A Survey on Bergman Completeness

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Abstract We provide a survey of results on Bergman completeness of open complex manifolds

Keywords Bergman completeness

1 Introduction

Let M be a complex manifold of dimension n. Let $\mathscr{H}$ be the Hilbert space of holomorphic $n-$forms f on M satisfying

$$\left|\int_M f \wedge \bar{f}\right| < \infty.$$

Let $h_1, h_2, \ldots$ be a complete orthonormal basis for $\mathscr{H}$. We may define the Bergman kernel (form) K_M of M as

$$K_M(z, w) = \sum_j h_j(z) \wedge \overline{h_j(w)}.$$

Let $(z_1, z_2, \ldots, z_n)$ be a local coordinate system in M. Let

$$K_M(z) := K_M(z, z) = K^*(z) dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$$

where $K^*(z)$ is a locally defined function. If K^* is positive, then we may define the Bergman metric ds^2_M of M as

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$$ds_M^2 = \sum_{\alpha,\beta} \frac{\partial^2 \log K^*}{\partial z_\alpha \partial \bar{z}_\beta} dz_\alpha d\bar{z}_\beta.$$

We say that M possesses the Bergman metric if ds_M^2 is a Kähler metric on M. The Bergman distance d_B is the distance with respect to ds_M^2. A complex manifold is said to be Bergman complete if d_B is complete.

In contrast with compact complex manifolds, the *quantitative* complex analysis on *open* complex manifolds is far from well-developed, even in the case of Riemann surfaces! As the Bergman kernel and metric depend only on the complex structure, and they are invariant under biholomorphic transformations, thus they should occupy a central place in the study of open complex manifolds. This is essentially the theme of Kobayashi's ground-breaking paper [Kobayashi59], although it is titled *geometry of bounded domains*.

It is not difficult to see that M possesses the Bergman metric if and only if the holomorphic mapping

$$\tau : M \to \mathbb{P}(\mathscr{H}), \quad z \mapsto [h_1(z) : h_2(z) : \cdots]$$

is an immersion, where $\mathbb{P}(\mathscr{H})$ stands for the complex projective space of $\mathscr{H}$. Kobayashi's decisive observation is

$$ds_M^2 = \tau^*(ds_{FS}^2)$$

where ds_{FS}^2 is the Fubini-Study metric of $\mathbb{P}(\mathscr{H})$. It follows that for any given distinct points $z, w \in M$, the Bergman distance d_B and the Fubini-Study distance d_{FS} satisfy

$$d_B(z, w) \geq d_{FS}(\tau(z), \tau(w)).$$

Since

$$d_{FS}(\tau(z), \tau(w)) = \arccos \frac{|\sum_j h_j^*(z)\overline{h_j^*(w)}|}{\sqrt{\sum_j |h_j^*(z)|}\sqrt{\sum_j |h_j^*(w)|}}$$

where h_j^* is a local representation of h_j, we have

$$d_B(z, w) \geq \arccos \frac{|h_1^*(z)|}{\sqrt{\sum_j |h_j^*(z)|^2}} \geq \sqrt{1 - \frac{|h_1^*(z)|^2}{\sum_j |h_j^*(z)|^2}} = \sqrt{1 - \frac{|h_1^*(z)|^2}{K^*(z)}} \quad (1.1)$$

provided that we choose $\{h_j\}$ such that $h_j(w) = 0$ for all $j \geq 2$. From this Kobayashi reached the following

Kobayashi's criterion (cf. [Kobayashi59, Kobayashi61]). *Suppose there is a dense subset S of $\mathscr{H}$ such that for every $f \in S$ and for any infinite sequence $\{p_k\}$ of points in M which has no adherent point in M, there is a subsequence $\{p_{k_j}\}$ such that*

$$\frac{f(p_{k_j}) \wedge \overline{f(p_{k_j})}}{K_M(p_{k_j})} \to 0 \quad as \quad j \to \infty. \tag{1.2}$$

Then M is Bergman complete.

Let us give a short proof of Kobayashi's criterion. Suppose M is not Bergman complete, i.e. there is a d_B−Cauchy sequence $\{p_k\}$ which has no adherent point in M. Let $k_0 \in \mathbb{Z}^+$ satisfy

$$d_B(p_k, p_l) < 1/2 \qquad \forall k, l \geq k_0.$$

Since S is dense in $\mathscr{H}$, we may construct by using the Gram-Schmidt procedure on S a complete orthonormal basis $\{\tilde{h}_j\}$ of $\mathscr{H}$ such that every $\tilde{h}_j$ enjoys the same property as $f \in S$. Put $w = p_{k_0}$ in (1.1). We may write $h_1 = \sum_j a_j \tilde{h}_j$ with $\sum_j |a_j|^2 = 1$. Choose $j_0 \in \mathbb{Z}^+$ (depending only on p_{k_0}), such that $\sum_{j>j_0} |a_j|^2 \leq 1/4$. Put $h_{1,j_0} = \sum_{j=1}^{j_0} a_j \tilde{h}_j$. By the Cauchy-Schwarz inequality, we have

$$(h_1 - h_{1,j_0}) \wedge \overline{(h_1 - h_{1,j_0})} \leq \sum_{j>j_0} |a_j|^2 \sum_{j>j_0} \tilde{h}_j \wedge \overline{\tilde{h}_j} \leq \frac{1}{4} K_M,$$

so that

$$\frac{h_1 \wedge \overline{h_1}}{K_M} \leq \frac{2h_{1,j_0} \wedge \overline{h_{1,j_0}}}{K_M} + \frac{1}{2}.$$

Let $\{p_{k_j}\}$ be a subsequence of $\{p_k\}$ such that (1.2) is verified for h_{1,j_0}. Then $\frac{h_1 \wedge \overline{h_1}}{K_M}(p_{k_j}) < 3/4$ provided j sufficiently large. On the other hand, it follows from (1.1) that

$$\frac{h_1(p_{k_j}) \wedge \overline{h_1(p_{k_j})}}{K_M(p_{k_j})} > \frac{3}{4},$$

and we get a contradiction.

The goal of this article is to survey some results concerning Bergman completeness, built on Kobayashi's criterion. Due to my personal taste, I am not able to cover all interesting results in this direction. I must apologize to those authors whose papers are not mentioned here. One may consult the nice books of Jarnicki and Pflug [JarnickiPflug, JarnickiPflug2] for more references.

Nevertheless, Bergman completeness is only the first step to understand the geometry of the Bergman metric, much more works need to be done in future.

2 Bergman Completeness for Domains in $\mathbb{C}^n$

The first result concerning Bergman completeness was given by Bremermann:

Theorem 2.1 (cf. [Bremermann]) *Every bounded Bergman complete domain in* $\mathbb{C}^n$ *is pseudoconvex.*

Obviously, the converse is not true (e.g., the punctured disc). Thus it is natural to ask

Problem 2.1 (*cf.* [Kobayashi59]) Which bounded pseudoconvex domain in $\mathbb{C}^n$ is Bergman complete?

By using his criterion, Kobayashi showed that every bounded analytic polyhedron is Bergman complete. A useful consequence of Kobayashi's criterion is that

$$H^\infty(\Omega) \text{ lies dense in } \mathscr{H} \text{ and } \lim_{z\to\partial\Omega} K_\Omega(z) = \infty \tag{2.1}$$

implies Bergman completeness, where $H^\infty(\Omega)$ stands for the set of bounded holomorphic functions on Ω. For the sake of simplicity, we say that a bounded domain Ω is Bergman exhaustive if $\lim_{z\to\partial\Omega} K_\Omega(z) = \infty$.

The first general result toward Problem 2.1 is due to Ohsawa:

Theorem 2.2 (cf. [Ohsawa81]) *Every bounded pseudoconvex domain in* $\mathbb{C}^n$ *with a* C^1 *boundary is Bergman complete.*

The Bergman exhaustiveness follows from the following result of Pflug:

Theorem 2.3 (cf. [Pflug75]) *Let* Ω *be a bounded pseudoconvex domain in* $\mathbb{C}^n$ *and* $p \in \partial\Omega$. *Suppose there exist a sequence* $\{z^\nu\} \subset \mathbb{C}^n\backslash\Omega$, *and positive numbers* $\beta \geq 1$, $r \leq 1$ *such that* $z^\nu \to p$ *and*

$$B(z^\nu, r|z^\nu - p|^\beta) \cap \Omega = \emptyset.$$

Then Ω *is Bergman exhaustive.*

It is difficult to verify that $H^\infty(\Omega)$ lies dense in $\mathscr{H}$, yet it is easy to verify this property *locally*. Thus the following localization principle of the Bergman metric becomes important:

Proposition 2.1 (cf. [Ohsawa84]) *Let* Ω *be a bounded pseudoconvex domain in* $\mathbb{C}^n$. *Let* $p \in \partial\Omega$ *and let* $V \subset\subset U$ *be two bounded neighborhoods of* p. *Then there are constants* $C_1, C_2 > 0$ *such that*

$$C_1 ds^2_\Omega(z) \leq ds^2_{\Omega\cap U}(z) \leq C_2 ds^2_\Omega(z), \qquad \forall\, z \in \Omega \cap V.$$

This proposition may be proved by a standard application of Hörmander's L^2 estimates for the $\bar{\partial}-$operator (cf. [HormanderBook]). Without any regularity assumption on the boundary, Jarnicki and Pflug [JarnickiPflug89] proved that every bounded balanced domain is Bergman complete.

It follows from the work of Kerzman and Rosay [KerzmanRosay] that every bounded pseudoconvex domain with a C^1 boundary is *hyperconvex*, i.e., there exists a continuous plurisubharmonic (psh) function $\rho : \Omega \to [-1, 0)$ such that $\{\rho < -c\} \subset\subset \Omega$ for all $c > 0$. Another important class of hyperconvex domains are Teichmüller spaces of compact Riemann surfaces of genus ≥ 2 (cf. [Krushkal]). Blocki and Pflug [BlockiPflug] and Herbort [HerbortHyperconvex] proved independently the following result which has been a longstanding conjecture due to Kobayashi (see e.g., [KobayashiBook98]):

Theorem 2.4 *Every bounded hyperconvex domain in $\mathbb{C}^n$ is Bergman complete.*

Earlier, Ohsawa [OhsawaHyperconvex] has proved that every hyperconvex domain is Bergman exhaustive, which also initiates a program of studying asymptotic behavior of L^2 holomorphic objects through investigating the Green function (see also [Ohsawa95]).

Recall that the *pluricomplex Green function* $g_\Omega(z, w)$ of Ω is defined as

$$g_\Omega(z, w) = \sup\{u(z) : u < 0, u \in PSH(\Omega), u(z) \leq \log|z - w| + O(1) \text{ near } w\}$$

where $PSH(\Omega)$ stands for the set of psh functions on Ω.

The following result was discovered independently by Herbort and myself, and suggests that pluripotential theory would be essential for the study of Bergman completeness:

Proposition 2.2 (cf. [Chen99, HerbortHyperconvex]) *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Suppose there is a constant $c > 0$ such that*

$$\lim_{w \to \partial\Omega} |\{g_\Omega(\cdot, w) < -c\}| = 0 \tag{2.2}$$

where $|\cdot|$ stands for the (Euclidean) volume. Then Ω is Bergman complete.

Let me explain briefly the idea of proving the proposition. It suffices to verify Kobayashi's criterion. Given $f \in \mathscr{H}$ and $w \in \Omega$, we look for a new function $\tilde{f} \in \mathscr{H}$ (which actually depends on w) such that $\tilde{f}(w) = f(w)$ and $\|\tilde{f}\|_{L^2}$ tends to zero as $w \to \partial\Omega$. Since $K_\Omega(w) \geq |\tilde{f}(w)|^2/\|\tilde{f}\|^2_{L^2}$, it follows that

$$\frac{|f(w)|^2}{K_\Omega(w)} \leq \|\tilde{f}\|^2_{L^2} \to 0 \quad \text{as } w \to \partial\Omega.$$

The desired function $\tilde{f}$ is given by

$$\tilde{f} = \chi(\log(-g_\Omega(\cdot, w)))f - u$$

where χ is a standard cut-off function such that

$$\operatorname{supp}\chi(\log(-g_\Omega(\cdot,w)))\subset\{g_\Omega(\cdot,w)<-c\}.$$

Note that $\bar\partial\tilde f=0$ if and only if

$$\bar\partial u=f\bar\partial\chi(\log(-g_\Omega(\cdot,w))).$$

Thanks to the L^2−estimates of Donnelly and Fefferman [DonnellyFefferman], we may find a solution u satisfying

$$\begin{aligned}
&\int_\Omega|u|^2e^{-2ng_\Omega(\cdot,w)}\\
&\le\text{const.}\int_\Omega|f|^2|\chi'(\cdot)|^2|\bar\partial\log(-g_\Omega(\cdot,w))|^2_{i\partial\bar\partial\log(-g_\Omega(\cdot,w)+1)}e^{-2ng_\Omega(\cdot,w)}\\
&\le\text{const.}\int_{\{g_\Omega(\cdot,w)<-c\}}|f|^2.
\end{aligned}$$

Since u is holomorphic in a neighborhood of w, we see that $u(w)=0$. Thus $\tilde f(w)=f(w)$ and

$$\begin{aligned}
\|\tilde f\|^2_{L^2}&\le2\int_{\{g_\Omega(\cdot,w)<-c\}}|f|^2+2\int_\Omega|u|^2\\
&\le\text{const.}\int_{\{g_\Omega(\cdot,w)<-c\}}|f|^2\to0\quad\text{as }w\to\partial\Omega.
\end{aligned}$$

Thus we are done. To make the argument rigorous, we need to smooth $g_\Omega(\cdot,w)$ by a standard approximating procedure.

To prove Theorem 2.5, it suffices to verify (2.2) for bounded hyperconvex domains. Blocki and Pflug used the following results due to Blocki:

Proposition 2.3 (cf. [Blocki93]) *Let Ω be a bounded domain in $\mathbb{C}^n$. Assume that u,v are non-positive psh functions such that $u=0$ on $\partial\Omega$. Then*

$$\int_\Omega|u|^n(dd^cv)^n\le n!\|v\|_\infty^{n-1}\int_\Omega|v|(dd^cu)^n.\tag{2.3}$$

Theorem 2.5 (cf. [Blocki96]) *Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$. Then there exists a solution ϕ of the following Monge-Ampere equation*

$$\det\left(\frac{\partial^2\phi}{\partial z_\alpha\partial\bar z_\beta}\right)=1,\quad\phi\in C(\overline{\Omega})\text{ and }\phi|_{\partial\Omega}=0.$$

Put $u=g_\Omega(\cdot,w)$ and $v=\phi$ in (2.3), one gets

$$
\begin{aligned}
|\{g_\Omega(\cdot, w) < -1\}| &\le \int_\Omega |g_\Omega(\cdot, w)|^n (dd^c\phi)^n \\
&\le n!\|\phi\|_\infty^{n-1} \int_\Omega |\phi|(dd^c g_\Omega(\cdot, w))^n \\
&\le \text{const.}|\phi(w)| \to 0
\end{aligned}
$$

as $w \to \partial\Omega$, for $(dd^c g_\Omega(\cdot, w))^n = \delta_w$ (cf. Demailly [Demailly82]).

Remark 2.1 Recently, the property of the function $\lambda(t) := |\{g_\Omega(\cdot, w) < -t\}|$ defined on $(0, \infty)$, in particular, the asymptotic behavior of $\lambda(t)$ as $t \to \infty$, has attracted much attention (see e.g., [BlockiSuita, BlockiBourgain, BerndtssonLempert]).

On the other side, there are many non-hyperconvex, Bergman complete domains (cf. [Chen99, HerbortHyperconvex, PflugZwonek03, PflugZwonek05]). For instance, one has the following

Proposition 2.4 (cf. [ChenEssay]) *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let $\varphi > 0$ be a continuous psh function on D satisfying*

$$\liminf_{z \to \partial D} \frac{\varphi(z)}{\log 1/\delta_D(z)} = \infty.$$

Then (2.2) holds for the Hartogs domain $\Omega := \{(z, w) \in D \times \mathbb{C} : |w| < e^{-\varphi(z)}\}$, in particular, it is Bergman complete.

This result suggests that condition (2.2) is almost optimal for Bergman completeness, e.g., let D be a punctured disc and $\varphi(z)$ be psh on D satisfying $\varphi(z) \sim N \log 1/|z|$ as $z \to 0$, where N is a positive integer, then Ω would not be Bergman complete.

It is important to obtain *quantitative* lower estimates on the Bergman distance which implies completeness. Diederich and Ohsawa proved the following

Theorem 2.6 (cf. [DiederichOhsawa]) *Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain with a C^2 boundary and let $z^0 \in \Omega$. Then the Bergman distance d_B satisfies*

$$d_B(z^0, z) \ge \text{const.} \log|\log \delta_\Omega(z)|$$

for all $z \in \Omega$ sufficiently close to $\partial\Omega$. Here δ_Ω stands for the (Euclidean) boundary distance.

The key idea of [DiederichOhsawa] is to use the following strengthening of Kobayashi' observation:

Proposition 2.5 (cf. [DiederichOhsawa]) *Let p_1, p_2 be distinct points in a bounded domain $\Omega \subset \mathbb{C}^n$. Suppose there exists a constant $C > 0$ such that for any $f \in \mathscr{H}$ with $\|f\|_{L^2} = 1$ there is another $\tilde{f} \in \mathscr{H}$ satisfying $\tilde{f}(p_1) = 0$, $\tilde{f}(p_2) = f(p_2)$, and $\|\tilde{f}\|_{L^2} \le C$, then $d_B(p_1, p_2) \ge C'$ where C' is a positive constant depending only on C.*

Proof Recall from (1.1) that

$$d_B(p_1, p_2) \geq \sqrt{1 - \frac{|h_1(p_2)|^2}{K_\Omega(p_2)}}$$

where $\{h_j\}$ is a complete orthonormal basis of $\mathscr{H}$ satisfying $h_j(p_1) = 0$ for all $j \geq 2$. If $|h_1(p_2)|^2 \leq \frac{1}{2}K_\Omega(p_2)$, then we have $d_B(p_1, p_2) \geq 1/\sqrt{2}$; otherwise, we may choose h_2 satisfying $|h_2(p_2)| \geq |h_1(p_2)|/C$, so that

$$\begin{aligned} d_B(p_1, p_2) \geq \sqrt{1 - \frac{|h_1(p_2)|^2}{K_\Omega(p_2)}} &= \sqrt{\frac{\sum_{j=2}^{\infty} |h_j(p_2)|^2}{K_\Omega(p_2)}} \\ &\geq \frac{|h_2(p_2)|}{\sqrt{K_\Omega(p_2)}} \geq \frac{|h_1(p_2)|}{C\sqrt{K_\Omega(p_2)}} \geq \frac{1}{\sqrt{2}C}. \end{aligned}$$

□

Built on the previous proposition, we may prove the following result through a similar argument as the proof of Proposition 2.6:

Proposition 2.6 (cf. [BlockiGreen], see also [ChenZhang]) *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Suppose that p_1, p_2 are distinct points in Ω satisfying*

$$\{g_\Omega(\cdot, p_1) < -1\} \cap \{g_\Omega(\cdot, p_2) < -1\} = \emptyset,$$

then $d_B(p_1, p_2) \geq \text{const}_n$.

Blocki improved substantially the result of Diederich-Ohsawa as follows

Theorem 2.7 (cf. [BlockiGreen]) *One has*

$$d_B(z^0, z) \geq \text{const.} |\log \delta_\Omega(z)| / \log |\log \delta_\Omega(z)|$$

for all $z \in \Omega$ sufficiently close to $\partial\Omega$.

The proof of Theorem 2.7 relies on Proposition 2.6 and the following quantitative estimate of g_Ω, which is also useful for other purposes (see e.g., [ChenFu11]):

Proposition 2.7 (cf. [BlockiGreen], see also [HerbortGreen] for a weaker result) *Let $\Omega \subset\subset \mathbb{C}^n$ be a pseudoconvex domain. Suppose there is a negative psh function ρ on Ω satisfying*

$$C_1\delta_\Omega^a(z) \leq -\rho(z) \leq C_2\delta_\Omega^b(z), \quad z \in \Omega$$

where $C_1, C_2 > 0$ and $a \geq b \geq 0$ are constants. Then there are positive numbers δ_0, C such that

$$\{g_\Omega(\cdot, w) < -1\} \subset \{C^{-1}\delta_\Omega(w)^{\frac{a}{b}} |\log \delta_\Omega(w)|^{-\frac{1}{b}} \leq \delta_\Omega \leq C\delta_\Omega(w)^{\frac{b}{a}} |\log \delta_\Omega(w)|^{\frac{n}{a}}\}$$

holds for any $w \in \Omega$ *with* $\delta_\Omega(w) \le \delta_0$.

For planar domains, I showed the following

Theorem 2.8 (cf. [Chen00]) *Let* Ω *be a bounded domain in* $\mathbb{C}$. *If* Ω *is Bergman exhaustive, then it is Bergman complete.*

The converse does not hold. Zwonek [ZwonekExample] has constructed a Bergman complete Zalcman type domain, which is not Bergman exhaustive. By a Zalcman type domain we mean a planar domain defined by

$$\Delta \backslash \left(\bigcup_j \overline{\Delta}_j \cup \{0\} \right)$$

where $\{\Delta_j\}$ is a sequence of disjoint discs in the unit disc Δ. Zwonek's example also disproved an old conjecture due to Kobayashi [Kobayashi59] that Bergman completeness implies

$$\lim_{z \to \partial\Omega} |f(z)|^2 / K_\Omega(z) = 0$$

for *all* $f \in \mathscr{H}$. It is still unclear whether the converse of Kobayashi's criterion fails.

A characterization in terms of logarithmic capacity for Bergman exhaustive planar domains was given by Zownek:

Theorem 2.9 (cf. [ZwonekWiener]) *Let* Ω *be a bounded domain in* $\mathbb{C}$ *and* $p \in \partial\Omega$. *Then*

$$\lim_{z \to p} K_\Omega(z) = \infty$$

if and only if

$$\gamma_\Omega(z) := \int_{\delta_\Omega(z)}^{1/2} \frac{dt}{t^3 |\log(\operatorname{cap}(\overline{\Delta_t(z)} \backslash D))|} \to \infty \quad \text{as } z \to p.$$

Here $\Delta_t(z)$ *stands for the disc with center* z *and radius* t.

Similar results on the Bergman metric were obtained in Pflug and Zwonek [PflugZwonek03]. By using these results, Wang [XuWang] was able to show that Bergman completeness is not a *quasiconformal* invariant for bounded planar domains. It is a classical result that (Green) hyperbolicity is a quasiconformal invariant for open Riemann surfaces.

It is well-known that every hyperbolic planar domain admits a canonical complete conformally invariant metric: the Poincaré metric of constant curvature −1. The following question is of classical interest:

Problem 2.2 What are relationships between the Bergman metric and the Poincaré metric?

I have not learnt any example that the Bergman metric is not dominated by the Poincaré metric. On the positive side, one has the following

Theorem 2.10 (cf. [ChenEssay]) *The Bergman metric and the Poincaré metric are equivalent on uniformly perfect domains. Both distances grow like* $|\log \delta_\Omega|$ *near* $\partial\Omega$.

A hyperbolic domain $\Omega \subset \mathbb{C}$ is said to be *uniformly perfect* if there exists a constant $c > 0$ such that for any boundary point $p \in \partial\Omega$ and $0 < r < \mathrm{diam}\partial\Omega$ there is a point $q \in \partial\Omega$ such that $cr \leq |q - p| \leq r$. For instance, the complement of the $\frac{1}{3}-$Cantor set in Δ is uniformly perfect. There are many equivalent definitions of uniform perfectness, as well as various interesting examples, among them of particular interest is the complement in $\mathbb{P}^1$ of the Julia set of a rational function of degree at least two (cf. [SugawaPerfect]).

We refer to [Wolpert, NikolovPflugZwonek] for various interesting results concerning the comparison of the Bergman metric with other invariant metrics on higher dimensional domains (usually with a highly complicated boundary).

The Bergman kernel and metric are deeply studied for some unbounded domains, e.g., Siegel domains of the second kind. Another interesting class of unbounded domains are model domains defined by

$$\Omega_\psi := \{(z', z_n) \in \mathbb{C}^n : \mathrm{Im}\, z_n > \psi(z')\}$$

where ψ is a psh function in $\mathbb{C}^{n-1}$.

Problem 2.3 When is Ω_ψ Bergman complete?

The answer is positive when ψ satisfies $\psi > 0$ and

$$\lim_{|z'| \to +\infty} \psi(z') = +\infty$$

(cf. [ChenKamimotoOhsawa], see also [PflugZwonek05] for related results). The case when ψ has singularities is more complicated and interesting. For instance, we have $K_{\Omega_\psi} = 0$ if $\psi(z') = \log|z'|$, whereas $K_\psi > 0$ if

$$\psi(z') \sim \log|z'| \text{ as } z' \to 0 \quad \text{and} \quad \psi(z') \sim |z| \text{ as } |z'| \to +\infty.$$

Recently, Ahn-Gaussier-Kim obtained a closely related result:

Theorem 2.11 (cf. [AhnKim]) *Let* Ω_{KN} *be the Kohn-Nirenberg domain defined by*

$$\Omega_{\mathrm{KN}} = \{(z_1, z_2) \in \mathbb{C}^2 : \mathrm{Im}\, z_2 > P_{2k}(z_1)\}$$

where P_{2k} *is a real-valued polynomial in* z_1 *and* $\bar{z}_1$ *satisfying (1)* $P_{2k}(tz) = t^{2k} P_{2k}(z)$ *for any* $t \in \mathbb{R}$ *and* $z \in \mathbb{C}$. *(2)* $\partial^2 P_{2k}/\partial z \partial \bar{z} > 0$ *on* $\mathbb{C}^* = \mathbb{C} - \{0\}$. *Then* Ω_{KN} *is complete with respect to the Carathéodory and Bergman metrics.*

3 Bergman Completeness for Open Complex Manifolds

For complex manifolds, one has to deal at first with the existence problem of the Bergman kernel or metric. The following is a classical one:

Theorem 3.1 (cf. [AhlforsSario]) *Every non-planar Riemann surface admits a nonzero square integrable holomorphic* $1-$*form, i.e., the Bergman kernel does not vanish.*

One of the most interesting class of open complex manifolds are universal coverings of a compact complex manifold with an infinite fundamental group. Suppose $\tilde{M}$ is a complex manifold and Γ is a free, properly discontinuous subgroup of the automorphism group $\mathrm{Aut}(\tilde{M})$ of $\tilde{M}$ such that $M := \tilde{M}/\Gamma$ is compact. The first Chern number c_1 of M is negative provided that $\tilde{M}$ possesses the Bergman metric. From the opposite direction, one may propose the following

Problem 3.1 Let M be a compact complex $n-$manifold with an infinite fundamental group and $c_1 < 0$. Is the Bergman kernel of the universal covering $\tilde{M}$ of M nonvanishing?

The answer is positive when $n \le 2$. The case $n = 1$ is trivial. The proof for $n = 2$ is due to Claudon [Claudon]. It follows from Atiyah's L^2 index theorem and Miyaoka-Yau's inequality $c_2 \ge c_1^2/3$:

$$h^{2,0}_{(2)}(\tilde{M}) - h^{1,0}_{(2)}(\tilde{M}) + h^{0,0}_{(2)}(\tilde{M}) = \chi_{(2)}(\mathscr{O}_{\tilde{M}}) = \chi(\mathscr{O}_M) = \frac{c_1^2 + c_2}{12} \ge \frac{c_1^2}{9} > 0.$$

Every L^2 holomorphic function f on $\tilde{M}$ has to be constant in view of a L^p-Liouville theorem of Yau [Yau76]. Since $\tilde{M}$ is of infinite volume, f has to be zero, i.e., $h^{0,0}_{(2)}(\tilde{M}) = 0$, so that $h^{2,0}_{(2)}(\tilde{M}) > 0$, i.e., there exists a nonzero holomorphic $2-$form on $\tilde{M}$.

Conversely, I would like to ask

Problem 3.2 Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with $n \ge 2$ (e.g. a bounded symmetric domain) and Γ a free, properly discontinuous subgroup of the automorphism group $\mathrm{Aut}(\Omega)$ of Ω. When does Ω/Γ possess a nonzero square integrable holomorphic $n-$form?

Kobayashi proposed the following criterion for the existence of the Bergman metric:

Proposition 3.1 (cf. [Kobayashi59]) *A complex manifold* M *possesses the Bergman metric provided the following two conditions are verified:*

(1) *For every* $w \in M$, *there exists a* $n-$*form* $f \in \mathscr{H}$ *such that* $f(w) \neq 0$;
(2) *For every* $w \in M$, *there are* $n-$*forms* $f_1, \dots, f_n$ *in* $\mathscr{H}$ *satisfying* $f_\alpha(w) = 0$, $\frac{\partial f_\alpha^*}{\partial z_\beta}(w) = \delta_{\alpha\beta}$ *(Kronecker delta) for* $1 \leq \alpha, \beta \leq n$. *Here* f_α^*, $1 \leq \alpha \leq n$, *are local representations of* f.

The concept of the pluricomplex Green function may be extended to complex manifolds, which depends only on the complex structure of the manifold. A criterion in terms of the pluricomplex Green function can be given as follows:

Proposition 3.2 (cf. [ChenZhang]) *Let* M *be a Stein manifold. Suppose for any* $w \in M$ *there is a positive number* $c > 0$ *such that* $\{g_M(\cdot, w) < -c\}$ *is relatively compact in* M. *Then* M *possesses the Bergman metric.*

From this we immediately know that every hyperbolic Riemann surface possesses the Bergman metric. Combining with a theorem of Carleson on removable singularities of L^2 holomorphic functions (see e.g., [Conway95]), we know that for any $\Omega \subset \mathbb{C}$

$$
\begin{aligned}
K_\Omega > 0 \text{ at one point } \iff K_\Omega > 0 \text{ everywhere} \\
\iff ds_\Omega^2 \text{ exists } \iff \Omega \text{ is hyperbolic.}
\end{aligned}
$$

The situation is completely different for higher dimensional domains. Rosay and Rudin [RosayRudin] constructed a domain $\Omega \subset \mathbb{C}^2$ with finite volume, whereas there exists a surjective, locally biholomorphic map $F : \mathbb{C}^2 \to \Omega$. It follows that $K_\Omega(z) \geq 1/|\Omega| > 0$, i.e., $\partial\bar{\partial} \log K_\Omega$ is well-defined, whereas

$$
g_\Omega(z, w) \leq \inf \left\{ g_{\mathbb{C}^2}(\zeta, \eta) : \zeta \in F^{-1}(z),\ \eta \in F^{-1}(w) \right\} = -\infty
$$

for all $z, w \in \Omega$. It is unclear whether Ω can be made to be Bergman complete.

Let D be a parabolic domain and $U \subset\subset D$ a small disc. The domain $\Omega = D \backslash \overline{U}$ is hyperbolic so that it possesses the Bergman metric, which is not complete in view of Carleson's theorem. It is reasonable to ask

Problem 3.3 [1] Let M be a parabolic Riemann surface and $U \subset\subset M$ a local coordinate disc. Is $M' := M \backslash \overline{U}$ always Bergman incomplete?

In their famous book [GreeneWuBook], Greene and Wu suggested to study the Bergman metric through Riemannian geometry. They proved the following

Theorem 3.2 (cf. [GreeneWuBook]) *Let* (M, g) *be a Kählerian Cartan-Hadamard manifold, let* o *be a fixed point in* M *and let* r *be the distance from* o. *Then*

[1] Recently, I got a counterexample.

(1) *If the inequality*

$$\text{sectional curvature} \leq \frac{-A}{r^2(\log r)^{1-\varepsilon}}$$

holds outside a compact subset of M, where ε and A are positive constants, then M possesses the Bergman metric.

(2) *Suppose*

$$-\frac{B}{r^2} \leq \text{sectional curvature} \leq -\frac{A}{r^2}$$

holds outside a compact subset of M for some positive constants A and B, then the Bergman metric ds_M^2 satisfies $ds_M^2 \geq \text{const.}(1+r^2)^{-1}g$. In particular, M is Bergman complete.

(3) *Suppose*

$$-B \leq \text{sectional curvature} \leq -A$$

for some positive constants A and B, then $ds_M^2 \geq \text{const.}g$. In particular, M is Bergman complete.

Recall that a Cartan-Hadamard manifold is a complete, simply-connected Riemannian manifold of nonpositive sectional curvature. Greene-Wu conjectured that the hypothesized lower bound in (2) or (3) is unnecessary for the lower estimate of the Bergman metric, they even conjectured that M is Bergman complete under the assumptions in part (1).

In attempt to solve these conjectures, Zhang and I proved the following

Theorem 3.3 (cf. [ChenZhang]) *Let M be a Kählerian Cartan-Hadamard manifold, let o be a fixed point in M and let r be the distance from o. Then*

(1) *Suppose*

$$\text{sectional curvature} \leq -\frac{A}{r^2}$$

outside a compact subset of M for suitable positive constant A, then the Bergman distance d_B satisfies

$$d_B(o,x) \geq \text{const.}\log r(x).$$

(2) *Suppose*

$$\text{sectional curvature} \leq -A$$

for some positive constant A, then

$$d_B(o,x) \geq \text{const.}r(x).$$

Greene and Wu [GreeneWuBook] also showed that under the following weaker assumption

$$\text{sectional curvature} \le \frac{-(1+\varepsilon)}{r^2 \log r}$$

outside a compact set, M has to be a *hyperconvex* manifold, i.e., there is a smooth strictly psh function $\rho : M \to [-1, 0)$ such that $\{\rho < -c\} \subset\subset M$ for all $c > 0$. Thus it is worthwhile to extend Theorem 2.5 as follows:

Theorem 3.4 (cf. [ChenHyperconvex]) *Every hyperconvex manifold is Bergman complete.*

Below we list some examples of hyperconvex manifolds beyond hyperconvex domains:

(a) Closed complex submanifolds of a hyperconvex domain Ω; these manifolds can be highly complicated even when Ω is the unit ball!
(b) Bounded pseudoconvex domains in $\mathbb{P}^n$ with a C^2 boundary (cf. [OhsawaSibony]).
(c) Sufficiently small neighborhoods of a totally real C^1 submanifold in a complex manifold (cf. [HarveyWells]).
(d) Regular coverings of a hyperconvex manifold (cf. [Vajaitu]).

Although the proof of the previous theorem is not basically different from [BlockiPflug], it still requires a few additional observations. Indeed, the following modified criterion for Bergman completeness was implicitly used:

Proposition 3.3 (cf. [ChenEssay]) *Let M be a Stein manifold which possesses the Bergman metric. Suppose that for any infinite sequence of points $\{p_k\}$ in M which has no adherent point in M, there are a subsequence $\{p_{k_j}\}$, a number $c > 0$ and a continuous volume form dV on M such that for any compact subset K of M, the related volume*

$$|K \cap \{g_M(\cdot, p_{k_j}) < -c\}|$$

tends to zero as $j \to \infty$, then M is Bergman complete.

Even for bounded hyperconvex domains, this criterion has the advantage of avoiding any use of the solution of the Monge-Ampere equation. Furthermore, it was used in [ChenEssay] to show that every Stein subvariety in a complex manifold admits a fundamental family of *Bergman complete* Stein neighborhoods, which improves a famous result of Siu [SiuNeighborhood].

As is well-known, every Stein manifold can be embedded holomorphically as a closed complex submanifold of some $\mathbb{C}^n$. It is natural to ask

Problem 3.4 Which closed complex submanifold of $\mathbb{C}^n$ is Bergman complete?

For instance, the preimage $\pi^{-1}(S) \subset \mathbb{C}^n$ of a smooth ample divisor S in an Abelian variety A is Bergman complete, where $\pi : \mathbb{C}^n \to A$ is the covering map (see e.g., [ChenEssay]). When $n > 2$, there is no nonconstant bounded holomorphic functions on $\pi^{-1}(S)$; I guess that the related pluricomplex Green function equals $-\infty$.

Even for a smooth analytic hypersurface M defined by $f = 0$ where f is an entire function in $\mathbb{C}^n$, it is still of great interest to find a criterion for Bergman completeness of M in terms of the function f.

Finally, let us look at Riemann surfaces from a different viewpoint. Consider at first an orientable surface M, i.e., a two-dimensional differentiable manifold. Let

$$ds^2 = E(x, y)dx^2 + 2F(x, y)dxdy + G(x, y)dy^2$$

where $EG - F^2 > 0,\ E > 0$, be a (smooth) Riemannian metric defined in local coordinates (x, y) of M. It is easy to see that every (paracompact) surface carries a (complete) Riemannian metric by means of patching up together local metrics through a partition of unity. By *isothermal parameters* we mean local coordinates (ξ, ζ) with $\xi = \xi(x, y),\ \zeta = \zeta(x, y)$, such that

$$ds^2 = \lambda(\xi, \zeta)(d\xi^2 + d\zeta^2), \quad \lambda(\xi, \zeta) > 0.$$

Such isothermal parameters are known to exist by the famous Korn-Lichtenstein theorem, which goes back to Gauss. Thus M carries local complex coordinates $z = \xi + \zeta i$ so that it becomes a Riemann surface in classical sense. This observation is significant since the complex structure of a surface is often unknown, whereas the Riemannian metric can be analyzed through general theory of Riemannian geometry. From this viewpoint, assumptions relying on the complex structure are unnatural.

Now I formulate a basic problem:

Problem 3.5 Let M be an open Riemann surface with a complex structure induced by some complete Riemannian metric ds^2. Under which condition on ds^2 is the surface M Bergman complete?

As is well-known, popular conditions in Riemannian geometry are curvature, volume, etc. These are not strong enough for giving a criterion for Bergman completeness. Certain *global* condition is needed.

A nice global property of Riemannian manifolds is *isoperimetric inequalities*. Suppose M is a complete Riemannian $n-$manifold. Let $\mathscr{F}$ denote the set of precompact domains $\Omega \subset M$ with a smooth boundary. For $0 < \nu \leq \infty$, the $\nu-$dimensional isoperimetric constant $I_\nu(M)$ of M is defined by

$$I_\nu(M) = \inf_{\Omega \in \mathscr{F}} |\partial\Omega|/|\Omega|^{1-1/\nu}.$$

Recently, I obtained the following

Theorem 3.5 (cf. [ChenRiemann]) *Let M be a complete Riemannian surface with the Gauss curvature bounded below by a constant. Let o be a point in M and r be the distance from o. Suppose either of the following conditions is verified:*

(1) $I_\nu(M) > 0$, *for* some $2 < \nu < \infty$;
(2) $I_\infty(M) > 0$ *and* $\inf_{x\in M} |B_1(x)| > 0$, *where* $B_a(x)$ *stands for the geodesic ball with center* x *and radius* a.

Then the Bergman distance d_B *satisfies*

$$d_B(o, x) \geq \text{const.}\, r(x).$$

Remark 3.1 (1) For the flat complex plane, one has $I_2(\mathbb{C}) > 0$, whereas $\mathbb{C}$ does not possess the Bergman metric. (2) For the punctured disc Δ^* with the Poincaré metric, one has $I_\infty(\Delta^*) > 0$, whereas the Bergman metric is not complete.

How to realize these assumptions? With respect the Poincaré metric, every uniformly perfect domain Ω has bounded geometry and $I_\infty(\Omega) > 0$ (cf. [SugawaPerfect]). Recall that a complete Riemannian manifold M has *bounded geometry* if the Ricci curvature is bounded below by a constant, and the injectivity radius is positive. Thus Ω satisfies the assumption in part (2). The point is that one may construct from Ω many open Riemannian surfaces verifying this assumption, based on the following beautiful discovery of Kanai:

Theorem 3.6 (cf. [KanaiRough]) *Let* M_1, M_2 *be complete Riemannian manifolds with bounded geometries such that they are roughly isometric to each other. Let* $\nu \geq \max\{\dim M_1, \dim M_2\}$. *Then* $I_\nu(M_1) > 0$ *if and only if* $I_\nu(M_2) > 0$.

Recall that a map $F : M_1 \to M_2$ between two Riemannian manifolds M_1 and M_2 is called a *rough isometry* if there are constants $a \geq 1$ and $b \geq 0$ such that

$$a^{-1} d_1(x, y) - b \leq d_2(F(x), F(y)) \leq a d_1(x, y) + b$$

for all $x, y \in M_1$, and F is ε*−full* for some number $\varepsilon > 0$, i.e.,

$$\bigcup_{x\in M_1} B_\varepsilon(F(x)) = M_2.$$

For instance, we learn from Kanai's theorem that every 2−dimensional jungle gym in $\mathbb{R}^n$ with $n > 2$ has a positive n−dimensional isoperimetric constant; similarly, every 2−dimensional jungle gym in a Cartan-Hadamard n−manifold ($n \geq 2$) with sectional curvature $\leq -A$ ($A > 0$) has a positive infinite-dimensional isoperimetric constant.

Problem 3.6 Let M be an open real surface. Does there always exist a complex structure on M such that the related Bergman metric is complete?

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