, \frac{a_2}{a_1}, \frac{a_3}{a_2}, \ldots, \frac{a_{s-1}}{a_{s-2}}$$

and it is then evident from this definition of α_1 that

$$a_{q+1} - \alpha_1 a_q \geq 0 \ \ (q = 0, 1, 2, \ldots, s-2; a_0 = 1).$$

Multiplication of equation (A) by $k - \alpha_1$, results in

$$k^s + (a_1 - \alpha_1)k^{s-1} + (a_2 - \alpha_1 a_1)k^{s-2} + \cdots + (a_{s-1} - \alpha_1 a_{s-2})k - \alpha_1 a_{s-1} = 0 \tag{B}$$

and if we substitute $\rho(\cos\phi + i\sin\phi)$ for k, where ρ is the absolute value of k, then ρ and ϕ must satisfy the equations

$$\rho^s \cos s\phi + (a_1 - \alpha_1)\rho^{s-1}\cos(s-1)\phi + (a_2 - \alpha_1 a_1)\rho^{s-2}\cos(s-2)\phi + \cdots + (a_{s-1} - \alpha_1 a_{s-2})\rho\cos\phi - \alpha_1 a_{s-1} = 0,$$

$$\rho^s \sin s\phi + (a_1 - \alpha_1)\rho^{s-1}\sin(s-1)\phi + (a_2 - \alpha_1 a_1)\rho^{s-2}\sin(s-2)\phi + \cdots + (a_{s-1} - \alpha_1 a_{s-2})\rho\sin\phi = 0. \tag{C}$$

We now show that if $\rho < \alpha_1$, equation (C) can not hold, regardless of the value of ϕ. Indeed, all coefficients $a_1 - \alpha_1, a_2 - \alpha_1 a_1, \ldots, a_{s-1} - \alpha_1 a_{s-2}$ are positive, so the

[1] Härledning af en allmän formel för antalet pensionärer, som vid en godtyeklig tidpunkt förefinnas inom en sluten pensionslcassa; *Öfversigt af Vetenskaps-Akademiens Förhandlingar* (Stockholm), 50, 1893, pp. 405–415. The resulting theorem was stated by me, also in *L'intermédiaire des Mathématiciens* 2, 1895, p. 418, and in *Jahrbuch über die Fortschritte der Mathematik* 25 (1893–1894), p. 360, and also mentions the problem of the theory of pensions to which I alluded in the text.

left side can not be greater than

$$\rho^s + (a_1 - \alpha_1)\rho^{s-1} + (a_2 - \alpha_1 a_1)\rho^{s-2} + \cdots + (a_{s-1} - \alpha_1 a_{s-2})\rho - \alpha_1 a_{s-1}$$

and this expression can be written as

$$\rho^{s-1}(\rho - \alpha_1) + a_1\rho^{s-2}(\rho - \alpha_1) + \cdots + a_{s-2}\rho(\rho - \alpha_1) + a_{s-1}(\rho - \alpha_1),$$

which is negative if $\rho < \alpha_1$. The left side of equation (C) is therefore negative for $\rho < \alpha_1$. It follows that the absolute value of each root of equation (A) is greater than or equal to α_1.

In a similar way we can show that with α_2 as the largest of the quantities

$$a_1, \frac{a_2}{a_1}, \frac{a_3}{a_2}, \ldots, \frac{a_{s-2}}{a_{s-1}},$$

the absolute value of each root of the equation (A) must be less than or equal to α_2.

For this proof, replace k with k^{s-1} [in equation (A)]. Then multiply the new equation by $k - 1/\alpha_2$ and we easily find that the absolute value of k can never be less than $1/\alpha_2$, from which it follows immediately that the value of k can not be greater than α_2. We now have

$$\alpha_1 \le |k_i| \le \alpha_2, \quad (i = 0, 1, 2, \ldots, s-1).$$

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Chapter 9
Starlikeness, Convexity and Close-to-convexity of Harmonic Mappings

Sumit Nagpal and V. Ravichandran

9.1 Introduction

Let $\mathcal{H}$ denote the class of all complex-valued harmonic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0 = f_z(0) - 1$. Let $\mathcal{S}_H$ be the subclass of $\mathcal{H}$ consisting of univalent and sense-preserving functions. Such functions can be written in the form $f = h + \bar{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \tag{9.1}$$

are analytic and $|g'(z)| < |h'(z)|$ in $\mathbb{D}$. Let $\mathcal{S}_H^0 := \{f \in \mathcal{S}_H : f_{\bar{z}}(0) = 0\}$. Observe that $\mathcal{S}_H$ reduces to $\mathcal{S}$, the class of normalized univalent analytic functions, if the co-analytic part of f is zero. In 1984, Clunie and Sheil-Small [5] investigated the class $\mathcal{S}_H$ as well as its geometric subclasses. Let $\mathcal{S}_H^*$, $\mathcal{K}_H$ and $\mathcal{C}_H$ (resp. $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{C}$) be the subclasses of $\mathcal{S}_H$ (resp. $\mathcal{S}$) mapping $\mathbb{D}$ onto starlike, convex and close-to-convex domains, respectively. Denote by $\mathcal{S}_H^{*0}$, $\mathcal{K}_H^0$ and $\mathcal{C}_H^0$, the class consisting of those functions f in $\mathcal{S}_H^*$, $\mathcal{K}_H$ and $\mathcal{C}_H$ respectively, for which $f_{\bar{z}}(0) = 0$.

Recall that convexity and starlikeness are not hereditary properties for univalent harmonic mappings. In [13], the authors introduced the notion of fully starlike mappings of order β and fully convex mappings of order β $(0 \le \beta < 1)$ that are characterized by the conditions

$$\frac{\partial}{\partial\theta} \arg f(re^{i\theta}) > \beta \quad \text{and} \quad \frac{\partial}{\partial\theta}\left(\arg\left\{\frac{\partial}{\partial\theta} f(re^{i\theta})\right\}\right) > \beta, \quad (0 \le \theta < 2\pi, 0 < r < 1)$$

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respectively. For $\beta = 0$ these classes were studied by Chuaqui et al. [3]. Let $\mathcal{FS}^*_H(\beta)$ and $\mathcal{FK}_H(\beta)$ $(0 \leq \beta < 1)$ denote the subclasses of $\mathcal{S}^*_H$ and $\mathcal{K}_H$ respectively consisting of fully starlike functions of order β and fully convex functions of order β. Set $\mathcal{FS}^{*0}_H(\beta) = \mathcal{FS}^*_H(\beta) \cap \mathcal{S}^{*0}_H$ and $\mathcal{FK}^0_H(\beta) = \mathcal{FK}_H(\beta) \cap \mathcal{K}^0_H$.

Clunie and Sheil-Small [5] gave a sufficient condition for a harmonic function to be univalent close-to-convex.

Lemma 1 [5, Lemma 5.15, p. 19] *Suppose that H, G are analytic in $\mathbb{D}$ with $|G'(0)| < |H'(0)|$ and that $H + \epsilon G$ is close-to-convex for each $|\epsilon| = 1$. Then $F = H + \overline{G}$ is harmonic univalent and close-to-convex in $\mathbb{D}$.*

Making use of Lemma 1, Clunie and Sheil-Small [5] proved that if $f = h + \overline{g}$ is sense-preserving in $\mathbb{D}$ and $h + \epsilon g$ is convex for some ϵ $(|\epsilon| \leq 1)$, then f is harmonic univalent and close-to-convex in $\mathbb{D}$. A particular case of this result is the following.

Lemma 2 *Let $f = h + \overline{g} \in \mathcal{H}$ be sense-preserving and $h \in \mathcal{K}$. Then $f \in \mathcal{C}^0_H$.*

The conditions in the hypothesis of Lemma 2 can't be relaxed, that is, if $f = h + \overline{g} \in \mathcal{H}$ is sense-preserving and h is non-convex, then f need not be even univalent. Similarly the conclusion of Lemma 2 can't be strengthened, that is, if $f = h + \overline{g} \in \mathcal{H}$ is sense-preserving and $h \in \mathcal{K}$, then f need not map $\mathbb{D}$ onto a starlike or convex domain. These two statements are illustrated by examples in Sect. 2 of the paper. In addition, we will consider the cases under which a sense-preserving harmonic function $f = h + \overline{g} \in \mathcal{H}$ with $h \in \mathcal{K}$ belongs to $\mathcal{FS}^{*0}_H(\beta)$ or $\mathcal{FK}^0_H(\beta)$. The following lemma will be needed in our investigation.

Lemma 3 [15] *Let $f = h + \bar{g} \in \mathcal{H}$ where h and g are given by* (9.1) *with $b_1 = g'(0) = 0$. Suppose that $\lambda \in (0, 1]$.*

(i) If $\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \leq \lambda$ then f is fully starlike of order $2(1-\lambda)/(2+\lambda)$.
(ii) If $\sum_{n=2}^{\infty} n^2(|a_n| + |b_n|) \leq \lambda$ then f is fully starlike of order $2(2-\lambda)/(4+\lambda)$. Moreover, f is fully convex of order $2(1-\lambda)/(2+\lambda)$.

All these results are sharp.

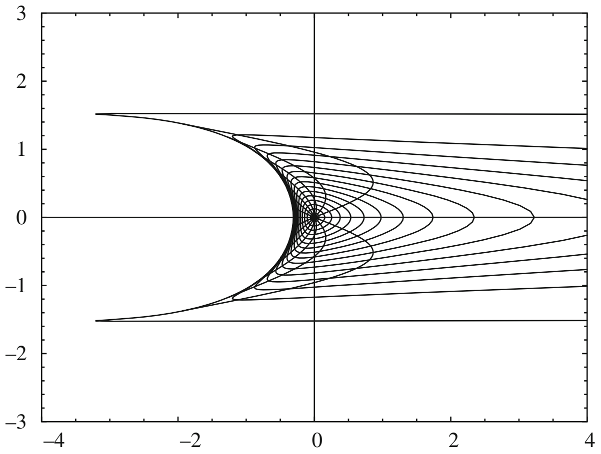
For $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, let $\mathcal{M}(\alpha)$ denote the set of all harmonic functions $f = h + \bar{g} \in \mathcal{H}$ that satisfy

$$g'(z) = \alpha z h'(z) \quad \text{and} \quad Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2} \text{for all } z \in \mathbb{D}.$$

In [12], Mocanu conjectured that the functions in the class $\mathcal{M}(1)$ are univalent in $\mathbb{D}$. In [4], Bshouty and Lyzzaik proved this conjecture by establishing that $\mathcal{M}(1) \subset \mathcal{C}^0_H$. Recently, this result is extended in [2] by proving that $\mathcal{M}(\alpha) \subset \mathcal{C}^0_H$ for $|\alpha| = 1$. In fact, $\mathcal{M}(\alpha) \subset \mathcal{C}^0_H$ for each $|\alpha| \leq 1$. The coefficient estimates, growth results, area theorem and convolution properties for the class $\mathcal{M}(\alpha)$ $(|\alpha| \leq 1)$ are obtained in the last section of the paper. The bounds for the radius of starlikeness and convexity of the class $\mathcal{M}(\alpha)$ are also determined. The bound for the radius of convexity turns out to be sharp for the class $\mathcal{M}(1)$ with the extremal function

$$F(z) := \text{Re}\frac{z}{(1-z)^2} + i\text{Im}\frac{z}{1-z} \in \mathcal{M}(1).$$

Fig. 9.1 Graph of the function $f(z) = 2\mathrm{Re}\frac{z}{1-z} + \log(1-\overline{z})$.



The radius of starlikeness of F is also determined. The convolution properties of F with certain right-half plane mappings are also discussed.

9.2 Sufficient Conditions for Starlikeness and Convexity

Neither the conditions in the hypothesis of Lemma 2 can be relaxed nor the conclusion of Lemma 2 can be strengthened. The first two examples of this section verify the truth of this statement.

Example 1 Let $h(z) = z - z^2/2 \in \mathcal{S}^*$ and $g(z) = z^2/2 - z^3/3$ so that $h'(z) = zg'(z)$. Then h is non-convex and $f = h + \overline{g}$ is sense-preserving in $\mathbb{D}$. But f is not even univalent in $\mathbb{D}$ since $f(z_0) = f(\overline{z}_0) = 3/4$ where $z_0 = (3 + \sqrt{3}i)/4 \in \mathbb{D}$.

Example 2 Let $h(z) = z/(1-z) \in \mathcal{K}$ and $g'(z) = zh'(z)$. Then the function

$$f(z) = h(z) + \overline{g(z)} = \frac{z}{1-z} + \overline{\frac{z}{1-z} + \log(1-z)}$$

belongs to $\mathcal{C}_H^0$ by Lemma 2. The image of the unit disk under f is shown in Fig. 9.1 as plots of the images of equally spaced radial segments and concentric circles. Clearly $f(\mathbb{D})$ is a non-starlike domain.

Now we will consider certain cases under which the conclusion of Lemma 2 can be extended to fully starlike mappings of order β and fully convex mappings of order β $(0 \leq \beta < 1)$.

Theorem 1 *Let $f = h + \overline{g} \in \mathcal{H}$ where h and g are given by (9.1), and let $\alpha \in \mathbb{C}$. Further, assume that*

$$g'(z) = \alpha z h'(z) \quad (z \in \mathbb{D}) \quad \textit{and} \quad \sum_{n=2}^{\infty} n^2 |a_n| \leq 1.$$

If $|\alpha| \leq 1$ *then* f *is univalent close-to-convex. If* $|\alpha| \leq 1/3$ *then* f *is fully starlike of order* $2(1-3|\alpha|)/(5+3|\alpha|)$.

Proof The coefficient inequality $\sum_{n=2}^{\infty} n^2|a_n| \leq 1$ implies that $h \in \mathcal{K}$ (see [1]). So $f \in \mathcal{C}_H^0$ by Lemma 2 if $|\alpha| \leq 1$. The relation $g'(z) = \alpha z h'(z)$ gives $b_1 = 0$ and

$$b_n = \alpha \frac{n-1}{n} a_{n-1}, \quad (n \geq 2;\ a_1 = 1).$$

Consider

$$\begin{aligned}\sum_{n=2}^{\infty} n(|a_n| + |b_n|) &= \sum_{n=2}^{\infty} (n|a_n| + (n-1)|\alpha||a_{n-1}|) \\ &= (1+|\alpha|)\sum_{n=2}^{\infty} n|a_n| + |\alpha| \\ &\leq \frac{1}{2}(1+|\alpha|)\sum_{n=2}^{\infty} n^2|a_n| + |\alpha| \\ &\leq \frac{1}{2}(1+|\alpha|) + |\alpha| = \frac{1+3|\alpha|}{2}.\end{aligned}$$

By applying Lemma 3(i), it follows that $f \in \mathcal{FS}_H^{*0}(2(1-3|\alpha|)/(5+3|\alpha|))$ if $|\alpha| \leq 1/3$.

Theorem 2 *Let* $f = h + \overline{g} \in \mathcal{H}$ *where* h *and* g *are given by* (9.1), *and let* $\alpha \in \mathbb{C}$. *Further, assume that*

$$g'(z) = \alpha z h'(z) \quad (z \in \mathbb{D}) \quad \textit{and} \quad \sum_{n=2}^{\infty} n^3|a_n| \leq 1.$$

If $|\alpha| \leq 2/11$ *then* f *is fully starlike of order* $2(6-11|\alpha|)/(18+11|\alpha|)$. *Moreover,* f *is fully convex of order* $2(2-11|\alpha|)/(10+11|\alpha|)$.

Proof To apply Lemma 3(ii), consider the sum

$$\begin{aligned}\sum_{n=2}^{\infty} n^2(|a_n| + |b_n|) &= \sum_{n=2}^{\infty} (n^2|a_n| + n(n-1)|\alpha||a_{n-1}|) \\ &= (1+|\alpha|)\sum_{n=2}^{\infty} n^2|a_n| + 2|\alpha| + |\alpha|\sum_{n=2}^{\infty} n|a_n| \\ &\leq \frac{1}{2}(1+|\alpha|)\sum_{n=2}^{\infty} n^3|a_n| + 2|\alpha| + \frac{1}{4}|\alpha|\sum_{n=2}^{\infty} n^3|a_n| \\ &\leq \frac{1}{2}(1+|\alpha|) + 2|\alpha| + \frac{1}{4}|\alpha| = \frac{2+11|\alpha|}{4}.\end{aligned}$$

Hence $f \in \mathcal{FS}_H^{*0}(2(6-11|\alpha|)/(18+11|\alpha|)) \cap \mathcal{FK}_H^0(2(2-11|\alpha|)/(10+11|\alpha|))$.

Remark 1 If $h \in \mathcal{K}$ then the harmonic function $f = h + \epsilon\overline{h}$ is univalent and fully convex of order 0 for each $|\epsilon| < 1$. Similarly, if $h \in \mathcal{S}^*$ then the harmonic function $f = h + \epsilon\overline{h}$ is fully starlike of order 0 for each $|\epsilon| < 1$.

9.3 Class $\mathcal{M}(\alpha)$ ($|\alpha| \leq 1$)

In this section, we will investigate the properties of functions in the class $\mathcal{M}(\alpha)$.

Theorem 3 *Let $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$. Then we have the following.*

(i) $\mathcal{M}(\alpha) \subset \mathcal{C}_H^0$.
(ii) *(Coefficient estimates) If $f = h + \bar{g} \in \mathcal{M}(\alpha)$ where h and g are given by* (9.1), then $b_1 = g'(0) = 0$,

$$|a_n| \leq \frac{n+1}{2} \quad \text{and} \quad |b_n| \leq \frac{n-1}{2}|\alpha|$$

for $n = 2, 3, \ldots$. Moreover, these bounds are sharp for each α.
(*iii*) *(Growth theorem) The inequality*

$$|f(z)| \leq \frac{|z|}{(1-|z|)^2}\left[1 - \frac{1}{2}(1-|\alpha|)|z|\right], \quad z \in \mathbb{D},$$

holds for every function $f \in \mathcal{M}(\alpha)$. This bound is sharp.
(*iv*) *(Area theorem) The area of the image of each function $f \in \mathcal{M}(\alpha)$ is greater than or equal to $\pi(1-|\alpha|^2/2)$ and this minimum is attained only for the function $g_\alpha(z) = z + \alpha\bar{z}^2/2$.*

Proof Proof of (i) follows by a minor modification of [4, Theorem 1, p. 768]. For the convenience of the reader, we include its details here. Let $f = h + \bar{g} \in \mathcal{M}(\alpha)$. Since $|g'(0)| < |h'(0)|$, it suffices to show that the analytic functions $F_\epsilon = h + \epsilon g$ are close-to-convex in $\mathbb{D}$ for each $|\epsilon| = 1$, in view of Lemma 1. It is easy to verify that

$$\begin{aligned}
\mathrm{Re}\left(1 + \frac{zF_\epsilon''(z)}{F_\epsilon'(z)}\right) &= \mathrm{Re}\frac{\alpha\epsilon z}{1+\alpha\epsilon z} + \mathrm{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) \\
&= \frac{1}{2} - \frac{1}{2}\frac{1-|\alpha\epsilon z|^2}{|1+\alpha\epsilon z|^2} + \mathrm{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) \\
&= \frac{1}{2} - \frac{1}{2}P_\zeta(\theta) + \mathrm{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}P_\zeta(\theta)
\end{aligned}$$

where $z = re^{i\theta}$ $(0 < r < 1)$, $\zeta = -\bar{\alpha}\bar{\epsilon}r$ and $P_\zeta(\theta) = (1-|\zeta|^2)/|e^{i\theta} - \zeta|^2$ $(|\zeta| < 1)$ is the Poisson Kernel. Fix θ_1, θ_2 with $0 < \theta_2 - \theta_1 < 2\pi$. Then

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left(1 + \frac{re^{i\theta}F_\epsilon''(re^{i\theta})}{F_\epsilon'(re^{i\theta})}\right) d\theta \geq -\frac{1}{2}\int_{\theta_1}^{\theta_2} P_\zeta(\theta)\, d\theta = -\pi.$$

By the well-known Kaplan's theorem [11], it follows that each F_ϵ is close-to-convex in $\mathbb{D}$ and hence $f \in \mathcal{C}_H^0$. This proves (i).

For the proof of (ii), note that the function h is close-to-convex of order $1/2$ and hence its coefficients satisfy $|a_n| \leq (n+1)/2$ for $n = 2, 3, \ldots$ (see [9, 16]). Regarding the bound for b_n, note that the relation $g'(z) = \alpha z h'(z)$ gives

$$(n+1)b_{n+1} = n\alpha a_n, \quad n = 1, 2, \ldots$$

so that $|b_n| \leq (n-1)|\alpha|/2$. For sharpness, consider the functions

$$f_\alpha(z) = \frac{1}{2}\left(\frac{z}{1-z} + \frac{z}{(1-z)^2}\right) - \overline{\frac{1}{2}\alpha\left(\frac{z}{1-z} - \frac{z}{(1-z)^2}\right)} \quad z \in \mathbb{D}, \quad |\alpha| \leq 1. \tag{9.2}$$

The functions $f_\alpha \in \mathcal{M}(\alpha)$ for each $|\alpha| \leq 1$ and

$$f_\alpha(z) = z + \sum_{n=2}^{\infty} \frac{n+1}{2} z^n + \overline{\sum_{n=2}^{\infty} \frac{n-1}{2} \alpha z^n},$$

showing that the bounds are best possible. Figure 9.2 depicts the image domain $f_\alpha(\mathbb{D})$ for $\alpha = \pm 1, \pm i$.

Using the estimates for $|a_n|$ and $|b_n|$, it follows that

$$\begin{aligned}
|f(z)| &\leq |z| + \sum_{n=2}^{\infty} |a_n||z|^n + \sum_{n=2}^{\infty} |b_n||z|^n \\
&\leq |z| + \frac{1}{2}\sum_{n=2}^{\infty}(n+1)|z|^n + \frac{1}{2}|\alpha|\sum_{n=2}^{\infty}(n-1)|z|^n \\
&= |z| + \frac{1}{2}(1+|\alpha|)\sum_{n=2}^{\infty} n|z|^n + \frac{1}{2}(1-|\alpha|)\sum_{n=2}^{\infty}|z|^n \\
&= \frac{|z|}{(1-|z|)^2}\left[1 - \frac{1}{2}(1-|\alpha|)|z|\right].
\end{aligned}$$

The bound is sharp with equality holding for the function f_α given by (9.2). This proves (iii).

For the last part of the theorem, suppose that $f = h + \bar{g} \in \mathcal{M}(\alpha)$, where h and g are given by (9.1). Then the area of the image $f(\mathbb{D})$ is

$$\begin{aligned}
A &= \iint_{\mathbb{D}} (|h'(z)|^2 - |g'(z)|^2)\, dx\, dy \\
&= \iint_{\mathbb{D}} |h'(z)|^2\, dx\, dy - |\alpha|^2 \iint_{\mathbb{D}} |zh'(z)|^2\, dx\, dy \\
&= \pi\left(1 + \sum_{n=2}^{\infty} n|a_n|^2\right) - |\alpha|^2\left(\frac{\pi}{2} + \pi\sum_{n=2}^{\infty}\frac{n^2}{n+1}|a_n|^2\right),
\end{aligned}$$

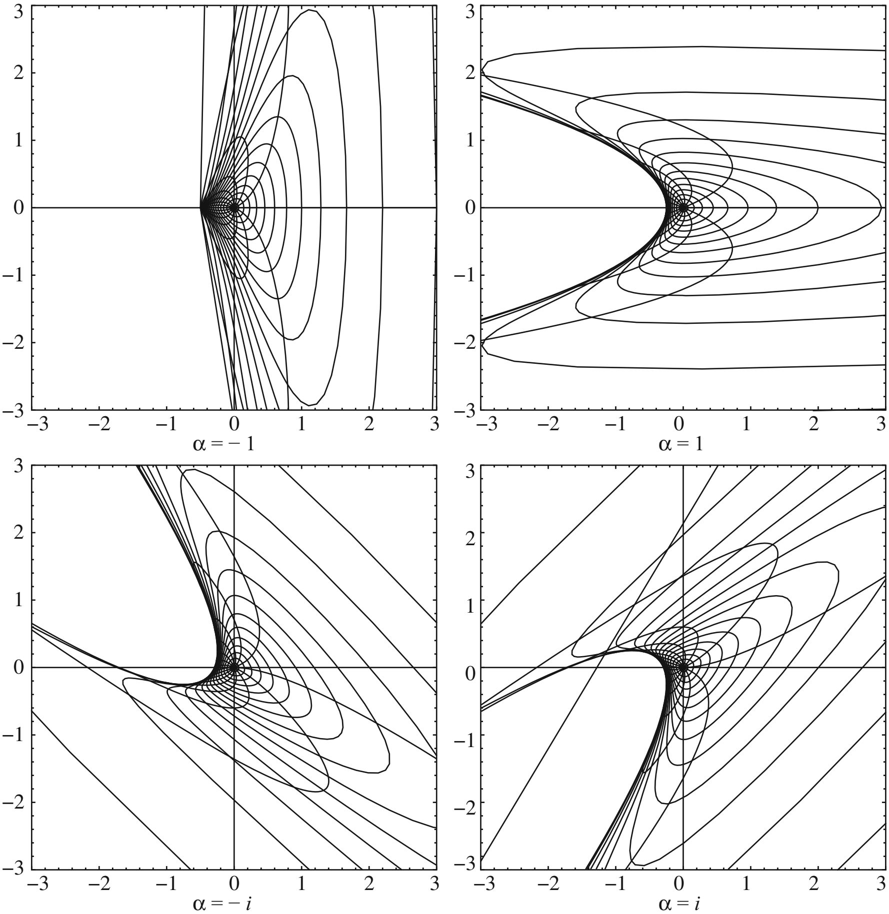


Fig. 9.2 Graph of the function f_α for different values of α

$$= \pi \left(1 - \frac{1}{2}|\alpha|^2\right) + \pi \sum_{n=2}^{\infty} \left(n - \frac{n^2}{n+1}|\alpha|^2\right) |a_n|^2.$$

The last sum is minimized by choosing $a_n = 0$ for $n = 2, 3, \ldots$. This gives $h(z) = z$ so that $g(z) = \alpha z^2/2$. This completes the proof of the theorem.

If $\alpha = 0$, then the family $\mathcal{M}(\alpha)$ reduces to the class of normalized analytic functions f with $f(0) = 0 = f'(0) - 1$ satisfying Re $(1 + zf''(z)/f'(z)) > -1/2$ for all $z \in \mathbb{D}$. Ozaki [17] independently proved that the functions in the class $\mathcal{M}(0)$ are univalent in $\mathbb{D}$.

For analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$, their convolution (or Hadamard product) is defined as $(f * F)(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n$. In the harmonic case, with

$$f = h + \bar{g} = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}, \quad \text{and}$$

$$F = H + \bar{G} = z + \sum_{n=2}^{\infty} A_n z^n + \overline{\sum_{n=1}^{\infty} B_n z^n}.$$

their harmonic convolution is defined as

$$f * F = h * H + \overline{g * G} = z + \sum_{n=2}^{\infty} a_n A_n z^n + \overline{\sum_{n=1}^{\infty} b_n B_n z^n}.$$

Results regarding harmonic convolution can be found in [5–7, 15, 13, 14].

Remark 2 Fix α with $|\alpha| \leq 1$. It is easy to see that the Hadamard product of two functions in $\mathcal{M}(\alpha)$ need not necessarily belong to $\mathcal{M}(\alpha)$. For instance, consider the function f_α given by (9.2). The coefficients of the product $f_\alpha * f_\alpha$ are too large for this product to be in $\mathcal{M}(\alpha)$ in view of Theorem 3(ii).

In [5], Clunie and Sheil-Small showed that if $\varphi \in \mathcal{K}$ and $f \in \mathcal{K}_H$ then the functions $(\beta\overline{\varphi} + \varphi) * f \in \mathcal{C}_H$ $(|\beta| \leq 1)$. The result is even true if $\mathcal{K}_H$ is replaced by $\mathcal{M}(\alpha)$.

Theorem 4 *Let $\varphi \in \mathcal{K}$ and $f \in \mathcal{M}(\alpha)$ $(|\alpha| \leq 1)$. Then the functions $(\beta\overline{\varphi}+\varphi)*f \in \mathcal{C}_H^0$ for $|\beta| \leq 1$.*

Proof Writing $f = h + \bar{g}$ we have

$$(\beta\overline{\varphi} + \varphi) * f = \varphi * h + \overline{\bar{\beta}(\varphi * g)} = H + \overline{G},$$

where $H = \varphi * h$ and $G = \bar{\beta}(\varphi * g)$ are analytic in $\mathbb{D}$ with $|G'(0)| < |H'(0)|$. Setting $F = H + \epsilon G = \varphi * (h + \bar{\beta}\epsilon g)$ where $|\epsilon| = 1$, we note that F is close-to-convex in $\mathbb{D}$ since $h + \bar{\beta}\epsilon g \in \mathcal{C}$, $\varphi \in \mathcal{K}$ and $\mathcal{K} * \mathcal{C} \subset \mathcal{C}$. By Lemma 1, it follows that $H + \overline{G}$ is harmonic close-to-convex, as desired.

Remark 3 The function $f_{-1} \in \mathcal{M}(-1)$ given by (9.2) is the harmonic half-plane mapping

$$L(z) := f_{-1}(z) = \text{Re}\left(\frac{z}{1-z}\right) + i\,\text{Im}\left(\frac{z}{(1-z)^2}\right) \tag{9.3}$$

constructed by shearing the conformal mapping $l(z) = z/(1-z)$ vertically with dilatation $w(z) = -z$ (see Fig. 9.2a). Note that

$$(L * L)(z) = z + \sum_{n=2}^{\infty} \left(\frac{n+1}{2}\right)^2 z^n + \overline{\sum_{n=2}^{\infty} \left(\frac{n-1}{2}\right)^2 z^n}, \quad z \in \mathbb{D}$$

is univalent in $\mathbb{D}$ by [7, Theorem 3]. In fact, the image of the unit disk $\mathbb{D}$ under $L * L$ is $\mathbb{C} \backslash (-\infty, -1/4]$ which shows that $L * L \in \mathcal{S}_H^{*0}$. Since $f_\alpha * f_{\bar{\alpha}} = L * L$ for each

$|\alpha| = 1$, where the functions f_α are given by (9.2), it follows that the Hadamard product $f_\alpha * f_{\bar{\alpha}}$ is univalent and starlike in $\mathbb{D}$ for each $|\alpha| = 1$.

The next theorem determines the bounds for the radius of starlikeness and convexity of the class $\mathcal{M}(\alpha)$.

Theorem 5 *Let $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$.*

(a) Each function in $\mathcal{M}(\alpha)$ maps the disk $|z| < 2 - \sqrt{3}$ onto a convex domain.
(b) Each function in $\mathcal{M}(\alpha)$ maps the disk $|z| < 4\sqrt{2} - 5$ onto a starlike domain.

Proof Let $f = h + \bar{g} \in \mathcal{M}(\alpha)$. Then the analytic functions $F_\lambda = h + \lambda g$ are close-to-convex in $\mathbb{D}$ for each $|\lambda| = 1$ (see the proof of Theorem 3(i)).

Since the radius of convexity in close-to-convex analytic mappings is $2 - \sqrt{3}$, the functions F_λ are convex in $|z| < 2 - \sqrt{3}$. In view of [13, Theorem 2.3, p. 89], it follows that f is fully convex (of order 0) in $|z| < 2 - \sqrt{3}$. This proves (a).

Similarly, since the radius of starlikeness for close-to-convex analytic mappings is $4\sqrt{2} - 5$, it follows that each F_λ is starlike in $|z| < 4\sqrt{2} - 5$. By [13, Theorem 2.7, p. 91], f is fully starlike (of order 0) in $|z| < 4\sqrt{2} - 5 \approx 0.65685$.

Now, we shall show that the bound $2 - \sqrt{3}$ for the radius of convexity is sharp for the class $\mathcal{M}(1)$. To see this, consider the function f_1 given by (9.2), which may be rewritten as

$$F(z) := f_1(z) = \mathrm{Re}\left(\frac{z}{(1-z)^2}\right) + i\,\mathrm{Im}\left(\frac{z}{1-z}\right). \tag{9.4}$$

Its worth to note that the function F may be constructed by shearing the conformal mapping $l(z) = z/(1-z)$ horizontally with dilatation $w(z) = z$. In [10], it has been shown that $F(\mathbb{D}) = \{u+iv : v^2 > -(u+1/4)\}$ (see Fig. 9.2b). In particular, $F \notin \mathcal{S}_H^{*0}$. For instance, $z_0 = -1 - i \in F(\mathbb{D})$ but $z_0/2 \notin F(\mathbb{D})$. In fact, $z_0/2 \in \partial F(\mathbb{D})$. Thus $\mathcal{M}(1) \not\subset \mathcal{S}_H^{*0}$.

The next example determines the radius of convexity of the mapping F by employing a calculation similar to the one carried out in [8, Sect. 3.5].

Example 3 For the purpose of computing the radius of convexity of F, it is enough to study the change of the tangent direction

$$\Psi_r(\theta) = \arg\left\{\frac{\partial}{\partial\theta}F(re^{i\theta})\right\}$$

of the image curve as the point $z = re^{i\theta}$ moves around the circle $|z| = r$. Note that

$$\frac{\partial}{\partial\theta}F(re^{i\theta}) = A(r,\theta) + iB(r,\theta),$$

where

$$|1-z|^6 A(r,\theta) = -r[(1-6r^2+r^4)\sin\theta + r(1+r^2)\sin 2\theta]$$

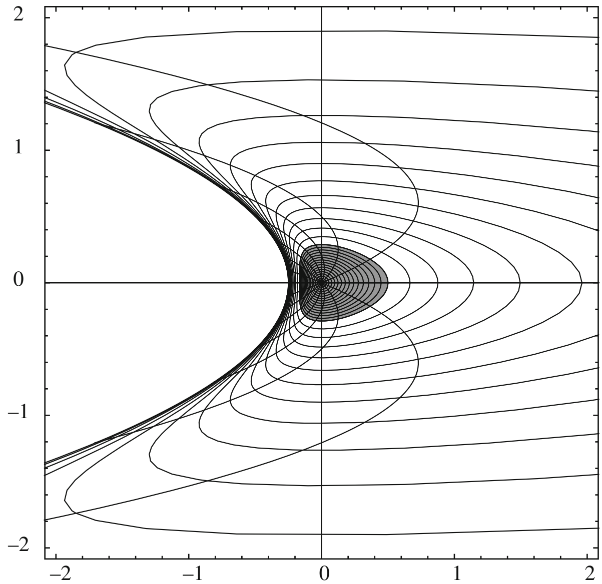


Fig. 9.3 $2-\sqrt{3}$ - the radius of convexity of F

and

$$|1-z|^4 B(r,\theta) = r[(1-r^2)\cos\theta - 2r].$$

The problem now reduces to finding the values of r such that the argument of the tangent vector, or equivalently

$$\tan\Psi_r(\theta) = \frac{B(r,\theta)}{A(r,\theta)} = \frac{(1-2r\cos\theta+r^2)[2r-(1-r^2)\cos\theta]}{(1-6r^2+r^4)\sin\theta + r(1+r^2)\sin 2\theta}$$

is a non-decreasing function of θ for $0<\theta<\pi$. A lengthy calculation leads to an expression for the derivative in the form

$$[(1-6r^2+r^4)+2r(1+r^2)u]^2(1-u^2)\frac{\partial}{\partial\theta}\tan\Psi_r(\theta) = p(r,u),$$

where $u=\cos\theta$ and

$$p(r,u) = 1+4r^2-26r^4+4r^6+r^8-6ur(1+r^2)(1+r^4-6r^2) \\ -12r^2u^2(1+r^2)^2+4ru^3(1+r^2)(1+r^4).$$

Observe that the roots of $p(r,u)=0$ in $(0,1)$ are increasing as a function of $u\in[-1,1]$. Consequently, it follows that $p(r,u)\geq 0$ for $-1\leq u\leq 1$ if and only if

$$p(r,-1) = (1+r)^6(1-4r+r^2)\geq 0.$$

This inequality implies that $r \leq 2 - \sqrt{3}$. This proves that the tangent angle $\Psi_r(\theta)$ increases monotonically with θ if $r \leq 2-\sqrt{3}$ but is not monotonic for $2-\sqrt{3} < r < 1$. Thus, the harmonic mapping F given by (9.4) sends each disk $|z| < r \leq 2-\sqrt{3}$ to a convex domain, but the image is not convex when $2 - \sqrt{3} < r < 1$ (see Fig. 9.3).

Combining Theorem 5 and Example 3, it immediately follows that

Theorem 6 *The radius of convexity of the class* $\mathcal{M}(1)$ *is* $2 - \sqrt{3}$. *Moreover, the bound* $2 - \sqrt{3}$ *is sharp.*

The next example determines the radius of starlikeness of the mapping F given by (9.4).

Example 4 The harmonic mapping F given by (9.2) sends each disk $|z| < r \leq r_0$ to a starlike domain, but the image is not starlike when $r_0 < r < 1$, where r_0 is given by

$$r_0 = \frac{1}{3}\sqrt{\frac{1}{3}(37 - 8\sqrt{10})} \approx 0.658331. \tag{9.5}$$

In this case, one needs to study the change of the direction $\Phi_r(\theta) = \arg F(re^{i\theta})$ of the image curve as the point $z = re^{i\theta}$ moves around the circle $|z| = r$. A direct calculation gives

$$F(re^{i\theta}) = C(r,\theta) + i\,D(r,\theta),$$

where

$$|1-z|^4\, C(r,\theta) = r[(1+r^2)\cos\theta - 2r] \quad \text{and} \quad |1-z|^2 D(r,\theta) = r\sin\theta.$$

For our assertion, it suffices to show that

$$\tan\Phi_r(\theta) = \frac{D(r,\theta)}{C(r,\theta)} = \frac{\sin\theta(1 - 2r\cos\theta + r^2)}{(1+r^2)\cos\theta - 2r}$$

is a nondecreasing function of θ. A straightforward calculation leads to an expression for the derivative in the form

$$[(1+r^2)u - 2r]^2 \frac{\partial}{\partial\theta}\tan\Phi_r(\theta) = q(r,u),$$

where $u = \cos\theta$ and

$$q(r,u) = (1-r^2)^2 - 2ru(1+r^2) + 8r^2u^2 - 2r(1+r^2)u^3.$$

The problem is now to find the values of the parameter r for which the polynomial $q(r,u)$ is non-negative in the whole interval $-1 \leq u \leq 1$. Observe that

$$q(r,-1) = (1+r)^4 > 0 \quad \text{and} \quad q(r,1) = (1-r)^4 > 0.$$

Also, differentiation gives

$$\frac{\partial}{\partial u}q(r,u) = -2r(1+r^2) + 16r^2u - 6r(1+r^2)u^2.$$

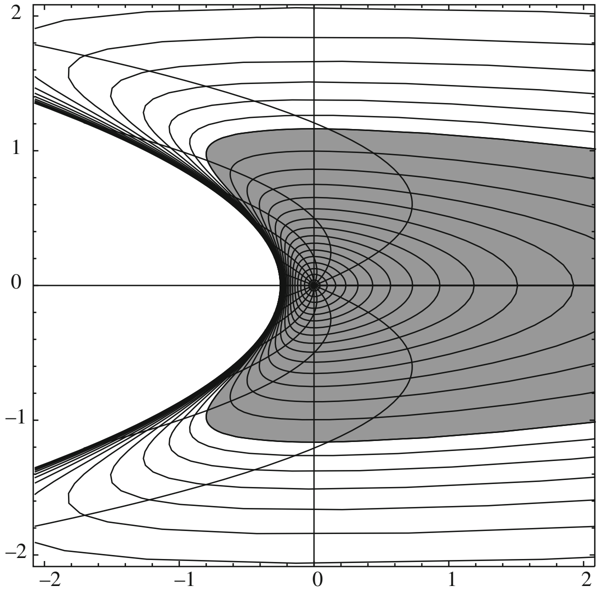


Fig. 9.4 $\frac{1}{3}\sqrt{\frac{1}{3}(37-8\sqrt{10})}$ - the radius of starlikeness of F

This shows that $q(r,u)$ is a decreasing function of u for $0 < r \leq 1/\sqrt{3}$. Since $q(r,1) = (1-r)^4 > 0$, it follows that $q(r,u) \geq 0$ for all $u \in [-1,1]$ and for all $r \in (0, 1/\sqrt{3}]$. For $r \geq 1/\sqrt{3}$, $q(r,u)$ has a local minimum at $u = (4r - \sqrt{-3+10r^2-3r^4})/(3(1+r^2))$ and a local maximum at $u = (4r + \sqrt{-3+10r^2-3r^4})/(3(1+r^2))$. Using these observations, we deduce that $q(r,u) \geq 0$ for $-1 \leq u \leq 1$ if and only if

$$q\left(r, \frac{4r - \sqrt{-3+10r^2-3r^4}}{3(1+r^2)}\right) = \frac{1}{27(1+r^2)^2} h(r) \geq 0,$$

where $h(r) := 27 - 72r^2 + 58r^4 - 72r^6 + 27r^8 + 4r(3 + 10r^2 + 3r^4)\sqrt{-3+10r^2-3r^4}$. The function h is a decreasing function of $r \in [1/\sqrt{3}, 1)$ and $h(r_0) = 0$, where r_0 is given by (9.5). Thus the inequality $h(r) \geq 0$ is satisfied provided $r \leq r_0$. This proves that the angle $\Phi_r(\theta)$ increases monotonically with θ if $r \leq r_0$ and hence the harmonic mapping F sends each disk $|z| < r \leq r_0$ to a starlike domain, but the image is not starlike when $r_0 < r < 1$ (see Fig. 9.4).

Combining Theorem 5 and Example 4, we have

Theorem 7 *If r_S is the radius of starlikeness of $\mathcal{M}(1)$, then*

$$4\sqrt{2} - 5 \leq r_S \leq \frac{1}{3}\sqrt{\frac{1}{3}(37 - 8\sqrt{10})}.$$

By Remark 3, $F * F$ is univalent and starlike in $\mathbb{D}$. However, the product $L * F$ where L is the harmonic half-plane mapping given by 9.3 is not even univalent, although it is sense-preserving in $\mathbb{D}$. In fact, the convolution of F with certain right-half plane mappings is sense-preserving in $\mathbb{D}$. This is seen by the following theorem.

Theorem 8 *Let $f = h + \bar{g} \in \mathcal{K}_H^0$ with $h(z) + g(z) = z/(1-z)$ and $w(z) = g'(z)/h'(z) = e^{i\theta}z^n$, where $\theta \in \mathbb{R}$. If $n = 1, 2$ then $F * f$ is locally univalent in $\mathbb{D}$, F being given by* (9.2).

Proof We need to show that the dilatation $\widetilde{w}$ of $F * f$ satisfies $|\widetilde{w}(z)| < 1$ for all $z \in \mathbb{D}$. It is an easy exercise to derive the expression of dilatation $\widetilde{w}$ in the form

$$\widetilde{w}(z) = z\frac{w^2(z) + [w(z) - \frac{1}{2}w'(z)z] + \frac{1}{2}w'(z)}{1 + [w(z) - \frac{1}{2}w'(z)z] + \frac{1}{2}w'(z)z^2}, \quad z \in \mathbb{D}.$$

The rest of the proof is similar to [7, Theorem 3].

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Chapter 10
On Generalized p-valent Non-Bazilevic Type Functions

Khalida Inayat Noor

10.1 Introduction

Let $\mathcal{A}_p(m)$ be the class of analytic functions f of the form

$$f(z) = z^p + \sum_{n=m}^{\infty} a_{n+p} z^{n+p}, \quad (n, p \in N = \{1, 2, \dots\}) \tag{10.1}$$

and analytic in the open unit disc $E = \{z \in \mathcal{C}, |z| < 1\}$.

Let $f(z)$ and $F(z)$ be analytic in E, then we say that the function $f(z)$ is subordinate to $F(z)$ in E if there exists an analytic function $w(z)$ in E such that $|w(z)| \leq |z|$ and $f(z) = F(w(z))$. In this case, we write $f \prec F$ or $f(z) \prec F(z)$. If $F(z)$ is univalent in E, then the subordination is equivalent to $f(0) = F(0)$ and $f(E) \subset F(E)$, see [6, 9].

Let $P(m)$ be the class of functions h of the form

$$h(z) = 1 + c_m z^m + c_{m+1} z^{m+1} + \dots, \tag{10.2}$$

which are analytic in E and satisfy $Reh(z) > 0$ for $z \in E$.

A function $f \in \mathcal{A}_p(m)$ is said to be p-valently starlike of order β if and only if there exists $h \in P(m)$ such that

$$\frac{zf'(z)}{f(z)} = (p - \beta)h(z) + \beta,$$

for some $\beta(0 \leq \beta < p)$, and for all $z \in E$.

We denote by $S^*(p, m, \beta)$ the subclass of $\mathcal{A}_p(m)$ consisting of functions of p-valently starlike of order β. For $\beta = 0$, we have $S^*(p, m, 0) = S^*(p, m)$.

Let $h(z)$ be analytic in E with $h(\text{o}) = 1$. Then $h \in P[m; A, B]$ if and only if

$$h(z) \prec \frac{1 + Az^m}{1 + Bz^m}, \quad z \in E, \quad -1 \leq B < A \leq 1.$$

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It can easily be seen that

$$P[1; A, B] \subset P\left(\frac{1-A}{1-B}\right) \subset P(1) = P.$$

We have the following.

Definition 1 Let $P_k[m; A, B]$ denote the class of functions $h(z)$ that are analytic in E with $h(0) = 1$ and are represented by

$$h(z0 = \left(\frac{k}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z), \tag{10.3}$$

where $h_1, h_2 \in P[m; A, B], \quad -1 \leq B < A \leq 1$ and $k \geq 2$.

We note that

$$P_k[m; A, B] \subset P_k[1; 1, -1] = P_k,$$

where P_k is the class introduced and studied in [8].

We shall assume throughout, unless stated otherwise, that $-1 \leq B < A \leq 1, \quad k \geq 2, \quad p, m \in N = \{1, 2, \ldots\}$ and $z \in E$.

We now define the class $\mathcal{N}_p[k, \mu, \alpha; A, B]$ of analytic functions as follows.

Definition 2 Let $\alpha \in (0, 1), \mu$ complex, and $f \in \mathcal{A}_p(m)$. Then $f \in \mathcal{N}_p[k, \mu, \alpha; A, B]$, if and only if

$$\left\{(1+\mu)\left(\frac{z^p}{f(z)}\right)^{\alpha} - \mu \frac{zf'(z)}{pf(z)}\left(\frac{z^p}{f(z)}\right)^{\alpha}\right\} \in P_k[m; A, B].$$

As a special case, with $B = \mu = -1, p = m = A = 1, k = 2$, we have the class of non-Bazilevic functions introduced and studied in [7]. See also [1, 5, 11] for the recent developments. For $k = 2$, we shall denote the class $\mathcal{N}_p[3, \mu, \alpha; A, B]$ as $\mathcal{N}_p(\mu, \alpha; A, B)$.

10.2 Preliminary Results

To establish our main results, we shall need the following Lemmas.

Lemma 1 *Let $h(z)$ be analytic and convex univalent function in E with $h(0) = 1$. Also, let the function $\phi(z)$, given by* (10.2)*, be analytic in E. If*

$$\left\{\phi(z) + \frac{z\phi'(z)}{\delta}\right\} \prec h(z), \quad Re(\delta) \geq 0 \quad (\delta \neq 0), \tag{10.4}$$

then

$$\phi(z) \prec \Psi_0(z) = \frac{\delta}{m} z^{\frac{-\delta}{m}} \int_0^z t^{\frac{\delta}{m}-1} h(t)dt, \tag{10.5}$$

and $\Psi_0(z)$ is the best dominant of (10.4).

Lemma 2 [10] *Let* $f(z) = \sum_{n=1}^{\infty} a_n z^n$ *be analytic in* E, $g(z) = \sum_{n=1}^{\infty} b_n z^n$ *be analytic and convex in* E. *If* $f(z) \prec g(z)$, *then* $|a_n| \leq |b_1|$ *for* $n = 1, 2, \ldots$.

This result can easily be extended to p-valent functions.

Lemma 3 [3] *Let* $-1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$, *then*

$$\frac{1 + A_2 z^m}{1 + B_2 z^m} \prec \frac{1 + A_1 z^m}{1 + B_1 z^m}.$$

Lemma 4 ([12], Chap. 14) *For real or complex numbers* $a, b, c (c \notin \mathcal{Z}_0^-)$, *we have*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} {}_2F_1(a, b; c; z) \quad (Re(c) > Re(b) > 0), \tag{10.6}$$

$${}_2F_1(a, b; c; z) = (1-z)^{-a} {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right) \tag{10.7}$$

$$(b+1) {}_2F_1(1, b; b+1; z) = (b+1) + bz {}_2F_1(1, b+1; b+2; z), \tag{10.8}$$

where ${}_2F_1$ *represents a hypergeometric function.*

10.3 Main Results

Theorem 1 *Let* $f \in \mathcal{N}_p[k, \mu, \alpha; A, B]$, $\quad Re\mu > 0$. *Then*

$$\left(\frac{z^p}{f(z)}\right)^{\alpha} \in P_k[m; A, B] \quad in \quad E.$$

Proof Let

$$\left(\frac{z^p}{f(z)}\right)^{\alpha} = h(z), \tag{10.9}$$

where $h(z)$ is given by (10.2) and is represented by (10.3). From (10.9), we obtain

$$(1+\mu)\left(\frac{z^p}{f(z)}\right)^{\alpha} - \mu \frac{zf'(z)}{pf(z)} \left(\frac{z^p}{f(z)}\right)^{\alpha} = h(z) + \frac{\mu m}{\alpha p} zh'(z). \tag{10.10}$$

Since $f \in \mathcal{N}_p[k, \mu, \alpha; A, B]$, it follows from (10.3) and (10.10) that, for $i = 1, 2$,

$$\left\{ h_i(z) + \frac{zh_i'(z)}{\left(\frac{\alpha p}{\mu m}\right)} \right\} \prec \frac{1 + Az^m}{1 + Bz^m}.$$

Using Lemma 1, we have

$$h_i(z) \prec q_i(z) = \frac{p\alpha}{m\mu} z^{\frac{-p\alpha}{m\mu}} \int_0^z t^{\frac{p\alpha}{m\mu}-1} \left(\frac{1+At^m}{1+Bt^m}\right) dt$$
$$= \frac{A}{B} + \left(1 - \frac{A}{B}\right) \left(1 + Bz^m\right)^{-1} {}_2F_1\left(1, 1; \frac{\alpha p}{\mu m} + 1; \frac{Bz^m}{1+Bz^m}\right), \quad (B \neq 0)$$
$$= 1 + \frac{p\alpha}{p\alpha + m\mu} Az \quad (B = 0), \tag{10.11}$$

where we have made a change of variables and then used Lemma 4 with $a = 1$, $b = \frac{p\alpha}{m\mu}$ and $c = b + 1$.

We shall show that $q_i(z)$ is the best dominant of (10.11), and also

$$Re\{q_i(z)\} > \rho, \quad z \in E, \tag{10.12}$$

where

$$\rho = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right) (1 - B)_2^{-1} F_1\left(1, 1; \frac{\alpha p}{\mu m} + 1; \frac{B}{B-1}\right), & (B \neq 0) \\ \\ 1 - \frac{\alpha p}{\alpha p + \mu m} A, & (B = 0) \end{cases}$$

The estimate in (10.12) is the best possible. To prove (10.12), it is sufficient to show that

$$\inf_{|z|<1} \{Re\{q_i(z)\}\} = q_i(-1).$$

Now, for $|z| \leq r < 1$,

$$Re\left\{\frac{1+Az^m}{1+Bz^m}\right\} \geq \frac{1-Ar^m}{1-Br^m}.$$

We take

$$Q_i(z, s) = \frac{1+Asz^m}{1+Bsz^m},$$

and

$$d(\nu(s)) = \frac{\alpha p}{\mu m} s^{\frac{\alpha p}{\mu m}-1} ds, \quad 0 \leq s < 1,$$

which is a positive measure on the closed interval. Then, we have

$$q_i(z) = \int_0^1 Q_i(z, s) d\nu(s),$$

so that

$$Re\{q_i(z)\} \geq \int_0^1 \frac{1-Asr^m}{1-Bsr^m} d\nu(s) = q_i(-r^m), \quad |z| \leq r^m < 1.$$

Letting $r \longrightarrow 1^-$ in the above inequality, we obtain the assertion (10.12).

Since $q_i(z)$ is the best dominant of (10.11), the estimate (10.12) is best possible. Using (10.11) and (10.12) in (10.10) together with (10.3), we obtain the required result. □

For $k = 2, p = 1$, and $m = 1$, this result reduces to one proved in [10]. By choosing different values of the parameters, we can derive several interesting results from Theorem 1. We obtain two of these special cases as follows.

Corollary 1 *Let $f \in \mathcal{N}_1(\mu, \alpha; 1, -1)$. Then, from Theorem 1, it follows that*

$$f \in \mathcal{N}_1\left(0, \alpha, 1 - \frac{2\beta}{p}, -1\right), \quad z \in E,$$

where

$$\beta = {}_2F_1\left(1, 1; \frac{\alpha + \mu}{\mu}, \frac{1}{2}\right) - 1.$$

Corollary 2 *When $p = m = 1, \quad A = 1, \quad B = -1$, and $\mu = \alpha \in (0, 1)$, $f \in \mathcal{N}_1$ $[k, \alpha, \alpha; 1, -1]$. That is*

$$\left\{(1 + \alpha)\left(\frac{z}{f(z)}\right)^{\alpha} - \alpha\frac{zf'(z)}{pf(z)}\left(\frac{z}{f(z)}\right)^{\alpha}\right\} \in P_k, \quad z \in E,$$

implies

$$\left(\frac{z}{f(z)}\right)^{\alpha} \in P_k\left[1 - \frac{2\delta_1}{p}, -1\right],$$

where δ_1 is given by

$$\delta_1 = {}_2F_1\left(1, 1; 2, \frac{1}{2}\right) - 1 \approx 2\ln 2 - 1.$$

Theorem 2 *Let $\alpha \in (0, 1), \quad \mu_2 \geq \mu_1 \geq 0, \quad -1 \leq B_1 \leq B_2 < A_2 \leq A_1 \leq 1$. Then*

$$\mathcal{N}_p\,[k, \mu_2, \alpha; A_2, B_2] \subset \mathcal{N}_p[k, \mu_1, \alpha; A_1, B_1].$$

Proof Let $f \in \mathcal{N}_p[k, \mu_2, \alpha; A_2, B_2]$. Then $f \in \mathcal{A}_p(m)$ and

$$\left\{(1 + \mu_2)\left(\frac{z^p}{f(z)}\right)^{\alpha} - \mu_2\frac{zf'(z)}{pf(z)}\left(\frac{z^p}{f(z)}\right)^{\alpha}\right\} \in P_k[m; A_2, B_2].$$

The case $\mu_2 = \mu_1 \geq 0$ follows trivially from Lemma 1. We suppose that $\mu_2 > \mu_1 \geq 0$. From Lemma 3, we note that

$$P[m; A_2, B_2] \subset P[m; A_1, B_1]$$

and, by using Definition 1, it can easily be derived that

$$P_k[m; A_2, B_2] \subset P_k[m; A_1, B_1].$$

Therefore, we have

$$\left\{(1+\mu_2)\left(\frac{z^p}{f(z)}\right)^{\alpha} - \mu_1 \frac{zf'(z)}{pf(z)}\left(\frac{z^p}{f(z)}\right)^{\alpha}\right\} \in P_k[m; A_1, B_1]. \tag{10.13}$$

Now

$$\begin{aligned}
&(1+\mu_1)\left(\frac{z^p}{f(z)}\right)^{\alpha} - \mu_1 \frac{zf'(z)}{pf(z)}\left(\frac{z^p}{f(z)}\right)^{\alpha} \\
&= \left(1-\frac{\mu_1}{\mu_2}\right)\left(\frac{z^p}{f(z)}\right)^{\alpha} + \frac{\mu_1}{\mu_2}\left\{(1+\mu_2)\left(\frac{z^p}{f(z)}\right)^{\alpha} - \mu_2 \frac{zf'(z)}{pf(z)}\left(\frac{z^p}{f(z)}\right)^{\alpha}\right\} \\
&= \left(1-\frac{\mu_1}{\mu_2}\right) H_1(z) + \frac{\mu_1}{\mu_2} H_2(z),
\end{aligned}$$

where $H_1, H_2 \in P_k[m; A_1, B_1]$, and since $P_k[m; A_1, B_1]$ is a convex set, see [2], it follows that $H \in P_k[m; A_1, B_1]$ in E. This proves the result. □

We shall now deal with a converse case of Theorem 1 as follows.

Theorem 3 *Let* $f \in \mathcal{N}_p[k, 0, \alpha; 1 - \frac{2\beta}{p}, -1]$. *Then* $f \in \mathcal{N}_p[k, \mu, \alpha; 1 - \frac{2\beta}{p}, -1]$ *for* $|z| < r_0$, *where*

$$r_0 = \left[\frac{\alpha p}{\mu m + \sqrt{\alpha^2 p^2 + \mu^2 m^2}}\right]^{\frac{1}{m}}. \tag{10.14}$$

This result is best possible.

Proof Let

$$\begin{aligned}
\left(\frac{z^p}{f(z)}\right)^{\alpha} &= (p-\beta)h(z) + \beta \\
&= \left(\frac{k}{4}+\frac{1}{2}\right)\{(p-\beta)h_1(z)+\beta\} - \left(\frac{k}{4}-\frac{1}{2}\right)\{(p-\beta)h_2(z)+\beta\}
\end{aligned} \tag{10.15}$$

Then $h_1, h_2 \in P(m)$ in E and $0 \le \beta < p$. Proceeding as in Theorem 1, we have

$$\begin{aligned}
&\frac{1}{p-\beta}\left[(1+\mu)\left(\frac{z^p}{f(z)}\right)^{\alpha} - \mu \frac{zf'(z)}{pf(z)}\left(\frac{z^p}{f(z)}\right)^{\alpha} - \beta\right] \\
&= h(z) + \frac{\mu m}{\alpha p} zh'(z) \\
&= \left(\frac{k}{4}+\frac{1}{2}\right)\left\{h_1(z) + \frac{\mu m}{\alpha p} zh_1'(z)\right\} - \left(\frac{k}{4}-\frac{1}{2}\right)\left\{h_2(z) + \frac{\mu m}{\alpha p} zh_2'(z)\right\}.
\end{aligned} \tag{10.16}$$

Now, by using the estimate [4],

$$\frac{|zh'(z)|}{Re\{h_i'(z)\}} \le \frac{2mr}{1-r^{2m}}$$

in (10.16), we have

$$\begin{aligned} Re\left\{h_i(z) + \frac{\mu m}{\alpha p} z h_i'(z)\right\} &\ge Re h_i(z)\left\{1 - \frac{2\mu m}{\alpha p}\frac{r^m}{1-r^{2m}}\right\} \\ &= Re h_i(z)\left\{\frac{\alpha p - 2\mu m r^m - \alpha p r^{2m}}{\alpha p(1-r^{2m})}\right\}. \end{aligned} \tag{10.17}$$

The right hand side of (10.17) is positive for $r \le r_0$, where r_0 is given by (10.14).

To show that the bound r_0 is best possible, we consider

$$h_1(z) = \frac{1+z^m}{1-z^m}, \quad h_2(z) = \frac{1-z^m}{1+z^m}$$

in (10.15) and this completes the proof. □

For $\mu m = \alpha p$, we have the bound for radius $r_0 = \left(\frac{1}{1+\sqrt{2}}\right)^{\frac{1}{m}}$.

Let $f \in \mathcal{A}_p(m)$ and define

$$\begin{aligned} I_\gamma(f(z)) = F^\gamma(z) &= \frac{\delta + p\gamma}{z^\delta}\int_0^z t^{\delta-1} f^\gamma(t)dt \quad (\delta > -p) \\ &= \left(z^p + \sum_{n=1}^{\infty}\frac{\delta+p\gamma}{\delta+\gamma p+n} z^{p+n}\right) \star f^\gamma(z) \\ &= z^p {}_2F_1(1, \delta+p\gamma; \delta+p\gamma+1; z) \star f^\gamma(z), \quad (z \in E) \end{aligned} \tag{10.18}$$

where $\star$ denotes convolution (Hadamard product). The operator $I_\gamma(f)$ is a generalized form of the well-known Bernardi integral operator.

10.4 The Class $\mathcal{N}_p(\mu, \alpha; A, B)$

Theorem 4 *Let $f \in \mathcal{A}_p(m)$ and be given by* (10.1). *Let F, defined by* (10.18), *belong to $\mathcal{N}_p(\mu, \alpha; 1, -1)$ for $z \in E$. If $\left|\frac{f(z)}{F(z)} - 1\right| < 1$ in E, then $f \in S^\star(p, m)$ for $|z| < R_0$, where*

$$R_0 = \left\{\frac{2\alpha p}{m(1+2\alpha p) + \sqrt{m^2(1+2\alpha p)^2 + 4\alpha p(\alpha p + m)}}\right\}^{\frac{1}{m}}. \tag{10.19}$$

Proof Since $F \in \mathcal{N}_p(\mu, \alpha; 1, -1)$, it follows from Theorem 1 that

$$\left(\frac{z^p}{F(z)}\right)^\alpha \in P(m) \quad \text{in} \quad E.$$

Now we set

$$h(z) = \frac{f(z)}{F(z)} - 1 = c_m z^m + c_{m+1} z^{m+1} + \dots , \tag{10.20}$$

and we can write

$$h(z) = z^m \Psi(z), \tag{10.21}$$

where Ψ is a Schwartz function with $|\Psi(z)| \leq 1$ in E.

In (10.20), we use (10.21) and have

$$1 + z^m \Psi(z) = \frac{f(z)}{F(z)}. \tag{10.22}$$

Differentiating (10.22) logarithmically, we obtain

$$\frac{zf'(z)}{f(z)} = \frac{zF'(z)}{F(z)} + \frac{z^m\{m\Psi(z) + z\Psi'(z)\}}{p[1 + z^m \Psi(z)]}. \tag{10.23}$$

We write

$$\left(\frac{z^p}{F(z)}\right)^\alpha = q(z), \quad q \in P(m) \quad \text{in} \quad E.$$

Therefore, we have

$$\alpha p \left(1 - \frac{zF'(z)}{F(z)}\right) = \frac{zq'(z)}{q(z)}. \tag{10.24}$$

Using (10.23), (10.24) and the known [4] estimates

$$\left|\frac{zq'(z)}{q(z)}\right| \leq \frac{2mr^m}{1 - r^{2m}}$$

$$\left|\frac{m\Psi(z) + z\Psi'(z)}{1 + z^m \Psi(z)}\right| \leq \frac{m}{1 - r^m}, \quad |z| = r < 1,$$

we have

$$\begin{aligned} Re\{\frac{zf'(z)}{f(z)}\| &\geq \alpha p - \alpha p \left|\frac{zq'(z)}{q(z)}\right| - \left|\frac{z^m\{m\Psi(z) + z\Psi'(z)\}}{1 + z^m\Psi(z)}\right| \\ &\geq \alpha p - \frac{2\alpha p m r^m}{1 - r^{2m}} - \frac{mr^m}{1 - r^m} \\ &= \frac{\alpha p - m(1 + 2\alpha p) r^m - (\alpha p + m) r^{2m}}{1 - r^{2m}}. \end{aligned} \tag{10.25}$$

The right hand side of (10.25) is positive for $|z| < R_0^m$, where R_0 is given by (10.19). This completes the proof. □

Theorem 5 *Let $f \in \mathcal{N}_p(\mu, \alpha; A, B)$. Then*

$$\left(\frac{\alpha p}{\mu m}\int_0^1 \frac{1-As}{1-Bs} s^{\frac{\alpha p}{\mu m}-1} ds\right) < Re\left(\frac{z^p}{f(z)}\right)^{\alpha} < \left(\frac{\alpha p}{\mu m}\int_0^1 \frac{1+As}{1+Bs} s^{\frac{\alpha p}{\mu m}-1} ds\right), \tag{10.26}$$

and inequality (10.26) *is sharp with the extremal function $f_\star(z) \in \mathcal{N}_p(\mu, \alpha; A, B)$, defined by*

$$f_\star(z) = z^p\left(\frac{\alpha p}{\mu m}\int_0^1 \frac{1+Az^m s}{1+Bz^m s} s^{\frac{\alpha p}{\mu m}-1} ds\right)^{\frac{-1}{\alpha}}. \tag{10.27}$$

Proof From Theorem 1 (with $k = 2$), it follows that

$$\left(\frac{z^p}{f(z)}\right)^{\alpha} \prec \frac{\alpha p}{\mu m}\int_0^1 \frac{1+Az^m s}{1+Bz^m s} s^{\frac{\alpha p}{\mu m}-1} ds.$$

Therefore, by subordination, it follows that

$$\begin{aligned} Re\left\{\left(\frac{z^p}{f(z)}\right)^{\alpha}\right\} &< \sup_{z\in E} Re\left\{\frac{\alpha p}{\mu m}\int_0^1 \frac{1+Az^m s}{1+Bz^m s} s^{\frac{\alpha p}{\mu m}-1} ds\right\} \\ &\leq \frac{\alpha p}{\mu m}\left(\int_0^1 \sup_{z\in E} Re\left\{\frac{1+Az^m s}{1+Bz^m s}\right\} s^{\frac{\alpha p}{\mu m}-1} ds\right) \\ &< \frac{\alpha p}{\mu m}\int_0^1 \frac{1+As}{1+Bs} s^{\frac{\alpha p}{\mu m}-1} ds. \end{aligned}$$

Similarly

$$\begin{aligned} Re\left\{\left(\frac{z^p}{f(z)}\right)^{\alpha}\right\} &> \inf_{z\in E} Re\left\{\frac{\alpha p}{\mu m}\int_0^1 \frac{1+Az^m s}{1+Bz^m s} s^{\frac{\alpha p}{\mu m}-1} ds\right\} \\ &\geq \frac{\alpha p}{\mu m}\left(\int_0^1 \inf_{z\in E} Re\left\{\frac{1+Az^m s}{1+Bz^m s}\right\} s^{\frac{\alpha p}{\mu m}-1} ds\right) \\ &> \frac{\alpha p}{\mu m}\int_0^1 \frac{1-As}{1-Bs} s^{\frac{\alpha p}{\mu m}-1} ds. \end{aligned}$$

□

Corollary 3 *Since, for $h \in P(m)$,*

$$(Reh(z))^{\frac{1}{2}} \leq Reh^{\frac{1}{2}}(z) \leq |h(z)|^{\frac{1}{2}},$$

it follows, from Theorem 5, that

$$\left(\frac{\alpha p}{\mu m}\int_0^1 \frac{1-As}{1-Bs} s^{\frac{\alpha p}{\mu m}-1} ds\right)^{\frac{1}{2}} < Re\left\{\left(\frac{z^p}{f(z)}\right)^{\frac{\alpha}{2}}\right\} < \left(\frac{\alpha p}{\mu m}\int_0^1 \frac{1+As}{1+Bs} s^{\frac{\alpha p}{\mu m}-1} ds\right)^{\frac{1}{2}}.$$

Equality holds for $f_\star(z)$ defined by (10.27).

Corollary 4 *Let* $f \in \mathcal{N}_p(\mu, \alpha; A, B,)$.

(i) *If* $\mu = 0$, *for* $|z| = r^m < 1$, *we have*

$$r^p \left(\frac{1 + Br^m}{1 + Ar^m}\right)^{\frac{1}{\alpha}} \leq |f(z)\| \leq r^p \left(\frac{1 - Br^m}{1 - Ar^m}\right)^{\frac{1}{\alpha}},$$

and equality holds for

$$f_1(z) = z^p \left[\frac{1 + Bz^m}{1 + Az^m}\right]^{\frac{1}{\alpha}}. \tag{10.28}$$

(ii) *For* $\mu \neq 0$, $\quad |z| = r < 1$, *we have*

$$r^p \left(\frac{\alpha p}{\mu m} \int_0^1 \frac{1 + Ar^m s}{1 + Br^m s} s^{\frac{\alpha p}{\mu m} - 1} ds\right)^{\frac{-1}{\alpha}}$$
$$\leq |f(z)| \leq r^p \left(\frac{\alpha p}{\mu m} \int_0^1 \frac{1 - Ar^m s}{1 - Br^m s} s^{\frac{\alpha p}{\mu m} - 1} ds\right)^{\frac{-1}{\alpha}}.$$

Equality is attained by the function $f_\star(z)$ *defined by* (10.27).

Theorem 6 *Let* $\alpha \in (0, 1)$, μ *complex,* $-1 \leq B \leq 1$, $A \neq B$, A *real,*

$$f(z) = z^p + \sum_{n=p+m}^{\infty} a_n z^n \in \mathcal{N}_p(\mu, \alpha; A, B).$$

Then

$$\left|a_{p+m}\right| \leq \frac{|A - B|}{|\mu m + \alpha p|}. \tag{10.29}$$

The inequality (10.29) *is sharp with the extremal function defined by the function*

$$f_0(z) = z^p \left(\frac{\alpha p}{\mu m} \int_0^1 \frac{1 + Az^m u}{1 + Bz^m u} u^{\frac{\alpha p}{\mu m} - 1} du\right)^{\frac{-1}{\alpha}}.$$

Proof We can write

$$(1 + \mu)\left(\frac{z^p}{f(z)}\right)^{\alpha} - \mu \frac{zf'(z)}{pf(z)} \left(\frac{z^p}{f(z)}\right)^{\alpha} = 1 + (-\mu m - \alpha p)a_{m+p} z^{m+p} + \dots. \tag{10.30}$$

Now

$$h(z) = \frac{1 + Az^m}{1 + Bz^m}, \quad -1 \leq B \leq 1, \quad B \neq A, \quad A \in \mathcal{R}$$

is convex univalent in E.

In fact,

$$1 + \frac{zh''(z)}{h'(z)} = m\left[1 - \frac{2Bz^m}{1 + Bz^m}\right],$$

and so

$$Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > -m + \frac{2m}{1 + |B|} \geq 0, \quad z \in E.$$

Since $f \in \mathcal{N}_p(\mu, \alpha; A, B)$, we use Lemma 2 and (10.30) to obtained the required result. □

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Chapter 11
Integral Mean Estimates for a Polynomial with Restricted Zeros

A. Liman and W. M. Shah

11.1 Introduction

Let $\mathbf{P_n}$ be the class of polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree n. For $P \in \mathbf{P_n}$, define

$$\|P\|_q := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right\}^{\frac{1}{q}}, \ \ 0 < q < \infty,$$

and

$$\|P\|_\infty := \max_{|z|=1} |P(z)|.$$

It was shown by Turan [14], that if $P \in \mathbf{P_n}$ and $P(z)$ has all its zeros in $|z| \le 1$, then

$$n\|P\|_\infty \le 2\|P^{'}\|_\infty\cdot \tag{11.1}$$

The result is best possible and equality holds for $P(z) = \alpha z^n + \beta$, where $|\alpha| = |\beta|$.

As a generalization of inequality (11.1), Malik [12] proved that if $P \in \mathbf{P_n}$ and $P(z) = 0$ in $|z| \le k$, where $k \le 1$ then

$$n\|P\|_\infty \le (1+k)\|P^{'}\|_\infty\cdot \tag{11.2}$$

The result is best possible and equality holds for $P(z) = (z+k)^n$.

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Later on Malik [13] obtained another generalization of inequality (11.1) in the sense that the left hand side was replaced by a factor involving the integral mean of $|P(z)|$ on $|z| = 1$. In fact he proved:

Theorem A 1 *If $P \in \mathbf{P_n}$ and has all its zeros in $|z| \le 1$, then for each $q > 0$,*

$$n\|P\|_q \le \|1 + z\|_q \|P^{'}\|_\infty \cdot \tag{11.3}$$

The result is sharp and equality holds for $P(z) = (\alpha z + \beta)^n$, where $|\alpha| = |\beta|$.

On the other hand as an extension of inequality (11.2), Aziz [1] proved the following:

Theorem B 1 *If $P \in \mathbf{P_n}$ and $P(z)$ has all its zeros in $|z| \le k$ where $k \le 1$, then for each $q > 0$,*

$$n\left\|\frac{P}{P'}\right\|_q \le \|1 + kz\|_q \cdot \tag{11.4}$$

The result is sharp and equality holds for $P(z) = (\alpha z + \beta k)^n$, where $|\alpha| = |\beta|$.

In this paper, we consider a more general class of polynomials

$$\mathbf{P}_{n,\mu} := \left\{P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j} z^{n-j},\ \ 1 \le \mu \le n\right\}\ \ with\ \ \mathbf{P}_{n,1} = \mathbf{P}_n,$$

and prove some results which generalize the above integral inequalities and provide improvements of some polynomial inequalities as well. We first prove

Theorem 1 *If $P \in \mathbf{P}_{\mathbf{n},\mu}$ has all its zeros in $|z| \le k$ where $k \le 1$ and $m := \min_{|z|=k}|P(z)|$, then for every real or complex number β with $|\beta| < 1$ and each $q > 0$,*

$$n\left\|\frac{P - \frac{m\beta z^n}{k^n}}{P' - \frac{mn\beta z^{n-1}}{k^n}}\right\|_q \le \|1 + t_{k,\mu} z\|_q, \tag{11.5}$$

where

$$t_{k,\mu} = \left\{\frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n|a_n|k^{\mu-1} + \mu|a_{n-\mu}|}\right\}. \tag{11.6}$$

The result is best possible and equality in (11.5) holds for $P(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, where n is a multiple of μ.

A result of Aziz and Rather [1, Theorem 2] is a special case of Theorem 1 when $\beta = 0$, whereas Theorem A immediately follows from this result when $\beta = 0$ and $k = 1$. Also, since $\frac{\mu}{n}\left|\frac{a_{n-\mu}}{a_n}\right| \le k^\mu$ (see lemma), it can be easily verified that $t_{k,\mu} \le k^\mu$ and thus

$$\|1 + t_{k,\mu} z\|_q \le \|1 + k^\mu z\|_q \cdot \tag{11.7}$$

Using these observations in Theorem 1, we have the following:

Corollary 1 *If $P \in \mathbf{P}_{\mathbf{n},\mu}$ has all its zeros in $|z| \leq k$ where $k \leq 1$ and $m := \min_{|z|=k}|P(z)|$, then for every real or complex number β with $|\beta| < 1$ and each $q > 0$,*

$$n\left\| \frac{P - \frac{m\beta z^n}{k^n}}{P' - \frac{mn\beta z^{n-1}}{k^n}} \right\|_q \leq \|1 + k^\mu z\|_q \cdot \tag{11.8}$$

The result is best possible and equality holds for the polynomial $P(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, where n is a multiple of μ.

If we take $\beta = 0$ in Corollary 1, we immediately get a result recently proved by Aziz and Shah [5], whereas for $\beta = 0, \ \mu = 1$, Corollary 1 reduces to a result earlier proved by Aziz [1, Theorem 2]. Again, since $|P^{'}(e^{i\theta})| \leq \|P^{'}\|_\infty \ \ for \ \ 0 \leq \theta < 2\pi$, as a consequence of Theorem 1, we have the following:

Corollary 2 *If $P \in \mathbf{P}_{\mathbf{n},\mu}$ has all its zeros in $|z| \leq k$ where $k \leq 1$ and $m := \min_{|z|=k}|P(z)|$, then for every β with $|\beta| < 1$ and each $q > 0$,*

$$n\left\| P - \frac{m\beta z^n}{k^n} \right\|_q \leq \|1 + t_{k,\mu} z\|_q \left\| P^{'} - \frac{mn\beta z^{n-1}}{k^n} \right\|_\infty, \tag{11.9}$$

where $t_{k,\mu}$ is defined by (11.6).

A result due to Aziz and Rather [4, Corollary 5] is a special case of Corollary 2 when $\beta = 0$. Also using the fact that $1 \leq t_{k,\mu} \leq k^\mu, 1 \leq \mu \leq n$, we conclude that Corollary 2 is a refinement of a result due to Aziz and Shah [5], when $\beta = 0$. If we take $\mu = 1$ and make $q \to \infty$ in Corollary 2, we get the following:

Corollary 3 *If $P \in \mathbf{P}_{\mathbf{n},\mu}$ and $P(z)$ has all its zeros in $|z| \leq k$ where $k \leq 1$ and $m := \min_{|z|=k}|P(z)|$, then for every β with $|\beta| < 1$,*

$$n\left\| P - \frac{m\beta z^n}{k^n} \right\|_\infty \leq \|1 + t_{k,1} z\|_\infty \left\| P^{'} - \frac{mn\beta z^{n-1}}{k^n} \right\|_\infty, \tag{11.10}$$

where

$$t_{k,1} = \left\{ \frac{n|a_n|k^2 + |a_{n-1}|}{n|a_n| + |a_{n-1}|} \right\}.$$

By taking $\beta = 0$ in (11.10), we obtain a result due to Govil, Rahman and Schmeisser [9, Corollary 2], whereas for $k = 1, \ \beta = 0$, Corollary 3 reduces to inequality (11.1).

Remark 1 Using the fact that $1 \leq t_{k,\mu} \leq k^\mu, \ \ 1 \leq \mu \leq n$, it follows from Corollary 3 after suitable choice of β, that if $P \in \mathbf{P}_\mathbf{n}$ and $P(z) = 0$ in $|z| \leq k$ where $k \leq 1$, then

$$\|P^{'}(z)\|_\infty \geq \frac{n}{1+k}\|P\|_\infty + \frac{n}{k^{n-1}(1+k)}\min_{|z|=k}|P(z)|. \tag{11.11}$$

The result was independently proved by Govil [8, Theorem 2]. Whereas for $k = 1$, Corollary 3 reduces to a result of Aziz and Dawood [2, Theorem 4].

Next, by using Holder's inequality we establish the following result and obtain generalizations of Theorem B and a result [4, Theorem 2] due to Aziz and Rather.

Theorem 2 *Let* $P \in \mathbf{P_{n,\mu}}$ *has all its zeros in* $|z| \le k$ *where* $k \le 1$. *If* $m := \min_{|z|=k}|P(z)|$, *then for each* $q > 0$, $s > 1$, $r > 1$ *with* $r^{-1} + s^{-1} = 1$, *and for any* β *with* $|\beta| < 1$,

$$n\left\| P - \frac{m\beta z^n}{k^n}\right\|_q \le \|1 + t_{k,\mu}z\|_{qr}\left\| P' - \frac{mn\beta z^{n-1}}{k^n}\right\|_{qs}, \tag{11.12}$$

where $t_{k,\mu}$ *is defined by* (11.6).

Choosing $\beta = 0$ in (11.12), we get Theorem B. Corollary 2 can also be obtained from Theorem 2 by letting $s \to \infty$ (so that $r \to 1$).

11.2 Proofs of the Theorems

For the proofs of these theorems, we need the following lemma.

Lemma 1 *If* $P \in \mathbf{P_{n,\mu}}$ *and* $P(z)$ *has all its zeros in* $|z| \le k$ *where* $k \le 1$ *and* $Q(z) = z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$, *then*

$$|Q'(z)| \le t_{k,\mu}|P'(z)| \quad for \quad |z| = 1, \tag{11.13}$$

where

$$t_{k,\mu} = \left\{\frac{n|a_n|k^{2\mu} + \mu|a_{n-\mu}|k^{\mu-1}}{n|a_n|k^{\mu-1} + \mu|a_{n-\mu}|}\right\} \quad and \quad \frac{\mu}{n}\left|\frac{a_{n-\mu}}{a_n}\right| \le k^{\mu}.$$

This lemma is due to Aziz and Rather [4].

Proof By hypothesis the polynomial $P(z) = a_n z^n + \sum_{j=\mu}^{n} a_{n-j}z^{n-j}$ has all its zeros in $|z| \le k \le 1$. If $P(z)$ has a zero on $|z| = k$, then $\min_{|z|=k}|P(z)| = 0$ and the result follows from Theorem A in this case. Henceforth, we suppose that all the zeros of $P(z)$ lie in $|z| < k$ where $k \le 1$, so that $m > 0$. Now $m \le |P(z)|$ for $|z| = k$, therefore, if β is any real or complex number such that $|\beta| < 1$, then $\left|\frac{m\beta z^n}{k^n}\right| < |P(z)|$ for $|z| = k$. Since all the zeros of $P(z)$ lie in $|z| < k$, it follows by Rouche's theorem, that all the zeros of $F(z) = P(z) - \frac{m\beta z^n}{k^n}$ also lie in $|z| < k$. If $G(z) = z^n\overline{F\left(\frac{1}{\bar{z}}\right)} = Q(z) - \frac{m\bar{\beta}}{k^n}$ then it can be easily verified that for $|z| = 1$,

$$|F'(z)| = |nG(z) - zG'(z)|. \tag{11.14}$$

As $F(z)$ has all its zeros in $|z| < k \le 1$, the above lemma in conjunction with inequality (11.14) gives,

$$|G'(z)| \le t_{\mu,k} \ |nG(z) - zG'(z)| \quad for \quad |z| = 1, \ 1 \le \mu \le n, \tag{11.15}$$

where $t_{\mu,k}$ is defined in (11.6). By Gauss–Lucas theorem all the zeros of the polynomial $F^{'}(z) = P^{'}(z) - \frac{mn\beta z^{n-1}}{k^n}$ lie in $|z| < k \leq 1$. Therefore, the polynomial $z^{n-1}\overline{F^{'}\left(\frac{1}{\bar{z}}\right)} = nG(z) - zG^{'}(z)$ has all its zeros in $|z| > \frac{1}{k} \geq 1$. Hence it follows that the function

$$W(z) = \frac{zG^{'}(z)}{t_{\mu,k}\{nG(z) - zG^{'}(z)\}} \tag{11.16}$$

is analytic for $|z| \leq 1$, $|W(z)| \leq 1$ for $|z| = 1$ and $W(0) = 0$. Thus the function $1 + t_{\mu,k}W(z)$ is subordinate to the function $1 + t_{\mu,k}z$ for $|z| \leq 1$. By a well-known property of subordination [10, p. 422], we have for each $q > 0$, and $|z| = 1$,

$$\int_0^{2\pi} |1 + t_{\mu,k}W(e^{i\theta})|^q d\theta \leq \int_0^{2\pi} |1 + t_{\mu,k}e^{i\theta}|^q d\theta. \tag{11.17}$$

Now by (11.16), we have

$$|1 + t_{\mu,k}W(z)| = \left|\frac{nG(z)}{nG(z) - zG^{'}(z)}\right|$$

$$= \frac{n\left|Q(z) - \frac{m\bar{\beta}}{k^n}\right|}{\left|P^{'}(z) - \frac{mn\beta z^{n-1}}{k^n}\right|}$$

$$= \frac{n\left|P(z) - \frac{m\beta z^n}{k^n}\right|}{\left|P^{'}(z) - \frac{mn\beta z^{n-1}}{k^n}\right|}. \tag{11.18}$$

From (11.17) and (11.18), we conclude that for every real or complex number β with $|\beta| < 1$ and for each $q > 0$,

$$n^q \int_0^{2\pi} \left|\frac{P(e^{i\theta}) - \frac{m\beta e^{in\theta}}{k^n}}{P^{'}(e^{i\theta}) - \frac{mn\beta e^{i(n-1)\theta}}{k^n}}\right|^q d\theta \leq \int_0^{2\pi} |1 + t_{k,\mu}e^{i\theta}|^q d\theta,$$

which is equivalent to (11.5) and this completes the proof of Theorem 1. □

Proof Proceeding similarly as in the proof of Theorem 1, we have from (11.18) for every real or complex number β with $|\beta| < 1$ and for each $q > 0$,

$$n^q \int_0^{2\pi} \left|P(e^{i\theta}) - \frac{m\beta e^{in\theta}}{k^n}\right|^q d\theta = \int_0^{2\pi} \left\{|1 + t_{k,\mu}W(e^{i\theta})|\left|P^{'}(e^{i\theta}) - \frac{mn\beta e^{i(n-1)\theta}}{k^n}\right|\right\}^q d\theta.$$

This gives with the help of Holder's inequality for $s > 1$, $r > 1$ with $r^{-1} + s^{-1} = 1$,

$$n^q \int_0^{2\pi} \left| P(e^{i\theta}) - \frac{m\beta e^{in\theta}}{k^n} \right|^q d\theta \leq \left\{ \int_0^{2\pi} |1 + t_{k,\mu} W(e^{i\theta})|^{qr} d\theta \right\}^{\frac{1}{r}} \tag{11.19}$$

$$\left\{ \int_0^{2\pi} \left| P^{'}(e^{i\theta}) - \frac{mn\beta e^{i(n-1)\theta}}{k^n} \right|^{qs} d\theta \right\}^{\frac{1}{s}}.$$

Using Inequality (11.17) with q replaced by qr in (11.2), we obtain for each $q > 0,\ s > 1,\ \ r > 1$ with $r^{-1} + s^{-1} = 1$,

$$n^q \int_0^{2\pi} \left| P(e^{i\theta}) - \frac{m\beta e^{in\theta}}{k^n} \right|^q d\theta \leq \left\{ \int_0^{2\pi} |1 + t_{k,\mu} W(e^{i\theta})|^{qr} d\theta \right\}^{\frac{1}{r}} \tag{11.20}$$

$$\left\{ \int_0^{2\pi} \left| P^{'}(e^{i\theta}) - \frac{mn\beta e^{i(n-1)\theta}}{k^n} \right|^{qs} d\theta \right\}^{\frac{1}{s}}.$$

Equivalently

$$n \left\| P - \frac{m\beta z^n}{k^n} \right\|_q \leq \|1 + t_{k,\mu} z\|_{qr} \left\| P^{'} - \frac{mn\beta z^{n-1}}{k^n} \right\|_{qs},$$

which is Inequality (11.12) and this completes the proof of Theorem 2. □

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Chapter 12
Uniqueness Results of Meromorphic Functions Concerning Small Functions

Xiao-Min Li, Kai-Mei Wang and Hong-Xun Yi

12.1 Introduction and Main Results

In this chapter, by meromorphic functions, we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [2, 4, 11]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any nonconstant meromorphic function $h(z)$, we denote by $S(r,h)$, any quantity satisfying $S(r,h) = o(T(r,h))\,(r \to \infty, r \notin E)$. It also will be convenient to let E_1 denote any set of positive real numbers, such that $E_1 \subset (1, +\infty)$ and $\int_{E_1} d\log\log r < +\infty$, we denote by $S_1(r,h)$, any quantity satisfying $S_1(r,h) = o(T(r,h))\,(r \to \infty, r \notin E_1)$. Let f be a nonconstant meromorphic function, a meromorphic function a satisfying $T(r,a) = S(r,f)$ is called a small function related to f, and let $S(f)$ be the set of meromorphic functions which are small functions related to f. Obviously, $C \subset S(f)$ and $S(f)$ is a field (see [3]). Let f and g be two nonconstant meromorphic functions and let $a \in \{S(f) \cap S(g)\} \cup \{\infty\}$. Next we denote by $\overline{N}_0(r,a,f,g)$, the reduced counting function of the common zeros of $f-a$ and $g-a$ in $|z| < r$, where each point in $\overline{N}_0(r,a,f,g)$ is counted only once, $f - \infty$ means $1/f$. Let

$$\overline{N}_{12}(r,a,f,g) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{g-a}\right) - 2\overline{N}_0(r,a,f,g), \qquad (12.1)$$

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where $\overline{N}_{12}(r, a, f, g)$ denotes the reduced counting function of the different zeros of $f - a$ and $g - a$ in $|z| < r$. Let

$$\lambda(a, f, g) = 1 - \limsup_{r\to\infty} \frac{\overline{N}_{12}(r, a, f, g)}{T(r, f) + T(r, g)}. \tag{12.2}$$

Then, $0 \le \lambda(a, f, g) \le 1$ (see [8]). If $\overline{N}_{12}(r, a, f, g) = 0$, we say that f and g share a IM. If $\overline{N}_{12}(r, a, f, g) = S(r, f) + S(r, g)$, we say that f and g share a "IM," which can be found, e.g., in [12].

In 1929, Nevanlinna [8] proved the following famous theorem:

Theorem 1 ([8]) *If f and g are meromorphic functions sharing a_1, a_2, a_3, a_4, a_5 IM, where a_1, a_2, a_3, a_4, a_5 are five distinct elements in $C \cup \{\infty\}$, then $f \equiv g$.*

Regarding Theorem 1, one may ask, whether it is possible to extend Theorem A to the case when a_1, a_2, a_3, a_4, a_5 are five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$ (see [8]). In this direction, many results, not until the beginning of 1990s, were obtained, see, e.g., in the references [5, 6, 9, 10, 12, 13, 15]. In 2000, Li and Qiao [7] affirmatively answered this question and proved the following result:

Theorem 2 ([7, *Theorem* 1]) *If f and g are meromorphic functions sharing a_1, a_2, a_3, a_4, a_5 IM, where a_1, a_2, a_3, a_4, a_5 are five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$, then $f \equiv g$.*

Later on, Yi [13] proved the following results to improve Theorem 2:

Theorem 3 ([13, Theorem 4.2]) *Let f and g be nonconstant meromorphic functions, and let a_1, a_2, a_3, a_4, a_5 be five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$. If*

$$\sum_{j=1}^{5} \lambda(a_j, f, g) > \frac{43}{9}, \tag{12.3}$$

then $f \equiv g$.

Theorem 4 ([13, Theorem 1.2]) *Let f and g be nonconstant meromorphic functions, and let a_1, a_2, a_3, a_4, a_5 be five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$. If f and g share a_1, a_2, a_3 "IM," and if*

$$\lambda(a_4, f, g) + \lambda(a_5, f, g) > \frac{16}{9}, \tag{12.4}$$

then $f \equiv g$.

Regarding Theorems 3 and 4, one may ask, is it possible to relax the assumption (12.3) in Theorem 3 and the assumption (12.4) in Theorem 4? In this direction, we will prove the following result to improve Theorems 3 and 4:

Theorem 5 *Let f and g be nonconstant meromorphic functions, and let a_1, a_2, a_3, a_4, a_5 be five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$. If*

$$\sum_{j=1}^{5} \lambda(a_j, f, g) > \frac{14}{3}, \tag{12.5}$$

then $f \equiv g$.

By (12.2) and Theorem 5, we can get the following results to improve and supplement Theorem 6.3 in [13]:

Corollary 1 *Let f and g be nonconstant meromorphic functions, and let a_1, a_2, a_3, a_4, a_5 be five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$. If f and g share a_1 "IM," and if $\lambda(a_2, f, g) + \lambda(a_3, f, g) + \lambda(a_4, f, g) + \lambda(a_5, f, g) > 11/3$, then $f \equiv g$.*

Corollary 2 *Let f and g be nonconstant meromorphic functions, and let a_1, a_2, a_3, a_4, a_5 be five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$. If f and g share a_1, a_2 "IM," and if $\lambda(a_3, f, g) + \lambda(a_4, f, g) + \lambda(a_5, f, g) > 8/3$, then $f \equiv g$.*

Corollary 3 *Let f and g be nonconstant meromorphic functions, and let a_1, a_2, a_3, a_4, a_5 be five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$. If f and g share a_1, a_2, a_3 "IM," and if $\lambda(a_4, f, g) + \lambda(a_5, f, g) > 5/3$, then $f \equiv g$.*

Corollary 4 *Let f and g be nonconstant meromorphic functions, and let a_1, a_2, a_3, a_4, a_5 be five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$. If f and g share a_1, a_2, a_3, a_4 "IM," and if $\lambda(a_5, f, g) > 2/3$, then $f \equiv g$.*

We recall the following result proved by Yi [13]:

Theorem 6 ([13, *Theorem* 6.4]) *Let f and g be two distinct nonconstant meromorphic functions, and let a_1, a_2, a_3, a_4, a_5 be five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$. If f and g share a_1, a_2, a_3, a_4 "IM," and if*

$$\overline{N}\left(r, \frac{1}{g - a_5}\right) = S(r, g), \tag{12.6}$$

then

$$2T(r, f) \le 5\overline{N}\left(r, \frac{1}{f - a_5}\right) + S(r, f). \tag{12.7}$$

We will prove the following result to improve Theorem 6:

Theorem 7 *Let f and g be two distinct nonconstant meromorphic functions, and let a_1, a_2, a_3, a_4, a_5 be five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$. If f and g share a_1, a_2, a_3, a_4 "IM," and if*

$$\overline{N}\left(r, \frac{1}{g - a_5}\right) = S(r, g), \tag{12.8}$$

then for every positive number ε, we have

$$(3 - \varepsilon)T(r, f) \le 5\overline{N}\left(r, \frac{1}{f - a_5}\right) + O(1) \tag{12.9}$$

for any positive number r *excluding some set* $E_1 \subset (1, +\infty)$ *with* $\int_{E_1} d \log \log r < +\infty$.

In 1998, Ishizaki and Toda [3] proved the following theorem:

Theorem 8 ([3, *Theorem* 3]) *Let* f *and* g *be transcendental meromorphic functions such that* f *and* g *share* a_1, a_2, a_3, a_4 *IM, where* a_1, a_2, a_3, a_4 *are four distinct elements in* $\{S(f) \cap S(g)\}$. *If* $\overline{N}(r, f)$ *and* $\overline{N}(r, g)$ *satisfy one of the following conditions (a), (b), (c), and (d):*

(a) $\overline{N}(r, f) = S(r, f)$ *and* $\overline{N}(r, g) = S(r, g)$;
(b) $\overline{N}(r, g) = S(r, g)$, $\overline{N}(r, f) \neq S(r, f)$ *and* $\overline{N}(r, f) \leq uT(r, f) + S(r, f)$ *for some* $u \in (0, 1/19)$;
(c) $\overline{N}(r, f) = S(r, f)$, $\overline{N}(r, g) \neq S(r, g)$ *and* $\overline{N}(r, g) \leq vT(r, g) + S(r, g)$ *for some* $v \in (0, 1/19)$;
(d) $\overline{N}(r, f) \neq S(r, f)$, $\overline{N}(r, g) \neq S(r, g)$, $\overline{N}(r, f) \leq uT(r, f) + S(r, f)$ *and* $\overline{N}(r, g) \leq vT(r, g) + S(r, g)$ *for some* $u, v \in (0, 1)$ *satisfying either*

(i) $0 < u < 1/19$ *and* $0 < v < (2 - 19u)/(20 - 19u)$ *or*
(ii) $0 < v < 1/19$ *and* $0 < u < (2 - 19v)/(20 - 19v)$;

then $f \equiv g$.

Later on, Yi [13] proved the following result to improve Theorem 8:

Theorem 9 ([13, *Theorem* 7.1]) *Let* f *and* g *be transcendental meromorphic functions such that* f *and* g *share* a_1, a_2, a_3, a_4 *"IM," where* a_1, a_2, a_3, a_4 *are four distinct elements in* $\{S(f) \cap S(g)\}$. *Suppose that* $\overline{N}(r, f)$ *and* $\overline{N}(r, g)$ *satisfy one of the following conditions (a), (b), and (c):*

(a) $\overline{N}(r, g) = S(r, g)$, $\overline{N}(r, f) \neq S(r, f)$ *and* $\overline{N}(r, f) \leq uT(r, f) + S(r, f)$ *for some* $u \in [0, 2/5)$;
(b) $\overline{N}(r, f) = S(r, f)$, $\overline{N}(r, g) \neq S(r, g)$ *and* $\overline{N}(r, g) \leq vT(r, g) + S(r, g)$ *for some* $v \in [0, 2/5)$;
(c) $\overline{N}(r, f) \neq S(r, f)$, $\overline{N}(r, g) \neq S(r, g)$, $\overline{N}(r, f) \leq uT(r, f) + S(r, f)$ *and* $\overline{N}(r, g) \leq vT(r, g) + S(r, g)$ *for some* $u, v \in (0, 1)$ *satisfying either*

(i) $0 < u < 2/9$ *and* $0 < v < (4 - 9u)/(11 - 9u)$ *or*
(ii) $0 < v < 2/9$ *and* $0 < u < (4 - 9v)/(11 - 9v)$.

Then $f \equiv g$.

We will prove the following result to improve Theorem 9:

Theorem 10 *Let* f *and* g *be transcendental meromorphic functions such that* f *and* g *share* a_1, a_2, a_3, a_4 *"IM," where* a_1, a_2, a_3, a_4 *are four distinct elements in* $\{S(f) \cap S(g)\}$. *Suppose that* $\overline{N}(r, f)$ *and* $\overline{N}(r, g)$ *satisfy one of the following conditions (a), (b), and (c):*

(a) $\overline{N}(r, g) = S(r, g)$.
(b) $\overline{N}(r, f) = S(r, f)$.

(c) $\overline{N}(r, f) \neq S(r, f)$, $\overline{N}(r, g) \neq S(r, g)$, $\overline{N}(r, f) \leq uT(r, f) + S(r, f)$ *and* $\overline{N}(r, g) \leq vT(r, g) + S(r, g)$ *for some* $u, v \in (0, 1)$ *satisfying either*

(i) $0 < u < 2/3$ *and* $0 < v < \frac{2}{3} - u$ *or*
(ii) $0 < v < 2/3$ *and* $0 < u < \frac{2}{3} - v$.

Then $f \equiv g$.

From Theorem 10, we get the following result to improve Corollary 7.1 in [13]:

Corollary 5 *Let* f *and* g *be nonconstant meromorphic functions such that* $\overline{N}(r, f) \leq uT(r, f) + S(r, f)$ *and* $\overline{N}(r, g) \leq vT(r, g) + S(r, g)$ *for some finite complex numbers* u *and* v *satisfying* $(u, v) \in [0, 2/3) \times [0, 2/3)$. *If there exist four distinct elements* a_1, a_2, a_3, a_4 *in* $S(f) \cap S(g)$, *such that* f *and* g *share* a_1, a_2, a_3, a_4 *"IM," then* $f \equiv g$.

12.2 Some Lemmas

In this section, we introduce some important results which are used to prove the main results in this chapter:

Lemma 1 ([13, *Theorem* 3.2]) *Let* f *and* g *be nonconstant meromorphic functions, and let* a_1, a_2, a_3, a_4, a_5 *be five distinct elements in* $\{S(f) \cap S(g)\} \cup \{\infty\}$. *If* $f \not\equiv g$, *then*

$$\overline{N}_0(r, a_5, f, g) \leq \sum_{j=1}^{4} \overline{N}_{12}(r, a_j, f, g) + S(r, f) + S(r, g) \tag{12.10}$$

and

$$\overline{N}\left(r, \frac{1}{f - a_5}\right) + \overline{N}\left(r, \frac{1}{g - a_5}\right) \leq 2\sum_{j=1}^{4} \overline{N}_{12}(r, a_j, f, g) + \overline{N}_{12}(r, a_5, f, g) + S(r, f) + S(r, g). \tag{12.11}$$

Lemma 2 ([1, *Theorem* 2.3], or [14, *Theorem* 1]) *Let* f *be a nonconstant meromorphic function, and let* $a_1, a_2, \cdots, a_q$ $(q \geq 3)$ *be* q *mutually distinct small functions with respect to* f. *Then for any* $\varepsilon > 0$,

$$(q - 2 - \varepsilon)T(r, f) \leq \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f - a_j}\right) + O(1)$$

for any positive number r *excluding some set* $E_1 \subset (1, +\infty)$ *with* $\int_{E_1} d \log \log r < +\infty$.

The following result plays an important role in proving Theorem 5 of this chapter, which improves Theorem 4.1 in [13]:

Lemma 3 *Let f and g be nonconstant meromorphic functions and let a_1, a_2, a_3, a_4, a_5 be five distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$. If $f \not\equiv g$, then*

$$T(r,f)+T(r,g) \le 3\sum_{j=1}^{5}\overline{N}_{12}(r,a_j,f,g)+\varepsilon(T(r,f) + T(r,g))+S_1(r,f)+S_1(r,g). \tag{12.12}$$

Proof First of all, by Lemma 1, we have from (12.11) that

$$\overline{N}\left(r,\frac{1}{f-a_k}\right)+\overline{N}\left(r,\frac{1}{g-a_k}\right) \le 2\sum_{j=1}^{5}\overline{N}_{12}\left(r,a_j,f,g\right)-\overline{N}_{12}(r,a_k,f,g) + S(r,f)+S(r,g) \tag{12.13}$$

for $k = 1, 2, 3, 4, 5$. By taking the summation over $k = 1,2,3,4,5$ in the inequality (12.13), we have

$$\sum_{k=1}^{5}\overline{N}\left(r,\frac{1}{f-a_k}\right)+\sum_{k=1}^{5}\overline{N}\left(r,\frac{1}{g-a_k}\right) \le 9\sum_{j=1}^{5}\overline{N}_{12}(r,a_j,f,g) + S(r,f)+S(r,g). \tag{12.14}$$

By Lemma 2, we have

$$3(T(r,f)+T(r,g)) \le \sum_{j=1}^{5}\overline{N}\left(r,\frac{1}{f-a_j}\right)+\sum_{j=1}^{5}\overline{N}\left(r,\frac{1}{g-a_j}\right) + \varepsilon(T(r,f)+T(r,g))+S_1(r,f)+S_1(r,g). \tag{12.15}$$

From (12.14) and (12.15) we have

$$T(r,f)+T(r,g) \le 3\sum_{j=1}^{5}\overline{N}_{12}(r,a_j,f,g)+\frac{\varepsilon}{3}(T(r,f) + T(r,g))+S_1(r,f)+S_1(r,g),$$

which reveals (12.12). This completes the proof of Lemma 3.

Lemma 4 ([13, *Lemma* 2.2]) *Let f and g be nonconstant meromorphic functions sharing a_1, a_2, a_3 "IM," where a_1, a_2 and a_3 are three distinct elements in $\{S(f) \cap S(g)\} \cup \{\infty\}$. Then $S(r,f) = S(r,g)$.*

The following result improves Theorem 2 in [2]:

Lemma 5 *Let f and g be two distinct nonconstant meromorphic functions such that f and g share a_1, a_2, a_3, a_4 "IM," where a_1, a_2, a_3, a_4 are four distinct elements in $S(f) \cap S(g)$. Then, for any $\varepsilon > 0$, we have*

$$T(r, f) \le T(r, g) + \varepsilon T(r, f) + O(1) \tag{12.16}$$

and

$$T(r, g) \le T(r, f) + \varepsilon T(r, g) + O(1) \tag{12.17}$$

for any positive number r excluding some set $E_1 \subset (1, +\infty)$ with $\int_{E_1} d \log \log r < +\infty$.

Proof Without loss of generality, we suppose that $a_j \not\equiv \infty$ for $j = 1, 2, 3, 4$. Then, by Lemma 2 and the assumptions of Lemma 5, we have

$$\begin{aligned} 2T(r, f) &\le \sum_{j=1}^{4} \overline{N}\left(r, \frac{1}{f - a_j}\right) + \varepsilon T(r, f) \\ &\le N\left(r, \frac{1}{f - g}\right) + \varepsilon T(r, f) \\ &\le T(r, f - g) + \varepsilon T(r, f) + O(1) \\ &\le T(r, f) + T(r, g) + \varepsilon T(r, f) + O(1), \end{aligned} \tag{12.18}$$

for any positive number r excluding some set $E_1 \subset (1, +\infty)$ with $\int_{E_1} d \log \log r < +\infty$, where ε is any positive number. From (12.18), we have (12.16). Similarly, we can get (12.17). This completes the proof of Lemma 5.

12.3 Proof of Theorems

Proof of Theorem 1.1 Suppose that $f \not\equiv g$. Then, from (12.2), (12.12), and Lemma 3, we deduce

$$\sum_{j=1}^{5} \lambda(a_j, f, g) \le \frac{14}{3},$$

which contradicts (12.5), and so we get the conclusion of Theorem 5.

Proof of Theorem 1.2 First of all, from (12.1), Lemma 4, and the assumption

$$\overline{N}\left(r, \frac{1}{g - a_5}\right) = S(r, g),$$

we get

$$\overline{N}_{12}(r, a_5, f, g) = \overline{N}\left(r, \frac{1}{f - a_5}\right) + S(r, f) + S(r, g). \tag{12.19}$$

From (12.11), (12.19), Lemma 1, and the assumption that f and g share a_1, a_2, a_3, a_4 "IM," we have for $k = 1, 2, 3, 4$ that

$$2\overline{N}\left(r, \frac{1}{f - a_k}\right) = \overline{N}\left(r, \frac{1}{f - a_k}\right) + \overline{N}\left(r, \frac{1}{g - a_k}\right)$$
$$\leq 2\overline{N}\left(r, \frac{1}{f - a_5}\right) + S(r, f) + S(r, g),$$

i.e.,

$$\overline{N}\left(r, \frac{1}{f - a_k}\right) \leq \overline{N}\left(r, \frac{1}{f - a_5}\right) + S(r, f) + S(r, g) \quad \text{for} \quad k = 1, 2, 3, 4. \tag{12.20}$$

By Lemma 5 and the assumption that f and g share a_1, a_2, a_3, a_4 "IM," we have (12.16) and (12.17). From (12.16), (12.17), and (12.20), we get the conclusion of Theorem 7.

Proof of Theorem 1.3 Suppose that $f \not\equiv g$. We discuss the following three cases.

Case 1 Suppose that $\overline{N}(r, g) = S(r, g)$. Then, by Lemma 2 and Lemma 5, we have

$$(3 - \varepsilon)T(r, g) \leq \overline{N}(r, g) + \sum_{j=1}^{4} \overline{N}\left(r, \frac{1}{g - a_j}\right) + O(1)$$
$$\leq N\left(r, \frac{1}{f - g}\right) + S(r, g)$$
$$\leq T(r, f) + T(r, g) + S(r, g)$$
$$\leq 2T(r, g) + \varepsilon T(r, f) + O(1)$$

and

$$(3 - \varepsilon)T(r, f) \leq 2T(r, f) + \varepsilon T(r, g) + O(1),$$

and so

$$T(r, f) + T(r, g) \leq 2\varepsilon[T(r, f) + T(r, g)] + O(1) \tag{12.21}$$

for any positive number r excluding some set $E_1 \subset (1, +\infty)$ with $\int_{E_1} d \log \log r < +\infty$, where ε is any given positive number. From (12.21), we can find that f and g are constants, which is impossible.

Case 2 Suppose that $\overline{N}(r, f) = S(r, f)$. Then, in the same manner as in Case 1, we can get a contradiction.

Case 3 Suppose that $\overline{N}(r, f) \neq S(r, f)$, $\overline{N}(r, g) \neq S(r, g)$ and

$$\overline{N}(r, f) \leq uT(r, f) + S(r, f), \quad \overline{N}(r, g) \leq vT(r, g) + S(r, g) \tag{12.22}$$

for some $u, v \in (0, 1)$. We discuss the following two subcases.

Subcase 3.1 Suppose that $0 < u < 2/3$ and $0 < v < \frac{2}{3} - u$. First of all, from (12.1), we have

$$\overline{N}_{12}(r, \infty, f, g) \leq \overline{N}(r, f) + \overline{N}(r, g). \tag{12.23}$$

From Lemma 3 and the assumption that f and g share a_1, a_2, a_3, a_4 "IM," we get

$$\begin{aligned} T(r, f) + T(r, g) \leq 3\overline{N}_{12}(r, \infty, f, g) + \varepsilon(T(r, f) + T(r, g)) \\ + S_1(r, f) + S_1(r, g), \end{aligned} \tag{12.24}$$

where ε is an arbitrary positive number. From (12.23) and (12.24), we have

$$\begin{aligned} T(r, f) + T(r, g) \leq 3(\overline{N}(r, f) + \overline{N}(r, g)) + \varepsilon(T(r, f) + T(r, g)) \\ + S_1(r, f) + S_1(r, g). \end{aligned} \tag{12.25}$$

From Lemma 4, we have $S(r, f) = S(r, g)$ and $S_1(r, f) = S_1(r, g)$. From Lemma 5, we have (12.16) and (12.17.) From (12.22) and (12.25), we have

$$T(r, f) + T(r, g) \leq 3uT(r, f) + 3vT(r, g) + \varepsilon(T(r, f) + T(r, g)),$$

i.e.,

$$(1 - 3u - \varepsilon)T(r, f) \leq (3v - 1 + \varepsilon)T(r, g). \tag{12.26}$$

From (12.16), (12.17), (12.26), and the above supposition, we have $1 - 3u \leq 3v - 1$ and so

$$v \geq \frac{2}{3} - u,$$

which contradicts the above supposition.

Subcase 3.2 Suppose that $0 < v < \frac{2}{3}$ and $0 < u < \frac{2}{3} - v$. Then, in the same manner as in Subcase 3.1, we can get a contradiction. This completes the proof of Theorem 10.

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Chapter 13
Maximal Polynomial Ranges for a Domain of Intersection of Two Circular Disks

Chinta Mani Pokhrel

13.1 Introduction and Basic Theory

Let $\Omega \subseteq \mathbb{C}$ with $0 \in \Omega$ be a domain and $\mathcal{P}_n^0(\Omega)$ the set of all polynomials whose degree is $\leq n$ and are normalized by the condition $P(0) = 0$, with geometric constraint $P(\mathbb{D}) \subseteq \Omega$, where $\mathbb{D} = \{z : \ |z| \ < 1\}$ is the unit disk. More precisely,

$$\mathcal{P}_n^0(\Omega) := \{P \ : \ P \text{ is a polynomial of degree} \leq n, \ P(0) = 0, \ P(\mathbb{D}) \subseteq \Omega\}.$$

Definition 1 The maximal range of the family $\mathcal{P}_n^0(\Omega)$, denoted by Ω_n, is defined as

$$\Omega_n := \bigcup_{P \in \mathcal{P}_n^0(\Omega)} P(\mathbb{D}).$$

Example The maximal polynomial range of the domain $\Omega := \mathbb{C} \setminus \{1\}$ is

$$\Omega_n := \bigcup_{P \in \mathcal{P}_n^0(\Omega)} P(\mathbb{D}) = P_n(\mathbb{D}),$$

where $P_n(z) = 1 - (1 + z)^n, \ n \in \mathbb{N}$. This is the best known nontrivial example for which a range of the single polynomial $P_n(z) = 1 - (1 + z)^n$ describes the whole maximal range Ω_n of the family $\mathcal{P}_n^0(\Omega)$. However, this will not be true in general.

Definition 2 A polynomial $P \in \mathcal{P}_n^0(\Omega)$ is said to be an extremal polynomial for Ω_n if

$$P(\overline{\mathbb{D}}) \cap (\partial\Omega_n \setminus \partial\Omega) \neq \phi$$

Definition 3 A point $\zeta \in \partial\mathbb{D}$ is called a point of contact of a polynomial $P \in \mathcal{P}_n^0(\Omega)$ if $P(\zeta) \in \partial\Omega$.

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Remark Points on $(\partial\Omega_n \setminus \partial\Omega)$ turn out to be the most interesting ones.
The following results, concerning extremal polynomials and the description of the maximal range Ω_n, have been proved by Cordova and Ruscheweyh in [1, 2, 4, 5].

Theorem 1 *For every point $\omega \in (\partial\Omega_n \setminus \partial\Omega)$, there exists at least one extremal polynomial $P \in P_n^0(\Omega)$ such that $\omega = P(1)$. Moreover, every extremal polynomial with $P(1) \in (\partial\Omega_n \setminus \partial\Omega)$ satisfies the following conditions:*

(1) P' has all zeroes in $\partial\mathbb{D}$.
(2) If we denote all these zeroes by $e^{i\psi_j}$, $j = 1, 2, 3 \ldots, n-1$, ordered as $0 < \psi_1 \leq \psi_2 \leq \ldots \leq \psi_{n-1} < 2\pi$, then there exist at least n points of contact $e^{i\theta_k}$, $k = 1, 2, \ldots, n$ (multiplicities counted) such that $0 < \theta_1 \leq \psi_1 \leq \theta_2 \leq \ldots < \psi_{n-1} \leq \theta_n < 2\pi$.
(3) P is univalent in $\mathbb{D}$ if Ω is simply connected.

If the given domain Ω is convex, then Theorem 1 can be remarkably refined as follows:

Theorem 2 *If Ω is a convex domain, then in addition to (1)–(3) in Theorem 1, we have:*

4. If $\omega \in (\partial\Omega_n \setminus \partial\Omega)$, then there exists a unique extremal polynomial $P \in P_n^0(\Omega)$ such that $\omega = P(1)$
5. If θ_1, θ_n from Theorem 1 are chosen in such a way that no θ in $[0, \theta_1) \cup (\theta_n, 2\pi]$ corresponds to a point of contact, then the arc

$$\left\{P(e^{it}) \text{ for } \theta_n - 2\pi < \theta < \theta_1\right\}$$

is a connected component of $\partial\Omega_n \setminus \partial\Omega$.

13.2 Polynomials Having All Zeroes of its Derivatives on $\partial\mathbb{D}$

A polynomial having the property that all the zeroes of its derivative lie on $\partial\mathbb{D}$ will play a central role to construct extremal polynomials and maximal polynomial ranges Ω_n of the given domain Ω. We shall start this section with the following lemma [3].

Lemma 1 *Let $P(z) = \sum_{k=1}^{n} a_k z^k \in \mathcal{P}_n^0$ with $a_n \neq 0$ and suppose all zeroes of P' lie on $\partial\mathbb{D}$. Then the coefficients of P satisfy the following condition:*

$$\overline{a}_1 k a_k = n a_n (n+1-k) \overline{a}_{n+1-k}, \; k = 1, \ldots, n. \tag{13.1}$$

Proof Let us denote the zeroes of $P'(z)$ by e^{is_k}, $k = 1, \ldots, n-1$. Then we can write

$$P'(z) = n a_n \prod_{k=1}^{n-1} \left(z - e^{is_k}\right). \tag{13.2}$$

Then

$$\overline{a}_1 = \overline{P'(0)} = n\overline{a}_n \prod_{k=1}^{n-1} \left(-e^{-is_k}\right). \tag{13.3}$$

Also,

$$\begin{aligned}
\overline{a}_1 P'(z) - na_n z^{n-1}\overline{P'(\overline{z}^{-1})} &= \overline{a}_1 n\, a_n \prod_{k=1}^{n-1}\left(z - e^{is_k}\right) - na_n z^{n-1} n\overline{a}_n \prod_{k=1}^{n-1}\left(z^{-1} - e^{-is_k}\right)\\
&= \overline{a}_1 n\, a_n \prod_{k=1}^{n-1}\left(z - e^{is_k}\right) - na_n n\overline{a}_n \prod_{k=1}^{n-1}\left(1 - ze^{-is_k}\right)\\
&= \overline{a}_1 n\, a_n \prod_{k=1}^{n-1}\left(z - e^{is_k}\right) - na_n n\overline{a}_n \prod_{k=1}^{n-1}\left(z - e^{is_k}\right)\left(-e^{-is_k}\right)\\
&= na_n \prod_{k=1}^{n-1}\left(z - e^{is_k}\right)\left(\overline{a}_1 - n\overline{a}_n \prod_{k=1}^{n-1}\left(-e^{-is_k}\right)\right).
\end{aligned}$$

By using Eq. (13.3), we have

$$\overline{a}_1 P'(z) - na_n z^{n-1}\overline{P'(\overline{z}^{-1})} = na_n \prod_{k=1}^{n-1}\left(z - e^{is_k}\right)\left(\overline{a}_1 - n\overline{a}_n \prod_{k=1}^{n-1}\left(-e^{-is_k}\right)\right) \equiv 0. \tag{13.4}$$

On the other hand,

$$\begin{aligned}
\overline{a}_1 P'(z) - na_n z^{n-1}\overline{P'(\overline{z}^{-1})} &= \overline{a}_1 \sum_{k=1}^{n} ka_k z^{k-1} - na_n z^{n-1} \sum_{k=1}^{n} k\overline{a}_k z^{-(k-1)}\\
&= \overline{a}_1 \sum_{k=1}^{n} ka_k z^{k-1} - na_n \sum_{k=1}^{n} k\overline{a}_k z^{n-k}\\
&= \overline{a}_1 \sum_{k=1}^{n} ka_k z^{k-1} - na_n \sum_{k=1}^{n} (n+1-k)\overline{a}_{n+1-k} z^{k-1}\\
&= \sum_{k=1}^{n} \left(\overline{a}_1 ka_k - na_n(n+1-k)\overline{a}_{n+1-k}\right) z^{k-1}.
\end{aligned}$$

By then,

$$\overline{a}_1 P'(z) - na_n z^{n-1}\overline{P'(\overline{z}^{-1})} = \sum_{k=1}^{n} \left(\overline{a}_1 ka_k - na_n(n+1-k)\overline{a}_{n+1-k}\right) z^{k-1} \equiv 0. \tag{13.5}$$

Equating each coefficient of Eq. (13.5) to zero, we get the Eq. (13.1). This completes the proof. We shall use Lemma 1 to prove the following result [6].

Theorem 3 *Let* $P(z) = \sum_{k=1}^{n} a_k z^k$ *with* $a_n \neq 0$ *and suppose all zeroes of* P' *lie on* $\partial\mathbb{D}$. *Then there exists* $\phi \in \mathbb{R}$ *such that*

$$P(e^{i(t+\phi)}) = a_0 + e^{i\frac{n+1}{2}t}\left[S\left(\cos\frac{t}{2}\right) + \sin\frac{t}{2}\,T\left(\cos\frac{t}{2}\right)\right.$$
$$\left. + \frac{2i}{n+1}\frac{d}{dt}\left(S\left(\cos\frac{t}{2}\right) + \sin\frac{t}{2}\,T\left(\cos\frac{t}{2}\right)\right)\right].$$

Where S *and* T *are real symmetric polynomials of degree* $(n-1)$ *and* $(n-2)$ *respectively given by*

$$S(x) := \frac{1}{2}\sum_{k=1}^{n}\frac{n+1}{n+1-k}\alpha_k \cos\left((n+1-2k)\arccos x\right), \tag{13.6}$$

$$T(x) := \frac{1}{2}\sum_{k=1}^{n}\frac{n+1}{n+1-k}\beta_k \frac{\sin\left((n+1-2k)\arccos x\right)}{\sin\left(\arccos x\right)} \tag{13.7}$$

and satisfy the conditions $S(-x) = (-1)^{(n-1)}S(x)$ *and* $T(-x) = (-1)^{(n-2)}T(x)$.

Proof From Eq. (13.3), it is possible to choose a real number ϕ so that

$$e^{i(n+1)\phi} = \frac{\overline{a}_1}{na_n}. \tag{13.8}$$

Now, if we choose

$$b_k := a_k e^{ik\phi}, \quad k = 1, \ldots, n, \tag{13.9}$$

we get

$$P(e^{i\phi}z) = \sum_{k=1}^{n} b_k z^k \text{ and } \overline{b}_1 = nb_n.$$

Then from Lemma 1, coefficients b_k satisfy the relation

$$kb_k = (n+1-k)\overline{b}_{n+1-k}. \tag{13.10}$$

It follows that

$$P(e^{i\phi}z) = a_0 + \frac{1}{2}\sum_{k=1}^{n}\left(b_k z^k + b_{n+1-k}z^{n+1-k}\right)$$
$$= a_0 + \frac{1}{2}\sum_{k=1}^{n}\left(b_k z^k + \frac{k}{n+1-k}\overline{b}_k z^{n+1-k}\right).$$

Again, if we choose $l := \frac{n+1}{2}$, $z = e^{it}$, we have

$$\begin{aligned}
P(e^{i(t+\phi)}) &= a_0 + \frac{1}{2}\sum_{k=1}^{n}\left(b_k e^{ikt} + \frac{k}{n+1-k}\overline{b}_k\, e^{i(n+1-k)t}\right) \\
&= a_0 + \frac{1}{2}e^{ilt}\sum_{k=1}^{n}\left(b_k e^{i(k-l)t} + \frac{k}{n+1-k}\overline{b}_k\,\overline{e^{i(k-l)t}}\right) \\
&= a_0 + \frac{1}{2}e^{ilt}\sum_{k=1}^{n}\frac{1}{n+1-k}\left[l\left(b_k e^{i(k-l)t} + \overline{b}_k\,\overline{e^{i(k-l)t}}\right)\right. \\
&\quad \left. + (l-k)\left(b_k e^{i(k-l)t} - \overline{b}_k\,\overline{e^{i(k-l)t}}\right)\right].
\end{aligned}$$

If we write $b_k = \alpha_k + i\beta_k$ with $\alpha_k, \beta_k \in \mathbb{R}$, we get

$$\begin{aligned}
P(e^{i(t+\phi)}) &= a_0 + \frac{1}{2}e^{ilt}\sum_{k=1}^{n}\frac{1}{n+1-k}\left[2l(\alpha_k\cos((k-l)t) - \beta_k\sin((k-l)t))\right. \\
&\quad \left. + 2\,(l-k)i(\alpha_k\sin((k-l)t) - \beta_k\cos((k-l)t))\right] \\
&= a_0 + \frac{1}{2}e^{i\frac{n+1}{2}t}\sum_{k=1}^{n}\frac{n+1}{n+1-k}\left[\alpha_k\left(\cos(n+1-2k)\frac{t}{2}\right)\right. \\
&\quad + \beta_k\left(\sin(n+1-2k)\frac{t}{2}\right) + \frac{2i}{n+1}\frac{d}{dt}\left(\alpha_k\left(\cos(n+1-2k)\frac{t}{2}\right)\right. \\
&\quad \left.\left. + \beta_k\left(\sin(n+1-2k)\frac{t}{2}\right)\right)\right] \\
&= a_0 + e^{i\frac{n+1}{2}t}\left[S\left(\cos\frac{t}{2}\right) + \sin\frac{t}{2}T\left(\cos\frac{t}{2}\right)\right. \\
&\quad \left. + \frac{2i}{n+1}\frac{d}{dt}S\left(\cos\frac{t}{2}\right) + \sin\frac{t}{2}T\left(\cos\frac{t}{2}\right)\right],
\end{aligned}$$

where $S(\cos\frac{t}{2})$ and $T(\cos\frac{t}{2})$ are polynomials as defined in Eqs. (13.6) and (13.7). Because S is a linear combination of Chebyshev polynomilas of degree $|n+1-2k| \le (n-1)$, it is a polynomial of degree $n-1$ and satisfies the condition $S(-x) = (-1)^{n-1}S(x)$. Similarly, T is a polynomial of degree $n-2$ and satisfies $T(-x) = (-1)^{n-2}T(x)$. This completes the proof.

In our work, we shall concentrate on domains which are symmetric with respect to the real axis and on extremal polynomials having real coefficients. In such a situation, if a point e^{is_k} is a zero of P', then the point e^{-is_k} is also a zero of P' and we have

$$P'(z) = na_n\prod_{k=1}^{n-1}(z - e^{is_k}) = \sum_{k=1}^{n} ka_k z^{k-1} \tag{13.11}$$

$$a_1 = P'(0) = na_n \prod_{k=1}^{n-1} (-e^{is_k}) = na_n(-1)^{\nu_1}, \tag{13.12}$$

where ν_1 is the multiplicity of the zero of P' at 1. Without loss of generality, we may assume that $P'(1) \neq 0$ so that $\nu_1 = 0$ and we have

$$a_1 = na_n. \tag{13.13}$$

In such a case, we have $\phi = 0$ in Eq. (13.8). Consequently, from Eq. (13.9), we have $\alpha_k + i\,\beta_k := b_k = a_k e^{ik\phi} = a_k \in \mathbb{R}$, and so $\beta_k = 0$ for all k. Hence the polynomial T is identically zero. Therefore polynomial P takes the form

$$P(e^{it}) = a_0 + e^{i\frac{n+1}{2}t}\left[S\left(\cos\frac{t}{2}\right) + \frac{2i}{n+1}\frac{d}{dt}S\left(\cos\frac{t}{2}\right)\right]. \tag{13.14}$$

Here, one more special case will occur when n is odd so that $\frac{n+1-2k}{2} \in \mathbb{N}$ for $k \in \mathbb{N}$. In this case, we have

$$S(x) = \frac{1}{2}\sum_{k=1}^{n} \frac{n+1}{n+1-k}\,\alpha_k\,\cos\left(\frac{(n+1-2k)}{2}\arccos x\right), \tag{13.15}$$

which is a polynomial of degree $\frac{n-1}{2}$. Therefore if n is odd, polynomial P becomes

$$P(e^{it}) = a_0 + e^{i\frac{n+1}{2}t}\left[S(\cos t) + \frac{2i}{n+1}\frac{d}{dt}S(\cos t)\right]. \tag{13.16}$$

Moreover, if we differentiate Eq. (13.16) with respect to t, we get

$$\begin{aligned}
\frac{d}{dt}P(e^{it}) &= i\,\frac{n+1}{2}\,e^{i\frac{n+1}{2}t}\left[S(\cos t) + \frac{2i}{n+1}\frac{d}{dt}S(\cos t)\right] \\
&\quad + e^{i\frac{n+1}{2}t}\left[\frac{d}{dt}S(\cos t) + \frac{2i}{n+1}\frac{d^2}{dt^2}S(\cos t)\right] \\
&= i\,\frac{n+1}{2}\,e^{i\frac{n+1}{2}t}S(\cos t) - e^{i\frac{n+1}{2}t}\,\frac{d}{dt}S(\cos t) \\
&\quad + e^{i\frac{n+1}{2}t}\frac{d}{dt}\,S(\cos t) + i\,e^{i\frac{n+1}{2}t}\,\frac{2}{n+1}\frac{d^2}{dt^2}S(\cos t) \\
&= ie^{i\frac{n+1}{2}t}\left[\frac{n+1}{2}S(\cos t) + \frac{2}{n+1}\frac{d^2}{dt^2}S(\cos t)\right].
\end{aligned}$$

Therefore, we have

$$\frac{d}{dt}P(e^{it}) = ie^{i\frac{n+1}{2}t}r(t), \tag{13.17}$$

where $r(t)$ is a real function. From this, we conclude that the polynomial P has the property that the tangent vectors at $P(e^{it})$ turn monotonically with constant speed of $\frac{n+1}{2}$, provided $r(t)$ is not 0 except for the zeroes of P' where the direction is reversed.

13.3 Construction of Extremal Polynomials for a Domain Ω Which is an Intersection of Two Circular Disks

Here we take a domain of intersection of two circular disks of equal radii. The centers of the disks are on the real axis at equidistant but on opposite sides of the origin. Without loss of generality, we may further assume that the centers of the disks are at -1 and 1. So $\Omega := B(-1;r) \cap B(1;r)$, $r(>1) \in \mathbb{R}$ and $\mathcal{P}_n$ the set of all complex polynomials of degree $\leq n$. Also,

$$\mathcal{P}_n^0 := \{P \in \mathcal{P}_n : P(0) = 0, P(\mathbb{D}) \subset \Omega\},$$

and

$$\Omega_n = \bigcup_{P \in \mathcal{P}_n^0} P(\mathbb{D}).$$

If P is an extremal polynomial for Ω_n, then by definition, there exists some $\omega \in (\partial\Omega_n \setminus \partial\Omega)$. Without loss of generality, we may also assume that $\omega = P(1)$. Theorem 1 guarantees that P' has all the zeroes on $\partial\mathbb{D}$. Let $e^{is_j},\ j = 1, 2, 3 \ldots, n-1$, denote those zeroes with $0 < s_1 \leq s_2 \leq \ldots \leq s_{n-1} < 2\pi$, then there exist at least n points of contact $e^{it_k},\ k = 1, 2, \ldots, n$ satisfying the condition

$$0 < t_1 \leq s_1 \leq t_2 \leq \ldots \leq s_{n-1} \leq t_n < 2\pi. \tag{13.18}$$

Since our domain Ω is convex, by Theorem 2, $P(e^{it}) \in (\partial\Omega_n \setminus \partial\Omega)$ for $t_n - 2\pi < t < t_1$; here e^{it_n} and e^{it_1} are those contact points such that $P'(e^{it}) \neq 0$ for $t_n - 2\pi < t < t_1$. Then from Eq. (13.17), we conclude that the tangent vectors at $P(e^{it})$ turn monotonically with constant speed $\frac{n+1}{2}$. This means that the curvature of the extremal curve $P(e^{it}),\ t_n - 2\pi < t < t_1$ remains constant. Therefore from the convexity of the domain Ω, the images of such contact points $p(e^{it_n})$ and $p(e^{it_1})$ must lie one on γ_1 and another on γ_2, where γ_1 and γ_2 are subsets of $\partial\Omega$. More precisely,

$$\gamma_1 := \{z \in \partial B(-1;\ r) \cap B(1;\ r)\} \text{ and } \gamma_2 := \{z \in B(-1, r) \cap \partial B(1;\ r)\}.$$

Since Ω is convex, so is the maximal range Ω_n [6]. From the convexity of Ω_n, the boundary $\partial\Omega_n$ coincides with the curves γ_1 and γ_2 near the points z_1 and z_2, where z_1 and z_2 are the point of intersection of $\partial B(-1;\ r)$ and $\partial B(1;\ r)$.

Since we are dealing only with the extremal polynomial P with real coefficients, two cases arise:

Case I (n odd):

In this case,

$$P(e^{it}) = e^{i\frac{n+1}{2}t}\left[S(\cos t) + \frac{2i}{n+1}\frac{d}{dt}S(\cos t)\right], \tag{13.19}$$

where S is a real symmetric polynomial of degree $\frac{n-1}{2} = q$.

The parametric equation of the curve γ_1 may be taken as

$$\gamma_1(\tau) = -1 + re^{i\tau}, \quad -\frac{\pi}{4} \le \tau \le \frac{\pi}{4}. \tag{13.20}$$

The polynomial P has at least $n = 2q + 1$ points of contact with $\partial\Omega$. Without loss of generality, we may assume that $P'(1) \neq 0$ and so, because of symmetry, it has at least $q + 1$ points of contact with the curve $\gamma_1 \subset \partial\Omega$. Therefore there exist $q + 1$ values of t_k and corresponding τ_k with the property

$$P(e^{it_k}) = \gamma_1(\tau_k), \tag{13.21}$$

and

$$\frac{\frac{d}{dt}P(e^{it_k})}{\gamma_1'(\tau_k)} \in \mathbb{R}. \tag{13.22}$$

Now from Eq. (13.17), we get

$$\frac{\frac{d}{dt}P(e^{it_k})}{\gamma_1'(\tau_k)} = \frac{ie^{(q+1)t_k}R(t)}{ire^{\tau_k}} = e^{i((q+1)t_k-\tau_k)}R_1(t),$$

where R_1 is a real polynomial. Therefore

$$(q+1)t_k - \tau_k = m\pi,$$

and the values of t_k's are given by the equation

$$t_k = \frac{m\pi}{q+1} + \frac{\tau_k}{q+1}, \quad k = 0, 1, \ldots, q. \tag{13.23}$$

Finally, with the help of Eqs. (13.21) and (13.17), we get

$$P(e^{it_k}) = \gamma_1(\tau_k),$$

$$e^{i(k\pi+\tau_k)}\left[S(\cos t_k) + \frac{i}{q+1}\frac{d}{dt}S(\cos t_k)\right] = -1 + re^{i\tau_k},$$

$$S(\cos t_k) + \frac{i}{q+1}\frac{d}{dt}S(\cos t_k) = -e^{-i(k\pi+\tau_k)} + re^{-ik\pi},$$

$$S(\cos t_k) + \frac{i}{q+1}\frac{d}{dt}S(\cos t_k) = (-1)^{k+1}e^{-i\tau_k} + r(-1)^k,$$

$$S(\cos t_k) + \frac{i}{q+1}\frac{d}{dt}S(\cos t) = (-1)^k(r - \cos\tau_k) + i(-1)^{k+1}\sin\tau_k. \tag{13.24}$$

Equating the real parts of Eq. (13.24), we get

$$S(\cos t_k) = (-1)^k (r - \cos \tau_k),\ \ k = 0, 1, 2, \ldots, q,$$

and the polynomial S satisfies the condition

$$S(-\cos t_k) = (-1)^q S(\cos t_k).$$

Case I (a): Suppose q is even. Then we have $q = 2p$, so $n = 4p+1$ and $S(-\cos t_k) = (-1)^{2p} S(\cos t_k) = S(\cos t_k)$. Therefore the points $\pm\cos t_k,\ k = 0, 1, 2, \ldots, p$ are located symmetrically with respect to 0. If we write the polynomial S in Lagrange's interpolation form (because of symmetry, we have considered $2p + 2$ nodes instead of $2p + 1$), it becomes

$$S(x) = \sum_{k=0}^{p} (-1)^k (r - \cos \tau_k) \frac{\omega(x) 2 \cos t_k}{\omega'(\cos t_k)(x^2 - \cos^2 t_k)}, \tag{13.25}$$

where

$$\omega(x) = \prod_{k=0}^{p} \left(x^2 - \cos^2 t_k\right) \ \text{ with } \ t_k = \frac{m\pi}{2p+1} + \frac{\tau_k}{2p+1}.$$

Case I(b): If $q = 2p + 1$ is odd, then $n = 4p + 3$ and the polynomial S is of degree $2p + 1$. So, as usual, we choose $2p + 2$ nodes. The Lagrange's interpolation form for the polynomial S takes the form

$$S(x) = \sum_{k=0}^{p} (-1)^k (r - \cos \tau_k) \frac{\omega(x)\, 2x}{\omega'(\cos t_k)(x^2 - \cos^2 t_k)}, \tag{13.26}$$

where

$$\omega(x) = \prod_{k=0}^{p} \left(x^2 - \cos^2 t_k\right) \ \text{ with } \ t_k = \frac{m\pi}{2p+2} + \frac{\tau_k}{2p+2}.$$

From here, we can say that the polynomial P has two extremal curves near the two points of intersection $\{z_1, z_2\} = \partial B(-1; r) \cap \partial B(1; r)$.

Case II (n is even): In this case also, as in case I, we can construct the maximal polynomial. However, in this case, the polynomial P has the form

$$P(e^{it}) = e^{i\frac{n+1}{2}t} \left[S(\cos\left(\frac{t}{2}\right) + \frac{2i}{n+1} \frac{d}{dt} S(\cos\left(\frac{t}{2}\right)\right], \tag{13.27}$$

on the boundary; where S is a real symmetric polynomial of degree $n - 1$.
The parametric equation of the curve γ_1 may take the form

$$\gamma_1(\tau) = -1 + re^{i\tau},\ \ -\frac{\pi}{4} \le \tau \le \frac{\pi}{4}. \tag{13.28}$$

By our assumption, $P(z)$ has real coefficients, and so, $P(\mathbb{D})$ is symmetric with respect to the real axis and it has at least $n = 2p$ points of contact with the boundary of

Ω_n. Then the curve $\gamma_1(\tau)$ has at least $\frac{n}{2} = p$ points of contact. This means that there exist p points t_k, $k = 0, 1, 2, \ldots, p-1$ and corresponding point τ_k satisfying the conditions

$$P(e^{it_k}) = \gamma_1(\tau_k) \tag{13.29}$$

$$\frac{\frac{d}{dt}P(e^{it_k})}{\gamma_1'(\tau_k)} \in \mathbb{R}. \tag{13.30}$$

Since the P' is a polynomial of odd degree $n-1$, it has at least one real zero. Since we assumed that $P'(1) \neq 0$, so we must have $P'(-1) = 0$ and each extremal polynomial has only one extremal curve. Finally using the Eqs. (13.29) and (13.30), we can conclude that

$$t_k = \frac{2}{n+1}\left[m\pi + \tau_k\right] \tag{13.31}$$

and

$$S\left(\cos\frac{t_k}{2}\right) = (-1)^k\left[r - \cos\tau_k\right]. \tag{13.32}$$

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