Chapter 8
Statistical Approximation of Positive Linear Operators

In this chapter, we present some Korovkin-type approximation theorems for functions of two variables via statistical convergence, A-statistical convergence, and statistical A-summability. We also study rates of A-statistical convergence of a double sequence of positive linear operators. Through some concrete examples, we show that the results present in this chapter are stronger than the classical results.

8.1 Introduction

Let $F(\mathbb{R})$ denote the linear space of all real-valued functions defined on $\mathbb{R}$. Let $C(\mathbb{R})$ be the space of all functions f continuous on $\mathbb{R}$. We know that $C(\mathbb{R})$ is a normed space with the norm

$$\|f\|_\infty := \sup_{x\in\mathbb{R}}\left|f(x)\right|, \quad f \in C(\mathbb{R}).$$

We denote by $C_{2\pi}(\mathbb{R})$ the space of all 2π-periodic functions $f \in C(\mathbb{R})$, which is a normed spaces with

$$\|f\|_{2\pi} = \sup_{t\in\mathbb{R}}\left|f(t)\right|.$$

The classical Korovkin first and second theorems are stated as follows [59, 60].

Theorem I *Let* (T_n) *be a sequence of positive linear operators from* $C[0,1]$ *into* $F[0,1]$. *Then* $\lim_n \|T_n(f,x) - f(x)\|_\infty = 0$ *for all* $f \in C[0,1]$ *if and only if* $\lim_n \|T_n(f_i,x) - e_i(x)\|_\infty = 0$ *for* $i = 0,1,2$, *where* $e_0(x) = 1$, $e_1(x) = x$, *and* $e_2(x) = x^2$.

Theorem II *Let* (T_n) *be a sequence of positive linear operators from* $C_{2\pi}([0,1])$ *into* $F([0,1])$. *Then* $\lim_n \|T_n(f,x) - f(x)\|_\infty = 0$ *for all* $f \in C_{2\pi}([0,1])$ *if and only*

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if $\lim_n \|T_n(f_i, x) - f_i(x)\|_\infty = 0$ *for* $i = 0, 1, 2$, *where* $f_0(x) = 1$, $f_1(x) = \cos x$, *and* $f_2(x) = \sin x$.

Several mathematicians have worked on extending or generalizing the Korovkin theorems in many ways and to several settings, including function spaces, abstract Banach lattices, Banach algebras, Banach spaces, and so on. This theory is very useful in real analysis, functional analysis, harmonic analysis, measure theory, probability theory, summability theory and partial differential equations. But the foremost applications are concerned with constructive approximation theory, which uses it as a valuable tool. Even today, the development of Korovkin-type approximation theory is far from complete. Note that the first and second theorems of Korovkin are actually equivalent to the algebraic and trigonometric versions, respectively, of the classical Weierstrass approximation theorem [5]. For some recent work on this topic, we refer to [76].

8.2 Korovkin-Type Theorem via Statistical A-Summability

By $C(K)$ we denote the space of all continuous real-valued functions on any compact subset of the real two-dimensional space. Then $C(K)$ is a Banach space with the norm $\|\cdot\|_{C(K)}$ defined as

$$\|f\|_{C(K)} := \sup_{(x,y)\in K} \big|f(x, y)\big| \quad \big(f \in C(K)\big).$$

Before proceeding further, we recall the classical and statistical forms of Korovkin-type theorems studied in [37] and [124].

Theorem 8.1 [124] *Let* $\{L_{ij}\}$ *be a double sequence of positive linear operators acting from* $C(K)$ *into itself. Then, for all* $f \in C(K)$,

$$P\text{-}\lim_{m,n}\big\|L_{ij}(f) - f\big\|_{C(K)} = 0$$

if and only if

$$P\text{-}\lim_{m,n}\big\|L_{ij}(f_r) - f_r\big\|_{C(K)} = 0 \quad (r = 0, 1, 2, 3),$$

where $f_0(x, y) = 1$, $f_1(x, y) = x$, $f_2(x, y) = y$, $f_3(x, y) = x^2 + y^2$.

Theorem 8.2 [37] *Let* $A = (a_{ij}^{mn})$ *be a nonnegative RH-regular summability matrix. Let* $\{L_{ij}\}$ *be a double sequence of positive linear operators acting from* $C(K)$ *into itself. Then, for all* $f \in C(K)$,

$$\mathcal{S}_A\text{-}\lim_{m,n}\big\|L_{ij}(f) - f\big\|_{C(K)} = 0$$

if and only if

$$\mathcal{S}_A\text{-}\lim_{m,n} \left\| L_{ij}(f_r) - f_r \right\|_{C(K)} = 0 \quad (r = 0, 1, 2, 3),$$

where $f_0(x, y) = 1,\ f_1(x, y) = x,\ f_2(x, y) = y,\ f_3(x, y) = x^2 + y^2$.

By using the concept of statistical A-summability for single sequences, Korovkin-type theorems are proved in [35] and [36]. Now, we prove the following.

Theorem 8.3 *Let $A = (a_{ij}^{mn})$ be a nonnegative RH-regular summability matrix method. Let $\{L_{ij}\}$ be a double sequence of positive linear operators acting from $C(K)$ into itself. Then, for all $f \in C(K)$,*

$$\mathcal{S}\text{-}\lim_{m,n} \left\| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f) - f \right\|_{C(K)} = 0 \tag{8.1}$$

if and only if

$$\mathcal{S}\text{-}\lim_{m,n} \left\| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f_r) - f_r \right\|_{C(K)} = 0 \quad (r = 0, 1, 2, 3) \tag{8.2}$$

where $f_0(x, y) = 1,\ f_1(x, y) = x,\ f_2(x, y) = y,\ f_3(x, y) = x^2 + y^2$.

Proof Condition (8.2) follows immediately from condition (8.1) since each $f_r \in C(K)$ $(r = 0, 1, 2, 3)$. Let us prove the converse. By the continuity of f on the compact set K, we can write $|f(x, y)| \le M$, where $M = \|f\|_{C(K)}$. Also, since $f \in C(K)$, for every $\epsilon > 0$, there is a number $\delta > 0$ such that $|f(u, v) - f(x, y)| < \epsilon$ for all $(u, v) \in K$ satisfying $|u - x| < \delta$ and $|v - y| < \delta$. Hence, we get

$$\left| f(u, v) - f(x, y) \right| < \epsilon + \frac{2M}{\delta^2} \left\{ (u - x)^2 + (v - y)^2 \right\}. \tag{8.3}$$

Since $L_{i,j}$ is linear and positive, from (8.3) we obtain that, for any $m, n \in \mathbb{N}$,

$$\begin{aligned} &\left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f; x, y) - f(x, y) \right| \\ &\quad \le \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}\big(\left| f(u, v) - f(x, y) \right|; x, y \big) \\ &\qquad + \left| f(x, y) \right| \left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f_0; x, y) - f_0(x, y) \right| \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}\left(\epsilon + \frac{2M}{\delta^2}\left[(u-x)^2 + (v-y)^2\right]; x, y\right) \\
&\quad + \left|f(x,y)\right| \left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f_0; x, y) - f_0(x,y) \right| \\
&\leq \epsilon + (\epsilon + M) \left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f_0; x, y) - f_0 \right| \\
&\quad + \frac{2M}{\delta^2} \Bigg\{ \left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f_3; x, y) - f_3(x,y) \right| \\
&\quad + 2|x| \left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f_1; x, y) - f_1(x,y) \right| \\
&\quad + 2|y| \left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f_2; x, y) - f_2(x,y) \right| \\
&\quad + \left(x^2 + y^2\right) \left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f_0; x, y) - f_0(x,y) \right| \\
&\leq \epsilon + \left(\epsilon + M + \frac{2M}{\delta^2}\left(C^2 + D^2\right)\right) \left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f_0; x, y) - f_0(x,y) \right| \\
&\quad + \frac{2M}{\delta^2} \left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f_3; x, y) - f_3(x,y) \right| \\
&\quad + \frac{4MC}{\delta^2} \left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f_1; x, y) - f_1(x,y) \right| \\
&\quad + \frac{4MD}{\delta^2} \left| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f_2; x, y) - f_2(x,y) \right|,
\end{aligned}
$$

where $C := \max|x|$ and $D := \max|y|$. Taking the supremum over $(x, y) \in K$, we get

$$
\left\| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f) - f \right\| \leq \epsilon + B \sum_{r=0}^{3} \left\| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f_r; x, y) - f_r(x,y) \right\|,
$$

where

$$B := \max\left\{\epsilon + M + \frac{2M}{\delta^2}\left(C^2 + D^2\right), \frac{2M}{\delta^2}, \frac{4MC}{\delta^2}, \frac{4MD}{\delta^2}\right\}.$$

Now for a given $\sigma > 0$, choose $\epsilon > 0$ such that $\epsilon < \sigma$ and define

$$E := \left\{(m,n) \in \mathbb{N}^2 : \left\| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f;x,y) - f(x,y) \right\| \geq \sigma \right\},$$

$$E_r := \left\{(m,n) \in \mathbb{N}^2 : \left\| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f_r;x,y) - f_r(x,y) \right\| \geq \frac{\sigma - \epsilon}{4B} \right\},$$

$$r = 0, 1, 2, 3.$$

Then $E \subset \bigcup_{r=0}^{3} E_r$, and so $\delta_2(E) \leq \sum_{r=0}^{3} \delta_2(E_r)$. By considering this inequality and using (8.2) we obtain (8.1), which completes the proof. □

Example 8.4 Now we will show that Theorem 8.3 is stronger than its classical and statistical forms. Let $A = (a_{ij}^{mn})$ be a four-dimensional Cesàro matrix, i.e.,

$$a_{ij}^{mn} = \begin{cases} 1/mn & \text{if } i \leq m \text{ and } j \leq n, \\ 0 & \text{otherwise,} \end{cases}$$

and let $x = (x_{ij})$ be defined as

$$x_{ij} = (-1)^j \quad \text{for all } i.$$

Then this sequence is neither P-convergent nor A-statistically convergent, but $\mathcal{S}\text{-}\lim Ax = 0$.

Now, consider the Bernstein operators (see [119]) defined for $f \in C(K)$ by

$$B_{ij}(f;x,y) = \sum_{k=0}^{i} \sum_{l=0}^{j} f\left(\frac{k}{i}, \frac{l}{j}\right) C(i,k) x^k (1-x)^{i-k} C(j,l) y^j (1-y)^{j-l} \quad (8.4)$$

for $(x, y) \in K = [0, 1] \times [0, 1]$. By using these operators, define the following positive linear operators on $C(K)$:

$$L_{ij}(f;x,y) = (1 + x_{ij}) B_{ij}(f;x,y), \quad (x,y) \in K, f \in C(K). \quad (8.5)$$

Then observe that

$$L_{ij}(f_0;x,y) = (1 + x_{ij}) f_0(x,y),$$

$$L_{ij}(f_1;x,y) = (1 + x_{ij}) f_1(x,y),$$

$$L_{ij}(f_2;x,y) = (1 + x_{ij}) f_2(x,y),$$

$$L_{ij}(f_3; x, y) = (1 + x_{ij})\left(f_3(x, y) + \frac{x - x^2}{i} + \frac{y - y^2}{j} \right),$$

where $f_0(x, y) = 1$, $f_1(x, y) = x$, $f_2(x, y) = y$, $f_3(x, y) = x^2 + y^2$. Since $\mathcal{S}\text{-}\lim Ax = 0$, we obtain

$$\mathcal{S}\text{-}\lim_{m,n} \left\| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f_r) - f_r \right\|_{C(K)} = \mathcal{S}\text{-}\lim_{m,n} \frac{1}{mn} \left\| \sum_{i,j=1,1}^{\infty,\infty} L_{ij}(f_r) - f_r \right\|_{C(K)} = 0$$

for $r = 0, 1, 2, 3$. Hence, by Theorem 8.3 we conclude that

$$\mathcal{S}\text{-}\lim_{m,n} \left\| \sum_{i,j=1,1}^{\infty,\infty} a_{ij}^{mn} L_{ij}(f) - f \right\|_{C(K)} = 0$$

for any $f \in C(K)$.

However, since the P-limit and the statistical limit of the double sequence (x_{ij}) are not zero, it follows that, for $r = 0, 1, 2, 3$, $\|L_{ij}(f_i) - f_i\|_{C(K)}$ is neither P-convergent nor statistically convergent to zero. So, Theorems 8.1 and 8.2 do not work for our operators defined by (8.4).

8.3 Korovkin-Type Theorem via *A*-Statistical Convergence

Boyanov and Veselinov [17] have proved the Korovkin theorem on $C[0, \infty)$ by using the test functions 1, e^{-x}, e^{-2x}. In this section, we first extend the result of Boyanov and Veselinov for functions of two variables by using the notion of P-convergence and further generalize for A-statistical convergence.

Theorem 8.5 *Let $(T_{j,k})$ be a double sequence of positive linear operators from $C(I^2)$ into $C(I^2)$. Then, for all $f \in C(I^2)$,*

$$P\text{-}\lim_{j,k\to\infty} \left\| T_{j,k}(f; x, y) - f(x, y) \right\|_\infty = 0 \tag{8.6}$$

if and only if

$$P\text{-}\lim_{j,k\to\infty} \left\| T_{j,k}(1; x, y) - 1 \right\|_\infty = 0, \tag{8.7}$$

$$P\text{-}\lim_{j,k\to\infty} \left\| T_{j,k}\left(e^{-s}; x, y\right) - e^{-x} \right\|_\infty = 0, \tag{8.8}$$

$$P\text{-}\lim_{j,k\to\infty} \left\| T_{j,k}\left(e^{-t}; x, y\right) - e^{-y} \right\|_\infty = 0, \tag{8.9}$$

$$P\text{-}\lim_{j,k\to\infty} \left\| T_{j,k}\left(e^{-2s} + e^{-2t}; x, y\right) - \left(e^{-2x} + e^{-2y}\right) \right\|_\infty = 0. \tag{8.10}$$

Proof Since each of the functions $1, e^{-x}, e^{-y}, e^{-2x}+e^{-2y}$ belongs to $C(I^2)$, conditions (8.7)–(8.10) follow immediately from (8.6). Let $f \in C(I^2)$. There exists a constant M such that $|f(x,y)| \le M$ for all $(x,y) \in I^2$, where $M = \|f\|_\infty$. Therefore,

$$\left|f(s,t)-f(x,y)\right| \le 2M, \quad 0 \le s,t,x,y<\infty. \tag{8.11}$$

It is easy to prove that for given $\varepsilon>0$, there is $\delta>0$ such that

$$\left|f(s,t)-f(x,y)\right| < \varepsilon \tag{8.12}$$

whenever $|e^{-s}-e^{-x}|<\delta$ and $|e^{-t}-e^{-y}|<\delta$ for all $(x,y)\in I^2$.

Using (8.11), (8.12) and putting $\psi_1=\psi_1(s,x)=(e^{-s}-e^{-x})^2$ and $\psi_2=\psi_2(t,y)=(e^{-t}-e^{-y})^2$, we get

$$\left|f(s,t)-f(x,y)\right| < \varepsilon + \frac{2M}{\delta^2}(\psi_1+\psi_2) \quad \forall |s-x|<\delta \text{ and } |t-y|<\delta,$$

that is,

$$-\varepsilon - \frac{2M}{\delta^2}(\psi_1+\psi_2) < f(s,t)-f(x,y) < \varepsilon + \frac{2M}{\delta^2}(\psi_1+\psi_2).$$

Now, operate $T_{j,k}(1;x,y)$ to this inequality. Since $T_{j,k}(f;x,y)$ is monotone and linear, we obtain

$$\begin{aligned} T_{j,k}(1;x,y)\left(-\varepsilon - \frac{2M}{\delta^2}(\psi_1+\psi_2)\right) &< T_{j,k}(1;x,y)\big(f(s,t)-f(x,y)\big) \\ &< T_{j,k}(1;x,y)\left(\varepsilon + \frac{2M}{\delta^2}(\psi_1+\psi_2)\right). \end{aligned}$$

Note that x and y are fixed, and so $f(x,y)$ is a constant number. Therefore, by simple calculations we get

$$\begin{aligned} &\left|T_{j,k}(f;x,y)-f(x,y)\right| \\ &\quad \le \varepsilon + (\varepsilon+M)\left|T_{j,k}(1;x,y)-1\right| + \frac{2M}{\delta^2}\left|e^{-2x}+e^{-2y}\right|\left|T_{j,k}(1;x,y)-1\right| \\ &\qquad + \frac{2M}{\delta^2}\left|T_{j,k}\big(e^{-2s}+e^{-2t};x,y\big)-\big(e^{-2x}+e^{-2y}\big)\right| \\ &\qquad + \frac{4M}{\delta^2}\left|e^{-x}\right|\left|T_{j,k}\big(e^{-s};x,y\big)-e^{-x}\right| + \frac{4M}{\delta^2}\left|e^{-y}\right|\left|T_{j,k}\big(e^{-t};x,y\big)-e^{-y}\right| \\ &\quad \le \varepsilon + \left(\varepsilon + M + \frac{4M}{\delta^2}\right)\left|T_{j,k}(1;x,y)-1\right| \\ &\qquad + \frac{2M}{\delta^2}\left|T_{j,k}\big(e^{-2s}+e^{-2t};x,y\big)-\big(e^{-2x}+e^{-2y}\big)\right| \end{aligned}$$

$$+\frac{4M}{\delta^2}\left|T_{j,k}\left(e^{-s};x,y\right)-e^{-x}\right|+\frac{4M}{\delta^2}\left|T_{j,k}\left(e^{-t};x,y\right)-e^{-y}\right|. \tag{8.13}$$

Since $|e^{-x}|, |e^{-y}| \leq 1$ for all $x, y \in I$, taking $\sup_{(x,y)\in I^2}$, we get

$$\begin{aligned}&\left\|T_{j,k}(f;x,y)-f(x,y)\right\|_\infty\\&\quad\leq \varepsilon+K\Big(\left\|T_{j,k}(1;x,t)-1\right\|_\infty+\left\|T_{j,k}\left(e^{-s};x,y\right)-e^{-x}\right\|_\infty\\&\qquad+\left\|T_{j,k}\left(e^{-t};x,y\right)-e^{-y}\right\|_\infty+\left\|T_{j,k}\left(e^{-2s}+e^{-2t};x,y\right)-\left(e^{-2x}+e^{-2y}\right)\right\|_\infty\Big),\end{aligned} \tag{8.14}$$

where $K=\max\{\varepsilon+M+\frac{4M}{\delta^2}, \frac{4M}{\delta^2}, \frac{2M}{\delta^2}\}$. Taking P-lim as $j,k\to\infty$ and using (8.7)–(8.10), we get

$$P\text{-}\lim_{p,q\to\infty}\left\|T_{j,k}(f;x,y)-f(x,y)\right\|_\infty=0, \quad \text{uniformly in } m,n. \qquad \square$$

In the following theorem, we use the notion of almost convergence of double sequences to generalize the above theorem. We also give an example showing its importance.

Theorem 8.6 *Let $(T_{j,k})$ be a double sequence of positive linear operators from $C(I^2)$ into $C(I^2)$. Then, for all $f\in C(I^2)$,*

$$\mathcal{S}_A\text{-}\lim_{j,k\to\infty}\left\|T_{j,k}(f;x,y)-f(x,y)\right\|_\infty=0 \tag{8.15}$$

if and only if

$$\mathcal{S}_A\text{-}\lim_{j,k\to\infty}\left\|T_{j,k}(1;x,y)-1\right\|_\infty=0, \tag{8.16}$$

$$\mathcal{S}_A\text{-}\lim_{j,k\to\infty}\left\|T_{j,k}\left(e^{-s};x,y\right)-e^{-x}\right\|_\infty=0, \tag{8.17}$$

$$\mathcal{S}_A\text{-}\lim_{j,k\to\infty}\left\|T_{j,k}\left(e^{-t};x,y\right)-e^{-y}\right\|_\infty=0, \tag{8.18}$$

$$\mathcal{S}_A\text{-}\lim_{j,k\to\infty}\left\|T_{j,k}\left(e^{-2s}+e^{-2t};x,y\right)-\left(e^{-2x}+e^{-2y}\right)\right\|_\infty=0. \tag{8.19}$$

Proof For a given $r>0$, choose $\epsilon>0$ such that $\epsilon<r$. Define the following sets:

$$D:=\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\left\|T_{j,k}(f;x,y)-f(x,y)\right\|_\infty\geq r\right\},$$

$$D_1:=\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\left\|T_{j,k}(1;x,t)-1\right\|_\infty\geq\frac{r-\epsilon}{4K}\right\},$$

$$D_2:=\left\{(j,k)\in\mathbb{N}\times\mathbb{N}:\left\|T_{j,k}\left(e^{-s};x,y\right)-e^{-x}\right\|_\infty\geq\frac{r-\epsilon}{4K}\right\},$$

$$D_3 := \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \left\|T_{j,k}\left(e^{-t}; x, y\right) - e^{-y}\right\|_\infty \geq \frac{r-\epsilon}{4K}\right\},$$

$$D_4 := \left\{(j,k) \in \mathbb{N}\times\mathbb{N} : \left\|T_{j,k}\left(e^{-2s}+e^{-2t}; x, y\right) - \left(e^{-2x}+e^{-2y}\right)\right\|_\infty \geq \frac{r-\epsilon}{4K}\right\}.$$

Then by (8.14) it follows that $D \subset D_1 \cup D_2 \cup D_3 \cup D_4$. Hence, $\delta_A^{(2)}(D) \leq \delta_A^{(2)}(D_1) + \delta_A^{(2)}(D_2) + \delta_A^{(2)}(D_3) + \delta_A^{(2)}(D_4)$. Using (8.16)–(8.19), we get $\delta_A^{(2)}(D) = 0$, i.e.,

$$\mathcal{S}_A\text{-}\lim_{j,k\to\infty}\left\|T_{j,k}(f; x, y) - f(x, y)\right\|_\infty = 0.$$

□

In the following example, we construct a double sequence of positive linear operators that satisfies the conditions of Theorem 8.6 but does not satisfy the conditions of Theorem 8.5, that is, Theorem 8.6 is stronger than Theorem 8.5.

Example 8.7 Consider the sequence of classical Baskakov operators of two variables [48]

$$B_{m,n}(f; x, y) := \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} f\left(\frac{j}{m}, \frac{k}{n}\right)\binom{m-1+j}{j}\binom{n-1+k}{k} \times x^j(1+x)^{-m-j}y^k(1+y)^{-n-k} \tag{8.20}$$

for $0 \leq x, y < \infty$.

Take A as in Example 8.4. Define a double sequence $z = (z_{mn})$ by

$$z_{mn} = \begin{cases} 1 & \text{if } m \text{ and } n \text{ are squares,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $L_{m,n} : C(I^2) \to C(I^2)$ be defined by

$$L_{m,n}(f; x, y) = (1 + z_{mn})B_{m,n}(f; x, y).$$

Since

$$B_{m,n}(1; x, y) = 1,$$

$$B_{m,n}\left(e^{-s}; x, y\right) = \left(1 + x - xe^{-\frac{1}{m}}\right)^{-m},$$

$$B_{m,n}\left(e^{-t}; x, y\right) = \left(1 + y - ye^{-\frac{1}{n}}\right)^{-n},$$

$$B_{m,n}\left(e^{-2s}+e^{-2t}; x, y\right) = \left(1 + x - xe^{-\frac{2}{m}}\right)^{-m} + \left(1 + y - ye^{-\frac{2}{n}}\right)^{-n},$$

we have that the sequence $(L_{m,n})$ satisfies conditions (8.16)–(8.19). Hence, by Theorem 8.6 we have

$$\mathcal{S}_A\text{-}\lim_{m,n\to\infty}\left\|L_{m,n}(f; x, y) - f(x, y)\right\|_\infty = 0.$$

On the other hand, we get $L_{m,n}(f;0,0)=(1+z_{mn})f(0,0)$ since $B_{m,n}(f;0,0)=f(0,0)$, and hence,

$$\left\|L_{m,n}(f;x,y)-f(x,y)\right\|_{\infty}\geq\left|L_{m,n}(f;0,0)-f(0,0)\right|=z_{mn}\left|f(0,0)\right|.$$

We see that $(L_{m,n})$ does not satisfy the conditions of Theorem 8.5 since P-$\lim_{m,n\to\infty}z_{mn}$ does not exist.

8.4 A-Statistical Approximation for Periodic Functions and Rate of A-Statistical Convergence

In this section, we present a Korovkin-type approximation theorem for periodic functions via A-statistical convergence and also study the rate of A-statistical convergence of a double sequence of positive linear operators defined from $C^*(\mathbb{R}^2)$ into $C^*(\mathbb{R}^2)$, where $C^*(\mathbb{R}^2)$ is the space of all 2π-periodic and real-valued continuous functions on $\mathbb{R}^2$ (see Demirci and Dirik [34] and Duman and Erkus [38]).

Theorem 8.8 *Let $A=(a_{jkmn})$ be a nonnegative RH-regular summability matrix, and let (L_{mn}) be a double sequence of positive linear operators acting from $C^*(\mathbb{R}^2)$ into $C^*(\mathbb{R}^2)$. Then, for all $f\in C^*(\mathbb{R}^2)$,*

$$\mathcal{S}_A\text{-}\lim_{m,n\to\infty}\left\|L_{mn}(f)-f\right\|_{C^*(\mathbb{R}^2)}=0 \tag{8.21}$$

if and only if

$$\mathcal{S}_A\text{-}\lim_{m,n\to\infty}\left\|L_{mn}(f_i)-f_i\right\|_{C^*(\mathbb{R}^2)}=0,\quad i=0,1,2,3,4, \tag{8.22}$$

where $f_0(x,y)=1$, $f_1(x,y)=\sin x$, $f_2(x,y)=\sin y$, $f_3(x,y)=\cos x$, and $f_4(x,y)=\cos y$.

Proof Since each of the functions f_0, f_1, f_2, f_3, f_4 belongs to $C^*(\mathbb{R}^2)$, the necessity follows immediately from (8.21). Let conditions (8.22) hold, and let $f\in C^*(\mathbb{R}^2)$. Let I and J be closed intervals of length 2π. Fix $(x,y)\in I\times J$. By the continuity of f at (x,y) it follows that for given $\varepsilon>0$, there is a number $\delta>0$ such that, for all $(u,v)\in\mathbb{R}^2$,

$$\left|f(u,v)-f(x,y)\right|<\varepsilon \tag{8.23}$$

whenever $|u-x|,|v-y|<\delta$. Since f is bounded, it follows that

$$\left|f(u,v)-f(x,y)\right|\leq M_f=\|f\|_{C^*(\mathbb{R}^2)} \tag{8.24}$$

for all $(u,v)\in\mathbb{R}^2$.

For all $(u, v) \in (x-\delta, 2\pi + x - \delta] \times (y-\delta, 2\pi + y - \delta]$, it is well known that

$$\left|f(u,v) - f(x,y)\right| < \varepsilon + \frac{2M_f}{\sin^2\frac{\delta}{2}}\psi(u,v), \tag{8.25}$$

where $\psi(u,v) = \sin^2(\frac{u-x}{2}) + \sin^2(\frac{v-y}{2})$. Since the function $f \in C^*(\mathbb{R}^2)$ is 2π-periodic, inequality (8.25) holds for $(u, v) \in \mathbb{R}^2$. Then, we obtain

$$\begin{aligned}
&\left|L_{mn}(f;x,y) - f(x,y)\right| \\
&\quad \le L_{mn}\big(\left|f(u,v) - f(x,y)\right|; x,y\big) + \left|f(x,y)\right|\left|L_{mn}(f_0;x,y) - f_0(x,y)\right| \\
&\quad \le \left|L_{mn}\left(\varepsilon + \frac{2M_f}{\sin^2\frac{\delta}{2}}\psi(u,v); x,y\right)\right| + M_f\left|L_{mn}(f_0;x,y) - f_0(x,y)\right| \\
&\quad \le \varepsilon + (\varepsilon + M_f)\left|L_{mn}(f_0;x,y) - f_0(x,y)\right| + \frac{M_f}{\sin^2\frac{\delta}{2}}\big\{2\left|L_{mn}(f_0;x,y) - f_0(x,y)\right| \\
&\qquad + |\sin x|\left|L_{mn}(f_1;x,y) - f_1(x,y)\right| + |\sin y|\left|L_{mn}(f_2;x,y) - f_3(x,y)\right| \\
&\qquad + |\cos x|\left|L_{mn}(f_3;x,y) - f_3(x,y)\right| + |\cos y|\left|L_{mn}(f_4;x,y) - f_4(x,y)\right|\big\} \\
&\quad < \varepsilon + K\sum_{i=0}^{4}\left|L_{mn}(f_i;x,y) - f_i(x,y)\right|,
\end{aligned} \tag{8.26}$$

where $K := \varepsilon + M_f + \frac{2M_f}{\sin^2\frac{\delta}{2}}$. Now, taking $\sup_{(x,y)\in I\times J}$, we get

$$\left\|L_{mn}(f) - f\right\|_{C^*(\mathbb{R}^2)} < \varepsilon + K\sum_{i=0}^{4}\left\|L_{mn}(f_i) - f_i\right\|_{C^*(\mathbb{R}^2)}. \tag{8.27}$$

Now for a given $r > 0$, choose $\varepsilon' > 0$ such that $\varepsilon' < r$. Define the following sets:

$$D = \left\{(m,n) : \left\|L_{mn}(f) - f\right\|_{C^*(\mathbb{R}^2)} \ge r\right\},$$

$$D_i = \left\{(m,n) : \left\|L_{mn}(f_i) - f_i\right\|_{C^*(\mathbb{R}^2)} \ge \frac{r-\varepsilon'}{5K}\right\},$$

where $i = 0, 1, 2, 3, 4$. Then, by (8.27),

$$D \subseteq \bigcup_{i=0}^{4} D_i,$$

and so

$$\sum_{(m,n)\in D} a_{jkmn} \le \sum_{i=0}^{4}\sum_{(m,n)\in D_i} a_{jkmn},$$

i.e.,

$$\delta_A^{(2)}(D) \leq \sum_{i=0}^{4} \delta_A^{(2)}(D_i).$$

Now, using (8.22), we get

$$\mathcal{S}_A\text{-}\lim_{m,n\to\infty} \left\| L_{mn}(f) - f \right\|_{C^*(\mathbb{R}^2)} = 0.$$

□

Remark 8.9 If we replace the matrix A by the identity matrix for four-dimensional matrices in Theorem 8.8, then we immediately get the following result in Pringsheim's sense.

Corollary 8.10 *Let $A = (a_{jkmn})$ be a nonnegative RH-regular summability matrix, and let (L_{mn}) be a double sequence of positive linear operators acting from $C^*(\mathbb{R}^2)$ into $C^*(\mathbb{R}^2)$. Then, for all $f \in C^*(\mathbb{R}^2)$,*

$$P\text{-}\lim_{m,n\to\infty} \left\| L_{mn}(f) - f \right\|_{C^*(\mathbb{R}^2)} = 0 \tag{8.28}$$

if and only if

$$P\text{-}\lim_{m,n\to\infty} \left\| L_{mn}(f_i) - f_i \right\|_{C^*(\mathbb{R}^2)} = 0, \quad i = 0, 1, 2, 3, 4. \tag{8.29}$$

Example 8.11 Now we present an example of double sequences of positive linear operators, showing that Corollary 8.10 does not work but our approximation theorem works. We consider the double sequence of Fejér operators on $C^*(\mathbb{R}^2)$

$$\sigma_{mn}(f; x, y) = \frac{1}{(n\pi)} \cdot \frac{1}{(n\pi)} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(u, v) F_m(u) F_n(v)\, du\, dv, \tag{8.30}$$

where

$$F_m(u) = \frac{\sin^2(m(u-x)/2)}{\sin^2((u-x)/2)} \quad \text{and} \quad \frac{1}{\pi} \int_{-\pi}^{\pi} F_m(u)\, du = 1.$$

Observe that

$$\begin{aligned} \sigma_{mn}(f_0; x, y) &= f_0(x, y), \qquad \sigma_{mn}(f_1; x, y) = \frac{m-1}{m} f_1(x, y), \\ \sigma_{mn}(f_2; x, y) &= \frac{n-1}{n} f_2(x, y), \qquad \sigma_{mn}(f_3; x, y) = \frac{m-1}{m} f_3(x, y), \\ &\text{and} \quad \sigma_{mn}(f_4; x, y) = \frac{n-1}{n} f_4(x, y). \end{aligned} \tag{8.31}$$

Now take $A = (C, 1, 1)$ and define the double sequence $\alpha = (\alpha_{mn})$ by

$$\alpha_{mn} = \begin{cases} 1 & \text{if } m \text{ and } n \text{ are squares}, \\ 0 & \text{otherwise}. \end{cases}$$

We observe that $\alpha = (\alpha_{mn})$ is not P-convergent but

$$\mathcal{S}_{(C,1,1)}\text{-}\lim \alpha = 0. \tag{8.32}$$

Let us define the operators $L_{mn} : C^*(\mathbb{R}^2) \to C^*(\mathbb{R}^2)$ by

$$L_{mn}(f; x, y) = (1 + \alpha_{mn})\sigma_{mn}(f; x, y). \tag{8.33}$$

Then, observe that the double sequence of positive linear operators (L_{mn}) defined by (8.33) satisfies all hypotheses of Theorem 8.8. Hence, by (8.31) we have that, for all $f \in C^*(\mathbb{R}^2)$,

$$\mathcal{S}_A\text{-}\lim_{m,n\to\infty} \left\| L_{mn}(f) - f \right\|_{C^*(\mathbb{R}^2)} = 0.$$

Since (α_{mn}) is not P-convergent, the sequence (L_{mn}) given by (8.33) does not converge uniformly to the function $f \in C^*(\mathbb{R}^2)$. So, we conclude that Corollary 8.10 does not work for the operators (L_{mn}) given by (8.33) while Theorem 8.8 still works. Hence, we conclude that the $\mathcal{S}_A$-version is stronger than the P-version.

Definition 8.12 Let $A = (a_{jkmn})$ be a nonnegative RH-regular summability matrix. Let (β_{mn}) be a positive nonincreasing double sequence. We say that a double sequence $x = (x_{mn})$ is A-statistically convergent to the number L with the rate $o(\beta_{mn})$ if for every $\varepsilon > 0$,

$$P\text{-}\lim_{j,k\to\infty} \frac{1}{\beta_{jk}} \sum_{(m,n)\in K(\epsilon)} a_{jkmn} = 0,$$

where $K(\epsilon) := \{(m,n) \in \mathbb{N} \times \mathbb{N} : |x_{mn} - \ell| \geq \epsilon\}$. In this case, we write $x_{mn} - L = st_A^{(2)} - o(\beta_{mn})$ as $m, n \to \infty$.

Now, we recall the notion of modulus of continuity. The modulus of continuity of $f \in C^*(\mathbb{R}^2)$, denoted by $\omega(f, \delta)$ for $\delta > 0$, is defined by

$$\omega(f,\delta) = \sup\left\{ \left| f(u,v) - f(x,y) \right| : (u,v), (x,y) \in \mathbb{R}^2, \sqrt{(u-x)^2 + (v-y)^2} \leq \delta \right\}.$$

It is well known that

$$\begin{aligned} \left| f(u,v) - f(x,y) \right| &\leq \omega\left(f, \sqrt{(u-x)^2 + (v-y)^2}\right) \\ &\leq \omega(f,\delta)\left(\frac{\sqrt{(u-x)^2 + (v-y)^2}}{\delta} + 1 \right). \end{aligned} \tag{8.34}$$

Then we have the following result.

Theorem 8.13 *Let $A = (a_{jkmn})$ be a nonnegative RH-regular summability matrix, and let (L_{mn}) be a double sequence of positive linear operators acting from $C^*(\mathbb{R}^2)$ into $C^*(\mathbb{R}^2)$. Let (α_{mn}) and (β_{mn}) be two positive nonincreasing sequences. Suppose that*

(i) $\|L_{mn}(f_0) - f_0\|_{C^*(\mathbb{R}^2)} = \mathcal{S}_A\text{-}o(\alpha_{mn})$,
(ii) $\omega(f, \lambda_{mn}) = \mathcal{S}_A\text{-}o(\beta_{mn})$, *where* $\lambda_{mn} = \sqrt{\|L_{mn}(\varphi)\|_{C^*(\mathbb{R}^2)}}$ *with*

$$\varphi(u, v) = \sin^2\left(\frac{u-x}{2}\right) + \sin^2\left(\frac{v-y}{2}\right) \quad \textit{for } (u, v), (x, y) \in \mathbb{R}^2.$$

Then, for all $f \in C^*(\mathbb{R}^2)$,

$$\left\| L_{mn}(f) - f \right\|_{C^*(\mathbb{R}^2)} = \mathcal{S}_A\text{-}o(\gamma_{mn}), \tag{8.35}$$

where $\gamma_{mn} = \max\{\alpha_{mn}, \beta_{mn}\}$.

Proof Let $f \in C^*(\mathbb{R}^2)$ and $(x, y) \in [-\pi, \pi] \times [-\pi, \pi]$. Let $\delta > 0$. We have following cases.

Case I. If $\delta < |u - x| \le \pi$, $\delta < |v - y| \le \pi$, then $|u - x| \le \pi |\sin \frac{u-x}{2}|$ and $|v - y| \le \pi |\sin \frac{v-y}{2}|$. Therefore, by (8.34) we have

$$\left| f(u, v) - f(x, y) \right| \le \omega(f, \delta)\left(\pi^2 \frac{\sin^2(\frac{u-x}{2}) + \sin^2(\frac{v-y}{2})}{\delta^2} + 1\right). \tag{8.36}$$

Case II. $|u - x| > \pi$, $|v - y| \le \pi$. Let k be an integer such that $|u + 2k\pi - x| \le \pi$. Then

$$\begin{aligned}\left| f(u, v) - f(x, y) \right| &= \left| f(u + 2k\pi, v) - f(x, y) \right| \\ &\le \omega(f, \delta)\left(\pi^2 \frac{\sin^2(\frac{u+2k\pi-x}{2}) + \sin^2(\frac{v-y}{2})}{\delta^2} + 1\right) \\ &= \omega(f, \delta)\left(\pi^2 \frac{\sin^2(\frac{u-x}{2}) + \sin^2(\frac{v-y}{2})}{\delta^2} + 1\right).\end{aligned}$$

Similarly, in other two cases where $|u - x| \le \pi$, $|v - y| > \pi$ and $|u - x| > \pi$, $|v - y| > \pi$, we obtain (8.36).

Now, using the definition of modulus of continuity and the linearity and positivity of the operators L_{mn}, we get

$$\begin{aligned}&\left| L_{mn}(f; x, y) - f(x, y) \right| \\ &\quad \le L_{mn}\left(\left| f(u, v) - f(x, y) \right|; x, y\right) + \left| f(x, y) \right| \left| L_{mn}(f_0; x, y) - f_0(x, y) \right| \\ &\quad \le \omega(f, \delta) L_{mn}(f_0; x, y) + \pi^2 \frac{\omega(f, \delta)}{\delta^2} L_{mn}(\varphi; x, y) \\ &\qquad + \left| f(x, y) \right| \left| L_{mn}(f_0; x, y) - f_0(x, y) \right|.\end{aligned}$$

Taking the supremum over (x, y) on both sides of the above inequality and

$$\delta := \delta_{mn} = \sqrt{\left\| L_{mn}(\varphi) \right\|_{C^*(\mathbb{R}^2)}},$$

we obtain

$$\begin{aligned}\left\|L_{mn}(f)-f\right\|_{C^*(\mathbb{R}^2)} &\le \omega(f,\delta_{mn})\left\|L_{mn}(f_0)-f_0\right\|_{C^*(\mathbb{R}^2)} \\ &\quad + \left(1+\pi^2\right)\omega(f,\delta_{mn}) + M\left\|L_{mn}(f_0)-f_0\right\|_{C^*(\mathbb{R}^2)},\end{aligned} \tag{8.37}$$

where $M := \|f\|_{C^*(\mathbb{R}^2)}$. Now, for a given $\varepsilon > 0$, define the following sets:

$$D = \left\{(m,n) : \left\|L_{mn}(f)-f\right\|_{C^*(\mathbb{R}^2)} \ge \varepsilon\right\},$$

$$D_1 = \left\{(m,n) : \left\|L_{mn}(f_0)-f_0\right\|_{C^*(\mathbb{R}^2)} \ge \frac{\varepsilon}{3}\right\},$$

$$D_2 = \left\{(m,n) : \omega(f,\delta_{mn}) \ge \frac{\varepsilon}{3(1+\pi^2)}\right\},$$

$$D_3 = \left\{(m,n) : \left\|L_{mn}(f)-f\right\|_{C^*(\mathbb{R}^2)} \ge \frac{\varepsilon}{3M}\right\}.$$

Then $D \subset D_1 \cup D_2 \cup D_3$. Further, defining

$$D_4 = \left\{(m,n) : \omega(f,\delta_{mn}) \ge \sqrt{\frac{\varepsilon}{3}}\right\},$$

$$D_5 = \left\{(m,n) : \left\|L_{mn}(f)-f\right\|_{C^*(\mathbb{R}^2)} \ge \sqrt{\frac{\varepsilon}{3}}\right\},$$

we see that $D_1 \subset D_4 \cup D_5$. Therefore, $D \subset \bigcup_{i=2}^{5} D_i$. Therefore, since $\gamma_{mn} = \max\{\alpha_{mn}, \beta_{mn}\}$, we conclude that, for every $(j,k) \in \mathbb{N} \times \mathbb{N}$,

$$\begin{aligned}\frac{1}{\gamma_{jk}} \sum_{(m,n)\in D} a_{mnjk} &\le \frac{1}{\alpha_{jk}} \sum_{(m,n)\in D_1} a_{mnjk} + \frac{1}{\beta_{jk}} \sum_{(m,n)\in D_2} a_{mnjk} \\ &\quad + \frac{1}{\alpha_{jk}} \sum_{(m,n)\in D_3} a_{mnjk} + \frac{1}{\beta_{jk}} \sum_{(m,n)\in D_4} a_{mnjk}.\end{aligned}$$

Letting $j,k \to \infty$ and using conditions (i) and (ii), we get

$$\left\|L_{mn}(f)-f\right\|_{C^*(\mathbb{R}^2)} = \mathcal{S}_A\text{-}o(\gamma_{mn}). \qquad \square$$

8.5 Exercises

1 Prove a Korovkin-type approximation theorem via A-statistical convergence of double sequences by using the test functions $1, x, y, x^2+y^2$.

2 Prove Theorem 8.3 by using the test functions 1, $\frac{x}{1-x}$, $\frac{y}{1-y}$, $(\frac{x}{1-x})^2 + (\frac{y}{1-y})^2$.

3 Prove Theorem 8.3 by using the test functions 1, $\frac{x}{1+x}$, $\frac{y}{1+y}$, $(\frac{x}{1+x})^2 + (\frac{y}{1+y})^2$.

4 Prove Theorem 8.5 via statistical A-summability of double sequences.

5 Prove Theorem 8.6 via statistical A-summability of double sequences.

6 Prove Theorem 8.8 via statistical A-summability of double sequences.