Fluid Queue Driven by an *M*/*M*/1 Queue Subject to Catastrophes

K. V. Vijayashree and A. Anjuka

Abstract In this paper, we present the stationary analysis of a fluid queueing model modulated by an $M/M/1$ queue subject to catastrophes. The explicit expressions for the joint probability of the state of the system and the content of the buffer under steady state are obtained in terms of modified Bessel function of first kind using continued fraction methodology.

Keywords Buffer content distribution · Continued fractions · Laplace transform · Modified Bessel function of first kind

1 Introduction

In recent years, fluid queues have been widely accepted as appropriate models for modern telecommunication [4] and manufacturing systems [7]. This modelling approach ignores the discrete nature of the real information flow and treats it as a continuous stream. In addition, fluid models are often useful as approximate models for certain queueing and inventory systems where the flow consists of discrete entities, but the behaviour of individuals is not important to identify the performance analysis [3].

Steady-state behaviour of Markov-driven fluid queues has been extensively studied in the literature. Parthasarathy et al. [8] present an explicit expression for the buffer content distribution in terms of modified Bessel function of first kind using Laplace transforms and continued fractions. Silver Soares and Latouche [11]

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expressed the stationary distribution of a fluid queue with finite buffer as a linear combination of matrix exponential terms using matrix analytic methods. Besides, fluid queues also have successful applications in the field of congestion control [14] and risk processes [10]. Fluid models driven by an $M/M/1/N$ queue with single and multiple exponential vacations were recently studied by Mao et al. [5, 6] using spectral method.

In this paper, we analyse fluid queues driven by an $M/M/1$ queue subject to catastrophes. The effect of catastrophes in queueing models induces the system to be instantly reset to zero at some random times. Such models find a wide range of applications in diverse fields [12]. More specifically, birth–death stochastic models subject to catastrophes are popularly used in the study of population dynamics [1, 13], biological process [2], etc. With the arrival of a negative customer into the system (*referred to as catastrophe*), it induces the positive customers, if any, to immediately leave the system. For example, in computer systems which are not supported by power backup, the voltage fluctuations will lead to the system shut down momentarily and in that process all the work in progress and the jobs waiting to be completed will be lost and the system begins afresh. Such situations can be modelled as a single server queueing model with catastrophes. Under steady-state conditions, explicit analytical expressions for the joint system size probabilities of the modulating process and the buffer content distribution are obtained using continued fraction methodology.

2 Model Description

Let $X(t)$ denote a number of customer in the background queueing model at time t wherein customers arrive according to a Poisson process at an average rate λ and the service times are exponentially distribution with parameter μ. Further, the catastrophes are assumed to occur according to a Poisson process with rate γ. For such a model, the steady-state probabilities p_j's are given by

$$p_0 = 1 - \rho \quad \text{and} \quad p_j = (1-\rho)\rho^j, \quad j = 1, 2, 3\ldots,$$

where

$$\rho = \frac{\lambda + \mu + \gamma - \sqrt{\lambda^2 + \mu^2 + \gamma^2 + 2\lambda\gamma + 2\mu\gamma - 2\lambda\mu}}{2\mu}$$

See Nelson [9].

Let $\{C(t),\, t \geq 0\}$ represent the buffer content process where $C(t)$ denotes the content of the buffer at time t. During the busy period of the server, the fluid accumulates in an infinite capacity buffer at a constant rate $r > 0$. The buffer depletes the fluid during the idle periods of the server at a constant rate $r_0 < 0$ as long as the buffer is nonempty. Hence, the dynamics of the buffer content process is given by

$$\frac{\mathrm{d}C(t)}{\mathrm{d}t} = \begin{cases} 0, & \text{if } C(t) = 0, \quad X(t) = 0 \\ r_0, & \text{if } C(t) > 0, \quad X(t) = 0 \\ r, & \text{if } C(t) > 0, \quad X(t) > 0 \end{cases}$$

Clearly, the two-dimensional process $\{(X(t), C(t)), t \geq 0\}$ constitutes a Markov process, and it possesses a unique stationary distribution under a suitable stability condition. To ensure the stability of the process $\{(X(t), C(t)), t \geq 0\}$, we assume the mean aggregate input rate to be negative, that is, $r_0 p_0 + r \sum_{i=1}^{\infty} p_i < 0$.

3 Solution Methodology

Letting

$$F_j(t,x) = \Pr\{X(t) = j;\ C(t) \leq x\},\ t, x \geq 0,\ j = 0, 1, 2, \ldots,$$

the Kolmogorov forward equations for the Markov process $\{X(t), C(t)\}$ are given by

$$\frac{\partial F_0(t,x)}{\partial t} + r_0 \frac{\partial F_0(t,x)}{\partial x} = -\lambda F_0(t,x) + \mu F_1(t,x) + \gamma \sum_{k=1}^{\infty} F_k(t,x)$$

and for $j = 1, 2, 3, \ldots,$

$$\frac{\partial F_j(t,x)}{\partial t} + r \frac{\partial F_j(t,x)}{\partial x} = \lambda F_{j-1}(t,x) - (\lambda + \mu + \gamma) F_j(t,x) + \mu F_{j+1}(t,x).$$

Assume that the process is in equilibrium so that $\frac{\partial F_j(t,x)}{\partial t} \equiv 0$ and $\lim_{t \to \infty} F_j(t,x) \equiv F_j(x)$. The above system then reduces to a system of ordinary differential equations given by

$$r_0 \frac{\mathrm{d}F_0(x)}{\mathrm{d}x} = -\lambda F_0(x) + \mu F_1(x) + \gamma \sum_{k=1}^{\infty} F_k(x), \text{ and} \tag{1}$$

$$r \frac{\mathrm{d}F_j(x)}{\mathrm{d}x} = \lambda F_{j-1}(x) - (\lambda + \mu + \gamma) F_j(x) + \mu F_{j+1}(x) \quad j = 1, 2, 3, \ldots. \tag{2}$$

When the net input rate of fluid flow into the buffer is positive, the buffer content increases and the buffer cannot stay empty. Hence, it follows that the solution to Eqs. (1) and (2) must satisfy the boundary conditions,

$$F_j(0) = 0, \quad j = 1, 2, \ldots, \text{ and}$$
$$F_0(0) = a, \quad \text{for some constant } 0 < a < 1.$$

The condition $F_0(0) = a$ suggests that with some positive probability, say a, the buffer content remains empty when the server in the background queuing model is idle.

Taking Laplace transform of Eq. (1) leads to

$$r_0\left[s\hat{F}_0(s) - F_0(0)\right] = -\lambda\hat{F}_0(s) + \mu\hat{F}_1(s) + \gamma\sum_{k=1}^{\infty}\hat{F}_k(s) \tag{3}$$

which upon simplification yields

$$\hat{F}_0(s) = \frac{a}{s + \frac{\lambda}{r_0} - g(s)}, \tag{4}$$

where

$$g(s) = \frac{\mu\hat{F}_1(s)}{r_0\hat{F}_0(s)} + \frac{\gamma\sum_{k=1}^{\infty}\hat{F}_k(s)}{r_0\hat{F}_0(s)}.$$

Again, Laplace transform of Eq. (2) gives

$$(rs + \lambda + \mu + \gamma)\hat{F}_j(s) - \mu\hat{F}_{j+1}(s) = \lambda\hat{F}_{j-1}(s), \quad j = 1, 2, 3, \ldots$$

which leads to the continued fraction representation as

$$\frac{\hat{F}_j(s)}{\hat{F}_{j-1}(s)} = \frac{\lambda\mu}{\mu(rs + \lambda + \mu + \gamma)-} \quad \frac{\lambda\mu}{rs + \lambda + \mu + \gamma-} \quad \frac{\lambda\mu}{rs + \lambda + \mu + \gamma-}\cdots.$$

Assume

$$f(s) = \frac{\lambda\mu}{rs + \lambda + \mu + \gamma-} \quad \frac{\lambda\mu}{rs + \lambda + \mu + \gamma-}\cdots.$$

Then, $\frac{\hat{F}_j(s)}{\hat{F}_{j-1}(s)} = \frac{1}{\mu}f(s)$ and hence,

$$\hat{F}_j(s) = \frac{f(s)}{\mu}\hat{F}_{j-1}(s) = \left(\frac{f(s)}{\mu}\right)^j\hat{F}_0(s). \tag{5}$$

Also $f(s)$ can be rewritten as,

$$f(s) = \frac{\lambda\mu}{rs + \lambda + \mu + \gamma - f(s)} = \frac{\frac{\lambda\mu}{r}}{s + \frac{\lambda+\mu+\gamma}{r} - \frac{f(s)}{r}}$$

which leads to the quadratic equation given by

$$\frac{f(s)^2}{r} - \left(s + \frac{\lambda + \mu + \gamma}{r}\right)f(s) + \frac{\lambda\mu}{r} = 0.$$

Upon solving the above equation, we get

$$f(s) = \frac{p - \sqrt{p^2 - \alpha^2}}{\frac{2}{r}} \tag{6}$$

where $p = s + \frac{\lambda+\mu+\gamma}{r}$ and $\alpha = \frac{2\sqrt{\lambda\mu}}{r}$. Now, substituting for $\hat{F}_k(s)$ in $g(s)$ from Eq. (5), we get

$$g(s) = \frac{f(s)}{r_0}\left[1 + \frac{\gamma}{\mu - f(s)}\right].$$

Therefore, from Eq. (4), we get

$$\hat{F}_0(s) = \frac{a}{s + \frac{\lambda}{r_0} - g(s)} = \frac{a}{s + \frac{\lambda}{r_0}} + a\sum_{k=1}^{\infty}\frac{g(s)^k}{\left(s + \frac{\lambda}{r_0}\right)^{k+1}}.$$

which on inversion leads to

$$F_0(x) = a\exp^{\frac{-\lambda x}{r_0}} + a\sum_{k=1}^{\infty}\left(\frac{x^k}{k!}\exp^{-\frac{\lambda}{r_0}x}\right) * h(x) \tag{7}$$

where

$$h(x) = \frac{1}{r_0^k}\sum_{n=0}^{k}\binom{k}{n}\left(\frac{\gamma}{\mu}\right)^n\sum_{j=0}^{\infty}\binom{n+j-1}{j}\left(\frac{r}{2}\right)^{j+k}\frac{1}{\mu^j}\left(\frac{(j+k)I_{j+k}(\alpha x)\alpha^{j+k}}{x}\right)\exp^{-\left(\frac{\lambda+\mu+\gamma}{r}\right)x}.$$

The other steady-state probabilities are computed from Eq. (5) as follows

$$\begin{aligned}\hat{F}_j(s) &= \left(\frac{f(s)}{\mu}\right)^j\hat{F}_0(s)\\ &= \left(\frac{p - \sqrt{p^2 - \alpha^2}}{\frac{2\mu}{r}}\right)^j\hat{F}_0(s)\\ &= \left(\frac{r}{2\mu}\right)^j\left[p - \sqrt{p^2 - \alpha^2}\right]^j\hat{F}_0(s)\end{aligned}$$

which on the inversion yields,

$$F_j(x) = \left(\frac{r}{2\mu}\right)^j\left(\frac{jI_j(\alpha x)\alpha^j}{x}\right)\exp^{-\left(\frac{\lambda+\mu+\gamma}{r}\right)x} * F_0(x) \tag{8}$$

where $F_0(x)$ is explicitly given by Eq. (7). It still remains to determine the constant a which represents $F_0(0)$. Towards this end, adding the Eqs. (1) and (2) yields

$$r_0\frac{\mathrm{d}F_0(x)}{\mathrm{d}x} + \sum_{j=1}^{\infty}r\frac{\mathrm{d}F_j(x)}{\mathrm{d}x} = 0. \tag{9}$$

Integrating (9) from zero to infinity gives

$$r_0(F_0(0) - F_0(\infty)) + \sum_{j=0}^{\infty} r\big(F_j(0) - F_j(\infty)\big) = 0.$$

Note that $F_j(\infty) = \lim_{t\to\infty} \Pr(X(t) = j, C(t) < \infty) = \lim_{t\to\infty} \Pr(X(t) = j) = p_j$, where p_j's are the steady-state probabilities of the background queueing model. Using the boundary conditions, we obtain

$$F_0(0) = \frac{r_0 p_0 + \sum_{j=1}^{\infty} r p_j}{r_0} = \frac{(r_0 - r)p_0 + r}{r_0}.$$

Therefore, the constant a is explicitly given by

$$F_0(0) = a = \frac{(r_0 - r)(1-\rho) + r}{r_0} \tag{10}$$

Remark 1 When the catastrophe parameter $\gamma = 0$, after some direct calculations, it can be shown that $\hat{F}_0(s)$ and $\hat{F}_j(s)$ becomes

$$\hat{F}_0(s) = \frac{a}{s + \frac{\lambda}{r_0}} + a \sum_{k=1}^{\infty} \left(\frac{r}{2r_0}\right)^k \frac{\left(p - \sqrt{p^2 - \alpha^2}\right)^k}{\left(s + \frac{\lambda}{r_0}\right)^{k+1}}$$

and

$$\hat{F}_j(s) = a\left(\frac{r}{2\mu}\right)^j \sum_{k=0}^{\infty} \left(\frac{r}{2r_0}\right)^k \frac{\left(p - \sqrt{p^2 - \alpha^2}\right)^{k+j}}{\left(s + \frac{\lambda}{r_0}\right)^{k+1}} \quad j = 1, 2, \ldots$$

which is evidently the same as Eqs. (15) and (19) in Parthasarathy et al. [8].

Theorem 1 *The stationary buffer content distribution of the fluid model under consideration is given by*

$$\begin{aligned} F(x) = \Pr\{C < x\} &= \sum_{j=0}^{\infty} F_j(x) \\ &= \int_0^x exp^{-\left(\frac{\lambda+\mu+\gamma}{r}\right)u} \sum_{j=0}^{\infty} \left(\frac{r}{2\mu}\right)^j \frac{jI_j(\alpha u)\alpha^j}{u} F_0(x-u)\,\mathrm{d}u, \end{aligned}$$

where $F_0(x)$ is given by Eq. (7).

4 Conclusion

We provide explicit analytical expressions for the joint probability of the number of the customer in the background queueing model and the content of the buffer under steady state in terms of modified Bessel function of first kind. Also, the stationary buffer content distribution is obtained in closed form and is shown to coincide with the result of [8] as a special case when the catastrophe parameter γ equals to zero. To the best of our knowledge, the present paper is the first of its kind to analyse fluid models driven by queues subject to disasters, although earlier several authors have studied fluid queues modulated by vacation queueing models. Further extension to the present work can possibly include to analyse the model in time-dependent regime and also to study the process by considering the repair time which arise due to catastrophe.

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