Chapter 4
Multiple Periodic Solutions of Nonlinear Functional Differential Equations

In this chapter[1], we present results on the existence of two positive periodic solutions of the first order functional differential equation

$$x'(t) = a(t)x(t) - f(t, x(h(t))), \tag{4.1}$$

where $a, h \in C(R, R_+)$ and $a(t + T) = a(t)$, $T > 0$ is a real number, $f : R \times R_+ \to R_+$, and $f(t + T, x) = f(t, x)$. If $h(t) = t - \tau(t)$ and $\tau \in C(R, R_+)$, $\tau(t + T) = \tau(t)$ with $\tau(t) \leq t$, then (4.1) takes the form

$$x'(t) = a(t)x(t) - f(t, x(t - \tau(t))). \tag{4.2}$$

From results on the existence of positive periodic solutions of (4.1), we can find from the arguments in the succeeding sections that some similar results can be derived for (4.2). The results obtained in [1, 5, 7, 12–14] can be applied to (4.1). One may observe from the sufficient conditions assumed in Chaps. 2 and 3, that the function f needs to be unimodal, that is, the function f first increases and then it decreases eventually. This is because of the choice of a constant c_4 needed in the use of the Leggett-Williams multiple fixed point Theorem 1.2.2, for the existence of three fixed points of an operator which, in turn, is equivalent to the existence of three positive periodic solutions of (4.1) or (4.2). The above choices of functions exclude many important class of growth functions arising in various mathematical models, such as:

(i) Logistic equation of multiplicative type with several delays

$$x'(t) = x(t)\left[a(t) - \prod_{i=1}^{n} b_i(t)x(t - \tau_i(t))\right], \tag{4.3}$$

where $a, b_i, \tau_i \in C(R, R_+)$ are T-periodic functions;

(ii) Generalized Richards single species growth model

[1] Some of the results in this chapter are taken from Padhi et al. [9–11].

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$$x'(t) = x(t)\left[a(t) - \left(\frac{x(t-\tau(t))}{E(t)}\right)^{\theta}\right], \tag{4.4}$$

where $a, E, \tau \in C(R, R_+)$ are T-periodic functions and $\theta > 0$ is a constant;

(iii) Generalized Michaelis-Menton type single species growth model

$$x'(t) = x(t)\left[a(t) - \sum_{i=1}^{n} \frac{b_i(t)x(t-\tau_i(t))}{1 + c_i(t)x(t-\tau_i(t))}\right], \tag{4.5}$$

where a, b_i, c_i, and $\tau_i \in C(R, R_+)$, $i = 1, 2, \ldots, n$, are T-periodic functions.

In this chapter, we attempt to study the existence of two positive T-periodic solutions of the Eq. (4.1). Then we apply the obtained result to find sufficient conditions for the existence of two positive T-periodic solutions of the models (4.3)–(4.5). To prove the results, we use the Leggett-Williams multiple fixed point Theorem 1.2.1.

The following open problem was proposed by Kuang [6, open Problem 9.2]: *Obtain sufficient conditions for the existence of positive periodic solutions of the equation*

$$x'(t) = x(t)[a(t) - b(t)x(t) - c(t)x(t-\tau(t)) - d(t)x'(t-\sigma(t))]. \tag{4.6}$$

Liu et al. [8] gave a partial answer to the above problem by using a fixed point theorem for strict set-contractions. They proved that (4.6) has at least one positive T-periodic solution. Freedman and Wu [3] studied the existence and global attractivity of a positive periodic solution of (4.6) with $d(t) \equiv 0$. In this chapter, we apply the Leggett-Williams multiple fixed point Theorem 1.2.1 to show that (4.6) has at least two positive T-periodic solutions (See Example 4.2.1) when $\mathrm{d}(t) \equiv 0$.

The results of this chapter can be extended to

$$x'(t) = a(t)x(t) - f(t, x(h_1(t)), ..., x(h_n(t))), \tag{4.7}$$

where $h_i(t) \geq 0$, $i = 1, \ldots, n$, and $f \in C(R \times R_+^n, R_+)$ is periodic with respect to the first variable.

Observe that (4.1) is equivalent to

$$x(t) = \int_{t}^{t+T} G(t, s) f(s, x(h(s)))\, \mathrm{d}s,$$

where

$$G(t, s) = \frac{e^{-\int_t^s a(\theta)\, \mathrm{d}\theta}}{1 - e^{-\int_0^T a(\theta)\, \mathrm{d}\theta}}$$

is the Green's kernel. The lower bound, being positive, is used for defining a cone. The Green's kernel $G(t, s)$ satisfies the property

$$0 < \alpha = \frac{\delta}{1-\delta} \le G(t, s) \le \frac{1}{1-\delta} = \beta, \quad s \in [t, t+T],$$

where $\delta = e^{-\int_0^T a(\theta)\,\mathrm{d}\theta} < 1$.

Let

$$X = \{x \in C(R, R) : x(t) = x(t+T)\}$$

with the norm $\|x\| = \sup_{t\in[0,T]} |x(t)|$; then X is a Banach space with the norm $\|\cdot\|$. Define a cone K in X by

$$K = \{x \in X : x(t) \ge \delta\|x\|,\ t \in [0, T]\}$$

and an operator A on X by

$$(Ax)(t) = \int_t^{t+T} G(t, s) f(s, x(h(s)))\,\mathrm{d}s. \tag{4.8}$$

If we proceed along the lines of Lemma 2.1.1 and Lemma 2.1.2 in Chap. 2, we can prove that $A(K) \subset K$, $A : K \to K$ is completely continuous, and the existence of a positive periodic solution of (4.1) is equivalent to the existence of a fixed point of A in K.

4.1 Positive Periodic Solutions of the Equation $x'(t) = a(t)x(t) - f(t, x(h(t)))$

In this section, we shall obtain some sufficient conditions for the existence of at least two positive T-periodic solutions of (4.1).

Denote

$$f^{\theta} = \limsup_{x\to\theta} \frac{f(t, x)}{a(t)x} \quad \text{and} \quad F^{\theta} = \limsup_{x\to\theta} \frac{f(t, x)}{x}.$$

Theorem 4.1.1 *Assume that there exist constants c_1 and c_2 with $0 < c_1 < c_2$ such that*

$$(\mathrm{H}_{26}) \quad \int_0^T f(t, x(h(t)))\,\mathrm{d}t > \frac{c_2}{\alpha} \ \ for\ x \in K,\ c_2 \le x \le \frac{c_2}{\delta},\ and\ 0 \le t \le T,$$

and

(H_{27}) $\displaystyle\int_0^T f(t,x))\,\mathrm{d}t < \frac{c_1}{\beta}$ *for* $x \in K,\ 0 \le x \le c_1,$ *and* $0 \le t \le T.$

Then Eq. (4.1) *has at least two positive* T*-periodic solutions.*

Proof Define a nonnegative concave continuous functional ψ on K by $\psi(x) = \min_{t\in[0,T]} x(t)$. Then $\psi(x) \le \|x\|$. Set $c_3 = \frac{c_2}{\delta}$ and $\phi_0(t) = \phi_0 = \frac{c_2+c_3}{2}$. Then $\phi_0 \in \{x \in K(\psi, c_2, c_3) : \psi(x) > c_2\}$. Furthermore, for $x \in K(\psi, c_2, c_3)$, (H_{26}) implies

$$\begin{aligned}\psi(Ax) &= \min_{0\le t\le T} \int_t^{t+T} G(t,s) f(s, x(h(s)))\,\mathrm{d}s\\ &\ge \alpha \int_0^T f(s, x(h(s)))\,\mathrm{d}s\\ &> c_2.\end{aligned}$$

Now let $x \in \overline{K}_{c_1}$. Then, from (H_{27}),

$$\begin{aligned}\|Ax\| &= \sup_{0\le t\le T} \int_t^{t+T} G(t,s) f(s, x(h(s)))\,\mathrm{d}s\\ &\le \beta \int_0^T f(s, x(h(s)))\,\mathrm{d}s\\ &< c_1.\end{aligned}$$

Next, suppose that $x \in \overline{K}_{c_3}$ with $\|Ax\| > c_3$. Then,

$$\begin{aligned}\psi(Ax) &= \min_{0\le t\le T} \int_t^{t+T} G(t,s) f(s, x(h(s)))\,\mathrm{d}s\\ &\ge \alpha \int_0^T f(s, x(h(s)))\,\mathrm{d}s\end{aligned}$$

and

$$
\begin{aligned}
c_3 < \|Ax\| &\leq \beta \int_0^T f(s, x(h(s)))\,\mathrm{d}s \\
&= \frac{\alpha}{\delta} \int_0^T f(s, x(h(s)))\,\mathrm{d}s \\
&\leq \frac{1}{\delta}\psi(Ax)
\end{aligned}
$$

imply that

$$\psi(Ax) > \frac{c_2}{c_3}\|Ax\|.$$

Hence, by Theorem 1.2.1, Eq. (4.1) has at least two positive T-periodic solutions. This completes the proof of the theorem. □

Theorem 4.1.2 *Assume that there exist constants c_1 and c_2 with $0 < c_1 < c_2$ such that*

(H_{28}) $f(t, x(h(t))) > \dfrac{c_2}{\alpha T}$ *for* $x \in K,\ c_2 \leq x \leq \dfrac{c_2}{\delta}$, *and* $0 \leq t \leq T$

and

(H_{29}) $f(t, x(h(t))) < \dfrac{c_1}{\beta T}$ *for* $x \in K,\ 0 \leq x \leq c_1$, *and* $0 \leq t \leq T$.

Then Eq. (4.1) *has at least two positive T-periodic solutions.*

The proof of the theorem follows from Theorem 4.1.1. Indeed, (H_{26}) and (H_{27}) follow from (H_{28}) and (H_{29}), respectively.

Theorem 4.1.3 *Let*

(H_{30}) $\min\limits_{0\leq t\leq T} f^{\infty} = \infty$

and

(H_{31}) $\max\limits_{0\leq t\leq T} f^{0} = 0.$

Then Eq. (4.1) *has at least two positive T-periodic solutions.*

Proof From (H_{30}), it follows that there exists $c_2 > 0$ large enough such that $f(t, x) \geq a(t)x$ for $c_2 \leq x \leq \frac{c_2}{\delta}$. Define ψ as in the proof of Theorem 4.1.1 and set

$c_3 = \frac{c_2}{\delta}$ and $\phi_0(t) = \phi_0 = \frac{c_2+c_3}{2}$. Then $\phi_0 \in \{x \in K(\psi, c_2, c_3) : \psi(x) > c_2\}$. For $x \in K(\psi, c_2, c_3)$, we have

$$\begin{aligned}\psi(Ax) &= \min_{0\le t\le T} \int_t^{t+T} G(t,s) f(s, x(h(s)))\,\mathrm{d}s \\ &\ge \min_{0\le t\le T} \int_t^{t+T} G(t,s) a(s) x(s)\,\mathrm{d}s \\ &\ge c_2 \min_{0\le t\le T} \int_t^{t+T} a(s) G(t,s)\,\mathrm{d}s \\ &= c_2.\end{aligned}$$

Next, by (H_{31}), there exists ξ, $0 < \xi < c_2$ such that $f(t,x) < a(t)x$ for $0 < x < \xi$. Set $c_1 = \xi$. Then $c_1 < c_2$ and $f(t,x) < a(t)c_1$ for $0 < x < c_1$. Now, for $x \in \overline{K}_{c_1}$, we have

$$\begin{aligned}\|Ax\| &= \sup_{0\le t\le T} \int_t^{t+T} G(t,s) f(s, x(h(s)))\,\mathrm{d}s \\ &\le \sup_{0\le t\le T} \int_t^{t+T} G(t,s) a(s) x(s)\,\mathrm{d}s \\ &< c_1 \sup_{0\le t\le T} \int_t^{t+T} a(s) G(t,s)\,\mathrm{d}s \\ &= c_1.\end{aligned}$$

In addition, for $x \in \overline{K}_{c_3}$ with $\|Ax\| > c_3$, we have

$$\begin{aligned}\psi(Ax) &= \min_{0\le t\le T} \int_t^{t+T} G(t,s) f(s, x(h(s)))\,\mathrm{d}s \\ &\ge \alpha \int_0^{T} f(s, x(h(s)))\,\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned} c_3 < \|Ax\| &\le \beta \int_0^T f(s, x(h(s)))\,\mathrm{d}s \\ &= \frac{\alpha}{\delta} \int_0^T f(s, x(h(s)))\,\mathrm{d}s \\ &\le \frac{1}{\delta}\psi(Ax). \end{aligned}$$

The above inequalities imply that

$$\psi(Ax) > \frac{c_2}{c_3}\|Ax\|.$$

Hence, by Theorem 1.2.1, Eq. (4.1) has at least two positive T-periodic solutions. This completes the proof of the theorem. □

Theorem 4.1.4 *Suppose that there exists a constant μ, $0 < \mu \le 1$ such that*

(H_{32}) $f^{\infty} > \dfrac{1}{\mu}$

and

(H_{33}) $f^0 < \mu$.

Then there exist at least two positive T-periodic solutions of Eq. (4.1).

Proof Since (H_{32}) holds, there exists $c_2 > 0$ such that

$$f(t, x) > \frac{a(t)x}{\mu} \quad \text{for} \quad c_2 \le x \le \frac{c_2}{\delta}.$$

Define the nonnegative concave continuous functional ψ on K by $\psi(x) = \min_{t\in[0,T]} x(t)$. Take $c_3 = \frac{c_2}{\delta}$ and $\phi_0(t) = \frac{c_2+c_3}{2}$. This shows that $\phi_0(t) \in \{x : x \in K(\psi, c_2, c_3), \psi(x) > c_2\} \ne \emptyset$. Then for $x \in K(\psi, c_2, c_3)$, we have

$$\begin{aligned} \psi(Ax) &= \min_{0\le t\le T} \int_t^{t+T} G(t, s) f(s, x(h(s)))\,\mathrm{d}s \\ &> \min_{0\le t\le T} \int_t^{t+T} G(t, s)\frac{a(s)x(s)}{\mu}\,\mathrm{d}s \end{aligned}$$

$$\geq \frac{c_2}{\mu} \min_{0 \leq t \leq T} \int_t^{t+T} a(s)G(t,s)\,\mathrm{d}s$$
$$> c_2.$$

From (H_{33}), there exists a real ξ, $0 < \xi < c_2$ such that $f(t,x) < a(t)\mu x$ for $0 < x \leq \xi$. Set $c_1 = \xi$; then $c_1 < c_2$. For $x \in \overline{K}_{c_1}$, we have

$$\begin{aligned}\|Ax\| &= \sup_{0 \leq t \leq T} \int_t^{t+T} G(t,s) f(s, x(h(s)))\,\mathrm{d}s \\ &< \sup_{0 \leq t \leq T} \int_t^{t+T} G(t,s)a(s)\mu\|x\|\,\mathrm{d}s \\ &\leq \mu c_1 \\ &< c_1.\end{aligned}$$

The rest of the proof is similar to that of Theorem 4.1.1 and is omitted. □

Corollary 4.1.1 *If* $f^0 < 1$ *and* $f^\infty > 1$, *then* Eq. (4.1) *has at least two positive* T*-periodic solutions.*

Theorem 4.1.5 *If*

(H_{34}) $\max_{t \in [0,T]} F^0 = \alpha_1 \in \left(0, \frac{1}{\beta T}\right)$

and there exists a constant $c_2 > 0$ *such that*

(H_{35}) $f(t,x) > \frac{1}{\alpha \delta T} x$ *for* $c_2 \leq x \leq \frac{c_2}{\delta}$,

then Eq. (4.1) *has at least two positive* T*-periodic solutions.*

Remark 4.1.1 The conditions in Theorems 4.1.1–4.1.4, Corollary 4.1.1, and Theorem 4.1.5 improve the results in [4, 8, 15, 16].

Theorem 4.1.6 *Suppose that*
(H_{36}) f *is nondecreasing with respect to* x

and there are constants $0 < c_1 < c_2$ *such that*

(H_{37}) $$\frac{\int_0^T f(t,c_1)\,\mathrm{d}t}{(1-\delta)c_1} < 1 < \frac{\delta \int_0^T f(t,\delta c_2)\,\mathrm{d}t}{(1-\delta)c_2}.$$

Then Eq. (4.1) *has at least two positive* T*-periodic solutions.*

Proof Set $c_3 = \frac{c_2}{\delta}$, define ψ as in the proof of Theorem 4.1.1, and let $\phi_0(t) = \phi_0 = \frac{c_2+c_3}{2}$. Then $\phi_0 \in \{x \in K(\psi, c_2, c_3) : \psi(x) > c_2\}$. For $x \in K(\psi, c_2, c_3)$, applying (H_{36}) and (H_{37}), we obtain

$$\begin{aligned}
\psi(Ax) &= \min_{0 \le t \le T} \int_t^{t+T} G(t,s) f(s, x(h(s)))\, ds \\
&\ge \frac{\delta}{1-\delta} \int_0^T f(s, x(h(s)))\, ds \\
&\ge \frac{\delta}{1-\delta} \int_0^T f(s, \delta c_2)\, ds \\
&> c_2.
\end{aligned}$$

Next, for $x \in \overline{K}_{c_1}$, we have

$$\begin{aligned}
\|Ax\| &= \sup_{0 \le t \le T} \int_t^{t+T} G(t,s) f(s, x(h(s)))\, ds \\
&\le \frac{1}{1-\delta} \int_0^T f(s, \|x\|)\, ds \\
&\le \frac{1}{1-\delta} \int_0^T f(s, c_1)\, ds \\
&< c_1
\end{aligned}$$

by using (H_{36}) and (H_{37}). Finally, for $x \in \overline{K}_{c_3}$ with $\|Ax\| > c_3$, we have

$$\begin{aligned}
\psi(Ax) &= \min_{0 \le t \le T} \int_t^{t+T} G(t,s) f(s, x(h(s)))\, ds \\
&\ge \frac{\delta}{1-\delta} \int_0^T f(s, x(h(s)))\, ds
\end{aligned}$$

and

$$c_3 < \|Ax\| \le \frac{1}{1-\delta}\int_0^T f(s, x(h(s)))\,\mathrm{d}s$$
$$\le \frac{1}{\delta}\psi(Ax),$$

which together imply

$$\psi(Ax) > \frac{c_2}{c_3}\|Ax\|.$$

Thus, all the conditions of Theorem 1.2.1 are satisfied and so Eq. (4.1) has at least two positive T-periodic solutions. This completes the proof of the theorem. □

Theorem 4.1.7 *Suppose that* (H_{36}) *holds and there are constants* $0 < c_1 < c_2$ *such that*

$$(H_{38}) \quad \frac{T \max\limits_{t\in[0,T]} f(t, c_1)}{(1-\delta)c_1} < 1 < \frac{\delta T \min\limits_{t\in[0,T]} f(t, \delta c_2)}{(1-\delta)c_2}.$$

Then Eq. (4.1) *has at least two positive* T*-periodic solutions.*

Proof Take ψ as in the proof of Theorem 4.1.1 and let $\phi_0(t) = \frac{c_2+c_3}{2}$, where $c_3 = \frac{c_2}{\delta}$. Then $\phi_0(t) \in \{x \in K(\psi, c_2, c_3) : \psi(x) > c_2\} \neq \phi$. Now using (H_{36}) and (H_{38}), we have for $x \in K(\psi, c_2, c_3)$,

$$\psi(Ax) = \min_{0\le t\le T}\int_t^{t+T} G(t,s) f(s, x(h(s)))\,\mathrm{d}s$$
$$\ge \frac{\delta}{1-\delta}\int_0^T f(s, \delta c_2)\,\mathrm{d}s$$
$$\ge \frac{\delta}{1-\delta}\min_{0\le t\le T} f(t, \delta c_2)\,T$$
$$> c_2.$$

For $x \in \overline{K}_{c_1}$, we can use (H_{36}) and (H_{38}) to obtain

$$\|Ax\| = \sup_{0\le t\le T}\int_t^{t+T} G(t,s) f(s, x(h(s)))\,\mathrm{d}s$$
$$\le \frac{1}{1-\delta}\int_0^T f(s, \|x\|)\,\mathrm{d}s$$

$$\leq \frac{1}{1-\delta} \max_{0 \leq t \leq T} f(t, c_1)\, T$$
$$< c_1.$$

The last part of the proof is similar to that of Theorem 4.1.1 and hence is omitted. Therefore, (4.1) has at least two positive T-periodic solutions and this completes the proof of the theorem. □

Wang [12] considered the differential equation

$$x'(t) = a(t)g(x(t))x(t) - \lambda b(t) f(x(t - \tau(t))), \tag{4.9}$$

where $\lambda > 0$ is a positive parameter, $a, b \in C(R, [0, \infty))$ are T-periodic functions, $\int_0^T a(t)\,\mathrm{d}t > 0$, $\int_0^T b(t)\,\mathrm{d}t > 0$, $\tau \in C(R, R)$ is a T-periodic function, $f, g : [0, \infty) \to [0, \infty)$ are continuous, $0 < l \leq g(x) < L < \infty$ for $x \geq 0$, l, L are positive constants and $f(x) > 0$ for $x > 0$. In developing sufficient conditions for the existence of positive T-periodic solutions he introduced the notations

$$i_0 = \textit{number of zeros in the set } \{\overline{f}_0, \overline{f}_\infty\}$$

and

$$i_\infty = \textit{number of infinities in the set } \{\overline{f}_0, \overline{f}_\infty\},$$

where

$$\overline{f}_0 = \lim_{x \to 0^+} \frac{f(x)}{x} \quad \textit{and} \quad \overline{f}_\infty = \lim_{x \to \infty} \frac{f(x)}{x}.$$

In what follows, we apply Theorem 1.2.2 to Eq. (4.9) to obtain some new results different from those in [12]. The Banach space X and a cone K are same as defined earlier in the chapter but the operator A is replaced by

$$(A_\lambda x)(t) = \lambda \int_t^{t+T} G_x(t, s) b(s) f(x(s - \tau(s)))\,\mathrm{d}s,$$

where

$$G_x(t, s) = \frac{e^{-\int_t^s a(\theta) g(x(\theta))\,\mathrm{d}\theta}}{1 - e^{-\int_0^T a(\theta) g(x(\theta))\,\mathrm{d}\theta}}$$

is the Green's kernel. The Green's kernel $G_x(t, s)$ satisfies the property

$$\frac{\delta^L}{1 - \delta^L} \leq G_x(t, s) \leq \frac{1}{1 - \delta^l}.$$

Proceeding as in the proof of Theorem 4.1.6, we obtain the following result.

Theorem 4.1.8 *Let* (H_{36}) *hold. Further, assume that there are constants* $0 < c_1 < c_2$ *such that*

$$(H_{39})\quad \frac{(1-\delta^L)c_2}{\delta^L f(c_2)\int\limits_0^T b(s)\,\mathrm{d}s} < \lambda < \frac{(1-\delta^l)c_1}{f(c_1)\int_0^T b(s)\,\mathrm{d}s}.$$

Then Eq. (4.9) *has at least two positive* T*-periodic solutions.*

Section 2.2 of Chap. 2 deals with the existence of at least three positive T-periodic solutions of the Eq. (4.1) with a parameter λ. Some of the results can be extended to Eq. (4.9). In the following, we apply Theorem 1.2.2 to Eq. (4.9) to obtain a different sufficient condition for the existence of at least three positive T-periodic solutions.

Theorem 4.1.9 *Let* $\overline{f}_0 < 1-\delta^l$ *and* $\overline{f}_\infty < 1-\delta^l$ *hold. Assume that there exists a constant* $c_2 > 0$ *such that*

$$(H_{40})\quad f(x) > \frac{(1-\delta^L)}{\delta^{2L}}c_2 \;\; for \;\; c_2 \le x \le \frac{(1-\delta^L)}{\delta^L(1-\delta^l)}c_2.$$

Then Eq. (4.9) *has at least three positive* T*-periodic solutions for*

$$\frac{\delta^L}{\int\limits_0^T b(t)\,\mathrm{d}t} < \lambda < \frac{1}{\int\limits_0^T b(t)\,\mathrm{d}t}.$$

Proof Since $\bar{f}^\infty < 1-\delta^l$, there exist $0 < \epsilon < 1-\delta^l$ and $\xi > 0$ such that $f(x) \le \epsilon x$ for $x \ge \xi$. Let $\gamma = \max\limits_{0\le x\le\xi, 0\le t\le T} f(x)$. Then $f(x) \le \epsilon x + \gamma$ for $x \ge 0$.

Choose $c_4 > 0$ such that

$$c_4 > \max\left\{\frac{\gamma}{(1-\delta^l)-\epsilon}, \frac{1-\delta^L}{\delta^L(1-\delta^l)}c_2\right\}.$$

Then, for $x \in \overline{K}_{c_4}$,

$$\begin{aligned}\|A_\lambda x\| &= \sup_{0\le t\le T} \lambda \int\limits_t^{t+T} G(t,s)b(s)f(x(s-\tau(s)))\,\mathrm{d}s \\ &\le \frac{1}{1-\delta^l}\lambda \int\limits_0^T b(s)f(x(s-\tau(s)))\,\mathrm{d}s\end{aligned}$$

$$\leq \frac{1}{1-\delta^l}\lambda \int_0^T b(s)(\epsilon\|x\| + \gamma)\,\mathrm{d}s$$
$$\leq \frac{1}{1-\delta^l}(\epsilon c_4 + \gamma)$$
$$< c_4,$$

that is, $A : \overline{K}_{c_4} \to \overline{K}_{c_4}$.

Now, we define a nonnegative concave continuous functional ψ on K by $\psi(x) = \min_{t\in[0,T]} x(t)$. Then $\psi(x) \leq \|x\|$. Set $c_3 = \frac{1-\delta^L}{\delta^L(1-\delta^l)}c_2$ and $\phi_0(t) = \phi_0 = \frac{c_2+c_3}{2}$. Then $c_2 < c_3$ and $\phi_0 \in \{x \in K(\psi, c_2, c_3) : \psi(x) > c_2\}$. For $x \in K(\psi, c_2, c_3)$, it follows from (H_{40}) that

$$\psi(A_\lambda x) = \min_{0\leq t\leq T} \lambda \int_t^{t+T} G(t,s)b(s)f(x(s-\tau(s)))\,\mathrm{d}s$$
$$\geq \frac{\delta^L}{1-\delta^L}\lambda \int_0^T b(s)f(x(s-\tau(s)))\,\mathrm{d}s$$
$$\geq \frac{\delta^L}{1-\delta^L}\lambda \int_0^T b(s)\,\frac{1-\delta^L}{\delta^{2L}}\,c_2\,\mathrm{d}s$$
$$> c_2.$$

Next, since $\bar{f}^0 < 1-\delta^l$, there exists a positive $\sigma < c_2$ such that

$$f(x) < (1-\delta^l)x \quad \text{for} \quad 0 < x \leq \sigma.$$

Set $c_1 = \sigma$; then $c_1 < c_2$. For $x \in \overline{K}_{c_1}$, we have

$$\|A_\lambda x\| = \sup_{0\leq t\leq T} \lambda \int_t^{t+T} G(t,s)b(s)f(x(s-\tau(s)))\,\mathrm{d}s$$
$$\leq \frac{1}{1-\delta^l}\lambda \int_0^T b(s)(1-\delta^l)\|x\|\,\mathrm{d}s$$
$$\leq \frac{1}{1-\delta^l}\lambda \int_0^T b(s)(1-\delta^l)c_1\,\mathrm{d}s$$
$$< c_1.$$

Finally, for $x \in K(\psi, c_2, c_4)$ with $\|A_\lambda x\| > c_3$, we have

$$c_3 < \|A_\lambda x\| \leq \frac{1}{1-\delta^l}\lambda \int_0^T b(s) f(x(s-\tau(s)))\,\mathrm{d}s,$$

which, in turn implies that

$$\begin{aligned}\psi(A_\lambda x) &\geq \frac{\delta^L}{1-\delta^L}\lambda \int_0^T b(s) f(x(s-\tau(s)))\,\mathrm{d}s \\ &> \frac{\delta^L}{1-\delta^L}(1-\delta^l)c_3 \\ &= c_2.\end{aligned}$$

Hence, by Theorem 1.2.2, Eq. (4.9) has at least three positive T-periodic solutions. □

Corollary 4.1.2 *If* $i_0 = 2$ *and there exists a constant* $c_2 > 0$ *such that* (H_{40}) *holds, then* Eq. (4.9) *has at least three positive* T*-periodic solutions.*

Remark 4.1.2 Wang [12] obtained three different results for the existence of at least one positive periodic solution of (4.9) using fixed point index theory [2]. In Corollary 4.1.2, it has been shown that (4.9) has at least three positive T-periodic solutions when $i_0 = 2$.

It would be interesting to obtain sufficient conditions for the existence of at least two or three positive periodic solutions of (4.9) when $i_0 \in \{0, 1\}$ and $i_0 \in \{0, 1, 2\}$ by using the Leggett-Williams multiple fixed point theorems. Bai and Xu [1] obtained a sufficient condition ([1, Theorem 3.2]) for the existence of three nonnegative T-periodic solutions of (4.9). Although the condition $i_0 = 2$ holds both in [1, Theorem 3.2] and in Corollary 4.1.2 above, condition (H_{40}) and the condition (H_5) in [1] are different. Accordingly, the ranges on the parameter λ are also different.

Finally, we generalize some of the above results to the scalar differential equation of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -A(t)x(t) + f(t, x(t)), \tag{4.10}$$

where $A \in C(R, R)$ and $f \in C(R \times R, R)$ satisfy $A(t+T) = A(t)$ and $f(t+T, x) = f(t, x)$. We shall apply Theorem 1.2.1 to obtain the existence of at least two positive periodic solutions of (4.10).

Lemma 4.1.1 *If* $x(t)$ *is a T—periodic solution of* (4.10) *then it satisfies the integral equation*

$$x(t) = \int_t^{t+T} G(t, s) f(s, x(s))\,\mathrm{d}s \tag{4.11}$$

where $G(t, s)$ *is the Green's function given by*

$$G(t, s) = \frac{\exp\left(\int_t^s A(\theta)\mathrm{d}\theta\right)}{\exp\left(\int_0^T A(\theta)\mathrm{d}\theta\right) - 1}, \quad t,\ s \in R. \tag{4.12}$$

Now, let us define

$$\delta = \exp\left(\int_0^T A(\theta)\mathrm{d}\theta\right). \tag{4.13}$$

Observe that $\delta > 1$ if

$$\int_0^T A(\theta)\mathrm{d}\theta > 0. \tag{4.14}$$

Under the assumption (4.14), the Green's function (4.12) satisfies

$$0 < \frac{1}{\delta - 1} < G(t, s) < \frac{\delta}{\delta - 1}, \quad s \in [t, t + T]. \tag{4.15}$$

We know that the set

$$X = \{x \in C([0, T], R) : x(0) = x(T)\} \tag{4.16}$$

endowed with the norm

$$\|x\| = \sup_{0 \le t \le T} x(t) \tag{4.17}$$

is a Banach space where $C[0, T]$ is the set of all continuous functions defined on $[0, T]$.

Theorem 4.1.10 *Let* $\int_0^T A(s)\mathrm{d}s > 0$. *Assume:*

(H_{41}) *there exists* $c_3 > 0$ *such that* $\int_0^T f(s, x)\mathrm{d}s > 0$ *if* $0 < x \le c_3$ *and*

$$\int_0^T f(s, x)\mathrm{d}s \ge \frac{\delta - 1}{\delta} c_3 \quad \textit{if} \ \frac{c_3}{\delta} \le x \le c_3; \tag{4.18}$$

(H_{42}) $\lim\limits_{\|x\| \to 0} \frac{1}{\|x\|} \int_0^T f(s, x)\mathrm{d}s < \frac{\delta - 1}{\delta}$.

Then Eq. (4.10) *has at least two positive T-periodic solutions in* $\overline{K}_{c_3}$.

Proof Let us consider the Banach space X endowed with the sup norm as defined in (4.16)–(4.17). Define a cone K on X by

$$K = \{x \in X : x(t) > 0\}. \tag{4.19}$$

Let c_3 be a positive constant satisfying the conditions in the hypotheses. Define an operator $E : \overline{K}_{c_3} \to K$ by

$$(Ex)(t) = \int_t^{t+T} G(t,s) f(s, x(s))\, \mathrm{d}s. \tag{4.20}$$

It is clear that the existence of a fixed point of E is equivalent to the existence of a positive periodic solution of (4.10).

We shall apply Leggett-Williams multiple fixed point theorem to the above operator E to prove the existence of at least two positive periodic solutions for the Eq. (4.10).

It can be easily verified that E is well defined, completely continuous on $\overline{K}_{c_3}$, and $E(\overline{K}_{c_3}) \subset K$. Consider the nonnegative concave continuous functional ψ defined on K by

$$\psi(x) = \min_{0 \le t \le T} x(t). \tag{4.21}$$

For $c_2 = \frac{c_3}{\delta}$ and $\phi_0 = \frac{1}{2}(c_2 + c_3)$ we have, $c_2 < \phi_0 < c_3$ and so

$$\{x \in K(\psi, c_2, c_3) : \psi(x) > c_2\} \neq \emptyset.$$

For $x(t) \in K(\psi, c_2, c_3)$,

$$\begin{aligned}
\psi(Ex) &= \min_{0 \le t \le T} \int_t^{t+T} G(t,s) f(s, x(s))\, \mathrm{d}s \\
&> \frac{1}{\delta - 1} \int_0^T f(s, x(s))\, \mathrm{d}s \quad \text{(from (4.15))} \\
&\ge \frac{1}{\delta - 1} \frac{\delta - 1}{\delta} c_3 \quad (\text{from } \mathrm{H}_{41}) \\
&= \frac{c_3}{\delta}.
\end{aligned}$$

Hence, condition (i) of Theorem 1.2.1 is satisfied.

Now, we show that condition (ii) of Theorem 1.2.1 holds. From condition (H_{42}), there exists $\xi,\ 0 < \xi < c_2$, such that

$$\int_0^T f(s, x(s))\, ds < \frac{\delta - 1}{\delta} \|x\| \text{ for } 0 \le \|x\| \le \xi. \tag{4.22}$$

Choose $c_1 = \xi$. Then we have $0 < c_1 < c_2$ and for $0 \le x(s) \le c_1$, applying (4.15) and (4.22), we obtain

$$\begin{aligned}
\|Ex\| &= \sup_{0 \le t \le T} \int_0^T G(t, s) f(s, x(s))\, ds \\
&< \frac{\delta}{\delta - 1} \int_0^T f(s, x(s))\, ds \\
&\le \frac{\delta}{\delta - 1} \frac{\delta - 1}{\delta} \|x\| \\
&\le c_1.
\end{aligned}$$

Hence, condition (ii) in Theorem 1.2.1 is established.

Now from (4.15),

$$\begin{aligned}
\psi(Ex) &= \min_{0 \le t \le T} \int_t^{t+T} G(t, s) f(s, x(s))\, ds \\
&> \frac{1}{\delta - 1} \int_0^T f(s, x(s))\, ds.
\end{aligned} \tag{4.23}$$

Let $0 < x(t) \le c_3$ be such that $\|Ex\| > c_3$. For such a choice of $x(t)$, we have

$$\begin{aligned}
c_3 < \|Ex\| &= \sup_{0 \le t \le T} \int_0^T G(t, s) f(s, x(s))\, ds \\
&< \frac{\delta}{\delta - 1} \int_0^T f(s, x(s))\, ds \\
&\le \delta \frac{1}{\delta - 1} \int_0^T f(s, x(s))\, ds \\
&< \delta \psi(Ex)
\end{aligned}$$

by (4.23). Therefore, $\psi(Ex) > \frac{1}{\delta}\|Ex\|$ and this implies that $\psi(Ex) > \frac{c_2}{c_3}\|Ex\|$ for $0 < x(t) \le c_3$ satisfying $\|Ex\| > c_3$. Hence, condition (iii) of Theorem 1.2.1 is also satisfied. Therefore, by Theorem 1.2.1, the operator (4.20) has at least two fixed points in $\overline{K}_{c_3}$, so Eq. (4.10) admits at least two positive T-periodic solutions. This completes the proof. □

Corollary 4.1.3 *Let* $\int\limits_0^T A(s)\,ds > 0$. *Assume there exists a positive constant* c_3 *such that*

$$\int\limits_0^T f(t,x)\mathrm{d}t > 0 \ \textit{for} \ 0 < x \le c_3, \tag{4.24}$$

$$\int\limits_0^T f(s,x)\mathrm{d}s = \frac{\delta-1}{\delta}x \ \textit{for} \ x = c_3, \tag{4.25}$$

$$\int\limits_0^T f(s,x)\mathrm{d}s > \frac{\delta-1}{\delta}c_3 \ \textit{for} \ \frac{c_3}{\delta} \le x < c_3, \tag{4.26}$$

and

(H_{42}^*) $\qquad \lim\limits_{x\to 0} \frac{1}{x}\int\limits_0^T f(s,x)\mathrm{d}s < \frac{\delta-1}{\delta}.$

Then Eq. (4.10) *has at least two positive T-periodic solutions in* $\overline{K}_{c_3}$.

Proof Assume that there exists $c_3 > 0$ such that (4.24)–(4.26) hold. This implies that

$$\int\limits_0^T f(s,x)\,\mathrm{d}s \ge \frac{\delta-1}{\delta}c_3 \ \text{ for } \ \frac{c_3}{\delta} \le x \le c_3,$$

and hence (H_{42}^*) implies (H_{42}).

Now, let us assume

$$\lim_{x\to 0}\int\limits_0^T \frac{f(s,x)}{x}\,\mathrm{d}s < \frac{\delta-1}{\delta}. \tag{4.27}$$

We have $\frac{1}{\|x\|}\int_0^T f(s,x)\,\mathrm{d}s = \int_0^T \frac{f(s,x)}{\|x\|}\,\mathrm{d}s \le \int_0^T \frac{f(s,x)}{x(s)}\,\mathrm{d}s$ for $s \in [0,T]$. Observe that $\|x\| \to 0$ if and only if $x(s)$ also tends to zero for all $s \in [0,T]$. Therefore, in view of (4.27) we have

$$\lim_{\|x\|\to 0} \frac{1}{\|x\|} \int_0^T f(s, x(s))\,\mathrm{d}s \le \lim_{x(s)\to 0} \int_0^T \frac{f(s, x(s))}{x(s)}\,\mathrm{d}s < \frac{\delta - 1}{\delta}, \text{ for all } s \in [0, T].$$

Therefore, condition (H_{42}^*) implies (H_{42}), and this completes the proof of the theorem. □

4.2 Applications to Some Mathematical Models

Ye et al. [15] and Zhang et al. [16] showed that the models (4.3)–(4.6) have at least one positive periodic solution. In the following section, we apply some of the results obtained in Sect. 4.1 to obtain sufficient conditions for the existence of at least two positive periodic solutions of the models (4.3)–(4.6).

Example 4.2.1 The generalized logistic model for a single species

$$x'(t) = x(t)[a(t) - b(t)x(t) - c(t)x(t - \tau(t))] \tag{4.28}$$

has at least two positive T-periodic solutions, where $a(t)$, $b(t)$ and $c(t)$ are nonnegative continuous periodic functions.

To see this, set $f(t, x) = x(t)[b(t)x(t) + c(t)x(t - \tau(t))]$. Since

$$\max_{t\in[0,T]} \frac{f(t, x)}{a(t)x} \le \max_{t\in[0,T]} \left\{\frac{b(t)}{a(t)}\right\} \|x\| + \max_{t\in[0,T]} \left\{\frac{c(t)}{a(t)}\right\} \|x\| \to 0 \text{ as } x \to 0,$$

we see that (H_{31}) is satisfied. Moreover,

$$\min_{t\in[0,T]} \frac{f(t, x)}{a(t)x} \ge \min_{t\in[0,T]} \delta \left\{\frac{b(t)}{a(t)}\right\} \|x\| \to \infty \text{ as } x \to \infty;$$

so (H_{30}) is satisfied. Thus, by Theorem 4.1.3, Eq. (4.28) has at least two positive T-periodic solutions.

Example 4.2.2 The logistic equation for a single species

$$x'(t) = x(t)\left[a(t) - \sum_{i=1}^{n} b_i(t)x(t - \tau_i(t))\right] \tag{4.29}$$

has at least two positive T-periodic solutions, where a, b_i, $\tau_i \in C(R, R_+)$ are T-periodic functions.

Example 4.2.3 The logistic equation with several delays (4.3) has at least two positive T-periodic solutions.

Example 4.2.4 The generalized Richards single species growth model (4.4) has at least two positive T-periodic solutions.

The verification of Examples 4.2.2–4.4.4 are similar to that of Example 4.2.1.

Applying Corollary 4.1.1 to the generalized Michaelis-Menton type single species growth model (4.5), we obtain the following result.

Example 4.2.5 If

$$\min_{t\in[0,T]} \sum_{i=1}^{n} \frac{b_i(t)}{a(t)c_i(t)} > 1,$$

then (4.5) has at least two positive T-periodic solutions.

Now, we assume that the population is subject to harvesting. Under the catch-per-unit-effort hypothesis [15], the harvested population's growth model becomes

$$x'(t) = x(t)\left[a(t) - \frac{b(t)x(t)}{1+c(t)x(t)}\right] - qEx, \tag{4.30}$$

where q and E are positive constants denoting the catch-ability coefficients and harvesting effort, respectively. Ye et al. [15] proved that if $0 < qE < \frac{1-\delta}{T}$ and $\left(\frac{b^m}{c} + qE\right) > \frac{1-\delta}{\delta^2 T}$, then (4.30) has at least one positive T-periodic solution, where $b^m = \min_{0\le t\le T} b(t)$ and $0 < c(t) \le c$.

Theorem 4.2.1 *Suppose that* $0 < qE < \frac{1-\delta}{T}$ *and* $\frac{\delta\int_0^T b(t)\,dt}{c} + qE > \frac{1-\delta}{\delta^2 T}$. *Then* Eq. (4.30) *has at least two positive T-periodic solutions.*

Proof Set $f(t,x) = \frac{b(t)x^2}{1+c(t)x} + qEx$. Then $qE < \frac{1-\delta}{T}$ implies the condition (H_{34}). Choose $c_2 = \frac{\delta(1-qE\alpha\delta T)}{\alpha\delta^2 T\int_0^T b(t)\,\mathrm{d}t - c(1-qE\alpha\delta T)}$; then $\frac{c_2\alpha\delta^2 T\int_0^T b(t)\,\mathrm{d}t}{\delta+cc_2} + qE\alpha\delta T = 1$. Setting $c_3 = \frac{c_2}{\delta} = \frac{1-qE\alpha\delta T}{\alpha\delta^2 T\int_0^T b(t)\,\mathrm{d}t - c(1-qE\alpha\delta T)}$, we have $c_2 < c_3$. Now for $c_2 \le x \le \frac{c_3}{\delta}$, we have

$$\begin{aligned} f(t,x) &> \frac{c_2^2\int_0^T b(t)\,\mathrm{d}t}{1+c\frac{c_2}{\delta}} + qEc_2 \\ &= \frac{c_2}{\alpha\delta T}\left[\frac{c_2\alpha\delta^2 T\int_0^T b(t)\,\mathrm{d}t}{\delta+cc_2} + qE\alpha\delta T\right] \\ &\ge \frac{c_2}{\alpha\delta T}, \end{aligned}$$

that is, (H_{35}) holds. Hence, by Theorem 4.1.5, (4.30) has at least two positive T-periodic solutions. □

Remark 4.2.1 There are very few results in the literature on the existence of two periodic solutions of (4.1) with its application to the models (4.3)–(4.6). Hence, simple results on the existence of two periodic solutions of the above equations are of immense importance.

4.3 Application to Renewable Resource Dynamics

In this section, we apply Theorem 4.1.10 and Corollary 4.1.3 to investigate the existence of positive T-periodic solutions of the ordinary differential equation

$$x' = a(t)x(x - b(t))(c(t) - x) \tag{4.31}$$

representing dynamics of a renewable resource that is subjected to Allee effects. The transformation $y(t) = c(t)x(t)$ transforms Eq. (4.31) in to

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\left(a(t)c^2(t)k(t) + \frac{c'(t)}{c(t)}\right) y + a(t)c^2(t)\left((1 + k(t)) - y\right) y^2 \tag{4.32}$$

where

$$k(t) = \frac{b(t)}{c(t)} < 1. \tag{4.33}$$

Note that (4.32) is a particular case of a general scalar differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -A(t)y(t) + f(t, y(t)) \tag{4.34}$$

where $A \in C(R, R)$ and $f \in C(R\times R, R)$ satisfy $A(t+T) = A(t)$ and $f(t+T, x) = f(t, x)$. Comparing (4.32) with (4.34) we have

$$A(t) = \left(a(t)c^2(t)k(t) + \frac{c'(t)}{c(t)}\right) \tag{4.35}$$

and

$$f(t, x) = a(t)c^2(t)\left((1 + k(t)) - x\right) x^2. \tag{4.36}$$

Let us consider a Banach space X as defined in (4.16)–(4.17). We have $f(t, 0) = 0$, $f(t, x(t)) > 0$ for $0 < x(t) < 1 + k_m$, and $f(t, x(t)) < 0$ for $x(t) > 1 + k_M$ where $k_m = \min_{0\le t\le T} k(t)$ and $k_M = \max_{0\le t\le T} k(t)$.

Henceforth, we let

$$M = \int_0^T a(s)c^2(s)\,\mathrm{d}s \quad \text{and} \quad N = \int_0^T a(s)c^2(s)k(s)\,\mathrm{d}s. \tag{4.37}$$

Since $0 < k(t) < 1$ (from 4.33), we have $M > N > 0$. From (4.36) we observe that $\lim_{x \to 0} \frac{1}{x} \int_0^T f(s, x)\,\mathrm{d}s = 0$ and hence (H_{42}^*) of Corollary 4.1.3 is satisfied for Eq. (4.32). We then have the following theorem.

Theorem 4.3.1 *If*

$$\frac{(M+N) + \sqrt{(M+N)^2 - 4M(\frac{e^N - 1}{e^N})}}{2M} > \frac{e^{2N} - \frac{1}{e^N}}{M+N} \tag{4.38}$$

then Eq. (4.31) *has at least two positive T-periodic solutions.*

Proof We shall use Corollary 4.1.3 to prove this theorem. From (4.35), it is easy to observe that $\int_0^T A(s)\,\mathrm{d}s = N > 0$. To complete the proof of the theorem, it suffices to find the existence of a positive constant $c_3 > 0$ such that (4.24)–(4.26) hold.

Take

$$c_3 = \frac{(M+N) + \sqrt{(M+N)^2 - 4M(\frac{\delta - 1}{\delta})}}{2M} \tag{4.39}$$

and define $c_2 = \frac{c_3}{\delta}$. Clearly $0 < c_2 < c_3$. It is easy to verify that $p = c_3$ is a solution of

$$-Mp^2 + (M+N)p - \frac{\delta - 1}{\delta} = 0. \tag{4.40}$$

A simple calculation shows that (4.40) is equivalent to

$$(1-p)p \int_0^T a(s)c^2(s)\,\mathrm{d}s \; + \; p \int_0^T a(s)c^2(s)k(s)\,\mathrm{d}s = \frac{\delta - 1}{\delta}. \tag{4.41}$$

The above equation can be rewritten as

$$\int_0^T f(s, p)\,\mathrm{d}s = \frac{\delta - 1}{\delta} p. \tag{4.42}$$

That is, $p = c_3$ satisfies

$$\int_0^T f(s, c_3)\,\mathrm{d}s = \frac{\delta - 1}{\delta} c_3.$$

Next, we consider the inequality

$$\int_0^T f\left(s, \frac{c_3}{\delta}\right)\mathrm{d}s > \frac{\delta - 1}{\delta} c_3. \tag{4.43}$$

Substituting for f, we obtain

$$\int_0^T a(s)c^2(s)\left(1 + k(s) - \frac{c_3}{\delta}\right)\frac{c_3^2}{\delta^2}\,\mathrm{d}s > \frac{\delta - 1}{\delta} c_3. \tag{4.44}$$

The above inequality is equivalent to

$$-Mc_3^2 + (M + N)\delta c_3 - \delta^2(\delta - 1) > 0.$$

Since $p = c_3$ is a solution of (4.40), the above inequality yields

$$c_3 > \frac{\delta^2 - \frac{1}{\delta}}{(M + N)}. \tag{4.45}$$

Therefore, (4.43) will be satisfied if the root $p = c_3$ of (4.40) satisfies the inequality (4.45). Thus, (4.24)–(4.26) will be satisfied if the parameters of the associated Eq. (4.32) satisfies (4.45), that is, (4.38) holds. Hence, the proof is complete. □

Observe that Theorem 4.3.1 verifiable only if M and N satisfy the inequality

$$(M + N)^2 - 4M\left(\frac{e^N - 1}{e^N}\right) > 0. \tag{4.46}$$

Note that the left hand side of the inequality (4.46) is an implicit expression in M and N and the Fig. 4.1 presents the region in the (M, N) space where the inequality is satisfied. From this figure we observe that (4.46) is valid in the interior of the positive quadrant of (M, N) space. Since we have both M and N to be positive from (4.37), we see that (4.46) is always satisfied for the model (4.31).

Figure 4.2 presents the region in the $(M,\ N)$ space where the inequality (4.38) is satisfied. This figure helps in identifying the coefficient functions that ensure the existence of at least two T-periodic solutions for (4.31). Let us choose the functions $a(t)$, $b(t)$ and $c(t)$ to be the 2π periodic functions

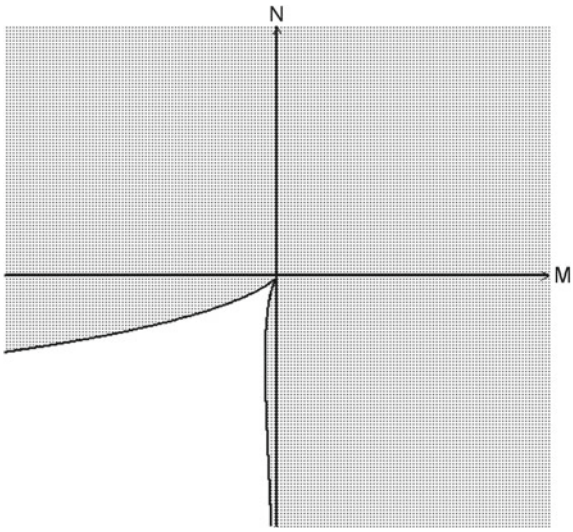

Fig. 4.1 The shaded region represents the portion in the (M, N) space that satisfies (4.46)

$$a(t) = (1.2 + \sin t)^2, \quad b(t) = \frac{1.2 + \cos t}{12(1.2 + \sin t)}, \quad \text{and} \quad c(t) = \frac{1}{1.2 + \sin t}. \tag{4.47}$$

From (4.33), we have $k(t) = \frac{b(t)}{c(t)} = \frac{1.2+\cos t}{12} < 1$. According to (4.37), we have $M = 6.28, \ N = 0.628$. Clearly, $M > N > 0$. Also, we have

$$\frac{(M+N) + \sqrt{(M+N)^2 - 4M\left(\frac{e^N - 1}{e^N}\right)}}{2M} = 1.0277$$

and

$$\frac{e^{2N} - \frac{1}{e^N}}{M+N} = 0.4310.$$

Therefore (4.38) is satisfied and hence (4.31) admits at least two positive solutions with $a(t)$, $b(t)$, and $c(t)$ as given in (4.47). The existence of 2π periodic solutions can also be ascertained from Fig. 4.2 by observing presence of the point $(M, N) = (6.28, 0.628)$ in the region that satisfies (4.38). In fact, Fig. 4.2 indicates that if the positive T-periodic coefficient functions $a(t)$, $b(t)$ and $c(t)$ with $b(t) < c(t)$ are so chosen such that the corresponding M and N in (4.37) belong to the shaded region, this implies that the model (4.31) admits at least two positive T-periodic solutions.

In this section, we examined the existence of at least two positive T-periodic solutions for a scalar differential equation representing the dynamics of a renewable resource that is subjected to Allee effects. This study is physically relevant as it takes into account the seasonally dependent (cyclic) behavior in the intrinsic growth rate, Allee threshold, and carrying capacity for the renewable resource. While the

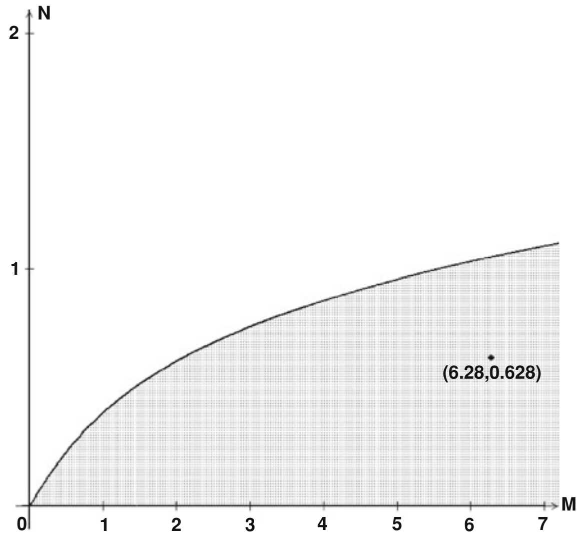

Fig. 4.2 The shaded region represents the portion in the (M, N) space that satisfies (4.38)

equation with constant coefficients (independent of strict periodicity) admits exactly two positive equilibrium solutions, the study undertaken in this section reveals that the equation with periodic coefficients admits at least two positive periodic solutions.

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