Chapter 6
Semi-inner Product: Application to Frame Theory and Numerical Range of Operators

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Abstract This paper deals with the theory of semi-inner product, its generalizations, and applications to frame theory and numerical range of operators. The notion of frames is introduced in classical and generalized semi-inner product spaces. Numerical range of two operators is also studied in semi-inner product spaces.

1 Semi-inner Product

An inner product is a handy and powerful tool to study the geometrical properties of Hilbert space. It is difficult to build Hilbert space-like theory in Banach spaces because of the absence of inner product. A semi-inner product is a generalization of inner product. It was introduced by Lumer [11] for the purpose of extending Hilbert space-like arguments to Banach spaces. It plays a vital role in describing the geometry on Banach spaces. The formal definition of semi-inner product due to Lumer is as follows:

Definition 1.1 (Lumer [11])
Let X be a vector space over the real or complex field F. A semi-inner product $[.,.]$ on X is a real or complex valued functional defined on $X \times X$, which satisfies the following properties:

1. $[x + y, z] = [x, z] + [y, z]$
   $[\lambda x, y] = \lambda[x, y]$ for all $x, y, z \in X$ and $\lambda \in F$;

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2. $[x, x] > 0$ for $x \neq 0$ for all $x \in X$;
3. $|[x, y]|^2 \leq [x, x][y, y]$ for all $x, y \in X$.

The vector space X endowed with $[., .]$ is called a semi-inner product space.

Lumer proved that a semi-inner product space is a normed linear space with the norm $\|x\| = [x, x]^{\frac{1}{2}}$. Every normed linear space can be made into semi-inner product space in many ways. An inner product space is a semi-inner product space where the inner product plays the role of semi-inner product. Conversely, a semi-inner product is an inner product if and only if the norm induced by the semi-inner product obeys the parallelogram law. It was Giles [7] who put forward some decisive structural modifications to the notion of semi-inner product. He imposed the additional homogeneity property in the Definition 1.1 of Lumer semi-inner product. That is, $[x, \lambda y] = \overline{\lambda}[x, y]$ for all $\lambda \in F$, where $\overline{\lambda}$ denotes the conjugate of λ. The imposition of this property adds much convenience without causing any significant restriction. He proved that every normed linear space is a semi-inner product space with the homogeneity property.

Definition 1.2 (Giles [7])
A semi-inner product $[., .]$ is continuous, if it satisfies

$$\lim_{\lambda \to 0} Re[y, x + \lambda y] \to Re[y, x] \text{ for all } x, y \in X \text{ and } \lambda \in \mathbb{R}.$$

The corresponding space X is called continuous semi-inner product space. If the involved limit is uniform, then it is called uniformly continuous semi-inner product space.

Giles also defined the orthogonality relation in semi-inner product space.

Definition 1.3 Let X be a semi-inner product space. For $x, y \in X$, x is said to be normal to y and y is said to be transversal to x if $[y, x] = 0$. A vector $x \in X$ is normal to a subspace S of X and S is transversal to x if x is normal to all vectors $y \in S$.

Definition 1.4 A normed linear space X is said to be G$\hat{a}$teaux differentiable or smooth if for all $x, y \in X$ and real λ, $\lim_{\lambda \to 0} \dfrac{\|x + \lambda y\| - \|x\|}{\lambda}$ exists.

Giles proved that the continuity restriction on the semi-inner product is equivalent to the G$\hat{a}$teaux differentiability of the norm.

To extend Hilbert space-type argument to the theory of the dual of a semi-inner product space, one has to impose more restriction on the semi-inner product to guarantee the existence of normals to closed vector subspaces. For that, one has to restrict the normed space.

Definition 1.5 A normed space X is strictly convex if whenever $\|x\| + \|y\| = \|x + y\|$, where $x, y \neq 0$, then $y = \lambda x$ for some real $\lambda > 0$.

Definition 1.6 A normed space X is uniformly convex if given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that for $x, y \in X$ with $\|x\| = \|y\| = 1$, we have $\frac{\|x+y\|}{2} \leq 1 - \delta(\varepsilon)$ when $\|x - y\| > \varepsilon$.

It is true that uniform convexity implies strict convexity. It is also proved that a semi-inner product space is strictly convex if and only if the equality $[x, y] = \|x\|\|y\|$, where $x, y \neq 0$, implies that $y = \lambda x$ for some real $\lambda > 0$ (see Berkson [2]).

In Hilbert space, the representation theorem for continuous linear functionals sets up a natural correspondence between vectors and continuous linear functionals by means of the inner product. This correspondence was discovered by the famous mathematician Riesz and is known as the Riesz representation theorem. There is a similar representation theorem named as the generalized Riesz representation theorem in a continuous semi-inner product space which is a uniformly convex Banach space.

Theorem 1.1 [Generalized Riesz representation theorem] (Giles [7])
Let X be a continuous semi-inner product space which is uniformly convex and complete in its norm. Let X^* be the dual space of X. Then for every continuous linear functional $f \in X^*$ there exists a unique vector $y \in X$ such that $f(x) = [x, y]$ for all $x \in X$.

Definition 1.7 A uniform semi-inner product space is a uniformly continuous semi-inner product space where the induced normed space is uniformly convex and complete.

Theorem 1.2 (Giles [7])
If X is a uniform semi-inner product space, then the dual space X^* is also a uniform semi-inner product space with respect to the semi-inner product defined by $[f_x, f_y]_{X^*} = [y, x]$, where $[., .]_{X^*}$ denotes the semi-inner product in X^*.

Giles also proved that every finite dimensional strictly convex, continuous semi-inner product space is a uniform semi-inner product space. We have the following examples of uniform semi-inner product spaces:

Example 1.1 The real Banach space $L_p(X, \rho, \mu)$ for $1 < p < \infty$ is a uniform semi-inner product space with the semi-inner product defined as

$$[y, x] = \frac{1}{\|x\|_p^{p-2}} \int_X y|x|^{p-1}\,\mathrm{sgn}(x)\mathrm{d}\mu.$$

Example 1.2 The real sequence space l^p for $1 < p < \infty$ is a uniform semi-inner product space with the semi-inner product defined as

$$[x, y] = \frac{1}{\|y\|_p^{p-2}} \sum_i x_i y_i |y_i|^{p-2}.$$

If the vector space is a uniformly convex smooth Banach space, then there is unique semi-inner product.

The notion of generalized adjoint of a bounded linear operator in a semi-inner product space was introduced by Koehler [10]. Let X be a uniformly convex smooth Banach space. If A is a bounded linear operator from X to itself, then the map $g_y : X \to F(\mathbb{R}\ or\ \mathbb{C})$, defined by $g_y(x) = [Ax, y]$ is a continuous linear functional. By the generalized Riesz representation theorem, it follows that there is a unique vector $A^\dagger(y)$ such that $[Ax, y] = [x, A^\dagger y]$ for all $x \in X$. The operator $A^\dagger$ is called the generalized adjoint of A. This generalized adjoint operator is not usually linear but still it has some interesting properties. The following properties are investigated by Koehler [10] for the generalized adjoint operator:

Theorem 1.3 *Let A and B be two bounded linear functionals on a uniformly convex smooth Banach space X and λ be a scalar. Then,*

1. $(\lambda A)^\dagger = \overline{\lambda} A^\dagger$*;*
2. $(AB)^\dagger = B^\dagger A^\dagger$*;*
3. $A^\dagger$ *is one-to-one if and only if the range of A is dense in X;*
4. *If the norm of X is strongly (Frechet) differentiable, then $A^\dagger$ is continuous.*

1.1 Semi-inner Product Space of Type (p)

Nath [13] generalized the concept of semi-inner product introduced by Lumer [11], by replacing the Schwarz's inequality with the Holder's inequality. The similar type of semi-inner product is called semi-inner product of type (p), and is defined as follows:

Definition 1.8 Let X be a vector space over the field F of real or complex numbers. The functional $[.,.] : X \times X \to F$ satisfying

1. $[x + y, z] = [x, z] + [y, z]$ for all $x, y, z \in X$;
2. $[\lambda x, y] = \lambda[x, y]$ for all $\lambda \in F$ and $x, y \in X$;
3. $[x, x] > 0$ for all $x \neq 0$;
4. $|[x, y]| \leq [x, x]^{\frac{1}{p}} [y, y]^{\frac{p-1}{p}}$ for all $x, y \in X$ and $1 < p < \infty$;
   is called a semi-inner product of type (p) on X. The space equipped with $[.,.]_p$ is called the semi-inner product space of type (p).

The semi-inner product of type (p) induces a norm by setting $\|x\| = [x, x]^{\frac{1}{p}}$. Also, for every normed space we can construct semi-inner product of type (p) in many ways. Pap and Pavlovic [14] discovered the adjoint theorem for maps on semi-inner product spaces of type (p). They proved some properties of the generalized adjoint operator similar to the properties established by Koehler [10] in semi-inner product spaces. El-Sayyad and Khaleelulla [6] introduced the semi-inner product algebras of type (p). They found some interesting results on the generalized adjoint of an operator defined on this space.

Theorem 1.4 (El-Sayyad and Khaleelulla [6])
Let T be a bounded linear operator defined on a semi-inner product space of type (p) and $T^{\dagger}$ be its generalized adjoint. Then,

(i) $\|T\| = \|T^{\dagger}\|^{p-1}$,
(ii) $\|T^{\dagger}T\|^{p-1} = \|T\|^{p}$.

1.2 Generalized Semi-inner Product

With a view to study regularized learning in general Banach spaces, Zhang and Zhang [18] introduced the concept of generalized semi-inner product.

To define generalized semi-inner product, one has to know the notion of gauge function. A gauge function ϕ is a map $\phi : \mathbb{R}^{+} \to \mathbb{R}^{+}$ such that ϕ is continuous, surjective, and strictly increasing with $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = +\infty$. The definition of generalized semi-inner product is as follows:

Definition 1.9 Let X be a vector space over the field F of real or complex numbers. Let ϕ and ψ be two gauge functions with $\phi(t)\psi(t) = t$ for all positive real numbers t. The map $[.,.]_{\phi} : X \times X \to F$ satisfying

1. $[\alpha x + \beta y, z]_{\phi} = \alpha[x, z]_{\phi} + \beta[y, z]_{\phi}$ for all $\alpha, \beta \in F$ and $x, y, z \in X$;
2. $[x, x]_{\phi} > 0$ for all $x \in X \setminus \{0\}$;
3. $|[x, y]_{\phi}| \le \phi([x, x]_{\phi})\ \psi([y, y]_{\phi})$ for all $x, y \in X$ and the equality holds when $x = y$;

is called a generalized semi-inner product on X. The space X equipped with $[.,.]_{\phi}$ is called a generalized semi-inner product space.

When $\phi(t) = t^{\frac{1}{p}}$ and $\psi(t) = t^{\frac{1}{q}}$, $p, q \in (1, +\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, the generalized semi-inner product reduces to the semi-inner product of type (p) introduced by Nath [13]. Again if $\phi(t) = t^{\frac{1}{2}}$ and $\psi(t) = t^{\frac{1}{2}}$ then the generalized semi-inner product reduces to the classical semi-inner product introduced by Lumer [11]. Zhang and Zhang [18] proved that if $[.,.]_{\phi}$ is a generalized semi-inner product on a vector space X then $\|x\| = \Phi([x, x]_{\phi})$ defines a norm on X. Conversely, if Φ is surjective onto $\mathbb{R}^{+}$ then for any normed space X, there exists a generalized semi-inner product on it such that $\|x\| = \Phi([x, x]_{\phi})$. The Riesz representation of continuous linear functionals is also true in this generalized semi-inner product space.

2 Bessel Sequence and Frame in Semi-inner Product Space

Frames are redundant signal representations having a wide range of applications in signal and image processing, wavelet analysis, data transmission with erasures, wireless communication, data transmission, and many more new applications arising every year. In this section, we define Bessel sequence and frame in Banach spaces by using semi-inner product. The notion of frames was introduced by Duffin and Schaeffer [5] in 1952 while studying the nonharmonic Fourier series. Frames in L^p spaces and other Banach spaces are effective tools for modeling a variety of natural signals and images. There is a plethora of literature available for frames in Banach spaces. For classical frame theory in Banach spaces, one may refer to Casazza and Christensen [3], Christensen and Heil [4], Gröchenig [8], Kaushik [9], and Stoeva [15]. To smoothen the study of frames in Banach spaces, Zhang and Zhang [19] defined this notion by taking the help of semi-inner product.

Here we assume that X is a uniformly convex smooth Banach space. In particular, we concentrate on the spaces l^p and L^p, where $1 < p < \infty$. It is seen that those spaces are semi-inner product spaces with uniquely defined semi-inner product (Giles [7]). Our definition is completely different from those Banach space frames available in the literature. In the remainder of this section, we assume that X is a real uniformly convex smooth Banach space with norm $\|.\|_p$ and semi-inner product $[.,.]$.

Definition 2.1 A set of elements $f = \{f_i\}_{i=1}^{\infty} \subseteq X$ is called a Bessel sequence if there exists a constant $B > 0$, such that

$$\sum_{i=1}^{\infty} |[f_i, x]|^q \leq B(\|x\|_p)^q, \ \forall x \in X,$$

where $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. The number B is called Bessel bound.

Definition 2.2 A sequence of elements $\{f_i\}_{i=1}^{\infty}$ in X is called a frame if there exist positive constants A and B such that

$$A(\|x\|_p)^q \leq \sum_{i=1}^{\infty} |[f_i, x]|^q \leq B(\|x\|_p)^q, \ \ \forall x \in X,$$

where $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. A and B are called lower and upper frame bound, respectively.

If $A = B$ then the frame is called a tight frame, and if $A = B = 1$ then the frame is called a Parseval frame. A frame is called a normalized frame if each frame element has unit norm. We have proved the following necessary and sufficient condition for a sequence of elements in X to be a Bessel sequence.

Theorem 2.1 *Let $f = \{f_i\}_{i=1}^{\infty}$ be a sequence in X. Then, the sequence f is a Bessel sequence if and only if $T : \{c_i\}_{i=1}^{\infty} \to \sum_{i=1}^{\infty} c_i f_i \frac{|[f_i, x]|^{q-2}}{\|\{[f_i, x]\}\|^{q-2}}$ is a well-defined and bounded operator from l^q into X.*

Our main focus is on Parseval frame and tight frame because the reconstruction formula naturally holds true without any assumptions. The following two results establish the reconstruction formulae for Parseval frames and tight frames:

Theorem 2.2 *A set of elements $\{f_i\}_{i=1}^{\infty}$ is a Parseval frame for X if and only if*

$$x = \sum_{i=1}^{\infty} \frac{|[f_i, x]|^{q-2}}{\|\{[f_i, x]\}\|^{q-2}} [f_i, x] f_i, \quad \forall x \in X. \tag{1}$$

Theorem 2.3 *A set of elements $\{f_i\}_{i=1}^{\infty}$ is a tight frame with bound A for X if and only if*

$$x = \sum_{i=1}^{\infty} \frac{1}{A^{\frac{2}{q}}} \frac{|[f_i, x]|^{q-2}}{\|\{[f_i, x]\}\|^{q-2}} [f_i, x] f_i \ \ \forall x \in X. \tag{2}$$

Definition 2.3 A tight frame is said to be a normalized tight frame if each of its element has unit norm.

Definition 2.4 An operator T on X is said to be a co-isometry if its generalized adjoint is an isometry.

The following theorem tells about the invariance of frame under a co-isometry operator.

Theorem 2.4 (a) *Let $\{f_i\}_{i=1}^{\infty}$ be a frame for the space X and T be a co-isometry, then $\{Tf_i\}_{i=1}^{\infty}$ is a frame. Moreover, $\{Tf_i\}_{i=1}^{\infty}$ is a normalized tight frame if $\{f_i\}_{i=1}^{\infty}$ is a normalized tight frame.*

(b) *Let $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ be Parseval frames for X and T be a bounded linear operator defined by $Tg_i = f_i$. Then T is a co-isometry.*

3 Bessel Sequence and Frame in Generalized Semi-inner Product Space

Let X be a generalized semi-inner product space with generalized semi-inner product $[.,.]_\phi$ and norm $\|.\|_X$. Let X_d be an associated BK-space with norm $\|.\|_{X_d}$. Suppose that X^* and X_d^* are the dual spaces of X and X_d, respectively. We define X_d-Bessel sequence and X_d^*-Bessel sequence in a generalized semi-inner product space X, and prove that the space of all X_d^*-Bessel sequences form a Banach space.

Definition 3.1 A sequence of elements $\{f_j\} \subseteq X$ is called an X_d-Bessel sequence in X if $\{[x, f_j]_\phi\} \in X_d$, and there exists a positive real constant B such that

$$\|\{[x, f_j]_\phi\}\|_{X_d} \leq B\ \phi([x, x]_\phi), \ \forall x \in X,$$

where $\phi : (0, \infty) \to (0, \infty)$ is a continuous, nondecreasing function with $\phi(0) = 0$ and $\phi(t) \to \infty$ as $t \to \infty$.

Definition 3.2 Let $\{f_j\} \subseteq X$. Then $\{f_j^*\} \subseteq X^*$ is an X_d^*-Bessel sequence for X^* if $\{[f_j, x]_\phi\} \in X_d^*$, and there exists a positive real constant B such that

$$\|\{[f_j, x]_\phi\}\|_{X_d^*} \leq B\ \phi([x, x]_\phi),\ \forall x \in X.$$

We define X_d-frame and X_d^*-frame in this prospective.

Definition 3.3 Let X be a generalized semi-inner product space with compatible generalized semi-inner product $[.,.]_\phi$. The sequence $\{f_j\} \subseteq X$ is said to be an X_d-frame for X if $\{[f, f_j]_\phi\} \in X_d$ for all $x \in X$, and there exist two positive constants A, B such that

$$A\ \phi([f, f]_\phi) \leq \|\{[f, f_j]_\phi\}\|_{X_d} \leq B\ \phi([f, f]_\phi),\ \ \forall f \in X. \tag{3}$$

Definition 3.4 Let $\{f_j\} \subseteq X$. Then $\{f_j^*\}$ is an X_d^*-frame for X^* if $\{[f_j, f]_\phi\} \in X_d^*$ for all $f \in X$, and there exist two positive constants A, B such that

$$A\ \phi([f, f]_\phi) \leq \|\{[f_j, f]_\phi\}\|_{X_d^*} \leq B\ \phi([f, f]_\phi),\ \ \forall f \in X. \tag{4}$$

In this section, we also define Riesz basis in a generalized semi-inner product space. Likewise X_d-frame and X_d^*-frame, we have X_d-Riesz basis and X_d^*-Riesz basis.

Definition 3.5 A sequence of elements $\{f_j\} \subseteq X$ is an X_d-Riesz basis for X if $\overline{span\{f_j\}} = X$, $\sum_{j\in I} c_j f_j$ converges in X for all $c \in X_d$, and there exist positive finite real numbers $A,\ B$ with $A \leq B$, such that

$$A\phi([c, c]_{X_d}) \leq \Big\| \sum_{j\in I} c_j f_j \Big\|_X \leq B\phi([c, c]_{X_d})\ \text{ for all } c \in X_d. \tag{5}$$

Definition 3.6 A sequence of elements $\{f_j^*\} \subseteq X^*$ is an X_d^*-Riesz basis for X^* if $\overline{span\{f_j^*\}} = X^*$, $\sum_{j\in I} d_j f_j^*$ converges in X^* for all $d \in X_d^*$, and there exist positive finite real numbers $A,\ B$ with $A \leq B$, such that

$$A\phi([d, d]_{X_d^*}) \leq \Big\| \sum_{j\in I} d_j f_j^* \Big\|_{X^*} \leq B\phi([d, d]_{X_d^*})\ \text{ for all } d \in X_d^*. \tag{6}$$

We can show that Riesz basis automatically generates a frame for the dual space.

4 Numerical Range of Two Operators in Semi-inner Product Spaces

Quadratic forms are quite useful in linear algebra. The numerical range is a natural extension of quadratic forms in vector spaces. Like the spectrum, the numerical range of a linear operator is a subset of the scalar field. It is structured in such a way that it is related to both algebraic as well as norm structures of the operator. Whereas the spectrum of an operator is related only to algebraic structure of the operator. One can extract much information about the operator through numerical range.

Lumer [11] discussed the numerical range for a linear operator in a Banach space by using semi-inner product. Williams [17] studied the spectra of products of two linear operators and their numerical ranges. To study the generalized eigenvalue problem $Tx = \lambda Ax$, Amelin [1] introduced the concept of numerical range for two linear operators in Hilbert space. The numerical range of two nonlinear operators in a semi-inner product space was defined by Nanda [12].

4.1 Numerical Range of Two Linear Operators

Let X be a uniformly convex smooth Banach space equipped with norm $\|.\|$ and semi-inner product $[.,.]$. Let T and A be two linear operators defined on X.

Definition 4.1 The numerical range $W(T, A)$ of the two linear operators T and A is defined as $W(T, A) := \{[Tx, Ax] : \|Ax\| = 1,\ \ x \in \mathrm{D}(T) \cap \mathrm{D}(A)\}$, where $D(T)$ and $D(A)$ are denoted as the domain of T and the domain of A, respectively. The numerical radius $w(T, A)$ is defined as $w(T, A) = \sup\{|\lambda| \ :\ \lambda \in W(T, A)\}$.

Definition 4.2 The coupled numerical range $W_A(T)$ of T with respect to A is defined as

$$W_A(T) := \left\{\frac{[ATx, x]}{[Ax, x]} : \|x\| = 1,\ [Ax, x] \neq 0\right\}. \tag{7}$$

In the above definition, we have assumed that $\mathrm{Dom}(A) \cap \mathrm{Range}(T) \neq \phi$. We can easily prove the following properties of the numerical range of two linear operators:

Theorem 4.1 *Let T_1, T_2, T, A be linear operators and α, μ, λ be scalars. Then,*

(i) $W(T_1 + T_2, A) \subseteq W(T_1, A) + W(T_2, A)$,
(ii) $W(\alpha T, A) = \alpha W(T, A)$,
(iii) $W(T, \mu A) = \overline{\mu} W(T, A)$,
(iv) $W(T - \lambda A, A) = W(T, A) - \{\lambda\}$,
(v) $w(T_1 + T_2, A) \leq w(T_1, A) + w(T_2, A)$,
(vi) $w(\lambda T, A) = |\lambda|\, w(T, A)$.

Theorem 4.2 *Let T_1, T_2, T, A be linear operators and α be a scalar. Then,*

(i) $W_A(T_1 + T_2) \subseteq W_A(T_1) + W_A(T_2)$,
(ii) $W_A(\alpha T) = \alpha W_A(T)$,
(iii) $W_{\alpha A}(T) = W_A(T)$.

Proof (i) Let $x, y \in \text{Dom}(T_1) \cap \text{Dom}(T_2)$. We assume that $\text{Dom}(A) \cap \text{Range}(T_1) \neq \phi$ and $\text{Dom}(A) \cap \text{Range}(T_2) \neq \phi$.
Then

$$\frac{[A(T_1 + T_2)x, x]}{[Ax, x]} = \frac{[AT_1x + AT_2x, x]}{[Ax, x]} = \frac{[AT_1x, x]}{[Ax, x]} + \frac{[AT_2x, x]}{[Ax, x]}.$$

Therefore $W_A(T_1 + T_2) \subseteq W_A(T_1) + W_A(T_2)$.
(ii) If $\text{Dom}(A) \cap \text{Range}(T) \neq \phi$, then

$$\frac{[A(\alpha T)x, x]}{[Ax, x]} = \frac{[\alpha ATx, x]}{[Ax, x]} = \alpha \frac{[ATx, x]}{[Ax, x]}.$$

Hence $W_A(\alpha T) = \alpha W_A(T)$.
(iii) If $\text{Dom}(A) \cap \text{Range}(T) \neq \phi$, then

$$\frac{[(\alpha A)Tx, x]}{[\alpha Ax, x]} = \frac{\alpha[ATx, x]}{\alpha[Ax, x]} = \frac{[ATx, x]}{[Ax, x]}.$$

As a result $W_{\alpha A}(T) = W_A(T)$.

Definition 4.3 The spectrum $\sigma(T, A)$ of the two linear operators T and A is defined as

$$\sigma(T, A) := \{\lambda \in \mathbb{C} : (T - \lambda A) \text{ is not invertible}\}. \tag{8}$$

The spectral radius $r(T, A)$ is defined as $r(T, A) = \sup\{|\lambda| \ : \ \lambda \in \sigma(T, A)\}$.

Definition 4.4 The eigen spectrum or point spectrum $e(T, A)$ of two linear operators T and A is defined as

$$e(T, A) := \{\lambda \in \mathbb{C} : Tx = \lambda Ax \ \text{ for } x \neq 0\}. \tag{9}$$

Definition 4.5 The approximate point spectrum $\pi(T, A)$ of two linear operators T and A is defined as
$\pi(T, A) := \{\lambda \in \mathbb{C}$ such that there exists a sequence $x_n \in X$ with $\|Ax_n\| = 1$ and $\|Tx_n - \lambda Ax_n\| \to 0$ as $n \to \infty\}$.

Definition 4.6 The compression spectrum $\sigma_0(T, A)$ of two linear operators T and A is defined as

$$\sigma_0(T, A) := \{\lambda \in \mathbb{C} : \text{Range}(T - \lambda A) \text{ is not dense in } X\}. \tag{10}$$

One can establish the inclusion relations among spectrum, eigen spectrum, compression spectrum, approximate point spectrum, and numerical range of two linear operators.

4.2 Numerical Range of Two Nonlinear Operators

Let X be a normed space, and T be an operator defined on X. Then T is said to be Lipschitz if there exists a constant $M > 0$ such that $\|Tx - Ty\| \leq M\|x - y\|$ for all $x, y \in X$. Let $Lip(X)$ denote the set of all Lipschitz operators on X. Suppose that $T \in Lip(X)$, and $x, y \in \mathrm{Dom}(T)$ with $x \neq y$. The generalized Lipschitz norm $\|T\|_L$ of a nonlinear operator T on a Banach space X is defined as $\|T\|_L = \|T\| + \|T\|_l$, where $\|T\| = \sup_x \dfrac{\|Tx\|}{\|x\|}$ and $\|T\|_l = \sup_{x \neq y} \dfrac{\|Tx - Ty\|}{\|x - y\|}$. If there exists a finite constant M such that $\|T\|_L < M$, then the operator T is called the generalized Lipschitz operator (see Verma [16]). Let $G_L(X)$ be the class of all generalized Lipschitz operators.

Definition 4.7 The numerical range $V_L(T, A)$ of two nonlinear operators T and A is defined as

$$V_L(T, A) := \left\{ \frac{[Tx, Ax] + [Tx - Ty, Ax - Ay]}{\|Ax\|^2 + \|Ax - Ay\|^2} \ : \ x, y \in D(T) \cap D(A),\ x \neq y \right\}, \tag{11}$$

where $D(T)$ and $D(A)$ are the domains of the operators T and A, respectively. The numerical radius $w_L(T, A)$ is defined as $w_L(T, A) = \{\sup |\lambda| \ : \ \lambda \in V_L(T, A)\}$.

We have the following elementary properties for the numerical range of two nonlinear operators.

Theorem 4.3 *Let X be a Banach space over $\mathbb{C}$. If T, A, T_1, T_2 be nonlinear operators defined on X and λ, μ be scalars, then*

(i) $V_L(\lambda T, A) = \lambda V_L(T, A)$,
(ii) $V_L(T, \mu A) = \frac{1}{\mu} V_L(T, A)$,
(iii) $V_L(T_1 + T_2, A) \subseteq V_L(T_1, A) + V_L(T_2, A)$,
(iv) $V_L(T - \lambda A, A) = V_L(T, A) - \{\lambda\}$.

Proof (i) We see that for any $x, y \in X$,

$$\frac{[\lambda Tx, Ax] + [\lambda Tx - \lambda Ty, Ax - Ay]}{\|Ax\|^2 + \|Ax - Ay\|^2} = \lambda \frac{[Tx, Ax] + [Tx - Ty, Ax - Ay]}{\|Ax\|^2 + \|Ax - Ay\|^2}.$$

Hence $V_L(\lambda T, A) = \lambda V_L(T, A)$.

(ii) For any $x, y \in X$,

$$\begin{aligned}
\frac{[Tx, \mu Ax] + [Tx - Ty, \mu Ax - \mu Ay]}{\|\mu Ax\|^2 + \|\mu Ax - \mu Ay\|^2} &= \frac{\overline{\mu}[Tx, Ax] + \overline{\mu}[Tx - Ty, \mu Ax - \mu Ay]}{|\mu|^2(\|Ax\|^2 + \|Ax - Ay\|^2)} \\
&= \frac{\overline{\mu}}{|\mu|^2} \frac{[Tx, Ax] + [Tx - Ty, Ax - Ay]}{\|Ax\|^2 + \|Ax - Ay\|^2} \\
&= \frac{1}{\mu} \frac{[Tx, Ax] + [Tx - Ty, Ax - Ay]}{\|Ax\|^2 + \|Ax - Ay\|^2}.
\end{aligned}$$

Hence $V_L(T, \mu A) = \frac{1}{\mu} V_L(T, A)$.

(iii) Let $x, y \in Dom(T_1) \cap Dom(T_2)$.
Then

$$\begin{aligned}
&\frac{[(T_1+T_2)x, Ax] + [(T_1+T_2)x - (T_1+T_2)y, Ax - Ay]}{\|Ax\|^2 + \|Ax - Ay\|^2} \\
&= \frac{[T_1x, Ax] + [T_2x, Ax] + [T_1x - T_1y, Ax - Ay] + [T_2x - T_2y, Ax - Ay]}{\|Ax\|^2 + \|Ax - Ay\|^2} \\
&= \frac{[T_1x, Ax] + [T_1x - T_1y, Ax - Ay]}{\|Ax\|^2 + \|Ax - Ay\|^2} + \frac{[T_2x, Ax] + [T_2x - T_2y, Ax - Ay]}{\|Ax\|^2 + \|Ax - Ay\|^2}.
\end{aligned}$$

Therefore $V_L(T_1 + T_2, A) \subseteq V_L(T_1, A) + V_L(T_2, A)$. Thus (iii) is proved.

(iv) For any $x, y \in X$,

$$\begin{aligned}
&\frac{[(T-\lambda A)x, Ax] + [(T-\lambda A)x - (T-\lambda A)y, Ax - Ay]}{\|Ax\|^2 + \|Ax - Ay\|^2} \\
&= \frac{[Tx, Ax] - \lambda\|Ax\|^2 + [Tx - Ty, Ax - Ay] - \lambda\|Ax - Ay\|^2}{\|Ax\|^2 + \|Ax - Ay\|^2} \\
&= \frac{[Tx, Ax] + [Tx - Ty, Ax - Ay]}{\|Ax\|^2 + \|Ax - Ay\|^2} - \lambda.
\end{aligned}$$

This implies that $V_L(T - \lambda A, A) = V_L(T, A) - \{\lambda\}$.

We give example of two nonlinear operators in a semi-inner product space and compute their numerical range and numerical radius.

Example 4.1 Consider the real sequence space l^p, $1 < p < \infty$.
Let $x = (x_1, x_2, ...)$, $y = (y_1, y_2, ...) \in l^p$. Consider the two nonlinear operators $T, A : l^p \to l^p$ defined by $Tx = (\|x\|, x_1, x_2, ...)$ and $Ax = (\|x\|, 0, 0, ...)$. The unique semi-inner product on the real sequence space l^p is defined as

$$[x, y] = \frac{1}{\|y\|^{p-2}} \sum_{n=1}^{\infty} |y_n|^{p-2} y_n x_n, \ \ \forall x = \{x_n\}, y = \{y_n\} \in l^p.$$

One can easily compute that $\|Ax\| = \|x\|$, $\|Ax - Ay\| = |\|x\| - \|y\||$,
$[Tx, Ax] = \|x\|^2$ and

$$\begin{aligned}
[Tx - Ty, Ax - Ay] &= \frac{1}{\|Ax - Ay\|^{p-2}} \{|(\|x\| - \|y\|)|^{p-2}(\|x\| - \|y\|)^2\} \\
&= \frac{1}{|(\|x\| - \|y\|)|^{p-2}} |(\|x\| - \|y\|)|^p = |(\|x\| - \|y\|)|^2.
\end{aligned}$$

One can calculate that

$$\frac{[Tx, Ax] + [Tx - Ty, Ax - Ay]}{\|Ax\|^2 + \|Ax - Ay\|^2} = \frac{\|x\|^2 + |(\|x\| - \|y\|)|^2}{\|x\|^2 + |(\|x\| - \|y\|)|^2} = 1, \ \forall x, y \in l^p.$$

Therefore $V_L(T, A) = \{1\}$ and $w_L(T, A) = 1$.

5 Conclusion

Researchers usually take the help of bounded linear functionals to establish Hilbert space-like theory in Banach spaces. Without using arbitrary bounded linear functionals, we have taken the help of semi-inner product to study frames and numerical range of operators in Banach spaces. The main benefits of this approach are threefold. It is computationally easy. We can avoid the inconvenience of using arbitrary bounded linear functionals. It helps in constructing concrete examples.

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