Chapter 20
Subjective Random Discounting and Intertemporal Choice

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Abstract This chapter provides an axiomatic foundation for a particular type of preference shock model called the *random discounting representation* where a decision maker believes that her discount factors change randomly over time. For this purpose, we formulate an infinite horizon extension of Dekel, Lipman, and Rustichini (Econometrica 69:891–934, 2001), and identify the behavior that reduces all subjective uncertainties to those about future discount factors. We also show uniqueness of subjective belief about discount factors. Moreover, a behavioral comparison about preference for flexibility characterizes the condition that one's subjective belief second-order stochastically dominates the other. Finally, the resulting model is applied to a consumption-savings problem.

Keywords Random discounting • Preference for flexibility • Subjective states

1 Introduction

1.1 Objective and Outline

In intertemporal decision making, a decision maker (DM) faces two kinds of trade-offs among alternatives. The first is a trade-off from the difference of alternatives within a time period and the second is an intertemporal trade-off between different

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time periods. Anticipating intertemporal trade-offs seems more difficult than anticipating trade-offs within a period. Thus, we consider a DM who is certain about ranking of alternatives within a period, and at the same time is uncertain about future intertemporal discount rates.

In addition, several authors have mentioned psychological reasons for uncertainty about discount factors. As Yaari (1965) and Blanchard (1985) point out, a discount factor admits an interpretation as a probability of death. Depending on the future prospects of diseases, armed conflicts, and discoveries in medical treatments, the probabilities of death will change over time. An alternative interpretation is to think not of an agent but of a dynasty, in which case a discount factor is regarded as a degree of altruism. The bequest motives of the current generations may fluctuate over time because they may die without descendants. Becker and Mulligan (1997) suggest a model in which a discount factor depends on the quantity of resources the DM invests into making future pleasures less remote. For example, schooling may focus students' attention on the future, and parents would spend resources on teaching their children to make better future plan. The choice of investment or effort level in these activities is affected by economic variables, for instance, interest rates or the DM's wealth, which are uncertain by their own nature. Thus, these uncertainties may lead to random discounting.[1]

Moreover, random discounting has been used in several macroeconomic models with infinite horizon since it is a useful device for generating heterogeneity among agents, in particular, for realistic wealth heterogeneity in quantitative models. For example, see Krusell and Smith (1998) and Chatterjee et al. (2007). However, preference shock to discounting is often postulated in an ad hoc way since shock cannot be observed directly by analysts. The reliance on these unobservable entities seems problematic.

In this chapter, we provide an axiomatic foundation for the random discounting model, in which the DM believes that her discount factors change randomly over time. Therefore, we demonstrate that there exists a behavior which can, in principle, pin down expected shocks to discount factors. For this purpose, we extend the two-period framework of Kreps (1979, 1992) and Dekel et al. (2001) (hereafter DLR) to an infinite horizon setting. They axiomatize a preference shock model by considering preference over menus (opportunity sets) of alternatives. If a DM is aware of uncertainties regarding her future preference over alternatives, then ranking of menus reflects how she perceives those uncertainties. Kreps and DLR derive the set of future preferences from the ranking of menus.

Behavioral characterization of random discounting shows that uncertainty about future preferences, whether it is about future discount factors or other aspects of preference, leads to a demand for flexibility—larger menus are preferred. This observation is made on the basis of Kreps and DLR. However, if uncertainty is only about discount factors, then flexibility has value only in limited cases. The

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[1]See Mehra and Sah (2002, Section 1.1, pp. 871–873) for more examples about fluctuations in subjective parameters.

behavioral characterization of random discounting takes the form of identifying primarily the instances where flexibility has no value (see the example that follows shortly for elaboration).

To analyze sequential decision making, we adopt the same domain of choice used by Gul and Pesendorfer (2004). Let C be the outcome space (consumption set), which is a compact metric space. There exists a compact metric space $\mathscr{Z}$ such that $\mathscr{Z}$ is homeomorphic to $\mathscr{K}(\Delta(C \times \mathscr{Z}))$, where $\Delta(C \times \mathscr{Z})$ is the set of lotteries, that is, all Borel probability measures over $C \times \mathscr{Z}$ and $\mathscr{K}(\cdot)$ denotes the set of all non-empty compact subsets of "$\cdot$". An element of $\mathscr{Z}$, called a menu, is an opportunity set of lotteries over pairs of current consumption and future menus. Preference $\succsim$ is defined on $\mathscr{Z} \simeq \mathscr{K}(\Delta(C \times \mathscr{Z}))$.

We consider the following timing of decisions: In period 0, the DM chooses a menu x. In period 1^-, current discount factor α becomes known to the DM, and she chooses a lottery l out of the menu x in period 1. In period 1^+, the DM receives a pair (c, x') according to realization of the lottery l, and consumption c takes place. The DM expects another discount factor α' to be realized in the following period. Subsequently, she chooses another lottery l' out of the menu x', and so on.

Notice that our primitive $\succsim$ is preference in period 0. Thus, beyond period 0, the time line in Fig. 20.1 is not part of the formal model. However, if the DM has in mind this time line and anticipates uncertain discount factors to be resolved over time, then $\succsim$ should reflect the DM's perception of those uncertainties. Hence, our domain can capture the expectation of random discounting.

We provide an axiomatic foundation for the following functional form, called the *random discounting representation*: there exists a non-constant, continuous, mixture linear function $u : \Delta(C) \to \mathbb{R}$, and a probability measure μ over $[0, 1]$ with $\mathbb{E}_\mu[\alpha] < 1$, such that $\succsim$ is represented numerically by the functional form,

$$U(x) = \int_{[0,1]} \max_{l \in x} \left((1 - \alpha) u(l_c) + \alpha \int_{\mathscr{Z}} U(z) \, dl_z \right) d\mu(\alpha),$$

where l_c and l_z are the marginal distributions of l on C and $\mathscr{Z}$, respectively.

The above functional form can be interpreted as follows: the DM behaves as if she has in mind the time line described above, and anticipates a discount factor α to be realized with probability μ in every time period. After the realization of α, the DM evaluates a lottery by the weighted sum of its instantaneous expected utility $u(l_c)$ and its expected continuation value $\int U(z) \, dl_z$. The same representation
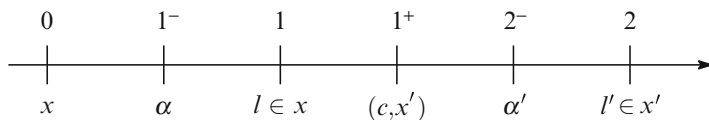

Fig. 20.1 Timing of decisions

U is used to evaluate menus at all times. Hence, the representation has a stationary recursive structure and the DM's belief about future discount factors is constant over time.

We show uniqueness of the DM's belief μ. That is, the components (u, μ) of the representation are uniquely derived from preference. This result is in stark contrast to that of Kreps (1979, 1992) and DLR. Since future preference is state-dependent in their model, arbitrary manipulations on subjective probabilities are possible. In our model, a future utility function over $\Delta(C \times \mathscr{Z})$ is state-dependent as in DLR (α corresponds to a subjective state). However, notice that both the instantaneous expected utility u and the utility U over menus are independent of subjective states, and state-dependent components, $1 - \alpha$ and α, add up to one. Hence, the representation cannot be maintained under arbitrary manipulations. A combination of the additive recursive structure and the normalization of discount factors ensures uniqueness.

Owing to uniqueness result, it is meaningful to compare subjective beliefs among agents. We provide a behavioral condition capturing a situation where one agent is more uncertain about discount factors than the other. For objective uncertainty, second-order stochastic dominance is widely used to describe such a comparison (Rothschild and Stiglitz 1970). If agent 2 perceives greater uncertainty about discount factors than agent 1, the former is more reluctant to make a commitment to a specific plan than the latter. This greater demand for flexibility is the behavioral manifestation of greater uncertainty about future discount factors. An implication of the behavioral comparison similar to DLR is also investigated and it shows contrasting results.

The resulting model is applied to a consumption-savings problem and it analyzes how uncertainty about discount factors affects savings behavior. By assuming an instantaneous utility function to be CRRA with the parameter $\sigma < 1$ (or $\sigma > 1$), savings rates increase (or decrease) when the DM becomes more uncertain about future discount factors in the sense of second-order stochastic dominance. This result can be interpreted based on the DM's attitude toward flexibility. It is noted that uncertainty about discount factors has an opposite effect on savings when compared to uncertainty about interest rates.

1.2 Motivating Example

To understand the behavior that characterizes the random discounting model, we consider a simple example as follows: Let C stand for a set of monetary payoffs.

Suppose that DM faces uncertainty about future discount factors. As pointed out by Kreps (1979) and DLR, she may keep her options open until the uncertainty is resolved, that is, she may exhibit preference for flexibility. On considering two alternatives, $(\$60, \{(\$100, z)\})$ and $(\$100, \{(\$50, z)\})$ in $\Delta(C \times \mathscr{Z})$, chosen in period 1, there might be a difference in consumption levels between periods 1 and 2. However, from period 3 onward, both alternatives guarantee the same

opportunity set z. We assume that $\{(\$60, \{(\$100, z)\})\} \succsim \{(\$100, \{(\$50, z)\})\}$. This ranking under commitment reflects the DM's ex ante perspective on random discount factors. The DM, on average, believes that she will be patient in period 1 and prefer $(\$60, \{(\$100, z)\})$ to $(\$100, \{(\$50, z)\})$. However, the DM may still prefer keeping $(\$100, \{(\$50, z)\})$ as an option if there is a possibility of becoming impatient in period 1, in which case $(\$100, \{(\$50, z)\})$ would be more attractive than $(\$60, \{(\$100, z)\})$. Hence, the DM may exhibit preference for flexibility as

$$\{(\$60, \{(\$100, z)\}), (\$100, \{(\$50, z)\})\} \succ \{(\$60, \{(\$100, z)\})\}$$
$$\succsim \{(\$100, \{(\$50, z)\})\}.$$

However, if the DM is uncertain *only* about future discount factors, then other forms of flexibility may not be valued. In that case, the DM would be sure of her preference over consumption in the next period (in this example, consumption is scalar, and hence, greater consumption is preferred to less), and also other preference over menus for the rest of the horizon. Therefore, uncertainty is relevant for rankings only when an intertemporal trade-off must be made, as in comparing $(\$60, \{(\$100, z)\})$ and $(\$100, \{(\$50, z)\})$, between consumption for period 1 and the menu for period 2 onward. Hence, some forms of flexibility are *not* valuable on uncertainty related to future discount factors.

To illustrate further, consider a lottery l, yielding

$$(\$0, \{(\$0, z)\}) \text{ or } (\$120, \{(\$200, z)\})$$

with an equal probability of one-half. For current consumption, l induces the lottery yielding \$0 or \$120 with a probability of one-half, while it induces the lottery over menus with an equal chance of $\{(\$0, z)\}$ or $\{(\$200, z)\}$. If \$60 and $\{(\$100, z)\}$ are both preferred to these induced lotteries respectively, then the DM does not face an intertemporal trade-off between $(\$60, \{(\$100, z)\})$ and l. Irrespective of how patient she will be in the next period, l will not be chosen over $(\$60, \{(\$100, z)\})$. Since there is no benefit in keeping l as an option with $(\$60, \{(\$100, z)\})$, the DM will exhibit

$$\{(\$60, \{(\$100, z)\}), l\} \sim \{(\$60, \{(\$100, z)\})\} \succsim \{l\}.$$

In a later section, we formally provide axioms consistent with these behavior.

1.3 Related Literature

1.3.1 Macroeconomics

Random discounting in a number of infinite-horizon macroeconomic models, where its role broadly appears, generates suitable heterogeneity across agents. Models of

wealth inequality based on standard and identical preferences and on uninsurable shocks to income can explain only a small part of the observed wealth inequality. Krusell and Smith (1998) consider shocks to discount factors and succeed in relating wealth heterogeneity predicted by the model to the observed data in the United States. Dutta and Michel (1998) use random discounting to model imperfect altruism to future generations, and derive a stationary wealth distribution where fewer agents hold higher levels of wealth. Karni and Zilcha (2000) prove that if the agents have random discount factors, in a steady-state competitive equilibrium, agents other than the most patient agents hold capital. This contrasts with the result in deterministic economies where only the most patient agents hold capital (see Becker 1980). Chatterjee et al. (2007) construct a general equilibrium model where agents with random discounting are allowed to default. They are able to match a default rate consistent with data partly because agents with low discount factors tend to consume more and default more frequently.

In models of monetary economics, random discounting also plays an important role. In a two-period model with random discounting, Goldman (1974) shows the possibility that an agent holds money that yields a lower interest than other interest-bearing assets. If the discount factor is random, after finding the discount factor, the agent may be willing to change her portfolio consisting of money and other assets. Since the transaction cost of money is lower than that of other assets, money allows the agent to change the portfolio more easily, and hence, can be valuable for the agent with random discounting.

Atkeson and Lucas (1992) and Farhi and Werning (2007) consider an intergenerational model, where each generation is composed of a continuum of agents who live for one-period and are altruistic to a descendant. Agents are ex ante identical but they experience taste-shocks to the degree of altruism (or discount factor), which are private information. The authors investigate the property of the second-best allocations of consumption. In these papers, agent's private information about taste-shocks is elicited through an incentive-compatible mechanism, while in the present chapter, the information is elicited indirectly from the observable behavior of the agent.

1.3.2 Axiomatic Models

To provide a foundation for random discounting, we follow studies of preference on the opportunity set approach. Koopmans (1964) first introduces an opportunity set as a choice object to model sequential decision making and emphasizes that intertemporal choice may be essentially different from a once-and-for-all decision making. He points out that if a DM perceives uncertainty about future preferences, she may strictly prefer to leave some options open rather than to choose a completely specified future plan.

Kreps (1979, 1992) interprets uncertain future preferences as subjective uncertainties of the DM, and provides an axiomatic foundation for the subjective state space. Dekel et al. (2001) refine Kreps's idea and show uniqueness of the

subjective state space. Furthermore, Dekel et al. (2007) modify the argument of DLR surrounding the additive representation with subjective states. In this line of research, our result can be viewed as an infinite-horizon extension of DLR, where the DM's subjective state space is specified to the set of sequences of discount factors.

Several authors provide sequential choice models consistent with preference for flexibility. Rustichini (2002) follows the same idea as DLR and considers closed subsets of C^∞ as choice objects. In this framework, all subjective uncertainties are resolved one period ahead. Kraus and Sagi (2006) follow the dynamic model of Kreps and Porteus (1978) and consider a sequence of preferences without the completeness axiom. Each incomplete preference is represented by the decision rule of the form, where one choice object is preferred to another if the former is unanimously preferred to the latter with respect to a set of uncertain future preferences. This uncertainty leads to preference for flexibility. Takeoka (2007) introduces objective states into DLR's model and considers preference over menus of menus of Anscombe-Aumann acts, which is viewed as a three-period extension of DLR. He derives a subjective decision tree and a subjective probability measure on it as components of representation.

2 Model

2.1 Domain

Let C be the outcome space (consumption set), which is assumed to be compact and metric. Let $\Delta(C)$ be the set of lotteries, that is, all Borel probability measures over C. Under the weak convergence topology, $\Delta(C)$ is also compact and metric. Gul and Pesendorfer (2004) show that there exists a compact metric space $\mathscr{Z}$ such that $\mathscr{Z}$ is homeomorphic to $\mathscr{K}(\Delta(C \times \mathscr{Z}))$, where $\mathscr{K}(\cdot)$ denotes the set of non-empty compact subsets of "$\cdot$".[2] Generic elements of $\mathscr{Z}$ are denoted by $x, y, z, \cdots$. Each such object is called a menu (or an opportunity set) of lotteries over pairs of current consumption and menu for the rest of the horizon.

Preference $\succsim$ is defined on $\mathscr{Z} \simeq \mathscr{K}(\Delta(C \times \mathscr{Z}))$. We have in mind the timing of decisions as mentioned in Fig. 20.1.

An important subdomain of $\mathscr{Z}$ is the set $\mathscr{L}$ of perfect commitment menus where DM is committed in every period. We identify a singleton menu with its only element. Then a perfect commitment menu can be viewed as a multistage lottery, considered by Epstein and Zin (1989), that is, $\mathscr{L}$ is a subdomain of $\mathscr{Z}$ satisfying $\mathscr{L} \simeq \Delta(C \times \mathscr{L})$. A formal treatment is found in Appendix section "Perfect Commitment Menus".

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[2]The set $\mathscr{K}(\Delta(C \times \mathscr{Z}))$ is endowed with the Hausdorff metric. Details are relegated to Appendix section "Hausdorff Metric".

The following examples illustrate that the recursive domain $\mathscr{Z}$ can accommodate sequential decision problems.

Example 1 (Consumption-Savings Problem) Given a constant interest rate $r > 0$ and an initial savings $s > 0$, DM decides a current consumption c and savings s' carried over to the next period within the wealth of $(1 + r)s$. That is, the DM faces the budget constraint,

$$B(s) = \{(c, s') \in \mathbb{R}_+^2 \,|\, c + s' = (1 + r)s\},$$

which is translated into the menu $x(s) = \{(c, x(s'))| (c, s') \in B(s)\}$. If r is random, $x(s)$ can be modified to a set of lotteries.

Example 2 (Durable Goods) A durable good provides flow of services, $c = \{c_t\}_{t=1}^{\infty}$ over certain time periods. The durability of goods depends both on the physical property and intensive use of goods. Let $\{c = \{c_t\}_{t=1}^{\infty} | f(c) \leq 0\}$ be the feasible set of flow of services associated with a durable good, where f is a technology frontier. Given the history of consumptions up to period $t - 1$, denoted by $c^{t-1} = (c_1, \cdots, c_{t-1})$, c_t is a feasible consumption in period t if and only if $f(c^{t-1}, c_t, c) \leq 0$ for some c. In other words, the DM faces the menu as

$$x_f(c^{t-1}) = \{(c_t, x_f(c^{t-1}, c_t)) \,|\, f(c^{t-1}, c_t, c) \leq 0 \text{ for some sequence c}\}.$$

The DM prefers one durable good f to another g if and only if $x_f(c^0) \succsim x_g(c^0)$.

Example 3 (Sampling Problem) Given a wage offer w, which is a random sample from a distribution F, a DM has to decide whether to accept or reject the offer. If the DM accepts w, she receives current payoff from w and nothing for the rest of the horizon. Otherwise, the DM continues sampling, and receives a new random sample w' in the next period. Sampling is repeated until the DM accepts an offer. This decision problem can be described formally as follows: given a current offer w, define the menu $x(w) \equiv \{\text{accept} w, \text{continue}\}$. The object "accept w" is the consumption stream $(w, \{(0, \{(0, \cdots)\})\})$, and "continue" is the lottery over menus of the form $x(w') = \{\text{accept} w', \text{continue}\}$, where w' is given according to the distribution F.

2.2 Random Discounting Representations

By taking any non-constant, continuous, mixture linear function $u : \Delta(C) \to \mathbb{R}$ and any Borel probability measure μ over $[0, 1]$ with the mean $\bar{\alpha} \equiv \mathbb{E}_\mu[\alpha] < 1$ we consider the functional form $U : \mathscr{Z} \to \mathbb{R}$ defined as

$$U(x) = \int_{[0,1]} \max_{l \in x} \left((1 - \alpha)u(l_c) + \alpha \int_{\mathscr{Z}} U(z) \, \mathrm{d}l_z \right) \mathrm{d}\mu(\alpha), \qquad (20.1)$$

where l_c and l_z denote the marginal distributions of l on C and $\mathscr{Z}$, respectively.

The functional form (20.1) can be interpreted as follows: the DM behaves as if she has in mind the time line described in Fig. 20.1 and anticipates uncertainty about discount factors, which is captured by μ over $[0, 1]$. On the other hand, she is certain about future risk preference, u over $\Delta(C)$. Moreover, she is also certain about future ranking of menus, which is identical with the current ranking U. That is, the representation has a stationary and recursive structure. After considering the realization of discount factor α, the DM chooses a lottery out of the menu to maximize the "ex post" utility function,

$$(1 - \alpha)u(l_c) + \alpha \int_{\mathscr{Z}} U(z) \, dl_z, \tag{20.2}$$

which is the weighted sum of expected utilities from current consumption and the opportunity set for the rest of the horizon. The functional form (20.1) states that the DM evaluates a menu x by taking the expected value of these maximum values with respect to her subjective belief μ over discount factors.

Definition 1 Preference $\succsim$ on $\mathscr{Z}$ admits a random discounting representation if $\succsim$ can be represented numerically by the functional form U as given by (20.1) with components (u, μ).

A random discounting representation coincides with a stationary cardinal utility function on the subdomain $\mathscr{L}$ of perfect commitment menus. Since the DM has no opportunity for choice, random discounting does not matter on this subdomain. The functional form (20.1) reduces to

$$U(l) = (1 - \bar{\alpha})u(l_c) + \bar{\alpha} \int_{\mathscr{L}} U(l') \, dl_L(l'),$$

where l_c and l_L denote the marginal distributions of $l \in \Delta(C \times \mathscr{L})$ on C and $\mathscr{L}$, respectively. This is a standard stationary recursive utility with a deterministic discount factor $\bar{\alpha} < 1$.

Apart from the difference of choice objects,[3] a random discounting representation is a special case of DLR's additive representation of the form that

$$U(x) = \int_S \max_{l \in x} V(l, s) \, d\mu(s), \tag{20.3}$$

where S is a state space, μ is a non-negative measure on S, and $V(\cdot, s)$ is a state-dependent expected utility function. Indeed, the ex post utility function (20.2) can be written as

$$V(l, s) = (1 - \alpha(s))u(l_c) + \alpha(s) \int_{\mathscr{Z}} U(z) \, dl_z \tag{20.4}$$

with an index $s \in S$.

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[3]DLR consider preference over $\mathscr{K}(\Delta(C))$ with finite set C.

DLR have a model of the form (20.3) with a signed measure μ, where choice based on some subjective states may be negatively evaluated from the ex ante perspective. The DM having such a representation does not necessarily desire flexibility. Similarly, it is possible to consider functional form (20.1) by assuming μ as a signed measure. For example, Gul and Pesendorfer (2004) correspond to the case where μ has one "regular" discount factor $\alpha > 0$ with a positive weight and a completely myopic discount factor $\alpha = 0$ with a negative weight.[4] In this chapter, however, we do not focus on this general model. First, unless some further restriction is imposed such as in Gul and Pesendorfer (2004), a model with a signed measure does not necessarily have a clear implication of how choice behavior evolves over time. Such a model is not appropriate as a dynamic model, while the random discounting representation can generate a stochastic choice according to the probability measure. Second, as Koopmans (1964) and Kreps (1979) point out, a dynamic model consistent with preference for flexibility is of its own interest.

3 Foundations

3.1 Axioms

The axioms which we consider on $\succsim$ are the following. The first two axioms are standard and need no explanation.

Axiom 1 (Order) $\succsim$ is complete and transitive.

Axiom 2 (Continuity) For all $x \in \mathscr{Z}$, $\{z \in \mathscr{Z} | x \succsim z\}$ and $\{z \in \mathscr{Z} | z \succsim x\}$ are closed.

For any $l \in \Delta(C \times \mathscr{Z})$, l_c and l_z denote the marginal distributions of l on C and on $\mathscr{Z}$, respectively.

Axiom 3 (Nondegeneracy) There exist $l, l' \in \Delta(C \times \mathscr{Z})$ such that $l_c \neq l'_c$, $l_z = l'_z$, and $\{l\} \succ \{l'\}$.

The lotteries l and l' differ only in the distribution of current consumption. Thus, strict preference for l over l' presumably reveals that the DM's risk preference over C is not constant.

The next three axioms are the same as those in Gul and Pesendorfer (2004).

Axiom 4 (Commitment Independence) For all $l, l', l'' \in \Delta(C \times \mathscr{Z})$ and for all $\lambda \in (0, 1)$,

$$\{l\} \succ \{l'\} \implies \{\lambda l + (1 - \lambda)l''\} \succ \{\lambda l' + (1 - \lambda)l''\}.$$

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[4]A sophisticated DM, who is fully aware of time-inconsistency caused by hyperbolic discounting, may be viewed as a limiting case of their model, where the DM never exercises self-control at the moment of choice.

For all $l, l' \in \Delta(C \times \mathscr{Z})$, $\{l\} \succsim \{l'\}$ means commitment preference, which reflects the DM's ex ante perspective on future preference over lotteries. Axiom 4 states that commitment preference over lotteries satisfies the vNM independence.

Axiom 5 (Stationarity) For all $x, y \in \mathscr{Z}$ and $c \in C$, $\{(c, x)\} \succsim \{(c, y)\} \Leftrightarrow x \succsim y$.

Since current consumption is the same, the ranking between $\{(c, x)\}$ and $\{(c, y)\}$ reflects how the DM evaluates x and y in the next period. Thus, Stationarity means that the ranking over menus is identical across time.

In general, belief about future discount factors may depend on the history of consumptions and realizations of discount factors up to that period. Stationarity, however, excludes such history-dependent beliefs: the DM is sure that her belief about discount factors will not change over time. We adopt Stationarity because it seems sensible as a first step and because the general model seems much more difficult to characterize and is beyond our grasp at this time.

For any $(c, x), (c', x') \in C \times \mathscr{Z}$ and $\lambda \in [0, 1]$, the notation

$$\lambda \circ (c, x) + (1 - \lambda) \circ (c', x')$$

denotes the lottery over $C \times \mathscr{Z}$ yielding (c, x) with probability λ and yielding (c', x') with probability $1 - \lambda$.

For any $x, x' \in \mathscr{Z}$ and $\lambda \in [0, 1]$, define the mixture of two menus by considering the mixtures element by element between x and x', that is,

$$\lambda x + (1 - \lambda)x' \equiv \{\lambda l + (1 - \lambda)l' | l \in x, \ l' \in x'\} \in \mathscr{Z}.$$

If the DM identifies a two-stage lottery $\lambda \circ l + (1 - \lambda) \circ l'$ with its reduced lottery $\lambda l + (1 - \lambda)l'$, $\lambda x + (1 - \lambda)x'$ can also be viewed as a set of two-stage lotteries.

Axiom 6 (Timing Indifference) For all $x, x' \in \mathscr{Z}$, $c \in C$, and $\lambda \in (0, 1)$,

$$\{\lambda \circ (c, x) + (1 - \lambda) \circ (c, x')\} \sim \{(c, \lambda x + (1 - \lambda)x')\}.$$

Notice that $\lambda \circ (c, x) + (1-\lambda) \circ (c, x')$ is the lottery yielding (c, x) with probability λ and yielding (c, x') with probability $1-\lambda$, while $(c, \lambda x + (1-\lambda)x')$ is the degenerate lottery that assigns the pair of consumption c and menu $\lambda x + (1 - \lambda)x'$ of two-stage lotteries. Hence, these two lotteries differ in timing of resolution of randomization λ. For the former, the DM makes choice out of a menu (either x or x') after the resolution of λ, while for the latter, this order is reversed, that is, the choice out of the menu $\lambda x + (1-\lambda)x'$ is made before the resolution of λ. Timing Indifference suggests that the DM does not care about this difference in timing. Timing Indifference can be justified by the same argument as in DLR. Suppose that a DM is uncertain about future preference over $\Delta(C \times \mathscr{Z})$, yet she surely anticipates that it will satisfy the expected utility axioms. Let l and l' be a rational choice from x and x', respectively, with respect to a future preference. Therefore, $(c, \lambda l + (1 - \lambda)l')$

is the expected choice from $\lambda \circ (c, x) + (1 - \lambda) \circ (c, x')$. On the other hand, if the future preference satisfies the expected utility axioms, $(c, \lambda l + (1 - \lambda)l')$ is a rational choice from $(c, \lambda x + (1 - \lambda)x')$ as well. Therefore, irrespective of the future preference, the two menus will ensure indifferent consequences.[5]

Axioms 1, 2, and 4–6 appear in Gul and Pesendorfer (2004).[6] They consider a DM facing self-control problems. Such a DM may be better off by restricting available options and, hence, exhibits preference for commitment rather than for flexibility. A key axiom of their model is called *Set Betweenness*: for any $x, y \in \mathscr{Z}$,

$$x \succsim y \implies x \succsim x \cup y \succsim y.$$

Even if $x \succsim y$, the DM may rank x over $x \cup y$ because y may contain a tempting option and choosing from $x \cup y$ may require costly self-control to the DM.

We adopt the following two axioms, which distinguish our model from theirs. As mentioned in Sect. 1.2, the DM facing uncertainty about her future preferences may want to keep options open as much as possible. This is because flexibility allows the DM to make a decision contingent upon realization of her future preference. This informational advantage leads to preference for flexibility. Such a DM would rank $x \cup y$ over x even though $x \succsim y$. To accommodate such behavior, we follow Kreps and DLR, and assume (instead of Set Betweenness):

Axiom 7 (Monotonicity) For all $x, y \in \mathscr{Z}$, $y \subset x \Rightarrow x \succsim y$.

This axiom states that a bigger menu is always weakly preferred. That is, Monotonicity is consistent with preference for flexibility.[7]

Monotonicity is consistent with any kind of uncertainty about future preferences. To identify behavior that reduces uncertainty about future preferences to that about future discount factors, we need to impose a qualification on the attitude toward flexibility. The DM facing random discount factors is sure how she evaluates consumption in the next period and a menu from that time period onward. Thus, uncertainty is relevant only when an intertemporal trade-off must be made. As mentioned in Sect. 1.2, such a DM should not value flexibility provided by "dominated lotteries", which are now described formally.

Let $l_c \otimes l_z$ denote the product measure on $C \times \mathscr{Z}$ that consists of marginal distributions $l_c \in \Delta(C)$ and $l_z \in \Delta(\mathscr{Z})$. We define dimension-wise dominance as follows:

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[5] The DM may care about timing of resolution of risk and prefer earlier or later resolution of multistage lotteries. Such distinction is examined in Kreps and Porteus (1978). Epstein et al. (2007) argue against Timing Indifference and provide a model with nonlinear future preferences.

[6] Their Nondegeneracy axiom requires the existence of menus x, y with $x \succ y$ and $x \subset y$. That is, this axiom captures preference for commitment—a DM may prefer a smaller menu.

[7] A sophisticated DM with hyperbolic discounting exhibits preference for commitment rather than for flexibility. Thus, such a DM is excluded by this axiom.

Definition 2 For all $l, l' \in \Delta(C \times \mathscr{Z})$, l dominates l' if $\{l_c \otimes l'_z\} \succsim \{l'_c \otimes l'_z\}$ and $\{l'_c \otimes l_z\} \succsim \{l'_c \otimes l'_z\}$, where l_c (resp. l_z) denotes the marginal distribution of l on C (resp. $\mathscr{Z}$).

If the DM is certain about her risk preferences over $\Delta(C)$ and over $\Delta(\mathscr{Z})$ in future, the commitment rankings appearing in the above definition should reflect those preferences. Since $l_c \otimes l'_z$ and $l'_c \otimes l'_z$ differ only in marginal distributions on C, the ranking $\{l_c \otimes l'_z\} \succsim \{l'_c \otimes l'_z\}$ reflects that l_c is preferred to l'_c in terms of the future risk preference over $\Delta(C)$. Similarly, the ranking $\{l'_c \otimes l_z\} \succsim \{l'_c \otimes l'_z\}$ should reveal the DM's future preference for l_z over l'_z. If l dominates l', the marginal distributions of l on C and $\mathscr{Z}$ are both preferred to those of l'. Hence, l will definitely be chosen over l' by the DM who is certain about her future risk preferences over C and $\mathscr{Z}$.

For any $l \in \Delta(C \times \mathscr{Z})$, let $O(l)$ be the set of all lotteries dominated by l, that is,

$$O(l) \equiv \{l' \in \Delta(C \times \mathscr{Z}) \,|\, l \text{ dominates } l'\}. \tag{20.5}$$

If $\succsim$ satisfies Order, $l \in O(l)$. Thus, a DM having preference for flexibility weakly prefers $O(l)$ to $\{l\}$. However, there is no reason to choose a dominated lottery $l' \in O(l)$ over l. Hence, $O(l)$ should be indifferent to $\{l\}$.

The same intuition should hold between a general menu x and the set

$$O(x) \equiv \bigcup_{l \in x} O(l), \tag{20.6}$$

that is, $O(x)$ is the set of all lotteries dominated by some lottery in x. Lemma 3 (i) in Appendix section "Proof of Theorem 1" shows that $O(x)$ is a well-defined choice object, that is, $O(x) \in \mathscr{Z}$ for all x.

Axiom 8 (Marginal Dominance) For all $x \in \mathscr{Z}$, $x \sim O(x)$.

Marginal Dominance states that the DM should not care about dominated lotteries. Since $x \subset O(x)$ when $\succsim$ satisfies Order, the DM having preference for flexibility weakly prefers $O(x)$ to x. Thus, this axiom is a counterpoint to Monotonicity, and shows that it is not useful to keep dominated lotteries within the menu, that is, $x \succsim O(x)$. Such behavior can be justified if the DM believes that her future risk preference over $\Delta(C)$ is separated from her future ranking of menus, and these two preferences are known to the DM without uncertainty. Then, dominated lotteries are definitely useless because they give less utilities in the future, both immediate and remote, and hence the DM exhibits $x \sim O(x)$.

Marginal Dominance involves a form of separability of preferences between immediate and remote future. Two remarks are in order: First, under this axiom, the DM cares only about the marginal distributions on C and $\mathscr{Z}$—the correlation between immediate consumption and the future opportunity set does not matter. Second, Marginal Dominance is stronger than the Separability axiom stated by Gul and Pesendorfer (2004), which requires a form of separability only in the singleton sets. That is, for any $c, c' \in C$ and $x, x' \in \mathscr{Z}$,

$$\left\{ \frac{1}{2} \circ (c, x) + \frac{1}{2} \circ (c', x') \right\} \sim \left\{ \frac{1}{2} \circ (c', x) + \frac{1}{2} \circ (c, x') \right\}.$$

If $\succsim$ satisfies Marginal Dominance, for all $l \in \Delta(C \times \mathscr{Z})$, we have $\{l\} \sim O(l) = O(l_c \otimes l_z) \sim \{l_c \otimes l_z\}$. Thus, both the above singleton menus are indifferent to

$$\left\{ \left(\frac{1}{2} \circ c + \frac{1}{2} \circ c' \right) \otimes \left(\frac{1}{2} \circ x + \frac{1}{2} \circ x' \right) \right\},$$

which results in Separability.

3.2 Representation Results

It is now appropriate to state the main theorem.

Theorem 1 *If preference $\succsim$ satisfies Order, Continuity, Nondegeneracy, Commitment Independence, Stationarity, Timing Indifference, Monotonicity, and Marginal Dominance, then there exists a random discounting representation (u, μ).*

Conversely, for any pair (u, μ) with $\bar{\alpha} < 1$, there exists a unique functional form U that satisfies functional equation (20.1) and the preference it represents satisfies all the axioms.

The above theorem is closely related to DLR's study, and the role of the axioms may be well understood when compared with their axioms. DLR show that preference over menus of lotteries admits the additive representation (20.3) with a non-negative measure if and only if it satisfies Order, Continuity, Monotonicity, and the following axiom[8]:

Axiom 9 (Independence) For all x, y, z and $\lambda \in (0, 1]$,

$$x \succ y \implies \lambda x + (1 - \lambda)z \succ \lambda y + (1 - \lambda)z.$$

Indeed, Commitment Independence, Stationarity, and Timing Indifference imply Independence.[9] Marginal Dominance plays a key role in restricting subjective states (future preferences) to differ only in intertemporal trade-offs between the immediate and remote future. The recursive form of the representation is due to Stationarity. For an outline of the proof of sufficiency, see Sect. 3.4. A formal proof is relegated to Appendix section "Proof of Theorem 1".

According to the above argument, a natural strategy to obtain a random discounting representation would be to establish the additive representation (20.3) on

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[8]Dekel et al. (2007) fill a gap in DLR surrounding this representation result.

[9]See Gul and Pesendorfer (2004, p. 125, footnote 7) for more details.

$\mathcal{K}(\Delta(C \times \mathcal{Z}))$, and to manipulate the representation to convert the subjective state space S to the set of discount factors $[0, 1]$ using the additional axioms (especially Marginal Dominance). However, we do not follow this strategy mainly because the first step is not immediate: DLR consider menus of lotteries over finite alternatives as choice objects and, hence, can regard the compact set of expected utility functions over lotteries as the subjective state space, while in this chapter, choice objects are menus of lotteries over a compact set. Thus, instead of dealing with the set of all mixture linear functions over the compact set, we start off with the subjective state space $[0, 1]$ of discount factors and establish our functional form by adapting DLR's argument. For an outline of the proof of sufficiency, see Sect. 3.4.

The next result considers uniqueness of the representation. If preference admits two distinct random discounting representations, say (u, μ) and (u', μ'), we cannot know which belief actually captures the DM's subjective uncertainty about discount factors. Therefore, we have the following uniqueness result. A proof can be found in Appendix section "Proof of Theorem 2".

Theorem 2 *If two random discounting representations, U and U', with components* (u, μ) *and* (u', μ') *respectively, represent the same preference, then:*

 (i) u and u' are cardinally equivalent; and
(ii) $\mu = \mu'$.

Theorem 2 pins down a subjective probability measure μ over the set of future discount factors, which is interpreted as the set of subjective states of the DM. Our result is in contrast to Kreps (1979, 1992) and DLR, where probability measures over subjective states are not identified; since the ex post utility functions are state-dependent as shown in (20.3), probabilities assigned to those states can be manipulated arbitrarily. Formally, let ν be a probability measure which is absolute continuous with respect to μ. Then there is a function f such that $V'(l, s) = V(l, s)f(s)$ and

$$U(x) = \int_S \max_{l \in x} V'(l, s) \, d\nu(s),$$

which means that μ cannot be identified. On the other hand, our representation has an additive recursive structure, that is, the ex post utility functions are specified as shown in (20.4). Notice that both the instantaneous expected utility u and the utility U over menus are independent of subjective states, and state-dependent components, $1-\alpha(s)$ and $\alpha(s)$, add up to one for all s. Under the above manipulation, the representation is maintained only when $f(s) = 1$ almost surely, and thus $\nu = \mu$. Therefore, it is a combination of the additive recursive structure and the normalization of discount factors that ensures uniqueness of subjective beliefs.[10]

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[10]To prevent arbitrary manipulations, DLR (p. 912) suggest that probability measures can be identified if some aspect of the ex post utility functions is state-independent. Such a condition is satisfied in our model.

3.3 Special Case: Deterministic Discounting

We imagine a "standard" DM with deterministic discounting, who is not anticipating any uncertainty about future discount factors. Such a DM should not care about flexibility, and should evaluate a menu by its best element according to a fixed weak order over singleton sets. That is,

$$x \succsim y \Leftrightarrow \{l^x\} \succsim \{l^y\}, \tag{20.7}$$

where $\{l^x\} \succsim \{l\}$ and $\{l^y\} \succsim \{l'\}$ for all $l \in x$ and $l' \in y$. Kreps (1979) characterizes such a standard DM based on the next axiom:

Axiom 10 (Strategic Rationality) For all $x, y \in \mathscr{Z}$, $x \succsim y \Rightarrow x \sim x \cup y$.

Strategic Rationality states that as long as x is preferred to y, the DM does not care whether options in y are added into x or not. This axiom is more restrictive than Monotonicity, and excludes preference for flexibility.[11]

Strategic Rationality is not enough to characterize deterministic discounting because it does not impose any restriction on the commitment ranking. The next axiom requires the dimension-wise dominance on singleton sets.

Axiom 11 (Commitment Marginal Dominance) For all $l \in \Delta(C \times \mathscr{Z})$, $\{l\} \sim O(l)$.

This axiom is weaker than Marginal Dominance, but the intuition is the same as before.

The next corollary of Theorem 3.1 characterizes deterministic discounting. Appendix section "Proofs of Corollary 1 and Proposition 1" can be referred for a proof.

Corollary 1 *Preference $\succsim$ satisfies Order, Continuity, Nondegeneracy, Commitment Independence, Stationarity, Timing Indifference, Strategic Rationality, and Commitment Marginal Dominance if and only if $\succsim$ admits a random discounting representation (u, μ) such that μ is degenerate.*

As mentioned above, Strategic Rationality implies Monotonicity. To verify that the set of axioms in Corollary 1 implies Marginal Dominance, we provide a further perspective on Strategic Rationality. A standard DM, who surely anticipates her preference in the next period, will rank menus according to the decision rule (20.7). Consequently, she should be indifferent between committing to a lottery $l \in \Delta(C \times \mathscr{Z})$ and having its "lower contour set"

$$O^*(l) \equiv \{l' \in \Delta(C \times \mathscr{Z}) \mid \{l\} \succsim \{l'\}\},$$

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[11]Strategic Rationality implies Monotonicity. Indeed, assume $y \subset x$. Arguing by contradiction, suppose $y \succ x$. Strategic Rationality implies $x = x \cup y \sim y \succ x$, which is a contradiction.

which is the set of all lotteries that are no more desired than l with respect to commitment ranking. Accordingly, arranging $O^*(x) \equiv \cup_{l \in x} O^*(l)$ for all $x \in \mathcal{Z}$, the standard DM will satisfy the next axiom:

Axiom 12 (Dominance) For all $x \in \mathcal{Z}$, $x \sim O^*(x)$.

This axiom states that DM does not care about keeping a lottery which is no more desired than some lottery in the menu in terms of commitment ranking. Even if $\{l\} \succsim \{l'\}$, the support of l may be different from that of l', that is, these lotteries may differ in intertemporal trade-offs, and hence, the DM facing uncertainty about discount factors may be better off by keeping l' as an option. However, Dominance implies that if $\{l\} \succsim \{l'\}$, the DM surely anticipates not to choose l' in the next period, and does not care about flexibility regarding intertemporal trade-offs in the future. Hence, this axiom is a necessary condition for deterministic discounting.

As one might imagine, Dominance has close relations to Strategic Rationality and Marginal Dominance.

Proposition 1 *Assume that $\succsim$ satisfies Order and Continuity.*

(i) Strategic Rationality is equivalent to Dominance.
(ii) Dominance and Commitment Marginal Dominance imply Marginal Dominance.

See Appendix section "Proofs of Corollary 1 and Proposition 1" for a proof. From this proposition, Strategic Rationality and Commitment Marginal Dominance together with the other axioms imply the set of axioms in Theorem 1.

3.4 Proof Sketch for Sufficiency of Theorem 1

As mentioned in Sect. 3.2, Commitment Independence, Stationarity, and Timing Indifference imply Independence. Focusing on the subdomain $\mathcal{Z}_1 \subset \mathcal{Z}$ consisting of convex menus, the mixture space theorem delivers a mixture linear representation $U : \mathcal{Z}_1 \to \mathbb{R}$. We have to show that U can be rewritten as the desired form.

Marginal Dominance implies that the DM is certain about her future risk preferences over C and $\mathcal{Z}$. Let $u : \Delta(C) \to \mathbb{R}$ and $W : \Delta(\mathcal{Z}) \to \mathbb{R}$ be

$$u(l_c) \equiv U(\{l_c \otimes \underline{l}_z\}) \text{ and } W(l_z) \equiv U(\{\underline{l}_c \otimes l_z\}),$$

where $\underline{l} \in \Delta(C \times \mathcal{Z})$ is a minimal lottery in terms of commitment ranking. These two functions should represent those future preferences.

Monotonicity captures preference for flexibility, which presumably reflects uncertainty about future preferences. Since u and W are sure for the DM, all the uncertainties about future preferences are effectively reduced to those about future discount factors. The DM should expect her future preference over $\Delta(C \times \mathcal{Z})$ to have the form of

$$(1 - \alpha)u(l_c) + \alpha W(l_z),$$

where $\alpha \in [0, 1]$ is a subjective weight between u and W.

We identify a menu x with its "support function" $\sigma_x : [0, 1] \to \mathbb{R}$, defined by

$$\sigma_x(\alpha) \equiv \max_{l \in x} (1 - \alpha)u(l_c) + \alpha W(l_z), \text{ for all } \alpha \in [0, 1].$$

That is, $\sigma_x = \sigma_y \iff x = y$. This identification ensures that the mapping σ embeds the set of menus into the space of real-valued continuous functions on $[0, 1]$, and hence, the functional $V(f) = U(\sigma^{-1}(f))$ is well-defined on the image of σ. Following the similar argument by DLR (and Dekel et al. 2007), we show that there exists a unique probability measure μ over $[0, 1]$ such that $V(f)$ can be written as $\int f(\alpha) \, d\mu(\alpha)$, and hence,

$$U(x) = V(\sigma_x) = \int_{[0,1]} \left(\max_{l \in x} (1 - \alpha)u(l_c) + \alpha W(l_z) \right) d\mu(\alpha).$$

The remaining step is to show that U has a stationary and recursive form as desired. Since $W(l_z)$ is a mixture linear function, it has the expected utility form $\int_{\mathscr{Z}} W(z) \, dl_z$. By Stationarity, W and U must represent the same preference. Moreover, Timing Indifference implies that W is mixture linear with respect to the mixture operation over menus. Hence, $W(z)$ can be written as an affine transformation of $U(z)$. Manipulating the functional form appropriately, we obtain the desired representation. Finally, Continuity and Nondegeneracy imply $\bar{\alpha} < 1$.

4 Greater Demand for Flexibility and Greater Uncertainty

We would like to analyze the situation where one agent is more uncertain about discount factors than another. We provide behavioral comparisons about preference for flexibility and characterize intuitive properties of subjective beliefs.

Consider two agents: Agent i has preference $\succsim^i$ on $\mathscr{Z}$, $i = 1, 2$. Since we are interested in comparing preference for flexibility, we focus on agents having identical commitment rankings. Recall that $\mathscr{L}$ is the set of multistage lotteries. If an element of $\mathscr{L}$ is chosen, there remains no opportunity for choice over the rest of the horizon. We say that $\succsim^1$ and $\succsim^2$ are equivalent on $\mathscr{L}$ if, for all $l, l' \in \mathscr{L}$,

$$\{l\} \succ^1 \{l'\} \iff \{l\} \succ^2 \{l'\}.$$

A DM's preference for flexibility is captured by the Monotonicity axiom, that is, the DM prefers bigger menus. Hence, one can say that agent 2 has greater demand for flexibility than agent 1 if agent 2 strictly prefers a bigger menu whenever agent 1 does so. Such a behavioral comparison is provided by DLR. Formally:

Definition 3 Agent 2 *desires more flexibility than* agent 1 if, for all $x, y \in \mathcal{Z}$ with $y \subset x, x \succ^1 y \Rightarrow x \succ^2 y$.

In a two-period model, DLR show that, under Definition 3, the subjective state space of agent 2 is bigger than that of agent 1. That is, greater demand for flexibility reflects greater uncertainty about future contingencies. In particular, in the case of their additive representation, the subjective state space corresponds to a support of non-negative measure. By analogy of DLR, one might expect that greater demand for flexibility reflects a bigger support of the subjective belief about discount factors.

Since a menu in $\mathcal{Z}$ is an infinite horizon decision problem, preference for flexibility on $\mathcal{Z}$ reflects the agent's belief about sequence of discount factors over the rest of the horizon, while μ is her belief about discount factors only in the immediate future. To obtain a characterization result, it is relevant to specify preference for flexibility attributable solely to belief about the immediate future. To formalize the idea, define "the two-period domain" as

$$\mathcal{Z}^1 \equiv \mathcal{K}(\mathcal{L}) \subset \mathcal{Z}.$$

Holding a menu $x \in \mathcal{Z}^1$, the agent can postpone a decision only until period 1, from which point on she has to make a commitment. A comparison of preference for flexibility on $\mathcal{Z}^1$ corresponds to DLR's two-period case.

Definition 4 Agent 2 *desires more flexibility in the two-period model than* agent 1 if, for all $x, y \in \mathcal{Z}^1$ with $y \subset x, x \succ^1 y \Rightarrow x \succ^2 y$.

The next theorem is a counterpart of DLR. A proof is found in Appendix section "Proof of Theorem 3".

Theorem 3 *Assume that $\succsim^i$, $i = 1, 2$, satisfy all the axioms of Theorem 1 and are equivalent on $\mathcal{L}$. The following conditions are equivalent:*

(a) Agent 2 desires more flexibility in the two-period model than agent 1.
(b) There exist random discounting representations U^i with (u^i, μ^i), $i = 1, 2$, such that (i) $u^1 = u^2$, and (ii) the support of μ^2 set-theoretically includes that of μ^1.

Two remarks are in order: First, although μ^i is constant over time, condition (b) does not imply that agent 2 desires more flexibility than agent 1. Indeed, under condition (b), we may find two menus z and z' such that $U^1(z) > U^1(z')$ and $U^2(z) \leq U^2(z')$. Considering $x \equiv \{(c, z), (c, z')\}$ and $y \equiv \{(c, z')\}$ for some $c \in C$, we have $y \subset x$ and $U^1(x) > U^1(y)$, yet $U^2(x) = U^2(y)$. Second, although our functional form is a special case of DLR's, Theorem 3 does not follow directly from Theorem 2 (p. 910) of DLR regarding the characterization of Definition 4. DLR consider menus of lotteries over finite outcomes, while in our study, choice objects are menus of lotteries on a compact outcome space.

We next consider another behavioral comparison about preference for flexibility. If agent 2 faces more uncertainty about discount factors than agent 1, agent 2 is presumably more averse to making a commitment to a specific plan than agent 1 is. That is,

Definition 5 Agent 2 is *more averse to commitment than* agent 1 if, for all $x \in \mathscr{X}$ and $l \in \mathscr{L}$, $x \succ^1 \{l\} \Rightarrow x \succ^2 \{l\}$.

This condition states that if agent 1 strictly prefers a menu x to a completely spelled-out future plan $\{l\}$, so does agent 2. Since l does not necessarily belong to x, Definition 5 is independent of Definition 3.

In a similar way to Definition 4, the above condition can be restricted to the two-period domain.

Definition 6 Agent 2 is *more averse to commitment in the two-period model than* agent 1 if, for all $x \in \mathscr{X}^1$ and $l \in \mathscr{L}$, $x \succ^1 \{l\} \Rightarrow x \succ^2 \{l\}$.

Several authors adopt conditions identical to Definitions 5 and 6 in different contexts. Ahn (2008) considers preference over subsets of lotteries and interprets those subsets as ambiguous objects. Since singleton sets are then regarded as options without ambiguity, a similar comparison with Definition 5 shows that agent 1 is more ambiguity averse than agent 2. By taking preference over menus of lotteries, Sarver (2008) models a DM who anticipates regret from choice in the future and, hence, may prefer smaller menus. In his model, the identical comparison is interpreted as agent 1 being more regret prone than agent 2.[12]

Now the implication of the above behavioral comparison is considered. We show that Definition 5 characterizes second-order stochastic dominance in terms of subjective beliefs. In case of objective uncertainty, second-order stochastic dominance has been widely used to describe increasing uncertainty since Rothschild and Stiglitz (1970).

Definition 7 Consider probability measures μ^1 and μ^2 over $[0, 1]$. Say that μ^1 exhibits second-order stochastic dominance over μ^2 if, for all continuous and concave functions $v : [0, 1] \to \mathbb{R}$,[13]

$$\int_{[0,1]} v(\alpha) \, d\mu^1(\alpha) \geq \int_{[0,1]} v(\alpha) \, d\mu^2(\alpha).$$

Rothschild and Stiglitz (1970) show that the above condition holds if and only if μ^2 is obtained as μ^1 plus some "noise".[14] Thus, second-order stochastic dominance is a natural ordering on probability measures to describe increasing uncertainty. One immediate observation is that $\mathbb{E}_{\mu^1}[\alpha] = \mathbb{E}_{\mu^2}[\alpha]$ if μ^1 exhibits second-order stochastic dominance over μ^2 because $v(\alpha) = \alpha$ is a convex and concave function.

We may now state a characterization result.

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[12]In literature on ambiguity in the Savage-type model, Epstein (1999) and Ghirardato and Marinacci (2002) adopt closely related conditions to capture comparative attitudes toward ambiguity aversion. They compare an arbitrary act with an unambiguous act instead of comparing an arbitrary menu with a commitment menu.

[13]Notice that continuity is not redundant because a concave function is continuous in the interior of the domain. In the original definition by Rothschild and Stiglitz (1970), continuity is not imposed.

[14]Their argument for this equivalence works even when continuity is imposed on v.

Theorem 4 *Assume that $\succsim^i$, $i = 1, 2$, satisfy all the axioms of Theorem 1. Then the following conditions are equivalent:*

(a) Agent 2 is more averse to commitment than agent 1.
(b) Agent 2 is more averse to commitment in the two-period model than agent 1.
(c) There exist random discounting representations U^i with (u^i, μ^i), $i = 1, 2$, such that (i) $u^1 = u^2$, and (ii) μ^1 exhibits second-order stochastic dominance over μ^2.

A formal proof is relegated to Appendix section "Proof of Theorem 4". By definition, condition (a) implies condition (b). The intuition behind (b)⟹(c) is as follows: Definition 6 implies that $\succsim^1$ and $\succsim^2$ are equivalent on $\mathscr{L}$, and hence part (i) is obtained. Furthermore, together with this observation, Definition 6 implies that $U^2(x) \geq U^1(x)$ for all $x \in \mathscr{Z}^1$. Since, for all x, the function

$$\max_{l \in x} \left((1 - \alpha)u(l_c) + \alpha \int_{\mathscr{Z}} U^i(z) \, dl_z \right) \tag{20.8}$$

is convex in α, the ranking $U^2(x) \geq U^1(x)$ means that the integral of a convex function of the form (20.8) with respect to μ^2 is always bigger than that corresponding to μ^1. As the last step, we show that any continuous convex function v on $[0, 1]$ can be arbitrarily approximated by a function of the form (20.8) if an affine transformation of u is chosen appropriately.

Unlike Theorem 3, condition (c) implies condition (a). That is, condition (c) is sufficient to show relative aversion to commitment with respect to not only "two-period menus" $x \in \mathscr{Z}^1$ but also all infinite horizon decision problems $x \in \mathscr{Z}$. In the proof, by exploiting the property of second-order stochastic dominance, we show that condition (c) ensures $U^2(x) \geq U^1(x)$ for all $x \in \mathscr{Z}$, which implies $U^2(x) \geq U^1(x) > U^1(\{l\}) = U^2(\{l\})$ as desired.

5   Consumption-Savings Decisions Under Random Discounting

In this section, we apply the resulting model to a consumption-savings problem and analyze how random discounting affects consumption-savings decisions. We focus on the situation where the DM becomes more uncertain about discount factors in the sense of second-order stochastic dominance. We will characterize savings behavior when the DM has a CRRA utility function on consumption.

Recall Example 1 in Sect. 2.1. Assume that an interest rate r is constant as in the example. Given the savings s from the previous period, the DM evaluates $x(s)$ according to the random discounting representation,

$$U(x(s)) = \int \max_{(c,x(s')) \in x(s)} \Big((1-\alpha)u(c) + \alpha U(x(s')) \Big) \, d\mu(\alpha). \qquad (20.9)$$

Throughout this section, the DM is assumed to have a CRRA utility function over instantaneous consumption, that is, $u(c) = c^{1-\sigma}/(1-\sigma)$ for $\sigma > 0$, $\sigma \neq 1$. As is well-known, the inverse of σ is the elasticity of intertemporal substitution.

We examine the effect of the DM being more uncertain about future discount factors. Suppose that the DM changes her belief μ^1 to μ^2, where μ^1 second-order stochastically dominates μ^2. Let U^i denote the random discounting representation with components (u, μ^i), $i = 1, 2$.

After realization of $\alpha \in (0, 1)$, the DM faces the following problem:

$$\begin{aligned} V_{\mu^i}(s, \alpha) &\equiv \max_{(c,x(s')) \in x(s)} (1-\alpha)u(c) + \alpha U^i(x(s')) \\ &= \max_{(c,s') \in B(s)} (1-\alpha)u(c) + \alpha U^i(x(s')). \end{aligned} \qquad (20.10)$$

Here, the current discount factor is known as α and the DM believes discount factors to follow distribution μ^i over the rest of the horizon. From (20.9) and (20.10), the Bellman equation is obtained as

$$V_{\mu^i}(s, \alpha) = \max_{(c,s') \in B(s)} \left((1-\alpha)u(c) + \alpha \int V_{\mu^i}(s', \alpha') \, d\mu^i(\alpha') \right). \qquad (20.11)$$

Let $g_{\mu^i}(s, \alpha)$ denote the savings function which solves problem (20.11).

We state the main result in this section. A proof is relegated to Appendix section "Proof of Theorem 5".

Theorem 5 *Assume that μ^1 second-order stochastically dominates μ^2 and $\bar{\alpha} \equiv \mathbb{E}_{\mu^1}[\alpha] = \mathbb{E}_{\mu^2}[\alpha] < 1/(1+r)^{1-\sigma}$. Then:*

(i) the DM saves a constant fraction of wealth, that is,

$$g_{\mu^i}(s, \alpha) = SR_{\mu^i}(\alpha)(1+r)s,$$

where the savings rate $SR_{\mu^i}(\alpha) \in (0, 1)$ is uniquely determined, and;
(ii) for all $\alpha \in (0, 1)$, $SR_{\mu^1}(\alpha) \lessgtr SR_{\mu^2}(\alpha)$ if $\sigma \lessgtr 1$.

Part (i) is a characterization of the savings function, and is based on the assumption that u is a CRRA utility function. Owing to part (i), we can focus on the savings rate rather than the savings function to analyze the savings behavior of the DM. Part (ii) concerns a comparative analysis. Depending on the relative size of σ compared to one, the savings rate increases or decreases as the DM becomes more uncertain about discount factors.

To obtain the intuition behind part (ii), for each s, define the number $\theta^i(s)$ by

$$V_{\bar{\alpha}}(\theta^i(s)) = \int V_{\mu^i}(s, \alpha') \, \mathrm{d}\mu^i(\alpha'), \tag{20.12}$$

where $V_{\bar{\alpha}}$ is the value function of the savings problem when a discount factor is constant over time and equal to the average $\bar{\alpha}$, that is, for all s, $V_{\bar{\alpha}}(s)$ is defined as

$$V_{\bar{\alpha}}(s) = \max_{(c,s') \in B(s)} (1 - \bar{\alpha}) u(c) + \bar{\alpha} V_{\bar{\alpha}}(s'). \tag{20.13}$$

Let $\mathbf{c}^i = (c^i_1, c^i_2, \cdots)$ be a solution to (20.13) that attains the maximum value $V_{\bar{\alpha}}(\theta^i(s))$. Then, the discounted sum of $\mathbf{c}^i$ is equal to $\theta^i(s)$, that is, $\theta^i(s) = \sum_{t=1}^{\infty} c^i_t/(1+r)^t$, and (20.12) is equivalent to saying that $\{\mathbf{c}^i\} \sim^i x(s)$. Since the DM desires flexibility, $\theta^i(s)$ must be greater than s so as to compensate the DM for being committed to $\mathbf{c}^i$. Hence, the ratio $\phi^i \equiv \theta^i(s)/s$ is interpreted as *the commitment premium*.[15] As uncertainty increases, the DM becomes more averse to commitment, and hence, ϕ^i increases. From (20.12), maximization problem (20.10) is rewritten as

$$\max_{(c,s') \in B(s)} (1 - \alpha) u(c) + \alpha V_{\bar{\alpha}}(\phi^i s').$$

That is, increasing uncertainty has the same effect as if the rate of return from savings increases in the consumption-savings model with no uncertainty. Therefore, the substitution and income effects lead to the desired result.

Part (ii) of Theorem 5 includes, as a special case, a comparison between random and deterministic discounting with the same mean. According to the theorem, the savings increase or decrease depending on parameter σ when the DM becomes uncertain about discount factors, which implies that observed over-savings or under-savings behavior may be explained by subjective uncertainty about discount factors. Salanié and Treich (2006) provide the same observation in a three-period model.

Instead of uncertainty about discount factors, uncertainty about interest rates has been discussed in studies on consumption-savings, for example, Levhari and Srinivasan (1969), Sandmo (1970) and Rothschild and Stiglitz (1971). They report that increasing uncertainty will decrease (increase) savings in case of $\sigma < (>)1$, which is the opposite to Theorem 5 (ii). Under risk aversion, the certainty equivalent of an uncertain interest rate always decreases as uncertainty increases. Hence, increasing uncertainty has the same effect as if the interest rate decreases in the consumption-savings problem with no uncertainty, while the commitment premium increases as discount factors become more uncertain. Thus, substitution and income effects lead to opposite implications.

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[15] Since u is CRRA, ϕ^i is independent of s.

Appendices

Hausdorff Metric

Let X be a compact metric space with a metric d. Let $\mathscr{K}(X)$ be the set of all non-empty compact subsets of X. For $x \in X$ and $A, B \in \mathscr{K}(X)$, let

$$d(x, B) \equiv \min_{x' \in B} d(x, x'), \ d(A, B) \equiv \max_{x \in A} d(x, B).$$

For all $A, B \in \mathscr{K}(X)$, define *the Hausdorff metric d_H* by

$$d_H(A, B) \equiv \max[d(A, B), d(B, A)].$$

Then, d_H satisfies (i) $d_H(A, B) \geq 0$, (ii) $A = B \Leftrightarrow d_H(A, B) = 0$, (iii) $d_H(A, B) = d_H(B, A)$, and (iv) $d_H(A, B) \leq d_H(A, C) + d_H(C, B)$. Moreover, $\mathscr{K}(X)$ is compact under the Hausdorff metric.

Perfect Commitment Menus

We follow the construction of menus by Gul and Pesendorfer (2004, Appendix A) (hereafter GP) and define the set $\mathscr{L}$ of perfect commitment menus. Then we show that $\mathscr{L}$ is homeomorphic to $\Delta(C \times \mathscr{L})$. That is, a perfect commitment menu can be viewed as a multistage lottery.

Let C denote the outcome space (consumption set), which is a compact metric space. We define the set of one-period consumption problems as $\mathscr{Z}_1 \equiv \mathscr{K}(\Delta(C))$. For $t > 1$, define the set of t-period consumption problems inductively as $\mathscr{Z}_t \equiv \mathscr{K}(\Delta(C \times \mathscr{Z}_{t-1}))$. Let $\mathscr{Z}^* \equiv \Pi_{t=1}^{\infty} \mathscr{Z}_t$. A menu is a *consistent* element of $\mathscr{Z}^*$.

Formally, define $G_1 : C \times \mathscr{Z}_1 \to C$, $F_1 : \Delta(C \times \mathscr{Z}_1) \to \Delta(C)$, and $\overline{F}_1 : \mathscr{K}(\Delta(C \times \mathscr{Z}_1)) \to \mathscr{K}(\Delta(C))$ as follows:

$$G_1(c, z_1) \equiv c, \ F_1(\mu_2)(E) \equiv \mu_2(G_1^{-1}(E)) \text{ and } \overline{F}_1(z_2) \equiv \{F_1(\mu_2) \mid \mu_2 \in z_2\},$$

for E in the Borel $\sigma-$algebra of C. For $t > 1$, we define inductively $G_t : C \times \mathscr{Z}_t \to C \times \mathscr{Z}_{t-1}$, $F_t : \Delta(C \times \mathscr{Z}_t) \to \Delta(C \times \mathscr{Z}_{t-1})$, and $\overline{F}_t : \mathscr{K}(\Delta(C \times \mathscr{Z}_t)) \to$

$\mathscr{K}(\Delta(C \times \mathscr{Z}_{t-1}))$ by

$$G_t(c, z_t) \equiv (c, G_{t-1}(z_t)), \ F_t(\mu_{t+1})(E) \equiv \mu_{t+1}(G_t^{-1}(E)), \text{ and}$$

$$\overline{F}_t(z_{t+1}) \equiv \{F_t(\mu_{t+1}) \mid \mu_{t+1} \in z_{t+1}\},$$

for E in the Borel σ−algebra of $C \times \mathscr{Z}_{t-1}$. Finally, we define $\{z_t\}_{t=1}^{\infty} \in \mathscr{Z}^*$ is *consistent* if $z_{t-1} = \overline{F}_{t-1}(z_t)$ for every $t > 1$.

We identify a singleton menu with its only element by slightly abusing notation. Let $\mathscr{L}_1 \equiv \Delta(C) \subset \mathscr{Z}_1$. An element of $\mathscr{L}_1$ is a one-period "commitment" consumption problem. For $t > 1$, we define $\mathscr{L}_t$ inductively as $\mathscr{L}_t \equiv \Delta(C \times \mathscr{L}_{t-1}) \subset \mathscr{Z}_t$. An element of $\mathscr{L}_t$ is a t-period "commitment" consumption problem. Let $\mathscr{L}^* \equiv \Pi_{t=1}^{\infty} \mathscr{L}_t$. We define the set of perfect commitment menus as $\mathscr{L} \equiv \mathscr{Z} \cap \mathscr{L}^*$. Thus, an element in $\mathscr{L}$ is a menu in which the DM is committed in every period.

Proposition 2 $\mathscr{L}$ *is homeomorphic to* $\Delta(C \times \mathscr{L})$.

Proof GP construct a homeomorphism $f : \mathscr{Z} \to \mathscr{K}(\Delta(C \times \mathscr{Z}))$. Note that $\mathscr{L}$ is compact since $\mathscr{L}_t$ is compact for every t. It is sufficient to check that $f(\mathscr{L}) = \Delta(C \times \mathscr{L})$.

Definition 8 Let $Y_1 \equiv \hat{L}_1 \equiv \Delta(C)$ and for $t > 1$ let $Y_t \equiv \Delta(C \times \Pi_{n=1}^{t-1} \mathscr{Z}_n)$ and $\hat{L}_t \equiv \Delta(C \times \Pi_{n=1}^{t-1} \mathscr{L}_n)$. Define $Y^{kc} \equiv \{\{\hat{\mu}_t\} \in \Pi_{t=1}^{\infty} Y_t \mid \text{marg}_{C \times \Pi_{n=1}^{t-1} \mathscr{Z}_n} \hat{\mu}_{t+1} = \hat{\mu}_t\}$. Let $\hat{L}^{kc} = Y^{kc} \cap \Pi_{t=1}^{\infty} \hat{L}_t$.

GP show that for every $\{\hat{\mu}_t\} \in Y^{kc}$ there exists a unique $\mu \in \Delta(C \times \mathscr{Z}^*)$ such that $\text{marg}_C \mu = \hat{\mu}_1$ and $\text{marg}_{C \times \Pi_{n=1}^{t} \mathscr{Z}_n} \mu = \hat{\mu}_t$. Then they define $\psi : Y^{kc} \to \Delta(C \times \mathscr{Z}^*)$ as the mapping that associates this μ with the corresponding $\{\hat{\mu}_t\}$.

Step 1: $\psi(\hat{L}^{kc}) = \Delta(C \times \mathscr{L}^*)$.

Note that, for a sequence $\{\hat{l}_t\} \in \hat{L}^{kc}$, it holds that

$$\text{marg}_{C \times \Pi_{n=1}^{t-1} \mathscr{L}_n} \hat{l}_{t+1} = \text{marg}_{C \times \Pi_{n=1}^{t-1} \mathscr{Z}_n} \hat{l}_{t+1} = \hat{l}_t.$$

The same argument of Lemma 3 in GP shows that there exists a homeomorphism $\psi' : \hat{L}^{kc} \to \Delta(C \times \mathscr{L}^*)$ such that $\text{marg}_C \psi'(\{\hat{l}_t\}) = \hat{l}_1$ and $\text{marg}_{C \times \Pi_{n=1}^{t-1} \mathscr{L}_n} \psi'(\{\hat{l}_t\}) = \hat{l}_t$. The uniqueness part of the Kolmogorov's Existence Theorem implies that $\psi' = \psi|_{\hat{L}^{kc}}$. Step 1 thus follows.

Definition 9 Let $D_t \equiv \{(z_1, \ldots, z_t) \in \Pi_{n=1}^{t} \mathscr{Z}_n \mid z_k = \overline{F}_k(z_{k+1}), \ \forall k = 1, \ldots, t-1\}$ and $D_t^L \equiv D_t \cap \Pi_{n=1}^{t} \mathscr{L}_n$. Define $M^c \equiv \{\{\mu_t\} \in \Delta(C) \times \Pi_{t=1}^{\infty} \Delta(C \times \mathscr{Z}_t) \mid F_t(\mu_{t+1}) = \mu_t, \ \forall t \geq 1\}$. Let $Y^c \equiv \{\{\hat{\mu}_t\} \in Y^{kc} \mid \hat{\mu}_{t+1}(C \times D_t) = 1, \forall t \geq 1\}$ and $\hat{L}^c \equiv Y^c \cap \hat{L}^{kc}$.

Note that $\mathscr{L} = M^c \cap \mathscr{L}^*$. GP show that for every $\{\mu_t\} \in M^c$ there exists a unique $\{\hat{\mu}_t\} \in Y^c$ such that $\hat{\mu}_1 = \mu_1$ and $\text{marg}_{C \times \mathscr{Z}_{t-1}} \hat{\mu}_t = \mu_t$ for every $t \geq 2$.

Then they define $\phi : M^c \to Y^c$ as the mapping that associates this $\{\mu_t\}$ with the corresponding $\{\hat{\mu}_t\}$.

Step 2: $\phi(\mathscr{L}) = \hat{L}^c$.

It is straightforward from the definition of ϕ that $\phi(\mathscr{L}) \supset \hat{L}^c$. We show $\phi(\mathscr{L}) \subset \hat{L}^c$ or $\phi(\mathscr{L}) \subset \Pi_{t=1}^{\infty}\hat{L}_t$ by mathematical induction. Take $\{l_t\} \in \mathscr{L}$ and let $\{\hat{\mu}_t\} \equiv \phi(\{l_t\}) \in Y^c$. By definition, $\hat{\mu}_1 = l_1 \in \Delta(C) = \hat{L}_1$ and $\hat{\mu}_2 = \mathrm{marg}_{C \times \mathscr{L}_1}\hat{\mu}_2 = l_2 \in \Delta(C \times \mathscr{L}_1) = \hat{L}_2$.

Suppose that $\hat{\mu}_k \in \hat{L}_k$ for every $k = 1, 2, \ldots, t$. Since $\{\hat{\mu}_t\}$ is a Kolmogorov consistent sequence, $\mathrm{marg}_{C \times \Pi_{n=1}^{t-1}\mathscr{L}_n}\hat{\mu}_{t+1} = \hat{\mu}_t \in \hat{L}_t$. Thus, $\hat{\mu}_{t+1} \in \Delta(C \times \Pi_{n=1}^{t-1}\mathscr{L}_n \times \mathscr{L}_t)$. The definition of ϕ implies that $\mathrm{marg}_{C, \mathscr{L}_t}\hat{\mu}_{t+1} = l_{t+1} \in \Delta(C \times \mathscr{L}_t)$. Therefore, $\hat{\mu}_{t+1} \in \hat{L}_{t+1} = \Delta(C \times \Pi_{n=1}^{t}\mathscr{L}_n)$.

Step 3: $\psi(\hat{L}^c) = \{l \in \Delta(C \times \mathscr{L}^*) \mid l(C \times \mathscr{L}) = 1\}$.

Since $\hat{L}^c = \{\{\hat{l}_t\} \in \hat{L}^{kc} \mid \hat{l}_{t+1}(C \times D_t^L) = 1, \forall t \geqq 1\}$, Step 3 follows from the same argument of Lemma 5 in GP.

GP define $\xi : \mathscr{Z} \to \mathscr{K}(M^c)$ as $\xi(z) \equiv \{\{\mu_t\} \in M^c \mid \mu_t \in z_t, \ \forall t \geq 1\}$. Note that ξ is identity on $\mathscr{L}$. Finally, the homeomorphism $f : \mathscr{Z} \to \mathscr{K}(\Delta(C \times \mathscr{Z}))$ is given by $f(z) = \psi \circ \phi(\xi(z))$. Then the above steps imply that $f(\mathscr{L}) = \psi \circ \phi(\xi(\mathscr{L})) = \psi \circ \phi(\mathscr{L}) = \Delta(C \times \mathscr{L})$.

Proof of Theorem 1

Necessity

Necessity of the axioms is routine. We show that for any (u, μ) there exists U satisfying the functional equation.

Let $\mathscr{U}$ be the Banach space of all real-valued continuous functions on $\mathscr{Z}$ with the sup-norm metric. Define the operator $T : \mathscr{U} \to \mathscr{U}$ by

$$T(U)(x) \equiv \int_{[0,1]} \max_{l \in x} \left((1-\alpha)u(l_c) + \alpha \int_{\mathscr{Z}} U(z)\,\mathrm{d}l_z \right) \mathrm{d}\mu(\alpha).$$

Since $T(U)$ is continuous, the operator T is well-defined. To show T is a contraction mapping, it suffices to verify that (i) T is monotonic, that is, $T(U) \geq T(V)$ whenever $U \geq V$, and (ii) T satisfies the discounting property, that is, there exists $\delta \in [0, 1)$ such that for any U and $c \in \mathbb{R}$, $T(U + c) = T(U) + \delta c$.

Step 1: T is monotonic.

Take any $U, V \in \mathscr{U}$ with $U \geq V$. Since $\int U(z)\,\mathrm{d}l_z \geq \int V(z)\,\mathrm{d}l_z$ for all $l \in \Delta(C \times \mathscr{Z})$, we have

$$\max_{l \in x} \left((1 - \alpha) u(l_c) + \alpha \int_{\mathscr{Z}} U(z) \, dl_z \right) \geq \max_{l \in x} \left((1 - \alpha) u(l_c) + \alpha \int_{\mathscr{Z}} V(z) \, dl_z \right)$$

for all x and α, and hence $T(U)(x) \geq T(V)(x)$ as desired.

Step 2: T satisfies the discounting property.

Let $\delta \equiv \bar{\alpha}$. By assumption, $\delta \in [0, 1)$. For any $U \in \mathscr{U}$ and $c \in \mathbb{R}$,

$$T(U + c) = \int_{[0,1]} \max_{l \in x} \left((1 - \alpha) u(l_c) + \alpha \int_{\mathscr{Z}} (U(z) + c) \, dl_z \right) d\mu(\alpha)$$
$$= T(U) + \bar{\alpha} c = T(U) + \delta c.$$

By Steps 1 and 2, T is a contraction mapping. Thus, the fixed point theorem (See Bertsekas and Shreve 1978, p. 55) ensures that there exists a unique $U^* \in \mathscr{U}$ satisfying $U^* = T(U^*)$. This U^* satisfies Eq. (20.1).

Sufficiency

Lemma 1 *Commitment Independence, Stationarity, and Timing Indifference imply Independence, that is,*

$$x \succ y \implies \lambda x + (1 - \lambda) z \succ \lambda y + (1 - \lambda) z,$$

for all $x, y, z \in \mathscr{Z}$ and $\lambda \in (0, 1)$.

Proof Let $x \succ y$. From Stationarity, $\{(c, x)\} \succ \{(c, y)\}$. For any $\lambda \in (0, 1)$, Commitment Independence implies $\{\lambda \circ (c, x) + (1 - \lambda) \circ (c, z)\} \succ \{\lambda \circ (c, y) + (1 - \lambda) \circ (c, z)\}$. From Timing Indifference, $\{(c, \lambda x + (1 - \lambda) z)\} \succ \{(c, \lambda y + (1 - \lambda) z)\}$. Again, from Stationarity, $\lambda x + (1 - \lambda) z \succ \lambda y + (1 - \lambda) z$.   □

Let $\overline{co}(x)$ denote the closed convex hull of x. As in DLR, Order, Continuity, and Independence imply $x \sim \overline{co}(x)$. Hence we can pay attention to the sub-domain

$$\mathscr{Z}_1 \equiv \{x \in \mathscr{Z} | x = \overline{co}(x)\}.$$

Since $\mathscr{Z}_1$ is a mixture space, Order, Continuity, and Independence ensure that $\succsim$ can be represented by a mixture linear function $U : \mathscr{Z}_1 \to \mathbb{R}$. Nondegeneracy implies U is not constant. Since $C \times \mathscr{Z}$ is compact, there exist a maximal and a minimal lottery $\bar{l}, \underline{l} \in \Delta(C \times \mathscr{Z})$ with respect to $U(\{\cdot\})$. Without loss of generality, assume $U(\{\bar{l}\}) = 1$ and $U(\{\underline{l}\}) = 0$.

Define $u : \Delta(C) \to \mathbb{R}$ and $W : \Delta(\mathscr{Z}) \to \mathbb{R}$ by

$$u(l_c) \equiv U(\{l_c \otimes \underline{l}_z\}), \; W(l_z) \equiv U(\{\underline{l}_c \otimes l_z\}),$$

where $\underline{l}_c$ and $\underline{l}_z$ be the marginal distributions of $\underline{l}$ on C and $\mathscr{Z}$.

Lemma 2 *(i) For any $l_c, l'_c \in \Delta(C)$ and $l_z, l'_z \in \Delta(\mathscr{Z})$,*

$$u(l_c) \geq u(l'_c) \;\Leftrightarrow\; U(\{l_c \otimes l_z\}) \geq U(l'_c \otimes l_z),$$
$$W(l_z) \geq W(l'_z) \;\Leftrightarrow\; U(\{l_c \otimes l_z\}) \geq U(l_c \otimes l'_z).$$

(ii) u and W are mixture linear.

Proof (i) Consider the restriction of U on $\Delta(C \times \mathscr{Z})$. Let $U(c, z) \equiv U(\{(c, z)\})$. First we will claim that there exist $\bar{u} : C \to \mathbb{R}$ and $\bar{W} : \mathscr{Z} \to \mathbb{R}$ such that $U(c, z) = \bar{u}(c) + \bar{W}(z)$. Since

$$O\left(\frac{1}{2} \circ (c, z) + \frac{1}{2} \circ (c', z')\right) = O\left(\frac{1}{2} \circ (c', z) + \frac{1}{2} \circ (c, z')\right),$$

Marginal Dominance implies

$$U\left(\left\{\frac{1}{2} \circ (c, z) + \frac{1}{2} \circ (c', z')\right\}\right) = U\left(\left\{\frac{1}{2} \circ (c', z) + \frac{1}{2} \circ (c, z')\right\}\right).$$

Mixture linearity of U implies

$$U(c, z) + U(c', z') = U(c', z) + U(c, z').$$

Define $\bar{u}(c) \equiv U(c, z')$ and $\bar{W}(z) \equiv U(c', z) - U(c', z')$ for an arbitrarily fixed (c', z'). Then, $U(c, z) = \bar{u}(c) + \bar{W}(z)$.

By the above claim, for any $l \in \Delta(C \times Z)$,

$$U(\{l\}) = \int U(c, z) \, \mathrm{d}l(c, z) = \int (\bar{u}(c) + \bar{W}(z)) \, \mathrm{d}l(c, z) = \int \bar{u}(c) \, \mathrm{d}l_c(c)$$
$$+ \int \bar{W}(z) \, \mathrm{d}l_z(z).$$

Thus,

$$u(l_c) \geq u(l'_c) \Leftrightarrow U(\{l_c \otimes \underline{l}_z\}) \geq U(\{l'_c \otimes \underline{l}_z\})$$
$$\Leftrightarrow \int \bar{u}(c) \, \mathrm{d}l_c(c) + \int \bar{W}(z) \, \mathrm{d}\underline{l}_z(z) \geq \int \bar{u}(c) \, \mathrm{d}l'_c(c) + \int \bar{W}(z) \, \mathrm{d}\underline{l}_z(z)$$
$$\Leftrightarrow \int \bar{u}(c) \, \mathrm{d}l_c(c) \geq \int \bar{u}(c) \, \mathrm{d}l'_c(c)$$
$$\Leftrightarrow \int \bar{u}(c) \, \mathrm{d}l_c(c) + \int \bar{W}(z) \, \mathrm{d}l_z(z) \geq \int \bar{u}(c) \, \mathrm{d}l'_c(c) + \int \bar{W}(z) \, \mathrm{d}l_z(z)$$

$$\Leftrightarrow U(\{l_c \otimes l_z\}) \geq U(\{l'_c \otimes l_z\}).$$

The symmetric argument can be applied to W.

(ii) We want to show $u(\lambda l_c + (1 - \lambda)l'_c) = \lambda u(l_c) + (1 - \lambda)u(l'_c)$ for any l_c, l'_c and $\lambda \in [0, 1]$. Since

$$O((\lambda l_c + (1 - \lambda)l'_c) \otimes l_z) = O(\lambda l_c \otimes l_z + (1 - \lambda)l'_c \otimes l_z),$$

Marginal Dominance implies

$$U(\{(\lambda l_c + (1 - \lambda)l'_c) \otimes l_z\}) = U(\{\lambda l_c \otimes l_z + (1 - \lambda)l'_c \otimes l_z\}).$$

Since $U(\{\cdot\})$ is mixture linear,

$$
\begin{aligned}
u(\lambda l_c + (1 - \lambda)l'_c) &= U(\{(\lambda l_c + (1 - \lambda)l'_c) \otimes \underline{l}_z\}) \\
&= U(\{\lambda l_c \otimes \underline{l}_z + (1 - \lambda)l'_c \otimes \underline{l}_z\}) \\
&= \lambda U(\{l_c \otimes \underline{l}_z\}) + (1 - \lambda)U(\{l'_c \otimes \underline{l}_z\}) \\
&= \lambda u(l_c) + (1 - \lambda)u(l'_c).
\end{aligned}
$$

By the symmetric argument, we can show that W is mixture linear. ☐

Next we show several properties of the Marginal Dominance operator.

Lemma 3   (i) *For any $x \in \mathscr{Z}$, $O(x) \in \mathscr{Z}$.*
(ii) *If x is convex, so is $O(x)$.*
(iii) *$O : \mathscr{Z} \to \mathscr{Z}$ is Hausdorff continuous.*

Proof   (i) Since $\Delta(C \times \mathscr{Z})$ is compact, it suffices to show that $O(x)$ is a closed subset of $\Delta(C \times \mathscr{Z})$. Let $l^n \to l$ with $l^n \in O(x)$. By definition, there exists a sequence $\{\bar{l}^n\}$ with $\bar{l}^n \in x$ such that $\{\bar{l}^n_c \otimes l^n_z\} \succsim \{l^n_c \otimes l^n_z\}$ and $\{l^n_c \otimes \bar{l}^n_z\} \succsim \{l^n_c \otimes l^n_z\}$. Since x is compact, without loss of generality, we can assume that $\{\bar{l}^n\}$ converges to a limit $\bar{l} \in x$. Since $l^n_c \to l_c$ and $l^n_z \to l_z$, $\bar{l}^n_c \to \bar{l}_c$ and $\bar{l}^n_z \to \bar{l}_z$, Continuity implies $\{\bar{l}_c \otimes l_z\} \succsim \{l_c \otimes l_z\}$ and $\{l_c \otimes \bar{l}_z\} \succsim \{l_c \otimes l_z\}$. Hence, $l \in O(x)$.

(ii) Take $l, l' \in O(x)$ and $\lambda \in [0, 1]$. Let $l^\lambda \equiv \lambda l + (1 - \lambda)l'$. We want to show $l^\lambda \in O(x)$. By definition, there exist $\bar{l}, \bar{l}' \in x$ such that $\{\bar{l}_c \otimes l_z\} \succsim \{l_c \otimes l_z\}$, $\{l_c \otimes \bar{l}_z\} \succsim \{l_c \otimes l_z\}$, $\{\bar{l}'_c \otimes l'_z\} \succsim \{l'_c \otimes l'_z\}$, and $\{l'_c \otimes \bar{l}'_z\} \succsim \{l'_c \otimes l'_z\}$. Let $\bar{l}^\lambda \equiv \lambda \bar{l} + (1 - \lambda)\bar{l}' \in x$. From Commitment Independence,

$$\{\lambda \bar{l}_c \otimes l_z + (1 - \lambda)\bar{l}'_c \otimes l'_z\} \succsim \{\lambda l_c \otimes l_z + (1 - \lambda)l'_c \otimes l'_z\}.$$

Since $O(l_c \otimes l_z) = O(l)$, Marginal Dominance implies $\{l_c \otimes l_z\} \sim \{l\}$. By the same reason, $\{l'_c \otimes l'_z\} \sim \{l'\}$, $\{l^\lambda_c \otimes l^\lambda_z\} \sim \{l^\lambda\}$, and

$$\{(\lambda \bar{l}_c + (1-\lambda)\bar{l}'_c) \otimes (\lambda l_z + (1-\lambda)l'_z)\} \sim \{\lambda \bar{l}_c \otimes l_z + (1-\lambda)\bar{l}'_c \otimes l'_z\}.$$

Thus,

$$
\begin{aligned}
\{\bar{l}^\lambda_c \otimes l^\lambda_z\} &= \{(\lambda \bar{l} + (1-\lambda)\bar{l}')_c \otimes (\lambda \bar{l} + (1-\lambda)\bar{l}')_z\} \\
&= \{(\lambda \bar{l}_c + (1-\lambda)\bar{l}'_c) \otimes (\lambda l_z + (1-\lambda)l'_z)\} \\
&\sim \{\lambda \bar{l}_c \otimes l_z + (1-\lambda)\bar{l}'_c \otimes l'_z\} \succsim \{\lambda l_c \otimes l_z + (1-\lambda)l'_c \otimes l'_z\} \\
&\sim \{\lambda l + (1-\lambda)l'\} \sim \{l^\lambda_c \otimes l^\lambda_z\}.
\end{aligned}
$$

Similarly, $\{l^\lambda_c \otimes \bar{l}^\lambda_z\} \succsim \{l^\lambda_c \otimes l^\lambda_z\}$. Hence, $l^\lambda \in O(x)$.

(iii) Let $x^n \to x$. We want to show $O(x^n) \to O(x)$. By contradiction, suppose otherwise. Then, there exists a neighborhood $\mathscr{U}$ of $O(x)$ such that $O(x^\ell) \notin \mathscr{U}$ for infinitely many ℓ. Let $\{x^\ell\}_{\ell=1}^\infty$ be the corresponding subsequence of $\{x^n\}_{n=1}^\infty$. Since $x^n \to x$, $\{x^\ell\}_{\ell=1}^\infty$ also converges to x. Since $\{O(x^\ell)\}_{\ell=1}^\infty$ is a sequence in a compact metric space $\mathscr{Z}$, there exists a convergent subsequence $\{O(x^m)\}_{m=1}^\infty$ with a limit $y \neq O(x)$. As a result, now we have $x^m \to x$ and $O(x^m) \to y$. In the following argument, we will show that $y = O(x)$, which is a contradiction.

Step 1: $O(x) \subset y$.

Take any $l \in O(x)$. Then, there exists $\bar{l} \in x$ such that $\{\bar{l}_c \otimes l_z\} \succsim \{l_c \otimes l_z\}$ and $\{l_c \otimes \bar{l}_z\} \succsim \{l_c \otimes l_z\}$. Since $x^m \to x$, we can find a sequence $\{\bar{l}^m\}_{m=1}^\infty$ such that $\bar{l}^m \in x^m$ and $\bar{l}^m \to \bar{l}$.

Now we will construct a sequence $\{l^m\}_{m=1}^\infty$ with $l^m \in O(x^m)$ satisfying $l^m \to l$. Let $l^-_c \in \Delta(C)$ be a worst element with respect to u and $l^-_z \in \Delta(\mathscr{Z})$ be a worst element with respect to W. For all sufficiently large k, let $B_k(l)$ be the $1/k$-neighborhood of l with respect to the weak convergence topology. There exists $0 < \lambda^k < 1$ such that $l^k \equiv \lambda^k l + (1-\lambda^k)(l^-_c \otimes l^-_z) \in B_k(l)$. By construction, $l^k \to l$. Since u is mixture linear from Lemma 2 (ii), $u(l_c) > u(l^k_c)$ if $u(l_c) > u(l^-_c)$, and $u(l_c) = u(l^k_c)$ if $u(l_c) = u(l^-_c)$. In the case of former, since $\bar{l}^m \to \bar{l}$, by Continuity, there exists m^k_1 such that for all $m \geq m^k_1$, $u(\bar{l}^m_c) > u(l^k_c)$. In the case of latter, for all m, $u(\bar{l}^m_c) \geq u(l^-_c) = u(l^k_c)$. In both cases, we have $u(\bar{l}^m_c) \geq u(l^k_c)$ for all $m \geq m^k_1$. Since W is mixture linear from Lemma 2 (ii), by the same argument, there exists m^k_2 such that for all $m \geq m^k_2$, $W(\bar{l}^m_z) \geq W(l^k_z)$. Therefore, for all $m \geq m^k \equiv \max[m^k_1, m^k_2]$, $u(\bar{l}^m_c) \geq u(l^k_c)$ and $W(\bar{l}^m_z) \geq W(l^k_z)$, that is, $\{\bar{l}^m_c \otimes l^k_z\} \succsim \{l^k_c \otimes l^k_z\}$ and $\{l^k_c \otimes \bar{l}^m_z\} \succsim \{l^k_c \otimes l^k_z\}$. Hence, we have $l^k \in O(\bar{l}^m) \subset O(x^m)$ for all $m \geq m^k$. Since $m^{k+1} \geq m^k$ for all k, define $l^m \equiv l^k$ for all m satisfying $m^k \leq m < m^{k+1}$. Then, $\{l^m\}_{m=1}^\infty$ is a required sequence.

Since $l^m \to l$ and $O(x^m) \to y$ with $l^m \in O(x^m)$, we have $l \in y$. Thus, $O(x) \subset y$.

Step 2: $y \subset O(x)$.

Take any $l \in y$. Since $O(x^m) \to y$, we can find a sequence $l^m \in O(x^m)$ with $l^m \to l$. By definition, there is $\bar{l}^m \in x^m$ such that $\{\bar{l}_c^m \otimes l_z^m\} \succsim \{l_c^m \otimes l_z^m\}$ and $\{l_c^m \otimes \bar{l}_z^m\} \succsim \{l_c^m \otimes l_z^m\}$. Since $\Delta(C \times \mathscr{Z})$ is compact, we can assume $\{\bar{l}^m\}$ converges to a limit $\bar{l} \in \Delta(C \times \mathscr{Z})$. Since $\bar{l}^m \to \bar{l}$ and $x^m \to x$ with $\bar{l}^m \in x^m$, we have $\bar{l} \in x$. From Continuity, $\{\bar{l}_c \otimes l_z\} \succsim \{l_c \otimes l_z\}$ and $\{l_c \otimes \bar{l}_z\} \succsim \{l_c \otimes l_z\}$. Thus, $l \in O(x)$. $\square$

From Marginal Dominance, $x \sim O(x)$. Hence we can pay attention to the sub-domain,

$$\mathscr{Z}_2 \equiv \{x \in \mathscr{Z}_1 | x = O(x)\}.$$

From Lemma 3 (iii), $\mathscr{Z}_2$ is compact. Moreover, Lemma 3 (i) and (ii) imply that any $x \in \mathscr{Z}_2$ is compact and convex.

For each $x \in \mathscr{Z}_2$ and $\alpha \in [0, 1]$, define

$$\sigma_x(\alpha) \equiv \max_{l \in x}\Big((1 - \alpha)u(l_c) + \alpha W(l_z)\Big). \tag{20.14}$$

Let $\mathscr{C}([0, 1])$ be the set of real-valued continuous functions on $[0, 1]$ with the supnorm. The above formulation (20.14) defines the mapping $\sigma : \mathscr{Z}_2 \to \mathscr{C}([0, 1])$.

Lemma 4 *(i) σ is continuous.*
(ii) For all $x, y \in \mathscr{Z}_2$ and $\lambda \in [0, 1]$, $\lambda\sigma_x + (1 - \lambda)\sigma_y = \sigma_{O(\lambda x+(1-\lambda)y)}$.
(iii) σ is injective.

Proof (i) Let

$$V(x) \equiv \{(u, w) | u = u(l_c), w = W(l_z), l \in x\} \subset \mathbb{R}^2.$$

Since u and W are continuous and $C \times \mathscr{Z}$ is compact, there exists a compact set $L \subset \mathbb{R}^2$ such that $V(x) \subset L$ for all x. Hence, $V(x)$ is also compact and, moreover, convex because u and W are mixture linear. Let $\mathscr{K}(L)$ be the set of non-empty compact subsets of L with the Hausdorff metric.

Step 1: The map $V : \mathscr{Z}_2 \ni x \mapsto V(x) \in \mathscr{K}(L)$ is Hausdorff continuous.

Take a sequence $x^n \to x$ with $x^n, x \in \mathscr{Z}_2$. We want to show that $V(x^n) \to V(x)$. By contradiction, suppose otherwise. Then, there exists a neighborhood $\mathscr{U}$ of $V(x)$ such that $V(x^m) \notin \mathscr{U}$ for infinitely many m. Let $\{x^m\}_{m=1}^\infty$ be the corresponding subsequence of $\{x^n\}_{n=1}^\infty$. Since $x^n \to x$, $\{x^m\}_{m=1}^\infty$ also converges to x. Since $\{V(x^m)\}_{m=1}^\infty$ is a sequence in a compact metric space $\mathscr{K}(L)$, there exists a convergent subsequence $\{V(x^\ell)\}_{\ell=1}^\infty$ with a limit $z \neq V(x)$. As a result, now we have $x^\ell \to x$ and $V(x^\ell) \to z$.

In the following argument, we will show that $z = V(x)$, which is a contradiction. Take any $(\bar{u}, \bar{w}) \in V(x)$. There exists $\bar{l} \in x$ such that $\bar{u} = u(\bar{l}_c)$ and $\bar{w} = W(\bar{l}_z)$. Since $x^\ell \to x$, we can find $\{l^\ell\}_{\ell=1}^\infty$ such that $l^\ell \to \bar{l}$ with $l^\ell \in x^\ell$. Let $(u^\ell, w^\ell) \equiv$

$(u(l_c^\ell), W(l_z^\ell)) \in V(x^\ell)$. The conditions $(u^\ell, w^\ell) \to (\bar{u}, \bar{w})$ and $V(x^\ell) \to z$ with $(u^\ell, w^\ell) \in V(x^\ell)$ imply $(\bar{u}, \bar{w}) \in z$. Thus, $V(x) \subset z$.

For the other direction, take any $(\bar{u}, \bar{w}) \in z$. Since $V(x^\ell) \to z$, we can find $\{(u^\ell, w^\ell)\}_{\ell=1}^\infty$ such that $(u^\ell, w^\ell) \to (\bar{u}, \bar{w})$ with $(u^\ell, w^\ell) \in V(x^\ell)$. There exists $l^\ell \in x^\ell$ satisfying $(u^\ell, w^\ell) = (u(l_c^\ell), W(l_z^\ell))$. Since $\Delta(C \times \mathcal{Z})$ is compact, there exists a convergent subsequence $\{l^k\}_{k=1}^\infty$ with a limit $\bar{l}$. By continuity of u and W, $(u(\bar{l}_c), W(\bar{l}_z)) = (\bar{u}, \bar{w})$. Moreover, since $l^k \to \bar{l}, x^k \to x$ with $l^k \in x^k$, we have $\bar{l} \in x$. Thus $(\bar{u}, \bar{w}) \in V(x)$, which implies $z \subset V(x)$.

Step 2: $d_{\text{supnorm}}(\sigma_x, \sigma_y) \leq d_{\text{Hausdorff}}(V(x), V(y))$.

For any $\alpha \in [0, 1]$, by definition,

$$\left| \sigma_x(\alpha) - \sigma_y(\alpha) \right| = \left| \max_{l \in x} \Big((1-\alpha)u(l_c) + \alpha W(l_z) \Big) - \max_{h \in y} \Big((1-\alpha)u(l_c) + \alpha W(l_z) \Big) \right|$$

$$= \left| \max_{(u,w) \in V(x)} ((1-\alpha)u + \alpha w) - \max_{(u,w) \in V(y)} ((1-\alpha)u + \alpha w) \right|.$$

Let $(u^{\alpha x}, w^{\alpha x}) \in V(x)$ and $(u^{\alpha y}, w^{\alpha y}) \in V(y)$ be maximizers for the maximization problems, respectively. Without loss of generality, assume

$$(1-\alpha)u^{\alpha x} + \alpha w^{\alpha x} \geq (1-\alpha)u^{\alpha y} + \alpha w^{\alpha y}.$$

Let

$$H^{\alpha y} \equiv \{(u, w) | (1-\alpha)u + \alpha w = (1-\alpha)u^{\alpha y} + \alpha w^{\alpha y}\}$$

and $(u^*, w^*) \in H^{\alpha y}$ be a point solving

$$\min_{(u,w) \in H^{\alpha y}} \|(u, w) - (u^{\alpha x}, w^{\alpha x})\|.$$

Then, by the Schwarz inequality,

$$\left| \max_{(u,w) \in V(x)} ((1-\alpha)u + \alpha w) - \max_{(u,w) \in V(y)} ((1-\alpha)u + \alpha w) \right|$$

$$= |((1-\alpha)u^{\alpha x} + \alpha w^{\alpha x}) - ((1-\alpha)u^{\alpha y} + \alpha w^{\alpha y})|$$

$$= |((1-\alpha)u^{\alpha x} + \alpha w^{\alpha x}) - ((1-\alpha)u^* + \alpha w^*)|$$

$$= |(1-\alpha)(u^{\alpha x} - u^*) + \alpha(w^{\alpha x} - w^*)|$$

$$\leq \|(u^{\alpha x} - u^*, w^{\alpha x} - w^*)\| \|(1-\alpha, \alpha)\| \leq \|(u^{\alpha x} - u^*, w^{\alpha x} - w^*)\|$$

$$\leq \min_{(u,w) \in V(y)} \|(u^{\alpha x}, w^{\alpha x}) - (u, w)\| \leq d_{\text{Hausdorff}}(V(x), V(y)).$$

Since this inequality holds for all α,

$$d_{\text{supnorm}}(\sigma_x, \sigma_y) = \sup_{\alpha \in [0,1]} |\sigma_x(\alpha) - \sigma_y(\alpha)| \leq d_{\text{Hausdorff}}(V(x), V(y)).$$

From Steps 1 and 2, σ is continuous.

(ii) Fix $\alpha \in [0, 1]$. Let $l^x \in x$ and $l^y \in y$ satisfy

$$(1 - \alpha)u(l^x_c) + \alpha W(l^x_z) = \max_{l \in x}((1 - \alpha)u(l_c) + \alpha W(l_z)),$$

$$(1 - \alpha)u(l^y_c) + \alpha W(l^y_z) = \max_{l \in y}((1 - \alpha)u(l_c) + \alpha W(l_z)).$$

Since $\lambda l^x + (1 - \lambda)l^y \in \lambda x + (1 - \lambda)y$, mixture linearity of u and W implies

$$\lambda \sigma_x(\alpha) + (1 - \lambda)\sigma_y(\alpha)$$
$$= \lambda((1 - \alpha)u(l^x_c) + \alpha W(l^x_z)) + (1 - \lambda)((1 - \alpha)u(l^y_c) + \alpha W(l^y_z))$$
$$= (1 - \alpha)u(\lambda l^x_c + (1 - \lambda)l^y_c) + \alpha W(\lambda l^x_z + (1 - \lambda)l^y_z)$$
$$= \max_{l \in \lambda x + (1 - \lambda)y}((1 - \alpha)u(l_c) + \alpha W(l_z))$$
$$= \max_{l \in O(\lambda x + (1 - \lambda)y)}((1 - \alpha)u(l_c) + \alpha W(l_z)) = \sigma_{O(\lambda x + (1 - \lambda)y)}(\alpha).$$

(iii) Take $x, x' \in \mathscr{Z}_2$ with $x \neq x'$. Without loss of generality, assume $x \not\subset x'$. Take $\tilde{l} \in x \setminus x'$. Let $\tilde{u} = u(\tilde{l}_c)$ and $\tilde{w} = W(\tilde{l}_z)$. Let

$$V' \equiv \{(u, w) | u = u(l_c), w = W(l_z), l \in x'\} \subset \mathbb{R}^2.$$

We will claim that $(\{(\tilde{u}, \tilde{w})\} + \mathbb{R}^2_+) \cap V' = \emptyset$. Suppose otherwise. Then, there exists $l' \in x'$ such that $u(l'_c) \geq \tilde{u}$ and $W(l'_z) \geq \tilde{w}$. That is, $U(\{l'_c \otimes l_z\}) \geq U(\{\tilde{l}_c \otimes l_z\})$ and $U(\{l_c \otimes l'_z\}) \geq U(\{l_c \otimes \tilde{l}_z\})$. From Lemma C.2 (i), $U(\{l'_c \otimes l_z\}) \geq U(\{\tilde{l}_c \otimes \tilde{l}_z\})$ and $U(\{\tilde{l}_c \otimes l'_z\}) \geq U(\{\tilde{l}_c \otimes \tilde{l}_z\})$. Thus, $\tilde{l} \in O(l') \subset O(x')$. Since $O(x') = x'$, this is a contradiction.

Since the above claim holds, by the separating hyperplane theorem, there exists $\alpha \in [0, 1]$ and $\gamma \in \mathbb{R}$ such that $(1 - \alpha)\tilde{u} + \alpha \tilde{w} > \gamma > (1 - \alpha)u' + \alpha w'$ for all $(u', w') \in V'$. Equivalently,

$$(1 - \alpha)u(\tilde{l}_c) + \alpha W(\tilde{l}_z) > \gamma > (1 - \alpha)u(l'_c) + \alpha W(l'_z),$$

for all $l' \in x'$. Hence,

$$\sigma_x(\alpha) = \max_{l \in x}((1 - \alpha)u(l_c) + \alpha W(l_z)) \geq (1 - \alpha)u(\tilde{l}_c) + \alpha W(\tilde{l}_z)$$
$$> \max_{l' \in x'}((1 - \alpha)u(l'_c) + \alpha W(l'_z)) = \sigma_{x'}(\alpha).$$

Therefore, $\sigma_x \neq \sigma_{x'}$.   □

Let $C \subset \mathscr{C}([0, 1])$ be the range of σ.

Lemma 5 *(i)* *C is convex.*
(ii) The zero function belongs to C.
(iii) The constant function equal to a positive number $c > 0$ belongs to C.
(iv) The supremum of any two points $f, f' \in C$ belongs to C. That is, $\max[f(\alpha), f'(\alpha)]$ *belongs to C.*
(v) For all $f \in C, f \geq 0$.

Proof (i) Take any $f, f' \in C$ and $\lambda \in [0, 1]$. There are $x, x' \in \mathscr{Z}_2$ satisfying $f = \sigma_x$ and $f' = \sigma_{x'}$. From Lemma 4 (ii),

$$\lambda f + (1 - \lambda)f' = \lambda \sigma_x + (1 - \lambda)\sigma_{x'} = \sigma_{O(\lambda x + (1-\lambda)x')} \in \mathscr{Z}_2.$$

Hence, C is convex.

(ii) Let $x \equiv O(\underline{l}) \in \mathscr{Z}_2$. Then, for all α,

$$\sigma_x(\alpha) = \max_{l \in O(\underline{l})} (1 - \alpha)u(l_c) + \alpha W(l_z) = (1 - \alpha)u(\underline{l}_c) + \alpha W(\underline{l}_z)$$

$$= (1 - \alpha)U(\{\underline{l}_c \otimes \underline{l}_z\}) + \alpha U(\{\underline{l}_c \otimes \underline{l}_z\}) = 0.$$

(iii) Recall that $\bar{l}$ is a maximal element of $U(\{\cdot\})$. Without loss of generality, assume $u(\bar{l}_c) \geq W(\bar{l}_z)$. From Nondegeneracy, there exists l_z^* such that $W(l_z^*) > W(\underline{l}_z) = 0$. Since $u(\bar{l}_c) \geq W(l_z^*) > 0 = u(\underline{l}_c)$, continuity of u implies that there exists l_c^* such that $u(l_c^*) = W(l_z^*)$. Let $c \equiv W(l_z^*) > 0$ and $x \equiv O(l_c^* \otimes l_z^*) \in \mathscr{Z}_2$. Then, for all α,

$$\sigma_x(\alpha) = \max_{l \in O(l_c^* \otimes l_z^*)} (1 - \alpha)u(l_c) + \alpha W(l_z) = (1 - \alpha)u(l_c^*) + \alpha W(l_z^*) = c.$$

(iv) There exist $x', x \in \mathscr{Z}_2$ such that $f = \sigma_x$ and $f' = \sigma_{x'}$. Let $f'' \equiv \sigma_{O(\text{co}(x \cup x'))} \in C$. Then, $f''(\alpha) = \max[\sigma_x(\alpha), \sigma_{x'}(\alpha)]$.

(v) There exists $x \in \mathscr{Z}_2$ such that $f = \sigma_x$. Since $O(\underline{l}) \subset x$, Lemma 5 (ii) implies $f(\alpha) = \sigma_x(\alpha) \geq \sigma_{O(\underline{l})}(\alpha) = 0$, for any α.   □

Define $T : C \to \mathbb{R}$ by $T(f) \equiv U(\sigma^{-1}(f))$. Notice that $T(0) = 0$ and $T(c) = c$, where 0 and c are identified with the zero function and the constant function equal to $c > 0$, respectively. Since U and σ are continuous and mixture linear, so is T.

Lemma 6 $T(\beta f + \gamma f') = \beta T(f) + \gamma T(f')$ *as long as $f, f', \beta f + \gamma f' \in C$, where* $\beta, \gamma \in \mathbb{R}_+$.

Proof For any $\beta \in [0, 1]$, $T(\beta f) = T(\beta f + (1 - \beta)0) = \beta T(f) + (1 - \beta)T(0) = \beta T(f)$, where 0 is the zero function. For any $\beta > 1$, let $f'' \equiv \beta f$. Since

$$T\left(\frac{1}{\beta}f''\right) = \frac{1}{\beta}T(f''), \ \beta T(f) = T(\beta f). \ \text{Additivity follows from } T(f + f') =$$
$$2T\left(\frac{1}{2}f + \frac{1}{2}f'\right) = T(f) + T(f'). \qquad\qquad\qquad\qquad\qquad\qquad \square$$

By the same argument as in DLR, we will extend T to $\mathscr{C}([0, 1])$ step by step. For any $r \geq 0$, let $rC \equiv \{rf | f \in C\}$ and $H \equiv \cup_{r\geq 0} rC$. For any $f \in H \setminus 0$, there is $r > 0$ satisfying $(1/r)f \in C$. Define $T(f) \equiv rT((1/r)f)$. From linearity of T on C, $T(f)$ is well-defined. That is, even if there is another $r' > 0$ satisfying $(1/r')f \in C$, $rT((1/r)f) = r'T((1/r')f)$. It is easy to see that T on H is mixture linear. By the same argument in Lemma 6, T is also linear.

Let

$$H^* \equiv H - H = \{f_1 - f_2 \in \mathscr{C}([0, 1]) | f_1, f_2 \in H\}.$$

For any $f \in H^*$, there are $f_1, f_2 \in H$ satisfying $f = f_1 - f_2$. Define $T(f) \equiv T(f_1) - T(f_2)$. We can verify that $T : H^* \to \mathbb{R}$ is well-defined. Indeed, suppose that f_1, f_2, f_3 and f_4 in H satisfy $f = f_1 - f_2 = f_3 - f_4$. Since $f_1 + f_4 = f_2 + f_3$, $T(f_1) + T(f_4) = T(f_2) + T(f_3)$ by linearity of T on H.

Lemma 7 H^* *is dense in* $\mathscr{C}([0, 1])$.

Proof From the Stone-Weierstrass theorem, it is enough to show that (i) H^* is a vector sublattice, (ii) H^* separates the points of $[0, 1]$; that is, for any two distinct points $\alpha, \alpha' \in [0, 1]$, there exists $f \in H^*$ with $f(\alpha) \neq f(\alpha')$, and (iii) H^* contains the constant functions equal to one. By the exactly same argument as Lemma 11 (p. 928) in DLR, (i) holds. To verify condition (ii), take $\alpha, \alpha' \in [0, 1]$ with $\alpha \neq \alpha'$. Without loss of generality, $\alpha' > \alpha$. Let $x \equiv O(\bar{l}_c \otimes \underline{l}_z)$. Then, $\sigma_x \in C \subset H^*$. Since $u(\bar{l}_c) > 0$ and $W(\underline{l}_z) = 0$,

$$\sigma_x(\alpha) = (1 - \alpha)u(\bar{l}_c) + \alpha W(\underline{l}_z) > (1 - \alpha')u(\bar{l}_c) + \alpha' W(\underline{l}_z) = \sigma_x(\alpha').$$

Finally, condition (iii) directly follows from Lemma 5 (iii) and the definition of H.
$$\square$$

Lemma 8 *There exists a constant $K > 0$ such that $T(f) \leq K\|f\|$ for any $f \in H^*$.*

Proof We use the same argument as in Theorem 2 of Dekel et al. (2007).[16] First, we claim that T is increasing in the pointwise order. Indeed, take any $g', g \in H^*$ with $g' \geq g$. Since H^* is a vector space, $g' - g \in H^*$. Hence there exist $f, f' \in C$ and $r > 0$ such that $r(f' - f) = g' - g \geq 0$. Thus $f' \geq f$ pointwise. Since $T(f') \geq T(f)$ by Monotonicity, $T(r(f' - f)) \geq T(0) = 0$ implies $T(g' - g) \geq 0$. That is, we have $T(g') \geq T(g)$.

---

[16]They fix the argument (Lemma 12, p. 929) of DLR.

For all $f \in H^*$, we have $f \leq \|f\|\mathbf{1}$, where $\mathbf{1} \in H$ is the function identically equal to 1. Since T is increasing, $T(f) \leq \|f\|T(\mathbf{1})$. Thus $K \equiv T(\mathbf{1})$ is the desired object.

□

By Lemma 8 and the Hahn-Banach theorem, we can extend $T : H^* \to \mathbb{R}$ to $\overline{T} : \mathscr{C}([0, 1]) \to \mathbb{R}$ in a linear, continuous and increasing way. Since H^* is dense in $\mathscr{C}([0, 1])$ by Lemma 7, this extension is unique.

Now we have the following commutative diagram:

$$
\begin{array}{ccc}
\mathscr{L}_2 & \xrightarrow{\ U\ } & \mathbb{R} \\
\sigma \downarrow & \nearrow{\overline{T}} & \\
\mathscr{C}([0, 1]) & &
\end{array}
$$

Since $\overline{T}$ is a positive linear functional on $\mathscr{C}([0, 1])$, the Riesz representation theorem ensures that there exists a unique countably additive probability measure μ on $[0, 1]$ satisfying

$$
\overline{T}(f) = \int_{[0,1]} f(\alpha)\mathrm{d}\mu(\alpha),
$$

for all $f \in \mathscr{C}([0, 1])$. Thus we have

$$
U(x) = \overline{T}(\sigma(x)) = \int_{[0,1]} \max_{l \in x} \left((1 - \alpha)u(l_c) + \alpha W(l_z) \right) \mathrm{d}\mu(\alpha).
$$

For any $x \in \mathscr{L}$, let δ_x be the degenerate measure at x. Denote $W(\delta_x)$ by $W(x)$.

Lemma 9 $U(x) \geq U(y) \Leftrightarrow W(x) \geq W(y)$.

Proof First of all,

$$
W(l_z) = \int_{\Delta(\mathscr{L})} W(x)\mathrm{d}l_z(x).
$$

Since

$$
U(\{(c, x)\}) = \int \left((1 - \alpha)u(c) + \alpha W(x) \right) \mathrm{d}\mu(\alpha) = (1 - \bar{\alpha})u(c) + \bar{\alpha}W(x),
$$

Stationarity implies that $U(x) \geq U(y) \Leftrightarrow U(\{(c, x)\}) \geq U(\{(c, y)\}) \Leftrightarrow W(x) \geq W(y)$.

□

Lemma 10 *There exist $\beta > 0$ and $\zeta \in \mathbb{R}$ such that $W(x) = \beta U(x) + \zeta$.*

Proof Since U is mixture linear, there exists $(\underline{c}, \underline{z}) \in C \times \mathscr{Z}$ such that $\{\underline{l}\} \sim \{(\underline{c}, \underline{z})\}$. Thus $W(l_z) = U(\{\underline{c} \otimes l_z\})$. We have

$$U(\{\lambda \circ (\underline{c}, x) + (1 - \lambda) \circ (\underline{c}, y)\}) = U(\{\underline{c} \otimes (\lambda \circ x + (1 - \lambda) \circ y)\})$$
$$= W(\lambda \circ x + (1 - \lambda) \circ y),$$

and $U(\{(\underline{c}, \lambda x + (1 - \lambda)y)\}) = W(\lambda x + (1 - \lambda)y)$. Since W is mixture linear over $\Delta(\mathscr{Z})$, Timing Indifference implies

$$\lambda W(x) + (1 - \lambda)W(y) = W(\lambda \circ x + (1 - \lambda) \circ y) = W(\lambda x + (1 - \lambda)y).$$

Hence, W is mixture linear over $\mathscr{Z}_1$. From Lemma 9, we know $U(x)$ and $W(x)$ represent the same preference. Since both functions are mixture linear, there exist $\beta > 0$ and $\zeta \in \mathbb{R}$ such that $W(x) = \beta U(x) + \zeta$. $\qquad\square$

We will claim that β can be normalized to one. Define $W^* : \Delta(\mathscr{Z}) \to \mathbb{R}$ by $W^*(l_z) = W(l_z)/\beta$. For any $x \in \mathscr{D}$, define $\sigma_x^* : [0, 1] \to \mathbb{R}$ by

$$\sigma_x^*(\alpha) \equiv \max_{l \in x} \Big((1 - \alpha)u(l_c) + \alpha W^*(l_z) \Big).$$

Since W^* is continuous and mixture linear, the same arguments up to Lemma 9 work even for σ^*. Thus, there exists a probability measure μ^* on $[0, 1]$ such that

$$U(x) = \int_{[0,1]} \max_{l \in x} \big((1 - \alpha)u(l_c) + \alpha W^*(l_z) \big) \, d\mu^*(\alpha).$$

By definition, $W^*(z) = U(z) + \zeta/\beta$.

Lemma 11 $\bar{\alpha} < 1$, where $\bar{\alpha}$ is the mean of μ^*.

Proof Since U is not constant, there exist x and x' such that $U(x) > U(x')$. For any fixed c, let

$$x^t \equiv \{(c, \{(c, \{\cdots \{(c, x)\} \cdots\})\})\}, \quad x'^t \equiv \{(c, \{(c, \{\cdots \{(c, x')\} \cdots\})\})\}.$$

Then,

$$U(x^t) - U(x'^t) = (1 - \bar{\alpha})\bar{\alpha}^t U(x) - (1 - \bar{\alpha})\bar{\alpha}^t U(x') = (1 - \bar{\alpha})(U(x) - U(x'))\bar{\alpha}^t.$$

Since Continuity requires $U(x^t) - U(x'^t) \to 0$ as $t \to \infty$, we must have $\bar{\alpha} < 1$. $\quad\square$

Define $\zeta^* \equiv \zeta/\beta$ and

$$u^*(l_c) \equiv u(l_c) + \frac{\bar{\alpha}}{1 - \bar{\alpha}} \zeta^*.$$

Then

$$U(x) = \int_{[0,1]} \max_{l \in x} \left((1-\alpha)u(l_c) + \alpha \int_{\mathscr{Z}} W^*(z)\, dl_z(z) \right) d\mu^*(\alpha)$$

$$= \int_{[0,1]} \max_{l \in x} \left((1-\alpha)u(l_c) + \alpha \int_{\mathscr{Z}} (U(z) + \zeta^*)\, dl_z(z) \right) d\mu^*(\alpha)$$

$$= \int_{[0,1]} \max_{l \in x} \left((1-\alpha) \left(u(l_c) + \frac{\bar{\alpha}}{1-\bar{\alpha}} \zeta^* \right) + \alpha \int_{\mathscr{Z}} U(z)\, dl_z(z) \right) d\mu^*(\alpha)$$

$$= \int_{[0,1]} \max_{l \in x} \left((1-\alpha)u^*(l_c) + \alpha \int_{\mathscr{Z}} U(z)\, dl_z(z) \right) d\mu^*(\alpha).$$

Therefore the functional form U with components (u^*, μ^*) is the required representation.

Proof of Theorem 2

(i) Since mixture linear functions u and u' represent the same conditional preference over $\Delta(C)$, by the standard argument, u' is rewritten as an affine transformation of u. That is, u and u' are cardinally equivalent.

(ii) From (i), there exist $\gamma > 0$ and $\zeta \in \mathbb{R}$ such that $u' = \gamma u + \zeta$. Since U and U' are mixture linear functions representing the same preference, there exist $\gamma^* > 0$ and $\zeta^* \in \mathbb{R}$ such that $U' = \gamma^* U + \zeta^*$. Let x_c be the perfect commitment menu to c, that is, $x_c \equiv \{(c, \{(c, \{\cdots\})\})\}$. Since $U(x_c) = u(c)$ and $U'(x_c) = u'(c)$, we have $U'(x_c) = \gamma U(x_c) + \zeta$, which implies $\gamma^* = \gamma$ and $\zeta^* = \zeta$. Now we have

$$U'(x) = \int_{[0,1]} \max_{l \in x} \left((1-\alpha)u'(l_c) + \alpha \int_{\mathscr{Z}} U'(z) dl_z \right) d\mu'(\alpha)$$

$$= \int_{[0,1]} \max_{l \in x} \left((1-\alpha)(\gamma u(l_c) + \zeta) + \alpha \int_{\mathscr{Z}} (\gamma U(z) + \zeta) dl_z \right) d\mu'(\alpha)$$

$$= \gamma \int_{[0,1]} \max_{l \in x} \left((1-\alpha)u(l_c) + \alpha \int_{\mathscr{Z}} U(z)\, dl_z \right) d\mu'(\alpha) + \zeta.$$

Hence,

$$U''(x) \equiv \int_{[0,1]} \max_{l \in x} \left((1-\alpha)u(l_c) + \alpha \int_{\mathscr{Z}} U(z)\, dl_z \right) d\mu'(\alpha)$$

also represents the same preference. Since $U' = \gamma U + \zeta$ and $U' = \gamma U'' + \zeta$, we must have $U(x) = U''(x)$ for all x. For all $x \in \mathscr{Z}$ and $\alpha \in [0, 1]$, let

$$\sigma_x(\alpha) \equiv \max_{l \in x} \left((1-\alpha)u(l_c) + \alpha \int U(z)\,\mathrm{d}l_z(x) \right).$$

Then,

$$U(x) = \int \sigma_x(\alpha)\,\mathrm{d}\mu(\alpha) = \int \sigma_x(\alpha)\,\mathrm{d}\mu'(\alpha) = U''(x). \tag{20.15}$$

If x is convex, σ_x is its support function. Equation (20.15) holds also when σ_x is replaced with $a\sigma_x - b\sigma_y$ for any convex menus x, y and $a, b \geq 0$. From Lemma C.6, the set of all such functions is a dense subset of the set of real-valued continuous functions over $[0, 1]$. Hence, Eq. (20.15) holds when σ_x is replaced with any real-valued continuous function. Hence, the Riesz representation theorem implies $\mu = \mu'$.

Proofs of Corollary 1 and Proposition 1

First we show Proposition 1. For all x, let l^x denote a best element in x with respect to commitment ranking.

Lemma 12 $\succsim$ *satisfies Dominance if and only if, for all x, $x \sim \{l^x\}$.*

Proof If $\succsim$ satisfies Dominance, $\{l^x\} \sim O^*(l^x) = O^*(x) \sim x$. Conversely, by definition of $O^*(x)$, l^x is a best element in $O^*(x)$. Thus $x \sim \{l^x\} \sim O^*(x)$. □

 (i) By Lemma 12, it suffices to show that Strategic Rationality is equivalent to the condition that $x \sim \{l^x\}$ for all x. First suppose that $\succsim$ satisfies $x \sim \{l^x\}$. Since $x \succsim y$ implies $\{l^x\} \succsim \{l^y\}$, l^x is a best element of $x \cup y$ with respect to commitment ranking. Hence $x \sim \{l^x\} \sim x \cup y$.

Next suppose $\succsim$ satisfies Strategic Rationality. Take any finite menu x, denoted by $\{l_1, l_2, \cdots, l_N\}$. Without loss of generality, let $l^x = l_1$. Since $\{l_1\} \succsim \{l_2\}$, Strategic Rationality implies $\{l_1, l_2\} \sim \{l_1\}$. Since $\{l_1, l_2\} \sim \{l_1\} \succsim \{l_3\}$, again by Strategic Rationality, $\{l_1, l_2, l_3\} \sim \{l_1, l_2\} \sim \{l_1\}$. Repeating the same argument finite times, $x \sim \{l^x\}$. For any menu x, Lemma 0 of Gul and Pesendorfer (2001, p. 1421) shows that there exists a sequence of finite subsets x^n of x converging to x in the sense of the Hausdorff metric. Since l^x is a best element of x and $x^n \subset x$, applying the above claim, $x^n \cup \{l^x\} \sim \{l^x\}$. Thus, by Continuity, $x = x \cup \{l^x\} \sim \{l^x\}$ as $n \to \infty$.

(ii) For all x, choose any $l \in O(x)$. By definition, there exists $l' \in x$ such that $l \in O(l')$. From part (i), preference satisfies Monotonicity. Applying Commitment Marginal Dominance and Monotonicity, we have $\{l^x\} \succsim \{l'\} \sim O(l') \succsim \{l\}$. Hence, l^x is a best element in $O(x)$. Therefore, by Lemma 12, $x \sim \{l^x\} \sim O(x)$.

Turn to the proof of Corollary 1. If part: The representation has the form of

$$U(x) = \max_{l \in x} \left\{ (1-\alpha)u(l_c) + \alpha \int_{\mathscr{Z}} U(z)\,\mathrm{d}l_z \right\},$$

for some $\alpha \in [0, 1)$. Thus it is easy to verify that $U(x) \geq U(y)$ implies $U(x) = U(x \cup y)$.

Only-if part: From Proposition 1, $\succsim$ satisfies all the axioms of Theorem 1. Hence $\succsim$ admits a random discounting representation U with components (u, μ). By contradiction, suppose $\#\mathrm{supp}(\mu) \neq 1$. Then, there exist $\alpha', \alpha'' \in \mathrm{supp}(\mu)$ with $\alpha'' > \alpha'$. Let $u(\Delta(C))$ denote the image of $\Delta(C)$ under u. Let $U(\mathscr{L})$ denote the image of $\mathscr{L} \subset \mathscr{Z}$ under U. Since $U(\mathscr{L})$ and $u(\Delta(C))$ are non-degenerate intervals of $\mathbb{R}_+$, take $p_1 \in u(\Delta(C))$ and $p_2 \in U(\mathscr{L})$ from the relative interiors. Take two points $(p_1', p_2'), (p_1'', p_2'') \in \mathbb{R}_+^2$ such that $p_1'' > p_1 > p_1'$, $p_2' > p_2 > p_2''$, and

$$(1-\alpha')p_1' + \alpha'p_2' = (1-\alpha')p_1 + \alpha'p_2, \text{ and } (1-\alpha'')p_1'' + \alpha''p_2'' = (1-\alpha'')p_1 + \alpha''p_2. \tag{20.16}$$

Since p_1 belongs to the relative interior of $u(\Delta(C))$, p_1', p_1'' can be taken to be in $u(\Delta(C))$. Similarly, we can assume p_2', p_2'' belong to $U(\mathscr{L})$. Then we have

$$(1-\alpha')p_1' + \alpha'p_2' > (1-\alpha')p_1'' + \alpha'p_2'', \text{ and } (1-\alpha'')p_1'' + \alpha''p_2'' > (1-\alpha'')p_1' + \alpha''p_2'. \tag{20.17}$$

Indeed, by contradiction, suppose $(1-\alpha')p_1'' + \alpha'p_2'' \geq (1-\alpha')p_1' + \alpha'p_2'$. By (20.16), $(1-\alpha')p_1'' + \alpha'p_2'' \geq (1-\alpha')p_1 + \alpha'p_2$. Since $p_1'' > p_1$, $p_2'' < p_2$, and $\alpha'' > \alpha'$, we have $(1-\alpha'')p_1'' + \alpha''p_2'' > (1-\alpha'')p_1 + \alpha''p_2$, which contradicts (20.16). The same argument can be applied to the other case. Now take lotteries $l_c', l_c'' \in \Delta(C)$ and $l', l'' \in \mathscr{L}$ such that $u(l_c') = p_1'$, $u(l_c'') = p_1''$, $U(\{l'\}) = p_2'$, and $U(\{l''\}) = p_2''$. Taking (20.17) and continuity of the inner product together, there exist open neighborhoods $B(\alpha')$ and $B(\alpha'')$ satisfying

$$(1-\alpha)u(l_c') + \alpha U(\{l'\}) > (1-\alpha)u(l_c'') + \alpha U(\{l''\}), \text{ and}$$
$$(1-\tilde{\alpha})u(l_c'') + \tilde{\alpha}U(\{l''\}) > (1-\tilde{\alpha})u(l_c') + \tilde{\alpha}U(\{l'\}), \tag{20.18}$$

for all $\alpha \in B(\alpha')$ and $\tilde{\alpha} \in B(\alpha'')$. Since α', α'' belong to the support of μ, $\mu(B(\alpha')) > 0$ and $\mu(B(\alpha'')) > 0$. Thus, by (20.18) and the representation,

$$U(\{l_c' \otimes \{l'\}, l_c'' \otimes \{l''\}\}) > \max[U(\{l_c' \otimes \{l'\}\}), U(\{l_c'' \otimes \{l''\}\})],$$

which contradicts Strategic Rationality.

Proof of Theorem 3

((a)$\Rightarrow$(b)) Since $\succsim^1$ and $\succsim^2$ are equivalent on $\mathscr{L}$, we have condition (i). Let $u^i(\Delta(C))$ denote the image of $\Delta(C)$ under u^i. Let $U^i(\mathscr{L})$ denote the image of $\mathscr{L} \subset \mathscr{Z}$ under U^i. Let l_c^+ and l_c^- be a maximal and a minimal lottery with respect to u^i. Since $u^i(l_c^+) \geq U^i(\{l\}) \geq u^i(l_c^-)$ for all $l \in \mathscr{L}$, we have $U^1(\mathscr{L}) = u^1(\Delta(C)) = u^2(\Delta(C)) = U^2(\mathscr{L})$. Let $\text{supp}(\mu^i)$ denote the support of μ^i. By contradiction, suppose that there exists $\alpha^* \in \text{supp}(\mu^1)$ with $\alpha^* \notin \text{supp}(\mu^2)$. Since $\text{supp}(\mu^2)$ is a relative closed set of $[0, 1]$, there exists a relative open interval (α^a, α^b) of α^* such that $(\alpha^a, \alpha^b) \cap \text{supp}(\mu^2) = \emptyset$. Since $u^1(\Delta(C))$ is a non-degenerate interval of $\mathbb{R}_+$, take $p_1 \in u^1(\Delta(C))$ and $p_2 \in U^1(\mathscr{L})$ from the relative interior. Take (p_1^a, p_2^a) and (p_1^b, p_2^b) such that $p_1^a > p_1 > p_1^b, p_2^b > p_2 > p_2^a$,

$$(1 - \alpha^a)p_1^a + \alpha^a p_2^a = (1 - \alpha^a)p_1 + \alpha^a p_2, \text{ and}$$

$$(1 - \alpha^b)p_1^b + \alpha^b p_2^b = (1 - \alpha^b)p_1 + \alpha^b p_2.$$

Then we have

$$(1 - \alpha)p_1^b + \alpha p_2^b > \max[(1 - \alpha)p_1 + \alpha p_2, \ (1 - \alpha)p_1^a + \alpha p_2^a] \text{ for all } \alpha > \alpha^b,$$

$$(1 - \alpha)p_1 + \alpha p_2 > \max[(1 - \alpha)p_1^a + \alpha p_2^a, \ (1 - \alpha)p_1^b + \alpha p_2^b] \text{ for all } \alpha \in (\alpha^a, \alpha^b),$$

$$(1 - \alpha)p_1^a + \alpha p_2^a > \max[(1 - \alpha)p_1 + \alpha p_2, \ (1 - \alpha)p_1^b + \alpha p_2^b] \text{ for all } \alpha < \alpha^a.$$

Since (p_1^a, p_2^a) and (p_1^b, p_2^b) can be chosen sufficiently close to (p_1, p_2), assume without loss of generality that $p_1^a, p_1^b \in u^1(\Delta(C))$ and $p_2^a, p_2^b \in U^1(\mathscr{L})$. Thus there exist $l_c, l_c^a, l_c^b \in \Delta(C)$ and $l, l^a, l^b \in \mathscr{L}$ such that $u^i(l_c) = p_1, u^i(l_c^a) = p_1^a, u^i(l_c^b) = p_1^b, U^i(\{l\}) = p_2, U^i(\{l^a\}) = p_2^a$, and $U^i(\{l^b\}) = p_2^b$. Define $x \equiv \{l_c \otimes \{l\}, l_c^a \otimes \{l^a\}, l_c^b \otimes \{l^b\}\} \in \mathscr{Z}^1$ and $y \equiv \{l_c^a \otimes \{l^a\}, l_c^b \otimes \{l^b\}\} \in \mathscr{Z}^1$. Since $(\alpha^a, \alpha^b) \cap \text{supp}(\mu^2) = \emptyset$, $U^2(x) = U^2(y)$. On the other hand, since $\mu^1((\alpha^a, \alpha^b)) > 0$, $U^1(x) > U^1(y)$. This contradicts the assumption that $\succsim^2$ desires more flexibility in the two-period model than $\succsim^1$.

((b)$\Rightarrow$(a)) Assume that $u^1 = u^2$ and $\text{supp}(\mu^1) \subset \text{supp}(\mu^2)$. Since $\succsim^i, i = 1, 2$ are equivalent on $\mathscr{L}$, we have $\overline{\alpha}^1 = \overline{\alpha}^2$. Consequently, $U^1(\{l\}) = U^2(\{l\})$ for all $l \in \mathscr{L}$. Now take all $x, y \in \mathscr{Z}^1$ with $y \subset x$ and assume $x \succ^1 y$. There exists $\alpha^* \in \text{supp}(\mu^1)$ such that

$$\max_{l \in x}(1 - \alpha^*)u^1(l_c) + \alpha^* U^1(\{l_L\}) > \max_{l \in y}(1 - \alpha^*)u^1(l_c) + \alpha^* U^1(\{l_L\}). \quad (20.19)$$

By continuity of the representation, there exists an open neighborhood $O \subset [0, 1]$ of α^* such that strict inequality (20.19) holds for all $\alpha \in O$. Since $\alpha^* \in \text{supp}(\mu^1) \subset \text{supp}(\mu^2)$, $\mu^2(O) > 0$. Moreover, since $u^1 = u^2$ and $U^1(\{l\}) = U^2(\{l\})$ for all $l \in \mathscr{L}$,

$$\max_{l \in x}(1 - \alpha)u^2(l_c) + \alpha U^2(\{l_L\}) > \max_{l \in y}(1 - \alpha)u^2(l_c) + \alpha U^2(\{l_L\})$$

for all $\alpha \in O$, which implies $U^2(x) > U^2(y)$.

Proof of Theorem 4

By definition, (a) implies (b). We show that (b)$\Rightarrow$(c) and (c)$\Rightarrow$(a). To show (b)$\Rightarrow$(c), we prepare two lemmas.

Lemma 13 *Suppose that $\succsim^i$ satisfies all the axioms of Theorem 1. If agent 2 is more averse to commitment in the two-period model than agent 1, then $x \succsim^1 \{l\} \Rightarrow x \succsim^2 \{l\}$ for all $x \in \mathscr{X}^1$ and $l \in \mathscr{L}$.*

Proof It suffices to show that $x \sim^1 \{l\} \Rightarrow x \succsim^2 \{l\}$. If agent 1 strictly prefers l to the worst lottery $\underline{l}$, then $x \succ^1 \{\lambda l + (1 - \lambda)\underline{l}\}$ for all $\lambda \in (0, 1)$. By assumption, $x \succ^2 \{\lambda l + (1 - \lambda)\underline{l}\}$. Thus Continuity implies $x \succsim^2 \{l\}$ as $\lambda \to 1$. If l is indifferent to $\underline{l}$, consider the best lottery $\bar{l}$. Since $\{\bar{l}\} \succsim x$ for all $x \in \mathscr{X}^1$, mixture linearity of the representation implies $U^1(\lambda x + (1 - \lambda)\{\bar{l}\}) > U^1(\{l\})$ for all $\lambda \in (0, 1)$. By assumption, $\lambda x + (1-\lambda)\{\bar{l}\} \succ^2 \{l\}$. Thus Continuity implies $x \succsim^2 \{l\}$ as $\lambda \to 1$. $\square$

Lemma 14 *Agent 2 is more averse to commitment in the two-period model than agent 1 if and only if there exist random discounting representations U^i with (u^i, μ^i), $i = 1, 2$ such that (i) $u^1 = u^2$ and $\bar{\alpha}^1 = \bar{\alpha}^2$, and (ii) $U^1(x) \le U^2(x)$ for all $x \in \mathscr{X}^1$.*

Proof Necessity follows because $U^2(x) \ge U^1(x) > U^1(\{l\}) = U^2(\{l\})$ for all $x \in \mathscr{X}^1$. We prove sufficiency. By Lemma 13, $\succsim^1$ and $\succsim^2$ are equivalent on $\mathscr{L}$. Thus there exist random discounting representations satisfying (i), and hence $U^1(\{l\}) = U^2(\{l\})$ for all $l \in \mathscr{L}$. Note that for all $x \in \mathscr{X}^1$ there exists $l \in \mathscr{L}$ such that $x \sim^1 \{l\}$ or $U^1(x) = U^1(\{l\})$. By Lemma 13, $x \sim^1 \{l\}$ implies that $x \succsim^2 \{l\}$ or $U^2(x) \ge U^2(\{l\}) = U^1(\{l\}) = U^1(x)$. $\square$

We show that, for all continuous and convex functions v of α, there is a sequence $\{v_n\}$ of functions of the form (20.8) such that $v \ge v_n$,

$$\sup_\alpha |v(\alpha) - v_n(\alpha)| < \frac{1}{n}, \text{ and } \int v_n(\alpha)\, \mathrm{d}\mu^1(\alpha) \le \int v_n(\alpha)\, \mathrm{d}\mu^2(\alpha)$$

for all $n = 1, 2, \cdots$. Then the result follows from the dominated convergence theorem.

Let $v : [0, 1] \to \mathbb{R}$ be a continuous convex function. Then, for every $\hat{\alpha} \in [0, 1]$, there exists a vector $p_{\hat{\alpha}} \in \mathbb{R}^2$ such that for all $\alpha \in [0, 1]$,

$$v(\alpha) \ge (1 - \alpha)p_{\hat{\alpha},1} + \alpha p_{\hat{\alpha},2}$$

with equality for $\hat{\alpha}$. Fix n. Since $v(\alpha) - \{(1-\alpha)p_{\hat{\alpha},1} + \alpha p_{\hat{\alpha},2}\}$ is continuous with respect to α, there exists an open neighborhood $B(\hat{\alpha})$ of $\hat{\alpha}$ such that for every $\alpha \in B(\hat{\alpha})$

$$0 \leq v(\alpha) - \{(1-\alpha)p_{\hat{\alpha},1} + \alpha p_{\hat{\alpha},2}\} < \frac{1}{n}.$$

It follows from the compactness of $[0, 1]$ that there exists a finite set $\{\hat{\alpha}_i\}_{i=1}^M \subset [0, 1]$ such that $\{B(\hat{\alpha}_i)\}_{i=1}^M$ is a covering of $[0, 1]$.

We define $v_n : [0, 1] \to \mathbb{R}$ by

$$v_n(\alpha) = \max_i [(1-\alpha)p_{\hat{\alpha}_i,1} + \alpha p_{\hat{\alpha}_i,2}].$$

Then it is straightforward that $v(\alpha) \geq v_n(\alpha)$ for every $\alpha \in [0, 1]$. Moreover, we see that

$$\sup_{\alpha} |v(\alpha) - v_n(\alpha)| < \frac{1}{n}.$$

In fact, pick an arbitrary $\alpha \in [0, 1]$. Then there is $j \in M$ such that $\alpha \in B(\hat{\alpha}_j)$. This implies

$$0 \leq v(\alpha) - v_n(\alpha) \leq v(\alpha) - \{(1-\alpha)p_{\hat{\alpha}_j,1} + \alpha p_{\hat{\alpha}_j,2}\} < \frac{1}{n}.$$

Finally we see that

$$\int v_n(\alpha)\, d\mu^1(\alpha) \leq \int v_n(\alpha)\, d\mu^2(\alpha).$$

Since $u(\Delta(C))$ and $U^1(\mathcal{L}) = U^2(\mathcal{L})$ are closed intervals, we can assume, without loss of generality, that there exist $\{l_{c,i}\}_{i=1}^M \subset \Delta(C)$ and $\{l_i\}_{i=1}^M \subset \mathcal{L}$ satisfying

$$u^1(l_{c,i}) = u^2(l_{c,i}) = p_{\hat{\alpha}_i,1}, \text{ and} U^1(\{l_i\}) = U^2(\{l_i\}) = p_{\hat{\alpha}_i,2}.$$

Thus we can rewrite v_n by

$$v_n(\alpha) = \max_i (1-\alpha)u^1(l_{c,i}) + \alpha U^1(\{l_i\}) = \max_i (1-\alpha)u^2(l_{c,i}) + \alpha U^2(\{l_i\}).$$

Consider the menu $x^n = \{l_{c,i} \otimes \{l_i\} | i = 1, \cdots, M\} \in \mathcal{X}^1$. Then it follows from Lemma 14 that

$$\int v_n(\alpha)\, d\mu^1(\alpha) = U^1(x^n) \leq U^2(x^n) = \int v_n(\alpha)\, d\mu^2(\alpha),$$

which completes the proof.

((c)⇒(a)) Let $\mathscr{U}$ be the Banach space of all real-valued continuous functions on $\mathscr{Z}$. Define the operator $T^i : \mathscr{U} \to \mathscr{U}$ by

$$T^i(U)(x) \equiv \int \max_{l \in x} \left((1 - \alpha) u(l_c) + \alpha \int_{\mathscr{Z}} U(z) \, dl_z \right) d\mu^i(\alpha).$$

Pick $x \in \mathscr{Z}$ arbitrarily. Note that, for all $U \in \mathscr{U}$, $\max_{l \in x}\big((1 - \alpha) u(l_c) + \alpha \int_{\mathscr{Z}} U(z) \, dl_z\big)$ is continuous and convex with respect to α. Hence it holds that $T^1(U)(x) \leq T^2(U)(x)$ for all $x \in \mathscr{Z}$.

For $i = 1, 2$, let $T^{i,n}$ denote the operation defined as n-times iterations of T^i. We show, by mathematical induction, that $T^{1,n}(U)(x) \leq T^{2,n}(U)(x)$ for all $x \in \mathscr{Z}$ and $n = 1, 2, \cdots$. Assume that $T^{1,k}(U)(x') \leq T^{2,k}(U)(x')$ for all $x' \in \mathscr{Z}$. Pick $x \in \mathscr{Z}$ arbitrarily. Then it holds that $T^2(T^{1,k}(U))(x) \leq T^2(T^{2,k}(U))(x)$. Moreover, since $T^{1,k}(U)$ is in $\mathscr{U}$, we have $T^1(T^{1,k}(U))(x) \leq T^2(T^{1,k}(U))(x)$. These together imply $T^{1,k+1}(U)(x) \leq T^{2,k+1}(U)(x)$ for all $x \in \mathscr{Z}$. Therefore, it holds that $U^1(x) \leq U^2(x)$ since $T^{i,n}(U)$ converges to U^i. The desired result follows because $U^2(x) \geq U^1(x) > U^1(\{l\}) = U^2(\{l\})$ for all $x \in \mathscr{Z}$ and $l \in \mathscr{L}$.

Proof of Theorem 5

(i) We can solve (20.11) by the guess-and-verify method. Let

$$V_\mu(s, \alpha) \equiv A_\mu(\alpha) \frac{s^{1-\sigma}}{1 - \sigma}. \tag{20.20}$$

Considering the F.O.C. of

$$\max_{s'} \left((1 - \alpha) \frac{((1 + r)s - s')^{1-\sigma}}{1 - \sigma} + \alpha \int \left(A_\mu(\alpha') \frac{s'^{1-\sigma}}{1 - \sigma} \right) d\mu(\alpha') \right), \tag{20.21}$$

we have

$$(1 - \alpha)((1 + r)s - s')^{-\sigma} = \alpha \overline{A}_\mu s'^{-\sigma},$$

where $\overline{A}_\mu \equiv \int A_\mu(\alpha') \, d\mu(\alpha')$. By rearrangement, we can obtain the savings function

$$s' = SR(\alpha, \overline{A}_\mu)(1 + r)s, \text{ where } SR(\alpha, \overline{A}_\mu) \equiv \frac{(\alpha \overline{A}_\mu)^{\frac{1}{\sigma}}}{(1 - \alpha)^{\frac{1}{\sigma}} + (\alpha \overline{A}_\mu)^{\frac{1}{\sigma}}}. \tag{20.22}$$

Substituting (20.22) into (20.21) and comparing the coefficients with (20.20),

$$A_\mu(\alpha) = \left((1 - \alpha)^{\frac{1}{\sigma}} + (\alpha \overline{A}_\mu)^{\frac{1}{\sigma}} \right)^\sigma (1 + r)^{1-\sigma}. \tag{20.23}$$

For all $\alpha \in [0, 1]$ and $A \geq 0$, define $f(\alpha, A)$ and $F(A)$ as

$$f(\alpha, A) \equiv \left((1-\alpha)^{\frac{1}{\sigma}} + (\alpha A)^{\frac{1}{\sigma}}\right)^{\sigma} (1+r)^{1-\sigma}, \quad F(A) \equiv \int f(\alpha, A) \, d\mu(\alpha). \quad (20.24)$$

From (20.23), $\overline{A}_\mu$ is characterized as a solution of $\overline{A}_\mu = F(\overline{A}_\mu)$. We want to show that there exists a unique $\overline{A} > 0$ satisfying this equation. Note first that $F(0) = (1-\bar{\alpha})(1+r)^{1-\sigma} > 0$. Since $\lim_{A\to\infty} F(A) \to \infty$, the L'Hopital's rule implies

$$\lim_{A\to\infty} \frac{F(A)}{A} = \lim_{A\to\infty} F'(A).$$

Since

$$F'(A) = \int \frac{\partial f}{\partial A} \, d\mu(\alpha) = \int (1+r)^{1-\sigma} \alpha^{\frac{1}{\sigma}} \left((1-\alpha)^{\frac{1}{\sigma}} + (\alpha A)^{\frac{1}{\sigma}}\right)^{\sigma-1} A^{\frac{1}{\sigma}-1} d\mu(\alpha)$$

$$= \int (1+r)^{1-\sigma} \alpha^{\frac{1}{\sigma}} \left(\left(\frac{1-\alpha}{A}\right)^{\frac{1}{\sigma}} + \alpha^{\frac{1}{\sigma}}\right)^{\sigma-1} d\mu(\alpha),$$

we have $\lim_{A\to\infty} F'(A) = \bar{\alpha}(1+r)^{1-\sigma} < 1$. Hence, there exists a sufficiently large number $\tilde{A}$ such that $F(\tilde{A}) < \tilde{A}$. By continuity of F, there exists $\overline{A} > 0$ such that $\overline{A} = F(\overline{A})$. Finally, since

$$F'' = \int \frac{\partial^2 f}{\partial A^2} \, d\mu(\alpha)$$

$$= \frac{1-\sigma}{\sigma}(1+r)^{1-\sigma} \int \alpha^{\frac{2}{\sigma}} (1-\alpha)^{\frac{1}{\sigma}} \left((1-\alpha)^{\frac{1}{\sigma}} + (\alpha A)^{\frac{1}{\sigma}}\right)^{\sigma-2} A^{\frac{1}{\sigma}-2} \, d\mu(\alpha),$$

F is either strictly convex or concave depending on $\sigma \lessgtr 1$. Since $\lim_{A\to\infty} F'(A) < 1, \overline{A}$ must be unique.

(ii) From (20.22), it is easy to verify that $\frac{\partial SR(\alpha, A)}{\partial A} > 0$. Thus it suffices to show that $\overline{A}_{\mu^1} \lessgtr \overline{A}_{\mu^2}$ if $\sigma \lessgtr 1$. Note first that f defined as (20.24) is strictly convex or strictly concave in α according as $\sigma < 1$ or $\sigma > 1$. Indeed, for any $\alpha \in (0, 1)$ and $A > 0$,

$$\frac{\partial^2 f}{\partial \alpha^2} = (1+r)^{1-\sigma} \frac{1-\sigma}{\sigma} \left((1-\alpha)^{\frac{1}{\sigma}} + (\alpha A)^{\frac{1}{\sigma}}\right)^{\sigma-2}$$

$$\times \left(2(\alpha(1-\alpha))^{\frac{1}{\sigma}-1} A^{\frac{1}{\sigma}} + (1-\alpha)^{\frac{1}{\sigma}-2}(\alpha A)^{\frac{1}{\sigma}} + \alpha^{\frac{1}{\sigma}-2}((1-\alpha)A)^{\frac{1}{\sigma}}\right) \gtrless 0$$

whenever $\sigma \lessgtr 1$. Since μ^1 second-order stochastically dominates μ^2,

$$\overline{A}_{\mu^2} = \int f(\alpha, \overline{A}_{\mu^2}) \, d\mu^2(\alpha) \gtrless \int f(\alpha, \overline{A}_{\mu^2}) \, d\mu^1(\alpha) \quad (20.25)$$

depending on $\sigma \lessgtr 1$. Let $F^1(A) \equiv \int f(\alpha, A)\, d\mu^1(\alpha)$. We know from the proof of part (i) that $F^1(0) > 0$ and $\overline{A}_{\mu^1}$ is a unique positive solution of $F^1(A) = A$. Hence, $F^1(A) \gtrless A$ if $A \lessgtr \overline{A}_{\mu^1}$. Taking this observation and (20.25) together, $\overline{A}_{\mu^2} \gtrless \overline{A}_{\mu^1}$ if $\sigma \lessgtr 1$.

Addendum: Recent Developments[17]

If a DM is uncertain about her future preference, she may prefer to leave some options open rather than choose a completely spelled-out future plan. This behavior is called *preference for flexibility* and is admitted as an important aspect of sequential decision making. Kreps (1979, 1992) and Dekel et al. (2001) provide a behavioral foundation for preference for flexibility and derive the set of future preferences, called *the subjective state space*, from observable choice behavior. Although the preference for flexibility arises inherently in a dynamic setup, the derivation of the subjective state space has been considered within a two-period model. Higashi et al. (2009) extend their model to an infinite-horizon setting and specify the subjective state space to be the set of sequences of discount factors.

In Higashi et al. (2009), the belief of future discount factors is assumed to be constant, and hence, the DM's attitude toward flexibility is the same over time. Some recent works consider more general models for preference for flexibility in a dynamic setup. Krishna and Sadowski (2014) provide a complementary result to ours. To introduce their model, let S be a finite set of objective states. A *state-contingent infinite-horizon consumption problem (S-IHCP)* is a function specifying for each $s \in S$ an opportunity set of lotteries over pairs of current consumption and an S-IHCP in the next period. They show that there exists a compact metric space $\mathscr{F}$, which is linearly homeomorphic to $\mathscr{H}(\mathscr{K}(\Delta(C \times \mathscr{F})))$, where $\mathscr{H}(X)$ means the set of functions from S to a compact metric space X.[18] A generic element of $\mathscr{F}$ is denoted by f and for lottery $l \in \Delta(C \times \mathscr{F})$ its marginals on C and $\mathscr{F}$ are denoted by l_c and l_f, respectively. A preference $\succsim$ is defined on $\mathscr{F} \simeq \mathscr{H}(\mathscr{K}(\Delta(C \times \mathscr{F})))$.

Krishna and Sadowski (2014) consider the following representation. The DM has a subjective belief about objective states S captured by a Markov process, that is, a pair of a transition probability $\Pi : S \times S \to [0, 1]$ and a stationary distribution (or initial prior) π over S. The subjective state space of this model is the set of all vN-M functions over C denoted by $\mathscr{U} := \{u \in \mathbb{R}^C : \sum u_i = 0\}$. A belief about subjective states depends on objective states, that is, for each $s \in S$, μ_s is a probability measure on $\mathscr{U}$. Finally, let $\delta \in (0, 1)$ be a discount factor. A preference $\succsim$ on $\mathscr{F}$ admits a representation of Dynamic Preference for Flexibility (a DPF representation) with components $((\Pi, \pi), (\mu_s)_{s \in S}, \delta)$ if $V_0(f) \equiv \sum_s V(f, s)\pi(s)$ represents $\succsim$, where

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[17]This addendum has been newly written for this book chapter.

[18]In their model, C is assumed to be finite.

$V(\cdot, s) : \mathscr{F} \to \mathbb{R}$ is defined as

$$V(f, s) = \sum_{s' \in S} \Pi(s, s') \left[\int_{\mathscr{U}} \max_{l \in f(s')} [u(l_c) + \delta V(l_f, s')] \, \mathrm{d}\mu_{s'}(u) \right],$$

and $V(l_f, s') \equiv \int V(g, s') \, \mathrm{d}l_f(g)$. Since π is a stationary distribution, which satisfies $\pi(s') = \sum_s \pi(s) \Pi(s, s')$, the representation is rewritten as

$$V_0(f) = \sum_s \pi(s) \left[\int_{\mathscr{U}} \max_{l \in f(s)} [u(l_c) + \delta V(l_f, s)] \, \mathrm{d}\mu_s(u) \right].$$

In this representation, the Markov process represented by Π captures persistent shocks on objective states, and the probability measures $(\mu_s)_{s \in S}$ correspond to unobservable transitory shocks on future utilities. Therefore, an attitude toward flexibility may change according to realization of objective states. Krishna and Sadowski prove that a preference $\succsim$ satisfies suitable axioms if and only if it has a DPF representation. Moreover, they prove that $(\mu_s)_{s \in S}$ are unique up to a common scaling, and (Π, π) and δ are unique.[19]

There are two remarks related to our study (Higashi et al. 2009). First, their DPF representation allows persistent shocks on subjective states, while preference shocks are i.i.d. in our model. Second, a DPF representation can capture a random discounting by specifying $\mu_s(\{\lambda \bar{u} : \lambda \geq 0\}) = 1$ for some fixed $\bar{u} \in \mathscr{U}$. As a special case, a random discount factor representation is behaviorally characterized.

Another attempt to accommodate a changing preference for flexibility is made by Higashi et al. (2014). In this paper, we extend the previous model in order to allow the situation where a prior action affects future attitude toward flexibility. For example, imagine a DM who invested in self-improvement such as health investment or education is more likely to expect new information about her future preference, and hence may want to have greater demand for flexibility. More formally, we incorporate the histories of past consumption, $h = (c_{-T}, \cdots, c_{-1})$, into Higashi et al. (2009) and consider a set of preferences $\{\succsim_h\}_{h \in H}$. The following recursive representation is axiomatized: there exist a non-constant continuous function $u : C \to \mathbb{R}$ and a history-dependent probability measure μ_h on the set $[0, 1]$ of discount factors such that for all h, $\succsim_h$ on $\mathscr{Z}$ is represented by

$$V(x, h) = \int_{[0,1]} \max_{l \in x} \int_{C \times \mathscr{Z}} \Big((1 - \alpha) u(c) + \alpha V(z, hc) \Big) \mathrm{d}l(c, z) \, \mathrm{d}\mu_h(\alpha),$$

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[19]In the supplement to Krishna and Sadowski (2014), Krishna and Sadowski (2013) show a similar result for a preference $\succsim$ on $\mathscr{Z} \simeq \mathscr{K}(\Delta(C \times \mathscr{Z}))$.

where $hc = (c_{-T+1}, \cdots, c_{-1}, c)$ denotes an updated history of $h = (c_{-T}, \cdots, c_{-1})$. This representation can capture a changing future attitude toward flexibility from past consumption.

As an application of the random discounting, Higashi et al. (2014) investigate impatience comparisons within the random discounting model. Time preference has been measured as the magnitude of the discount factor, which is elicited from choices among consumption streams with a time trade-off. This elicitation implicitly assumes that choices are made under commitment. In sequential decision making, however, the degree of impatience may be affected by two potentially conflicting effects: one is pure time preference, which is a preference for early consumption, and the other is preference for flexibility, which is an attitude of leaving one's options open until the future.

In this paper, we consider preference over menus of consumption streams in two periods and provide behavioral definitions for impatience comparisons among menus having a time trade-off. If one menu includes more options allowing earlier consumption than another menu (such as $x = \{(100, 0), (70, 35)\}$ vs $y = \{(50, 60), (0, 120)\}$), an agent expecting to be more impatient in the future will tend to choose the former. Thus, if agent 2 is more impatient than agent 1, we require that

$$x \succsim^1 y \implies x \succsim^2 y.$$

This is a natural extension of impatience comparisons made under commitment. We show that in the random discounting model, the relative degree of impatience is measured as a probability shift in the *monotone likelihood ratio order (MLR)*, which is characterized via behavioral comparisons among menus defined as above.

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