Amoebas of Cuspidal Strata for Classical Discriminant

E.N. Mikhalkin, A.V. Shchuplev and A.K. Tsikh

Abstract An amoeba of an analytic set is the real part of its image in a logarithmic scale. Among all hypersurfaces A-discriminantal sets have the most simple amoebas. We prove that any singular cuspidal stratum of the classical discriminant can be transformed by a monomial change of variables into an A-discriminantal set and compute the contours of the amoebas of these strata.

Keywords Amoeba · A-discriminant · Cuspidal stratum

The notion of the amoeba of an algebraic hypersurface was introduced in 1994 in the book [GKZ94]. The study of the structure of amoebas began with the papers [FPT00, Mik00], and by now there are many interesting results related both to the description of amoebas [MR01, BT12], and to their applications in the study of dimer configurations [KOS06], extensions of non-Archimedean fields [EKL06], to mention but a few. The interest in amoebas is partly stimulated by the connections to real algebraic geometry [Mik00] and tropical arithmetic [EKL06, Stu02]. The extension of the notion of amoeba to non-algebraic complex analytic sets allows to use this language in thermodynamics and statistical physics in general, for example in problems with several Hamiltonians for a given physical system [PPT13, PT09]. In statistical physics amoebas appear when using asymptotical methods for studying integrals with integration over cycles on analytic sets [LPT08, BKT14].

Denote by $\mathbb{T}^n$ the complex algebraic torus $(\mathbb{C} \setminus 0)^n$ and consider the mapping $\mathrm{Log} : \mathbb{T}^n \to \mathbb{R}^n$ given by

E.N. Mikhalkin · A.V. Shchuplev · A.K. Tsikh (✉)
Institute of Mathematics and Computer Science, Siberian Federal University,
pr. Svobodny 79, 660041 Krasnoyarsk, Russia
e-mail: atsikh@sfu-kras.ru

E.N. Mikhalkin
e-mail: mikhalkin@bk.ru

A.V. Shchuplev
e-mail: ashchuplev@sfu-kras.ru

F. Bracci et al. (eds.), *Complex Analysis and Geometry*, Springer Proceedings
in Mathematics & Statistics 144, DOI 10.1007/978-4-431-55744-9_19

$$\mathrm{Log}z = (\log|z_1|, \ldots, \log|z_n|).$$

The *amoeba* of an algebraic set $V \subset \mathbb{T}^n$ is its image $\mathrm{Log}V \subset \mathbb{R}^n$. The amoeba of the set V will be denoted by $\mathscr{A}_V$, while its complement $\mathbb{R}^n \setminus \mathscr{A}_V$ will be denoted by ${}^c\mathscr{A}_V$. Since the map Log is proper, the complement of the amoeba is open. For a hypersurface V, i.e. for a set of codimension 1, the complement ${}^c\mathscr{A}_V$ consists of a finite number of connected components, each is open and convex. Indeed, if a hypersurface V is the zero set of a polynomial P, then for every connected component E of ${}^c\mathscr{A}_V$ the set $\mathrm{Log}^{-1}E$ is a domain of convergence for some Laurent series for $1/P$ centered at the origin:

$$\frac{1}{P(z)} = \sum_{\alpha\in\mathbb{Z}^n} a_\alpha z^\alpha,$$

and such domains are logarithmically convex.

In the case of arbitrary codimension $k = \mathrm{codim}_{\mathbb{C}}\, V$ the complement ${}^c\mathscr{A}_V$ has the property of being $(k-1)$-convex (the 0-convexity is the usual convexity) [BT12, Hen04].

In comparison to the case of hypersurfaces, amoebas of surfaces of codimension $k > 1$ are studied to a less extent. One of the reasons behind that is the absence of a simple analog of the Jensen-Ronkin function [PR04]. This paper deals with amoebas of singular strata of cuspidal type for the classical discriminant. An important role in this study is played by the Horn-Kapranov parametrization for the discriminant set (see Sect. 2). The implicit function theorem yields that singularities of an algebraic function, given by a polynomial equation, appear only in those points where the discriminant of the polynomial vanishes. It turns out that a general algebraic function, i.e. given by a polynomial with independent variable coefficients, is a hypergeometric function in the sense of Horn [Hor89]. The hallmark of this property is that one can explicitly parametrize the boundary of the domain of convergence of a hypergeometric series, i.e. parametrize the set of conjugate radii of convergence. This parametrization was obtained by J.Horn in 1889, and a hundred years later M. Kapranov noticed a miraculous fact: if in Horn's parametrization we omit the absolute value signs and let the parameters be complex it becomes the parametrization of the singular set of a hypergeometric function [Kap91]. The ideas of Horn and Kapranov were further developed in [AT12] to parametrize discriminantal sets for polynomial transformations of $\mathbb{C}^n$.

We proceed as follows. In Sect. 1 we define the contour of an amoeba and the logarithmic Gauss map and formulate a theorem that establishes a relationship between them (Theorem 1). In Sect. 2 we consider cuspidal strata for the classical discriminant and find their place in the hierarchy of all A-discriminantal sets (Theorem 2). Theorems 3 and 4 are necessary steps to justify the fact that amoebas of cuspidal strata have non-empty contours, which admit explicit parameterizations (Theorem 5).

1 The Contour of Amoeba and the Logarithmic Gauss Map

Definition 1 The *contour* of the amoeba $\mathscr{A}_V$ is the set $\mathscr{C}_V$ of critical values of the logarithmic mapping Log restricted to V, i.e. of the mapping $\mathrm{Log}: V \to \mathbb{R}^n$.

The structure of the contour of an amoeba can be described in terms of the logarithmic Gauss map. This mapping, introduced by Kapranov in [Kap91] for hypersurfaces, extends naturally to the case of surfaces V of any codimension k.

Definition 2 Let $\mathrm{Gr}(n, k)$ be the Grassmanian of k-dimensional complex subspaces in $\mathbb{C}^n$. *The logarithmic Gauss map* $\gamma: V \to \mathrm{Gr}(n, k)$ sends a smooth point $z \in \mathrm{reg} V$ to the normal subspace $\gamma(z)$ to $\mathrm{Log}_{\mathbb{C}} V$ at $\mathrm{Log}_{\mathbb{C}}(z)$, where $\mathrm{Log}_{\mathbb{C}}$ is the complex logarithm $\mathrm{Log}_{\mathbb{C}}: (z_1, \ldots, z_n) \to (\log z_1, \ldots, \log z_n)$.

If V is a hypersurface

$$V = \{z \in \mathbb{T}^n : P(z) = 0\}$$

(i.e. if $k = 1$ and $\mathrm{Gr}(n, 1) = \mathbb{CP}_{n-1}$) the logarithmic Gauss map $\gamma : V \to \mathbb{CP}_{n-1}$ has the following analytic expression

$$(z_1, \ldots, z_n) \to \left(z_1 \frac{\partial P}{\partial z_1} : \ldots : z_n \frac{\partial P}{\partial z_n} \right).$$

In this case it is known [Mik00, The02] that a point $z \in \mathrm{reg} V$ is critical for the map $\mathrm{Log}|_V$ if and only if its image $\gamma(z)$ under the logarithmic Gauss map lies in the real projective subspace $\mathbb{RP}_{n-1} \subset \mathbb{CP}_{n-1}$. So the contour $\mathscr{C}_V$ of the amoeba $\mathscr{A}_V$ of a hypersurface is the set $\mathrm{Log}(\gamma^{-1}(\mathbb{RP}_{n-1}))$.

Consider now an algebraic surface $V \subset \mathbb{T}^n$, $n > 1$. Assume that V is of pure complex dimension d, i.e. all irreducible components of V have the same dimension d. Denote by $k = n - d$ the codimension of V.

In a neighborhood of any its smooth point z_0 the set V is given by the system $P_1(z)=\ldots=P_k(z) = 0$ with the Jacobian matrix of rank k. Then the logarithmic Gauss map at this point is defined by the matrix

$$\gamma(z) = \begin{pmatrix} z_1 \dfrac{\partial P_1}{\partial z_1} & \cdots & z_n \dfrac{\partial P_1}{\partial z_n} \\ \vdots & & \vdots \\ z_1 \dfrac{\partial P_k}{\partial z_1} & \cdots & z_n \dfrac{\partial P_k}{\partial z_n} \end{pmatrix}.$$

The rows of this matrix form a basis for the normal space to the image $\mathrm{Log}_{\mathbb{C}} V$ at $\mathrm{Log}_{\mathbb{C}}(z_0)$.

Theorem 1 ([BT12]) *A point $z \in \operatorname{reg} V$ is critical for the mapping* Log *if and only if the image $\gamma(z)$ of the logarithmic Gauss map contains*

- *at least $n - 2d + 1$ linearly independent real vectors if $2d \leq n$,*
- *at least one real vector if $2d \geq n$.*

In particular, if V is a hypersurface or a curve, i.e. $d = n - 1$ or $d = 1$, a point z is critical if and only if $\gamma(z)$ is real.

Let us say some words on the essence of this statement. The mapping $\operatorname{Log}|_V : V \to \mathbb{R}^n$ is the composition of the complex logarithm

$$\operatorname{Log}_{\mathbb{C}}(z) = \operatorname{Log}(z) + i\operatorname{Arg}(z) : \ V \to \mathbb{R}^n \oplus i\mathbb{R}^n$$

and the projection onto the real part $\mathbb{R}^n$:

$$\operatorname{Log}|_V = \pi_{\mathbb{R}^n} \circ \operatorname{Log}_{\mathbb{C}}\big|_V .$$

The complex logarithm does not have critical point on $\operatorname{reg} V$ (it is locally biholomorphic in $\mathbb{T}^n$), therefore the critical points of $\operatorname{Log}|_V$ appear only as critical point of the projection

$$\pi_{\mathbb{R}^n} : \ \operatorname{Log}_{\mathbb{C}} V \to \mathbb{R}^n.$$

But the critical point of this projection are defined by the properties of its tangent map

$$d(\pi_{\mathbb{R}^n})|_{\operatorname{Log}_{\mathbb{C}} V} : \ T_w(\operatorname{Log}_{\mathbb{C}} V) \to T_{\operatorname{Re}(w)}(\mathbb{R}^n), \ \ w = \operatorname{Log}_{\mathbb{C}}(z).$$

As a matter of fact, the criterion for $\operatorname{Log}|_V$ to be critical at z can be formulated as follows

- if $2d \leq n$, the tangent map of the projection $\pi_{\mathbb{R}^n}$ is *not injective*,
- if $2d \geq n$, the tangent map of the projection $\pi_{\mathbb{R}^n}$ is *not surjective*.

The conditions of being non-injective or non-surjective are related to whether the normal space to $\operatorname{Log}_{\mathbb{C}} V$ is real or not (in some sense Fig. 1 clarifies that: in critical points of the projection $\pi_{\mathbb{R}^n}$ the normal subspace $\gamma(z)$ becomes 'horizontal' and does not have a real part, i.e. $\gamma(z)$ is real).

As an example, let us examine whether an amoeba of a complex line has a contour. Let the complex line V in $\mathbb{C}^n$ be given by

$$\begin{cases} z_2 = a_2 z_1 + b_2, \\ \dots \\ z_n = a_n z_1 + b_n, \end{cases} \tag{1}$$

where all $a_j, b_j \neq 0$. The logarithmic projection of V has the form

$$\operatorname{Log}(z)|_V = (\log|z_1|, \log|a_2 z_1 + b_2|, \dots, \log|a_n z_1 + b_n|).$$

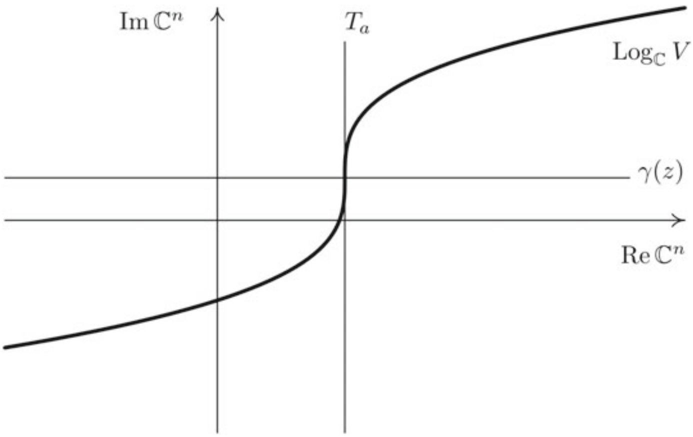

Fig. 1 An illustration to Theorem 1

Its Jacobian matrix equals

$$\frac{\partial(\mathrm{Log})}{\partial(z,\bar{z})} = \frac{1}{2}\begin{pmatrix} \frac{1}{z_1} & \frac{1}{\bar{z}_1} \\ \frac{a_2}{z_1} & \frac{\bar{a}_2}{\bar{z}_1} \\ \dots & \\ \frac{a_n}{z_1} & \frac{\bar{a}_n}{\bar{z}_1} \end{pmatrix}.$$

Denote

$$z_1 = x + iy, \qquad \frac{b_j}{a_j} = c_j + id_j,$$

then the condition for $z_1 = x + iy$ to be critical for the mapping $\mathrm{Log}|_V$ (i.e. when the rank of the Jacobian matrix is not maximal) can be written as

$$\begin{vmatrix} x & y \\ c_j & d_j \end{vmatrix} = 0, \quad j = 2, \dots, n,$$

$$\begin{vmatrix} c_k & d_k \\ x & y \end{vmatrix} + \begin{vmatrix} x & y \\ c_l & d_l \end{vmatrix} + \begin{vmatrix} c_k & d_k \\ c_l & d_l \end{vmatrix} = 0, \quad k,\ l = 2, \dots, n.$$

This system is consistent if and only if $c_k d_l = c_l d_k$ for all $k, l = 2, \dots, n$, but this condition is equivalent to

$$\frac{a_k b_l}{a_l b_k} \in \mathbb{R}, \quad k,\ l = 2, \dots, n. \tag{2}$$

Thus, we arrive at

Proposition *For $n \geq 3$ the contour of the amoeba of a complex line (1) is not empty if and only if the conditions (2) hold. In such case the contour of the amoeba is the image of the real line $d_2 x = c_2 y$ under the mapping* Log.

Consider two examples of lines in $\mathbb{T}^3$.

Example 1 For the complex line given by

$$\begin{cases} z_2 = z_1 + 1, \\ z_3 = z_1 + 1 + i, \end{cases}$$

the conditions (2) do not hold, therefore the contour of its amoeba is empty. The logarithmic projection of this line does not have critical points, and the line is diffeomorphic to its amoeba (see Fig. 2, left). In this case we say that the amoeba is not degenerate. At each point of the line the value $\gamma(z)$ of the logarithmic Gauss map has only one real vector (see Fig. 2, right).

Example 2 For the complex line

$$\begin{cases} z_2 = z_1 + 1, \\ z_3 = z_1 + 2 \end{cases}$$

the condition (2) holds: $\dfrac{a_2 b_3}{a_3 b_2} = 2 \in \mathbb{R}$. The amoeba is a surface with a corner in $\mathbb{R}^3$, each its interior point has two preimages on the line. Namely, for every non-real $z_1 = x + iy$ the images of

$$\mathrm{Log}(z_1, z_1 + 1, z_1 + 2) \text{ and } \mathrm{Log}(\overline{z}_1, \overline{z}_1 + 1, \overline{z}_1 + 2)$$

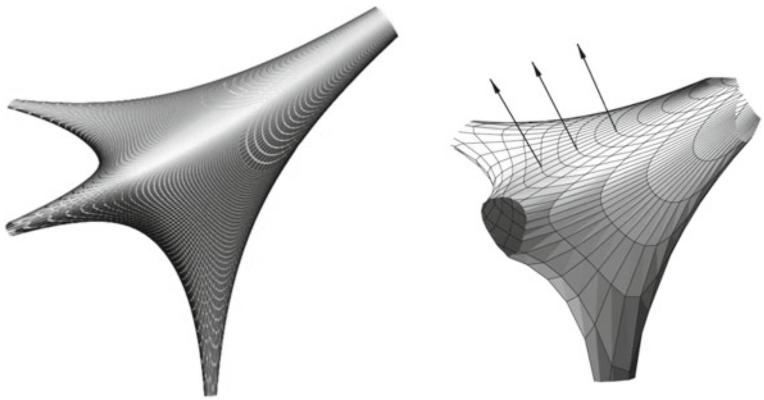

Fig. 2 The amoeba of the complex line of Example 1

Fig. 3 The amoeba of the complex line of Example 2

coincide. The real line $z_1 = x_1$ is mapped to the contour of the amoeba (its topological boundary), and the amoeba itself is the result of collapsing of a non-degenerate amoeba (see Fig. 3, left). At the points of the contour the logarithmic Gauss map $\gamma(z)$ contains a plane of real normal vectors (see Fig. 3, right).

2 Cuspidal Strata for Classical Discriminant

By a general algebraic equation we understand the equation

$$f(y) := a_0 + a_1 y + \ldots + a_{n-1} y^{n-1} + a_n y^n = 0 \tag{3}$$

with variable complex coefficients $a = (a_0, a_1, \ldots, a_n)$.

The classical discriminant is the polynomial $D(a)$ that vanishes if and only if the Eq. (3) has multiple roots. The zero set of the discriminant $D(a)$ we denote by ∇ and call *the discriminantal set* of the Eq. (3) or of the polynomial f.

Define subsets $\mathscr{M}^j \subset \nabla$ that comprise all $a \in \mathbb{C}^{n+1}$ for which the Eq. (3) has roots of multiplicity $\geq j$. They form a sequence of nested subsets

$$\nabla = \mathscr{M}^2 \supset \mathscr{M}^3 \supset \ldots \supset \mathscr{M}^n.$$

Each $\mathscr{M}^{j+1}$ is a subset of singular points sng$\mathscr{M}^j$, and the stratum $S^j = \mathscr{M}^j \setminus \mathscr{M}^{j+1}$ consists of points where either $\mathscr{M}^j$ is smooth or self intersects with its smooth components.Therefore we call $\mathscr{M}^j$ the *cuspidal strata*. Note that certain properties of these strata were studied in [Kat03].

Our recent result from the forthcoming paper [MT] states the following.

Theorem 2 *There exist monomial transformations that turn the strata $\mathscr{M}^2$, $\mathscr{M}^3, \ldots, \mathscr{M}^n$ into some A-discriminantal sets $\nabla_{A_2},\ \nabla_{A_3}, \ldots, \nabla_{A_n}$.*

Recall the definition of an A-discriminantal set (see [GKZ94], Chap. 9). Instead of Eq. (3) in one unknown y we consider an equation in k unknowns $y = (y_1, \ldots, y_k)$:

$$f(y_1, \dots, y_k) := \sum_{\alpha=(\alpha_1,\dots,\alpha_k)\in A} a_\alpha {y_1}^{\alpha_1} \dots {y_k}^{\alpha_k} = 0, \tag{4}$$

where $A \subset \mathbb{Z}^k$ is a fixed set of exponents that generate the lattice $\mathbb{Z}^k$ as an additive group, and the coefficients a_α are variables. The set of coefficients (same as the set of Eq. (4) and the set of Laurent polynomials f with exponents $\alpha \in A$) is $\mathbb{C}^A$, whose dimension is equal to the cardinality of A.

Definition 3 Let ∇° be the set of all $(a_\alpha) \in \mathbb{C}^A$ for which the Eq. (4) has critical roots $y \in (\mathbb{C} \setminus 0)^k$, i.e. the roots where the gradient of f vanishes. The closure $\overline{\nabla}^\circ$ of this set is called *an A-discriminantal set* and is denoted by ∇_A.

In the case $k = 1, \; A = \{0, 1, 2 \dots, n\} \subset \mathbb{Z}$ the set ∇_A is the classical discriminantal set ∇ of the Eq. (3). In Theorem 2 each ∇_{A_j} is an A_j-discriminantal set of an equation in $j - 1$ unknowns. Moreover, the cardinality of A_j is $n + 1$ and $\nabla_{A_2} = \nabla$.

For the proof of Theorem 2, the crucial thing is the Horn-Kapranov parametrization (see [PT04]) for the discriminantal set of a reduced equation

$$f(y) = 1 + x_1 y + \dots + x_{n-1} y^{n-1} + y^n = 0. \tag{5}$$

This parametrization $x = \Psi(s) : \; \mathbb{CP}^{n-2}_s \to \mathbb{C}^{n-1}_x$ is given by the formula

$$x_k = -\frac{n s_k}{\langle \alpha, s\rangle} \left(\frac{\langle \alpha, s\rangle}{\langle \beta, s\rangle} \right)^{\frac{k}{n}}, \quad k = 1, \dots, n-1, \tag{6}$$

where $\alpha, \; \beta$ are vectors of integers

$$\alpha = (n-1, \dots, 2, 1), \; \beta = (1, 2, \dots, n-1).$$

Notice that the Eq. (3) can be reduced differently, fixing coefficients of any pair of monomials y^p and y^q. The parametrization of the corresponding reduced discriminantal set ∇_{pq} will differ from formula (6) (i.e. the parametrization of ∇_{0n}), it will depend on different vectors α and β, and the root in the formula will be of degree $p - q$ instead of n [PT04].

Define the sequence of critical strata $\mathscr{C}^j$ of the parametrization (6). The first stratum $\mathscr{C}^1$ is defined as the set of critical values of the parametrization Ψ. It turns out that the critical points of Ψ constitute a hyperplane $L_1 \subset \mathbb{CP}^{n-2}$, consequently, the first critical stratum $\mathscr{C}^1$ is parametrized by the restriction of Ψ to L_1. Analogously, we define the stratum $\mathscr{C}^2$ of critical values of that restriction and proceed by induction. To formulate the result, introduce the following hyperplanes in $\mathbb{CP}^{n-2}$:

$$L_j = \left\{ s : \sum_{i=j}^{n-1} i(i-1)\cdots(i-(j-1))(n-i)s_i = 0 \right\},$$

where $s = (s_1 : \ldots : s_{n-1})$ is the homogeneous coordinates. The following theorem is proved by the direct computations.

Theorem 3 *The strata* $\mathscr{C}^j$ *are parametrized by the restrictions* $\Psi\Big|_{L^j}$ *on the planes* $L^j = L_1 \cap \ldots \cap L_j$.

The next theorem shows the relationship between the critical strata of Ψ with the reduced singular strata $\mathscr{M}_{0n}^j$ obtained from $\mathscr{M}^j$ by intersecting with the plane $a_0 = a_n = 1$.

Theorem 4 *The reduced singular strata* $\mathscr{M}_{0n}^{j+2} \subset \nabla_{0n}$ *coincide with the critical strata* $\mathscr{C}^j$ *of the parametrization* Ψ.

The proof of Theorem 4 goes as follows. First, we notice that the expression

$$t(s) = \left(\frac{\langle \beta, s\rangle}{\langle \alpha, s\rangle}\right)^{\frac{1}{n}}$$

involved in (6) is a root of the Eq. (5) of multiplicity ≥ 2 for $x = \Psi(s)$.

Let t be a root of the Eq. (5) of multiplicity $\geq \mu$, i.e.

$$f(y) = (y - t)^{\mu} f_{n-\mu}(y), \tag{7}$$

where

$$f_{n-\mu}(y) := \sum_{k=0}^{n-\mu} x_k^{(n-\mu)} y^k$$

is the result of division of f by $(y - t)^{\mu}$. Computing the coefficients $x_k^{(n-\mu)}$ in terms of the root t and the coefficients of x_k of the initial polynomial f, we prove that

$$f_{n-\mu}(t(s)) = 0 \Longleftrightarrow s \in L^{\mu-1}.$$

So, if $x = \Psi(s)$ then $y = t(s)$ is a root of multiplicity $\geq \mu - 2$ if and only if $s \in L^{\mu}$. From there, it is easy to finish the proof of Theorem 4.

To explain the proof of Theorem 2, recall the Horn-Kapranov parametrization for a reduced A-discriminantal set. In order to do that, with the set of exponents $\alpha^j \in A$ of (5) we associate the matrix

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_{11} & \alpha_{21} & \cdots & \alpha_{N1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{1k} & \alpha_{2k} & \cdots & \alpha_{Nk} \end{pmatrix}$$

(we denote this matrix by A too). For the Eq. (3) we have

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & \cdots & n-1 & n \end{pmatrix}$$

Let B be an integer right annulator of A of rank $m = N - k$. There are many such annulators and the choice of B gives a *reduction* of the Eq. (4) (see [GKZ94] or [Kap91]). Write this annulator in the form

$$B = \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{N1} & \cdots & b_{Nm} \end{pmatrix}.$$

The matrix B defines the mapping

$$\Psi_B : \mathbb{CP}^{m-1} \to (\mathbb{C}^*)^m, \quad s \to z = (Bs)^B, \tag{8}$$

where $s = (s_1 : \ldots : s_m)$ is the homogeneous coordinates in $\mathbb{CP}^{m-1}$. Coordinatewise the mapping Ψ_B has the form

$$z_k = \prod_{j=1}^{N} \langle b_j, s \rangle^{b_{jk}}, \quad k = 1, \ldots, m,$$

where $b_j = (b_{j1}, \ldots, b_{jm})$ are the rows of the matrix B. Since the first row of A is orthogonal to each column of B, the degree of homogeneity of these expressions in s is zero, therefore $(Bs)^B$ are correctly defined on $\mathbb{CP}^{m-1}$. The mapping $\Psi_B(s)$ defined by (8) is called *the Horn-Kapranov parametrization*. The importance of this mapping follows from Kapranov's theorem [Kap91] stating that

- The mapping $\Psi_B(s)$ is a parametrization of the reduced A-discriminantal set $\widetilde{\nabla}_A$.
- If $\widetilde{\nabla}_A$ is a hypersurface then $\Psi_B(s)$ is a birational isomorphism that coincide with the inversion of the logarithmic Gauss map for $\widetilde{\nabla}_A$.

Example 3 Let us sketch the idea of the proof of Theorem 2 by the example of the stratum $\mathscr{M}_{01}^3$ for the equation of degree 4. Consider a reduced equation of fourth degree

$$1 + y + z_2 y^2 + z_3 y^3 + z_4 y^4 = 0.$$

According to Kapranov's theorem the reduced discriminantal set ∇_{01} is parametrized by the mapping $\Psi : \mathbb{CP}^2 \to \mathbb{C}^3$ by

$$\begin{aligned} z_2 &= s_2(s_2 + 2s_3 + 3s_4)^1(-2s_2 - 3s_3 - 4s_4)^{-2} \\ z_3 &= s_3(s_2 + 2s_3 + 3s_4)^2(-2s_2 - 3s_3 - 4s_4)^{-3} \\ z_4 &= s_4(s_2 + 2s_3 + 3s_4)^3(-2s_2 - 3s_3 - 4s_4)^{-4}. \end{aligned}$$

The line $L^1 \subset \mathbb{CP}^2$ of its critical points has the equation $s_2 + 3s_3 + 6s_4 = 0$. Therefore, the reduced stratum $\mathscr{M}_{01}^3$ defined by the restriction $\Psi\big|_{L^1}$, in the homogeneous coordinates $s' = (s_3 : s_4)$ of this line is given by the formulas

$$\begin{aligned}
z_2 &= (-3s_3 - 6s_4)(-s_3 - 3s_4)^1(3s_3 + 8s_4)^{-2}\\
z_3 &= s_3(-s_3 - 3s_4)^2(3s_3 + 8s_4)^{-3}\\
z_4 &= s_4(-s_3 - 3s_4)^3(3s_3 + 8s_4)^{-4}.
\end{aligned}$$

The coefficients of five linear functions involved here define the matrix

$$B = \begin{pmatrix} -1 & -3 \\ 3 & 8 \\ -3 & -6 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The monomial change of variables $M\colon (z_1, z_2, z_3) \to (w_3, w_4)$ given by

$$w_3 = z_3 z_2^{-3},\ w_4 = z_4 z_2^{-6},$$

transforms the parametrization of $\mathscr{M}_{01}^3$ into

$$\begin{aligned}
w_3 &= s_3^1 s_4^0(-3s_3 - 6s_4)^{-3}(-s_3 - 3s_4)^{-1}(3s_3 + 8s_4)^3\\
w_4 &= s_3^0 s_4^1(-3s_3 - 6s_4)^{-6}(-s_3 - 3s_4)^{-3}(3s_3 + 8s_4)^8,
\end{aligned}$$

which has the form $w = (Bs')^B$. By Kapranov's theorem such a mapping parametrizes some reduced A-discriminantal set. In order to determine the set A it is enough to find a left integer annulator of B of the size 3×5 such that all elements of its first row are 1 and its columns generate $\mathbb{Z}^3$. In this case we can take

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 & 6 \end{pmatrix}.$$

Therefore, $w = (Bs')^B$ parametrizes a reduced A-discriminantal set of the equation

$$a_{10}y_1 + a_{01}y_2 + a_{00} + a_{31}y_1^3 y_2 + a_{63}y_1^6 y_2^3 = 0,$$

whose exponents are columns of the matrix A without the first row. The corresponding reduction of the equation is obtained if we fix $a_{10} = a_{01} = a_{00} = 1$ and denote $a_{31} =: w_3,\ a_{63} =: w_4$.

Notice that the chosen annulator A of B has all its elements non-negative. Following the general scheme (see Lemma below), we would have chosen the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -3 & -8 \end{pmatrix},$$

which is obtained from A by multiplication by a unimodular (3×3)-matrix.

The following lemma is a generalization of the observations of this example, it is an essential ingredient of the proof of Theorem 2. Here $p, q, i_0, \dots, i_{j-1}$ is an arbitrary sequence of pair-wise distinct integers from $\{0, 1, \dots, n\}$, and $1 \le j \le n-2$.

Lemma *Consider two sets of variables*

$$\begin{aligned} z &= (z_i), i \neq p, q, \\ w &= (w_k), k \neq p, q, i_0, \dots, i_{j-1}. \end{aligned}$$

The map $M : (\mathbb{C}^*)^{n-1}_z \to (\mathbb{C}^*)^{n-1-j}_w$ *defined by*

$$w_k = z_k \prod_{\nu=0}^{j-1} z_{i_\nu}^{-\frac{(k-p)(k-q)}{(i_\nu-p)(i_\nu-q)} \prod\limits_{m\neq\nu} \frac{k-i_m}{i_\nu-i_m}}, \quad k \neq p, q, i_0, \dots, i_{j-1}, \tag{9}$$

transforms the parametrization of the stratum $\mathscr{M}^{j+2}_{pq}$ *to the form* $w = (Bs')^B$, *where* B *is a rational* $(n+1) \times (n-1-j)$*-matrix of rank* $(n-1-j)$ *such that the sum of elements in a row is zero, and* $s' = (s_k)$, $k \neq p, q, i_0, \dots, i_{j-1}$.

For the proof of Theorem 2 it is convenient to take as p, q, $i_0, \dots, i_{j-1}$ the sequence $0,\ 1,\ 2, \dots,\ j+1$.

3 Amoebas of Reduced Cuspidal Strata for Classical Discriminant

Let us turn back to the reduced Eq. (5) and let $n = 4$. Consider the reduced discriminantal set ∇_{04} for this equation. According to Theorems 2 and 3, its stratum $\mathscr{M}^3_{04}$ is parametrized by the restriction of

$$\Psi : \mathbb{CP}^2 \to \nabla_{04} \subset \mathbb{C}^3$$

to the complex line of its critical points

$$L^1 = L_1 = \{(s_1 : s_2 : s_3) : \ 1 \cdot 3 \cdot s_1 + 2 \cdot 2 \cdot s_2 + 3 \cdot 1 \cdot s_3 = 0\},$$

where Ψ is defined by formula (6) for $n = 4$. Choosing s_1 as an affine coordinate in L_1, we see that the restriction $\Psi\big|_{L^1}$ is

$$\begin{aligned} x_1 &= -\frac{8s_1}{3s_1-1}\left(\frac{3s_1-1}{-s_1+3}\right)^{\frac{1}{4}} \\ x_2 &= 2\frac{3s_1+3}{3s_1-1}\left(\frac{3s_1-1}{-s_1+3}\right)^{\frac{1}{2}} \\ x_3 &= -\frac{8}{3s_1-1}\left(\frac{3s_1-1}{-s_1+3}\right)^{\frac{3}{4}}. \end{aligned}$$

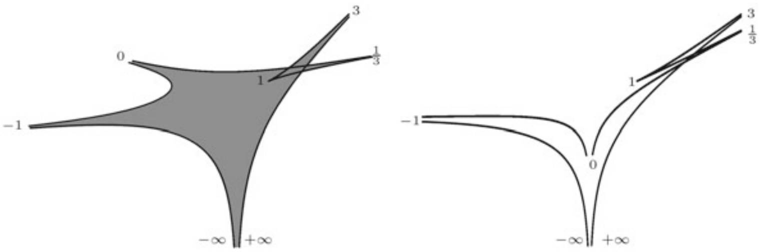

Fig. 4 The amoeba for the reduced stratum $\mathscr{M}^3_{04}$ (*left*) and its contour from a different angle (*right*)

The amoeba and its contour for the stratum $\mathscr{M}^3$, which admits this parametrization, is depicted on Fig. 4. One can see that the tentacles of the amoeba correspond to the values

$$s_1 = -\infty,\ -1,\ 0,\ \frac{1}{3},\ 3.$$

The value $s_1 = 1$ corresponds to the zero-dimensional stratum $\mathscr{M}^4_{04}$; this is a critical point of the parametrization. Thus, the contour of the amoeba for the zero-dimensional stratum $\mathscr{M}^4_{04}$ is a cuspidal point for the contour of the amoeba for the one-dimensional stratum $\mathscr{M}^3_{04}$ attached to it.

One has to be subtle when studying the attachement of the contours of the amoebas for the strata $\mathscr{M}^2_{04} = \nabla_{04}$ and $\mathscr{M}^3_{04}$. In the affine coordinates s_1, s_2 of $\mathbb{CP}^2$ the parametrization Ψ for $\mathscr{M}^2 = \widetilde{\nabla}$ looks like

$$\begin{aligned} x_1 &= \tfrac{-4s_1}{3s_1+2s_2+1}\left(\tfrac{3s_1+2s_2+1}{s_1+2s_2+3}\right)^{\frac{1}{4}} \\ x_2 &= \tfrac{-4s_2}{3s_1+2s_2+1}\left(\tfrac{3s_1+2s_2+1}{s_1+2s_2+3}\right)^{\frac{1}{2}} \\ x_3 &= \tfrac{-4}{3s_1+2s_2+1}\left(\tfrac{3s_1+2s_2+1}{s_1+2s_2+3}\right)^{\frac{3}{4}}. \end{aligned}$$

To draw the contour of the amoeba we need to compute the image of $\mathbb{R}^2 \subset \mathbb{RP}^2$ under the map $\mathrm{Log} \circ \Psi$. This map has four polar singularities on four lines (the fifth line $s_3 = 0$ lies at infinity of the chosen affine space):

$$s_1 = 0,\ s_2 = 0,\ 3s_1 + 2s_2 + 1 = 0,\ s_1 + 2s_2 + 3 = 0.$$

The contour of the amoeba for the stratum $\mathscr{M}^3_{04}$ is a curve of cuspidal points for the contour of the amoeba for the whole ∇_{04}, as shown on Fig. 5 (left). In a neighborhood of the edge of the contour of the amoeba for $\mathscr{M}^3_{04}$, which corresponds to $s_1 = 1$, the attachment of the contours forms 'the swallowtail'. It should be noticed that the contour of the amoeba of the discriminantal set contains the logarithmic image of the real part of the discriminantal set, which is the object of study in singularity theory.

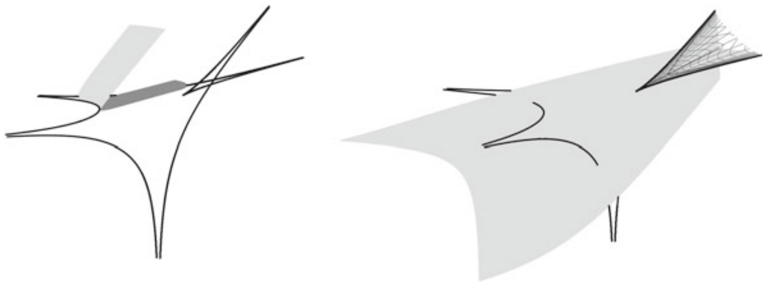

Fig. 5 Attachment of the contour of the amoeba for ∇_{04} to the contour of the amoeba for $\mathscr{M}^3_{04}$

The contour of the amoeba, however, is significantly larger, and its stratification is more complex.

Let us make now some observations based on studying the equation of degree 4. The contours of the amoebas for strata $\mathscr{M}^3_{04}$ and $\mathscr{M}^2_{04} = \nabla_{04}$ are parametrized by the restrictions of parameterizations $\Psi\Big|_{L^1}$ and $\Psi\Big|_{L^0} = \Psi$ (here $L^0 = \mathbb{CP}^2$) on the real parts of the planes L^1 and L^0. The mapping Ψ behaves continuously as the parameter $s \in L^0$ approach $L^1 \setminus L^2$ (where L^2 is the zero-dimensional subspace corresponding to the stratum $\mathscr{M}^4_{04}$). Note a sharp contrast of such a 'nice' behavior with the fact that the inverse $\Psi^{-1} : \nabla_{04} \to \mathbb{CP}^2$, which coincides with the logarithmic Gauss map, is not defined at singular points $\mathscr{M}^3_{04} \subset \nabla_{04}$. In general, similar arguments prove the following theorem.

Theorem 5 *The contours of the amoebas of all strata $\mathscr{M}^2_{0n} \supset \mathscr{M}^3_{0n} \supset \cdots \supset \mathscr{M}^n_{0n}$ are not empty and their preimages under the Log-projection are parametrized by the restrictions of the parametrization (6) to the real parts of the complex planes $L^0 \supset L^1 \supset \cdots \supset L^{n-2}$.*

In conclusion, we would like to raise a question about the distribution of values of the classical Gauss map for amoebas of complex curves $V \in \mathbb{T}^3$. In its smooth points the curve V admits a holomorphic parametrization $z = z(t)$, therefore the mapping $\mathrm{Log} z(t)$, which parametrizes the amoeba $\mathscr{A}_V$, is given by a triple of harmonic functions. If $t = u + iv$ were an isothermal coordinate for $\mathscr{A}_V$, the amoeba would be a minimal surface. According to the result of Fujimoto [Fuj97], the Gauss map for a minimal surface can not omit more than 4 points. In the case of amoebas the situation is quite different.

The line from Example 2 is parametrized by $t = z_1$ and the Gauss map is given by the formula

$$t \mapsto \frac{\left(-|t|^2,\ 2|t+1|^2,\ -|t+2|^2\right)}{\sqrt{|t|^4 + 4|t+1|^4 + |t+2|^4}}.$$

Fig. 6 An oval on the *sphere* is the image of the boundary of the amoeba

The image of the boundary of the amoeba $\mathscr{A}_V$ under this map is shown on Fig. 6. It is a smooth curve on the sphere, and the rest of the amoeba is mapped into the smaller spherical cap bounded by this curve. The Gauss map omits here a dense set of points of S^2.

Acknowledgments The first author is supported by the RFBR, research project no. 14-01-31265 mol_a. The second author is supported by the RFBR, research project no. 14-01-31239 mol_a. The third author is supported by the RFBR, research project no. 14-01-00544_a.
The research for this paper was carried out in Siberian Federal University within the framework of the research project 'Multidimensional Complex Analysis and Differential Equations' funded by the grant of the Russian Federation Government to support scientific research under the supervision of leading scientist, no. 14.Y26.31.0006.

References

[AT12] Antipova, I.A., Tsikh, A.K.: The discriminant locus of a system of *n* Laurent polynomials in *n* variables. Izv. Math. **76**(5), 881–906 (2012)

[BKT14] Bushueva, N.A., Kuzvesov, K., Tsikh, A.K.: On the asymptotics of homological solutions to linear multidimensional difference equations. J. Siberian Fed. Univ. Math. Phys. **7**(4), 417–430 (2014)

[BT12] Bushueva, N.A., Tsikh, A.K.: On amoebas of algebraic sets of higher codimension. Proc. Steklov Inst. Math. **279**(1), 52–63 (2012)

[EKL06] Einsiedler, M., Kapranov, M., Lind, D.: Non-archimedean amoebas and tropical varieties. J. Reine Angew. Math. **601**, 139–157 (2006)

[FPT00] Forsberg, M., Passare, M., Tsikh, A.: Laurent determinants and arrangements of hyperplane amoebas. Adv. Math. **151**, 54–70 (2000)

[Fuj97] Fujimoto, H.: Nevanlinna theory and minimal surfaces. Geom. V. Encyclopaedia Math. Sci. **90**, 95–151 (1997)

[GKZ94] Gelfand, I., Kapranov, M., Zelevinsky, A.: Discriminants, Resultants and Multidimensional Determinants. Birkhäuser, Boston (1994)

[Hen04] Henriques, A.: An analogue of convexity for complements of amoebas of varieties of higher codimension, an answer to a question asked by B. Sturmfels. Adv. Geom. **4**(1), 61–73 (2004)

[Hor89] Horn, J.: Über die Konvergenz der hypergeometrischen Riehen zweier und dreier Veranderlichen. Math. Ann. **34**, 544–600 (1889)

[Kap91] Kapranov, M.M.: A characterization of A-discriminantal hypersurfaces in terms of the logarithmic Gauss map. Math. Ann. **290**, 277–285 (1991)

[Kat03] Katz, G.: How tangents solve algebraic equations, or a remarkable geometry of discriminant varieties. Expo. Math. **21**, 219–261 (2003)

[KOS06] Kenyon, R., Okounkov, A., Sheffield, S.: Dimers and amoebae. Ann. Math. **163**, 1019–1056 (2006)

[LPT08] Leinartas, E.K., Passare, M., Tsikh, A.K.: Multidimensional versions of Poincare's theorem for difference equations. Sb. Math. **199**(10), 1505–1521 (2008)

[MT] Mikhalkin, E.N., Tsikh, A.K.: Singular strata of cuspidal type for classical discriminant. Sb. Math. **206**, 282–310 (2015)

[Mik00] Mikhalkin, G.: Real algebraic curves, the moment map and amoebas. Ann. Math. **151**, 309–326 (2000)

[MR01] Mikhalkin, G., Rullgård, H.: Amoebas of maximal area. Internat. Math. Res. Not. **9**, 441–451 (2001)

[PPT13] Passare, M., Pochekutov, D., Tsikh, A.: Amoebas of complex hypersurfaces in statistical thermodynamics. Math. Phys. Anal. Geomy. **16**, 89–108 (2013)

[PR04] Passare, M., Rullgård, H.: Amoebas, Monge-Ampére measures, and triangulations of the Newton polytope. Duke Math. J. **121**, 481–507 (2004)

[PT04] Passare, M., Tsikh, A.: Algebraic equations and hypergeometric series. In the book 'The legacy of Niels Henrik Abel', pp. 653–672 (2004)

[PT05] Passare, M., Tsikh, A.: Amoebas: their spines and their contours. Contemp. Math. **377**, 275–288 (2005)

[PT09] Pochekutov, D., Tsikh, A.K.: On the asymptotic of Laurent coefficients and its application in statistical mechanics. J. Siberian Fed. Univ. Math. Phys. **2**(4), 483–493 (2009)

[Stu02] Sturmfels, B.: Solving systems of polynomial equations. In: CBMS Regional Conferences Series, No. 97. American Mathematical Society, Providence, Rhode Island (2002)

[The02] Theobald, T.: Computing amoebas. Exp. Math. **11**, 513–526 (2002)