Structure Theorems for Compact Kähler Manifolds with Nef Anticanonical Bundles

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Abstract This survey presents various results concerning the geometry of compact Kähler manifolds with numerically effective first Chern class: structure of the Albanese morphism of such manifolds, relations tying semipositivity of the Ricci curvature with rational connectedness, positivity properties of the Harder-Narasimhan filtration of the tangent bundle.

Keywords Compact Kähler manifold · Anticanonical bundle · Semipositive Ricci curvature · Ricci flat manifold · Rationally connected variety · Holonomy principle

1 Introduction and Preliminaries

The goal of this survey is to present in a concise manner several recent results concerning the geometry of compact Kähler manifolds with numerically effective first Chern class. Especially, we give a rather complete sketch of currently known facts about the Albanese morphism of such manifolds, and study the relations that tie semipositivity of the Ricci curvature with rational connectedness. Many of the ideas are borrowed from [DPS96, BDPP] and the recent PhD thesis of Cao [Cao13a, Cao13b].

Recall that a compact complex manifold X is said to be rationally connected if any two points of X can be joined by a chain of rational curves. A line bundle L is said to be hermitian semipositive if it can be equipped with a smooth hermitian metric of semipositive curvature form. A sufficient condition for hermitian semipositivity is that some multiple of L is spanned by global sections; on the other hand, the hermitian semipositivity condition implies that L is numerically effective (nef) in the sense of [DPS94], which, for X projective algebraic, is equivalent to saying that $L \cdot C \geq 0$ for every curve C in X. Examples contained in [DPS94] show that all three conditions are different (even for X projective algebraic). Finally, let us recall that a line bundle

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$L \to X$ is said to be pseudoeffective if here exists a singular hermitian metric h on L such that the Chern curvature current $T = i\Theta_{L,h} = -i\partial\overline{\partial}\log h$ is non-negative; equivalently, if X is projective algebraic, this means that the first Chern class $c_1(L)$ belongs to the closure of the cone of effective $\mathbb{Q}$-divisors.

The (Chern-)Ricci curvature is the curvature of the anticanonical bundle $K_X^{-1} = \det(T_X)$, and by Yau's solution of the Calabi conjecture (see [Aub76, Yau78]), a compact Kähler manifold X has a hermitian semipositive anticanonical bundle K_X^{-1} if and only if X admits a Kähler metric ω with $\mathrm{Ricci}(\omega) \geq 0$. Let us first review some classical examples of varieties with K_X^{-1} nef.

(ZFCC) *Compact Kähler manifolds with zero first Chern class*

The celebrated Bogomolov-Kobayashi-Beauville theorem yields the structure of compact Kähler Ricci-flat manifolds ([Bog74a, Bog74b, Kob81, Bea83]) which, by Yau's theorem [Yau78], are precisely compact Kähler manifolds with zero first Chern class. Recall that a *hyperkähler manifold* X is a simply connected compact Kähler manifold admitting a holomorphic symplectic 2-form σ (i.e. a holomorphic 2-form of maximal rank $n = 2p = \dim_{\mathbb{C}} X$ everywhere; in particular $K_X = \mathcal{O}_X$). A *Calabi-Yau* manifold is a simply connected projective manifold with $K_X = \mathcal{O}_X$ and $H^0(X, \Omega_X^p) = 0$ for $0 < p < n = \dim X$. Sometimes, finite étale quotient of such manifolds are also included in these classes (so that $\pi_1(X)$ is finite and possibly non trivial).

1.1 Theorem ([Bea83]) *Let (X, ω) be a compact Ricci flat Kähler manifold. Then there exists a finite étale Galois cover $\widehat{X} \to X$ such that*

$$\widehat{X} = T \times \prod Y_j \times \prod S_k$$

where $T = \mathbb{C}^q/\Lambda = \mathrm{Alb}(\widehat{X})$ is the Albanese torus of $\widehat{X}$, and Y_j, S_k are compact simply connected Kähler manifolds of respective dimensions n_j, n'_k with irreducible holonomy, Y_j being Calabi-Yau manifolds (holonomy group $= \mathrm{SU}(n_j)$) and S_k holomorphic symplectic manifolds (holonomy group $= \mathrm{Sp}(n'_k/2)$).

(RC-NAC) *Rationally connected manifolds with nef anticanonical class*

A classical example of projective surface with K_X^{-1} nef is the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$ blown-up in 9 points $\{a_j\}_{1\leq j\leq 9}$. By a trivial dimension argument, there always exist a cubic curve $C = \{P(z) = 0\}$ containing the 9 points, and we assume that C is nonsingular (hence a smooth elliptic curve). Let $\mu : X \to \mathbb{P}^2$ the blow-up map, $E_j = \mu^{-1}(a_j)$ the exceptional divisors and $\widehat{C}$ the strict transform of C. One has

$$K_X = \mu^* K_{\mathbb{P}^2} \otimes \mathcal{O}_X(\textstyle\sum E_j),$$

thus

$$K_X^{-1} = \mu^* \mathscr{O}_{\mathbb{P}^2}(3) \otimes \mathscr{O}_X(-\sum E_j) = \mathscr{O}_X(\widehat{C}),$$
$$\widehat{L} := (K_X^{-1})_{|\widehat{C}} = (\mu_{|\widehat{C}})^* L$$

where $L := \mathscr{O}_C(3) \otimes \mathscr{O}_C(-\sum[a_j]) \in \mathrm{Pic}^0(C)$. As a consequence we have $K_X^{-1} \cdot \widehat{C} = (\widehat{C})^2 = 0$. For any other irreducible curve Γ in X, we find $K_X^{-1} \cdot \Gamma = \widehat{C} \cdot \Gamma \geq 0$, therefore K_X^{-1} is nef. There is a non trivial section in $H^0(\widehat{C}, \widehat{L}^{\otimes m})$ if and only if L is a m-torsion point in $\mathrm{Pic}^0(C)$ (i.e. iff L has rational coordinates with respect to the periods of $\widehat{C}$), and in that case, it is easy to see that this section extends to a section of $H^0(X, K_X^{-m})$ (cf. e.g. [DPS96]). This also means that there is an elliptic pencil $\alpha P(z)^m + \beta Q_m(z) = 0$ defined by a fibration

$$\pi_m = Q_m/P^m : X \to \mathbb{P}^1,$$

where $Q_m \in H^0(\mathbb{P}^2, \mathscr{O}(3m))$ vanishing at order m at all points a_j; the generic fiber of π_m is then a singular elliptic curve of multiplicity m at a_j, and we have $K_X^{-m} = (\pi_m)^* \mathscr{O}_{\mathbb{P}^1}(1)$, in particular K_X^{-m} is generated by its sections and possesses a real analytic metric of semipositive curvature. Now, when $L \notin \mathrm{Pic}^0(C)$ (corresponding to a generic position of the 9 points a_j on C), Ueda has analyzed the structure of neighborhoods of $\widehat{C}$ in X, and shown that it depends on a certain following diophantine condition for the point $\lambda \in H^1(C, \mathscr{O}_C)/H^1(C, \mathbb{Z})$ on the Jacobian variety of C associated with L (cf. [Ued82, p. 595], see also [Arn76]). This condition can be written

$$-\log d(m\lambda, 0) = O(\log m) \quad \text{as } m \to +\infty, \tag{1.1}$$

where d is a translation invariant geodesic distance on the Jacobian variety. Especially, (1.1) is independent of the choice of d and is satisfied on a set of full measure in $\mathrm{Pic}^0(C)$. When this is the case, Ueda has shown that $\widehat{C}$ possesses a "pseudoflat neighborhood", namely an open neighborhood U on which there exists a pluriharmonic function with logarithmic poles along $\widehat{C}$. Relying on this, Brunella [Bru10] has proven

1.2 Theorem *Let X, C, L be as above and assume that L is not a torsion point in* $\mathrm{Pic}^0(C)$. *Then*

(a) There exists on X a smooth Kähler metric with semipositive Ricci curvature if and only if $\widehat{C}$ admits a pseudoflat neighborhood in X.
(b) There does not exist on X a real analytic Kähler metric with semipositive Ricci curvature.

It seems likely (but is yet unproven) that $\widehat{C}$ does not possess pseudoflat neighborhoods when (0.2) badly fails, e.g. when the coordinates of λ with respect to periods are some sort of Liouville numbers like $\sum 1/10^{n!}$. Then, K_X^{-1} would be a nef line bundle without any smooth semipositive hermitian metric[1]. It might still be possible that there always exist singular hermitian metrics with zero Lelong numbers (and thus with trivial multiplier ideal sheaves) on such a rational surface, but this seems to be an open question as well. In general, the example of ruled surface over an elliptic curve given in [DPS94, Example 1.7] shows that such metrics with zero Lelong numbers need not always exist when K_X^{-1} is nef, but we do not know the answer when X is rationally connected. Studying in more depth the class of rationally connected projective manifolds with nef or semipositive anticanonical bundles is thus very desirable.

2 Criterion for Rational Connectedness

We give here a criterion characterizing rationally connected manifolds X in terms of positivity properties of invertible subsheaves contained in Ω_X^p or $(T_X^*)^{\otimes p}$; this is only a minor variation of Theorem 5.2 in [Pet06].

2.1 Critertion *Let X be a projective algebraic n-dimensional manifold. The following properties are equivalent.*

(a) X is rationally connected.
(b) For every invertible subsheaf $\mathscr{F} \subset \Omega_X^p := \mathscr{O}(\Lambda^p T_X^*)$, $1 \le p \le n$, $\mathscr{F}$ is not pseudoeffective.
(c) For every invertible subsheaf $\mathscr{F} \subset \mathscr{O}((T_X^*)^{\otimes p})$, $p \ge 1$, $\mathscr{F}$ is not pseudoeffective.
(d) For some (resp. for any) ample line bundle A on X, there exists a constant $C_A > 0$ such that

$$H^0(X, (T_X^*)^{\otimes m} \otimes A^{\otimes k}) = 0 \qquad \text{for all } m, k \in \mathbb{N}^* \text{ with } m \ge C_A k.$$

Proof Observe first that if X is rationally connected, then there exists an immersion $f : \mathbb{P}^1 \subset X$ (in fact, many of them) passing through any given finite subset of X, and such that $f^* T_X$ is ample, see e.g. [Kol96, Theorem 3.9, p. 203]. It follows easily from there that 1.1 (a) implies 1.1 (d). The only non trivial implication that remains to be proved is that 1.1 (b) implies 1.1 (a). First note that K_X is not pseudoeffective, as one sees by applying the assumption 1.1 (b) with $p = n$. Hence X is uniruled by [BDPP]. We consider the quotient with maximal rationally connected fibers (rational quotient or MRC fibration, see [Cam92, KMM92])

$$f : X \dashrightarrow W$$

[1] Added in proof. In a very recent manuscript, Takayuki Koike has established the existence of such nef and non semipositive configurations, cf. arXiv:1507.00109, "Ueda theory for compact curves with nodes".

to a smooth projective variety W. By [GHS01], W is not uniruled, otherwise we could lift the ruling to X and the fibers of f would not be maximal. We may further assume that f is holomorphic. In fact, assumption 1.1 (b) is invariant under blow-ups. To see this, let $\pi : \hat{X} \to X$ be a birational morphisms from a projective manifold $\hat{X}$ and consider a line bundle $\hat{\mathscr{F}} \subset \Omega^p_{\hat{X}}$. Then $\pi_*(\hat{\mathscr{F}}) \subset \pi_*(\Omega^p_{\hat{X}}) = \Omega^p_X$, hence we introduce the line bundle

$$\mathscr{F} := (\pi_*(\hat{\mathscr{F}}))^{**} \subset \Omega^p_X.$$

Now, if $\hat{\mathscr{F}}$ were pseudoeffective, so would be $\mathscr{F}$. Thus 1.1 (b) is invariant under π and we may suppose f holomorphic. In order to show that X is rationally connected, we need to prove that $p := \dim W = 0$. Otherwise $K_W = \Omega^p_W$ is pseudoeffective by [BDPP], and we obtain a pseudo-effective invertible subsheaf $\mathscr{F} := f^*(\Omega^p_W) \subset \Omega^p_X$, in contradiction with 1.1 (b). □

2.2 Remark By [DPS94], assumptions 1.1 (b) and (c) make sense on arbitrary compact complex manifolds and imply that $H^0(X, \Omega^2_X) = 0$. If X is assumed to be compact Kähler, then X is automatically projective algebraic by Kodaira [Kod54], therefore, 1.1 (b) or (c) also characterize rationally connected manifolds among all compact Kähler ones. □

3 A Generalized Holonomy Principle

Recall that the restricted holonomy group of a hermitian vector bundle (E, h) of rank r is the subgroup $H \subset \mathrm{U}(r) \simeq U(E_{z_0})$ generated by parallel transport operators with respect to the Chern connection ∇ of (E, h), along loops based at z_0 that are contractible (up to conjugation, H does not depend on the base point z_0). The standard holonomy principle (see e.g. [BY53]) admits a generalized "pseudoeffective" version, which can be stated as follows.

3.1 Theorem *Let E be a holomorphic vector bundle of rank r over a compact complex manifold X. Assume that E is equipped with a smooth hermitian structure h and X with a hermitian metric ω, viewed as a smooth positive $(1, 1)$-form $\omega = i \sum \omega_{jk}(z) dz_j \wedge d\overline{z}_k$. Finally, suppose that the ω-trace of the Chern curvature tensor $\mathrm{i}\Theta_{E,h}$ is semipositive, that is*

$$\mathrm{i}\Theta_{E,h} \wedge \frac{\omega^{n-1}}{(n-1)!} = B\,\frac{\omega^n}{n!}, \qquad B \in \mathrm{Herm}(E, E), \quad B \geq 0 \ \text{on } X,$$

and denote by H the restricted holonomy group of (E, h).

(a) *If there exists an invertible sheaf $\mathscr{L} \subset \mathscr{O}((E^*)^{\otimes m})$ which is pseudoeffective as a line bundle, then $\mathscr{L}$ is flat and $\mathscr{L}$ is invariant under parallel transport by the* *connection of* $(E^*)^{\otimes m}$ *induced by the Chern connection* ∇ *of* (E, h); *in fact,* H *acts trivially on* $\mathscr{L}$.

(b) *If* H *satisfies* $H = \mathrm{U}(r)$, *then none of the invertible subsheaves* $\mathscr{L}$ *of* $\mathscr{O}((E^*)^{\otimes m})$ *can be pseudoeffective for* $m \geq 1$.

Proof The semipositivity hypothesis on $B = \mathrm{Tr}_\omega \mathrm{i}\Theta_{E,h}$ is invariant by a conformal change of metric ω. Without loss of generality we can assume that ω is a Gauduchon metric, i.e. that $\partial\overline{\partial}\omega^{n-1} = 0$, cf. [Gau77]. We consider the Chern connection ∇ on (E, h) and the corresponding parallel transport operators. At every point $z_0 \in X$, there exists a local coordinate system $(z_1, \ldots, z_n)$ centered at z_0 (i.e. $z_0 = 0$ in coordinates), and a holomorphic frame $(e_\lambda(z))_{1\leq\lambda\leq r}$ such that

$$\langle e_\lambda(z), e_\mu(z)\rangle_h = \delta_{\lambda\mu} - \sum_{1\leq j,k\leq n} c_{jk\lambda\mu} z_j \overline{z}_k + O(|z|^3), \quad 1 \leq \lambda, \mu \leq r,$$

$$\Theta_{E,h}(z_0) = \sum_{1\leq j,k,\lambda,\mu\leq n} c_{jk\lambda\mu} dz_j \wedge d\overline{z}_k \otimes e_\lambda^* \otimes e_\mu, \quad c_{kj\mu\lambda} = \overline{c_{jk\lambda\mu}},$$

where $\delta_{\lambda\mu}$ is the Kronecker symbol and $\Theta_{E,h}(z_0)$ is the curvature tensor of the Chern connection ∇ of (E, h) at z_0.

Assume that we have an invertible sheaf $\mathscr{L} \subset \mathscr{O}((E^*)^{\otimes m})$ that is pseudoeffective. There exist a covering U_j by coordinate balls and holomorphic sections f_j of $\mathscr{L}_{|U_j}$ generating $\mathscr{L}$ over U_j. Then $\mathscr{L}$ is associated with the Čech cocycle g_{jk} in $\mathscr{O}_X^*$ such that $f_k = g_{jk} f_j$, and the singular hermitian metric $e^{-\varphi}$ of $\mathscr{L}$ is defined by a collection of plurisubharmonic functions $\varphi_j \in \mathrm{PSH}(U_j)$ such that $e^{-\varphi_k} = |g_{jk}|^2 e^{-\varphi_j}$. It follows that we have a globally defined bounded measurable function

$$\psi = e^{\varphi_j} \| f_j \|^2 = e^{\varphi_j} \| f_j \|^2_{h^{*m}}$$

over X, which can be viewed also as the ratio of hermitian metrics $(h^*)^m / e^{-\varphi}$ along $\mathscr{L}$, i.e. $\psi = (h^*)^m_{|\mathscr{L}} e^{\varphi}$. We are going to compute the Laplacian $\Delta_\omega \psi$. For simplicity of notation, we omit the index j and consider a local holomorphic section f of $\mathscr{L}$ and a local weight $\varphi \in \mathrm{PSH}(U)$ on some open subset U of X. In a neighborhood of an arbitrary point $z_0 \in U$, we write

$$f = \sum_{\alpha\in\mathbb{N}^m} f_\alpha\, e_{\alpha_1}^* \otimes \ldots \otimes e_{\alpha_m}^*, \qquad f_\alpha \in \mathscr{O}(U),$$

where (e_λ^*) is the dual holomorphic frame of (e_λ) in $\mathscr{O}(E^*)$. The hermitian matrix of (E^*, h^*) is the transpose of the inverse of the hermitian matrix of (E, h), hence we get

$$\langle e_\lambda^*(z), e_\mu^*(z)\rangle_h = \delta_{\lambda\mu} + \sum_{1\leq j,k\leq n} c_{jk\mu\lambda} z_j \overline{z}_k + O(|z|^3), \qquad 1 \leq \lambda, \mu \leq r.$$

On the open set U the function $\psi = (h^*)^m_{|\mathscr{L}} e^{\varphi}$ is given by

$$\psi = \Big(\sum_{\alpha \in \mathbb{N}^m} |f_\alpha|^2 + \sum_{\alpha,\beta \in \mathbb{N}^m,\, 1 \le j,k \le n,\, 1 \le \ell \le m} f_\alpha \overline{f_\beta}\, c_{jk\beta_\ell \alpha_\ell} z_j \overline{z}_k + O(|z|^3)|f|^2 \Big) e^{\varphi(z)}.$$

By taking $\mathrm{i}\partial\overline{\partial}(...)$ of this at $z = z_0$ in the sense of distributions (that is, for almost every $z_0 \in X$), we find

$$\begin{aligned} \mathrm{i}\partial\overline{\partial}\psi =& e^{\varphi}\Big(|f|^2 \mathrm{i}\partial\overline{\partial}\varphi + i\langle \partial f + f\partial\varphi, \partial f + f\partial\varphi\rangle \\ &+ \sum_{\alpha,\beta,j,k,1 \le \ell \le m} f_\alpha \overline{f_\beta}\, c_{jk\beta_\ell\alpha_\ell}\, i dz_j \wedge d\overline{z}_k \Big). \end{aligned}$$

Since $\mathrm{i}\partial\overline{\partial}\psi \wedge \frac{\omega^{n-1}}{(n-1)!} = \Delta_\omega \psi \frac{\omega^n}{n!}$ (we actually take this as a definition of Δ_ω), a multiplication by ω^{n-1} yields the fundamental inequality

$$\Delta_\omega \psi \ge |f|^2 e^{\varphi}(\Delta_\omega \varphi + m\lambda_1) + |\nabla_h^{1,0} f + f\partial\varphi|^2_{\omega,h^{*m}}\, e^{\varphi}$$

where $\lambda_1(z) \ge 0$ is the lowest eigenvalue of the hermitian endomorphism $B = \mathrm{Tr}_\omega \mathrm{i}\Theta_{E,h}$ at an arbitrary point $z \in X$. As $\partial\overline{\partial}\omega^{n-1} = 0$, we have

$$\int_X \Delta_\omega \psi\, \frac{\omega^n}{n!} = \int_X \mathrm{i}\partial\overline{\partial}\psi \wedge \frac{\omega^{n-1}}{(n-1)!} = \int_X \psi \wedge \frac{\mathrm{i}\partial\overline{\partial}(\omega^{n-1})}{(n-1)!} = 0$$

by Stokes' formula. Since $\mathrm{i}\partial\overline{\partial}\varphi \ge 0$, the above inequality implies $\Delta_\omega\varphi = 0$, i.e. $\mathrm{i}\partial\overline{\partial}\varphi = 0$, and $\nabla_h^{1,0} f + f\partial\varphi = 0$ almost everywhere. This means in particular that the line bundle $(\mathscr{L}, e^{-\varphi})$ is flat. In each coordinate ball U_j the pluriharmonic function φ_j can be written $\varphi_j = w_j + \overline{w}_j$ for some holomorphic function $w_j \in \mathscr{O}(U_j)$, hence $\partial\varphi_j = dw_j$ and the condition $\nabla_h^{1,0} f_j + f_j\partial\varphi_j = 0$ can be rewritten $\nabla_h^{1,0}(e^{w_j} f_j) = 0$ where $e^{w_j} f_j$ is a local holomorphic section. This shows that $\mathscr{L}$ must be invariant by parallel transport and that the local holonomy of the Chern connection of (E, h) acts trivially on $\mathscr{L}$. Statement 2.1 (a) follows.

Finally, if we assume that the restricted holonomy group H of (E, h) is equal to $\mathrm{U}(r)$, there cannot exist any holonomy invariant invertible subsheaf $\mathscr{L} \subset \mathscr{O}((E^*)^{\otimes m})$, $m \ge 1$, on which H acts trivially, since the natural representation of $\mathrm{U}(r)$ on $(\mathbb{C}^r)^{\otimes m}$ has no invariant line on which $\mathrm{U}(r)$ induces a trivial action. Property 2.1 (b) is proved. □

4 Structure Theorem for Compact Kähler Manifolds with K_X^{-1} Semipositive

In this context, the following generalization of the Bogomolov-Kobayashi-Beauville Theorem 1.1 holds.

4.1 Structure Theorem *Let X be a compact Kähler manifold with K_X^{-1} hermitian semipositive. Then there exists a finite étale Galois cover $\widehat{X} \to X$ such that*

$$\widehat{X} \simeq \mathbb{C}^q/\Lambda \times \prod Y_j \times \prod S_k \times \prod Z_\ell$$

where $\mathbb{C}^q/\Lambda = \mathrm{Alb}(\widehat{X})$ is the Albanese torus of $\widehat{X}$, and Y_j, S_k, Z_ℓ are compact simply connected Kähler manifolds of respective dimensions n_j, n'_k, n''_ℓ with irreducible holonomy, Y_j being Calabi-Yau manifolds (holonomy $\mathrm{SU}(n_j)$), S_k holomorphic symplectic manifolds (holonomy $\mathrm{Sp}(n'_k/2)$), and Z_ℓ rationally connected manifolds with $K_{Z_\ell}^{-1}$ semipositive (holonomy $\mathrm{U}(n''_\ell)$).

Proof The proof relies on our generalized holonomy principle, combined with De Rham's splitting theorem [DR52] and Berger's classification [Ber55]. Foundational background can be found in papers by Lichnerowicz [Lic67, Lic71], and Cheeger and Gromoll [CG71, CG72].

We suppose here that X is equipped with a Kähler metric ω such that $\mathrm{Ricci}(\omega) \geq 0$, and we set $n = \dim_{\mathbb{C}} X$. We consider the holonomy representation of the tangent bundle $E = T_X$ equipped with the hermitian metric $h = \omega$. Here

$$B = \mathrm{Tr}_\omega \mathrm{i}\Theta_{E,h} = \mathrm{Tr}_\omega \mathrm{i}\Theta_{T_X,\omega} \geq 0$$

is nothing but the Ricci operator. Let $\widetilde{X} \to X$ be the universal cover of X and

$$(\widetilde{X}, \omega) \simeq \prod (X_i, \omega_i)$$

be the De Rham decomposition of $(\widetilde{X}, \omega)$, induced by a decomposition of the holonomy representation in irreducible representations. Since the holonomy is contained in $\mathrm{U}(n)$, all factors (X_i, ω_i) are Kähler manifolds with irreducible holonomy and holonomy group $H_i \subset \mathrm{U}(n_i)$, $n_i = \dim X_i$. By Cheeger and Gromoll [CG71], there is possibly a flat factor $X_0 = \mathbb{C}^q$ and the other factors X_i, $i \geq 1$, are compact and simply connected. Also, the product structure shows that each $K_{X_i}^{-1}$ is hermitian semipositive. By Berger's classification of holonomy groups [Ber55] there are only three possibilities, namely $H_i = \mathrm{U}(n_i)$, $H_i = \mathrm{SU}(n_i)$ or $H_i = \mathrm{Sp}(n_i/2)$. The case $H_i = \mathrm{SU}(n_i)$ leads to X_i being a Calabi-Yau manifold, and the case $H_i = \mathrm{Sp}(n_i/2)$ implies that X_i is holomorphic symplectic (see e.g. [Bea83]). Now, if $H_i = \mathrm{U}(n_i)$, the generalized holonomy principle 2.1 (b) shows that none of the invertible subsheaves $\mathscr{L} \subset \mathscr{O}((T^*_{X_i})^{\otimes m})$ can be pseudoeffective for $m \geq 1$. Therefore X_i is rationally connected by Criterion 2.1.

It remains to show that the product decomposition descends to a finite cover $\widehat{X}$ of X. However, the fundamental group $\pi_1(X)$ acts by isometries on the product, and does not act at all on the rationally connected factors Z_ℓ which are simply connected. Thanks to the irreducibility, the factors have to be preserved or permuted by any element $\gamma \in \pi_1(X)$, and the group of isometries of the factors S_j, Y_j are finite (since $H^0(Y, T_Y) = 0$ for such factors and the remaining discrete group $\mathrm{Aut}(Y)/\mathrm{Aut}^0(Y)$

is compact). Therefore, there is a subgroup Γ_0 of finite index in $\pi_1(X)$ which acts trivially on all factors except $\mathbb{C}^q$. By Bieberbach's theorem, there is a subgroup Γ of finite index in Γ_0 that acts merely by translations on $\mathbb{C}^q$. After taking the intersection of all conjugates of Γ in $\pi_1(X)$, we can assume that Γ is normal in $\pi_1(X)$. Then, if Λ is the lattice of translations of $\mathbb{C}^q$ defined by Γ, the quotient $\widehat{X} = \widetilde{X}/\Gamma$ is the finite étale cover of X we were looking for. □

Thanks to the exact sequence of fundamental groups associated with a fibration, we infer

4.2 Corollary *Under the assumptions of Theorem 4.1, there is an exact sequence*

$$0 \to \mathbb{Z}^{2q} \to \pi_1(X) \to G \to 0$$

where G is a finite group, namely $\pi_1(X)$ is almost abelian and is an extension of a finite group G by the normal subgroup $\pi_1(\widehat{X}) \simeq \mathbb{Z}^{2q}$.

5 Compact Kähler Manifolds with Nef Anticanonical Bundles

In this section, we investigate the properties of compact Kähler manifolds possessing a numerically effective anticanonical bundle K_X^{-1}. A simple but crucial observation made in [DPS93] is

5.1 Proposition *Let X be compact Kähler manifold and $\{\omega\}$ a Kähler class on X. Then the following properties are equivalent:*

(a) K_X^{-1} *is nef.*

(b) For every $\varepsilon > 0$, there exists a Kähler metric $\omega_\varepsilon = \omega + i\partial\overline{\partial}\varphi_\varepsilon$ in the cohomology class $\{\omega\}$ such that $\mathrm{Ricci}(\omega_\varepsilon) \geq -\varepsilon\omega$.

(c) For every $\varepsilon > 0$, there exists a Kähler metric $\omega_\varepsilon = \omega + i\partial\overline{\partial}\varphi_\varepsilon$ in the cohomology class $\{\omega\}$ such that $\mathrm{Ricci}(\omega_\varepsilon) \geq -\varepsilon\omega_\varepsilon$.

Sketch of Proof The nefness of K_X^{-1} means that $c_1(X) = c_1(K_X^{-1})$ contains a closed $(1,1)$-form ρ_ε with $\rho_\varepsilon \geq -\varepsilon\omega$, so (b) implies (a); the converse is true by Yau's theorem [Yau78] asserting the existence of Kähler metrics $\omega_\varepsilon \in \{\omega\}$ with prescribed Ricci curvature $\mathrm{Ricci}(\omega_\varepsilon) = \rho_\varepsilon$. Since $\omega_\varepsilon \equiv \omega$, (c) implies

$$c_1(X) + \varepsilon\{\omega\} \ni \rho'_\varepsilon := \mathrm{Ricci}(\omega_\varepsilon) + \varepsilon\omega_\varepsilon \geq 0,$$

hence (c) implies (a). The converse (a) $\Rightarrow$ (c) can be seen to hold thanks to the solvability of Monge-Ampère equations of the form $(\omega + i\partial\overline{\partial}\varphi)^n = \exp(f + \varepsilon\varphi)$, due to Aubin [Aub76]. □

By using standard methods of Riemannian geometry such as the Bishop-Gage inequality for the volume of geodesic balls, one can then show rather easily that the fundamental group $\pi_1(X)$ has subexponential growth. This was improved by M. Păun in his PhD thesis, using more advanced tools (Gromov-Hausdorff limits and results of Cheeger and Colding [CC96, CC97], as well as the fundamental theorem of Gromov on groups of polynomial growth [Gr81a, Gr81b]).

5.2 Theorem ([Pau97, Pau98]) *Let X be a compact Kähler manifold with K_X^{-1} nef. Then $\pi_1(X)$ has polynomial growth and, as a consequence (thanks to Gromov) it possesses a nilpotent subgroup of finite index.*

We next study stability issues. Recall that the *slope* of a non zero torsion-free sheaf $\mathscr{F}$ with respect to a Kähler metric ω is

$$\mu_\omega(\mathscr{F}) = \frac{1}{\mathrm{rank}(\mathscr{F})} \int_X c_1(\mathscr{F}) \wedge \omega^{n-1}.$$

Moreover, $\mathscr{F}$ is said to be ω-stable (in the sense of Mumford-Takemoto) if $\mu_\omega(\mathscr{S}) < \mu_\omega(\mathscr{F})$ for every torsion-free subsheaf $\mathscr{S} \subset \mathscr{F}$ with $0 < \mathrm{rank}(\mathscr{S}) < \mathrm{rank}(\mathscr{F})$. In his PhD thesis [Cao13a, Cao13b], Junyan Cao observed the following important fact.

5.3 Theorem ([Cao13a, Cao13b]) *Let (X, ω) be a compact n-dimensional Kähler manifold such that K_X^{-1} is nef. Let*

$$0 = \mathscr{F}_0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_s = T_X$$

be a Harder-Narasimhan filtration of T_X with respect to ω, namely a filtration of torsion-free subsheaves such that $\mathscr{F}_i/\mathscr{F}_{i-1}$ is ω-stable with maximal slope in $T_X/\mathscr{F}_{i-1}$ [it is then well known that $i \mapsto \mu_\omega(\mathscr{F}_i/\mathscr{F}_{i-1})$ is a non increasing sequence]. Then

$$\mu_\omega(\mathscr{F}_i/\mathscr{F}_{i-1}) \geq 0 \quad \text{for all } i.$$

Proof First consider the case where the filtration is regular, i.e., all sheaves $\mathscr{F}_i$ and their quotients $\mathscr{F}_i/\mathscr{F}_{i-1}$ are vector bundles. By the stability condition, it is sufficient to prove that

$$\int_X c_1(T_X/\mathscr{F}_i) \wedge \omega^{n-1} \geq 0 \quad \text{for all } i.$$

By 4.1 (b), for each $\varepsilon > 0$, there is a metric $\omega_\varepsilon \in \{\omega\}$ such that $\mathrm{Ricci}(\omega_\varepsilon) \geq -\varepsilon\omega_\varepsilon$. This is equivalent to the pointwise estimate

$$\mathrm{i}\Theta_{T_X,\omega_\varepsilon} \wedge \omega_\varepsilon^{n-1} \geq -\varepsilon \cdot \mathrm{Id}_{T_X} \omega_\varepsilon^n.$$

Taking the induced metric on $T_X/\mathscr{F}_i$ (which we also denote by ω_ε), the second fundamental form contributes nonnegative terms on the quotient, hence the ω_ε-trace yields

$$\mathrm{i}\Theta_{T_X/\mathscr{F}_i,\omega_\varepsilon} \wedge \omega_\varepsilon^{n-1} \geq -\varepsilon\,\mathrm{rank}(T_X/\mathscr{F}_i)\,\omega_\varepsilon^n.$$

Therefore, putting $r_i = \mathrm{rank}(T_X/\mathscr{F}_i)$, we get

$$\begin{aligned}\int_X c_1(T_X/\mathscr{F}_i) \wedge \omega^{n-1} &= \int_X c_1(T_X/\mathscr{F}_i) \wedge \omega_\varepsilon^{n-1} \\ &\geq -\varepsilon r_i \int_X \omega_\varepsilon^n = -\varepsilon r_i \int_X \omega^n,\end{aligned}$$

and we are done. In case there are singularities, they occur only on some analytic subset $S \subset X$ of codimension 2. The first Chern forms calculated on $X \smallsetminus S$ extend as locally integrable currents on X and do not contribute any mass on S. The above calculations are thus still valid. □

By the results of Bando and Siu [BS94], all quotients $\mathscr{F}_i/\mathscr{F}_{i-1}$ possess a Hermite-Einstein metric h_i that is smooth in the complement of the analytic locus S of codimension at least 2 where the $\mathscr{F}_i$ are not regular subbundles of T_X. Assuming ω normalized so that $\int_X \omega^n = 1$, we thus have

$$\Theta_{\mathscr{F}_i/\mathscr{F}_{i-1},h_i} \wedge \omega^{n-1} = \mu_i \mathrm{Id}_{\mathscr{F}_i/\mathscr{F}_{i-1}} \omega^n$$

where $\mu_i \geq 0$ is the corresponding slope. Using this, one easily obtains:

5.4 Corollary *Let (X, ω) be a compact Kähler manifold with K_X^{-1} nef, and S the analytic set of codimension ≥ 2 in X where the Harder-Narasimhan filtration of T_X with respect to ω is not regular. If a section $\sigma \in H^0(X, (T_X^*)^{\otimes m})$ vanishes at some point $x \in X \smallsetminus S$, it must vanish identically.*

Proof By dualizing the filtration of T_X and taking the m-th tensor product, we obtain a filtration

$$0 = \mathscr{G}_0 \subset \mathscr{G}_1 \subset \cdots \subset \mathscr{G}_N = (T_X^*)^{\otimes m}$$

such that all slopes $\mu_i = \mu_\omega(\mathscr{G}_i/\mathscr{G}_{i-1})$ satisfy $0 \geq \mu_1 \geq \ldots \geq \mu_N$. Now, if u is a section of a hermitian vector bundle $(\mathscr{G}, h)$ of slope $\mu \leq 0$, a standard calculation shows that

$$\Delta_\omega(\log \|u\|_h^2) = \mathrm{i}\partial\overline{\partial} \log \|u\|_h^2 \wedge \frac{\omega^{n-1}}{(n-1)!} \geq \|\nabla_h u\|_h^2 \, \frac{\omega^n}{n!} \geq 0.$$

By the maximum principle $\|u\|_h$ must be constant, and also u must be h-parallel, and if $\mu < 0$, the strict inequality for the trace of the curvature implies in fact $u \equiv 0$.

For $\mu = 0$ and $u \not\equiv 0$, any equality $u(x) = 0$ at a point where h does not blow up would lead to a non constant subharmonic function $\log \|u\|_h$ with a $-\infty$ pole on $X \smallsetminus S$, contradiction. From this, we conclude by descending induction starting with $i = N - 1$ that the image of σ in $H^0(X, (T_X^*)^{\otimes m}/\mathcal{G}_i)$ vanishes identically, hence σ lies in fact in $H^0(X, \mathcal{G}_i)$, and we proceed inductively by looking at its image in $H^0(X, \mathcal{G}_i/\mathcal{G}_{i-1})$. □

The next result has been first proved by Zhang [Zha96] in the projective case, and by Păun [Pau12] in the general Kähler case. We give here a different proof based on the ideas of Junyan Cao (namely, on Theorem 5.3 and Corollary 5.4).

5.5 Corollary *Let (X, ω) be a compact Kähler manifold with nef anticanonical bundle. Then the Albanese map $\alpha : X \to \mathrm{Alb}(X)$ is surjective, and smooth outside a subvariety of codimension at least 2. In particular, the fibers of the Albanese map are connected and reduced in codimension 1.*

Proof Let $\sigma_1, \ldots, \sigma_q \in H^0(X, \Omega_X^1)$ be a basis of holomorphic 1-forms. The Albanese map is obtained by integrating the σ_j's and the differential of α is thus given by $d\alpha = (\sigma_1, \ldots, \sigma_q) : T_X \to \mathbb{C}^q$. Hence α is a submersion at a point $x \in X$ if and only if no non trivial linear combination $\sigma = \sum \lambda_j \sigma_j$ vanishes at x. This is the case if $x \in X \smallsetminus S$. In particular α has generic rank equal to q, and must be surjective and smooth in codimension 1. The connectedness of fibers is a standard fact (α cannot descend to a finite étale quotient because it induces an isomorphism at the level of the first homology groups). □

A conjecture attributed to Mumford states that a projective or Kähler manifold X is rationally connected if and only if $H^0(X, (T_X^*)^{\otimes m}) = 0$ for all $m \geq 1$. As an application of the above results of J. Cao, it is possible to confirm this conjecture in the case of compact Kähler manifolds with nef anticanonical bundles.

5.6 Proposition *Let X be a compact Kähler n-dimensional manifold with nef anti-canonical bundle. Then the following properties are equivalent*:

(a) X is projective and rationally connected;
(b) for every $m \geq 1$, one has $H^0(X, (T_X^*)^{\otimes m}) = 0$;
(c) for every $m = 1, \ldots, n$ and every finite étale cover $\widehat{X}$ of X, one has $H^0(\widehat{X}, \Omega_{\widehat{X}}^m) = 0$.

Proof As already seen, (a) implies (b) and (c) (apply 1.1 (d) and the fact that X is simply connected). Now, for any $p : 1$ cover $\widehat{X} \to X$, by taking a "direct image tensor product", a non zero section of $H^0(\widehat{X}, \Omega_{\widehat{X}}^m)$ would yield a non zero section of

$$(\Omega_X^m)^{\otimes p} \subset (T_X^*)^{\otimes mp},$$

thus (b) implies (c). It remains to show that (c) implies (a). Assume that (c) holds. In particular $H^0(X, \Omega_X^2) = 0$ and X must be projective by Kodaira. Fix an ample line bundle A on X and look at the Harder-Narasimhan filtration $(\mathscr{F}_i)_{0\le i\le s}$ of T_X with respect to any Kähler class ω. If all slopes are strictly negative, then for any $m \gg p > 0$ the tensor product $(T_X^*)^{\otimes m} \otimes A^p$ admits a filtration with negative slopes. In this circumstance, the maximum principle then implies that Criterion 2.1 (d) holds, therefore X is rationally connected. The only remaining case to be treated is when one of the slopes is zero, i.e. for every Kähler class there is a subsheaf $\mathscr{F}_\omega \subsetneq T_X$ such that $\int_X c_1(T_X/\mathscr{F}_\omega) \wedge \omega^{n-1} = 0$. Now, by standard lemmas on stability, these subsheaves $\mathscr{F}_\omega$ live in a finite number of families. Since the intersection number $\int_X c_1(T_X/\mathscr{F}) \wedge \omega^{n-1}$ does not change in a given irreducible component of such a family of sheaves, we infer (e.g. by Baire's theorem!) that there would exist a subsheaf $\mathscr{F} \subsetneq T_X$ and a set of Kähler classes $\{\omega\}$ with non empty interior in the Kähler cone, such that $\int_X c_1(T_X/\mathscr{F}) \wedge \omega^{n-1} = 0$ for all these classes. However, by taking variations of $(\omega + t\alpha)^{n-1}$ with $t > 0$ small, we conclude that the intersection product of the first Chern class $c_1(T_X/\mathscr{F})$ with any product $\omega^{n-2} \wedge \alpha$ vanishes. The Hard Lefschetz together with Serre duality now implies that $c_1(T_X/\mathscr{F})_{\mathbb{R}} \in H^2(X, \mathbb{R})$ is equal to zero. By duality, there is a subsheaf $\mathscr{G} \subset \Omega_X^1$ of rank $m = 1, \ldots, n$ such that $c_1(\mathscr{G})_{\mathbb{R}} = 0$. By taking $\mathscr{L} = \det(\mathscr{G})^{**}$, we get an invertible subsheaf $\mathscr{L} \subset \Omega_X^m$ with $c_1(\mathscr{L})_{\mathbb{R}} = 0$. Since $h^1(X, \mathscr{O}_X) = h^0(X, \Omega_X^1) = 0$, some power $\mathscr{L}^p$ is trivial and we get a finite cover $\pi : \widehat{X} \to X$ such that $\pi^*\mathscr{L}$ is trivial. This produces a non zero section of $H^0(\widehat{X}, \Omega_{\widehat{X}}^m)$, contradiction. □

The following basic question is still unsolved (cf. also [DPS96]).

5.7 Problem *Let X be a compact Kähler manifold with K_X^{-1} pseudoeffective. Is the Albanese map $\alpha : X \to \mathrm{Alb}(X)$ a (smooth) submersion? Especially, is this always the case when K_X^{-1} is nef?*

By [DPS96] or Theorem 4.1, the answer is affirmative if K_X^{-1} is semipositive. More generally, the generalized Hard Lefschetz theorem of [DPS01] shows that this is true if K_X^{-1} is pseudoeffective and possesses a singular hermitian metric of nonnegative curvature with trivial multiplier ideal sheaf. The general nef case seems to require a very delicate study of the possible degenerations of fibers of the Albanese map (so that one can exclude them in the end). In this direction, Cao and Höring [CH13] recently proved the following

5.8 Theorem ([CH13]) *Assuming X compact Kähler with K_X^{-1} nef, the answer to Problem* 4.7 *is affirmative in the following cases*:

(a) $\dim X \le 3$;
(b) $q(X) = h^0(X, \mathscr{O}_X) = \dim X - 1$;
(c) $q(X) = h^0(X, \mathscr{O}_X) \ge \dim X - 2$ *and X is projective;*
(d) *the general fiber F of $\alpha : X \to \mathrm{Alb}(X)$ is a weak Fano manifold, i.e. K_F^{-1} is nef and big.*

In general, a deeper understanding of the behavior of Harder-Narasimhan filtrations of the tangent bundle of a compact Kähler manifold would be badly needed.

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