Characterizations of Strongly Pseudoconvex Models in Almost Complex and CR Geometries

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Abstract In this paper, we introduce the Wong-Rosay theorem, R. Schoen's theorem and its generalization in almost complex geometry.

Keywords Almost CR manifolds · Pseudo-Hermitian manifolds · Infinitesimal automorphism

1 Introduction

The aim of this paper is to introduce (1) the Wong-Rosay theorem, a characterization of the unit ball by its holomorphic automorphism group, (2) Schoen's theorem, a characterization of the unit sphere and the Heisenberg group by their CR automorphism groups, and (3) their generalizations to the almost complex and CR manifolds, respectively.

1.1 The Characterization of the Unit Ball

The Riemann mapping theorem says that a simply connected proper domain in the complex plane $\mathbb{C}$ is biholomorphic to the unit disc Δ. Hence in Complex Analysis of one variable, it is important to understand the nature of the unit disc. But in multi-dimensional complex Euclidean spaces, the Riemann mapping theorem fails as H. Poincaré showed that the unit ball $\mathbb{B}^2 = \left\{z \in \mathbb{C}^2 : \|z\| < 1\right\}$ and the bidisc $\Delta^2 = \Delta \times \Delta$ are biholomorphically distinct. Moreover as showed in [BU78, GE82], the biholomorphic equivalence classes of simply connected domains in $\mathbb{C}^n$ $(n \geq 2)$ forms indeed an infinite dimensional space. Therefore it has been a fundamental problem in Several Complex Variables to classify bounded domains which can play

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the same rôle of model objects as the unit disc. A precondition for the rôle in Complex Analysis and Complex Geometry is to admit a noncompact automorphism group. While it is not possible to classify simply connected domains in $\mathbb{C}^n$, the classification of domains with noncompact automorphism groups seems to be possible since a generic bounded domain has no automorphism except the identity (see [GE82]). A typical classification is B. Wong's characterization of the unit ball $\mathbb{B}^n = \{z \in \mathbb{C}^n : \|z\| < 1\}$.

Theorem 1.1 ([WO77]) *A bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with noncompact automorphism group is biholomorphic to the unit ball $\mathbb{B}^n$.*

For a bounded domain Ω, the noncompactness of the automorphism group of Ω, denoted by $\mathrm{Aut}(\Omega)$, is equivalent to the existence of an automorphism orbit $\{\varphi_k(p)\}$ for some $\varphi_k \in \mathrm{Aut}(\Omega)$ and $p \in \Omega$ which is accumulating at a bounded point. In his paper [RO79], J. P. Rosay strengthened Wong's theorem as following:

Theorem 1.2 ([RO79, EF95, GA02]) *A domain in a complex manifold which admits an automorphism orbit accumulating at a strongly pseudoconvex boundary point is biholomorphic to the unit ball.*

Theorems 1.1 and 1.2 are usually called the Wong-Rosay theorem.

1.2 The Characterization of the Unit Sphere

In the confomal geometry, the Euclidean space $\mathbb{R}^n$ and the Euclidean sphere $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ are characterized as global homogeneous models as showed in [AL72, SC95, FE96]:

Theorem 1.3 *The conformal group of the Riemannian manifold (M^n, g) is essential if and only if M is conformally equivalent to either $\mathbb{R}^n$ or S^n.*

Here essential means that the conformal group can not be reduced to an isometry group of a metric in the conformal class. As in [AL72], if the conformal group of M is essential, then it acts improperly on M (A topological group G *acts improperly* on M if there is a compact subset K of M such that $G_K = \{\varphi \in G : \varphi(K) \cap K \neq \emptyset\}$ is noncompact). The main proof of Theorem 1.3 is to confirm D. V. Alekseevskiĭ's assertion: *if the conformal group acts improperly on M, then M is conformally equivalent to $\mathbb{R}^n$ or S^n.*

A strongly pseudoconvex real hypersurface in a complex manifold, especially a boundary of a strongly pseudoconvex domain, has a similar geometric structure to the conformal geometry, usually called the pseudo-conformal structure. A real hypersurface M in a complex manifold X admits a CR structure inherited by the complex structure of X. If M is strongly pseudoconvex, then its CR structure is determined by the conformal structure of its pseudo-hermitian metric. R. Schoen also gave the CR version of Theorem 1.3 in case of strongly pseudoconvex CR manifolds:

Theorem 1.4 ([SC95]) *Suppose that M^{2n+1} is a strongly pseudoconvex CR manifold whose CR automorphism group acts on M improperly. Then M is CR equivalent to either the unit sphere $S^{2n+1} = \{z \in \mathbb{C}^{n+1} : \|z\| = 1\}$ if M is compact or the Heisenberg group if M is noncompact.*

This is a CR counterpart of the Wong-Rosay theorem. In case of a bounded strongly pseudoconvex domain Ω, Fefferman's extension theorem ([FE74]) implies that each automorphism of Ω extends to a CR automorphism of the boundary $\partial\Omega$ which is a compact strongly pseudoconvex CR manifold. Thus the noncompactness of $\mathrm{Aut}(\Omega)$ implies that the CR automorphism group of $\partial\Omega$ is also noncompact, equivalently, it acts improperly (for a compact manifold, the improper action by a topological group G is the same as the noncompactness of G). In case of an unbounded domain, consider the *Siegel half plane*, $\mathbb{H}^{n+1} = \{(z^0, z^1, \ldots, z^n) \in \mathbb{C}^{n+1} : \mathrm{Re}\, z^0 + \sum_{\alpha=1}^{n} |z^\alpha|^2 < 0\}$ which is biholomorphic to the unit ball $\mathbb{B}^{n+1}$ by the Cayley transform. The group of affine automorphisms of $\mathbb{H}^{n+1}$ coincides with the CR automorphism group of the *Heisenberg group* $\partial\mathbb{H}^{n+1}$. Since $\mathscr{D}_s$ in (2.1) belongs to the isotropy subgroup at the origin, the CR automorphism group of $\partial\mathbb{H}^{n+1}$ is noncompact and moreover acts improperly.

1.3 Generalizations

Gaussier and Sukhov ([GA03]) showed that the Wong-Rosay theorem is also valid in almost complex manifolds of complex dimension 2. But in higher dimensional case, there is an exotic model (called a pseudo-Siegel domain) which admits an automorphism orbit accumulating at a strongly pseudoconvex boundary point and whose almost complex structure is non-integrable, so which is not biholomorphic to the unit ball with the standard complex structure. Thus the local version (Theorem 1.2) fails in almost complex manifolds. Gaussier and Sukhov [GA06] and the author [LK06] characterized the pseudo-Siegel domains: *a domain in almost complex manifold which admits an automorphism orbit accumulating at a strongly pseudoconvex boundary point is biholomorphic to a pseudo-Siegel domain* (Theorem 2.1). However as in [BY09], the global version (Theorem 1.1) is also valid in any dimension: *a relatively compact, strongly pseudoconvex domain in an almost complex manifold with a noncompact automorphism group is biholomorphic to the unit ball with the standard complex structure* (Theorem 2.2).

As in Sect. 2, a pseudo-Siegel domain is the Siegel half plane with a certain almost complex structure, so its boundary is noncompact. And its automorphism group is the same as the CR automorphism group of the boundary which acts improperly. Therefore the relationship between the Wong-Rosay theorem and Schoen's theorem makes us to expect:

Conjecture 1.1 A strongly pseudoconvex almost CR manifold M whose CR automorphism group action is improper is CR equivalent to either the standard sphere if M is compact or a boundary of a pseudo-Siegel domain if M is noncompact.

In this paper, we introduce the basic technique to get the Wong-Rosay theorem in almost complex structure as in [GA06, LK06] and a partial confirmation of the conjecture by the collaboration work with Joo [JO15].

Convention: Throughout this paper, Greek indices indicating coefficients of complex tensor run from 1 to n and Latin indices for real tensors run from 1 to $2n$. For Greek indices, the summation convention is always assumed. We will take the bar on Greek indices to denote the complex conjugation of the corresponding tensor coefficients: $\overline{Z}_\alpha = Z_{\bar\alpha}$, $\bar\omega^\alpha = \omega^{\bar\alpha}$, $\overline{R}_\beta{}^\alpha{}_{\lambda\bar\mu} = R_{\bar\beta}{}^{\bar\alpha}{}_{\bar\lambda\mu}$.

2 The Wong-Rosay Theorem in the Almost Complex Manifold

Let X be an almost complex manifold with an almost complex structure J. By an (holomorphic) *automorphism* of (X, J), we mean a biholomorphism of X onto itself with respect to J. The *automorphism group* $\mathrm{Aut}(X, J)$ of (X, J) is the topological group of automorphisms of (X, J) with the composition law and the compact-open topology.

Let us define the pseudo-Siegel domain as in [LK08]:

Definition 2.1 Consider the complex Eulidean space $\mathbb{C}^{n+1}$ with the standard coordinates $(z^0, z^1, \ldots, z^n)$. Let $P = (P_{\alpha\beta})_{\alpha,\beta=1,\ldots,n}$ be a $n \times n$ skew-symmetric complex matrix. The *model structure* J_P is the almost complex structure of $\mathbb{C}^{n+1}$ defined by the following $(1, 0)$-vector fields:

$$Z_0 = \frac{\partial}{\partial z^0}\ , \quad Z_\alpha = \frac{\partial}{\partial z^\alpha} - i P_{\alpha\beta} z^\beta \frac{\partial}{\partial z^{\bar 0}} \quad (\alpha = 1, \ldots, n)\ .$$

The pair $(\mathbb{H}^{n+1}, J_P)$ is called a *pseudo-Siegel domain* for the Siegel half plane $\mathbb{H}^{n+1}$ and the model structure J_P.

2.1 *Automorphisms of the Pseudo-Siegel Domain*

As mentioned, the Siegel-half plane $\mathbb{H} = \mathbb{H}^{n+1}$ with the standard complex structure J_{st} (the case of $P = 0$) is biholomorphic to the unit ball $(\mathbb{B}^{n+1}, J_{\mathrm{st}})$; thus the pseudo-Siegel domains can be considered as a deformation of the unit ball. The matrix P represents the torsion for the integrability of the structure, in the sense of $[Z_\alpha, Z_\beta] = -2i P_{\alpha\beta}\partial/\partial z^{\bar 0}$ so that J_P is always non-integrable except $P = 0$. For any choice of P, the boundary of $\mathbb{H}$ is always strongly pseudoconvex and $\mathbb{H}$ has the non-isotropic dilation

$$\mathscr{D}_s : (z^0, z^1, \ldots, z^n) \mapsto (e^s z^0, e^{s/2} z^1, \ldots, e^{s/2} z^n) \quad (s \in \mathbb{R}) \tag{2.1}$$

as its automorphism. This means that Theorem 1.2 fails in almost complex setting.

Moreover any pseudo-Siegel domain is homogeneous since it has the Heisenberg group as its holomorphic transformation group. The *Heisenberg group* is the group $\mathscr{H}_P = (\partial\mathbb{H}, *_P)$ whose binary operation $*_P$ is defined by

$$\zeta *_P \xi = \left(\zeta^0 + \xi^0 - 2\delta_{\alpha\bar{\beta}}\xi^\alpha\zeta^{\bar{\beta}} + iP_{\alpha\beta}\xi^\alpha\zeta^\beta + iP_{\bar{\alpha}\bar{\beta}}\xi^{\bar{\alpha}}\zeta^{\bar{\beta}}, \zeta' + \xi'\right), \qquad (2.2)$$

for $\zeta = (\zeta^0, \zeta'), \xi = (\xi^0, \xi') \in \partial\mathbb{H}$. Each element $\zeta \in \mathscr{H}_P$ generates an automorphism by $z \mapsto \zeta *_P z$; hence $\mathscr{H}_P$ can be considered as a subgroup of $\mathrm{Aut}(\mathbb{H}, J_P)$. Then one can easily see that the transformation group generated by $\mathscr{H}_P$ and $\{\mathscr{D}_s : s \in \mathbb{R}\}$ acts on $\mathbb{H}$ transitively.

In [LK08], the automorphism groups and the bihomorphic equivalence of pseudo-Siegel domains are completely described.

2.2 The Scaling Method in Almost Complex Manifold

Here, we introduce the scaling method to the almost complex manifold due to Gaussier and Sukhov [GA03, GA06].

Let Ω be a domain in an almost complex manifold (X, J) of complex dimension $n+1$. Suppose that there are $\varphi_k \in \mathrm{Aut}(\Omega, J)$ and $p \in \Omega$ such that

$$\varphi_k(p) \to q \in \partial\Omega \quad \text{as} \quad k \to \infty ,$$

where $\partial\Omega$ is smooth near q and strongly J-pseudoconvex at q.

Step 1 (a local coordinate system): Choosing a local coordinate system $\Phi : U \subset \mathbb{C}^{n+1} \to M$ about q with $\Phi(0) = q$, we can identity $q = 0$, $\Phi(U) = U$ and $d\Phi^{-1} \circ J \circ d\Phi = J$. For a suitable Φ, we may assume that

1. $J(0) = J_{\mathrm{st}}$ where J_{st} is the standard complex structure of $\mathbb{C}^{n+1}$,
2. $U \cap \Omega = \{z : \rho(z) < 0\}$ where $\rho(z) = \mathrm{Re}\, z^0 + \sum_{\alpha=1}^n |z^\alpha|^2 + o(\|z\|^2)$.

Step 2 (centering): We shall only consider sufficiently large k with $\varphi_k(p) \in U$. For each k, take $p_k^* \in U \cap \partial\Omega$ that realizes the Euclidean distant τ_k from $p_k = \varphi_k(p)$ to $U \cap \partial\Omega$. Then we consider a rigid motion L_k of $\mathbb{C}^{n+1}$ with $L_k(p_k^*) = 0$ and $L_k(p_k) = (-\tau_k, 0, \ldots, 0)$.

Step 3 (dilating): Now we let

$$\Lambda_k(z) = \left(\frac{z^0}{\tau_k}, \frac{z^1}{\sqrt{\tau_k}}, \ldots, \frac{z^n}{\sqrt{\tau_k}}\right) .$$

For $A_k = \Lambda_k \circ L_k$, the sequence $A_k(U \cap \Omega)$ of domains converges to the Siegel half plane $\mathbb{H}^{n+1} = \{\mathrm{Re}\, z^0 + \|z'\|^2 < 0\}$ in the sense of the Hausdorff set convergence.

Simultaneously, the sequence $dA_k \circ J \circ dA_k^{-1}$ of induced almost complex structures on $A_k(U \cap \Omega)$ converges to an almost complex structure J' of $\mathbb{H}$ for which $(\mathbb{H}, J')$ is biholomorphic to a pseudo-Siegel domain $(\mathbb{H}, J_P)$.

Finally one can get that $A_k \circ \varphi_k : \varphi_k^{-1}(U \cap \Omega) \subset \Omega \to \mathbb{C}^{n+1}$ has a subsequential limit F defined on the whole of Ω which is biholomorphism (Ω, J) to $(\mathbb{H}, J')$.

Theorem 2.1 ([GA06, LK06]) *Let Ω be a domain in an almost complex manifold (X, J). If Ω admits an automorphism orbit accumulating at a strongly J-pseudoconvex boundary point, then (Ω, J) is biholomorphic to a pseudo-Siegel domains.*

2.3 Bounded Realization of the Pseudo-Siegel Domain

For any non-integrable model structure, the induced structure by the Cayley transform on the unit ball has a singularity at the boundary point corresponding to the point at infinity. Thus it is natural to ask whether there is biholomorphism from the non-integrable pseudo-Siegel domain to a relatively compact domain in an almost complex manifold.

Let Ω be a relatively compact, strongly pseudconvex domain in an almost complex manifold (X, J). If $\mathrm{Aut}(\Omega, J)$ is noncompact, then by Theorem 2.1, there is a biholomorphism $F : (\Omega, J) \to (\mathbb{H}, J_P)$. Consider the point $-\mathbf{1} = (-1, 0, \ldots, 0) \in \mathbb{H}$ and the automorphism $\mathscr{D}_k$ as in (2.1) for $k = 1, 2, \ldots$. Since the automorphism orbit $\{\mathscr{D}_k(-\mathbf{1}) : k = 1, 2, \ldots\}$ is noncompact in $\mathbb{H}$, there is a subsequential limit $q \in \partial\Omega$ of the sequence $F^{-1}(\mathscr{D}_k(-\mathbf{1}))$. Applying the scaling method again to the automorphism orbit $\{F^{-1}(\mathscr{D}_k(-\mathbf{1}))\}$ with certain local coordinates about q, we can obtain a biholomorphism $G : (\Omega, J) \to (\mathbb{H}, J_P)$ with $G(q) = 0$ in the limit sense. Then $F^{-1} \circ G$ is the automorphism of $(\mathbb{H}, J_P)$ with $(F^{-1} \circ G)(0) = \infty$. But every automorphism of $(\mathbb{H}, J_P)$ is affine if $P \neq 0$ ([LK08]); thus $P = 0$ so J_P is integrable.

Theorem 2.2 (Byun et al. [BY09]) *A relative compact and strongly pseudoconvex domain in an almost complex manifold with a noncompact automorphism group is biholomorphic to the unit ball with the standard complex structure.*

3 Schoen's Theorem in Almost CR Manifolds

The scaling method in the Wong-Rosay theorem allows to rescale a given domain and its complex structure to a biholomorphically equivalent model. But in order to get the CR equivalence to a model, the local equivalence problem of CR structures must be considered since the CR structure is a local structure. For the CR equivalence in Theorem 1.4, R. Schoen used the pseudo-hermitian equivalence of Webster [WE78] via the CR Yamabe problem. In this section, we introduce the pseudo-hermitian equivalence problem, the Yamabe type problem and generalization of Theorem 1.4 in strongly pseudoconvex almost CR manifolds as studied in Joo and Lee [JO15].

3.1 Pseudo-hermitian Structure Equations

Let us consider an almost CR manifold M of real dimension $2n+1$ with a CR structure (H, J), that is, $H = \bigcup_{p\in M} H_p \subset TM$ is a hyperplane bundle with a smooth field of bundle isomorphisms $J : H \to H$ such that $J \circ J = -I$. By a *CR automorphism* of M, we mean a diffeomorphism φ of M onto itself with $d\varphi(H) = H$ and $J \circ d\varphi = d\varphi \circ J$. The *CR automorphism group* of M, simply denoted by $\mathrm{Aut}_{\mathrm{CR}}(M)$, is the topological group of CR automorphisms of M with the composition law and the compact-open topology.

The tensor field J decomposes the complexified bundle $\mathbb{C}H = \mathbb{C} \otimes_{\mathbb{R}} H$ by $\mathbb{C}H = H^{1,0} \oplus H^{0,1}$ where $H^{1,0} = \{v - iJv : v \in H\}$ and $H^{0,1} = \overline{H^{1,0}}$. The CR manifold is *strongly peudoconvex* if for an 1-form θ annihilating H, the *Levi form* L_θ defined by $L_\theta(Z, W) = 2i d\theta(Z, \overline{W})$ for $Z, W \in H^{1,0}$ is positively or negatively definite. This is independent of the choice of θ. Let $(Z_\alpha) = (Z_1, \ldots, Z_n)$ be a $(1,0)$-frame, a local frame filed to $H^{1,0}$. Then there is an *admissible* $(1,0)$*-coframe* $(\omega^\alpha) = (\omega^1, \ldots, \omega^n)$, a $\mathbb{C}^n$-valued 1-form which is dual to (Z_α) and satisfies

$$d\theta = 2i g_{\alpha\bar\beta}\omega^\alpha \wedge \omega^{\bar\beta} + p_{\alpha\beta}\omega^\alpha \wedge \omega^\beta + p_{\bar\alpha\bar\beta}\omega^{\bar\alpha} \wedge \omega^{\bar\beta}\ . \tag{3.1}$$

Here $(g_{\alpha\bar\beta})$ stands for the Levi form and $(p_{\alpha\beta})$ is uniquely determined by $p_{\alpha\beta} = -p_{\beta\alpha}$. We will use the Levi form $(g_{\alpha\bar\beta})$ and its inverse $(g^{\bar\beta\alpha})$ to lower and raise indices (e.g. ${\omega_\beta}^\gamma g_{\gamma\bar\alpha} = \omega_{\beta\bar\alpha}$). Then we can define the *pseudo-hermitian connection form* $({\omega_\beta}^\alpha)$, uniquely determined by $dg_{\alpha\bar\beta} - \omega_{\alpha\bar\beta} - \omega_{\bar\beta\alpha} = 0$ and

$$d\omega^\alpha = \omega^\beta \wedge {\omega_\beta}^\alpha + {T_\beta}^\alpha{}_\gamma \omega^\beta \wedge \omega^\gamma + {N_{\bar\beta}}^\alpha{}_{\bar\gamma}\omega^{\bar\beta} \wedge \omega^{\bar\gamma} + A^\alpha{}_{\bar\beta}\theta \wedge \omega^{\bar\beta} + B^\alpha{}_\beta \theta \wedge \omega^\beta\ . \tag{3.2}$$

The functions ${T_\beta}^\alpha{}_\gamma$, ${N_{\bar\beta}}^\alpha{}_{\bar\gamma}$, $A^\alpha{}_{\bar\beta}$, $B^\alpha{}_\beta$ are also fixed by ${T_\beta}^\alpha{}_\gamma = -{T_\gamma}^\alpha{}_\beta$, ${N_{\bar\beta}}^\alpha{}_{\bar\gamma} = -{N_{\bar\gamma}}^\alpha{}_{\bar\beta}$, $B_{\beta\bar\alpha} = B_{\bar\alpha\beta}$. The J-linear connection defined by $\nabla Z_\alpha = {\omega_\alpha}^\beta \otimes Z_\beta$ is the *pseudo-hermitian connection*. Then we have the *pseudo-hermitian curvature tensor* $({R_\beta}^\alpha{}_{\lambda\bar\mu})$ defined by

$$ {\Omega_\beta}^\alpha \equiv {R_\beta}^\alpha{}_{\lambda\bar\mu}\omega^\lambda \wedge \omega^{\bar\mu} \quad \mathrm{mod}\ \{\theta, \omega^\lambda \wedge \omega^\mu, \omega^{\bar\lambda} \wedge \omega^{\bar\mu}\}$$

for the curvature form ${\Omega_\beta}^\alpha = d{\omega_\beta}^\alpha - {\omega_\beta}^\gamma \wedge {\omega_\gamma}^\alpha$.

3.2 Pseudo-hermitian Equivalence Problem

Now we characterize a pseudo-hermitian structure of the boundary of the Siegel domain. First, we introduce an intrinsic form of the boundary.

Let $(t, z) = (t, z^1, \dots, z^n)$ be the standard coordinates of $\mathbb{R} \times \mathbb{C}^n$. A $n \times n$ skew-symmetric complex matrix $P = (P_{\alpha\beta})$ gives the Lie group structure $*_P$ to $\mathbb{R} \times \mathbb{C}^n$ by

$$(t, z) *_P (s, w) = (t + s + 2\mathrm{Im}\ \langle z, w\rangle - 2\mathrm{Re}\, P(z, w), z + w)$$

where $\langle z, w\rangle = \delta_{\alpha\bar{\beta}} z^\alpha w^{\bar{\beta}}$ and $P(z, w) = P_{\alpha\beta} z^\alpha w^\beta$. This is the induced operation from (2.2) under the natural projection $\pi : \partial\mathbb{H}^{n+1} \to \mathbb{R} \times \mathbb{C}^n$. We call $\mathscr{H}_P = (\mathbb{R} \times \mathbb{C}^n, *_P)$ a *Heisenberg group* associated to P. In fact all Heisenberg groups are Lie group isomorphic to each others (see [BY09]).

Each Heisenberg group has the contact distribution H_P annihilated by

$$\theta_P = dt + i\delta_{\alpha\bar{\beta}} z^\alpha dz^{\bar{\beta}} - i\delta_{\alpha\bar{\beta}} z^{\bar{\beta}} dz^\alpha + P_{\alpha\beta} z^\alpha dz^\beta + P_{\bar{\alpha}\bar{\beta}} z^{\bar{\alpha}} dz^{\bar{\beta}} \tag{3.3}$$

and the strongly pseudoconvex CR structure J_P on H_P whose the global $(1, 0)$-frame $(Z_1, \dots, Z_n)$ is defined by

$$Z_\alpha = \frac{\partial}{\partial z^\alpha} + \left(i\delta_{\alpha\bar{\beta}} z^{\bar{\beta}} + P_{\alpha\beta} z^\beta\right) \frac{\partial}{\partial t}\ , \quad \alpha = 1, \dots, n.$$

Then the Heisenberg group $\mathscr{H}_P$ acts transitively on itself as a CR transformation group of (H_P, J_P). We call the CR manifold $\mathbb{R} \times \mathbb{C}^n$ with the CR structure (H_P, J_P) a *Heisenberg group*, simply denoted by $\mathscr{H}_P$.

Since $[Z_\alpha, Z_\beta] = -2P_{\alpha\beta}\partial/\partial t$, the CR structure of $\mathscr{H}_P$ is non-integrable except $P = 0$. Each Heisenberg group $\mathscr{H}_P$ is CR equivalent to $(\partial\mathbb{H}^{n+1}, J_P)$ and admits the *CR dilation*,

$$\mathscr{D}_s : (t, z^1, \dots, z^n) \mapsto (e^s t, e^{s/2} z^1, \dots, e^{s/2} z^n) \quad (s \in \mathbb{R}) \tag{3.4}$$

as its CR automorphism. Therefore the CR automorphism group of $\mathscr{H}_P$ acts improperly, so Theorem 1.4 is not valid in the almost CR setting.

Let us consider the pseudo-hermitian structure equations of $\mathscr{H}_P$. For the contact form θ_P, we have $d\theta_P = 2i\delta_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}} + P_{\alpha\beta} dz^\alpha \wedge dz^\beta + P_{\bar{\alpha}\bar{\beta}} dz^{\bar{\alpha}} \wedge dz^{\bar{\beta}}$, so that $g_{\alpha\bar{\beta}} \equiv \delta_{\alpha\bar{\beta}}$, $p_{\alpha\beta} \equiv P_{\alpha\beta}$ for (3.1), and $(dz^1, \dots, dz^n)$ is the admissible coframe for θ_P. Since dz^α is closed, one can see that the connection form $({\omega_\beta}^\alpha)$ of (dz^α) vanishes identically. So all torsion tensors except $p_{\alpha\beta} \equiv P_{\alpha\beta}$ and curvature tensors are vanishing identically. This characterizes the Heisenberg model with θ_P:

Proposition 3.1 *Let (M, θ) be a pseudo-hermitian manifold. Suppose that there is an admissible coframe $(\omega^1, \dots, \omega^n)$ with the following vanishing tensors:*

$$p_{\alpha\beta;\gamma} \equiv T_{\beta}{}^{\alpha}{}_{\gamma} \equiv N_{\bar{\beta}}{}^{\alpha}{}_{\bar{\gamma}} \equiv A^{\alpha}{}_{\bar{\beta}} \equiv R_{\beta}{}^{\alpha}{}_{\lambda\bar{\mu}} \equiv 0 \,. \tag{3.5}$$

Then (M, θ) *is locally pseudo-hermitian equivalent to a Heisenberg group model* $(\mathscr{H}_P, \theta_P)$.

Here $p_{\alpha\beta;\gamma}$ stands for the coefficient of the covariant derivative of the tensor $(p_{\alpha\beta})$ by Z_γ's: $p_{\alpha\beta;\gamma} = Z_\gamma p_{\alpha\beta} - p_{\alpha\lambda}\omega_{\beta}{}^{\lambda}(Z_\gamma) - \omega_{\alpha}{}^{\lambda}(Z_\gamma)p_{\lambda\beta}$.

3.3 Sub-Riemannian Yamabe Problem

In order to use the pseudo-hermitian equivalence (Theorem 3.1), we need to find a contact form for which (3.5) holds. In his paper [SC95], R. Schoen uses the CR Yamabe problem for the Webstar scalar curvature, $R = R_{\alpha\bar{\beta}\lambda\bar{\mu}} g^{\alpha\bar{\beta}} g^{\lambda\bar{\mu}}$. Unlike the integrable case, the transformation formula of the Webster scalar curvature is much more complicated. It is not possible to be simplified as the CR Yamabe equation in the integrable pseudo-hermitian geometry. Thus in [JO15] we studied an auxiliary contact sub-Riemannian structure and its Yamabe problem to find a desired contact form.

Let (M^{2n+1}, H) be a contact manifold and θ be a contact form ($H = \ker\theta$). A positive quadratic form g on the contact distribution H is called a *sub-Riemannian metric* and the pair (θ, g) is called a *contact sub-Riemannian structure* of M.

For an orthonormal frame $(X_1, \dots, X_{2n})$ to H with respect to g, we have a $2n \times 2n$ skew-symmetric matrix (h_{ij}) defined by

$$h_{ij} = d\theta(X_i, X_j) \,.$$

Let X_0 be the *characteristic vector field* of the contact form θ, that is, the vector field uniquely determined by $\theta(X_0) = 1$ and $X_0 \lrcorner\, d\theta = 0$. For the dual coframe $(\theta, \theta^1, \dots, \theta^{2n})$ of $(X_0, X_1, \dots, X_{2n})$, we have $d\theta = h_{ij}\theta^i \wedge \theta^j$. Then we can define the *contact sub-Rimannian connection form* (θ_j^i) for (θ, g) (see [FV93, FV07]) which is uniquely determined by

$$d\theta^i = \theta^j \wedge \theta_j^i + \theta \wedge \tau^i \;, \quad \theta_j^i = -\theta_i^j \;, \quad \sum_i \tau^i \wedge \theta^i = 0 \,.$$

Moreover the *curvature form* (Θ_j^i) and the *curvature tensor* (R^i_{jkl}) for (θ, g) are defined by

$$\Theta_j^i = d\theta_j^i - \theta_j^k \wedge \theta_k^i \equiv R^i_{jkl}\theta^k \wedge \theta^l \mod \theta.$$

We call $R = \sum_{i,j} R^i_{jij}$ a *sub-Riemannian scalar curvature* of (θ, g). When we let $R_h = \sum_{i,j,k,l} R^i_{jkl} h^{ik} h^{jl}$ for the inverse (h^{ji}) of (h_{ij}), we call the amount

$$S = (2n+1)R - R_h$$

a *twisted scalar curvature* for (θ, g).

Assume that the contact sub-Riemannian structure (θ, g) is *orthogonal*, that is, $h = (h_{ij})$ is the orthogonal matrix. Then we get the Yamabe type transformation formula for the twisted scalar curvature:

Theorem 3.1 ([JO15]) *Let (M^{2n+1}, θ, g) be an orthogonal contact sub-Riemannian manifold. For a subconformal change $(\theta', g') = (u^{2/n}\theta, u^{2/n}g)$, let S and S' be the twisted scalar curvatures for (θ, g) and (θ', g'), respectively. Then u satisfies*

$$S' u^{\frac{2}{n}+1} = Lu \ , \tag{3.6}$$

where $L = 4(n+1)\Delta_b + S$ and Δ_b is the sub-laplacian operator *defined by $\Delta_b u = -\sum_i X_i(X_i u) + \sum_{i,j}(X_j u)\theta_i^j(X_i)$.*

For $p = 2 + 2/n$ and the volume form $dV = (1/n!)\theta \wedge d\theta^n$, the *sub-conformal Yamabe invariant* $Q(M)$ is defined by

$$Q(M) = \inf\left\{\int_M u\,Lu\,dV : \int_M u^p dV = 1,\ u \in C_c^\infty(M) \text{ and } u \geq 0\right\}$$

which is independent from the subconformal change of the contact sub-Riemannian structure. Then we can solve the subconformal Yamabe problem.

Theorem 3.2 ([JO15]) *Let (M, θ, g) be an orthogonal contact sub-Riemannian manifold.*

(1) *If M is compact and $Q(M) < Q(S^{2n+1})$, then there is a sub-conformal change $(\theta', g') = (u^{2/n}\theta, u^{2/n}g)$ whose twisted scalar curvature S' of (M, θ', g') is the constant $Q(M)$.*
(2) *If M is noncompact and $Q(M) \geq 0$ or $Q(M) < 0$, then there exists a sub-conformal change whose twisted scalar curvature is the constant 0 or -1, respectively.*

This is a generalization of the CR Yamabe problem for the Webster scalar curvature as in Jerison and Lee [JE87], Schoen [SC95].

3.4 A Generalization of Schoen's Theorem

Let M be a strongly pseudoconvex almost CR manifold with a CR structure (H, J). Suppose that there is a contact sub-Riemannian structure (θ, g) on (M, H) which is *associated to the almost CR structure of M*, that is, every CR automorphism of (M, J) is a subconformal transformation of (M, θ, g). Note that if (θ, g) is associated to (M, H, J), then a sub-conformal change $(u^{2/n}\theta, u^{2/n}g)$ for positive u is also associated to the almost CR structure.

Suppose that M is noncompact. Then by (2) of Theorem 3.2, we may assume that $S \equiv -1$ or $S \equiv 0$ for (θ, g).

Case 1 ($S \equiv -1$): For each CR automorphism φ of M, let u_φ be the positive function with $(\varphi^*\theta, \varphi^*g) = (u_\varphi^{2/n}\theta, u_\varphi^{2/n}g)$ satisfies $(4(n+1)\Delta_b - 1)u_\varphi = -u_\varphi^{2/n+1}$ from (3.6) since φ is the isometry from $(\varphi^*\theta, \varphi^*g)$ to (θ, g). Using the self-adjoint property of Δ_b to non-negative test functions, we get that $\int u_\varphi^{(n+2)/n}$ is locally bounded uniformly for $\varphi \in \mathrm{Aut}_{\mathrm{CR}}(M)$. Using the mean value inequality for sub-elliptic operator $4(n+1)\Delta_b - 1$, one can conclude that u_φ is locally bounded uniformly for $\varphi \in \mathrm{Aut}_{\mathrm{CR}}(M)$; so the CR automorphism group of M acts properly by the Arzela-Ascoli theorem.

Case 2 ($S \equiv 0$): Let θ be a contact form with $S \equiv 0$. If $\mathrm{Aut}_{\mathrm{CR}}(M)$ acts improperly, then there are a compact subset K of M and a sequence $\varphi_k \in \mathrm{Aut}_{\mathrm{CR}}(M)$ such that $\varphi_k(K) \cap K \neq \emptyset$ and $\sup_K u_k \to \infty$ where $\varphi_k^*\theta = u_k^{2/n}\theta$. Equiation (3.6) to each u_k is

$$\Delta_b u_k = 0\ . \tag{3.7}$$

Using the normal coordinates for the orthogonal contact sub-Rimannian structure (θ, g) about a point $p_k \in K$ with $\varphi_k(p_k) \in K$, we take a small open neighborhood V_k of p_k such that $\inf_{V_k} u_k \to \infty$ by the sub-elliptic Harnack Principle to (3.7), so $\varphi_k(V_k)$ increasingly exhausts M by passing a subsequence.

Now consider $T = (T_{\beta}{}^{\alpha}{}_{\gamma})$, the torsion tensors in (3.2) for θ and let $T_k = \varphi_k^* T$ be the corresponding one for $\varphi_k^*\theta$. Since φ_k is the pseudo-hermitian isometry from $\varphi_k^*\theta$ to θ, we have $\left\| T_k \circ \varphi_k^{-1} \right\|_{\varphi_k^*\theta} = \|T\|_\theta$, where $\|\cdot\|_\theta$ and $\|\cdot\|_{\varphi_k^*\theta}$ stand for tensor norms with respect to the pseudo-hermitian metrices of θ and $\varphi_k^*\theta$, respectively.

Take any point $q \in M$. We shall consider sufficiently large k such that $q \in \varphi_k(V_k)$. For $q_k = \varphi_k^{-1}(q) \in V_k$, we have $\|T_k(q_k)\|_{\varphi_k^*\theta} = \|T(q)\|_\theta$. The transformation formula for T under the pseudo-conformal change $\varphi_k^*\theta = u_k^{2/n}\theta$ (Proposition 4.12 in [JO15]) gives

$$\|T_k(q_k)\|^2_{\varphi_k^*\theta} \le C u_k(q_k)^{-2/n} \left(\|T(q_k)\|_\theta^2 + \frac{1}{(nu_k)^2} \|d_b u(q_k)\|_\theta^2 \right)$$

where $\|d_b u_k\|_\theta$ is the holomorphic gradient norm of u_k with respect to θ. By the sub-elliptic Schauder estimates for (3.7), we have a uniform bound of $\|d_b u_k\|_\theta / n u_k$ on the relatively compact subset $\cup_k V_k$ of M. Since $u_k(q_k)^{-2/n} \le (\inf_{V_k} u_k)^{-2/n} \to 0$, we have that $\|T(q_k)\|_{\varphi_k^*\theta} \to 0$, so $\|T(q)\|_\theta = 0$. This means that $T_{\beta}{}^{\alpha}{}_{\gamma} = 0$ at q. Following the same manner, we have Condition (3.5) for θ, so get a local pseudo-hermitian equivalence to $\mathscr{H}_P$ by Theorem 3.1. Taking a local CR diffeomorphism F_k from V_k to $\mathscr{H}_P$ and a CR dilation $\Lambda_k(t, z) = (\tau_k t, \sqrt{\tau_k} z)$ of $\mathscr{H}_P$ for some $\tau_k \to \infty$, we have a global CR diffeomorphism $F : M \to \mathscr{H}_P$ as a subsequential limit of $\Lambda_k \circ F_k \circ \varphi_k^{-1}$.

Theorem 3.3 ([JO15]) *Let M be a noncompact, strongly pseudoconvex, almost CR manifold with an associated orthogonal contact sub-Riemannian structure. If the CR automorphism group of M acts on M improperly, then M is CR equivalent to a Heisenberg group* $\mathscr{H}_P$.

If M is compact and $\mathrm{Aut}_{\mathrm{CR}}(M)$ acts improperly, then by the same way of Schoen [SC95], we have a point $p \in M$ such that there is a CR diffeomorphism $F : M \setminus \{p\} \to \mathscr{H}_P$. Then we show that the CR automorphism $\mathscr{D}_s$ of $M \setminus \{p\} \simeq \mathscr{H}_P$ as in (3.4) extends to the CR automorphism of the whole M. Since $F^{-1}(0)$ is a fixed point of each $\mathscr{D}_s$, $\{\mathscr{D}_s : s \in \mathbb{R}\}$ acts improperly on $M \setminus \{F^{-1}(0)\}$ which contains p. Form Therorem 3.3 and the homogeneity of $\mathscr{H}_P$, there is a CR diffeomorphism $G : M \setminus \{F^{-1}(0)\} \to \mathscr{H}_P$ with $G(p) = 0$. Thus the CR automorphism $G \circ F^{-1}$ of $\mathscr{H}_P \setminus \{0\}$ which can not extend on $\mathscr{H}_P$. This contacts to Proposition 3.3 in [JO15] if $P \neq 0$.

Theorem 3.4 ([JO15]) *Let* M^{2n+1} *be a compact, strongly pseudoconvex, almost CR manifold with an associated orthogonal contact sub-Riemannian structure. If the CR automorphism group of M is noncompact, then M is CR equivalent to the standard sphere* S^{2n+1}.

If $\dim M = 5$ or 7, M always admits an associated orthogonal contact sub-Riemannian structure. Thus we can partially confirm Conjecture 1.1.

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