Chapter 19
Some Geometric Properties of Generalized Cesàro–Musielak–Orlicz Sequence Spaces

Atanu Manna and P. D. Srivastava

Abstract A generalized Cesàro–Musielak–Orlicz sequence space $Ces_{\Phi}(q)$ equipped with the Luxemberg norm is introduced. It is proved that $Ces_{\Phi}(q)$ is a Banach space and also criteria for the coordinatewise uniformly Kadec–Klee property and the uniform Opial property are obtained.

Keywords Musielak–Orlicz function · Riesz weighted mean · Luxemberg norm · Coordinatewise Kadec–Klee property · Uniform Opial property

Mathematics Subject Classification (2010): 46B20, 46B45, 46A45, 46A80, 46E30

1 Introduction

In fixed point theory, geometrical properties of Banach space, such as Kadec–Klee property, Opial property, and their several generalizations play fundamental role. In particular, the Opial property of a Banach space has its applications in differential equations and integral equations, etc. On the other hand the Kadec–Klee property has several applications in Ergodic theory and many other branches of analysis [22].

The authors are very much thankful to the anonymous reviewers for their valuable comments, which improved the presentation of the paper. First author is grateful to CSIR, New Delhi, Government of India for the research fellowship with award no. 09/081(0988)/2009-EMR-I during this work.

A. Manna (✉) · P. D. Srivastava
Indian Institute of Technology Kharagpur, Kharagpur 721302, India
e-mail: atanumanna@maths.iitkgp.ernet.in

P. D. Srivastava
e-mail: pds@maths.iitkgp.ernet.in

R. N. Mohapatra et al. (eds.), *Mathematics and Computing 2013*,
Springer Proceedings in Mathematics & Statistics 91,
DOI: 10.1007/978-81-322-1952-1_19,

In recent times, the theory of Cesàro–Orlicz sequence spaces and Musielak–Orlicz sequence spaces and their geometric properties has been studied extensively. Some topological properties like absolute continuity, order continuity, separability, completeness, and relations between norm and modular as well as some geometrical properties like Fatou property, monotonicity, Kadec–Klee property, uniform Opial property, rotundity, local rotundity, property-β etc. are studied in [2–4, 6, 8, 13, 20, 21]. Recently, Khan (see [15, 16]) introduced Riesz–Musielak–Orlicz sequence spaces and studied some geometric properties of this space. Quite recently, Mongkolkeha, and Kumam [17] studied (H)-property and uniform Opial property of generalized Cesàro sequence spaces. Some topological properties of sequence spaces defined by using Orlicz function are also studied in [1, 5, 25]. This motivated us to introduce generalized Cesàro–Musielak–Orlicz sequence spaces, which include the well known Cesàro, generalized Cesàro [24], Cesàro-Orlicz, Cesàro–Musielak–Orlicz sequence spaces etc. in particular cases. In this paper, we have made an attempt to study some of the geometric properties in generalized Cesàro–Musielak–Orlicz sequence spaces.

Throughout the paper, we denote $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}^+$ as the set of natural numbers, real numbers, and nonnegative real numbers, respectively. Let $(X, \|.\|)$ be a Banach space and l^0 be the space of all real sequences $x = (x(i))_{i=1}^{\infty}$. Let $S(X)$ and $B(X)$ denote the unit sphere and closed unit ball, respectively. A sequence $(x_l) \subset X$ is said to be ε-separated sequence if separation of the sequence (x_l) denoted by $sep(x_l) = \inf\{\|x_l - x_m\| : l \neq m\} > \varepsilon$ for some $\varepsilon > 0$ [11].

A Banach space X is said to have the *Kadec–Klee property*, denoted by (H), if weakly convergent sequence on the unit sphere is strongly convergent, i.e., convergent in norm [12]. A Banach space X is said to possess *coordinatewise Kadec–Klee property*, denoted by (H_c) [7], if $x \in X$ and every sequence $(x_l) \subset X$ such that

$$\|x_l\| \to \|x\| \text{ and } x_l(i) \to x(i) \text{ for each } i, \text{ then } \|x_l - x\| \to 0.$$

It is known that $X \in (H_c)$ implies $X \in (H)$, because weak convergence in X implies the coordinatewise convergence. A Banach space X has the *coordinatewise uniformly Kadec–Klee property*, denoted by (UKK_c) [27], if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$(x_l) \subset B(X), sep(x_l) \geq \varepsilon, \|x_l\| \to \|x\| \text{ and } x_l(i) \to x(i) \text{ for each } i \text{ implies } \|x\| \leq 1 - \delta.$$

It is known that the property (UKK_c) implies property (H_c).

A Banach space X is said to have the *Opial property* [23] if for every weakly null sequence $(x_l) \subset X$ and every nonzero $x \in X$, we have

$$\liminf_{l\to\infty} \|x_l\| < \liminf_{l\to\infty} \|x_l + x\|.$$

A Banach space X is said to have the *uniform Opial property* [23] if for each $\varepsilon > 0$ there exists $\mu > 0$ such that for any weakly null sequence (x_l) in $S(X)$ and $x \in X$ with $\|x\| \geq \varepsilon$ the following inequality hold:

$$1 + \mu \le \liminf_{l \to \infty} \|x_l + x\|.$$

In any Banach space X an *Opial property* is important because it ensures that X has a weak fixed point property [9]. Opial in [19] has shown that the space $L_p[0, 2\pi]$ $(p \ne 2, 1 < p < \infty)$ does not have this property, but the Lebesgue sequence space $l_p(1 < p < \infty)$ has.

A map $\varphi : \mathbb{R} \to [0, \infty]$ is said to be an Orlicz function if it is an even, convex, left continuous on $[0, \infty)$, $\varphi(0) = 0$, not identically zero and $\varphi(u) \to \infty$ as $u \to \infty$. A sequence $\Phi = (\varphi_n)$ of Orlicz functions φ_n is called Musielak–Orlicz function [18]. For a Musielak–Orlicz function Φ, the complementary function $\Psi = (\psi_n)$ of Φ is defined in the sense of Young as

$$\psi_n(u) = \sup_{v \ge 0}\{\, |u|v - \varphi_n(v)\} \quad \text{for all } u \in \mathbb{R} \text{ and } n \in \mathbb{N}.$$

Given any Musielak–Orlicz function Φ and $x = (x(n))_{n=1}^{\infty} \in l^0$, a convex modular $I_\Phi : l^0 \to [0, \infty]$ is defined by

$$I_\Phi(x) = \sum_{n=1}^{\infty} \varphi_n\Big(|x(n)|\Big) \text{ and}$$

the linear space $l_\Phi = \{x \in l^0 : I_\Phi(rx) < \infty \text{ for some } r > 0\}$ is called Musielak–Orlicz sequence space. The space l_Φ equipped with functional $|||x|||_\Phi^L$ defined by

$$|||x|||_\Phi^L = \inf\left\{r > 0 : I_\Phi\left(\frac{x}{r}\right) \le 1\right\}$$

becomes a Banach space. This functional $|||x|||_\Phi^L$ is called Luxemberg norm and the corresponding Musielak–Orlicz sequence space is denoted by l_Φ^L. For the details about Musielak–Orlicz sequence spaces and their geometric properties we refer to the articles [3, 10, 13, 18]. The subspace of l_Φ defined as

$$\left\{x = (x(n)) \in l^0 : \forall r > 0\ \exists n_r \in \mathbb{N} \text{ such that } \sum_{n=n_r}^{\infty} \varphi_n\Big(r|x(n)|\Big) < \infty\right\},$$

equipped with the Luxemberg norm induced from l_Φ is denoted by h_Φ^L.

A Musielak–Orlicz function Φ is said to satisfy the δ_2^0-condition denoted by $\Phi \in \delta_2^0$ if there are positive constants a, K, a natural m and a sequence (c_n) of positive numbers such that $(c_n)_{n=m}^{\infty} \in l_1$ and the inequality

$$\varphi_n(2u) \le K\varphi_n(u) + c_n \tag{1}$$

holds for every $n \in \mathbb{N}$ whenever $\varphi_n(u) \leq a$. If a Musielak–Orlicz function Φ satisfies δ_2^0-condition with $m = 1$, then Φ is said to satisfy δ_2-condition [10, 18].

For any Musielak–Orlicz function Φ, h_Φ coincides with l_Φ if and only if Φ satisfies δ_2^0-condition [10].

A Musielak–Orlicz function $\Phi = (\varphi_n)_{n=1}^{\infty}$ satisfies the condition $(*)$ [13] if for any $\varepsilon \in (0, 1)$ there is a $\delta > 0$ such that

$$\varphi_n(u) < 1 - \varepsilon \text{ implies } \varphi_n((1+\delta)u) \leq 1 \text{ for all } n \in \mathbb{N} \text{ and } u \geq 0. \tag{2}$$

A Musielak–Orlicz function Φ is to said to vanishes only at zero, which is denoted by $\Phi > 0$ if $\varphi_n(u) > 0$ for any $n \in \mathbb{N}$ and $u > 0$.

2 Class $Ces_\Phi(q)$

Let $q = (q_n)_{n=1}^{\infty}$ be a sequence of real numbers with $q_k \geq 1$ for $k \in \mathbb{N}$, and $Q_n = \sum_{k=1}^{n} q_k$. We introduce the *Riesz weighted mean* map R^q on l^0 as $R^q : l^0 \to [0, \infty)$ such that $x \to R^q x$, where

$$R^q x = (R^q x(n))_{n=1}^{\infty}, \text{ with } R^q x(n) = \frac{1}{Q_n} \sum_{k=1}^{n} q_k |x(k)| \text{ for each } n = 1, 2, \ldots$$
$$\text{and } x \in l^0.$$

Using this *Riesz weighted mean* map and a Musielak–Orlicz function $\Phi = (\varphi_n)$, we define on l^0 a functional $\sigma_\Phi(x)$ by

$$\sigma_\Phi(x) = I_\Phi(R^q x) = \sum_{n=1}^{\infty} \varphi_n\Big(\frac{1}{Q_n} \sum_{k=1}^{n} q_k |x(k)|\Big).$$

Since Φ is convex, so it is easy to verify that $\sigma_\Phi(x)$ is a convex modular on l^0(for definition see [18]), i.e., it satisfies $\sigma_\Phi(x) = 0$ if and only if $x = 0$, $\sigma_\Phi(-x) = \sigma_\Phi(x)$, $\sigma_\Phi(\gamma x + \delta y) \leq \gamma \sigma_\Phi(x) + \delta \sigma_\Phi(y)$ whenever $x, y \in l^0$ and $\gamma, \delta \geq 0$ with $\gamma + \delta = 1$.

We now introduce the space $Ces_\Phi(q)$ as follows:

$$Ces_\Phi(q) = \{x \in l^0 : R^q x \in l_\Phi\} = \{x \in l^0 : \sigma_\Phi(rx) < \infty \text{ for some } r > 0\}.$$

Clearly, it is a linear space and also forms a normed linear space under the norm $\|x\|_\Phi^L = |||R^q x|||_\Phi^L$ introduced with the help of the norm on l_Φ. We call $Ces_\Phi(q)$ as the generalized Cesàro–Musielak–Orlicz sequence space.

The generalized class $Ces_\Phi(q)$ include the following classes in particular cases:

(i) When $q_n = 1$, $n = 1, 2, \ldots$, the $Ces_\Phi(q)$ reduces to the Cesàro–Musielak –Orlicz sequence space ces_Φ studied by Wangkeeree [26], where

$$ces_\Phi = \left\{x \in l^0 : \sum_{n=1}^{\infty} \varphi_n\left(\frac{r}{n}\sum_{k=1}^{n}|x(k)|\right) < \infty \text{ for some } r > 0\right\},$$

(ii) For $\varphi_n = \varphi$, $\forall n$ the ces_Φ becomes well-known Cesàro–Orlicz sequence space ces_φ studied recently by Cui et al. [2], Foralewski et al. [6], Petrot and Suantai [20],

(iii) For $\varphi_n(x) = |x|^{p_n}$, $p_n \geq 1$ $\forall n$ the $Ces_\Phi(q)$ reduces to the sequence space $Ces_{(p)}(q)$ studied by Mongkolkeha and Kumam [17] and when $\varphi_n(x) = |x|^{p_n}$ with $p_n = p \geq 1$ $\forall n$ then $Ces_\Phi(q)$ reduces to the sequence space $Ces_p(q)$ studied by Khan [14].

We consider the subspace $(Ces_\Phi^L(q))_a$ of $Ces_\Phi(q)$ as

$$(Ces_\Phi(q))_a = \left\{x \in Ces_\Phi(q) : \forall r > 0 \ \exists n_r \text{ such that} \sum_{n=n_r}^{\infty} \varphi_n\left(\frac{r}{Q_n}\sum_{k=1}^{n} q_k|x(k)|\right) < \infty\right\}.$$

In this article, we have introduced the generalized Cesàro–Musielak–Orlicz sequence space and have established the completeness property of the space and also obtained criteria for some geometric properties like coordinatewise Uniform Kadec–Klee property, uniform Opial property with respect to the Luxemberg norm.

Notations:

For any $x \in l^0$ and $i \in \mathbb{N}$, throughout the paper we use the following notations:

$x|_i = (x(1), x(2), x(3), \ldots, x(i), 0, 0, \ldots)$, called the truncation of x at i,

$x|_{\mathbb{N}-i} = (0, 0, 0, \ldots, 0, x(i+1), x(i+2), \ldots)$,

$x|_I = \{x = (x(i)) \in l^0 : x(i) \neq 0 \text{ for all } i \in I \subseteq \mathbb{N} \text{ and } x(i) = 0 \text{ for all } i \in \mathbb{N} \setminus I\}$,

For simplifying notations, we write $Ces_\Phi^L(q) = (Ces_\Phi(q), \|.\|_\Phi^L)$.

3 Main Results

This section contains main results of our work.

Theorem 1 *Let Φ be a Musielak–Orlicz function. Then the following statements are true:*

(i) $(Ces_\Phi(q), \|.\|_\Phi^L)$ *is a Banach space,*

(ii) $(Ces_\Phi^L(q))_a$ *is a closed subspace of* $Ces_\Phi^L(q)$,

(iii) if Φ *satisfies* δ_2*-condition then* $(Ces_\Phi^L(q))_a = Ces_\Phi^L(q)$.

Proof Let $(x^s)_{s=1}^{\infty}$ be a Cauchy sequence in $Ces_{\Phi}^{L}(q)$, where $x^s = (x^s(k))_{k=1}^{\infty}$ and $\varepsilon > 0$ be given. Then there exists a natural number T such that for every $\varepsilon > 0$ one can find r_ε with $r_\varepsilon < \varepsilon$, we have

$$\sigma_\Phi\left(\frac{x^s - x^t}{r_\varepsilon}\right) \leq 1 \text{ for all } s, t \geq T.$$

By definition of σ_Φ for each $l \in \mathbb{N}$, we have

$$\sum_{n=1}^{l} \varphi_n\left(\frac{1}{r_\varepsilon Q_n}\sum_{k=1}^{n} q_k|x^s(k) - x^t(k)|\right) \leq 1 \text{ for all } s, t \geq T, \tag{3}$$

which implies that for each $l \geq n \geq 1$

$$\varphi_n\left(\frac{1}{r_\varepsilon Q_n}\sum_{k=1}^{n} q_k|x^s(k) - x^t(k)|\right) \leq 1 \text{ for all } s, t \geq T. \tag{4}$$

Let p_n be the corresponding kernel of the Orlicz function φ_n for each n. We choose a constant $s_0 > 0$ and $\gamma > 1$ such that $\gamma \frac{s_0}{2} p_n(\frac{s_0}{2}) \geq 1$, for each $n \in \mathbb{N}$ (which is follows from $\varphi_n(\frac{s_0}{2}) = \int_0^{\frac{s_0}{2}} p_n(t)\mathrm{d}t$ and $s_0 > 0$).

By the integral representation of φ_n for each n, we have

$$\frac{1}{r_\varepsilon Q_n}\sum_{k=1}^{n} q_k|x^s(k) - x^t(k)| \leq \gamma s_0 \text{ for each } n \in \mathbb{N} \text{ and for all } s, t \geq T. \tag{5}$$

Otherwise, one can find a natural n with $\frac{1}{r_\varepsilon Q_n}\sum_{k=1}^{n} q_k|x^s(k) - x^t(k)| > \gamma s_0$ such that

$$\varphi_n\left(\sum_{k=1}^{n}\frac{q_k|x^s(k) - x^t(k)|}{r_\varepsilon Q_n}\right) \geq \int_{\frac{\gamma s_0}{2}}^{\sum_{k=1}^{n}\frac{q_k|x^s(k) - x^t(k)|}{r_\varepsilon Q_n}} p_n(t)\mathrm{d}t > \frac{\gamma s_0}{2} p_n(\frac{s_0}{2}),$$

which contradicts (4). Hence from (5), we have $(x^s(k))_{s=1}^{\infty}$ is a Cauchy sequence of real numbers for each k and hence converges for each k. Suppose for each $k \in \mathbb{N}$, $\lim_{t\to\infty} x^t(k) = x(k)$. Taking $t \to \infty$ in (3), we obtain for each $l \in \mathbb{N}$

$$\sum_{n=1}^{l} \varphi_n\left(\frac{1}{r_\varepsilon Q_n}\sum_{k=1}^{n} q_k|x^s(k) - x(k)|\right) \leq 1 \text{ for all } s \geq T,$$

which implies that $\sigma_\Phi\left(\frac{x^s-x}{r_\varepsilon}\right) \le 1$ for all $s \ge T$, i.e., $\|x^s - x\|_\Phi^L \le r_\varepsilon < \varepsilon$ for all $s \ge T$. Therefore $x^s \to x$ in $\|.\|_\Phi^L$ as $s \to \infty$. We omit the verification of $x \in Ces_\Phi^L(q)$ as it is easy to obtain. This finishes the proof of part (i).

(ii) Clearly $(Ces_\Phi^L(q))_a$ is a subspace $Ces_\Phi^L(q)$. It is sufficient to show that $(Ces_\Phi^L(q))_a$ is a closed subspace of $Ces_\Phi^L(q)$. For this, let $x_i = (x_i(k))_{k=1}^\infty \in (Ces_\Phi^L(q))_a$ for each $i \in \mathbb{N}$ and $\|x - x_i\|_\Phi^L \to 0$ as $i \to \infty$ and $x \in Ces_\Phi^L(q)$. We show that $x \in (Ces_\Phi^L(q))_a$. By the equivalent definition of norm and modular convergence, we have $\sigma_\Phi(r(x - x_i)) \to 0$ as $i \to \infty$ for all $r > 0$. So for all $r > 0$ there exists $J \in \mathbb{N}$ such that $\sigma_\Phi(2r(x - x_J)) < 1$. Since $x_J \in (Ces_\Phi^L(q))_a$ so there exists n_J such that $\sum_{n=n_J}^\infty \varphi_n\left(\frac{2r}{Q_n}\sum_{k=1}^n |q_k x_J(k)|\right) < \infty\ \forall r > 0$. We choose $n_r = n_J$, then we have

$$\begin{aligned}
&\sum_{n=n_J}^\infty \varphi_n\left(\frac{r}{Q_n}\sum_{k=1}^n q_k|x(k)|\right)\\
&\le \sum_{n=n_J}^\infty \varphi_n\left(\frac{r}{2Q_n}\sum_{k=1}^n 2q_k|x(k) - x_J(k)| + \frac{r}{2Q_n}\sum_{k=1}^n 2q_k|x_J(k)|\right)\\
&\le \frac{1}{2}\sum_{n=n_J}^\infty \varphi_n\left(\frac{2r}{Q_n}\sum_{k=1}^n q_k|x(k) - x_J(k)|\right) + \frac{1}{2}\sum_{n=n_J}^\infty \varphi_n\left(\frac{2r}{Q_n}\sum_{k=1}^n q_k|x_J(k)|\right)\\
&\le \frac{1}{2}\sigma_\Phi(2r(x - x_J)) + \frac{1}{2}\sum_{n=n_J}^\infty \varphi_n\left(\frac{2r}{Q_n}\sum_{k=1}^n q_k|x_J(k)|\right) < \infty.
\end{aligned}$$

Since r is arbitrary, we have $x \in (Ces_\Phi^L(q))_a$. This completes the proof.

(iii) We need to show here only the inclusion $Ces_\Phi^L(q) \subset (Ces_\Phi^L(q))_a$. Let $x \in Ces_\Phi^L(q)$. Then for some $t > 0$, $\sigma_\Phi(tx) < \infty$, i.e., $\sum_{n=1}^\infty \varphi_n\left(\frac{t}{Q_n}\sum_{k=1}^n q_k|x(k)|\right) < \infty$. We show that for any $r > 0$ there exists a $n_r \in \mathbb{N}$ such that

$$\sum_{n=n_r}^\infty \varphi_n\left(\frac{r}{Q_n}\sum_{k=1}^n q_k|x(k)|\right) < \infty.$$

If $r \in [0, t]$ then it is easily follows from

$$\sum_{n=n_r}^\infty \varphi_n\left(\frac{r}{Q_n}\sum_{k=1}^n q_k|x(k)|\right) \le \sum_{n=n_r}^\infty \varphi_n\left(\frac{t}{Q_n}\sum_{k=1}^n q_k|x(k)|\right) < \infty.$$

Now, we fix t and choose $r > t$. Since $x \in Ces_\Phi^L(q)$, i.e., for some $t > 0$, $\sigma_\Phi(tx) < \infty$, so there exists n_r and a constant a such that

$$\sum_{n=n_r}^{\infty} \varphi_n\Big(\frac{t}{Q_n}\sum_{k=1}^{n} q_k|x(k)|\Big) < \frac{a}{2}.$$

Therefore for each $n \geq n_r$, we have

$$\varphi_n\Big(\frac{t}{Q_n}\sum_{k=1}^{n} q_k|x(k)|\Big) < \frac{a}{2}.$$

Choose a sequence $(c_n)_{n=1}^{\infty}$ of positive real numbers such that $\sum_{n=1}^{\infty} c_n < \infty$. So for a given $\varepsilon > 0$, there exists a n_r such that $\sum_{n=n_r}^{\infty} c_n < \frac{\varepsilon}{2}$. Let $u = \frac{t}{Q_n}\sum_{k=1}^{n} q_k|x(k)|$, $K > 0$ be a constant and a is chosen above. Since $r > t$ so there is a $l \in \mathbb{N}$ such that $r \leq 2^l t$. Applying δ_2-condition for all $n \geq n_r$, we have

$$\varphi_n\Big(\frac{r}{Q_n}\sum_{k=1}^{n} q_k|x(k)|\Big) \leq \varphi_n\Big(\frac{2^l t}{Q_n}\sum_{k=1}^{n} q_k|x(k)|\Big) \leq K^l \varphi_n\Big(\frac{t}{Q_n}\sum_{k=1}^{n} q_k|x(k)|\Big) + \Big(\sum_{i=0}^{l-1} K^i\Big) c_n$$

Taking summation on both sides over $n \geq n_r$, we obtain

$$\sum_{n=n_r}^{\infty} \varphi_n\Big(\frac{r}{Q_n}\sum_{k=1}^{n} q_k|x(k)|\Big) \leq K^l \sum_{n=n_r}^{\infty} \varphi_n\Big(\frac{t}{Q_n}\sum_{k=1}^{n} q_k|x(k)|\Big) + \Big(\sum_{i=0}^{l-1} K^i\Big) \sum_{n=n_r}^{\infty} c_n < \infty.$$

Hence $x \in (Ces_{\Phi}^{L}(q))_a$.

We assume in the rest of this work that Musielak–Orlicz function $\Phi = (\varphi_n)$ with all φ_n being finitely valued. The following known lemmas are useful in the sequel:

Lemma 1 *Let $x \in (Ces_{\Phi}^{L}(q))_a$ be an arbitrary element. Then $\|x\|_{\Phi}^{L} = 1$ if and only if $\sigma_{\Phi}(x) = 1$.*

Proof The proof will run on the parallel lines of the proof of Lemma 2.1 in [2].

Lemma 2 *Suppose $\Phi \in \delta_2$ and $\Phi > 0$. Then for any sequence (x_l) in $Ces_{\Phi}^{L}(q)$, $\|x_l\|_{\Phi}^{L} \to 0$ if and only if $\sigma_{\Phi}(x_l) \to 0$.*

Proof For the proof of this lemma see [7, 13].

Lemma 3 *If $\Phi \in \delta_2$, i.e., (1), then for any $x \in Ces_{\Phi}^{L}(q)$,*

$$\|x\|_{\Phi}^{L} = 1 \text{ if and only if } \sigma_{\Phi}(x) = 1.$$

Proof Since $\Phi \in \delta_2$ implies $Ces^L_\Phi(q) = (Ces^L_\Phi(q))_a$. The proof follows from Lemma 1.

Lemma 4 *Let $\Phi \in \delta_2$, i.e., (1) and satisfies the condition $(*)$, i.e., (2). Then for any $x \in Ces^L_\Phi(q)$ and every $\varepsilon \in (0,1)$ there exists $\delta(\varepsilon) \in (0,1)$ such that $\sigma_\Phi(x) \le 1-\varepsilon$ implies $\|x\|^L_\Phi \le 1-\delta$.*

Proof The proof of this lemma will be in a way similar to that of the proof of Lemma 9 in [13].

Lemma 5 [13] *Let $(X, \|.\|)$ be normed space. If $f : X \to \mathbb{R}$ is a convex function in the set $K(0,1) = \{x \in X : \|x\| \le 1\}$ and $|f(x)| \le M$ for all $x \in K(0,1)$ and some $M > 0$ then f is almost uniformly continuous in $K(0,1)$; i.e., for all $d \in (0,1)$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|y\| \le d$ and $\|x-y\| < \delta$ implies $|f(x) - f(y)| < \varepsilon$ for all $x, y \in K(0,1)$.*

Lemma 6 *Let $\Phi \in \delta_2$, i.e., (1), $\Phi > 0$ and satisfies the condition $(*)$, i.e., (2). Then for each $d \in (0,1)$ and $\varepsilon > 0$ there exists $\delta = \delta(d,\varepsilon) > 0$ such that $\sigma_\Phi(x) \le d$, $\sigma_\Phi(y) \le \delta$ imply*

$$|\sigma_\Phi(x+y) - \sigma_\Phi(x)| < \varepsilon \text{ for any } x, y \in Ces^L_\Phi(q). \tag{6}$$

Proof Since $\Phi \in \delta_2$ and satisfies condition $(*)$, so by Lemma 4, there exists $d_1 \in (0,1)$ such that $\|x\|^L_\Phi \le d_1$. Also by Lemma 2, we find a $\delta > 0$ such that for every $\delta_1 > 0$, $\sigma_\Phi(y) \le \delta$ implies $\|y\|^L_\Phi \le \delta_1$ for any $y \in Ces^L_\Phi(q)$. So, if $\sigma_\Phi(x) \le d$ and $\sigma_\Phi(y) \le \delta$ then $\|x\|^L_\Phi \le d_1$ and $\|y\|^L_\Phi \le \delta_1$. Hence by Lemma 5, we have $|\sigma_\Phi(x+y) - \sigma_\Phi(x)| < \varepsilon$ because the functional σ_Φ satisfies all the assumptions of f defined in Lemma 5.

Lemma 7 *Let $\Phi \in \delta_2$, i.e., (1) and satisfies the condition $(*)$, i.e., (2) and $\Phi > 0$. Then for any $x \in Ces^L_\Phi(q)$ and any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\sigma_\Phi(x) \ge 1+\varepsilon$ implies $\|x\|^L_\Phi \ge 1+\delta$.*

Proof The proof of this lemma is parallel to the proof of the Lemma 4 in [3].

Theorem 2 *Let $\Phi > 0$ be a Musielak–Orlicz function satisfying condition δ_2, i.e., (1) and $(*)$, i.e., (2). Then sequence space $Ces^L_\Phi(q)$ has the UKK_c-property.*

Proof Since $\Phi > 0$ and it satisfies the condition δ_2, so by Lemma 2, for a given $\varepsilon > 0$ there exist a $\eta > 0$, we have

$$\|x\|^L_\Phi \ge \frac{\varepsilon}{4} \Rightarrow \sigma_\Phi(x) \ge \eta. \tag{7}$$

With this $\eta > 0$, by Lemma 4, one can find a $\delta \in (0,1)$ such that

$$\|x\|^L_\Phi > 1-\delta \Rightarrow \sigma_\Phi(x) > 1-\eta. \tag{8}$$

Let $(x_l) \subset B(Ces_\Phi^L(q))$, $\|x_l\|_\Phi^L \to \|x\|_\Phi^L$, $x_l(i) \to x(i)$ for all $i \in \mathbb{N}$ and $sep(x_l) \geq \varepsilon$. We show that there exists a $\delta > 0$ such that $\|x\|_\Phi^L \leq 1 - \delta$. If possible, let $\|x\|_\Phi^L > 1 - \delta$. Then one can select a finite set $I = \{1, 2, \ldots, N-1\}$ on which $\|x|_I\|_\Phi^L > 1 - \delta$. Since $x_l(i) \to x(i)$ for each $i \in \mathbb{N}$, so we obtain $x_l \to x$ uniformly on I. Consequently, by assumption $\|x_l\|_\Phi^L \to \|x\|_\Phi^L$ there exists $l_N \in \mathbb{N}$ such that

$$\|x_l|_I\|_\Phi^L > 1 - \delta \text{ and } \|(x_l - x_m)|_I\|_\Phi^L \leq \frac{\varepsilon}{2} \text{ for all } l, m \geq l_N.$$

Using Eq. (8), first one of the above inequalities implies that $\sigma_\Phi(x_l|_I) > 1 - \eta$ for $l \geq l_N$. Since $sep(x_l) \geq \varepsilon$, i.e., $\|x_l - x_m\|_\Phi^L \geq \varepsilon$, so second one of the above inequalities implies that $\|(x_l - x_m)|_{\mathbb{N}-I}\|_\Phi^L \geq \frac{\varepsilon}{2}$ for $l, m \geq l_N, l \neq m$. Hence for $N \in \mathbb{N}$ there exists a l_N such that $\|x_{l_N}|_{\mathbb{N}-I}\|_\Phi^L \geq \frac{\varepsilon}{4}$. Without loss of generality, we assume that $\|x_l|_{\mathbb{N}-I}\|_\Phi^L \geq \frac{\varepsilon}{4}$ for all $l, N \in \mathbb{N}$. Therefore by (7), we have $\sigma_\Phi(x_l|_{\mathbb{N}-I}) \geq \eta$.

By the integral representation of Musielak–Orlicz function Φ, we have $\varphi_n(u+v) \geq \varphi_n(u) + \varphi_n(v)$ for each n and all $u, v \in \mathbb{R}^+$. Using this, we obtain $\sigma_\Phi(x_l|_I) + \sigma_\Phi(x_l|_{\mathbb{N}-I}) \leq \sigma_\Phi(x_l) \leq 1$. This implies that $\sigma_\Phi(x_l|_{\mathbb{N}-I}) \leq 1 - \sigma_\Phi(x_l|_I) < 1 - (1 - \eta) = \eta$, i.e., $\sigma_\Phi(x_l|_{\mathbb{N}-I}) < \eta$, which contradicts to the fact that $\sigma_\Phi(x_l|_{\mathbb{N}-I}) \geq \eta$. This finishes the proof.

Theorem 3 *Let $\Phi > 0$ be a Musielak–Orlicz function satisfying condition δ_2, i.e.,* (1) *and* $(*)$*, i.e.,* (2)*. Then $Ces_\Phi^L(q)$ has the uniform Opial property.*

Proof Let $(x_l) \subset S(Ces_\Phi^L(q))$ be any weakly null sequence and $\varepsilon > 0$ be given. We show that for any $\varepsilon > 0$ there is a $\mu > 0$ such that

$$\liminf_{l \to \infty} \|x_l + x\|_\Phi^L \geq 1 + \mu,$$

for each $x \in Ces_\Phi^L(q)$ satisfying $\|x\|_\Phi^L \geq \varepsilon$. Since $\Phi \in \delta_2$ and $\Phi > 0$, so by Lemma 2, for each $\varepsilon > 0$ there is a number $\delta \in (0, 1)$ such that for each $x \in Ces_\Phi^L(q)$, we have $\sigma_\Phi(x) \geq \delta$. Since Φ (> 0) satisfies the condition δ_2, and the condition $(*)$, so by Lemma 6 for any $\varepsilon > 0$, there exists $\delta_1 \in (0, \delta)$ such that $\sigma_\Phi(u) \leq 1$, $\sigma_\Phi(v) \leq \delta_1$ imply

$$|\sigma_\Phi(u + v) - \sigma_\Phi(u)| < \frac{\delta}{6} \text{ for any } u, v \in Ces_\Phi^L(q). \tag{9}$$

Since $\sigma_\Phi(x) < \infty$, so there is a number $n_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0+1}^{\infty} \varphi_n\Big(\frac{1}{Q_n} \sum_{k=1}^{n} q_k |x(k)|\Big) \leq \frac{\delta_1}{6}. \tag{10}$$

From Eq. (10) it follows that

$$\begin{aligned}\delta &\le \sum_{n=1}^{n_0} \varphi_n\Big(\frac{1}{Q_n}\sum_{k=1}^{n} q_k|x(k)|\Big) + \sum_{n=n_0+1}^{\infty} \varphi_n\Big(\frac{1}{Q_n}\sum_{k=1}^{n} q_k|x(k)|\Big)\\ &\le \sum_{n=1}^{n_0} \varphi_n\Big(\frac{1}{Q_n}\sum_{k=1}^{n} q_k|x(k)|\Big) + \frac{\delta_1}{6},\end{aligned}$$

which implies $\sum_{n=1}^{n_0} \varphi_n\Big(\frac{1}{Q_n}\sum_{k=1}^{n} q_k|x(k)|\Big) \ge \delta - \frac{\delta_1}{6} > \delta - \frac{\delta}{6} = \frac{5\delta}{6}$. Since $x_l \to 0$ weakly, i.e., $x_l(i) \to 0$ for each i, so there exists a l_0 such that for all $l \ge l_0$, the last inequality yields

$$\sum_{n=1}^{n_0} \varphi_n\Big(\frac{1}{Q_n}\sum_{k=1}^{n} q_k|x_l(k)+x(k)|\Big) \ge \frac{5\delta}{6}. \tag{11}$$

Also by $x_l \to 0$ weakly, we can choose an n_0 such that $\sigma_\Phi(x_l|_{n_0}) \to 0$ as $l \to \infty$. So there exists a $l_1 > l_0$ such that $\sigma_\Phi(x_l|_{n_0}) \le \delta_1$ for all $l \ge l_1$. Since $(x_l) \subset S(Ces_\Phi^L(q))$, i.e., $\|x_l\|_\Phi^L = 1$, so by Lemma 3, we have $\sigma_\Phi(x_l) = 1$, which implies that there exists n_0 such that $\sigma_\Phi(x_l|_{\mathbb{N}-n_0}) \le 1$. Now choose $u = x_l|_{\mathbb{N}-n_0}$ and $v = x_l|_{n_0}$. Then $u, v \in Ces_\Phi^L(q)$, $\sigma_\Phi(u) \le 1$, $\sigma_\Phi(v) \le \delta_1$. So from (9), for all $l \ge l_1$ we have

$$\Big|\sigma_\Phi(x_l|_{\mathbb{N}-n_0} + x_l|_{n_0}) - \sigma_\Phi\big(x_l|_{\mathbb{N}-n_0}\big)\Big| < \frac{\delta}{6},$$

which implies that $\sigma_\Phi(x_l) - \frac{\delta}{6} < \sigma_\Phi\big(x_l|_{\mathbb{N}-n_0}\big)$ for all $l \ge l_1$, i.e., $\sum_{n=n_0+1}^{\infty} \varphi_n\Big(\frac{1}{Q_n}\sum_{k=1}^{n} q_k|x_l(k)|\Big) > 1-\frac{\delta}{6}$ for all $l \ge l_1$. Again, since $\sigma_\Phi\big(x_l|_{\mathbb{N}-n_0}\big) \le 1$ and $\sigma_\Phi\big(x|_{\mathbb{N}-n_0}\big) \le \frac{\delta_1}{6} < \delta_1$, so from the Eqs. (9) and (11), we obtain

$$\begin{aligned}\sigma_\Phi(x_l+x) &= \sum_{n=1}^{n_0} \varphi_n\Big(\frac{1}{Q_n}\sum_{k=1}^{n} q_k|x_l(k)+x(k)|\Big)\\ &\quad + \sum_{n=n_0+1}^{\infty} \varphi_n\Big(\frac{1}{Q_n}\sum_{k=1}^{n} q_k|x_l(k)+x(k)|\Big)\\ &> \sum_{n=1}^{n_0} \varphi_n\Big(\frac{1}{Q_n}\sum_{k=1}^{n} q_k|x_l(k)+x(k)|\Big)\\ &\quad + \sum_{n=n_0+1}^{\infty} \varphi_n\Big(\frac{1}{Q_n}\sum_{k=1}^{n} q_k|x_l(k)|\Big) - \frac{\delta}{6}\\ &> \frac{5\delta}{6} + \Big(1-\frac{\delta}{6}\Big) - \frac{\delta}{6} = 1 + \frac{\delta}{2}.\end{aligned}$$

Since $\Phi \in \delta_2$ and satisfies the condition $(*)$ and $\Phi > 0$, so by Lemma 7 there is a $\mu > 0$ depending only on δ such that $\|x_l + x\|_\Phi^L > 1 + \mu$. Hence $\liminf_{l\to\infty} \|x_l + x\|_\Phi^L \geq 1 + \mu$. This completes the proof.

Corollary 1 *(i) If $\varphi_n = \varphi$, $q_n = 1$ $\forall n$ and $\Phi \in \delta_2$, then Cesàro–Orlicz sequence space ces_φ^L [20] has the uniform Opial property.*

(ii) Suppose $q_n = 1$, $n = 1, 2, \ldots$ and $\varphi_n(u) = |u|^{p_n}$ for all $u \in \mathbb{R}$, $1 < p_n < \infty$ $\forall n$. Then it is easy to verify that $\Phi \in \delta_2$ if and only if $\limsup_{n\to\infty} p_n < \infty$. Therefore $ces_{(p)}^L$ [21] has the uniform Opial property.

(iii) If $\varphi_n(u) = |u|^{p_n}$, $1 \leq p_n < \infty$ $\forall n$ and $\limsup_{n\to\infty} p_n < \infty$, then $Ces_{(p)}^L(q)$ has the uniform Opial property [17].

4 Conclusion

In this study, we have obtained geometric properties such as coordinatewise uniformly Kadec–Klee property and uniform Opial property in the generalized Cesàro–Musielak–Orlicz sequence spaces, which include the well known Cesàro [24], generalized Cesàro [21], Cesàro–Orlicz [2], Cesàro–Musielak–Orlicz [26] classes of sequences in particular cases with respect to the Luxemberg norm. In future, our plan is to obtain these results for a more generalized class of sequences with respect to both the Luxemberg and Amemiya norm.

References

1. Altin, Y., Et, M., Tripathy, B.C.: The sequence space $|\bar{N}_p|(M, r, q, s)$ on seminormed spaces. Appl. Math. Comput. **154**, 423–430 (2004)
2. Cui, Y., Hudzik, H., Petrot, N., Suantai, S., Szymaszkiewicz, A.: Basic topological and geometric properties of Cesàro-Orlicz spaces. Proc. Indian Acad. Sci. (Math. Sci.) **115**(4), 461–476 (2005)
3. Cui, Y., Hudzik, H.: Maluta's coefficient and Opial's properties in Musielak-Orlicz sequence spaces equipped with the Luxemberg norm. Nonlinear Anal. **35**, 475–485 (1999)
4. Cui, Y., Hudzik, H.: On the uniform opial property in some modular sequence spaces. Func. Approx. Comment. **26**, 93–102 (1998)
5. Et, M., Altin, Y., Choudhary, B., Tripathy, B.C.: On some classes of sequences defined by sequences of Orlicz functions. Math. Ineq. Appl. **9**(2), 335–342 (2006)
6. Foralewski, P., Hudzik, H., Szymaszkiewicz, A.: Some remarks on Cesàro-Orlicz spaces. Math. Ineq. Appl. **13**(2), 363–386 (2010)
7. Foralewski, P., Hudzik, H.: On some geometrical and topological properties of generalized Calderón -Lozanovskiǐ sequence spaces. Houston J. Math. **25**(3), 523–542 (1999)
8. Foralewski, P., Hudzik, H., Szymaszkiewicz, A.: Local rotundity structure of Cesàro-Orlicz sequence spaces. J. Math. Anal. Appl. **345**(1), 410–419 (2008)
9. Gossez, J.P., Lami Dozo, E.: Some geometric properties related to fixed point theory for non expansive mappings. Pac. J. Math. **40**, 565–573 (1972)

10. Hudzik, H., Ye, Y.N.: Support functionals and smoothness in Musielak-Orlicz sequence spaces endowed with Luxemberg norm. Comment. Math. Univ. Carolinae **31**(4), 661–684 (1990)
11. Huff, R.: Banach spaces which are nearly uniformly convex. Rockey Mt. J. Math. **10**, 743–749 (1980)
12. Kadets, M.I.: The relation between some properties of convexity of the unit ball of a Banach space. Funct. Anal. Appl. **16**(3), 204–206 (1982) (translation)
13. Kaminska, A.: Uniform rotundity of Musielak-Orlicz sequence spaces. J. Approximation Theor. **47**(4), 302–322 (1986)
14. Khan, V.A.: Some geometric properties of generalized Cesaro sequence spaces. Acta Math. Univ. Comenianae **79**(1), 1–8 (2010)
15. Khan, V.A.: On Riesz-Musielak-Orlicz sequence spaces. Numer. Funct. Anal. Optim. **28**(7–8), 883–895 (2007)
16. Khan, V.A.: Some geometric properties of Riesz-Musielak-Orlicz sequence spaces. Thai J. Math. **8**(3), 565–574 (2010)
17. Mongkolkeha, C., Kumam, P.: On H-property and Uniform Opial Property of generalized Cesaro sequence spaces. J. Inequalities Appl. **2012**, 76 (2012)
18. Musielak, J.: Orlicz Spaces and Modular Spaces. Springer Lecture Notes in Mathematics, vol. 1034. Springer, Berlin (1983)
19. Opial, Z.: Weak convergence of the sequence of successive approximations for non expansive mappings. Bull. Amer. Math. Soc. **73**, 591–597 (1967)
20. Petrot, N., Suantai, S.: Some geometric properties in Cesàro-Orlicz Spaces. Sci. Asia **31**, 173–177 (2005)
21. Petrot, N., Suantai, S.: Uniform Opial properties in generalized Cesàro sequence spaces. Nonlinear Anal. **63**, 1116–1125 (2005)
22. Prus, S.: Geometrical background of metric fixed point theory. In: Kirk, W.A., Sims, B. (eds.) Handbook of Metric Fixed Point Theory, pp. 93–132. Kluwer Academic Publishers, Dordrecht (2001)
23. Prus, S.: Banach spaces with uniform Opial property. Nonlinear Anal. Theory Appl. **18**(8), 697–704 (1992)
24. Shiue, J.S.: Cesàro sequence spaces. Tamkang J. Math. **1**, 19–25 (1970)
25. Tripathy, B.C., Altin, Y., Et, M.: Generalized difference sequence spaces on seminormed spaces defined by Orlicz functions. Math. Slovaca **58**(3), 315–324 (2008)
26. Wangkeeree, R.: On property $(k\text{-}NUC)$ in Cesàro-Musielak-Orlicz sequence spaces. Thai J. Math. **1**, 119–130 (2003)
27. Zhang, T.: The coordinatewise uniformly Kadec-Klee property in some Banach spaces. Siberian Math. J. **44**(2), 363–365 (2003)