A Dual SBM Model with Fuzzy Weights in Fuzzy DEA

Jolly Puri and Shiv Prasad Yadav

Abstract The dual part of a SBM model in data envelopment analysis (DEA) aims to calculate the optimal virtual costs and prices (also known as weights) of inputs and outputs for the concerned decision-making units (DMUs). In conventional dual SBM model, the weights are found as crisp quantities. However, in real-world problems, the weights of inputs and outputs in DEA may have fuzzy essence. In this paper, we propose a dual SBM model with fuzzy weights for input and output data. The proposed model is then reduced to a crisp linear programming problem by using ranking function of a fuzzy number (FN). This model gives the fuzzy efficiencies and the fuzzy weights of inputs and outputs of the concerned DMUs as triangular fuzzy numbers (TFNs). The proposed model is illustrated with a numerical example.

Keywords Fuzzy DEA · Fuzzy SBM model · Fuzzy efficiency · Fuzzy weights

1 Introduction

Data envelopment analysis (DEA) [1] is a nonparametric and linear programming-based technique which evaluates the relative efficiency of homogeneous DMUs on the basis of multiple inputs and multiple outputs. Since the time DEA was proposed, it has got comprehensive attention both in theory and in applications. The beauty of DEA is its ability to measure relative efficiencies of DMUs without assuming prior weights on the inputs and outputs. The first model in DEA is the CCR model [1] which deals with proportional changes in inputs and outputs. The CCR efficiency score reflects the proportional maximum input reduction (or output augmentation)

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B. V. Babu et al. (eds.), *Proceedings of the Second International Conference on Soft Computing for Problem Solving (SocProS 2012), December 28–30, 2012*, Advances in Intelligent Systems and Computing 236, DOI: 10.1007/978-81-322-1602-5_34,

rate which is common to all inputs (outputs). But it neglects the slacks corresponding to inputs and outputs. To overcome this shortcoming of CCR model, Tone presented Slack-based Measure (SBM) model [15] in DEA, which puts aside the assumption of proportionate changes in inputs and outputs, and deals with slacks directly. The primal part of the SBM model directly deals with input excesses and output shortfalls of the concerned DMUs. On the other hand, the dual part of the SBM model can be interpreted as profit maximization model and it aims to calculate the optimal virtual costs and prices (also known as weights) of inputs and outputs for the concerned DMUs. Other theoretical extensions of SBM model can be seen in [4, 13].

The conventional DEA models are limited to only crisp input/output data and also their weights take only crisp values. However, in real-world problems, two situations can be possible: (1) Input/output data may have imprecision or fuzziness and (2) the weights of data may have fuzzy essence. To deal with imprecise data, the notion of fuzziness has been introduced in DEA. The DEA is extended to fuzzy DEA (FDEA) in which the imprecision is represented by fuzzy sets or FNs [7, 14]. The SBM efficiency in DEA is extended to fuzzy settings in [6, 11, 12]. Several approaches have been developed to deal with fuzzy data in FDEA. These approaches are as follows: (1) tolerance approach [14], (2) α-cut approach [7], (3) fuzzy ranking approach [5], and (4) possibility approach [8]. However, very less emphasis has been given to FDEA models with fuzzy weights. Mansourirad et al. [10] are the first who introduced fuzzy weights in fuzzy CCR model and proposed a method based on α-cut approach to evaluate weights for outputs in terms of TFNs. In this paper, we propose a dual SBM model with fuzzy weights corresponding to crisp input and output data. We reduce the proposed model into crisp linear programming problem (LPP) by using ranking function of an FN. The proposed model gives the fuzzy efficiencies and the fuzzy weights corresponding to inputs and outputs of the concerned DMUs as TFNs.

The paper is organized as follows: Section 2 presents preliminaries which include basic definitions. Section 3 presents the description of primal and dual parts of the SBM model. Section 4 presents the dual SBM model with fuzzy weights and its reduction to a crisp LPP. Section 5 presents the results and discussion of a numerical example to illustrate the proposed model. The last Section 6 concludes the findings of our study.

2 Preliminaries

The basic definitions in the fuzzy set theory can be seen from [16]. This section includes the definition of TFN and arithmetic operations on TFNs [2]. It also includes ranking function which maps FN to the real line [9].

2.1 Triangular Fuzzy Number

A TFN $\tilde{A}$, denoted by (a_1, a_2, a_3), is defined by the membership function $\mu_{\tilde{A}}$ given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 < x \le a_2, \\ 1, & x = a_2, \\ \dfrac{x - a_3}{a_2 - a_3}, & a_2 \le x < a_3, \\ 0, & \text{otherwise}, \end{cases}$$

$\forall x \in R$. In the present study, $\tilde{0} = (0, 0, 0)$, $\tilde{1} = (1, 1, 1)$ and $\tilde{a} = (a, a, a)$ where $a \in R$.

2.2 Arithmetic Operations on TFNs

Let $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ be two TFNs. Then,
Addition: $\tilde{A} \oplus \tilde{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$.
Subtraction: $\tilde{A} \ominus \tilde{B} = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$.
Scalar multiplication: $k\tilde{A} = \begin{cases} (ka_1, ka_2, ka_3), & k \ge 0, \\ (ka_3, ka_2, ka_1), & k < 0. \end{cases}$
Multiplication: $\tilde{A} \otimes \tilde{B} = (\min(a_1b_1, a_1b_3, a_3b_1, a_3b_3), a_2b_2, \max(a_1b_1, a_1b_3, a_3b_1, a_3b_3))$.

2.3 Ranking Function

Let $F(R)$ be the set of all FNs. A ranking function [9] $\Re$ is a mapping from $F(R)$ to the real line. The FNs can easily be compared by using ranking functions. The rank of TFN $\tilde{A} = (a_1, a_2, a_3)$, represented by $\Re(\tilde{A})$, is defined by $\Re(\tilde{A}) = (a_1 + 2a_2 + a_3)/4$.
Let $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ be two TFNs in $F(R)$. Then,

1. $\tilde{A}$ is said to be equal to $\tilde{B}$ based on ranking function $\Re$, written as $\tilde{A} \underset{\Re}{=} \tilde{B}$, iff $\Re(\tilde{A}) = \Re(\tilde{B})$.
2. $\tilde{A}$ is said to be less than or equal to $\tilde{B}$ based on ranking function $\Re$, written as $\tilde{A} \underset{\Re}{\le} \tilde{B}$, iff $\Re(\tilde{A}) \le \Re(\tilde{B})$.
3. $\tilde{A}$ is said to be greater than or equal to $\tilde{B}$ based on ranking function $\Re$, written as $\tilde{A} \underset{\Re}{\ge} \tilde{B}$, iff $\Re(\tilde{A}) \ge \Re(\tilde{B})$.

4. $\tilde{A}$ is said to be less than or equal to $\tilde{0}$ based on ranking function $\Re$, written as $\tilde{A} \underset{\Re}{\leq} \tilde{0}$, iff $\Re(\tilde{A}) \leq \Re(\tilde{0})$.

Theorem: $\Re(c\tilde{A}+\tilde{B}) = c\Re(\tilde{A})+\Re(\tilde{B})$, c is any constant. (*Linearity property* [11]).

3 Slack-based Measure Model

Assume that the performance of a set of n homogeneous DMUs (DMU$_j$; $j = 1, \ldots, n$) is to be measured. The performance of DMU$_j$ is characterized by a production process of m inputs (x_{ij}; $i = 1, \ldots, m$) to yield s outputs (y_{rj}; $r = 1, \ldots, s$). Let y_{rk} be the amount of the rth output produced by the kth DMU and x_{ik} be the amount of the ith input used by the kth DMU. Assume that input and output data are positive. The primal of SBM model [15] of the kth DMU, represented by SBM-P$_k$, is defined as

$$\text{SBM-P}_k \qquad \rho_k = \min \frac{1 - (1/m)\sum_{i=1}^{m} s_{ik}^{-}/x_{ik}}{1 + (1/s)\sum_{r=1}^{s} s_{rk}^{+}/y_{rk}}$$

$$\text{subject to}\; x_{ik} = \sum_{j=1}^{n} x_{ij}\lambda_{jk} + s_{ik}^{-}\ \forall i,$$

$$y_{rk} = \sum_{j=1}^{n} y_{rj}\lambda_{jk} - s_{rk}^{+}\ \forall r,$$

$$\lambda_{jk} \geq 0\ \forall j,\ s_{ik}^{-} \geq 0\ \forall i,\ s_{rk}^{+} \geq 0\ \forall r,$$

where s_{rk}^{+} is the slack in the rth output of the kth DMU; s_{ik}^{-} is the slack in the ith input of the kth DMU; λ_{jk}'s, i.e., $(\lambda_{j1}, \lambda_{j2}, \ldots, \lambda_{jn})$ are non-negative variables for $j = 1, 2, \ldots, n$. The kth DMU is SBM efficient if $\rho_k = 1$ and all $s_{ik}^{-} = 0$, $s_{rk}^{+} = 0$, i.e., no input excesses and no output shortfalls in any optimal solution.

SBM-P$_k$ can be transformed into LPP using Charnes–Cooper transformation given in [1]. Multiply a scalar $t_k > 0$ to both the denominator and the numerator of SBM-P$_k$. This causes no change in the value of ρ_k. The value of t_k can be adjusted in such a way that the denominator becomes 1. The SBM-P$_k$ model in LPP form becomes

$$\text{LPP-SBM-P}_k \quad \tau_k = \min t_k - \frac{1}{m}\sum_{i=1}^{m} S_{ik}^{-}/x_{ik}$$

$$\text{subject to}\; 1 = t_k + \frac{1}{s}\sum_{r=1}^{s} S_{rk}^{+}/y_{rk},$$

$$t_k x_{ik} = \sum_{j=1}^{n} x_{ij}\Lambda_{jk} + S_{ik}^{-}\forall i,$$

$$t_k y_{rk} = \sum_{j=1}^{n} y_{rj}\Lambda_{jk} - S_{rk}^{+}\forall r,$$

$$\Lambda_{jk} \geq 0\ \forall j,\ S_{ik}^{-} \geq 0\ \forall i,\ S_{rk}^{+} \geq 0\ \forall r,\ t_k > 0,$$

where $\rho_k = \tau_k, \lambda_{jk} = \Lambda_{jk}/t_k \forall j, s_{ik}^{-} = S_{ik}^{-}/t_k \forall i$ and $s_{rk}^{+} = S_{rk}^{+}/t_k \forall r$.

The dual of LPP-SBM-P$_k$, represented by SBM-D$_k$, can be expressed as follows:

$$\begin{aligned}
&\text{SBM-D}_k \qquad E_k = \max \xi_k \\
&\text{subject to}\ \xi_k + \textstyle\sum_{i=1}^{m} x_{ik} v_{ik} - \sum_{r=1}^{s} y_{rk} u_{rk} = 1, \\
&\textstyle\sum_{r=1}^{s} y_{rj} u_{rk} - \sum_{i=1}^{m} x_{ij} v_{ik} \leq 0\ \forall j, \\
&v_{ik} \geq \frac{1}{m x_{ik}}\ \forall i,\ u_{rk} \geq \frac{\xi_k}{s y_{rk}}\ \forall r.
\end{aligned}$$

where $\xi_k \in R$, $v_{ik}\ \forall i$, and $u_{rk}\ \forall r$ are the dual variables corresponding to LPP-SBM-P$_k$. The dual variables v_{ik} and u_{rk} are the weights associated with the ith input and the rth output, respectively. The E_k is the SBM efficiency of the kth DMU.

4 Dual SBM Model with Fuzzy Weights

In conventional SBM-D$_k$ model, the weights of inputs and outputs are found as crisp quantities. However, in real-world problems, the weights may have fuzzy essence. Therefore, in this paper, weights of inputs and outputs are taken as TFNs, and thus, the SBM-D$_k$ model becomes fuzzy SBM-D$_k$ (FSBM-D$_k$) model given by

$$\begin{aligned}
&\text{FSBM-D}_k \qquad \tilde{E}_k \underset{\Re}{=} \max \tilde{\xi}^k \\
&\text{subject to}\ \ \tilde{\xi}^k \oplus \textstyle\sum_{i=1}^{m} x_{ik} \tilde{v}^{ik}\ \Theta \sum_{r=1}^{s} y_{rk} \tilde{u}^{rk} \underset{\Re}{=} \tilde{1}, \\
&\textstyle\sum_{r=1}^{s} y_{rj} \tilde{u}^{rk} \Theta \sum_{i=1}^{m} x_{ij} \tilde{v}^{ik} \underset{\Re}{\leq} \tilde{0}\ \forall j, \\
&\tilde{v}^{ik} \underset{\Re}{\geq} \frac{1}{m x_{ik}} \tilde{1}\ \forall i,\ \tilde{u}^{rk} \underset{\Re}{\geq} \frac{1}{s y_{rk}} \tilde{\xi}^k\ \forall r, \\
&v_1^{ik} \leq v_2^{ik} \leq v_3^{ik}\ \forall i,\ u_1^{rk} \leq u_2^{rk} \leq u_3^{rk}\ \forall r,\ \xi_1^k \leq \xi_2^k \leq \xi_3^k.
\end{aligned}$$

where $\tilde{v}^{ik}$ and $\tilde{u}^{rk}$ are the triangular fuzzy weights associated with the ith input and the rth output, respectively. The $\tilde{E}_k$ is the fuzzy SBM efficiency of the kth DMU which is also found as a TFN. By using the ranking function of TFN, FSBM-D$_k$ model reduces to Model 1, which is as follows:

$$\begin{aligned}
&\text{Model-1}\quad \Re(\tilde{E}_k) = \max \Re(\tilde{\xi}^k) \\
&\text{subject to}\ \Re(\tilde{\xi}^k) + \textstyle\sum_{i=1}^{m} x_{ik} \Re(\tilde{v}^{ik}) - \sum_{r=1}^{s} y_{rk} \Re(\tilde{u}^{rk}) = \Re(\tilde{1}), \\
&\textstyle\sum_{r=1}^{s} y_{rj} \Re(\tilde{u}^{rk}) - \sum_{i=1}^{m} x_{ij} \Re(\tilde{v}^{ik}) \leq \Re(\tilde{0})\ \forall j, \\
&\Re(\tilde{v}^{ik}) \geq \frac{1}{m x_{ik}} \Re(\tilde{1})\ \forall i,\ \Re(\tilde{u}^{rk}) \geq \frac{1}{s y_{rk}} \Re(\tilde{\xi}^k)\ \forall r, \\
&v_1^{ik} \leq v_2^{ik} \leq v_3^{ik}\ \forall i,\ u_1^{rk} \leq u_2^{rk} \leq u_3^{rk}\ \forall r,\ \xi_1^k \leq \xi_2^k \leq \xi_3^k.
\end{aligned}$$

Table 1 Input and output data of six DMUs

Inputs and outputs	A	B	C	D	E	F
I_1	4	14	24	20	48	50
I_2	3	6	3	2	4	7.5
O_1	1	2	3	2	4	5
O_2	2	6	12	6	16	30

Source The input and output data are taken from [3]

Table 2 Fuzzy efficiencies $\tilde{\xi}^k = (\xi_1^k, \xi_2^k, \xi_3^k)$ and $\Re(\tilde{\xi}^k)$

$\tilde{\xi}^k$	A	B	C	D	E	F
ξ_1^k	0.3429	0.2930	0.6720	0.2733	0.2172	0.3247
ξ_2^k	0.8964	0.6113	0.7671	0.6183	0.4434	0.7321
ξ_3^k	1.8644	1.2535	1.7939	1.5758	2.2292	2.2112
$\Re(\tilde{\xi}^k)$	1.0000	0.6923	1.0000	0.7714	0.8333	1.0000

By putting the values of $\Re(\tilde{\xi}^k)$, $\Re(\tilde{v}^{ik})\ \forall i$, and $\Re(\tilde{u}^{rk})\ \forall r$, the Model-1 reduces to Model-2, which is crisp LPP.

$$\text{Model-2}\quad E_k = \max(\xi_1^k + 2\xi_2^k + \xi_3^k)/4$$

$$\text{subject to } (\xi_1^k + 2\xi_2^k + \xi_3^k) + \textstyle\sum_{i=1}^m x_{ik}(v_1^{ik} + 2v_2^{ik} + v_3^{ik})$$

$$- \textstyle\sum_{r=1}^s y_{rk}(u_1^{rk} + 2u_2^{rk} + u_3^{rk}) = 4,$$

$$\textstyle\sum_{r=1}^s y_{rj}(u_1^{rk} + 2u_2^{rk} + u_3^{rk}) - \sum_{i=1}^m x_{ij}(v_1^{ik} + 2v_2^{ik} + v_3^{ik}) \le 0\ \forall j,$$

$$\frac{v_1^{ik}+2v_2^{ik}+v_3^{ik}}{4} \ge \frac{1}{mx_{ik}}\ \forall i,\ u_1^{rk} + 2u_2^{rk} + u_3^{rk} \ge \frac{\xi_1^k+2\xi_2^k+\xi_3^k}{sy_{rk}}\ \forall r,$$

$$v_1^{ik} \le v_2^{ik} \le v_3^{ik}\ \forall i,\ u_1^{rk} \le u_2^{rk} \le u_3^{rk}\ \forall r,\ \xi_1^k \le \xi_2^k \le \xi_3^k.$$

5 Results and Discussion of a Numerical Example

In this section, we provide a numerical example to illustrate the proposed dual SBM model with fuzzy weights. Table 1 presents the performance evaluation problem of six DMUs with two inputs I_1 and I_2, and two outputs O_1 and O_2.

The fuzzy efficiencies of all DMUs are evaluated from Model-2, which are shown in Table 2. The results reveal that the rank of each fuzzy efficiency score lies between 0 and 1, i.e., $0 < \Re(\tilde{\xi}^k) \le 1$. The fuzzy weights corresponding to inputs and outputs of the concerned DMU are also evaluated by using Model-2, which are shown in Tables 3 and 4, respectively. These fuzzy weights provide additional information to the decision maker, which is not provided by crisp weights in crisp dual SBM model.

Table 3 Fuzzy weights corresponding to inputs

Fuzzy weights		A	B	C	D	E	F
$\tilde{v}^{1k}$	v_1^{1k}	4.4466	0.0220	3.8200	0.0080	0.0034	2.2823
	v_2^{1k}	10.6011	0.0543	8.4805	0.0203	0.0079	6.0048
	v_3^{1k}	35.0495	0.1515	26.6735	0.0514	0.0224	27.9433
$\tilde{v}^{2k}$	v_1^{2k}	4.1217	0.0262	10.2804	16.3350	33.8386	6.0896
	v_2^{2k}	10.0303	0.0645	23.4940	39.3018	78.3524	16.3315
	v_3^{2k}	31.1839	0.1781	61.7977	96.1473	175.6625	54.6475

Table 4 Fuzzy weights corresponding to outputs

Fuzzy weights		A	B	C	D	E	F
$\tilde{u}^{1k}$	u_1^{1k}	28.6998	0.1356	24.3567	16.7974	12.9102	11.6846
	u_2^{1k}	62.6096	0.3327	55.0749	40.0764	32.2595	32.5052
	u_3^{1k}	129.3616	0.8657	115.9649	93.9072	80.8491	89.4276
$\tilde{u}^{2k}$	u_1^{2k}	4.6918	0.0181	5.1196	0.0197	3.5286	3.6698
	u_2^{2k}	11.0246	0.0443	11.1800	0.0490	10.2253	10.0071
	u_3^{2k}	36.0650	0.1241	34.5782	0.1394	28.0860	42.3709

6 Conclusion

In this paper, we proposed a dual SBM model with fuzzy weights (FSBM-D$_k$) for crisp inputs and outputs. The FSBM-D$_k$ model is then reduced to crisp LPP by using ranking function. The proposed model evaluates the components of fuzzy efficiencies and fuzzy weights corresponding to inputs and outputs as TFNs. These fuzzy efficiencies and fuzzy weights provide additional information to the decision maker, which helps to deal with uncertainty in real-life problems.

Acknowledgments The first author is thankful to the University Grants Commission (UGC), Government of India, for financial assistance.

References

1. Charnes, A., Cooper, W.W., Rhodes, E.: Measuring the efficiency of decision making units. Eur. J. Oper. Res. **2**, 429–444 (1978)
2. Chen, S.M.: Fuzzy system reliability analysis using fuzzy number arithmetic operations. Fuzzy Set. Syst. **66**, 31–38 (1994)
3. Cooper, W. W., Seiford, L. M., Zhu, J.: Handbook on Data Envelopment Analysis. 2nd edn. International Series in Operations Research and Management Science, Springer, **164**, p. 200 (2011)
4. Goudarzi, M.R.M.: A slack-based model for estimating returns to scale under weight restrictions. Appl. Math. Sci. **6**(29), 1419–1430 (2012)

5. Hatami-Marbini, A., Saati, S., Makui, A.: An application of fuzzy numbers ranking in performance analysis. J. Appl. Sci. **9**(9), 1770–1775 (2009)
6. Jahanshahloo, G.R., Soleimani-damaneh, M., Nasrabadi, E.: Measure of efficiency in DEA with fuzzy input-output levels: A methodology for assessing, ranking and imposing of weights restrictions. Appl. Math. Comput. **156**, 175–187 (2004)
7. Kao, C., Liu, S.T.: Fuzzy efficiency measures in data envelopment analysis. Fuzzy Sets Syst. **113**, 427–437 (2000)
8. Lertworasirikul, S., Fang, S.C., Joines, J.A., Nuttle, H.L.W.: Fuzzy data envelopment analysis: A possibility approach. Fuzzy Set. Syst. **139**(2), 379–394 (2003)
9. Mahdavi-Amiri, N., Nasseri, S.H.: Duality in fuzzy number linear programming by use of a certain linear ranking function. Appl. Math. Comput. **180**, 206–216 (2006)
10. Mansourirad, E., Rizam, M.R.A.B., Lee, L.S., Jaafar, A.: Fuzzy weights in data envelopment analysis. Int. Math. Forum **5**(38), 1871–1886 (2010)
11. Puri, J., Yadav, S.P.: A concept of fuzzy input mix-efficiency in fuzzy DEA and its application in banking sector. Expert Syst. Appl. **40**, 1437–1450 (2013)
12. Saati, S., Memariani, A.: SBM model with fuzzy input-output levels in DEA. Aust. J. Basic Appl. Sci. **3**(2), 352–357 (2009)
13. Saen, R.F.: Developing a nondiscretionary model of slacks-based measure in data envelopment analysis. Appl. Math. Comput. **169**, 1440–1447 (2005)
14. Sengupta, J.K.: A fuzzy systems approach in data envelopment analysis. Comput. Math. Appl. **24**(8–9), 259–266 (1992)
15. Tone, K.: A slacks-based measure of efficiency in data envelopment analysis. Eur. J. Oper. Res. **130**, 498–509 (2001)
16. Zimmermann, H.J.: Fuzzy Set Theory and its Applications, 3rd edn. Kluwer-Nijhoff Publishing, Boston (1996)