# The Calabi Invariant and Central Extensions of Diffeomorphism Groups

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**Abstract** Let $D$ be a closed unit disc in dimension two and $G$ the group of symplectomorphisms on $D$. Denote by $G_\partial$ the group of diffeomorphisms on the boundary $\partial D$ and by $G_{\text{rel}}$ the group of relative symplectomorphisms. There exists a short exact sequence involving with those groups, whose kernel is $G_{\text{rel}}$. On such a group $G_{\text{rel}}$ one has a celebrated homomorphism called the Calabi invariant. By dividing the exact sequence by the kernel of the Calabi invariant, one obtains a central $\mathbb{R}$-extension, called the Calabi extension. We determine the resulting class of the Calabi extension in $H^2(G_\partial; \mathbb{R})$ and exhibit a transgression formula that clarify the relation among the Euler cocycle for $G_\partial$, the Thom class and the Calabi invariant.

**Keywords** Symplectomorphism · Euler class · Thom class · Cababi invarinat

## 1 Introduction

Let $D$ be a closed unit disc in $\mathbb{R}^2$ with a standard symplectic form $\omega = dx \wedge dy$ and $\text{Symp}(D)$ denote the group of symplectomorphisms on $D$. There exists the following short exact sequence of groups:

$$1 \longrightarrow G_{\text{rel}} \longrightarrow \text{Symp}(D) \longrightarrow G_\partial \longrightarrow 1.$$

Here we set $G_\partial = \text{Diff}_+(\partial D)$, the group of diffeomorphisms on the circle $\partial D$ that preserves the orientation, and $G_{\text{rel}} = \{g \in \text{Symp}(D) |\ g|_{\partial D} = \text{id}\}$, the group of relative symplectomorphisms on $D$. On such a group $G_{\text{rel}}$ one has a celebrated homomorphism $\text{Cal} : G_{\text{rel}} \to \mathbb{R}$ which is called the Calabi invariant. Thus, by dividing the sequence by the kernel of Cal, one obtains the Calabi extension:

$$0 \longrightarrow \mathbb{R} \longrightarrow \text{Symp}(D)/\ker(\text{Cal}) \longrightarrow G_\partial \longrightarrow 1,$$

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which turned out to be a central extension of $G_\partial$. In general, a central extension of $\Gamma$ determines an element in the cohomology group $H^2(\Gamma;\mathbb{R})$, which is called the Euler class. Thus it is natural to ask what the Euler class is in $H^2(G_\partial;\mathbb{R})$ of the Calabi extension. A result in Tsuboi [4] essentially gives an answer to the question even though he has not attained the Calabi extension. In this context his result is rephrased as follows: the Euler class of the Calabi extension is equal to that of a universal central extension of $G_\partial$ up to a constant multiptle. To be more precise, it is stated as follows.

**Theorem** (see Theorem 2) *Let $H$ be a universal covering space of $G_\partial$ and consider a central extension*

$$0 \longrightarrow \mathbb{Z} \longrightarrow H \longrightarrow G_\partial \longrightarrow 1,$$

*called a universal central extension of $G_\partial$. Let $e(G_\mathbb{R})$ denote the Euler class of $H$ with coefficients in $\mathbb{R}$, thus $e(G_\mathbb{R})$ belongs to $H^2(G_\partial;\mathbb{R})$. Then the Euler class of the Calabi extension is equal to $\pi^2 e(G_\mathbb{R})$.*

In order to prove the above theorem, we shall introduce a notion called a connection cochain, which is reminiscent of connection form on circle bundles. Applying this idea, the Euler class can be investigated at the cochain level and it turns out that the transgression image of the Calabi invariant is the Euler class of a universal central extension up to a constant multiple. Namely, the Euler class is equal to the negative coboundary $-\delta\tau$ of a connection cochain $\tau$ and the restriction of $\tau$ to $G_{\mathrm{rel}}$ coincides with the Calabi invariant.

Moreover, to clarify a topological significance of the Calabi invariant, we employ a double complex introduced by Bott. It provides a simplicial de Rham model for $DG$, which stands for the Borel construction of the disk $D$. It is a universal foliated disc bundle on the classifying space $BG$, where $G$ is the group Symp$(D)$ equipped with a discrete topology. Now due to the presence of an invariant volume form $\omega$, it is able to construct the Thom class $U$ in the simplicial model for $DG$. Moreover, the integration along the fiber $\pi_*$, which induces a homomorphism on cohomology groups from $H^n(DG)$ to $H^{n-2}(BG)$, is described in detail in terms of a simplicial model. It is then well known that a square of the Thom class $U \cup U$ is mapped to the Euler class. With the description of the integration along the fiber, we can prove on $DG$ that a negative coboudary of the connection cochain $\tau$ is exactly the image of $U \cup U$ and that the restriction of $\tau$ to the fiber $G_{\mathrm{rel}}$ coincides with the Calabi invariant. This is our transgression formula for the Calabi invariant (see Theorem 5).

The paper is organized as follows. In Sect. 2, we develop the theory of a connection cochain and the curvature. It turns out that there is a bijective correspondence between the sections and the connection cochains and the Euler cocycle given in terms of section is equal to the curvature of a corresponding connection cochain. In Sect. 3, we briefly review a simplicial de Rham model due to Bott. The Sects. 4 and 5 are the bulk of the paper. We define the Calabi extension and introduce a connection cochain. Then an explicit formula of the curvature is obtained and related to the Euler cocycle for a universal central extension. In the final section we obtain the transgression formula for the Calabi invariant.

## 2 The Euler Class and Connection Cochains

Let $\Gamma$ be a discrete group and $A$ an abelian group. Recall that a $p$-cochain $c : \Gamma^p \to A$ is a function on the $p$-tuple product and the coboundary is defined to be

$$\delta c(g_1, \ldots, g_{p+1}) = c(g_2, \ldots, g_{p+1}) - c(g_1 g_2, g_3, \ldots, g_{p+1}) + \cdots + (-1)^p c(g_1, \ldots, g_{p-1}, g_p g_{p+1}) + (-1)^{p+1} c(g_1, \ldots, g_p).$$

The cochain complex $(C^p(\Gamma; A), \delta)$ with coefficients in $A$ is given by the pair

$$C^p(\Gamma; A) = \{c : \Gamma^p \to A\}, \qquad \delta : C^p(\Gamma; A) \to C^{p+1}(\Gamma; A).$$

The cohomology group $H^*(C^\bullet(\Gamma; A))$ is called a cohomology group of $\Gamma$ with coefficients in $A$, which is denoted by $H^*(\Gamma; A)$. Let $B\Gamma$ be a classifying space of $\Gamma$. It is known that the singular cohomology group of $B\Gamma$ is isomorphic to that of $\Gamma$ with the same coefficients:

$$H^*(B\Gamma; A) \cong H^*(\Gamma; A).$$

On the other hand, there is an alternative definition for low-dimensional cohomology groups, in particular for dimension 2. Recall that a central $A$-extension of $\Gamma$ is a short exact sequence of groups

$$0 \longrightarrow A \longrightarrow G \longrightarrow \Gamma \longrightarrow 1$$

such that $A$ is contained in the center of $G$. It then turns out that the cohomology group $H^2(\Gamma; A)$ is isomorphic to the equivalence classes of central $A$-extensions of $\Gamma$;

$$H^2(\Gamma; A) \cong \{\text{central } A\text{-extensions of } \Gamma\}/\{\text{splitting extensions}\}.$$

The resulting class in $H^2(\Gamma; A) \cong H^2(B\Gamma; A)$, denoted by $e(G)$, is called the extension class or *the Euler class* of a central extension. It is defined in terms of a section $s : \Gamma \to G$, that is, a map satisfying $\pi \circ s = \mathrm{id}$ with $\pi : G \to \Gamma$ the surjective homomorphism in the central extension. Namely, $e(G)$ is defined by a 2-cocycle $\chi$ called a Euler cocycle:

$$\chi(\gamma_1, \gamma_2) = s(\gamma_1)s(\gamma_2)s(\gamma_1\gamma_2)^{-1} = s(\gamma_1\gamma_2)^{-1}s(\gamma_1)s(\gamma_2) \in A \qquad (1)$$

for $\gamma_1, \gamma_2 \in \Gamma$. Note that $A$ is contained in the center. It is also verified that the Euler class is independent of the choice of section.

There is the third method to define the Euler class, which is quite reminiscent of a characteristic class constructed from a connection form as in the Chern-Weil theory. In order to introduce this, we first define a notion, called a connection cochain.

**Definition 1** Suppose that there is a central $A$-extension $0 \longrightarrow A \longrightarrow G \longrightarrow \Gamma \longrightarrow 1$. A cochain $\tau : G \to A$ that satisfies the condition

$$\tau(ga) = \tau(g) + a \in A$$

for $g \in G, a \in A$ is called *a connection cochain*. Here we write the product multiplicatively in $G$ and additively in $A$. The coboundary $\delta\tau$ is called *a curvature* of $\tau$.

**Proposition 1** *The following holds.*

*(1) There exists a 2-cocycle $\sigma$ on $\Gamma$ such that $\delta\tau(g, h) = \sigma(\pi(g), \pi(h))$ for $g, h \in G$ with $\pi : G \to \Gamma$, in other words, the curvature is a basic cocycle.*
*(2) The Euler class $e(G)$ coincides with $[-\sigma] \in H^2(\Gamma; A)$.*
*(3) The cohomology class of a curvature in $H^2(\Gamma; A)$ is independent of the choice of connection cochain.*

*Proof* Let us take $a \in A$, which is central in $G$. One has

$$\delta\tau(ga, h) = \tau(h) - \tau(gha) + \tau(ga) = \tau(h) - \tau(gh) + \tau(g) = \delta\tau(g, h)$$

due to the property $\tau(ga) = \tau(g) + a$. Similarly $\delta\tau(g, ha) = \delta\tau(g, h)$. This proves that there is a cochain $\sigma$ on $\Gamma$ such that $\delta\tau = \pi^*\sigma$. It is straightforward to see $\delta\sigma = 0$.

Note that there is a bijective correspondence between the connection cochains and the sections. In fact, a section $s$ gives an identification of $G$ with the product $\Gamma \times A$ in such a way that $g \in G$ corresponds to $(\gamma, a) \in \Gamma \times A$ where $\gamma = \pi(g)$ and $g = s(\gamma)\, a = a\, s(\gamma)$. A connection cochain $\tau$ is then uniquely determined by the formula $\tau(g) = s(\gamma)^{-1} g$. In other words, a connection cochain $\tau$ is given by the projection map $\Gamma \times A \to A$. Take lifts $g_1, g_2 \in G$ of $\gamma_1, \gamma_2 \in \Gamma$, respectively. Since ${g_i}^{-1} s(\gamma_i)$ $(i = 1, 2)$ is central, one has

$$\begin{aligned}\chi(\gamma_1, \gamma_2) &= s(\gamma_1\gamma_2)^{-1} s(\gamma_1) s(\gamma_2) \\ &= {g_1}^{-1} s(\gamma_1) {g_2}^{-1} s(\gamma_2) s(\gamma_1\gamma_2)^{-1} g_1 g_2 \\ &= \tau(g_1)^{-1} \tau(g_2)^{-1} \tau(g_1 g_2),\end{aligned}$$

which is equal to $-\tau(g_2) + \tau(g_1 g_2) - \tau(g_1) = -\delta\tau(g_1, g_2)$ once written additively. Thus it proves $\chi(\gamma_1, \gamma_2) = -\sigma(\gamma_1, \gamma_2)$ and hence $e(G) = [-\sigma]$.

Let $\theta$ be another connection cochain. The property of connection cochain implies that there exists a cochain $\xi : \Gamma \to A$ such that $\xi(\gamma) = \tau(g) - \theta(g)$ with $\gamma = \pi(g)$. Thus one has $\delta\xi = \delta\tau - \delta\theta$, which proves that a curvatures is cohomologous to each other in the cochain complex of $\Gamma$. This completes the proof. □

The argument above also proves the following:

**Proposition 2** *Let us take a section $s : \Gamma \to G$ and the Euler cocycle $\chi$ as in (1). Let $\tau$ be a connection cochain given by the formula $\tau(g) = s(\gamma)^{-1} g$. Then one has $\chi = -\delta\tau$.*

*Remark 1* Let $B$ be an abelian group with a homomorphism $\iota : A \to B$. A mapping $\tau : G \to B$ satisfying $\tau(ga) = \tau(g) + \iota(a) \in B$ is called a connection cochain with values in $B$. In fact, given a central extension $0 \longrightarrow A \longrightarrow G \longrightarrow \Gamma \longrightarrow 1$, we can extend it to a central extension $0 \longrightarrow B \longrightarrow G_B \longrightarrow \Gamma \longrightarrow 1$ so that $G_B$ is the quotient $G \times B/\sim$ with the equivalence relation $(g, b) \sim (ga, b - \iota(a))$ for $(g, b) \in G \times B,\ a \in A$. The multiplication in $G_B$ is given by $(g_1, b_1) \cdot (g_2, b_2) = (g_1 g_2, b_1 + b_2)$ with $(g_i, b_i) \in G \times B\ (i = 1, 2)$. In terms of cohomology theory, the construction corresponds to a natural homomorphism $\iota_* : H^2(\Gamma; A) \to H^2(\Gamma; B)$ induced by $\iota$. It is easy to verify that $\tau : G \to B$ yields a connection cochain $\tau_B : G_B \to B$ defined by $\tau_B(g, b) = \tau(g) + b$. Thus, a connection cochain with values in $B$ determines the Euler class $e(G_B)$ in $H^2(\Gamma; B)$.

*Remark 2* Given a central extension $0 \longrightarrow \mathbb{Z} \longrightarrow G \longrightarrow \Gamma \longrightarrow 1$, there is a method to construct a fiber bundle $B\mathbb{Z} \to BG \to B\Gamma$ consisting of classifying spaces. This is a homotopy $S^1$-bundle on $B\Gamma$. It is known that the (topological) Euler class of such a bundle coincides with the extension class in $H^*(B\Gamma; \mathbb{Z}) \cong H^*(\Gamma; \mathbb{Z})$. This is the reason why they are called the Euler class. Thus, it is reasonable to consider a connection cochain as a counterpart of connection form on $S^1$-bundle.

*Example 1* (*A universal central extension for homoemorphisms*) Let $S^1$ be a circle, which is identified with the quotient space $\mathbb{R}/2\pi\mathbb{Z}$. Let $\mathrm{Homeo}_+(S^1)$ be the group of all homeomorphisms that preserve the orientation. There exists the following extension, called *a universal central extension* of $\mathrm{Homeo}_+(S^1)$:

$$0 \longrightarrow \mathbb{Z} \longrightarrow H \stackrel{\rho}{\longrightarrow} \mathrm{Homeo}_+(S^1) \longrightarrow 1.$$

Here $H$ denotes the universal covering space of a topological group $\mathrm{Homeo}_+(S^1)$, where an element $f \in H$ is considered as an homeomorphism of $\mathbb{R}$ satisfying $f(x + 2\pi) = f(x) + 2\pi$ for $x \in \mathbb{R}$. With $T$ the translation $T(x) = x + 2\pi$, it is equivalent to say $f \circ T(x) = T \circ f(x)$. In other words, $f$ is an orientation-preserving homeomorphism which is equivariant with $T$. It naturally induces a homeomrophism on $\mathbb{R}/2\pi\mathbb{Z}$ and thus yields a surjective homomorphism $\rho : H \to \mathrm{Homeo}_+(S^1)$. The kernel of $\rho$ consists of the translations by $2n\pi$ $(n \in \mathbb{Z})$, which is identified with the additive group $\mathbb{Z}$ in such a way that $T$ corresponds to $1 \in \mathbb{Z}$.

Then a connection cochain $\tau : H \to \mathbb{Z}$ is given by $\tau(f) = [f(0)/2\pi]$, where $[x]$ denotes the largest integer that does not exceed $x \in \mathbb{R}$. In fact, one has

$$\tau(f \circ T^n) = [f(2n\pi)/2\pi] = [f(0)/2\pi + n] = \tau(f) + n$$

since $f$ is equivariant. Thus, $\tau$ is a connection cochain once $n \in \mathbb{Z}$ is identified with $T^n$. Given $\gamma_1, \gamma_2 \in \mathrm{Homeo}_+(S^1)$, we take respective lifts $h_1, h_2$ in $H$. The Euler class of the universal central extension is then given by a cocycle

$$\chi(\gamma_1, \gamma_2) = [h_1 \circ h_2(0)/2\pi] - [h_1(0)/2\pi] - [h_2(0)/2\pi].$$

It is known that it coincides with the (topological) Euler class of a universal $\mathrm{Homeo}_+(S^1)^\delta$-bundle, where $\mathrm{Homeo}_+(S^1)^\delta$ denotes the homeomorphism group equipped with discrete topology. In addition, $\chi$ is a bounded cocycle. Such a property is crucial in the theory of bounded cohomology groups due to Gromov; see Ghys [5] for more detail.

*Example 2* (*A universal central extension for diffeomorphisms*) In a similar way to the above, one has a central extension for the orientation-preserving diffeomorphism group $\mathrm{Diff}_+(S^1)$:

$$0 \longrightarrow \mathbb{Z} \longrightarrow H \stackrel{\rho}{\longrightarrow} \mathrm{Diff}_+(S^1) \longrightarrow 1,$$

which is also called a universal central extension for the diffeomorphisms. Here an element $f \in H$ is an orientation-preserving diffeomorphism of $\mathbb{R}$ satisfying $f(x + 2\pi) = f(x) + 2\pi$ for $x \in \mathbb{R}$, and a surjective homomorphism $\rho : H \to \mathrm{Diff}_+(S^1)$ is defined as well. Then there exists a connection cochain $\tau : H \to \mathbb{R}$ such that

$$\tau(f) = \frac{1}{4\pi^2} \int_0^{2\pi} f(x)dx.$$

In fact, one has

$$\begin{aligned}\tau(f \circ T^n) =& \frac{1}{4\pi^2} \int_0^{2\pi} f(x + 2n\pi)dx \\ =& \frac{1}{4\pi^2} \int_0^{2\pi} (f(x) + 2n\pi)dx = \tau(f) + n.\end{aligned}$$

Thus, due to Remark 1, it determines $e(H_{\mathbb{R}})$ in $H^2(\mathrm{Diff}_+(S^1)^\delta; \mathbb{R})$, the Euler class of $H$ with coefficients in $\mathbb{R}$. With $h_1, h_2 \in H$ and $\gamma_1 = \pi(h_1), \gamma_2 = \pi(h_2)$, the Euler cocycle is then given by the formula

$$\chi(\gamma_1, \gamma_2) = \frac{1}{4\pi^2} \int_0^{2\pi} (h_1 \circ h_2(x) - h_1(x) - h_2(x))\, dx. \tag{2}$$

## 3 Simplicial de Rham Model of Classifying Spaces

In this section we briefly review a simplicial de Rham model due to Bott [1], which plays a key role in our study on the Calabi invariant. Let $M$ be a smooth manifold and $\Gamma$ a discrete group acting on $M$. Let $\Gamma^p$ denote the $p$-tuple product and $\Omega^q(M)$ the space of $q$-forms on $M$. We define $C^p(\Gamma, \Omega^q(M))$ to be the set of arbitrary mappings $c : \Gamma^p \to \Omega^q(M)$, which is called the space of group cochains of degree $p$ with values in $\Omega^q(M)$ or cochains of type $(p, q)$. We then introduce a double complex

$$C^{p,q} = C^p(\Gamma, \Omega^q(M))$$

with the (total) differential $D = \delta + (-1)^p d$ on $C^{p,q}$, where $d$ the exterior differential operator and $\delta$ a coboundary map for group cochains. To be precise, $\delta$ is given by

$$\begin{aligned}\delta c(g_1, \ldots, g_{p+1}) =& c(g_2, \ldots, g_{p+1}) - c(g_1 g_2, g_3, \ldots, g_{p+1}) + \cdots \\ &+ (-1)^p c(g_1, \ldots, g_{p-1}, g_p g_{p+1}) + (-1)^{p+1} c(g_1, \ldots, g_p)^{g_{p+1}},\end{aligned}$$

where $c : \Gamma^p \to \Omega^q(M)$ with $\Omega^q(M)$ a right $\Gamma$-module by the action on $M$. We also observe that there is a cup product given by

$$(c_1 \cup c_2)(g_1, \ldots, g_{p+r}) = c_1(g_1, \ldots, g_p)^{g_{p+1} g_{p+2} \cdots g_{p+r}} \wedge c_2(g_{p+1}, \ldots, g_{p+r})$$

for $c_1 \in C^{p,q}$, $c_2 \in C^{r,s}$. We know that the exterior differentiation is a skew-derivation and the same for the coboundary map:

$$\delta(c_1 \cup c_2) = \delta c_1 \cup c_2 + (-1)^p c_1 \cup \delta c_2.$$

Let $M\Gamma$ denote the Borel construction of $M$, namely, a quotient space

$$M\Gamma = (E\Gamma \times M)/\sim$$

obtained from the equivalence relation $(x, m) \sim (x\gamma^{-1}, \gamma m)$ with $(x, m) \in E\Gamma \times M$ and $\gamma \in \Gamma$. It turns out that $M\Gamma$ is a classifying space for foliated $M$-bundles with the structure group $\Gamma$. Now we assemble a total complex $\Omega^*(M\Gamma)$ in such a way that

$$\Omega^n(M\Gamma) = \bigoplus_{p+q=n} C^p(\Gamma, \Omega^q(M)).$$

It is called a simplicial de Rham model of $M\Gamma$ due to the following theorem by Bott:

**Theorem 1** [1] *Let $H^*(\Omega(M\Gamma))$ be the cohomology group of a simplicial de Rham model $\Omega^*(M\Gamma)$. Then there is an isomorphism*

$$H^*(\Omega(M\Gamma)) \cong H^*(M\Gamma),$$

*where $H^*(M\Gamma)$ stands for the singular cohomolgy group of $M\Gamma$.*

## 4 The Calabi Invariant

Let $D$ be a disk of radius 1 in $\mathbb{R}^2$, $D = \{(x, y) \in \mathbb{R}^2 | \, x^2 + y^2 \leq 1\}$ and take the standard symplectic form $\omega = dx \wedge dy$ on $D$. Let $\mathrm{Symp}(D)$ be the group of symplectomorphisms on $D$, which is simply denoted by $G$. We also denote by $G_{\mathrm{rel}}$

the group of relative symplectomorphisms; $G_{\mathrm{rel}} = \{g \in G|\ g|_{\partial D} = \mathrm{id}\}$, where $g|_{\partial D}$ denotes the restriction of $g$ to $\partial D$. Set $G_{\partial} = \mathrm{Diff}_{+}(\partial D)$, which is the group of orientation-preserving diffeomorphisms on $\partial D$. Then there exists a short exact sequence:

$$1 \longrightarrow G_{\mathrm{rel}} \longrightarrow G \longrightarrow G_{\partial} \longrightarrow 1. \tag{3}$$

We also choose a 1-from $\eta$ on $D$ such as $\omega = d\eta$ once for all.

**Definition 2** The Calabi invariant[1] is defined to be a mapping $\mathrm{Cal} : G_{\mathrm{rel}} \to \mathbb{R}$ such that

$$\mathrm{Cal}(g) = \int_D (\eta \cup \delta\eta)(g)$$

for $g \in G_{\mathrm{rel}}$, where the integrand is equal to

$$(\eta \cup \delta\eta)(g) = \eta^g \wedge (\eta - \eta^g) = \eta^g \wedge \eta.$$

Here we denote by $\eta^g$ the pullback $g^*\eta$ induced by $g$. In the sequel, we frequently use this notation even for functions: $\varphi^g(x) = \varphi(g(x))$.

**Proposition 3** *The Calabi invariant yields a homomorphism on* $G_{\mathrm{rel}}$.

*Proof* Note that

$$\delta\mathrm{Cal}(g, h) = \int_D (\delta\eta \cup \delta\eta)(g, h) = \int_D (\eta - \eta^g)^h \wedge (\eta - \eta^h).$$

Since $\omega$ is $G$-invariant, one has $d(\eta - \eta^g) = 0$ and $(\eta - \eta^g)|_{\partial D} = 0$ for $g \in G_{\mathrm{rel}}$. Recall that the relative cohomology group $H^1(D, \partial D)$ vanishes. Thus, there exists a smooth function $f_g \in C^\infty(D)$ such that $\delta\eta(g) = \eta - \eta^g = df_g$ and $f_g|_{\partial D} = 0$. This implies

$$\delta\mathrm{Cal}(g, h) = \int_D \delta\eta(g)^h \wedge \delta\eta(h) = \int_D (df_g)^h \wedge \delta\eta(h) = \int_{\partial D} f_g^h \wedge \delta\eta(h) = 0 \tag{4}$$

since $f_g|_{\partial D} = 0$. Therefore, one obtains

$$\delta\mathrm{Cal}(g, h) = \mathrm{Cal}(g) - \mathrm{Cal}(gh) + \mathrm{Cal}(h) = 0,$$

which proves that the Calabi invariant is a homomoprhism. □

It is possible to extend the Calabi invariant from $G_{\mathrm{rel}}$ to $G$ by the same formula. Namely, thers is a cochain $\tau : G \to \mathbb{R}$ defined by

$$\tau(g) = \int_D (\eta \cup \delta\eta)(g) = -\int_D \eta \wedge \eta^g \tag{5}$$

[1]The definition in McDuff-Salamon [6] is negative one half of the above and that in Tsuboi [4] coincides with ours.

for $g \in G$. However, there is a significant distinction between the Calabi invariant and $\tau$. It is known that the value of Calabi invariant is independent of the choice of $\eta$ such as $\omega = d\eta$; see McDuff-Salamon [6]. On the other hand, $\tau$ *does* depend on the choice of $\eta$ at the outside of $G_{\mathrm{rel}}$. Thus $\tau$ seems to be less interesting from the viewpoint of Symplectic Topology. Nevertheless, it turns out that $\tau$ is still relevant since it admits the following properties similar to a connection cochain.

**Proposition 4** *For $g \in G_{\mathrm{rel}}$ and $h \in G$, one has:*

*(1)* $\mathrm{Cal}(g) = \mathrm{Cal}(hgh^{-1})$;
*(2)* $\tau(gh) = \tau(h) + \mathrm{Cal}(g)$ *and* $\tau(hg) = \tau(h) + \mathrm{Cal}(g)$.

*Proof* Since $G_{\mathrm{rel}}$ is a normal subgroup, one has $hgh^{-1} \in G_{\mathrm{rel}}$. Note that $d(\eta - \eta^h) = \omega - \omega^h = 0$, thus there exist $f \in C^\infty(D)$ such that $\eta - \eta^h = df$ since $H^1(D; \mathbb{R}) = 0$. Then we obtain

$$\begin{aligned}
-\mathrm{Cal}(hgh^{-1}) &= \int_D \eta^h \wedge \eta^{hg} \\
&= \int_D (\eta - df) \wedge (\eta - df)^g \\
&= \int_D (\eta \wedge \eta^g - \eta \wedge df^g - df \wedge \eta^g + df \wedge df^g) \\
&= \int_D (\eta \wedge \eta^g + d(\eta \wedge f^g) - d\eta \wedge f^g - d(f \wedge \eta^g) + f \wedge d\eta^g + d(f \wedge df^g)) \\
&= \int_D (\eta \wedge \eta^g - d\eta \wedge f^g + f \wedge d\eta^g).
\end{aligned}$$

The last equality follows from the Stokes theorem and the property $g|_{\partial D} = \mathrm{id}$. Hence we finally have

$$\mathrm{Cal}(hgh^{-1}) = -\int_D (\eta \wedge \eta^g - \omega \wedge f^g + f \wedge \omega) = \mathrm{Cal}(g)$$

since $\omega^g = \omega$. This prove the first equation.

Recall the proof for the Eq. (4). The argument holds as long as either $g$ or $h$ is an element of $G_{\mathrm{rel}}$. Thus, in the case of $g \in G_{\mathrm{rel}}$ and $h \in G$, we obtain

$$\delta\tau(g, h) = \mathrm{Cal}(g) - \tau(gh) + \tau(h) = 0,$$

which prove the second equation. The same argument applies to the third. □

**Proposition 5** *Set* $K = \ker[\mathrm{Cal} : G_{\mathrm{rel}} \to \mathbb{R}]$. *It is a normal subgroup in* $G$.

*Proof* As observed, $G_{\mathrm{rel}}$ is a normal subgroup in $G$. Thus it is straightforward from the property (1) in Proposition 4.

Due to Proposition 4, a cochain $\tau : G \to \mathbb{R}$ can be considered as a connection cochain with values in $\mathbb{R}$ even though the exact sequence (3) is not central. In fact $\tau$ will amount to a connection cochain of the Calabi extension, which will be defined in the next section.

## 5 The Calabi Extension

Recall from Proposition 5 that $K = \ker(\mathrm{Cal})$ is a normal subgroup in $G = \mathrm{Symp}(D)$. By dividing $G$ by $K$, one obtains the following short exact sequence:

$$0 \longrightarrow G_{\mathrm{rel}}/K \longrightarrow G/K \longrightarrow G_\partial \longrightarrow 1.$$

Note that the quotient group $G_{\mathrm{rel}}/K$ is isomorphic to $\mathbb{R}$ since the Calabi invariant is surjective. With the identification to $\mathbb{R}$ fixed, we prove the following:

**Proposition 6** *The short exact sequence* $0 \longrightarrow \mathbb{R} \longrightarrow G/K \longrightarrow G_\partial \longrightarrow 1$ *is a central extension.*

*Proof* it suffices to show that $G_{\mathrm{rel}}/K$ is contained in the center of $G/K$. Let $g_i K$ be a coset of $G_{\mathrm{rel}}/K$ with $g_i \in G_{\mathrm{rel}}$ $(i = 1, 2)$. It is obvious that $g_1 K = g_2 K$ if and only if $\mathrm{Cal}(g_1) = \mathrm{Cal}(g_2)$. Thus, Proposition 4 implies $hgh^{-1}K = gK$ for $g \in G_{\mathrm{rel}}$, $h \in G$. Therefore, one obtains $hgK = hgh^{-1}K \cdot hK = gK \cdot hK = ghK$, which proves that $gK$ is central in $G/K$ for $g \in G_{\mathrm{rel}}$. □

**Definition 3** The central extension $0 \longrightarrow \mathbb{R} \longrightarrow G/K \longrightarrow G_\partial \longrightarrow 1$ is called *the Calabi extension.*

Let $\tau$ be a cochain defined in (5). It is obvious that $\tau$ induces a connection cochain on $G/K$ due to the second property in Proposition 4. We denote it by the same letter as $\tau : G/K \to \mathbb{R}$. Then a formula for the curvature will be derived. Recall that there exists a smooth function $f_g \in C^\infty(D)$ for $g \in G$ such that

$$\delta\eta(g) = \eta - \eta^g = df_g.$$

Thus, the curvature amounts to

$$\delta\tau(g, h) = \int_D \delta\eta \cup \delta\eta(g, h) = \int_D \delta\eta(g)^h \wedge \delta\eta(h) = \int_{\partial D} f_g^h df_h$$

by the Stokes theorem.

**Proposition 7** *Choose smooth functions $f_g$ as above.*

*(1) The curvature is given by*

$$\delta\tau(g,h) = \int_D \delta\eta \cup \delta\eta(g,h) = \int_{\partial D} f_g^h df_h.$$

*(2) Let $\varphi$ and $\psi$ be arbitrary smooth functions on $\partial D$ such that $(\eta - \eta^g)|_{\partial D} = d\varphi$, $(\eta - \eta^h)|_{\partial D} = d\psi$. It then follows*

$$\int_{\partial D} \varphi^h d\psi = \int_{\partial D} f_g^h df_h.$$

*Proof* We have already proved the first statement. For the second, note that there exist constants $a, b$ such that $\varphi = f_g|_{\partial D} + a$, $\psi = f_h|_{\partial D} + b$ by the assumption. Thus one has

$$\int_{\partial D} \varphi^h d\psi = \int_{\partial D} f_g^h df_h + \int_{\partial D} a df_h = \int_{\partial D} f_g^h df_h$$

due to the Stokes theorem. □

The second formula reminds us of a linking form. Functions $\varphi$ and $\psi$ are determined up to constant from $g, h \in G$, however, the value of integral has no ambiguity. In fact, it depends only on the restrictions $g|_{\partial D}$ and $h|_{\partial D}$, or $(\eta - \eta^g)|_{\partial D}$ and $(\eta - \eta^h)|_{\partial D}$ with $\eta$ fixed.

Recall that we have chosen $\eta$ such that $\omega = d\eta$. Now we deal with a specific $\eta$, nemely, $\eta = r^2 d\theta/2$ with the polar coordinate $(r, \theta)$ in $\mathbb{R}^2$. Given $\gamma \in G_\partial$, let $g \in G$ be a lift of $\gamma$, which is equivalent to say $\gamma = g|_{\partial D}$. We then obtain $(\eta - \eta^g)|_{\partial D} = (d\theta - d\theta^\gamma)/2$ on $\partial D$ ($d\theta$ denotes the angular form restricted to $\partial D$). In Example 2 we constructed a universal central extension of $G_\partial = \mathrm{Diff}_+(S^1)$:

$$0 \longrightarrow \mathbb{Z} \longrightarrow H \stackrel{\pi}{\longrightarrow} G_\partial \longrightarrow 1.$$

An element in $H$ is an orientation-preserving diffeomorphism $h : \mathbb{R} \to \mathbb{R}$ satisfying $h(x + 2\pi) = h(x) + 2n\pi$ for $x \in \mathbb{R}$, where $\partial D$ is identified with $\mathbb{R}/2\pi\mathbb{Z}$. Denote the respective coordinates by $\theta$ on $\partial D$ and $x$ on $\mathbb{R}/2\pi\mathbb{Z}$. Set $\phi_h(x) = x - h(x) + h(0)$ with $x \in \mathbb{R}$ (we consider $x$ as a function on $\mathbb{R}$), where $h$ is a lift of $\gamma \in G_\partial$ to $H$. For another lift $k \in H$ of $\gamma$, there exists an integer $n \in \mathbb{Z}$ such that $k(x) = h(x) + 2\pi n$. Hence one has $\phi_k(x) = x - (h(x) + 2\pi n) + (h(0) + 2\pi n) = \phi_h(x)$, which implies that $\phi_h(x)$ is independent of the choice of lift. We denote it by $\phi_\gamma(x)$ from now on. One further has $\phi_\gamma(x + 2\pi n) = x + 2\pi n - h(x + 2\pi n) + h(0) = \phi_\gamma(x)$ since $h$ is equivariant. This implies that $\phi_\gamma$ is a smooth function on $\mathbb{R}/2\pi\mathbb{Z}$. Shifting the coordinate from $x$ to $\theta \in \partial D$, one then obtains $d\phi_\gamma = d\theta - d\theta^\gamma$. Summarizing, we proved that there is a smooth function $\phi_\gamma(x) = x - h(x) + h(0)$ on $\partial D$, which depends only on $\gamma$, and that $(\eta - \eta^g)|_{\partial D} = d\phi_\gamma(x)/2$ with $g \in G$ and $g|_{\partial D} = \gamma$.

Employing Proposition 7, we obtain an explicit formula for a curvature of the Calabi extension. Set $\gamma_i = g_i|_{\partial D}$ for $g_i \in G$ and denote by $h_i$ a lift of $\gamma_i$ to $H$ $(i = 1, 2)$. One has

$$\begin{aligned}\delta\tau(g_1, g_2) &= \frac{1}{4}\int_{\partial D} (\phi_{\gamma_1})^{\gamma_2} d\phi_{\gamma_2} \\ &= \frac{1}{4}\int_0^{2\pi} (\theta - h_1(\theta) + h_1(0))^{h_2}(d\theta - d\theta^{h_2}) \\ &= \frac{1}{4}\int_0^{2\pi} (h_2(\theta)d\theta - h_1 \circ h_2(\theta)d\theta - \theta d\theta + h_1(\theta)d\theta) \\ &= \frac{1}{4}\int_0^{2\pi} (h_2(\theta)d\theta - h_1 \circ h_2(\theta)d\theta + h_1(\theta)d\theta) - \pi^2/2 \\ &= -\pi^2\chi(\gamma_1, \gamma_2) - \pi^2/2,\end{aligned}$$

where $\chi$ is the Euler cocyle in (2). Since a constant 2-cochain is a coboundary, it proves that $-\delta\tau$ is cohomologous to $\pi^2\chi$. Thus we have proved the following:

**Theorem 2** *Let $e(H_{\mathbb{R}})$ denote the Euler class in Example 2 of a universal central extension with coefficients in $\mathbb{R}$. Then the Euler class of the Calabi extension is equal to $\pi^2 e(H_{\mathbb{R}})$ in $H^2(G_\partial; \mathbb{R})$.*

*Remark 3* If one has the opposite signature with the Calabi invariant, the same for the identification to $\mathbb{R}$ and the Euler class is multiplied by negative one.

*Remark 4* Suppose that a symplectic form $\omega$ on $D$ has the form of $d\rho(r) \wedge d\theta$, where $\rho$ is a smooth function of radius $r$ with $\rho(0) = 0$ and the derivative $\rho'(0)$ vanishes. Then one can choose $\eta = \rho(r) \wedge d\theta$ and obtain

$$-\delta\tau = \pi^2\rho(1)^2\chi + \pi^2/2$$

by the same argument. Note that $2\pi\rho(1)$ is the symplectic volume of $D$. Thus, for such a symplectic form, the Euler class of the Calabi extension turns out to be equal to $\pi^2\rho(1)^2 e(G_{\mathbb{R}})$.

The identity above also proves the following:

**Theorem 3** *Recall a connection cochain $\tau : G \to \mathbb{R}$ in (5). Then the curvature $\delta\tau$ is bounded and thus $\tau$ gives rise to a quasi homomorphism on $G$, namely, $\delta\tau$ is a bounded cocycle.*

*Proof* Let $h : \mathbb{R} \to \mathbb{R}$ be an element of $H$ and set $n_h = [h(0)/2\pi]$. It is the largest integer that does not exceed $h(0)/2\pi$ as in Example 1. Then one can easily verify $2\pi n_h < \int_0^{2\pi} h(x)dx < 2\pi(n_h + 2)$ and $n_{h_1} + n_{h_2} \le n_{h_1 \circ h_2} < n_{h_1} + n_{h_2} + 2$. This implies that $|\chi(\gamma_1, \gamma_2)| < 2/\pi$, which proves that $\delta\tau$ is a bounded since $-\delta\tau = \pi^2\chi + \pi^2/2$. □

## 6 Transgression Formula for the Calabi Invariant

In this section we shall prove a transgression formula for the Calabi invariant. Let $\Omega^*(DG)$ be a simplicial de Rham model of $DG$, where $D$ is a closed unit disk and $G = \mathrm{Symp}(D)$. Denote also by $\Omega^*(\partial DG)$ a simplicial de Rham model of $\partial DG$, by letting $G$ act on the boundary $\partial D$ (the action is not faithful). Recall the definition of relative de Rham complex in Bott-Tu [2]. It is defined by

$$\Omega^n(DG, \partial DG) = \Omega^n(DG) \oplus \Omega^{n-1}(\partial DG), \qquad d(\alpha, \beta) = (D\alpha,\ \alpha|_{\partial D} - D\beta)$$

for $(\alpha, \beta) \in \Omega^n(DG) \oplus \Omega^{n-1}(\partial DG)$, where $D$ is the differential on a simplicial de Rham model. Denote by $H^*(\Omega(DG, \partial DG))$ a cohomology group of the relative complex and by $DG$ a flat disk bundle on the classifying space $BG$. Here we assume that $G$ is equipped with a discrete topology. Due to Theorem 1 by Bott [1], there is an isomorphism $H^*(\Omega(DG, \partial DG)) \cong H^*(DG, \partial DG; \mathbb{R})$, thus it plays the role of cohomology group for the Thom space of $DG$.

**Proposition 8** *Let $U$ denote the Thom class in $H^2(\Omega(DG, \partial DG))$. It is represented by cocycles $\frac{1}{\pi}(\omega, 0)$ and $-\frac{1}{\pi}(\delta\eta, \eta|_{\partial DG})$, which are cohomologous to each other.*

*Proof* It is easy to verify that $(\omega, 0)$ is a cocycle since $\omega$ is $G$-invariant. Note that $\omega = d\eta$. Thus one has $d(\eta, 0) = (\omega, 0) + (\delta\eta, \eta|_{\partial DG})$, which proves that above cocycles are cohomologous.

Recall that the Thom class is characterized by the property that $\int U = 1$, where $\int$ denotes the integration along the fiber, a homomorphism $H^n(DG, \partial DG; \mathbb{R}) \to H^{n-2}(BG; \mathbb{R})$. On a simplicial model, it is defined explicitly on a cochain level by the following formula

$$\int : \Omega^n(DG, \partial DG) \to C^{n-2}(G; \mathbb{R}), \qquad \int(\alpha, \beta) = \int_D \alpha - (-1)^r \int_{\partial D} \beta$$

with $\alpha$ of type $(p, q)$ and $\beta$ type $(r, s)$. Thus one has $\int(\omega, 0) = \int_D \omega = \pi$, which implies that the cohomology class of $(\omega, 0)/\pi$ is the Thom class. This completes the proof. □

The cup product on the relative complex is given by

$$(\alpha_1, \beta_1) \cup (\alpha_2, \beta_2) = (\alpha_1 \cup \alpha_2, \beta_1 \cup \alpha_2|_{\partial D}).$$

We often denote $\beta_1 \cup \alpha_2|_{\partial D}$ simply by $\beta_1 \cup \alpha_2$. The square of Thom class $U \cup U$ is then represented by a cocycle $(\delta\eta \cup \delta\eta, \eta \cup \delta\eta|_{\partial D})/\pi^2$, where $\eta \cup \delta\eta|_{\partial D}$ vanishes due to the dimensional reason. Thus, $U \cup U$ is represented by a cocycle $(\delta\eta \cup \delta\eta, 0)/\pi^2$. Let $\chi_{\mathrm{top}}$ be a 2-cocycle obtained as the image of $(\delta\eta \cup \delta\eta, 0)/\pi^2$ by the integration along the fiber. One then has

$$\chi_{\text{top}} = \frac{1}{\pi^2}\int(\delta\eta \cup \delta\eta, 0) = \frac{1}{\pi^2}\int_D \delta\eta \cup \delta\eta = \frac{1}{\pi^2}\delta\tau,$$

where $\tau : G \to \mathbb{R}$ the connection cochain in (5). Therefore, it turns out that $\chi_{\text{top}}$ is cohomologous to zero. On the other hand, it is known that the cohomology class of $\chi_{\text{top}}$ is equal to the Euler class of the disk bundle $DG \to BG$; see Milnor-Stasheff [3]. Thus this implies that the Euler class vanishes for $DG$. Summarizing, we have proved the following:

**Theorem 4** *Let $e(DG)$ denote the (topological) Euler class for a universal flat disc bundle $DG \to BG$. Then $e(DG)$ vanishes in $H^2(G;\mathbb{R}) \cong H^2(BG;\mathbb{R})$. In fact, a representative cocycle $\chi_{\text{top}}$ of $e(DG)$ is a coboundary of $\tau : G \to \mathbb{R}$ up to a constant multiple:*

$$\chi_{\text{top}} = \frac{1}{\pi^2}\int_D \delta\eta \cup \delta\eta = \frac{1}{\pi^2}\delta\tau.$$

*Remark 5* The vanishing of $e(DG)$ can be proved in a simple way. It is proved in Proposition 8 that the Thom class for $DG$ is represented by $(\omega, 0)/\pi$. Thus, a representative of $U \cup U$ is given by $(\omega, 0) \cup (\omega, 0)/\pi^2$, which vanishes due to the dimensional reason. Therefore, $e(DG)$ also vanishes by the same argument in the above.

The formula in Theorem 4 tells us that $\chi_{\text{top}}$ is a coboundary on $G$ such as $\pi^2\chi_{\text{top}} = \delta\tau$. However, it descends to a cocycle on the quotient group $G_\partial$ by Proposition 1, namely, there is a 2-cocycle $\sigma$ on $G_\partial$ such that $\rho^*\sigma = \delta\tau$ with $\rho : G \to G_\partial$ the surjective homomorphism. Moreover, we know that the cohomology class of $-\sigma$ is the Euler class of the Calabi extension, thus nontrivial. Putting these together, we proved the following transgression formula for $\sigma$:

**Theorem 5** (Transgression formula for the Calabi invariant) *Recall the short exact sequence in (3):*

$$1 \longrightarrow G_{\text{rel}} \longrightarrow G \stackrel{\rho}{\longrightarrow} G_\partial \longrightarrow 1.$$

*Let* Cal *be the Calabi invariant defined on $G_{\text{rel}}$ and $\tau : G \to \mathbb{R}$ the connection cochain in (5). Let $\sigma$ be a curvature of $\tau$ considerd as a cocycle on $G_\partial$. Then $\sigma$ is the transgression image of* Cal*, namely, we obtain the transgression formula:*

$$\sigma(\rho(g_1), \rho(g_2)) = \delta\tau(g_1, g_2) = \int_D \delta\eta \cup \delta\eta(g_1, g_2),$$
$$\mathrm{Cal}(h) = \tau(h) = \int_D \eta \cup \delta\eta(h)$$

*for $g_1, g_2 \in G$ and $h \in G_{\text{rel}}$.*

*Proof* The first identity is nothing but Proposition 7. The second one is obvious from the definition of $\tau$ in (5). □

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