Complete T-Dualization of a String in a Weakly Curved Background

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Abstract We apply the generalized Buscher procedure, to a subset of the initial coordinates of the bosonic string moving in the weakly curved background, composed of a constant metric and a linearly coordinate dependent Kalb-Ramond field with the infinitesimal strength. In this way we obtain the partially T-dualized action. Applying the procedure to the rest of the original coordinates we obtain the totally T-dualized action. This derivation allows the investigation of the relations between the Poisson structures of the original, the partially T-dualized and the totally T-dualized theory.

1 Bosonic String in the Weakly Curved Background

Let us consider the closed string moving in the coordinate dependent background, described by the action [1]

$$S[x] = \kappa \int_\Sigma d^2\xi\, \partial_+ x^\mu \Pi_{+\mu\nu}[x] \partial_- x^\nu. \tag{1}$$

The background is defined by the space-time metric $G_{\mu\nu}$ and the antisymmetric Kalb-Ramond field $B_{\mu\nu}$

$$\Pi_{\pm\mu\nu}[x] = B_{\mu\nu}[x] \pm \frac{1}{2} G_{\mu\nu}[x]. \tag{2}$$

The light-cone coordinates are

$$\xi^\pm = \frac{1}{2}(\tau \pm \sigma), \qquad \partial_\pm = \partial_\tau \pm \partial_\sigma, \tag{3}$$

and the action is given in the conformal gauge (the world-sheet metric is taken to be $g_{\alpha\beta} = e^{2F}\eta_{\alpha\beta}$).

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The world-sheet conformal invariance is required, as a condition of having a consistent theory on a quantum level. This leads to the space-time equations for the background fields, which equal

$$R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_{\nu}{}^{\rho\sigma} = 0, \quad D_\rho B^\rho{}_{\mu\nu} = 0, \tag{4}$$

in the lowest order in slope parameter α' and for the constant dilaton field $\Phi = const$. Here $B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ is the field strength of the field $B_{\mu\nu}$, and $R_{\mu\nu}$ and D_μ are Ricci tensor and covariant derivative with respect to the space-time metric.

We will consider a weakly curved background [2, 3], defined by

$$G_{\mu\nu}[x] = const,$$
$$B_{\mu\nu}[x] = b_{\mu\nu} + h_{\mu\nu}[x] = b_{\mu\nu} + \frac{1}{3} B_{\mu\nu\rho} x^\rho, \qquad b_{\mu\nu}, B_{\mu\nu\rho} = const. \tag{5}$$

Here, the constant $B_{\mu\nu\rho}$ is infinitesimal. The background (5) is the solution of the field equations (4) in the first order in $B_{\mu\nu\rho}$.

2 Partial T-Dualization

In the paper [3], we generalized the Buscher prescription for a construction of a T-dual theory. This prescription, unlike the standard one [4], is applicable to the string backgrounds depending on all the space-time coordinates, such as the weakly curved background. We performed the procedure along all the coordinates and obtained T-dual theory. The noncommutativity of the T-dual coordinates we investigated in [5]. In the present paper we consider the partial T-dualization, i.e. the application of the procedure to some without subset of the coordinates. We construct the partially T-dualized theory. The noncommutativity of the coordinates in similar theories was considered in [6].

Let us mark the T-dualization along the coordinate x^μ by T_μ, and separate the coordinates into two subsets (x^i, x^a) with $i = 0, \ldots, d-1$ and $a = d, \ldots, D-1$ and mark the T-dualizations along these subsets of coordinates by

$$T^i \equiv T_0 \circ \cdots \circ T_{d-1}, \quad T^a \equiv T_d \circ \cdots \circ T_{D-1}. \tag{6}$$

In this section we will find the partially T-dualized action performing T-dualization along coordinates x^a, $\mathcal{T}^a : S$.

The closed string action in the weakly curved background has a global symmetry

$$\delta x^\mu = \lambda^\mu. \tag{7}$$

Let us localize this symmetry for the coordinates x^a

$$\delta x^a = \lambda^a(\tau,\sigma), \quad a = d, \ldots, D-1, \tag{8}$$

by introducing the gauge fields v_α^a and substituting the ordinary derivatives with the covariant ones

$$\partial_\alpha x^a \to D_\alpha x^a = \partial_\alpha x^a + v_\alpha^a. \tag{9}$$

The gauge invariance of the covariant derivatives is obtained by imposing the following transformation law for the gauge fields

$$\delta v_\alpha^a = -\partial_\alpha \lambda^a. \tag{10}$$

Also, substitute x^a in the argument of the background fields with its invariant extension, defined by

$$\begin{aligned} \Delta x_{inv}^a &\equiv \int_P d\xi^\alpha\, D_\alpha x^a = \int_P (d\xi^+ D_+ x^a + d\xi^- D_- x^a) \\ &= x^a - x^a(\xi_0) + \Delta V^a, \end{aligned} \tag{11}$$

where

$$\Delta V^a \equiv \int_P d\xi^\alpha v_\alpha^a = \int_P (d\xi^+ v_+^a + d\xi^- v_-^a). \tag{12}$$

The line integral is taken along the path P, from the initial point $\xi_0^\alpha(\tau_0,\sigma_0)$ to the final one $\xi^\alpha(\tau,\sigma)$. To preserve the physical equivalence between the gauged and the original theory, one introduces the Lagrange multiplier y_a and adds the term $\frac{1}{2} y_a F_{+-}^a$ to the Lagrangian, which will force the field strength $F_{+-}^a \equiv \partial_+ v_-^a - \partial_- v_+^a = -2F_{01}^a$ to vanish. In this way, we obtain the gauge invariant action

$$\begin{aligned} S_{inv} = \kappa \int d^2\xi \Big[&\partial_+ x^i \Pi_{+ij}[x^i, \Delta x_{inv}^a]\partial_- x^j + \partial_+ x^i \Pi_{+ia}[x^i, \Delta x_{inv}^a] D_- x^a \\ &+ D_+ x^a \Pi_{+ai}[x^i, \Delta x_{inv}^a]\partial_- x^i + D_+ x^a \Pi_{+ab}[x^i, \Delta x_{inv}^a] D_- x^b \\ &+ \frac{1}{2}(v_+^a \partial_- y_a - v_-^a \partial_+ y_a)\Big], \end{aligned} \tag{13}$$

where the last term is equal to $\frac{1}{2} y_a F_{+-}^a$ up to the total divergence. Now, we can use the gauge freedom to fix the gauge $x^a(\xi) = x^a(\xi_0)$. The gauge fixed action equals

$$S_{fix} = \kappa \int d^2\xi \Big[\partial_+ x^i \Pi_{+ij}[x^i, \Delta V^a] \partial_- x^j + \partial_+ x^i \Pi_{+ia}[x^i, \Delta V^a] v^a_- \\ + v^a_+ \Pi_{+ai}[x^i, \Delta V^a] \partial_- x^i + v^a_+ \Pi_{+ab}[x^i, \Delta V^a] v^b_- \\ + \frac{1}{2}(v^a_+ \partial_- y_a - v^a_- \partial_+ y_a) \Big]. \tag{14}$$

The equations of motion for the Lagrange multiplier y_a, $\partial_+ v^a_- - \partial_- v^a_+ = 0$, have a solution $v^a_\pm = \partial_\pm x^a$, which turns the gauge fixed action to the initial one.

2.1 The Partially T-Dualized Action

The partially T-dualized action will be obtained after elimination of the gauge fields from the gauge fixed action (14), using their equations of motion. Varying over the gauge fields $v^a_\pm$ one obtains

$$\Pi_{\pm ai}[x^i, \Delta V^a] \partial_\mp x^i + \Pi_{\pm ab}[x^i, \Delta V^a] v^b_\mp + \frac{1}{2} \partial_\mp y_a = \pm \beta^\pm_a [x^i, V^a], \tag{15}$$

where $\beta^\pm_a[x^i, V^a]$ is the infinitesimal contribution from the background fields argument. Using the inverse of the background fields composition $2\kappa \Pi_{\pm ab}$, defined by $\tilde{\Theta}^{ab}_\pm \equiv -\frac{2}{\kappa}(\tilde{G}^{-1}_E)^{ac} \Pi_{\pm cd} (\tilde{G}^{-1})^{db}$, where $\tilde{G}_{ab} \equiv G_{ab}$ and $\tilde{G}_{Eab} \equiv G_{ab} - 4B_{ac}(\tilde{G}^{-1})^{cd} B_{db}$, we can extract the gauge fields $v^a_\pm$ from Eq. (15)

$$v^a_\mp = -2\kappa \tilde{\Theta}^{ab}_\mp [x^i, \Delta V^a] \Big[\Pi_{\pm bi}[x^i, \Delta V^a] \partial_\mp x^i + \frac{1}{2} \partial_\mp y_b \mp \beta^\pm_b [x^i, V^a] \Big]. \tag{16}$$

Substituting (16) into the action (14), we obtain the partially T-dualized action

$$S_\pi[x^i, y_a] = \kappa \int d^2\xi \Big[\partial_+ x^i \bar{\Pi}_{+ij}[x^i, \Delta V^a(x^i, y^a)] \partial_- x^j \\ + \frac{\kappa}{2} \partial_+ y_a \tilde{\Theta}^{ab}_- [x^i, \Delta V^a(x^i, y^a)] \partial_- y_b \\ - \kappa\, \partial_+ x^i \Pi_{+ia}[x^i, \Delta V^a(x^i, y^a)] \tilde{\Theta}^{ab}_- [x^i, \Delta V^a(x^i, y^a)] \partial_- y_b \\ + \kappa\, \partial_+ y_a \tilde{\Theta}^{ab}_- [x^i, \Delta V^a(x^i, y^a)] \Pi_{+bi}[x^i, \Delta V^a(x^i, y^a)] \partial_- x^i \Big], \tag{17}$$

where

$$\bar{\Pi}_{+ij} \equiv \Pi_{+ij} - 2\kappa \Pi_{+ia} \tilde{\Theta}^{ab}_- \Pi_{+bj}. \tag{18}$$

In order to find the explicit value of the background fields argument $\Delta V^a(x^i, y^a)$, one substitutes the zeroth order of the equations of motion (16) into (12) and obtains

$$\begin{aligned}\Delta V^{(0)a} &= -\kappa\Big[\tilde{\Theta}^{ab}_{0+}\Pi_{0-bi} + \tilde{\Theta}^{ab}_{0-}\Pi_{0+bi}\Big]\Delta x^{(0)i}\\ &- \kappa\Big[\tilde{\Theta}^{ab}_{0+}\Pi_{0-bi} - \tilde{\Theta}^{ab}_{0-}\Pi_{0+bi}\Big]\Delta \tilde{x}^{(0)i}\\ &- \frac{\kappa}{2}\Big[\tilde{\Theta}^{ab}_{0+} + \tilde{\Theta}^{ab}_{0-}\Big]\Delta y^{(0)}_b - \frac{\kappa}{2}\Big[\tilde{\Theta}^{ab}_{0+} - \tilde{\Theta}^{ab}_{0-}\Big]\Delta \tilde{y}^{(0)}_b,\end{aligned} \tag{19}$$

where $\tilde{\Theta}^{ab}_{0\pm}$ stands for the zeroth order value of $\tilde{\Theta}^{ab}_{\pm}$, which can be written as

$$\tilde{\Theta}^{ab}_{0\pm} \equiv -\frac{2}{\kappa}(\tilde{g}^{-1})^{ac}\,\Pi_{0\pm cd}(\tilde{G}^{-1})^{db} = \tilde{\theta}^{ab}_0 \mp \frac{1}{\kappa}(\tilde{g}^{-1})^{ab}, \tag{20}$$

where $\tilde{g}_{ab} = G_{ab} - 4b_{ac}(\tilde{G}^{-1})^{cd}b_{db}$; $\tilde{\theta}^{ab}_0 \equiv -\frac{2}{\kappa}(\tilde{g}^{-1})^{ac}\,b_{cd}(\tilde{G}^{-1})^{db}$ and

$$\Delta \tilde{y}^{(0)}_a = \int(d\tau y^{(0)\prime}_a + d\sigma \dot{y}^{(0)}_a), \quad \Delta \tilde{x}^{(0)i} = \int(d\tau x^{(0)\prime i} + d\sigma \dot{x}^{(0)i}). \tag{21}$$

Initial theory, the partially T-dualized theory and the totally T-dualized theory obtained in [3] are physically equivalent theories. In the next section we will partially T-dualize the partially T-dualized theory.

3 The Total T-Dualization of the Initial Action

The T-dual theory, derived in [3], a result of T-dualization of the initial action along all the coordinates, is given by

$${}^\star S[y] = \kappa\int d^2\xi\,\partial_+ y_\mu\,{}^\star\Pi^{\mu\nu}_+[\Delta V(y)]\,\partial_- y_\nu = \frac{\kappa^2}{2}\int d^2\xi\,\partial_+ y_\mu \Theta^{\mu\nu}_-[\Delta V(y)]\partial_- y_\nu, \tag{22}$$

with

$$\Theta^{\mu\nu}_\pm \equiv -\frac{2}{\kappa}(G_E^{-1}\Pi_\pm G^{-1})^{\mu\nu} = \theta^{\mu\nu} \mp \frac{1}{\kappa}(G_E^{-1})^{\mu\nu}, \tag{23}$$

where

$$G_{E\mu\nu} \equiv G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}, \quad \theta^{\mu\nu} \equiv -\frac{2}{\kappa}(G_E^{-1}BG^{-1})^{\mu\nu}. \tag{24}$$

The T-dual background fields are equal to

$$^\star G^{\mu\nu}[\Delta V(y)] = (G_E^{-1})^{\mu\nu}[\Delta V(y)], \quad ^\star B^{\mu\nu}[\Delta V(y)] = \frac{\kappa}{2}\theta^{\mu\nu}[\Delta V(y)]. \tag{25}$$

The argument of the background fields is given by

$$\Delta V^\mu(y) = -\kappa\theta_0^{\mu\nu}\Delta y_\nu + (g^{-1})^{\mu\nu}\Delta\tilde{y}_\nu, \tag{26}$$

where $\Delta y_\mu = y_\mu(\xi) - y_\mu(\xi_0)$ and $\tilde{y}_\mu = \int(d\tau y'_\mu + d\sigma\dot{y}_\mu)$, while $g_{\mu\nu} = G_{\mu\nu} - 4b^2_{\mu\nu}$ and $\theta_0^{\mu\nu} = -\frac{2}{\kappa}(g^{-1}bG^{-1})^{\mu\nu}$.

Let us now show that the same result will be obtained applying the T-dualization procedure to the coordinates x^i of the partially T-dualized theory (17), $\mathcal{T}^i$: $S_\pi[x^i, y_a]$. Substituting the ordinary derivatives $\partial_\pm x^i$ with the covariant derivatives

$$D_\pm x^i = \partial_\pm x^i + v^i_\pm, \tag{27}$$

where the gauge fields $v^i_\pm$ transform as $\delta v^i_\pm = -\partial_\pm\lambda^i$, and substituting the coordinates x^i in the background field arguments by

$$\Delta x^i_{inv} = \int_P (d\xi^+ D_+ x^i + d\xi^- D_- x^i), \tag{28}$$

we obtain the gauge invariant action, which after fixing the gauge by $x^i(\xi) = x^i(\xi_0)$ becomes

$$\begin{aligned} S^{fix}_\pi = \kappa \int d^2\xi \Big[& v^i_+ \bar{\Pi}_{+ij}[\Delta V^\mu] v^j_- + \frac{\kappa}{2}\partial_+ y_a \tilde{\Theta}^{ab}_-[\Delta V^\mu]\partial_- y_b \\ & -\kappa\, v^i_+ \Pi_{+ia}[\Delta V^\mu]\tilde{\Theta}^{ab}_-[\Delta V^\mu]\partial_- y_b + \kappa\,\partial_+ y_a \tilde{\Theta}^{ab}_-[\Delta V^\mu]\Pi_{+bi}[\Delta V^\mu] v^i_- \\ & + \frac{1}{2}(v^i_+\partial_- y_i - v^i_-\partial_+ y_i)\Big]. \end{aligned} \tag{29}$$

Here ΔV^i is defined by

$$\Delta V^i \equiv \int_P (d\xi^+ v^i_+ + d\xi^- v^i_-), \tag{30}$$

and ΔV^a is defined in (19), whose arguments are in this case ΔV^i and y^a.

The totally T-dualized action will be obtained by eliminating the gauge fields from the gauge fixed action, using their equations of motion. Varying the action (29) over the gauge fields $v^i_\pm$ one obtains

$$\bar{\Pi}_{\pm ij} v^j_\mp - \kappa\Pi_{\pm ia}\tilde{\Theta}^{ab}_\mp \partial_\mp y_b + \frac{1}{2}\partial_\mp y_i = \pm\beta^\pm_i. \tag{31}$$

Using the fact that the background field composition $\bar{\Pi}_{\pm ij}$ is inverse to $2\kappa\Theta_{\mp}^{ij}$, we can rewrite the equation of motion (31) expressing the gauge fields as

$$v_{\mp}^{i} = 2\kappa\Theta_{\mp}^{ij}\Big[\kappa\Pi_{\pm ja}\tilde{\Theta}_{\mp}^{ab}\partial_{\mp}y_b - \frac{1}{2}\partial_{\mp}y_j \pm \beta_j^{\pm}\Big]. \tag{32}$$

Using $\Pi_{\pm ab}\Theta_{\mp}^{bi} = -\Pi_{\pm aj}\Theta_{\mp}^{ji}$, we note that

$$\Theta_{\mp}^{ij}\Pi_{\pm ja}\tilde{\Theta}_{\mp}^{ab} = -\Theta_{\mp}^{ic}\Pi_{\pm ca}\tilde{\Theta}_{\mp}^{ab} = -\frac{1}{2\kappa}\Theta_{\mp}^{ib}, \tag{33}$$

and obtain

$$v_{\mp}^{i} = -\kappa\Theta_{\mp}^{i\mu}\partial_{\mp}y_{\mu} \pm 2\kappa\Theta_{\mp}^{ij}\beta_j^{\pm}. \tag{34}$$

Substituting (34) into (29), the action becomes

$$\begin{aligned} S = \kappa\int d^2\xi\Big[&\partial_+ y_i\Big(\kappa\Theta_-^{ij} - \kappa^2\Theta_-^{ik}\bar{\Pi}_{+kl}\Theta_-^{lj}\Big)\partial_- y_j \\ &+ \partial_+ y_a\Big(-\kappa^2\Theta_-^{aj}\bar{\Pi}_{+jk}\Theta_-^{ki} + \frac{\kappa}{2}\Theta_-^{ai} - \kappa^2\tilde{\Theta}_-^{ab}\Pi_{+bj}\Theta_-^{ji}\Big)\partial_- y_i \\ &+ \partial_+ y_i\Big(-\kappa^2\Theta_-^{ij}\bar{\Pi}_{+jk}\Theta_-^{ka} + \frac{\kappa}{2}\Theta_-^{ia} - \kappa^2\Theta_-^{ij}\Pi_{+jb}\tilde{\Theta}_-^{ba}\Big)\partial_- y_a \\ &+ \partial_+ y_a\Big(\frac{\kappa}{2}\tilde{\Theta}_-^{ab} - \kappa^2\Theta_-^{ai}\bar{\Pi}_{+ij}\Theta_-^{jb} - \kappa^2\Theta_-^{ai}\Pi_{+ic}\tilde{\Theta}_-^{cb} - \kappa^2\tilde{\Theta}_-^{ac}\Pi_{+ci}\Theta_-^{ib}\Big)\partial_- y_b\Big]. \end{aligned} \tag{35}$$

Using $\bar{\Pi}_{\pm ij}\Theta_{\mp}^{jk} = \Theta_{\mp}^{kj}\bar{\Pi}_{\pm ji} = \frac{1}{2\kappa}\delta_i^k$; $\tilde{\Pi}_{\pm ab}\Theta_{\mp}^{bc} = \Theta_{\mp}^{cb}\tilde{\Pi}_{\pm ba} = \frac{1}{2\kappa}\delta_a^c$; $\Pi_{\pm ab}\Theta_{\mp}^{bi} = -\Pi_{\pm aj}\Theta_{\mp}^{ji}$; $\Pi_{\pm ij}\Theta_{\mp}^{ja} = -\Pi_{\pm ib}\Theta_{\mp}^{ba}$ and $\Theta_{\mp}^{ci}\bar{\Pi}_{\pm ik} = -\tilde{\Theta}_{\mp}^{ca}\Pi_{\pm ak}$, one can rewrite this action as

$$S = \frac{\kappa^2}{2}\int d^2\xi\,\partial_+ y_{\mu}\Theta_-^{\mu\nu}\partial_- y_{\nu}. \tag{36}$$

In order to find the background fields argument ΔV^i, we consider the zeroth order of Eq. (34)

$$v_{0\mp}^{i} = -\kappa\Theta_{0\mp}^{i\mu}\partial_{\mp}y_{\mu}, \tag{37}$$

and conclude that

$$\Delta V^i = -\kappa\theta_0^{i\mu}\Delta y_{\mu} + (g^{-1})^{i\mu}\Delta\tilde{y}_{\mu}. \tag{38}$$

Using the integral form of the variables and the relations $\Pi_{\pm ac}\Theta_{\mp}^{cb} + \Pi_{\pm ai}\Theta_{\mp}^{ib} = \frac{1}{2\kappa}\delta_a^b$; $\Theta_{\mp}^{ib} = -2\kappa\bar{\Theta}_{\mp}^{ij}\Pi_{\pm ja}\Theta_{\mp}^{ab}$; $\Theta_{\mp}^{aj} = -2\kappa\tilde{\Theta}_{\mp}^{ab}\Pi_{\pm bi}\Theta_{\mp}^{ij}$, we obtain that $\Delta V^a(\Delta V^i, y^a)$ defined in (19) equals

$$\Delta V^a(\Delta V^i, y_a) = -\kappa\theta_0^{a\mu}\Delta y_\mu + (g^{-1})^{a\mu}\Delta\tilde{y}_\mu. \tag{39}$$

Therefore, we conclude that action (36) is the totally T-dualized action (22).

In this paper we performed the partial T-dualizations and obtained the T-duality chain

$$S[x^\mu] \xrightarrow{\mathcal{T}^a} S_\pi[x^i, y_a] \xrightarrow{\mathcal{T}^i} {}^\star S[y_\mu]. \tag{40}$$

The first action describes the geometrical background, while the second and the third describe the non-geometrical backgrounds with nontrivial fluxes. From this chain one can find the relations between the arbitrary two coordinates in the chain. These general T-duality coordinate transformation laws are used in the investigation of the relations between the Poisson structures of the original, the partially T-dualized and the totally T-dualized theory [5]. Their canonical form will be used in deriving the complete closed string non-commutativity relations, which are the important features of the non-geometrical backgrounds.

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