

A CR Proof for a Global Estimate of the Diederich–Fornaess Index of Levi-Flat Real Hypersurfaces

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Abstract Yet another proof is given for a global estimate of the Diederich–Fornaess index of relatively compact domains with Levi-flat boundary, namely, the index must be smaller than or equal to the reciprocal of the dimension of the ambient space. This proof reveals that this kind of estimate makes sense and holds also for abstract compact Levi-flat CR manifolds.

Keywords Diederich–Fornaess index · CR geometry · Levi-flat real hypersurface

1 Introduction

The *Diederich–Fornaess index* $\eta(\Omega)$ of a \mathcal{C}^∞ -smoothly bounded domain Ω in a complex manifold X is a numerical index on the strength of certain pseudoconvexity of its boundary $\partial\Omega$. In this paper, we consider the index in the sense that $\eta(\Omega)$ is defined to be the supremum of the exponents $\eta \in (0, 1]$ admitting a \mathcal{C}^∞ -smooth defining function of $\partial\Omega$, say $\rho : (\partial\Omega \subset)U \rightarrow \mathbb{R}$, so that $-|\rho|^\eta$ is strictly plurisubharmonic in $U \cap \Omega$; if no such η is allowed, we let $\eta(\Omega) = 0$.

For instance, if a defining function attains $\eta = 1$, it gives a strictly plurisubharmonic defining function of $\partial\Omega$ and the boundary is strictly pseudoconvex. The pseudoconvexity of $\partial\Omega$ is clearly necessary for $\eta(\Omega)$ to be positive; a much stronger condition is actually necessary and sufficient, the existence of a defining function ρ such that the complex hessian of $-\log|\rho|$ is bounded from below by a hermitian metric of X near the boundary $\partial\Omega$ as observed by Ohsawa and Sibony [OS].

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F. Bracci et al. (eds.), *Complex Analysis and Geometry*, Springer Proceedings
in Mathematics & Statistics 144, DOI 10.1007/978-4-431-55744-9_2

The Diederich–Fornaess index $\eta(\Omega)$ being positive means that the boundary $\partial\Omega$ is well approximated by a family of strictly pseudoconvex real hypersurfaces from inside. The original motivation of the study of Diederich and Fornaess [DF] was to construct such an approximation on pseudoconvex domains in Stein manifolds, and the index is considered to measure certain strength of the approximation. Since then, the meaning of the index has been intensively studied in relation to the global regularity in the $\bar{\partial}$ -Neumann problem, in particular, pathologies occurring on the worm domain. See for example [FS, §1], [Be] and references therein.

Under such circumstances, Fu and Shaw [FS] and Brinkschulte and the author [AB1] reached a general estimate for the Diederich–Fornaess index of weakly pseudoconvex domains. Here we state the result in a restricted form, for domains with \mathcal{C}^∞ -smooth Levi-flat boundary:

Theorem 1 ([FS], see also [AB1] and [A2]) *Let Ω be a relatively compact domain with \mathcal{C}^∞ -smooth Levi-flat boundary M in a complex manifold of dimension $(n+1) \geq 2$. Then the Diederich–Fornaess index $\eta(\Omega)$ must be $\leq 1/(n+1)$.*

The purpose of this paper is to give yet another proof of Theorem via an estimate on the Levi-flat boundary M without looking inside Ω directly. The idea is to identify the usual Diederich–Fornaess index $\eta(\Omega)$ with its counterpart $\eta(M)$ on the Levi-flat boundary based on the author’s previous work [A1].

Definition 1 Let M be an oriented \mathcal{C}^∞ -smooth Levi-flat CR manifold. The Diederich–Fornaess index $\eta(M)$ of M is defined to be the supremum of $\eta \in (0, 1)$ admitting a \mathcal{C}^∞ -smooth hermitian metric h^2 of the holomorphic normal bundle $N_M^{1,0}$ of M so that

$$i\Theta_h - \frac{\eta}{1-\eta}i\alpha_h \wedge \bar{\alpha}_h > 0$$

holds on M as quadratic forms on the holomorphic tangent bundle $T_M^{1,0}$ of M ; if no such η is allowed, we let $\eta(M) = 0$. Here the forms α_h and Θ_h denote the leafwise Chern connection form and its curvature form of $N_M^{1,0}$ with respect to h^2 respectively. (See Sect. 2 for their precise definitions.)

In our setting, $\eta(\Omega)$ agrees with $\eta(M)$ as we will see in Lemma 3, and Theorem follows from the following main lemma.

Lemma 1 *Let M be a compact \mathcal{C}^∞ -smooth Levi-flat CR manifold of dimension $(2n+1) \geq 3$. Then the Diederich–Fornaess index $\eta(M)$ must be $\leq 1/(n+1)$.*

The organization of this paper is as follows. In Sect. 2, we provide preliminaries on CR geometry and confirm that the two notions of Diederich–Fornaess index, $\eta(\Omega)$ and $\eta(M)$, actually coincide for Levi-flat real hypersurfaces based on previous works. In Sect. 3, after proving Lemma 1, we give a remark that the substantial content of Lemma 1 has been already pointed out by Bejancu and Deshmukh [BD] in manner of differential geometry, and conclude this paper with an open question.

2 Preliminaries

2.1 Levi-Flat CR Manifold

Let us recall the notion of Levi-flat CR manifold briefly. In the sequel, “smooth” means infinitely differentiable.

Let M be a smooth manifold of dimension $(2n + 1) \geq 3$. A CR structure (of hypersurface type) of M is given by a subbundle $T_M^{0,1} \subset \mathbb{C} \otimes_{\mathbb{R}} TM$ satisfying the following conditions:

- $T_M^{0,1}$ is a smooth \mathbb{C} -subbundle $T_M^{0,1} \subset \mathbb{C} \otimes_{\mathbb{R}} TM$ of $\text{rank}_{\mathbb{C}} n$;
- $T_M^{1,0} \cap T_M^{0,1} = \{0\}$ (the zero section) where $T_M^{1,0} := \{v \in \mathbb{C} \otimes_{\mathbb{R}} TM \mid \bar{v} \in T_M^{0,1}\}$;
- $[\Gamma(T_M^{0,1}), \Gamma(T_M^{0,1})] \subset \Gamma(T_M^{0,1})$

where $\Gamma(\cdot)$ denotes the space of smooth sections of the bundle, and the bracket means the Lie bracket of complexified vector fields. The pair $(M, T_M^{0,1})$ is called a CR manifold, which is regarded as an abstraction of real hypersurfaces in complex manifolds associated with their (anti-)holomorphic tangent bundles.

We say that a CR manifold $(M, T_M^{0,1})$ is Levi-flat if it satisfies further integrability condition

$$[\Gamma(T_M^{1,0} \oplus T_M^{0,1}), \Gamma(T_M^{1,0} \oplus T_M^{0,1})] \subset \Gamma(T_M^{1,0} \oplus T_M^{0,1}). \tag{1}$$

This is equivalent to say that the real distribution $H_M := \text{Re}(T_M^{1,0} \oplus T_M^{0,1}) \subset TM$ of $\text{rank}_{\mathbb{R}} 2n$ is integrable in the sense of Frobenius. It follows from Frobenius’ theorem and Newlander–Nirenberg’s theorem that the distribution H_M defines a smooth foliation \mathcal{F} by complex manifolds on M , namely, we have an atlas consisting of foliated charts. We call \mathcal{F} the Levi foliation.

For a Levi-flat CR manifold $(M, T_M^{0,1})$, we shall refer to $T_M^{1,0}$ as the holomorphic tangent bundle of M and call the quotient \mathbb{C} -line bundle $N_M^{1,0}$,

$$0 \rightarrow T_M^{1,0} \oplus T_M^{0,1} \rightarrow \mathbb{C} \otimes_{\mathbb{R}} TM \xrightarrow{\pi} N_M^{1,0} \rightarrow 0,$$

the holomorphic normal bundle. This is because $T_M^{1,0}$ agrees with the holomorphic tangent bundle of the leaves of the Levi foliation \mathcal{F} . Note that our holomorphic tangent bundle is distinct from $(\mathbb{C} \otimes_{\mathbb{R}} TM)/T_M^{0,1}$ and our (p, q) -form on M means a section of $\bigwedge^p (T_M^{1,0})^* \otimes \bigwedge^q (T_M^{0,1})^* \subset \bigwedge^{p+q} (T_M^{1,0} \oplus T_M^{0,1})^*$.

Now let us consider a Levi-flat CR manifold, simply denoted by M , and define the form α_h mentioned in Sect. 1. Fix a smooth hermitian metric h^2 of $N_M^{1,0}$; in our convention, we denote by $h : N_M^{1,0} \rightarrow \mathbb{R}$ the map given by the norm induced from h^2 on $(N_M^{1,0})_p$ for each $p \in M$. Pick a local smooth section ξ of $N_M^{1,0}$ around $p \in M$ so that it is both normalized by h^2 and real, i.e., $\bar{\xi} = \xi$, which is determined up to its sign. Using such a ξ , we define the $(1, 0)$ -form $\alpha_h : T_M^{1,0} \rightarrow \mathbb{C}$ so as to satisfy

$$\pi([v, \tilde{\xi}]_p) = -\alpha_h(v_p)\xi_p \quad (2)$$

for $v_p \in (T_M^{1,0})_p$ where $\tilde{\xi}$ and v are any lift and extension of ξ and v_p to local sections of $\mathbb{C} \otimes_{\mathbb{R}} TM$ respectively. Here we used the Levi-flatness (1) to assure that α_h is independent of the choice of ξ , $\tilde{\xi}$ and v . We define $\bar{\alpha}_h(\bar{v}_p) := \overline{\alpha_h(v_p)}$, the complex-conjugate (0, 1)-form of α_h .

Remark 2 The left hand side of (2) is the covariant derivative of ξ along v_p with respect to a complex Bott connection of the Levi foliation \mathcal{F} and the form α is considered to measure the size of infinitesimal holonomy of \mathcal{F} with respect to h^2 .

We give the (1, 1)-form $\Theta_h : T_M^{1,0} \otimes T_M^{0,1} \rightarrow \mathbb{C}$ by

$$\begin{aligned} \Theta_h(v_p \otimes \bar{w}_p) &:= v_p \alpha_h(\bar{w}) - \bar{w}_p \alpha_h(v) - \alpha_h([v, \bar{w}]_p) \\ &= -\bar{w}_p \alpha_h(v) - \alpha_h([v, \bar{w}]_p) \end{aligned}$$

where v and \bar{w} are arbitrary extensions of v_p and \bar{w}_p to local sections of $T_M^{1,0}$ and $T_M^{0,1}$ respectively. We again used the Levi-flatness (1) for the last term to be defined.

2.2 Description on Foliated Charts

Although we have defined the forms α_h and Θ_h in a coordinate-free manner, their descriptions on foliated charts are convenient in actual computations. Here we briefly introduce them.

Take a *foliated chart* $(U, (z_U, t_U))$ of the Levi-flat CR manifold M , a chart $(z_U, t_U) : U \rightarrow \mathbb{C}^n \times \mathbb{R}$ so that $T_M^{1,0}|_U$ agrees with the pull-back bundle of $T^{1,0}\mathbb{C}^n \subset \mathbb{C} \otimes_{\mathbb{R}} T(\mathbb{C}^n \times \mathbb{R})$. Any coordinate change between intersecting foliated charts, say $(U, (z_U, t_U))$ and $(V, (z_V, t_V))$, are of the form

$$z_U = z_U(z_V, t_V), \quad t_U = t_U(t_V)$$

where z_U is holomorphic in z_V . A *leaf* N of \mathcal{F} is a connected complex manifold injectively immersed in M such that z_U is holomorphic and t_U is locally constant on $U \cap N$ for any foliated chart $(U, (z_U, t_U))$. Our manifold M is decomposed into the direct sum of the leaves of \mathcal{F} . A *CR function* on M , a \mathbb{C} -valued function which is annihilated by vectors in $T_M^{0,1}$ by its definition, agrees with a function which is *leafwise holomorphic*, namely, holomorphic in z_U on any foliated chart $(U, (z_U, t_U))$.

On a foliated chart $(U, (z_U = (z_U^1, z_U^2, \dots, z_U^n), t_U))$, we may trivialize $T_M^{1,0}$ and $N_M^{1,0}$ by using

$$\left\{ \frac{\partial}{\partial z_U^1}, \frac{\partial}{\partial z_U^2}, \dots, \frac{\partial}{\partial z_U^n} \right\} \quad \text{and} \quad \frac{\partial}{\partial t_U}$$

respectively. This description illustrates that $T_M^{1,0}$ and $N_M^{1,0}$ are *locally trivial CR vector bundles*, smooth vectors bundles with local trivialization covers whose transition functions are CR. The transition functions of $N_M^{1,0}$ are much better; They are leafwise constant.

Some computations show that on a foliated chart $(U, (z_U, t_U))$, the forms α_h and Θ_h for a given hermitian metric h^2 of $N_M^{1,0}$ are described as

$$\alpha_h = \sum_{j=1}^n \frac{\partial \log h_U}{\partial z_U^j} dz_U^j,$$

$$\Theta_h = \sum_{j,k=1}^n \frac{\partial^2 (-\log h_U)}{\partial z_U^j \partial \bar{z}_U^k} dz_U^j \wedge d\bar{z}_U^k$$

where $h_U := h(\frac{\partial}{\partial t_U})$. We can see that α_h and Θ_h agree with the leafwise Chern connection and curvature form of $N_M^{1,0}$ with respect to h^2 respectively up to a positive multiplicative constant.

2.3 The Diederich–Fornaess Index

In this section, we confirm that the two notions of Diederich–Fornaess index given in Sect. 1 coincide for Levi-flat real hypersurfaces.

Let Ω be a relatively compact domain with smooth Levi-flat boundary M in a complex manifold of dimension ≥ 2 . We introduce here terms for intermediate notions that appeared in the definition of the Diederich–Fornaess indices. The *Diederich–Fornaess exponent* η_ρ of a fixed defining function $\rho : (\partial\Omega \subset)U \rightarrow \mathbb{R}$ of $\partial\Omega$ is the supremum of the exponents $\eta \in (0, 1]$ such that $-|\rho|^\eta$ is strictly plurisubharmonic in $U \cap \Omega$; if no such η is allowed, we let $\eta_\rho = 0$. We also define the *Diederich–Fornaess exponent* η_h of a fixed hermitian metric h^2 of $N_M^{1,0}$ in the same manner. The Diederich–Fornaess indices are clearly the supremum of the corresponding Diederich–Fornaess exponents.

Lemma 3 *We have $\eta(\Omega) = \eta(M)$.*

Proof It is proved in [A1, Theorem 1.1] that one can construct a smooth hermitian metric h_ρ^2 of $N_M^{1,0}$ from a given smooth defining function ρ of M with $\eta_\rho > 0$ so that $\eta_\rho = \eta_{h_\rho}$. Hence, $\eta(\Omega) \leq \eta(M)$.

To derive the other inequality, it suffices to show that any hermitian metric h^2 of $N_M^{1,0}$ with $\eta_h > 0$, which condition is equivalent to $i\Theta_h > 0$ as quadratic forms on $T_M^{1,0}$, can be obtained by the construction above from a defining function of M . This inverse construction originates from the work of Brunella [Br] where he proved that this is possible if the Levi foliation of M extends to a holomorphic foliation on a neighborhood of M . Although the extended holomorphic foliation

may not exist in our setting, we are able to apply refined constructions explained in [O, §1], [BI, Proposition 1], or [A2, Proposition 3.1] and finish the proof. \square

Remark 4 We have restricted ourselves not to formulate the results for Levi-flat real hypersurfaces with finite differentiability because we have a technical problem at this point. The construction from defining functions to hermitian metrics in [A1] loses one order in differentiability since taking its normal derivative, although the inverse constructions in [?] or [A2] do not give us a gain in differentiability. So we cannot simply state that any \mathcal{C}^k -smooth hermitian metric can be obtained from a \mathcal{C}^k or \mathcal{C}^{k+1} -smooth defining function for $2 \leq k < \infty$ unlike in the case $k = \infty$.

3 The Proof of Lemma 1 and a Remark

3.1 Proof of Lemma 1

Now we shall give the proof of Lemma 1.

Proof of Lemma 1 Suppose the contrary: $\eta(M) > 1/(n+1)$. By definition, there exists a smooth hermitian metric of $N_M^{1,0}$, say h^2 , such that

$$i\Theta_h - \frac{1}{n}i\alpha_h \wedge \bar{\alpha}_h > 0$$

as quadratic forms on $T_M^{1,0}$.

By taking a double covering of M if necessary, we may assume that M is oriented. We let $\eta := h_U dt_U$ where t_U is the transverse coordinate of a positively-oriented foliated chart $(U, (z_U, t_U))$ and $h_U := h(\frac{\partial}{\partial t_U})$. Then we see that η is a well-defined 1-form on M , and that $\Theta_h \wedge \eta$, $\alpha_h \wedge \eta$ and $\bar{\alpha}_h \wedge \eta$ make sense as differential forms on M regardless of the choice of extensions of α_h or Θ_h to tensors on $\mathbb{C} \otimes_{\mathbb{R}} TM$. Among these forms, we can show the equalities $(d\alpha_h) \wedge \eta = \Theta_h \wedge \eta$ and $d\eta = (\alpha_h + \bar{\alpha}_h) \wedge \eta$ from straightforward computation on the foliated chart.

Now we obtain by direct computation that

$$\begin{aligned} & d \left(\left(i\Theta_h - \frac{1}{n}i\alpha_h \wedge \bar{\alpha}_h \right)^{n-1} \wedge i\alpha_h \wedge \eta \right) \\ &= (n-1) \left(i\Theta_h - \frac{1}{n}i\alpha_h \wedge \bar{\alpha}_h \right)^{n-2} \wedge \frac{1}{n}i\Theta_h \wedge i\alpha_h \wedge \bar{\alpha}_h \wedge \eta \\ &\quad + \left(i\Theta_h - \frac{1}{n}i\alpha_h \wedge \bar{\alpha}_h \right)^{n-1} \wedge (i\Theta_h - i\alpha_h \wedge \bar{\alpha}_h) \wedge \eta \\ &= \left(i\Theta_h - \frac{1}{n}i\alpha_h \wedge \bar{\alpha}_h \right)^n \wedge \eta, \end{aligned}$$

and Stokes' theorem yields a contradiction:

$$\begin{aligned}
 0 &< \int_M (i\Theta_h - \frac{1}{n}i\alpha_h \wedge \overline{\alpha}_h)^n \wedge \eta \\
 &= \int_M d \left((i\Theta_h - \frac{1}{n}i\alpha_h \wedge \overline{\alpha}_h)^{n-1} \wedge i\alpha_h \wedge \eta \right) \\
 &= 0. \quad \square
 \end{aligned}$$

Remark 5 The proof shows in particular that $\int_M i\Theta_h \wedge \eta = \int_M i\alpha_h \wedge \overline{\alpha}_h \wedge \eta$ always holds when $\dim_{\mathbb{R}} M = 3$. This equality well explains the behavior of the Diederich–Fornaess exponent of an explicit example described in [A2, §5].

3.2 The Approach of Bejancu and Deshmukh

We give a remark that the substantial content of Lemma 1 has been already observed by Bejancu and Deshmukh [BD] in the context of differential geometry.

Remark 6 When $\dim_{\mathbb{R}} M = 3$, the integrand $(i\Theta_h - i\alpha_h \wedge \overline{\alpha}_h) \wedge \eta$ was used in [BD] to show that the totally real Ricci curvature of compact Levi-flat real hypersurfaces in Kähler surfaces cannot be everywhere positive.

Let us explain this coincidence. Suppose that we have an oriented smooth Levi-flat real hypersurface M in a Kähler surface (X, ω) . We restrict on M the Kähler metric ω as a Riemannian metric and consider its Levi-Civita connection ∇^M and Ricci curvature Ric^M . We also consider the Gauss–Kronecker curvature $G_{\mathcal{F}/M}$ of the leaves of the Levi foliation \mathcal{F} in M . Take the signed distance function to M with respect to the given Kähler metric ω and induce a hermitian metric h^2 of $N_M^{1,0}$ from it. Then, we can observe by direct computation that

$$\begin{aligned}
 4(i\Theta_h - i\alpha_h \wedge \overline{\alpha}_h) &= (\text{Ric}^M(\xi, \xi) - 2G_{\mathcal{F}/M}) \omega|_{T_M^{1,0}} \otimes T_M^{0,1} \\
 &= (\text{Ric}^M(\xi, \xi) - \frac{1}{2}\|d\eta\|^2 + \|\nabla^M \xi\|^2) \omega|_{T_M^{1,0}} \otimes T_M^{0,1}
 \end{aligned}$$

where ξ is the Reeb vector field of M chosen so that it is normalized and orthogonal to H_M with respect to ω and positively directed, and η is the metric dual of ξ . The last line is exactly the integrand used in [BD]. We leave the details of this computation to the reader, who can find the techniques needed in [AB2, BD].

3.3 Open Question

We conclude this paper with stating an open question explicitly.

Question Can we formulate the Diederich–Fornaess index for any CR manifold of hypersurface type? Needless to say, it should agree with the Diederich–Fornaess index of its complementary domain when it is realized as the boundary real hypersurface of a domain in a complex manifold. Can we prove the global estimate of Fu and Shaw, and Brinkschulte and the author for this index in its full generality?

Acknowledgments The author is partially supported by an NRF grant 2011-0030044 (SRC-GAIA) of the Ministry of Education, the Republic of Korea, and a JSPS Grant-in-Aid for Young Scientists (B) 26800057. The author gratefully acknowledges an enlightening discussion with J. Brinkschulte. He is also grateful to T. Inaba for his useful remarks.

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