Uniqueness and Stability for Double Crystals in the Plane

Eriko Shinkawa

Abstract We study a mathematical model of small double crystals, that is, two connected regions in $\mathbf{R}^{n+1}$ with prescribed volumes and with surface tension depending on the direction of the each point of the surface. Each double crystal is a critical point of the anisotropic surface energy which is the integral of the surface tension over the surface. We derive the first and the second variation formulas of the energy functional. For $n = 1$ and a certain special energy density function, we classify the double crystals in terms of symmetry and the given areas. Also, we prove that some of the double crystals are unstable, that is they are not local minimizers of the energy.

Keywords Anisotropic · Uniqueness · Stability · Double crystal

1 Introduction

There was a long-standing conjecture which was called the double bubble conjecture. It says that the standard double bubble provides the least-perimeter way to enclose and separate two given volumes, here the standard double bubble is consisting of three spherical caps meeting along a common circle at 120 degree angles. This conjecture had been believed since about 1870, and was proved in 2002. The existence of the minimizer was proved by Almgren [3] in 1976. (This paper proved, more general case, minimizing surface enclosing k prescribed volumes in $\mathbf{R}^{n+1}$, using geometric measure theory.) In 1993, the double bubble conjecture was proved in the plane by Foisy et al. [2] advised by Frank Morgan. For higher dimensional case, Hutchings [5] proved that any minimizer is axially symmetric and he also obtained a bound of the number of connected components of the two regions of a minimizer. Using these results, finally in 2002, the double bubble conjecture was proved by Hutchings et al. [6] in $\mathbf{R}^3$, and a student of Morgan extended it to higher dimensions [1].

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Double bubbles are a mathematical model of soap bubbles. The energy functional is the total area of the surface. On the other hand, when we think about a mathematical model of anisotropic substance like crystals, we need to consider the energy density function $\gamma : S^n \to \mathbf{R}^+$ depending on the normal direction N of the surface, where $S^n := \{X \in \mathbf{R}^{n+1} \mid \|X\| = 1\}$ is the n-dimensional unit sphere in $\mathbf{R}^{n+1}$. γ is called an anisotropic energy density function, and its integral $\mathscr{F} = \int_\Sigma \gamma(N)d\Sigma$ over the surface Σ is called an anisotropic (surface) energy. The surface is a constant anisotropic mean curvature (CAMC) surface if it is a critical point of the anisotropic energy for all volume preserving variations. CAMC surfaces are a generalization of CMC (constant mean curvature) surfaces.

In this paper, we extend the double bubble problem to a double crystal (DC) problem, that is, we minimize the anisotropic energy instead of the surface area. The solutions are a mathematical model of multiple crystals.

There were some previous researches relating to the DC problem. Gary [4] determined the energy-minimizer for the case where each anisotropic energy density function γ_i $(i = 0, 1, 2)$ is constant (we consider three surfaces, so we need three kinds of anisotropic energy density functions). Hence, his γ_i's are isotropic. His work also means that he gave a new proof of the double bubble conjecture. Besides, for $\gamma := \gamma_1 = \gamma_2 = \gamma_0$ that γ is a norm on $\mathbf{R}^2$, Morgan et al. [11] determined the shapes of the all minimizers for the case of $\gamma_i(\nu_1, \nu_2) = |\nu_1| + |\nu_2|$ $(i = 0, 1, 2)$ $((\nu_1, \nu_2) \in S^1)$.

Recall that there is a unique hypersurface that minimizes $\mathscr{F}$ among all closed hypersurfaces enclosing the same volume (cf. [13]). This surface is known as the Wulff shape. In this paper, we assume that the Wulff shape is smooth. We will derive the first variation formula for the anisotropic energy $\mathscr{F}$ (Theorem 1), and obtain the conditions for a surface Σ to be a double crystal (Theorem 2). Also, we will obtain the second variation formula for $\mathscr{F}$ (Theorem 3) and obtain the condition for a double crystal to be stable.

For $n = 1$, we will consider a special energy density function $\gamma := \gamma_1 = \gamma_2 = \gamma_0$ satisfying

$$\gamma(\nu_1, \nu_2) = \left(\nu_1^{2p} + \nu_2^{2p}\right)^{1-\frac{1}{2p}} / \sqrt{\nu_1^{4p-2} + \nu_2^{4p-2}}.$$

We classify the double crystals in terms of symmetry and the given areas. Also, we prove that some of the double crystals are unstable, that is they are not local minimizers of the energy.

We will explain our problem more precisely in Sect. 2. In Sect. 3, we derive the first and the second variation formulas of the anisotropic surface energy. In Sect. 4, we study the DC problem in the plane.

This paper is essentially a part of the author's doctoral dissertation [12].

2 Preliminaries

In this section, first we introduce some fundamental facts about CAMC surfaces (for details, see [8]). Then, we formulate the DC problem.

Let $\gamma : S^n \to \mathbf{R}^+$ be a positive smooth function on the unit sphere S^n in $\mathbf{R}^{n+1}$. We call this function γ an anisotropic energy density function. Let Σ be an n-dimensional oriented compact C^∞ manifold with or without boundary. And let $X : \Sigma \to \mathbf{R}^{n+1}$ be an immersion with Gauss map (unit normal) $N : \Sigma \to S^n$ be its Gauss map. The anisotropic energy of X is defined as

$$\mathscr{F}(X) = \int_\Sigma \gamma(N) d\Sigma,$$

where $d\Sigma$ is the volume form on Σ induced by X. Any smooth variation $\tilde{X} : \Sigma \times [-\varepsilon_0, \varepsilon_0] \to \mathbf{R}^{n+1}$ ($\varepsilon_0 > 0$) of X can be represented as $\tilde{X}(*, \varepsilon) = X_\varepsilon = X + \varepsilon(Z + \varphi N) + \mathscr{O}(\varepsilon^2)$, where Z is tangent to X. The first variation of $\mathscr{F}$ for this variation is (cf. Proof of Proposition 3.1 in [8])

$$\begin{aligned} \delta\mathscr{F} &:= \frac{d}{ds}\Big|_{\varepsilon=0} \mathscr{F}(X_\varepsilon) \\ &= \int_\Sigma \varphi(\mathrm{div}_\Sigma D\gamma - nH\gamma) d\Sigma + \oint_{\partial\Sigma} -\varphi\langle D\gamma, \nu\rangle + \gamma\langle Z, \nu\rangle \, ds, \end{aligned} \tag{1}$$

where D is the gradient on S^n, H is the mean curvature of X, ν is the outward pointing unit conormal of X along $\partial\Sigma$, and ds is the $(n-2)$-dimensional volume form of $\partial\Sigma$. $\Lambda := -\mathrm{div}_\Sigma D\gamma + nH\gamma$ is called the anisotropic mean curvature of X. X is called a Constant Anisotropic Mean Curvature (CAMC) hypersurface when $\Lambda \equiv$ constant. We remark that X is CAMC if and only if $\delta\mathscr{F} = 0$ for all compactly-supported $(n+1)$-dimensional-volume-preserving variations. For $\gamma \equiv 1$, we get $\Lambda = nH$. It means that CAMC surface is a generalization of CMC surface.

It is known that there is a unique (up to translation in $\mathbf{R}^{n+1}$) minimizer of $\mathscr{F}$ among all closed hypersurfaces enclosing the same volume (cf. [13]), and it is a rescaling of the so-called Wulff shape. The Wulff shape (we denote it by W) is a closed convex hypersurface defined by

$$W := \partial \bigcap_{N \in S^n} \{w \in \mathbf{R}^{n+1} | \langle w, N\rangle \le \gamma(N)\}$$

When the W is smooth and strictly convex (that is, all principal curvatures are positive with respect to the inward normal. This condition is equivalent to the condition that $A := D^2\gamma + \gamma \cdot 1$ is positive definite at earh $N \in S^n$, where $D^2\gamma$ is the Hessian of γ on S^n, and 1 is the identity map on $T_N S^n$. This condition is called the convexity condition), W can be parametrized as an embedding $\Phi : S^n \to W \subset \mathbf{R}^{n+1}$:

$$\Phi(N) = D\gamma + \gamma(N)N.$$

The anisotropic mean curvature of W is n with respect to the inward normal.

From now on, we assume that the convexity condition is satisfied.

For later use, we give a new representation of the 1st variation formula:

Lemma 1 *The first variation of $\mathscr{F}$ for the variation $X_\varepsilon = X + \varepsilon Y + \mathscr{O}(\varepsilon^2)$ is*

$$\delta\mathscr{F} = -\int_\Sigma \varphi\Lambda \, d\Sigma + \oint_{\partial\Sigma} \langle \Phi, -\varphi\nu + fN\rangle \, ds,$$

where $\varphi := \langle Y, N\rangle$ and $f := \langle Y, \nu\rangle$.

Proof We compute the integrand of the second term of (1).

$$\begin{aligned} -\varphi\langle D\gamma, \nu\rangle + \gamma\langle Z, \nu\rangle &= \langle -\varphi(D\gamma + \gamma N), \nu\rangle + \gamma f \\ &= \langle \Phi, -\varphi\nu\rangle + \langle D\gamma + \gamma N, N\rangle f \\ &= \langle \Phi, -\varphi\nu + fN\rangle, \end{aligned}$$

which proves the desired result. □

If $n = 1$, curves with constant anisotropic mean curvature are completely determined as follows:

Lemma 2 *Let $n = 1$ and $X : \mathbf{R} \supset I \to \mathbf{R}^2$ be a curve parametrized by arc-length. Then,*

$$\Lambda = \kappa/\kappa_W,$$

where Λ is the anisotropic mean curvature of X, κ is the curvature of X, and κ_W is the curvature of the Wulff shape W.

Proof We denote by θ a point $e^{i\theta}$ in S^1. Then, the Wulff shape W is represented by an embedding $\Phi : S^1 \to \mathbf{R}^2$ defined as

$$\Phi(\theta) = \gamma_\theta(\theta)(-\sin\theta, \cos\theta) + \gamma(\theta)(\cos\theta, \sin\theta).$$

Set $X(s) = (x(s), y(s))$. Then, the Gauss map N of X is

$$N(s) = (-y'(s), x'(s)) =: (\cos\theta(s), \sin\theta(s)).$$

Hence, the anisotropic mean curvature Λ of X is

$$\Lambda(s) = -\gamma_{\theta s} - \kappa\gamma = -\gamma_{\theta\theta}\theta_s - \kappa\gamma = -\kappa(\gamma_{\theta\theta} + \gamma). \tag{2}$$

On the other hand,

$$\frac{d\Phi}{d\theta} = (\gamma_{\theta\theta} + \gamma)(-\sin\theta, \cos\theta),$$

$$\frac{d^2\Phi}{d\theta^2} = (\gamma_{\theta\theta\theta} + \gamma_\theta)(-\sin\theta, \cos\theta) - (\gamma_{\theta\theta} + \gamma)(\cos\theta, \sin\theta).$$

Hence, by elementary calculations, the curvature κ_W of W with respect to the outward pointing unit normal is

$$\kappa_W = \frac{-1}{\gamma_{\theta\theta} + \gamma}. \tag{3}$$

(2) with (3) gives the desired formula. □

Proposition 1 *If the anisotropic mean curvature Λ of a curve X is constant, then either*

1. *X is (a part of) a straight line (when $\Lambda = 0$), or*

2. *X is a part of the Wulff shape up to translation and homothety (when $\Lambda \neq 0$).*

Proof By Lemma 2, the curvature of X is $\kappa = \Lambda\kappa_W$. Hence, by the fundamental theorem for plane curves, we obtain the desired result. □

Remark 1 For $n \geq 2$, we have great many varieties of CAMC hypersurfaces. For example, [10, Sect. 5] gives two parameter family of axisymmetric CAMC surfaces.

Let us explain our problem more precisely. Let Σ_1, Σ_2, Σ_0 be three piecewise smooth oriented connected compact hypersurfaces in $\mathbf{R}^{n+1}$ with common boundary C such that $\Sigma_1 \cup \Sigma_0$ (resp. $\Sigma_2 \cup \Sigma_0$) encloses a region R_1 (resp. R_2) with prescribed volume V_1 (resp. V_2), and let γ_i be energy density functions on Σ_i. We study the following anisotropic energy of the surface $\Sigma := \Sigma_1 \cup \Sigma_2 \cup \Sigma_0$:

$$\mathscr{F}(\Sigma) := \sum_{i=0}^{2} \int_{\Sigma_i} \gamma_i(N_i)\, d\Sigma_i, \tag{4}$$

where $N_i : \Sigma_i \to S^n$ is the unit normal vector field along Σ_i (we refer to Fig. 1 about the directions of N_i) and $d\Sigma_i$ is the n-dimensional volume form on Σ_i. The volumes V_i of the region R_i is given by

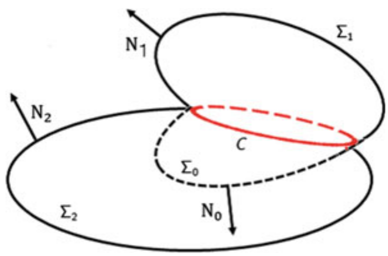

Fig. 1 An admissible surface Σ in $\mathbf{R}^3$. The *red curve* C is the common boundary of Σ_1, Σ_2 and Σ_0. We always assume that Σ_0 is in the *middle*

$$V_1 = \frac{1}{n+1}\left\{\int_{\Sigma_1}\langle x_1, N_1\rangle d\Sigma_1 + \int_{\Sigma_0}\langle x_0, N_0\rangle d\Sigma_0\right\},$$
$$V_2 = \frac{1}{n+1}\left\{\int_{\Sigma_2}\langle x_2, N_2\rangle d\Sigma_2 - \int_{\Sigma_0}\langle x_0, N_0\rangle d\Sigma_0\right\}.$$

Our problem is to study the minimizers of $\mathscr{F}$ among Σ's such that R_1, R_2 have prescribed volumes V_1, V_2, respectively.

3 Variation Formulas

Throughout this section, $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_0$ is such the union of smooth hypersurfaces Σ_0, Σ_1, and Σ_2 with common boundary C as in the last part of Sect. 2. We derive the first variation formula for the functional $\mathscr{F}$ defined by (4), and obtain the conditions for critical points.

Let $\tilde{X} : \Sigma \times [-\varepsilon_0, \varepsilon_0] \to \mathbf{R}^{n+1}$ ($\varepsilon_0 > 0$) be a variation of $X : \Sigma \to \mathbf{R}^{n+1}$. $\tilde{X}$ is called an admissible variation if the two volumes V_1, V_2 are preserved. Such $\tilde{X}$ can be represented as $\tilde{X}(x, \varepsilon) = X_\varepsilon = X + \varepsilon Y + \mathscr{O}(\varepsilon^2)$, and Y is called an admissible variation vector field of X. If Y is admissible, then

$$\delta V_1 := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} V_1(X_\varepsilon) = \int_{\Sigma_1}\langle Y, N_1\rangle d\Sigma_1 + \int_{\Sigma_0}\langle Y, N_0\rangle d\Sigma_0 = 0, \tag{5}$$
$$\delta V_2 := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} V_2(X_\varepsilon) = \int_{\Sigma_2}\langle Y, N_2\rangle d\Sigma_2 - \int_{\Sigma_0}\langle Y, N_0\rangle d\Sigma_0 = 0. \tag{6}$$

hold. By a suitable reparametrization of $\tilde{X}$, we may assume that, at each point on C, Y is orthogonal to the ($(n-1)$-dimensional) tangent space of C. Then, the boundary condition implies the following:

$$Y = \langle Y, N_1\rangle N_1 + \langle Y, \nu_1\rangle \nu_1 = \langle Y, N_2\rangle N_2 + \langle Y, \nu_2\rangle \nu_2 = \langle Y, N_0\rangle N_0 + \langle Y, \nu_0\rangle \nu_0 \tag{7}$$

hold on C, where ν_i is the outward pointing conormal vector for Σ_i along C.

Lemma 3 *Let φ_i, $f_i : \Sigma_i \to \mathbf{R}$ be smooth functions on Σ_i satisfying*

(i) $\int_{\Sigma_1} \varphi_1 \, d\Sigma_1 + \int_{\Sigma_0} \varphi_0 \, d\Sigma_0 = 0$, $\int_{\Sigma_2} \varphi_2 \, d\Sigma_2 - \int_{\Sigma_0} \varphi_0 \, d\Sigma_0 = 0$,
(ii) $\varphi_1 N_1 + f_1 \nu_1 = \varphi_2 N_2 + f_2 \nu_2 = \varphi_0 N_0 + f_0 \nu_0$ *on* C.

Then there exists an admissible variation such that the normal (resp. conormal to C) component of the variation vector field Y are $\varphi_i N_i$ (resp. $f_i \nu_i$).

Proof We give functions $h_i : \Sigma_i \to \mathbf{R}$ ($i = 1, 2$) such that $\int_{\Sigma_i} h_i d\Sigma_i \neq 0$ holds and each h_i has compact support on the interior of Σ_i. And we extend each function h_i to 0 on $\Sigma \setminus \Sigma_i$. On the other hand, set

$$Y := \varphi_i N_i + f_i \nu_i \quad \text{on } \Sigma_i,\ i = 0, 1, 2.$$

Then, Y gives a variation vector field of Σ. Set

$$X(s, t_1, t_2) := X + sY + t_1 h_1 N_1 + t_2 h_2 N_2,$$
$$V_i(s, t_1, t_2) := V_i(X(s, t_1, t_2)), \quad i = 1, 2.$$

Set $V_1^0 := V_1(0, 0, 0)$, $V_2^0 := V_2(0, 0, 0)$. Consider the following simultaneous equations.

$$V_1(s, t_1, t_2) = V_1^0, \quad V_2(s, t_1, t_2) = V_2^0.$$

Differentiate V_1, V_2 at $(s, t_1, t_2) = (0, 0, 0)$ to obtain

$$\frac{\partial V_1}{\partial s}(0,0,0) = \int_{\Sigma_1} \varphi_1 d\Sigma_1 + \int_{\Sigma_0} \varphi_0 d\Sigma_0 = 0, \quad \frac{\partial V_2}{\partial s}(0,0,0) = \int_{\Sigma_2} \varphi_2 d\Sigma_2 - \int_{\Sigma_0} \varphi_0 d\Sigma_0 = 0,$$

$$\frac{\partial V_i}{\partial t_j}(0,0,0) = \delta^i_j \int_{\Sigma_i} h_j \, d\Sigma_i \begin{cases} \neq 0, & i = j, \\ = 0, & i \neq j. \end{cases}$$

Therefore, by the implicit function theorem, in there exist a neighborhood I of $s = 0$ and smooth functions $t_1 = t_1(s)$, $t_2 = t_2(s)$ such that $t_1(0) = 0$, $t_2(0) = 0$, $\tilde{V}_1(s) := V_1(s, t_1(s), t_2(s)) = V_1^0$, $\tilde{V}_2(s) := V_1(s, t_1(s), t_2(s)) = V_2^0$ $(s \in I)$. Then,

$$0 = \tilde{V_i}'(s) = (V_i)_s + (V_i)_{t_1} t_1'(s) + (V_i)_{t_2} t_2'(s), \quad (i = 1, 2)$$

hold. Hence,

$$t_1'(0) = -\frac{(V_1)_s(0,0,0) + (V_1)_{t_2}(0,0,0) t_2'(0)}{(V_1)_{t_1}(0,0,0)} = 0,$$
$$t_2'(0) = -\frac{(V_2)_s(0,0,0) + (V_2)_{t_1}(0,0,0) t_1'(0)}{(V_2)_{t_2}(0,0,0)} = 0.$$

Consequently,

$$X(s, t_1(s), t_2(s)) = X + sY + t_1(s) h_1 N_1 + t_2(s) h_2 N_2 = X + sY + \mathscr{O}(s^2)$$

is an admissible variation of Σ, and so we obtain the desired result. □

Using Lemma 1, we immediately obtain the following:

Theorem 1 (First variation formula) *For a variation $X_\varepsilon = X + \varepsilon Y + \mathscr{O}(\varepsilon^2)$ of Σ, the first variation of the anisotropic energy $\mathscr{F}$ is*

$$\delta\mathscr{F} := \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathscr{F}(X_\varepsilon) = \sum_{i=0}^{2} \left[-\int_{\Sigma_i} \varphi_i \Lambda_i \, d\Sigma_i + (-1)^i \oint_C \langle \Phi_i, -\varphi_i \nu_i + f_i N_i \rangle \, dC \right], \quad (8)$$

where $\Phi_i = D\gamma_i + \gamma_i N_i$, $\varphi_i = \langle Y, N_i\rangle$, $f_i = \langle Y, \nu_i\rangle$ *on* C*, and the orientation of* C *is chosen so that it is the positive orientation for* Σ_1.

Definition 1 Each critical point of $\mathscr{F}$ for all admissible variations is called a *double crystal*.

Theorem 2 *A hypersurface* Σ *is a double crystal if and only if there hold:*

1. *For* $i = 0, 1, 2$*, the anisotropic mean curvature* Λ_i *is constant, and* $-\Lambda_1 + \Lambda_2 + \Lambda_0 = 0$ *holds, and*
2. *at each point* ζ *on* C*,* $\Phi_0 - \Phi_1 + \Phi_2$ *is in the* $(n-1)$*-dimensional linear subspace determined by the tangent space* $T_\zeta C$ *of* C *at* ζ.

Corollary 1 *Assume* $\gamma_i \equiv 1$, $i = 0, 1, 2$. *Then,* Σ *is a double bubble if and only if*

1. *For* $i = 0, 1, 2$*, the mean curvature* H_i *is constant, and* $-H_1 + H_2 + H_0 = 0$ *holds, and*
2. *at each point on* C*,* $N_0 - N_1 + N_2 = 0$.

Proof of Theorem 2 Assume that $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ is a double crystal. Then, Σ_1 is a critical point of $\mathscr{F}$ for all admissible variations that fix $\Sigma = \Sigma_0 \cup \Sigma_2$. Hence, Λ_1 is constant. Similarly, Λ_2 is constant. Now consider any variation $\Sigma_0(\varepsilon)$ of Σ_0 that fixes $\partial\Sigma_0$. Then, the variation vector field of $\Sigma_0(\varepsilon)$ can be extended to an admissible variation vector field of Σ. In fact, $\Sigma_0(\varepsilon)$ can be represented as

$$X_\varepsilon = X + \varepsilon\varphi_0 N_0 + \mathscr{O}(\varepsilon^2),$$

where $\varphi_0 = 0$ on C. It is obvious that we can find functions φ_1, φ_2, $f_1 = 0$, and $f_2 = 0$ satisfying (i) and (ii) in Lemma 3. So, by Lemma 3, there exists an admissible variation of Σ whose variation vector field is an extension of $Y_0 := \varphi_0 N_0$. We obtain, using Theorem 1, (5), and (6),

$$0 = \delta\mathscr{F} = \delta\mathscr{F} + \Lambda_1\delta V_1 + \Lambda_2\delta V_2 = \int_{\Sigma_0} (\Lambda_1 - \Lambda_2 - \Lambda_0)\langle Y_0, N_0\rangle \, d\Sigma_0.$$

Hence, $\Lambda_1 - \Lambda_2 - \Lambda_0 = 0$ holds, which proves the condition 1. Now, assume that the condition 2 does not hold. Then, there exists a non-empty open set U of C such that $(\Phi_0 - \Phi_1 + \Phi_2)(\zeta) \notin T_\zeta C$ for any $\zeta \in U$. Then, we can define a non-zero vector field $\tilde{Y}$ on C with support in U such that $\tilde{Y}$ is orthogonal to C at any $\zeta \in U$ and

$$\oint_C \langle \Sigma_{i=0}^2 (-1)^i \Phi_i, \tilde{Y}\rangle \, dC \neq 0$$

holds. Clearly, $\tilde{Y}$ can be represented as

$$\tilde{Y} = -\varphi_i \nu_i + f_i N_i, \quad i = 0, 1, 2,$$

and $Y := \varphi_i N_i + f_i \nu_i$ can be extended to an admissible variation vector field along Σ. Here we used Lemma 3 again as above. We obtain

$$0 = \delta\mathscr{F} = \delta\mathscr{F} + \Lambda_1 \delta V_1 + \Lambda_2 \delta V_2 = \oint_C \langle \Sigma_{i=0}^2 (-1)^i \Phi_i, \tilde{Y} \rangle \, dC \neq 0,$$

which is a contradiction.

Conversly, assume that the conditions 1 and 2 hold. Then, again by using Theorem 1, (5), and (6), for any admissible variation, we have

$$\delta\mathscr{F} = \delta\mathscr{F} + \Lambda_1 \delta V_1 + \Lambda_2 \delta V_2 = 0.$$

Hence, the hypersurface is a double crystal. □

Definition 2 A double crystal Σ is said to be stable if the second variation $\delta^2\mathscr{F}$ is nonnegative for all admissible variations, and otherwise it is said to be unstable.

Theorem 3 (Second variation formula) *Let $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_0$ be a double crystal. Then for any admissible variational vector field Y, the second variation of the anisotropic energy $\mathscr{F}$ is given by*

$$\delta^2\mathscr{F} = \sum_{i=0}^{2} \left[-\int_{\Sigma_i} \varphi_i L[\varphi_i] d\Sigma_i + (-1)^i \oint_C \varphi_i \langle A_i \nabla\varphi_i - f_i A_i dN_i(\nu_i), \nu_i \rangle \, dC \right], \quad (9)$$

where L is the self-adjoint Jacobi operator

$$L[\varphi_i] := \mathrm{div}(A_i \nabla\varphi_i) + \langle A_i dN_i, dN_i \rangle \varphi_i,$$

$A_i := D^2\gamma_i + \gamma_i \cdot 1$, *$\varphi_i = \langle Y, N_i \rangle$, and $f_i = \langle Y, \nu_i \rangle$ on C.*

Proof The first variation formula (Theorem 1) gives

$$\delta\mathscr{F} := \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathscr{F}(X_\varepsilon) = \sum_{i=0}^{2} \left[-\int_{\Sigma_i} \varphi_i \Lambda_i \, d\Sigma_i + (-1)^i \oint_C \langle \Phi_i, \tilde{Y} \rangle \, dC \right],$$

where $\tilde{Y} = -\varphi_i \nu_i + f_i N_i$. Hence, any volume-preserving variation, at a double crystal Σ, we obtain

$$\delta^2\mathscr{F} = \delta(\delta\mathscr{F} + \Lambda_1 \delta V_1 + \Lambda_2 \delta V_2) = \sum_{i=0}^{2} \left[-\int_{\Sigma_i} \varphi_i \delta\Lambda_i \, d\Sigma_i + (-1)^i \oint_C \langle \delta\Phi_i, \tilde{Y} \rangle \, dC \right].$$

Note that $\delta\Lambda_i = L[\varphi_i]$ holds (cf. [8]). Also, we compute, on C,

$$\begin{aligned} \langle \delta\Phi_i, -\varphi_i \nu_i + f_i N_i \rangle = -\varphi_i \langle \delta\Phi_i, \nu_i \rangle &= -\varphi_i \langle A_i(-\nabla\varphi_i + dN_i(f_i \nu_i)), \nu_i \rangle \\ &= -\varphi_i \langle -A_i \nabla\varphi_i + f_i A_i dN_i(\nu_i), \nu_i \rangle. \end{aligned}$$

□

4 Double Crystals in the Plane

In this section, we assume $n = 1$ and apply the above discussion to a certain special energy density function on S^1. The Wulff shape corresponding to this energy density function is a smooth square (see Fig. 2). We will discuss the critical points (i.e. double crystals) and their stability.

From Proposition 1 and Theorem 2, we immediately obtain the following:

Theorem 4 *For $n = 1$, $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ is a double crystal if and only if there hold:*

(i) Each Σ_i is, up to translation, a part of a rescaling of the Wulff shape corresponding to γ_i.
(ii) $\Phi_0 - \Phi_1 + \Phi_2 = 0$ on the common boundary C (C is a set of two points).

From now on, if we do not say anything special, we assume that the energy density functions $\gamma_i : S^1(\subset \mathbf{R}) \to \mathbf{R}$ are the following special ones.

$$\begin{aligned}\gamma(\nu_1, \nu_2) &:= \gamma_{(p)}(\nu_1, \nu_2) := \gamma_i(\nu_1, \nu_2)\\ &= \left(\nu_1^{2p} + \nu_2^{2p}\right)^{1-\frac{1}{2p}} / \sqrt{\nu_1^{4p-2} + \nu_2^{4p-2}}, \quad i = 0, 1, 2, \end{aligned} \tag{10}$$

where p is any fixed positive integer. Then the Wulff shape is given by

$$\Phi(\theta) := (\cos^{2p}\theta + \sin^{2p}\theta)^{-\frac{1}{2p}}(\cos\theta, \sin\theta).$$

4.1 Classifications of DC for a Special Energy Density Function

From now on, without loss of generality, we assume $V_1 \le V_2$. If $\Lambda_i \neq 0$, from (i) in Theorem 4, Σ_i is represented by

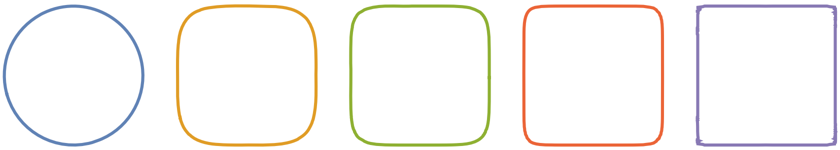

Fig. 2 The Wulff shapes $W_{(p)}$ for the energy density $\gamma_{(p)}$ in (10). $W_{(1)}$ is a *circle*. When p approaches infinity, $W_{(p)}$ converges to a *cube*

$$X_1(\theta) = -\frac{1}{\Lambda_1}(\cos^{2p}\theta + \sin^{2p}\theta)^{-\frac{1}{2p}}(\cos\theta, \sin\theta) + (a_1, b_1), \quad \alpha_1 \le \theta \le \beta_1, \tag{11}$$

$$X_2(\theta) = -\frac{1}{\Lambda_2}(\cos^{2p}\theta + \sin^{2p}\theta)^{-\frac{1}{2p}}(\cos\theta, \sin\theta) + (a_2, b_2), \quad \beta_2 \le \theta \le \alpha_2, \tag{12}$$

$$X_0(\theta) = -\frac{1}{\Lambda_0}(\cos^{2p}\theta + \sin^{2p}\theta)^{-\frac{1}{2p}}(\cos\theta, \sin\theta) + (a_0, b_0), \quad \beta_0 \le \theta \le \alpha_0, \tag{13}$$

where $\alpha_0, \alpha_1, \alpha_2$ correspond to one of the two points in the common boundary C, and β_0, β_1, β_2 correspond to the other point in C. By the second condition in Theorem 4, we have

$$\begin{cases} f(\theta_0)\cos\theta_0 - f(\theta_1)\cos\theta_1 + f(\theta_2)\cos\theta_2 = 0, \\ f(\theta_0)\sin\theta_0 - f(\theta_1)\sin\theta_1 + f(\theta_2)\sin\theta_2 = 0, \end{cases} \quad (\theta_i = \alpha_i, \beta_i), \tag{14}$$

where $f(\theta) = (\cos^{2p}\theta + \sin^{2p}\theta)^{-\frac{1}{2p}}$.

We can prove the following results about geometry of the double crystals.

Lemma 4 [7] *There are uniquely determined functions φ, ψ : $\mathbf{S}^1 \to \mathbf{R}$ such that $\theta_2 = \varphi(\theta_1)$ and $\theta_0 = \psi(\theta_1)$ satisfy (14).*

Lemma 5 [7] *For double crystals, we have the following results about the relationship between α_i and β_i.*

- *(I) If $\alpha_1 + \beta_1 = 2n_1\pi$ ($n_1 \in \mathbf{Z}$), then $\alpha_i + \beta_i = 2n_i\pi$, ($n_i \in \mathbf{Z}$, $i = 0, 2$).*
- *(II) If $\alpha_1 + \beta_1 = (2n_1 + 1/2)\pi$ ($n_1 \in \mathbf{Z}$), then $\alpha_i + \beta_i = (2n_i + 1/2)\pi$, ($n_i \in \mathbf{Z}$, $i = 0, 2$).*
- *(III) If $\alpha_1 + \beta_1 = (2n_1 + 1)\pi$ ($n_1 \in \mathbf{Z}$), then $\alpha_i + \beta_i = (2n_i + 1)\pi$, ($n_i \in \mathbf{Z}$, $i = 0, 2$).*
- *(IV) If $\alpha_1 + \beta_1 = (2n_1 + 3/2)\pi$ ($n_1 \in \mathbf{Z}$), then $\alpha_i + \beta_i = (2n_i + 3/2)\pi$, ($n_i \in \mathbf{Z}$, $i = 0, 2$).*
- *(V) If $\beta_1 = \alpha_1 + \pi$, then $\beta_0 = \alpha_0 - \pi$ and $\beta_2 = \alpha_2 - \pi$.*

Lemma 5 gives the following result about symmetry of double crystals:

Theorem 5 [7] *About the five types of the double crystals in Lemma 5, we have the following three types of symmetry (up to translation and homothety) (see Fig. 3).*

Type 1 *Symmetry with respect to either a horizontal line or a vertical line.*
Type 2 *Symmetry with respect to the $\pm\pi/4$ rotation of the horizontal line.*
Type 3 *Rotational symmetry with respect to the center point of the smallest cube. In this case, the two bigger Wulff shapes are double size of the smallest one.*

Actually, double crystals of Type (I) and (III) have Type 1 symmetry, double crystals of Type (II) and (IV) have Type 2 symmetry, and double crystals of Type (V) have Type 3 symmetry.

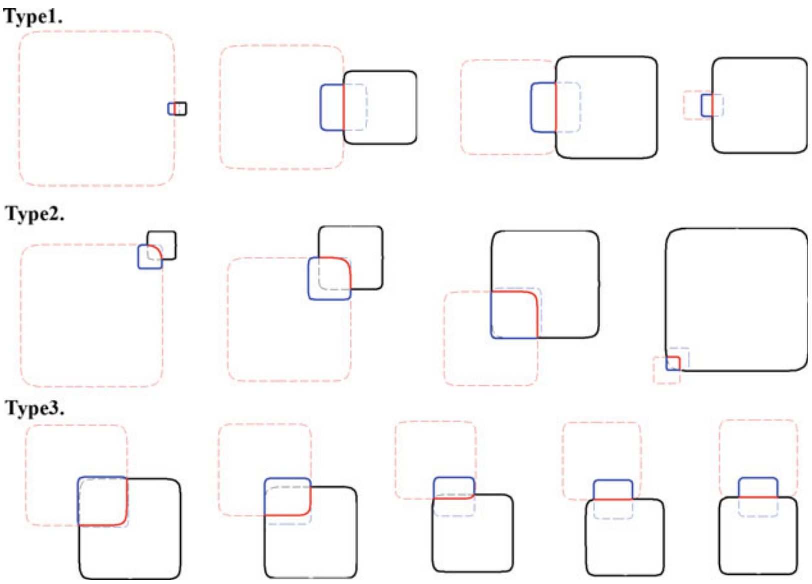

Fig. 3 These figures show the three types in Theorem 5 according to $\rho = V_2/V_1$

Remark 2 In Type 1 and 2, ρ can be any number bigger than or equal to 1. On the other hand, in Type 3, ρ can take numbers in the interval [3, 8]. In fact, in Type 3, two bigger Wulff shapes (black and red shape in Fig. 3) are double size of the smallest one (blue shape in Fig. 3).

4.2 Stability for Special Energy Density Function

In this section we discuss the stability of the three types of double crystals appeared in Theorem 5.

First we give a result about instability of some double crystals which was essentialy proved in [11].

Lemma 6 *Set* $\gamma_\infty(\nu_1, \nu_2) = |\nu_1| + |\nu_2|$ $(\nu_1, \nu_2) \in S^1$, *and consider an anisotropic surface energy* $\mathscr{F}(X) = \int_\Sigma \gamma_\infty(N)\, d\Sigma$. *Consider the three types of shapes in Fig. 4. Then we can decrease the anisotropic energy of these shapes without changing the enclosed areas (Fig. 5).*

Proof Note that the anisotropic energy of a horizontal or vertical edge is equal to its length, and the anisotropic energy of a diagonal edge is equal to $\sqrt{2}$ times its length. Figures 6, 7 and 8 show how the anisotropic energy is decreased without changing the enclosed area. □

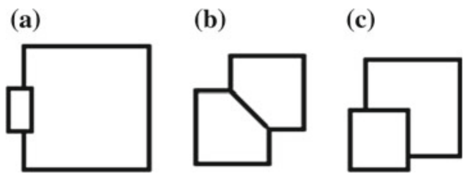

Fig. 4 The three types of shapes of which we can decrease the anisotropic energy without changing the enclosed areas

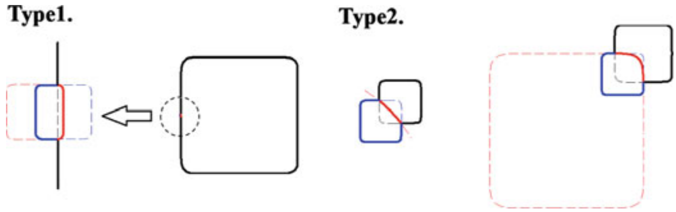

Fig. 5 Unstable examples corresponding to Fig. 4

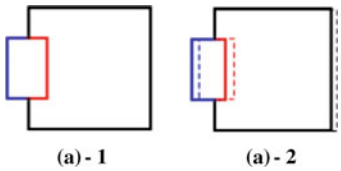

Fig. 6 The anisotropic energy of *(a)-1* is decreased without changing the enclosed area when it is changed like *(a)-2*

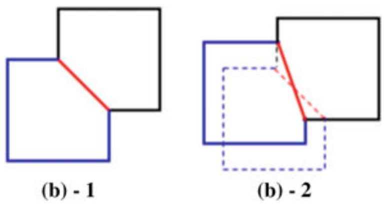

Fig. 7 The anisotropic energy of *(b)-1* is decreased without changing the enclosed area when it is changed like *(b)-2*

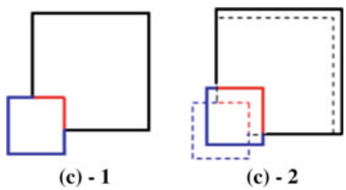

Fig. 8 The anisotropic energy of *(c)-1* is decreased without changing the enclosed area when it is changed like *(c)-2*

Let us think about the stability of the double crystals for our energy $\gamma = \gamma_{(p)}$ defined in (10). Define two angles ζ and η so that

$$f(\zeta)\cos(\zeta) = \frac{1}{2}, \quad f(\eta)\sin\left(\eta + \frac{\pi}{4}\right) = 2^{\frac{-1-p}{2p}}$$

holds.

Recall that the Wulff shape $W_{(p)}$ for the energy density $\gamma_{(p)}$ in (10) converges to the Wulff shape for γ_∞ when p approaches infinity (Fig. 2). By Lemma 6 and an approximation procedure (Fig. 5), we can show the following:

Proposition 2 *For sufficiently large p, we have the following result about the instability of double crystals. Double crystals of type 1 in Theorem 5 are unstable if $\frac{\pi}{4} < \alpha_0 < \zeta$. Double crystals of type 2 are unstable if $\frac{\pi}{4} \leq \alpha_0 < \frac{\pi}{2} - \eta$.*

We apply Theorem 3 (second variation formula) to the 2-dimensional case. For admissible variation vector field Y, We obtain

$$\delta^2 \mathscr{F} = -\sum_{i=0}^{2} \int_{\Sigma_i} q_i L[q_i]\, d\Sigma_i + [\Lambda_i p_i q_i - A_i(q_i)_t q_i]_a^b, \tag{15}$$

where

$p_i = \langle Y, \nu_i \rangle$ and $q_i = \langle Y, N_i \rangle$. We expect that we will be able to prove the following conjecture by using (15).

Conjecture 1 Except the cases in Proposition 2, double crystals of types 1–3 are stable.

Let the Wulff shape be a square. Then the energy minimizing shape is one of the three types in Fig. 9 according to $\rho = V_2/V_1$. We expect that, by using the variational method, we will be able to obtain not only the absolute minimum but also local minimums. It is important to get local minimums because the physical state sometimes takes a local minimum.

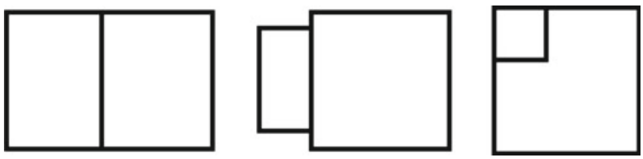

Fig. 9 The *right side* figure is $\rho \leq 2$ and both R_1 and R_2 are rectangular. The ratio of *middle* figure is $2 \leq \rho \leq \rho_0 := \frac{43+30\sqrt{2}}{16}$ and R_1 is square and R_2 is rectangular (this ratio of edge length is 1 : 2). The ratio of *left side* figure $\rho \leq \rho_0$, and both R_1 and R_2 are squares

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