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Lie Theory and Its Applications in Physics

Varna, Bulgaria, June 2013

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Editor

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Preface

The Workshop series ‘Lie Theory and Its Applications in Physics’ is designed to serve the community of theoretical physicists, mathematical physicists and mathematicians working on mathematical models for physical systems based on geometrical methods and in the field of Lie theory.

The series reflects the trend towards a geometrisation of the mathematical description of physical systems and objects. A geometric approach to a system yields in general some notion of symmetry which is very helpful in understanding its structure. Geometrisation and symmetries are meant in their widest sense, i.e., representation theory, algebraic geometry, infinite-dimensional Lie algebras and groups, superalgebras and supergroups, groups and quantum groups, noncommutative geometry, symmetries of linear and nonlinear PDE, special functions. Furthermore we include the necessary tools from functional analysis and number theory. This is a big interdisciplinary and interrelated field.

The first three workshops were organised in Clausthal (1995, 1997, 1999), the 4th was part of the 2nd Symposium ‘Quantum Theory and Symmetries’ in Cracow (2001), the 5th, 7th, 8th and 9th were organised in Varna (2003, 2007, 2009, 2011), the 6th was part of the 4th Symposium ‘Quantum Theory and Symmetries’ in Varna (2005), but has its own volume of Proceedings.

The 10th Workshop of the series (LT-10) was organized by the Institute of Nuclear Research and Nuclear Energy of the Bulgarian Academy of Sciences (BAS) in June 2013 (17–23), at the Guest House of BAS near Varna on the Bulgarian Black Sea Coast.

The overall number of participants was 71 and they came from 21 countries.

The scientific level was very high as can be judged by the speakers. The *plenary speakers* were: Loriano Bonora (Trieste), Branko Dragovich (Belgrade), Ludvig Faddeev (St. Petersburg), Malte Henkel (Nancy), Evgeny Ivanov (Dubna), Toshiyuki Kobayashi (Tokyo), Ivan Kostov (Saclay), Karl-Hermann Neeb (Erlangen), Eric Ragoucy (Annecy), Ivan Todorov (Sofia), Joris Van Der Jeugt (Ghent), George Zoupanos (Athens).

The topics covered the most modern trends in the field of the Workshop: Symmetries in String Theories and Gravity Theories, Conformal Field Theory, Integrable Systems, Representation Theory, Supersymmetry, Quantum Groups, Vertex Algebras and Superalgebras, Quantum Computing.

There is some similarity with the topics of preceding workshops, however, the comparison shows how certain topics evolve and that new structures were found and used. For the present workshop we mention more emphasis on: representation theory, quantum groups, integrable systems, vertex algebras and superalgebras, on conformal field theories, applications to the minimal supersymmetric standard model.

The International Organizing Committee was: V.K. Dobrev (Sofia) and H.-D. Doebner (Clausthal) in collaboration with G. Rudolph (Leipzig).

The Local Organizing Committee was: V.K. Dobrev (Chairman), V.I. Doseva, A.Ch. Ganchev, S.G. Mihov, D.T. Nedanovski, T.V. Popov, T.P. Stefanova, M.N. Stoilov, N.I. Stoilova, S.T. Stoimenov.

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Sofia, Bulgaria Vladimir Dobrev
May 2014

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Part I
Plenary Talks

Revisiting Trace Anomalies in Chiral Theories

Loriano Bonora, Stefano Giaccari, and Bruno Lima De Souza

Abstract This is a report on work in progress about gravitational trace anomalies. We review the problem of trace anomalies in chiral theories in view of the possibility that such anomalies may contain not yet considered CP violating terms. The research consists of various stages. In the first stage we examine chiral theories at one-loop with external gravity and show that a (CP violating) Pontryagin term appears in the trace anomaly in the presence of an unbalance of left and right chirality. However the imaginary coupling of such term implies a breakdown of unitarity, putting a severe constraint on such type of models. In a second stage we consider the compatibility of the presence of the Pontryagin density in the trace anomaly with (local) supersymmetry, coming to an essentially negative conclusion.

1 Introduction

We revisit trace anomalies in theories coupled to gravity, an old subject brought back to people's attention thanks to the importance acquired recently by conformal field theories both in themselves and in relation to the AdS/CFT correspondence. What has stimulated specifically this research is the suggestion by Nakayama [1] that trace anomalies may contain a CP violating term (the Pontryagin density). It is well known that a basic condition for baryogenesis is the existence of CP nonconserving reactions in an early stage of the universe. Many possible mechanisms for this have been put forward, but to date none is completely satisfactory. The appearance of a CP violating term in the trace anomaly of a theory weakly coupled to gravity may provide a so far unexplored new mechanism for baryogenesis.

Let us recall that the energy-momentum tensor in field theory is defined by $T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}$. Under an infinitesimal local rescaling of the matrix: $\delta g_{\mu\nu} = 2\sigma g_{\mu\nu}$ we have

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$$\delta S = \frac{1}{2}\int d^4x\sqrt{-g}T_{\mu\nu}\delta g^{\mu\nu} = -\int d^4x\sqrt{-g}\sigma T_\mu{}^\mu. \tag{1}$$

If the action is invariant, classically $T_\mu{}^\mu = 0$, but at one loop (in which case S is replaced by the one-loop effective action W) the trace of the e.m. tensor is generically nonvanishing. In D=4 it may contain, in principle, beside the Weyl density (square of the Weyl tensor)

$$\mathcal{W}^2 = \mathcal{R}_{nmkl}\mathcal{R}^{nmkl} - 2\mathcal{R}_{nm}\mathcal{R}^{nm} + \frac{1}{3}\mathcal{R}^2 \tag{2}$$

and the Gauss-Bonnet (or Euler) one,

$$E = \mathcal{R}_{nmkl}\mathcal{R}^{nmkl} - 4\mathcal{R}_{nm}\mathcal{R}^{nm} + \mathcal{R}^2, \tag{3}$$

another nontrivial piece, the Pontryagin density,

$$P = \frac{1}{2}\left(\epsilon^{nmlk}\mathcal{R}_{nmpq}\mathcal{R}_{lk}{}^{pq}\right) \tag{4}$$

Each of these terms appears in the trace with its own coefficient:

$$T_\mu{}^\mu = aE + c\mathcal{W}^2 + eP \tag{5}$$

The coefficient a and c are known at one-loop for any type of matter. The coefficient of (4) has not been sufficiently studied yet. The purpose of this paper is to fill up this gap. The plan of our research consists of three stages. To start with we analyse the one loop calculation of the trace anomaly in chiral models. Both the problem and the relevant results are not new: the trace anomaly contains beside the square Weyl density and the Euler density also the Pontryagin density. What is important is that the e coefficient is purely imaginary. This entails a violation of unitarity at one-loop and, consequently, introduces an additional criterion for a theory to be acceptable. The latter is similar to the analogous criterion for chiral gauge and gravitational anomalies, which is since long a selection criterion for acceptable theories. A second stage of our research concerns the compatibility between the appearance of the Pontryagin term in the trace anomaly and supersymmetry. Since it is hard to supersymmetrize the above three terms and relate them to one another in a supersymmetric context, the best course is to consider a conformal theory in 4D coupled to (external) $N = 1$ supergravity formulated in terms of superfields and find all the potential superconformal anomalies. This will allow us to see whether (4) can be accommodated in an anomaly supermultiplet as a trace anomaly member. The result of our analysis seems to exclude this possibility. Finally, a third stage of our research is to analyse the possibility that the Pontryagin density appears in the trace anomaly in a nonperturbative way, for instance via an AdS/CFT correspondence as suggested in [1].

In this contribution we will consider the first two issues above. In the next section we will examine the problem of the one-loop trace anomaly in a prototype chiral theory. Section 3 is devoted to the compatibility of the Pontryagin term in the trace anomaly with supersymmetry.

2 One-Loop Trace Anomaly in Chiral Theories

The model we will consider is the simplest possible one: a left-handed spinor coupled to external gravity in 4D. The action is

$$S = \int d^4x \sqrt{|g|}\, i \bar{\psi}_L \gamma^m (\nabla_m + \frac{1}{2}\omega_m)\psi_L \tag{6}$$

where $\gamma^m = e_a^m \gamma^a$, ∇ ($m, n, \ldots$ are world indices, $a, b, \ldots$ are flat indices) is the covariant derivative with respect to the world indices and ω_m is the spin connection:

$$\omega_m = \omega_m^{ab} \Sigma_{ab}$$

where $\Sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$ are the Lorentz generators. Finally $\psi_L = \frac{1+\gamma_5}{2}\psi$. Classically the energy-momentum tensor

$$T_{\mu\nu} = \frac{i}{2}\bar{\psi}_L \gamma_\mu \overleftrightarrow{\nabla}_\nu \psi_L \tag{7}$$

is both conserved on shell and traceless. At one loop to make sense of the calculations one must introduce regulators. The latter generally break both diffeomorphism and conformal invariance. A careful choice of the regularization procedure may preserve diff invariance, but anyhow breaks conformal invariance, so that the trace of the e.m. tensor takes the form (5), with specific nonvanishing coefficients a, c, e. There are various techniques to calculate the latter: cutoff, point splitting, Pauli-Villars, dimensional regularizations. Here we would like to briefly recall the heat kernel method utilized in [2] and in references cited therein (a more complete account will appear elsewhere). Denoting by D the relevant Dirac operator in (6) one can prove that

$$\delta W = -\int d^4x \sqrt{-g}\sigma T_\mu{}^\mu = -\frac{1}{16\pi^2}\int d^4x \sqrt{-g}\sigma b_4\left(x, x; D^\dagger D\right).$$

Thus

$$T_\mu{}^\mu = b_4\left(x, x; D^\dagger D\right) \tag{8}$$

The coefficient $b_4\left(x, x; D^\dagger D\right)$ appear in the heat kernel. The latter has the general form

$$K\left(t, x, y; \mathcal{D}\right) \sim \frac{1}{\left(4\pi t\right)^2} e^{-\frac{\sigma(x,y)}{2t}} \left(1 + t b_2\left(x, y; \mathcal{D}\right) + t^2 b_4\left(x, y; \mathcal{D}\right) + \cdots\right), \quad (9)$$

where $\mathcal{D} = D^\dagger D$ and $\sigma\left(x, y\right)$ is the half square length of the geodesic connecting x and y, so that $\sigma\left(x, x\right) = 0$. For coincident points we therefore have

$$K\left(t, x, x; \mathcal{D}\right) \sim \frac{1}{16\pi^2}\left(\frac{1}{t^2} + \frac{1}{t} b_2\left(x, x; \mathcal{D}\right) + b_4\left(x, x; \mathcal{D}\right) + \cdots\right). \quad (10)$$

This expression is divergent for $t \to 0$ and needs to be regularized. This can be done in various ways. The finite part, which we are interested in, has been calculated first by DeWitt [3], and then by others with different methods. The results are reported in [2]. For a spin $\frac{1}{2}$ left-handed spinor as in our example one gets

$$b_4\left(x, x; D^\dagger D\right) = \frac{1}{180 \times 16\pi^2} \int d^4x \sqrt{-g}\left(a\, E_4 + c\, W^2 + e\ \ P\right) \quad (11)$$

with

$$a = \frac{11}{4}, \qquad c = -\frac{9}{2}, \qquad e = \frac{15}{4} \quad (12)$$

This result was obtained with an entirely Euclidean calculation. Turning to the Minkowski the actual e.m trace at one loop is

$$T_\mu{}^\mu = \frac{1}{180 \times 16\pi^2}\left(\frac{11}{4}\, E + c\, \mathcal{W}^2 + i\, \frac{15}{4}\, P\right) \quad (13)$$

As pointed out above the important aspect of (13) is the i appearing in front of the Pontryagin density. The origin of this imaginary coupling is easy to trace. It comes from the trace of gamma matrices including a γ_5 factor. In 4D, while the trace of an even number of gamma matrices, which give rise to first two terms in the RHS of (13), is a real number, the trace of an even number of gamma's multiplied by γ_5 is always imaginary. The Pontryagin term comes precisely from the latter type of traces. It follows that, as a one loop effect, the energy momentum tensor becomes complex, and, in particular, since T_0^0 is the Hamiltonian density, we must conclude that unitarity is not preserved in this type of theories. Exactly as chiral gauge theories with nonvanishing chiral gauge anomalies are rejected as sick theories, also chiral models with complex trace anomalies are not acceptable theories. For instance the old-fashioned standard model with massless left-handed neutrinos is in this situation. This model, provided it has an UV fixed point, has a complex trace anomaly and breaks unitarity. This is avoided in the modern formulation of the electroweak interactions by the addition of a right-handed neutrino (for each flavor), or, alternatively, by using Majorana neutrinos. So, in hindsight, one could have predicted massive neutrinos.

In general we can say that in models with a chirality unbalance a problem with unitarity may arise due to the trace anomaly and has to be carefully taken into account.

3 Pontryagin Density and Supersymmetry

In this section we discuss the problem posed by the possible appearance of the Pontryagin term in the trace anomaly: is it compatible with supersymmetry? It is a well known fact that trace anomalies in supersymmetric theories are members of supermultiplets, to which also the Abelian chiral anomaly belongs. Thus one way to analyse this issue would be to try and supersymmetrize the three terms (2)–(4) and see whether they can be accommodated in supermultiplets. This direct approach, however, is far from practical. What we will do, instead, is to consider a conformal theory in 4D coupled to (external) supergravity formulated in terms of superfields, and find all the potential superconformal anomalies. This will allow us to see whether (4) can be accommodated in an anomaly supermultiplet as a trace anomaly member.

3.1 Minimal Supergravity

The most well known model of $N = 1$ supergravity in $D = 4$ is the so-called *minimal supergravity*, see for instance [4]. The superspace of $N = 1$ supergravity is spanned by the supercoordinates $Z^M = (x^m, \theta^\mu, \bar\theta_{\dot\mu})$. In this superspace one introduces a superconnection, a supertorsion and the relevant supercurvature. To determine the dynamics one imposes constraints on the supertorsion. Such constraints are not unique. A particular choice of the latter, the *minimal* constraints, define the minimal supergravity model, which can be formulated in terms of the superfields $R(z), G_a(z)$ and $W_{\alpha\beta\gamma}(z)$. R and $W_{\alpha\beta\gamma}$ are chiral while G_a is real. One also needs the antichiral superfields $R^+(z)$ and $\bar W_{\dot\alpha\dot\beta\dot\gamma}(z)$. $W_{\alpha\beta\gamma}$ is completely symmetric in the spinor indices $\alpha, \beta, \ldots$. These superfields satisfy themselves certain constraints. Altogether the independent degrees of freedom are 12 bosons + 12 fermions. One can define superconformal transformations in terms of a parameter superfield σ. For instance

$$\delta R = (2\bar\sigma - 4\sigma)R - \frac{1}{4}\nabla_{\dot\alpha}\nabla^{\dot\alpha}\bar\sigma$$

$$\delta G_a = -(\sigma + \bar\sigma)G_a + i\nabla_a(\bar\sigma - \sigma)$$

$$\delta W_{\alpha\beta\gamma} = -3\sigma W_{\alpha\beta\gamma}$$

To find the possible superconformal anomalies we use a cohomological approach. Having in mind a superconformal matter theory coupled to a $N = 1$ supergravity, we define the functional operator that implements these transformations, i.e.

$$\Sigma = \int_{x\theta} \delta\chi_i \, \frac{\delta}{\delta\chi_i}$$

where χ_i represent the various superfields in the game and $_{x\theta}$ denotes integration $d^4x d^4\theta$. This operator is nilpotent: $\Sigma^2 = 0$. As a consequence it defines a cohomology problem. The cochains are integrated local expressions of the superfields and their superderivatives, invariant under superdiffeomorphism and local superLorentz transformations. Candidates for superconformal anomalies are nontrivial cocycles of Σ which are not coboundaries, i.e. integrated local functionals Δ_σ, linear in σ, such that

$$\Sigma\,\Delta_\sigma = 0, \qquad \text{and} \qquad \Delta_\sigma \neq \Sigma\,\mathcal{C}$$

for any integrated local functional $\mathcal{C}$ (not containing σ).

The complete analysis of all the possible nontrivial cocycles of the operator Σ was carried out in [5]. It was shown there that the latter can be cast into the form

$$\Delta_\sigma = \int_{x\theta} \left[\frac{E(z)}{-8R(z)}\, \sigma(z)\, \mathcal{S}(z) + h.c. \right] \tag{14}$$

where $\mathcal{S}(z)$ is a suitable chiral superfield, and all the possibilities for $\mathcal{S}$ were classified. For supergravity alone (without matter) the only nontrivial possibilities turn out to be:

$$\mathcal{S}_1(z) = W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} \qquad \text{and} \qquad \mathcal{S}_2(z) = (\bar\nabla_{\dot\alpha}\bar\nabla^{\dot\alpha} - 8R)(G_a G^a + 2RR^+) \tag{15}$$

(the operator $(\bar\nabla_{\dot\alpha}\bar\nabla^{\dot\alpha} - 8R)$ maps a real superfield into a chiral one).

It is well-known that the (14) cocycles contain not only the trace anomaly, but a full supermultiplet of anomalies. The local expressions of the latter are obtained by stripping off the corresponding parameters from the integrals in (14).

In order to recognize the ordinary field content of the cocycles (15) one has to pass to the component form. This is done by choosing the lowest components of the supervielbein as follows:

$$E_M{}^A(z)\big|_{\theta=\bar\theta=0} = \begin{pmatrix} e_m{}^a(x) & \frac{1}{2}\psi_m{}^\alpha(x) & \frac{1}{2}\bar\psi_{m\dot\alpha}(x) \\ 0 & \delta_\mu{}^\alpha & 0 \\ 0 & 0 & \delta^{\dot\mu}{}_{\dot\alpha} \end{pmatrix}$$

where $e_m{}^a$ are the usual 4D vierbein and $\psi_m{}^\alpha(x), \bar\psi_{m\dot\alpha}(x)$ the gravitino field components. Similarly one identifies the independent components of the other superfields (the lowest component of R and G_a). For σ we have

$$\sigma(z) = \omega(x) + i\alpha(x) + \sqrt{2}\Theta^\alpha \chi_\alpha(x) + \Theta^\alpha \Theta_\alpha(F(x) + iG(x)) \tag{16}$$

where Θ^α are Lorentz covariant anticommuting coordinates [4]. The component fields of (16) identify the various anomalies in the cocycles (15). In particular ω is the parameter of the ordinary conformal transformations and α the parameter of the chiral transformations. They single out the corresponding anomalies. At this point it is a matter of algebra to write down the anomalies in component. Retaining for simplicity only the metric we obtain the *ordinary* form of the cocycles. This is

$$\Delta_\sigma^{(1)} \approx \int_x e\Big\{\omega\Big(\mathcal{R}_{nmkl}\mathcal{R}^{nmkl} - 2\mathcal{R}_{nm}\mathcal{R}^{nm} + \frac{1}{3}\mathcal{R}^2\Big) - \frac{1}{2}\,\alpha\,\epsilon^{nmlk}\mathcal{R}_{nmpq}\mathcal{R}_{lk}{}^{pq}\Big\} \tag{17}$$

for the first cocycle ($\approx$ denotes precisely the ordinary form), and

$$\Delta_\sigma^{(2)} \approx 4\int_x e\,\omega\big(\frac{2}{3}\mathcal{R}^2 - 2\mathcal{R}_{nm}\mathcal{R}^{nm}\big) \tag{18}$$

for the second. Taking a suitable linear combination of the two we get

$$\Delta_\sigma^{(1)} + \frac{1}{2}\Delta_\sigma^{(2)} \approx \tag{19}$$
$$\int_x e\Big\{\omega\big(\mathcal{R}_{nmkl}\mathcal{R}^{nmkl} - 4\mathcal{R}_{nm}\mathcal{R}^{nm} + \mathcal{R}^2\big) - \frac{1}{2}\,\alpha\,\epsilon^{nmlk}\mathcal{R}_{nmpq}\mathcal{R}_{lk}{}^{pq}\Big\}$$

We see that (17) contain $\mathcal{W}^2$ while (19) contains the Euler density in the terms proportional to ω (trace anomaly). They both contain the Pontryagin density in the term proportional to α (chiral anomaly).

In conclusion $\Delta_\sigma^{(1)}$ corresponds to a multiplet of anomalies, whose first component is the Weyl density multiplied by ω, accompanied by the Pontryagin density (the Delbourgo-Salam anomaly) multiplied by α. On the other hand $\Delta_\sigma^{(2)}$ does not contain the Pontryagin density and the part linear in ω is a combination of the Weyl and Gauss-Bonnet density. None of them contains the Pontryagin density in the trace anomaly part. Therefore we must conclude that, as far as $N = 1$ minimal supergravity is concerned, our conclusion about the compatibility between the Pontryagin density as a trace anomaly terms and local supersymmetry, is negative.

3.2 Other Nonminimal Supergravities

As previously mentioned the minimal model of supergravity is far from unique. There are many other choices of the supertorsion constraints, beside the minimal one. Most of them are connected by field redefinitions and represent the same theory. But there are choices that give rise to different dynamics. This is the case for the nonminimal $20+20$ and $16+16$ models. In the former case one introduces two new spinor superfields T_α and $\bar{T}_{\dot{\alpha}}$, while setting $R = R^+ = 0$. This model has $20 + 20$ degrees of freedom. The bosonic degrees of freedom are those of the minimal model, excluding R and R^+, plus ten additional ones which can be identified with the lowest components of the superfields $S = \mathcal{D}^\alpha T_\alpha - (n+1)T^\alpha T_\alpha$ and $\bar{S}$, $\bar{\mathcal{D}}_{\dot{\alpha}} T_\alpha$ and $\mathcal{D}_\alpha T_{\dot{\alpha}}$. The superconformal parameter is a generic complex superfield Σ constrained by the condition

$$(\mathcal{D}^\alpha \mathcal{D}_\alpha + (n+1)T^\alpha \mathcal{D}_\alpha)\left[3n(\bar{\Sigma} - \Sigma) - (\bar{\Sigma} + \Sigma)\right] = 0$$

where n is a numerical parameter. It is easy to find a nontrivial cocycle of this symmetry

$$\Delta^{(1)}_{n.m.} = \int_{x,\theta} E\, \Sigma\, W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} \frac{\bar{T}_{\dot{\alpha}} \bar{T}^{\dot{\alpha}}}{\bar{S}^2} + h.c.$$

and to prove that its ordinary component form is, up to a multiplicative factor,

$$\Delta^{(1)}_{\Sigma} \approx \frac{1}{4} \int_x e \left\{ \omega \left(\mathcal{R}_{nmkl} \mathcal{R}^{nmkl} - 2\mathcal{R}_{nm} \mathcal{R}^{nm} + \frac{1}{3} \mathcal{R}^2 \right) - \frac{1}{2}\, \alpha\, \epsilon^{nmlk} \mathcal{R}_{nmpq} \mathcal{R}_{lk}{}^{pq} \right\}$$

where $\omega + i\alpha$ is the lowest component of the superfield Σ. That is, the same ordinary form as $\Delta^{(1)}_\sigma$. As for other possible cocycles they can be obtained from the minimal supergravity ones by way of superfield redefinitions. To understand this point one should remember what was said above: different models of supergravity are defined by making a definite choice of the torsion constraints and, after such a choice, by identifying the dynamical degrees of freedom. This is the way minimal and nonminimal models are introduced. However it is possible to transform the choices of constraints into one another by means of linear transformations of the supervierbein and the superconnection [6, 7]:

$$E'_M{}^A = E_M{}^B X_B{}^A, \quad E'_A{}^M = X^{-1}{}_A{}^B E_B{}^M, \quad \Phi'_{MA}{}^B = \Phi_{MA}{}^B + \chi_{MA}{}^B$$

for suitable $X_A{}^B$ and $\chi_{MA}{}^B$. This was done in [8] and will not be repeated here. The result is a very complicated form for the cocycle $\Delta^{(2)}_{n.m.}$, derived from $\Delta^{(2)}_\sigma$. However the ordinary component form is the same for both.

As for the $16 + 16$ nonminimal supergravity, it is obtained from the $20 + 20$ model by imposing

$$T_\alpha = \mathcal{D}_\alpha \psi, \qquad T_{\dot{\alpha}} = \mathcal{D}_{\dot{\alpha}} \psi$$

where ψ is a (dimensionless) real superfield. The independent bosonic degrees of freedom are the lowest component of $S, \bar{S}$, $c_{\alpha\dot{\alpha}}$ and $G_{\alpha\dot{\alpha}}$, beside the metric. The superconformal transformation are expressed in terms of a real vector superfield L and an arbitrary chiral superfield Λ satisfying the constraint

$$(\mathcal{D}^\alpha \mathcal{D}_\alpha + (n+1)T^\alpha \mathcal{D}_\alpha)(2L + (3n+1)\Lambda) = 0.$$

The derivation of the nontrivial superconformal cocycles is much the same as for the previous model. The end result is two cocycles whose form, in terms of superfields, is considerably complicated, but whose ordinary form is the same as $\Delta_\sigma^{(1)}$ and $\Delta_\sigma^{(2)}$.

At this point we must clarify whether the cocycles we have found in 20+20 and 16+16 nonminimal supergravities are the only ones. In [8] a systematic cohomological search of such nontrivial cocycles has not been done, the reason being that when dimensionless fields, like ψ and $\bar{\psi}$, are present in a theory a polynomial analysis is not sufficient (and a non-polynomial one is of course very complicated). But we can argue as follows: consider a nontrivial cocycle in nonminimal or $16+16$ nonminimal supergravity; it can be mapped to a minimal cocycle which either vanishes or coincides with the ones classified in [5]. There is no other possibility because in minimal supergravity there are no dimensionless superfields (apart from the vielbein) and the polynomial analysis carried out in [5] is sufficient to identify all cocycles. We conclude that the $20+20$ and $16+16$ nonminimal nontrivial cocycles, which reduce in the ordinary form to a nonvanishing expression, correspond to $\Delta_\sigma^{(1)}$ and $\Delta_\sigma^{(2)}$ in minimal supergravity and only to them.

None of these cocycles contains the Pontryagin density in the trace anomaly part. Therefore we must conclude that, as far as $N = 1$ minimal and nonminimal supergravity is concerned, our conclusion about the compatibility between the Pontryagin density as a trace anomaly terms and local supersymmetry, is negative.

Conclusion

A component of the trace anomaly which appear in chiral theories (the Pontryagin density) may have interesting implications. It is a CP violating term and, as such, it could be an interesting mechanism for baryogenesis. At one loop, as we have seen, this term violates unitarity and the only use we can make of it is as a selection criterion for phenomenological models with an UV fixed point. If, on the other hand, by some other kind of mechanism still to be discovered, this term appears in the trace of the em tensor with a real coefficient, it may become very interesting as a CP violating term. In the last section we have seen that, however, this is incompatible with supersymmetry. In other words, if such mechanism exists, it can become effective only after supersymmetry breaking. The search for the P term continues.

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Complete T-Dualization of a String in a Weakly Curved Background

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Abstract We apply the generalized Buscher procedure, to a subset of the initial coordinates of the bosonic string moving in the weakly curved background, composed of a constant metric and a linearly coordinate dependent Kalb-Ramond field with the infinitesimal strength. In this way we obtain the partially T-dualized action. Applying the procedure to the rest of the original coordinates we obtain the totally T-dualized action. This derivation allows the investigation of the relations between the Poisson structures of the original, the partially T-dualized and the totally T-dualized theory.

1 Bosonic String in the Weakly Curved Background

Let us consider the closed string moving in the coordinate dependent background, described by the action [1]

$$S[x] = \kappa \int_\Sigma d^2\xi \; \partial_+ x^\mu \Pi_{+\mu\nu}[x] \partial_- x^\nu. \tag{1}$$

The background is defined by the space-time metric $G_{\mu\nu}$ and the antisymmetric Kalb-Ramond field $B_{\mu\nu}$

$$\Pi_{\pm\mu\nu}[x] = B_{\mu\nu}[x] \pm \frac{1}{2} G_{\mu\nu}[x]. \tag{2}$$

The light-cone coordinates are

$$\xi^\pm = \frac{1}{2}(\tau \pm \sigma), \qquad \partial_\pm = \partial_\tau \pm \partial_\sigma, \tag{3}$$

and the action is given in the conformal gauge (the world-sheet metric is taken to be $g_{\alpha\beta} = e^{2F}\eta_{\alpha\beta}$).

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The world-sheet conformal invariance is required, as a condition of having a consistent theory on a quantum level. This leads to the space-time equations for the background fields, which equal

$$R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_{\nu}{}^{\rho\sigma} = 0, \quad D_\rho B^\rho{}_{\mu\nu} = 0, \tag{4}$$

in the lowest order in slope parameter α' and for the constant dilaton field $\Phi = const$. Here $B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ is the field strength of the field $B_{\mu\nu}$, and $R_{\mu\nu}$ and D_μ are Ricci tensor and covariant derivative with respect to the space-time metric.

We will consider a weakly curved background [2, 3], defined by

$$\begin{aligned} &G_{\mu\nu}[x] = const, \\ &B_{\mu\nu}[x] = b_{\mu\nu} + h_{\mu\nu}[x] = b_{\mu\nu} + \frac{1}{3} B_{\mu\nu\rho} x^\rho, \qquad b_{\mu\nu}, B_{\mu\nu\rho} = const. \end{aligned} \tag{5}$$

Here, the constant $B_{\mu\nu\rho}$ is infinitesimal. The background (5) is the solution of the field equations (4) in the first order in $B_{\mu\nu\rho}$.

2 Partial T-Dualization

In the paper [3], we generalized the Buscher prescription for a construction of a T-dual theory. This prescription, unlike the standard one [4], is applicable to the string backgrounds depending on all the space-time coordinates, such as the weakly curved background. We performed the procedure along all the coordinates and obtained T-dual theory. The noncommutativity of the T-dual coordinates we investigated in [5]. In the present paper we consider the partial T-dualization, i.e. the application of the procedure to some without subset of the coordinates. We construct the partially T-dualized theory. The noncommutativity of the coordinates in similar theories was considered in [6].

Let us mark the T-dualization along the coordinate x^μ by T_μ, and separate the coordinates into two subsets (x^i, x^a) with $i = 0, \ldots, d-1$ and $a = d, \ldots, D-1$ and mark the T-dualizations along these subsets of coordinates by

$$T^i \equiv T_0 \circ \cdots \circ T_{d-1}, \quad T^a \equiv T_d \circ \cdots \circ T_{D-1}. \tag{6}$$

In this section we will find the partially T-dualized action performing T-dualization along coordinates x^a, $\mathcal{T}^a : S$.

The closed string action in the weakly curved background has a global symmetry

$$\delta x^\mu = \lambda^\mu . \tag{7}$$

Let us localize this symmetry for the coordinates x^a

$$\delta x^a = \lambda^a(\tau, \sigma), \quad a = d, \dots, D-1, \tag{8}$$

by introducing the gauge fields v_α^a and substituting the ordinary derivatives with the covariant ones

$$\partial_\alpha x^a \to D_\alpha x^a = \partial_\alpha x^a + v_\alpha^a . \tag{9}$$

The gauge invariance of the covariant derivatives is obtained by imposing the following transformation law for the gauge fields

$$\delta v_\alpha^a = -\partial_\alpha \lambda^a . \tag{10}$$

Also, substitute x^a in the argument of the background fields with its invariant extension, defined by

$$\begin{aligned} \Delta x_{inv}^a &\equiv \int_P d\xi^\alpha \, D_\alpha x^a = \int_P (d\xi^+ D_+ x^a + d\xi^- D_- x^a) \\ &= x^a - x^a(\xi_0) + \Delta V^a , \end{aligned} \tag{11}$$

where

$$\Delta V^a \equiv \int_P d\xi^\alpha v_\alpha^a = \int_P (d\xi^+ v_+^a + d\xi^- v_-^a). \tag{12}$$

The line integral is taken along the path P, from the initial point $\xi_0^\alpha(\tau_0, \sigma_0)$ to the final one $\xi^\alpha(\tau, \sigma)$. To preserve the physical equivalence between the gauged and the original theory, one introduces the Lagrange multiplier y_a and adds the term $\frac{1}{2} y_a F_{+-}^a$ to the Lagrangian, which will force the field strength $F_{+-}^a \equiv \partial_+ v_-^a - \partial_- v_+^a = -2F_{01}^a$ to vanish. In this way, we obtain the gauge invariant action

$$\begin{aligned} S_{inv} = \kappa \int d^2\xi \Big[&\partial_+ x^i \Pi_{+ij}[x^i, \Delta x_{inv}^a] \partial_- x^j + \partial_+ x^i \Pi_{+ia}[x^i, \Delta x_{inv}^a] D_- x^a \\ &+ D_+ x^a \Pi_{+ai}[x^i, \Delta x_{inv}^a] \partial_- x^i + D_+ x^a \Pi_{+ab}[x^i, \Delta x_{inv}^a] D_- x^b \\ &+ \frac{1}{2}(v_+^a \partial_- y_a - v_-^a \partial_+ y_a) \Big], \end{aligned} \tag{13}$$

where the last term is equal to $\frac{1}{2} y_a F_{+-}^a$ up to the total divergence. Now, we can use the gauge freedom to fix the gauge $x^a(\xi) = x^a(\xi_0)$. The gauge fixed action equals

$$S_{fix} = \kappa \int d^2\xi \Big[\partial_+ x^i \Pi_{+ij}[x^i, \Delta V^a] \partial_- x^j + \partial_+ x^i \Pi_{+ia}[x^i, \Delta V^a] v^a_- + v^a_+ \Pi_{+ai}[x^i, \Delta V^a] \partial_- x^i + v^a_+ \Pi_{+ab}[x^i, \Delta V^a] v^b_- + \frac{1}{2}(v^a_+ \partial_- y_a - v^a_- \partial_+ y_a) \Big]. \tag{14}$$

The equations of motion for the Lagrange multiplier y_a, $\partial_+ v^a_- - \partial_- v^a_+ = 0$, have a solution $v^a_\pm = \partial_\pm x^a$, which turns the gauge fixed action to the initial one.

2.1 The Partially T-Dualized Action

The partially T-dualized action will be obtained after elimination of the gauge fields from the gauge fixed action (14), using their equations of motion. Varying over the gauge fields $v^a_\pm$ one obtains

$$\Pi_{\pm ai}[x^i, \Delta V^a] \partial_\mp x^i + \Pi_{\pm ab}[x^i, \Delta V^a] v^b_\mp + \frac{1}{2} \partial_\mp y_a = \pm \beta^\pm_a [x^i, V^a], \tag{15}$$

where $\beta^\pm_a[x^i, V^a]$ is the infinitesimal contribution from the background fields argument. Using the inverse of the background fields composition $2\kappa \Pi_{\pm ab}$, defined by $\tilde{\Theta}^{ab}_\pm \equiv -\frac{2}{\kappa}(\tilde{G}^{-1}_E)^{ac} \Pi_{\pm cd} (\tilde{G}^{-1})^{db}$, where $\tilde{G}_{ab} \equiv G_{ab}$ and $\tilde{G}_{Eab} \equiv G_{ab} - 4B_{ac}(\tilde{G}^{-1})^{cd} B_{db}$, we can extract the gauge fields $v^a_\pm$ from Eq. (15)

$$v^a_\mp = -2\kappa \tilde{\Theta}^{ab}_\mp [x^i, \Delta V^a] \Big[\Pi_{\pm bi}[x^i, \Delta V^a] \partial_\mp x^i + \frac{1}{2} \partial_\mp y_b \mp \beta^\pm_b [x^i, V^a] \Big]. \tag{16}$$

Substituting (16) into the action (14), we obtain the partially T-dualized action

$$\begin{aligned} S_\pi[x^i, y_a] = \kappa \int d^2\xi \Big[& \partial_+ x^i \bar{\Pi}_{+ij}[x^i, \Delta V^a(x^i, y^a)] \partial_- x^j \\ & + \frac{\kappa}{2} \partial_+ y_a \tilde{\Theta}^{ab}_- [x^i, \Delta V^a(x^i, y^a)] \partial_- y_b \\ & - \kappa\, \partial_+ x^i \Pi_{+ia}[x^i, \Delta V^a(x^i, y^a)] \tilde{\Theta}^{ab}_- [x^i, \Delta V^a(x^i, y^a)] \partial_- y_b \\ & + \kappa\, \partial_+ y_a \tilde{\Theta}^{ab}_- [x^i, \Delta V^a(x^i, y^a)] \Pi_{+bi}[x^i, \Delta V^a(x^i, y^a)] \partial_- x^i \Big], \end{aligned} \tag{17}$$

where

$$\bar{\Pi}_{+ij} \equiv \Pi_{+ij} - 2\kappa \Pi_{+ia} \tilde{\Theta}^{ab}_- \Pi_{+bj}. \tag{18}$$

In order to find the explicit value of the background fields argument $\Delta V^a(x^i, y^a)$, one substitutes the zeroth order of the equations of motion (16) into (12) and obtains

$$\Delta V^{(0)a} = -\kappa\Big[\tilde{\Theta}^{ab}_{0+}\Pi_{0-bi} + \tilde{\Theta}^{ab}_{0-}\Pi_{0+bi}\Big]\Delta x^{(0)i} - \kappa\Big[\tilde{\Theta}^{ab}_{0+}\Pi_{0-bi} - \tilde{\Theta}^{ab}_{0-}\Pi_{0+bi}\Big]\Delta\tilde{x}^{(0)i} - \frac{\kappa}{2}\Big[\tilde{\Theta}^{ab}_{0+} + \tilde{\Theta}^{ab}_{0-}\Big]\Delta y^{(0)}_b - \frac{\kappa}{2}\Big[\tilde{\Theta}^{ab}_{0+} - \tilde{\Theta}^{ab}_{0-}\Big]\Delta\tilde{y}^{(0)}_b, \tag{19}$$

where $\tilde{\Theta}^{ab}_{0\pm}$ stands for the zeroth order value of $\tilde{\Theta}^{ab}_{\pm}$, which can be written as

$$\tilde{\Theta}^{ab}_{0\pm} \equiv -\frac{2}{\kappa}(\tilde{g}^{-1})^{ac}\,\Pi_{0\pm cd}(\tilde{G}^{-1})^{db} = \tilde{\theta}^{ab}_0 \mp \frac{1}{\kappa}(\tilde{g}^{-1})^{ab}, \tag{20}$$

where $\tilde{g}_{ab} = G_{ab} - 4b_{ac}(\tilde{G}^{-1})^{cd}b_{db}$; $\tilde{\theta}^{ab}_0 \equiv -\frac{2}{\kappa}(\tilde{g}^{-1})^{ac}\,b_{cd}(\tilde{G}^{-1})^{db}$ and

$$\Delta\tilde{y}^{(0)}_a = \int(d\tau y^{(0)\prime}_a + d\sigma\dot{y}^{(0)}_a), \quad \Delta\tilde{x}^{(0)i} = \int(d\tau x^{(0)\prime i} + d\sigma\dot{x}^{(0)i}). \tag{21}$$

Initial theory, the partially T-dualized theory and the totally T-dualized theory obtained in [3] are physically equivalent theories. In the next section we will partially T-dualize the partially T-dualized theory.

3 The Total T-Dualization of the Initial Action

The T-dual theory, derived in [3], a result of T-dualization of the initial action along all the coordinates, is given by

$$^\star S[y] = \kappa\int d^2\xi\,\partial_+ y_\mu\,{}^\star\Pi^{\mu\nu}_+[\Delta V(y)]\,\partial_- y_\nu = \frac{\kappa^2}{2}\int d^2\xi\,\partial_+ y_\mu\Theta^{\mu\nu}_-[\Delta V(y)]\partial_- y_\nu, \tag{22}$$

with

$$\Theta^{\mu\nu}_\pm \equiv -\frac{2}{\kappa}(G_E^{-1}\Pi_\pm G^{-1})^{\mu\nu} = \theta^{\mu\nu} \mp \frac{1}{\kappa}(G_E^{-1})^{\mu\nu}, \tag{23}$$

where

$$G_{E\mu\nu} \equiv G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}, \quad \theta^{\mu\nu} \equiv -\frac{2}{\kappa}(G_E^{-1}BG^{-1})^{\mu\nu}. \tag{24}$$

The T-dual background fields are equal to

$$ {}^\star G^{\mu\nu}[\Delta V(y)] = (G_E^{-1})^{\mu\nu}[\Delta V(y)], \quad {}^\star B^{\mu\nu}[\Delta V(y)] = \frac{\kappa}{2}\theta^{\mu\nu}[\Delta V(y)]. \tag{25} $$

The argument of the background fields is given by

$$ \Delta V^\mu(y) = -\kappa\theta_0^{\mu\nu}\Delta y_\nu + (g^{-1})^{\mu\nu}\Delta\tilde{y}_\nu, \tag{26} $$

where $\Delta y_\mu = y_\mu(\xi) - y_\mu(\xi_0)$ and $\tilde{y}_\mu = \int(d\tau y'_\mu + d\sigma\dot{y}_\mu)$, while $g_{\mu\nu} = G_{\mu\nu} - 4b^2_{\mu\nu}$ and $\theta_0^{\mu\nu} = -\frac{2}{\kappa}(g^{-1}bG^{-1})^{\mu\nu}$.

Let us now show that the same result will be obtained applying the T-dualization procedure to the coordinates x^i of the partially T-dualized theory (17), $\mathcal{T}^i : S_\pi[x^i, y_a]$. Substituting the ordinary derivatives $\partial_\pm x^i$ with the covariant derivatives

$$ D_\pm x^i = \partial_\pm x^i + v^i_\pm, \tag{27} $$

where the gauge fields $v^i_\pm$ transform as $\delta v^i_\pm = -\partial_\pm\lambda^i$, and substituting the coordinates x^i in the background field arguments by

$$ \Delta x^i_{inv} = \int_P (d\xi^+ D_+ x^i + d\xi^- D_- x^i), \tag{28} $$

we obtain the gauge invariant action, which after fixing the gauge by $x^i(\xi) = x^i(\xi_0)$ becomes

$$ \begin{aligned} S^{fix}_\pi = \kappa \int d^2\xi \Big[& v^i_+ \bar{\Pi}_{+ij}[\Delta V^\mu] v^j_- + \frac{\kappa}{2}\partial_+ y_a \tilde{\Theta}^{ab}_-[\Delta V^\mu]\partial_- y_b \\ & -\kappa\, v^i_+ \Pi_{+ia}[\Delta V^\mu]\tilde{\Theta}^{ab}_-[\Delta V^\mu]\partial_- y_b + \kappa\,\partial_+ y_a \tilde{\Theta}^{ab}_-[\Delta V^\mu]\Pi_{+bi}[\Delta V^\mu] v^i_- \\ & + \frac{1}{2}(v^i_+\partial_- y_i - v^i_-\partial_+ y_i)\Big]. \end{aligned} \tag{29} $$

Here ΔV^i is defined by

$$ \Delta V^i \equiv \int_P (d\xi^+ v^i_+ + d\xi^- v^i_-), \tag{30} $$

and ΔV^a is defined in (19), whose arguments are in this case ΔV^i and y^a.

The totally T-dualized action will be obtained by eliminating the gauge fields from the gauge fixed action, using their equations of motion. Varying the action (29) over the gauge fields $v^i_\pm$ one obtains

$$ \bar{\Pi}_{\pm ij} v^j_\mp - \kappa\Pi_{\pm ia}\tilde{\Theta}^{ab}_\mp \partial_\mp y_b + \frac{1}{2}\partial_\mp y_i = \pm\beta^\pm_i. \tag{31} $$

Using the fact that the background field composition $\bar{\Pi}_{\pm ij}$ is inverse to $2\kappa\Theta^{ij}_{\mp}$, we can rewrite the equation of motion (31) expressing the gauge fields as

$$v^i_{\mp} = 2\kappa\Theta^{ij}_{\mp}\Big[\kappa\Pi_{\pm ja}\tilde{\Theta}^{ab}_{\mp}\partial_{\mp}y_b - \frac{1}{2}\partial_{\mp}y_j \pm \beta^{\pm}_j\Big]. \tag{32}$$

Using $\Pi_{\pm ab}\Theta^{bi}_{\mp} = -\Pi_{\pm aj}\Theta^{ji}_{\mp}$, we note that

$$\Theta^{ij}_{\mp}\Pi_{\pm ja}\tilde{\Theta}^{ab}_{\mp} = -\Theta^{ic}_{\mp}\Pi_{\pm ca}\tilde{\Theta}^{ab}_{\mp} = -\frac{1}{2\kappa}\Theta^{ib}_{\mp}, \tag{33}$$

and obtain

$$v^i_{\mp} = -\kappa\Theta^{i\mu}_{\mp}\partial_{\mp}y_\mu \pm 2\kappa\Theta^{ij}_{\mp}\beta^{\pm}_j. \tag{34}$$

Substituting (34) into (29), the action becomes

$$\begin{aligned} S = \kappa\int d^2\xi\Big[&\partial_+y_i\Big(\kappa\Theta^{ij}_- - \kappa^2\Theta^{ik}_-\bar{\Pi}_{+kl}\Theta^{lj}_-\Big)\partial_-y_j \\ &+\partial_+y_a\Big(-\kappa^2\Theta^{aj}_-\bar{\Pi}_{+jk}\Theta^{ki}_- + \frac{\kappa}{2}\Theta^{ai}_- - \kappa^2\tilde{\Theta}^{ab}_-\Pi_{+bj}\Theta^{ji}_-\Big)\partial_-y_i \\ &+\partial_+y_i\Big(-\kappa^2\Theta^{ij}_-\bar{\Pi}_{+jk}\Theta^{ka}_- + \frac{\kappa}{2}\Theta^{ia}_- - \kappa^2\Theta^{ij}_-\Pi_{+jb}\tilde{\Theta}^{ba}_-\Big)\partial_-y_a \\ &+\partial_+y_a\Big(\frac{\kappa}{2}\tilde{\Theta}^{ab}_- - \kappa^2\Theta^{ai}_-\bar{\Pi}_{+ij}\Theta^{jb}_- - \kappa^2\Theta^{ai}_-\Pi_{+ic}\tilde{\Theta}^{cb}_- - \kappa^2\tilde{\Theta}^{ac}_-\Pi_{+ci}\Theta^{ib}_-\Big)\partial_-y_b\Big]. \end{aligned} \tag{35}$$

Using $\bar{\Pi}_{\pm ij}\Theta^{jk}_{\mp} = \Theta^{kj}_{\mp}\bar{\Pi}_{\pm ji} = \frac{1}{2\kappa}\delta^k_i$; $\tilde{\Pi}_{\pm ab}\Theta^{bc}_{\mp} = \Theta^{cb}_{\mp}\tilde{\Pi}_{\pm ba} = \frac{1}{2\kappa}\delta^c_a$; $\Pi_{\pm ab}\Theta^{bi}_{\mp} = -\Pi_{\pm aj}\Theta^{ji}_{\mp}$; $\Pi_{\pm ij}\Theta^{ja}_{\mp} = -\Pi_{\pm ib}\Theta^{ba}_{\mp}$ and $\Theta^{ci}_{\mp}\bar{\Pi}_{\pm ik} = -\tilde{\Theta}^{ca}_{\mp}\Pi_{\pm ak}$, one can rewrite this action as

$$S = \frac{\kappa^2}{2}\int d^2\xi\,\partial_+y_\mu\Theta^{\mu\nu}_-\partial_-y_\nu. \tag{36}$$

In order to find the background fields argument ΔV^i, we consider the zeroth order of Eq. (34)

$$v^i_{0\mp} = -\kappa\Theta^{i\mu}_{0\mp}\partial_{\mp}y_\mu, \tag{37}$$

and conclude that

$$\Delta V^i = -\kappa\theta^{i\mu}_0\Delta y_\mu + (g^{-1})^{i\mu}\Delta\tilde{y}_\mu. \tag{38}$$

Using the integral form of the variables and the relations $\Pi_{\pm ac}\Theta^{cb}_{\mp} + \Pi_{\pm ai}\Theta^{ib}_{\mp} = \frac{1}{2\kappa}\delta^b_a$; $\Theta^{ib}_{\mp} = -2\kappa\bar{\Theta}^{ij}_{\mp}\Pi_{\pm ja}\Theta^{ab}_{\mp}$; $\Theta^{aj}_{\mp} = -2\kappa\tilde{\Theta}^{ab}_{\mp}\Pi_{\pm bi}\Theta^{ij}_{\mp}$, we obtain that $\Delta V^a(\Delta V^i, y^a)$ defined in (19) equals

$$\Delta V^a(\Delta V^i, y_a) = -\kappa\theta_0^{a\mu}\Delta y_\mu + (g^{-1})^{a\mu}\Delta\tilde{y}_\mu. \quad (39)$$

Therefore, we conclude that action (36) is the totally T-dualized action (22).

In this paper we performed the partial T-dualizations and obtained the T-duality chain

$$S[x^\mu] \xrightarrow{\mathcal{T}^a} S_\pi[x^i, y_a] \xrightarrow{\mathcal{T}^i} {}^\star S[y_\mu]. \quad (40)$$

The first action describes the geometrical background, while the second and the third describe the non-geometrical backgrounds with nontrivial fluxes. From this chain one can find the relations between the arbitrary two coordinates in the chain. These general T-duality coordinate transformation laws are used in the investigation of the relations between the Poisson structures of the original, the partially T-dualized and the totally T-dualized theory [5]. Their canonical form will be used in deriving the complete closed string non-commutativity relations, which are the important features of the non-geometrical backgrounds.

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Modular Double of the Quantum Group $SL_q(2,\mathbb{R})$

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Abstract The term "quantum group", introduced by V. Drinfeld (Proceedings of ICM-86, Berkeley, vol. 1, p. 798. AMS, Providence, 1987), applies in fact to two dual objects: q-deformation of the algebra $\mathcal{A}$ of functions on the Lie group and that for the universal enveloping algebra $\mathcal{U}$ of the corresponding Lie algebra. See Faddeev [1] for the short history. It is instructive to stress, that the construction of q-deformation originates in the theory of the quantum integrable models and conformal field theory [see Faddeev [2]]. In this lecture I plan to survey some new developments on a representative example of the rang 1 $SL(2)$ case.

1 Definitions

$\mathcal{A}$—commutative algebra of functions of the matrix elements of matrix

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = 1$$

$\mathcal{U}$—noncommutative algebra of functions of the generators e, f, h with relations

$$[e,h] = e, \quad [f,h] = -f, \quad ef - fe = h$$

Corresponding q-deformations are given in terms of generators as follows

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Algebra $\mathcal{A}_q$

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$ab = qba, \quad ac = qca, \quad db = q^{-1}bd, \quad dc = q^{-1}cd \tag{1}$$

$$ad - da = (q - q^{-1})bc, \quad bc = cb, \quad ad - q^{-1}bc = 1$$

Algebra $\mathcal{U}_q$

$$E, F, K \quad KE = q^2 EK, \quad KF = q^{-2} FK, \tag{2}$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

These relations turn into the undeformed ones in the limit $q \to 1$ if we put $K = q^H$.

Algebra $\mathcal{U}_q$ appeared first in the paper [3] of Kulish and Reshetikhin. Corresponding commutation relations are those for the generalized spin operators appearing in the higher spin XXZ model. Parameter q is an anisotropy parameter, entering the definition of the model. The algebra $\mathcal{A}_q$ was introduced by L. Takhtajan and me [4] in course of definition of the quantum Liouville model. The relations (1) are those for the monodromy matrix of the corresponding quantized Lax matrix. Parameter q is defined via the coupling constant.

The mathematical side of the story began after short commentary of Sklyanin [5] on the connection of the developed formalism with the Hopf algebra (incidently, in 1982 Thierry-Mieg, who was a student of Les Houches summer school told me, that what I was presenting on my lectures is a Hopf Algebra). However a solid mathematical foundation was done by Drinfeld [6] and thereupon theory began to develop rapidly with important contribution of Jimbo [7], Bazhanov [8]. The approach of Leningrad group was described in detail in [9].

2 Weyl Pair

In the following we shall heavily use the Weyl pair u, v

$$uv = q^2 vu$$

It is convenient to use the notations from the theory of the automorphic functions

$$q = e^{i\pi\tau}, \quad \tau = \frac{\omega'}{\omega}, \quad \omega\omega' = -\frac{1}{4}, \quad \operatorname{Im}\tau \geq 0$$

Here ω, ω' are so called half periods and they lie in the upper half plane. We shall use u and v represented by operators acting on the functions $f(x)$, $x \in \mathbb{R}$ by multiplication and shift

$$uf(x) = e^{-i\pi x/\omega} f(x), \quad vf(x) = f(x + 2\omega'). \tag{3}$$

For general τ these are unbounded operators in $L_2(\mathbb{R})$ defined on a dense domain $\mathcal{D}$, consisting of the analytic functions on $\mathbb{C}$ rapidly vanishing along the lines parallel to the real axis.

There are two distinguished real structures:

1. $\tau > 0$, $\bar{\omega} = -\omega$, $\bar{\omega'} = -\omega'$, u, v—positive, essentially selfadjoint,
2. $\tau < 0$, ω, ω'—real, u, v—unitary.

It is instructive to observe, that the action of u and v is not irreducible. Indeed, a second pair of operators $\tilde{u}$, $\tilde{v}$, defined by formula similar to (3) after interchange of ω and ω'

$$\tilde{u}f(x) = e^{-i\pi x/\omega'} f(x), \quad \tilde{v}f(x) = f(x + 2\omega)$$

commute with u and v, which can be easily checked. Indeed we have

$$\begin{aligned} u\tilde{v}f &= e^{-i\pi x/\omega} f(x + 2\omega), \\ \tilde{v}uf &= e^{-i\pi(x+2\omega)/\omega} f(x + 2\omega) = e^{-2\pi i} u\tilde{v}f = u\tilde{v}f \end{aligned}$$

Formally we can say, that u, v and $\tilde{u}$, $\tilde{v}$ are connected as follows

$$\tilde{u} = u^{1/\tau}, \quad \tilde{v} = v^{1/\tau}.$$

I must confess, that I realized this feature rather late in my life. Amazing fact, that the algebra of operators in $L_2(\mathbb{R})$, which is an algebra for observables for the system with one degree of freedom, is factorized in two commuting factors, was so impressive, that I even have published a short note on this [10]. Later I realized, that this fact is known to some more learned people, in particular by Alain Connes [11], he likes to call u-v system as quantum torus but considers only the case of unitary operators u, v. In the factorization

$$\mathcal{B} = \mathcal{A}_\tau \otimes \mathcal{A}_{1/\tau}$$

for irrational τ algebras $\mathcal{A}_\tau$ and $\mathcal{A}_{1/\tau}$ are factors II_1. However the interpretation of this formula in general case is still lacking, because the von Neumann theory of C^* algebras is unapplicable. I believe that here we have an interesting mathematical problem.

From now on I consider u, v, $\tilde{u}$, $\tilde{v}$ as independent generators of algebra, which I propose to call “modular double”. This algebra has one more real structure for $|\tau| = 1$

$$u^* = \tilde{u}, \quad v^* = \tilde{v}$$

The involution $*$ interchanges $\mathcal{A}_\tau$ and $\mathcal{A}_{1/\tau}$, however

$$(*)^2 = \text{id}.$$

In conformal field theory of rang 1 appears a projective representation of the Virasoro algebra with central charge $c = 1 + 6(\tau + \frac{1}{\tau} + 2)$

1. $\tau > 0$, $c > 25$—weak coupling
2. $\tau < 0$, $c < 1$—minimal models for rational $\tau = -P/Q$
3. $|\tau| = 1$, $1 < c < 25$—strong coupling

so that the use of the modular double is indispensable in the case 3.

Now I say several words about irreducibility. The question when the commutativity

$$[A, u] = 0, \quad [A, v] = 0, \quad [A, \tilde{u}] = 0, \quad [A, \tilde{v}] = 0$$

leads to

$$A = \text{const}\, I$$

is not completely answered. It can be proved for the complex τ but for $\tau > 0$ one should require that τ is irrational. We shall comment on this later.

3 Explicit Realization

Generators of rank 1 algebras $\mathcal{A}_q$ and $\mathcal{U}_q$ can be expressed via a pair u, v and one central element. It is a generalization of Gelfand-Kirillov theorem on the structure of the universal enveloping algebra of a simple Lie algebra. Here are explicit formulas

$$\mathcal{A}_q : \quad T = \begin{pmatrix} u + v & \lambda t \\ \lambda^{-1} t & u^{-1} \end{pmatrix}, \quad t = q^{1/4} u^{-1/2} v^{1/2}$$

$$\lambda^2 = b/c\text{—central element}$$

$$\mathcal{U}_q : \quad E = \frac{i}{q - q^{-1}} e, \qquad e = q u^{-1} v + u^{-1} Z$$

$$F = \frac{i}{q - q^{-1}} f, \qquad f = u + q u v^{-1} Z^{-1}$$

$$K = v \qquad Z\text{—central element}$$

$$C = fe - qK - q^{-1}K^{-1} = Z + Z^{-1}$$

Operator C is a q-generalization of the corresponding Casimir operator.

The modular double of $\mathcal{A}_q$ and $\mathcal{U}_q$ are given by completion of these generators by those written similarly via $\tilde{u}$ and $\tilde{v}$. In the more explicit use of such formulas it is important to know the property

$$(u+v)^{1/\tau} = u^{1/\tau} + v^{1/\tau} \tag{4}$$

derived by A. Volkov [12].

4 Representation π_s

From now on we shall deal only with modular double of $\mathcal{U}_q$. In the case of real forms 1 and 3 we can speak about the modular double of quantum group $SL_q(2,\mathbb{R})$. Consider the first real form $\tau > 0$ and parametrize the central elements Z and $\tilde{Z}$ via real parameter s

$$Z = e^{i\pi s/\omega}, \quad \tilde{Z} = e^{i\pi s/\omega'}.$$

Then generators E, F, K and $\tilde{E}$, $\tilde{F}$, $\tilde{K}$ are positive and essentially selfadjoint in $L_2(\mathbb{R})$. We shall consider this case as a definition of the selfadjoint representation π_s of the modular double $SL_q(2,\mathbb{R})$. This representation was already introduced in my paper [13] with the definition of the modular double and investigated in detail by J. Teschner [14]. The main theorem on the decomposition of the tensor product $\pi_s \otimes \pi_{s'}$, due to R. Ponsot and J. Teschner [15] will be discussed below.

The case of the third real structure is also very interesting. We shall comment on it later.

The real variable s can be called the spin of the representation π_s. The reflection

$$s \to -s$$

is analogous to the Weyl reflection for the algebra $SL(2,\mathbb{R})$. Let us show, that the representations π_s and $\pi_{s'}$ are equivalent. We should find an intertwiner R such that

$$\begin{aligned} e(s)R &= Re(-s) \\ f(s)R &= Rf(-s) \\ KR &= RK \end{aligned} \tag{5}$$

and similarly for $\tilde{e}$, $\tilde{f}$, $\tilde{k}$. From commutativity of R with K and $\tilde{K}$ it follows, that it is a function of the generator v.

Introduce the Fourier transform

$$Ff(x) = \int_{-\infty}^{\infty} e^{-2\pi ixy} f(y)\, dy$$

so that

$$uF = Fv, \quad vF = Fu^{-1}$$

and look for R in the form

$$\hat{R} = F^{-1} R(v) F.$$

We have

$$\begin{aligned} F^{-1} e(s) F &= v^{-1}(qu^{-1} + Z) = (q^{-1}u^{-1} + Z)v^{-1} \\ F^{-1} f(s) F &= v(1 + qZ^{-1}u) = (1 + q^{-1}Z^{-1}u)v \end{aligned}$$

and

$$\hat{R} = \hat{R}(u).$$

Now from

$$(q^{-1}u^{-1} + Z)\hat{R}(q^2 u) = \hat{R}(u)(q^{-1}u^{-1} + Z^{-1})$$

we get a functional equation

$$\frac{\hat{R}(q^2 u)}{\hat{R}(u)} = \frac{1 + quZ^{-1}}{1 + quZ}.$$

Let $\Phi(u)$ satisfy the equation

$$\frac{\Phi(qu)}{\Phi(q^{-1}u)} = \frac{1}{1+u} \tag{6}$$

then

$$\hat{R}(u) = \frac{\Phi(Zu)}{\Phi(Z^{-1}u)}.$$

Due to the Volkov relation (4) the same Φ serves the second half of relations (5). Thus the intertwiner is constructed as soon as we know the solution of Eq. (6). We shall discuss the problem of solving (6) in the next section.

5 Quantum Dilogarithm

Equation (6) is solved by Euler q-exponent

$$e_q(u) = \prod_{k=0}^{\infty}(1+q^{2k+1}u)$$
$$= 1 + \sum_{n=1}^{\infty}\frac{q^{n(n+1)/2}u^n}{(q-q^{-1})\dots(q^n-q^{-n})} = \exp\sum\frac{(-1)^n u^n}{n(q^n-q^{-n})}. \tag{7}$$

The last formula, where we have in the denominator the integer n and q-analog of n, prompted me to introduce the term “quantum dilogarithm” [16]. However, the formula (7) is correct only for $|q| < 1$ or $\mathrm{Im}\tau > 0$. So we can not use it for our case of real structure 1. My proposal in the spirit of modular double was to modify the product in (7), adding the second factor with τ changed into $1/\tau$

$$\Phi(u) = \frac{e_q(u)}{e_{\tilde{q}}(\tilde{u})}, \quad \tilde{q} = e^{-i\pi/\tau}, \quad \tilde{u} = u^{1/\tau}. \tag{8}$$

One can consider each factor as a quasiconstant with respect to shift associated with τ or $1/\tau$.

The product can be realized via the integral

$$\gamma(x) = \exp -\frac{1}{4}\int_{-\infty+i0}^{\infty+i0}\frac{e^{ixt}}{\sin\omega t\ \sin\omega' t}\frac{dt}{t} \tag{9}$$

such that

$$\Phi(u) = \gamma(x), \quad u = e^{-i\pi x/\omega}.$$

Integral (9) has its own long history. It is related to the functions which we call Barnes double γ-function [17], which was in fact introduced in the end of nineteenth century by Russian mathematician Alexseievsky [18]. More exactly the function $\gamma(x)$ plays the role of argument of double γ-function, so it has a second name double sinus. The origin of this name is the functional equation

$$\frac{\gamma(x+\omega')}{\gamma(x-\omega')} = 1 + e^{-i\pi x/\omega}$$

and the dual one

$$\frac{\gamma(x+\omega)}{\gamma(x-\omega)} = 1 + e^{-i\pi x/\omega'},$$

which is equivalent to Eq. (6).

The function $\gamma(x)$ acquired a host of applications in the last 15 years and the term "quantum dilogarithm" was inherited by it. The beautiful paper by Volkov [12] contain a list of its important properties.

Let us observe, that contrary to the case of product (7), the integral (9) has no singularities for real τ. The singularities of numerator and denominator in (8) cancel each other. In particular, the problem of irrational τ disappears. Everything is smooth for the positive τ.

I shall conclude the lecture by describing the role of the quantum dilogarithm in the problem of decomposition of the tensor product of two representations π_{s_1} and π_{s_2} into direct sum of π_s.

6 Decomposition of $\pi_{s_1} \otimes \pi_{s_2}$

It is clear that the definition of $\pi_{s_1} \otimes \pi_{s_2}$ should be done according to comultiplication rule in $\mathcal{U}_q$, which is a deformation for those in $\mathcal{U}$

$$\begin{aligned} \Delta E &= E \otimes K + I \otimes E \\ \Delta F &= F \otimes I + K^{-1} \otimes F \\ \Delta K &= K \otimes K \end{aligned} \tag{10}$$

Teschner and Ponsot [15] has shown that

$$\pi_{s_1} \otimes \pi_{s_2} = \int_0^\infty \pi(s)\, d\mu(s), \quad d\mu(s) = -4 \sin\frac{\pi s}{\omega} \sin\frac{\pi s}{\omega'} ds,$$

so that in the decomposition for any pair of spins s_1, s_2 enter representations with all spins s_3.

To construct the intertwiner $S(x_1, x_2, x_3|s_1, s_2, s_3)$ we should solve the equations

$$\begin{aligned} e_{12} S &= S e_3' \\ f_{12} S &= S f_3' \\ K_{12} S &= S K_3', \end{aligned} \tag{11}$$

where $'$ means the transposition and e_{12}, f_{12}, K_{12} are defined according to (10) I present here shortly the way of solution, proposed by S. Derkachov and me [19].

Equation (11) can be rewritten more explicitly in the form

$$v_1 v_2 v_3 S = S$$

$$S(q^2 u_1, u_2, u_3) = \frac{1}{Z_2 Z_3} \cdot \frac{1 - \frac{u_3}{u_1}}{1 + \frac{Z_1}{q Z_2 Z_3} \frac{u_3}{u_1}} \cdot \frac{1 - Z_2 Z_3 \frac{u_1}{u_2}}{1 + \frac{q}{Z_1} \frac{u_1}{u_2}} S(u_1, u_2, u_3)$$

and similar for $S(u_1, u_2, q^{-2}u_3)$ and $S(u_1, q^{-2}u_2, u_3)$. Here we freely use variables u_1, u_2, u_3 instead of x_1, x_2, x_3. The use of property (6) leads to the explicit formula

$$S(x_1, x_2, x_3) = S_0 \exp\{-2\pi i (s_1 x_{23} + s_2 x_{31} + s_3 x_{21})\}$$
$$\times \frac{\gamma(x_{12} - s_1)}{\gamma(x_{12} + s_2 + s_3 + \omega'')} \cdot \frac{\gamma(x_{23} + s_3 - s_2 - \omega'')}{\gamma(x_{23} - s_1)} \cdot \frac{\gamma(x_{31} - \omega'')}{\gamma(x_{31} + s_1 - s_2 - s_3)}$$

$$x_{ij} = x_i - x_j, \quad \omega'' = \omega + \omega', \quad \omega'' \to \omega'' - i0.$$

One should prove the main properties of the intertwiner—orthogonality and completeness. To achieve this goal we use the spectral problem with the use of the Casimir for $\pi_{s_1} \otimes \pi_{s_2}$

$$C_{12} S(x_1, x_2, x_3) = (Z_3 + Z_3^{-1}) S(x_1, x_2, x_3) \tag{12}$$

The equation is in two variables x_1, x_2; s_1, s_2—parameters, s_3—eigenvalue, x_3—multiplicity.

The main tool—reduction to spectral problem

$$(u + u^{-1} + v)\psi = \lambda \psi, \tag{13}$$
$$\lambda = Z(s) + Z^{-1}(s),$$

which is solved by a generalized function

$$\psi(x, s) = \gamma(x - s - \omega'' + i0)\, \gamma(x + s - \omega'' + i0)\, e^{-i\pi(x - \omega'')^2}.$$

One should observe, that $\gamma(x)$ has a pole in $-\omega''$ so that

$$\gamma(x - \omega'') \sim \frac{c}{x}$$

and term $i0$ define the corresponding rule of treating this. It is instructive to observe, that the operator in the left hand side of (13) coincide with the trace of matrix T in (1).

This spectral problem (13) was analyzed in detail by Kashaev [20], who proved the orthogonality and completeness relation

$$\tau > 0, \quad \int \overline{\psi(x,s)}\psi(x,s')dx = \frac{1}{\rho(s)}\big(\delta(s-s') + \delta(s+s')\big)$$

$$\rho(s) = -4\sin\frac{\pi s}{\omega}\sin\frac{\pi s}{\omega'}$$

$$\int_0^\infty \psi(x,s)\overline{\psi(y,s)}\rho(s)\,ds = \delta(x-y)$$

Derkachov and me reduced the problem (12) to (13) by means of some "redressing". We found an intertwiner A such that

$$\hat{C}_{12} = A^{-1}C_{12}A, \quad A^{-1}K_{12}A = K_{12}$$

with rather simple expression for $\hat{C}_{12}$

$$\hat{C}_{12} = Z_2\frac{u_1}{u_2} + \frac{1}{Z_2}\frac{u_2}{u_1} + Z_1 v_2 + \frac{Z_1}{qZ_2}\frac{u_2}{u_1}v_1^{-1}$$

Now we use K_{12} to count the multiplicity. The new spectral problem looks as

$$\hat{C}_{12}\psi_p(x_1,x_2) = (Z_3 + Z_3)^{-1}\psi_p(x_1,x_2)$$

$$K_{12}\psi_p(x_1,x_2) = e^{i\pi p/\omega}\psi_p(x_1,x_2)$$

and the last equation allows to express v_1 via v_2. Thus we get $S(x_1,x_2,x_3)$ as a Fourier transform $\psi_p(x_1,x_2)$ over p and dressing A and Kashaev's properties allow to prove all necessary properties of S.

Conclusions

We have shown in this lecture the importance of combining the two q-deformations for dual expressions of the parameter q

$$q = e^{i\pi\tau}, \quad \tilde{q} = e^{-i\pi/\tau}$$

and illustrated its usefulness on one example. There are many other examples appearing in the construction of quantum Teichmüller space and theory of cluster algebras. There is no doubt, that the proper construction of the conformal blocks in the Liouville model should use modular double. So we have a lot of work ahead.

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Physical Ageing and New Representations of Some Lie Algebras of Local Scale-Invariance

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Abstract Indecomposable but reducible representations of several Lie algebras of local scale-transformations, including the Schrödinger and conformal Galilean algebras, and their applications in physical ageing are reviewed. The physical requirement of the decay of co-variant two-point functions for large distances is related to analyticity properties in the coordinates dual to the physical masses or rapidities.

1 Introduction

Scale-invariance is recognised as one of the main characteristics of phase transitions and critical phenomena. In addition, it has also become common folklore that given sufficiently local interactions, scale-invariance can be extended to larger Lie groups of coordinate transformations. Quite a few counter-examples exist, but the folklore carries on. Here, we are interested in the phenomenology of phase transitions, either at equilibrium or far from equilibrium and shall study situations when scale-invariance does indeed extend to conformal invariance or one of the generalisations appropriate for scale-invariant dynamics. We review recent results on indecomposable, but reducible ("logarithmic") representations and discuss sufficient conditions which guarantee they decay of co-variant two-point functions at large distances.

Consider the transformations in $(1 + d)$-dimensional time-space $\mathbb{R} \otimes \mathbb{R}^d$

$$t \mapsto t' = \frac{\alpha t + \beta}{\gamma t + \delta} \quad , \quad \mathbf{r} \mapsto \mathbf{r}' = \frac{\mathcal{R}\mathbf{r} + \mathbf{v}_{2\ell} t^{2\ell} + \ldots \mathbf{v}_1 t + \mathbf{v}_0}{(\gamma t + \delta)^{2\ell}} \quad ; \quad \alpha\delta - \beta\gamma = 1 \tag{1}$$

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where $\mathcal{R} \in SO(d)$, $\mathbf{v}_0, \ldots, \mathbf{v}_{2\ell} \in \mathbb{R}^d$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. The infinitesimal generators from these transformations only close in a Lie algebra if $\ell \in \frac{1}{2}\mathbb{N}$ (sometimes called "*spin-ℓ algebra*"). In $(1+1)$ dimensions, this algebra can be formulated in terms of two infinite families of generators $\mathfrak{sc}(1,\ell) := \langle X_n, Y_m \rangle_{n\in\mathbb{Z}, m+\ell\in\mathbb{Z}}$ of the form

$$X_n = -t^{n+1}\partial_t - (n+1)\ell t^n r\partial_r \ \ , \ \ Y_m = -t^{m+\ell}\partial_r \tag{2}$$

with the non-vanishing commutators [20]

$$[X_n, X_{n'}] = \left(n - n'\right) X_{n+n'} \ \ , \ \ [X_n, Y_m] = (\ell n - m)\, Y_{n+m} \tag{3}$$

and where $z := 1/\ell$ is the *dynamical exponent*. The maximal finite-dimensional sub-algebra is $\mathfrak{spi}(1,\ell) := \langle X_{\pm 1,0}, Y_{-\ell,\ldots,\ell-1,\ell} \rangle$, for $\ell \in \frac{1}{2}\mathbb{N}$. In analogy with conformal invariance, Ward identities must be formulated which will describe the action of these generators on scaling operators such that the co-variance under these transformations can be used to derive differential equations to be satisfied by n-point correlators. Alternatively, one may include the corresponding terms directly into the generators themselves, which has the advantage that the verification of the commutators guarantees the self-consistency of co-variance. However, the explicit representation (2) does not take into account any transformation properties of the (quasi-)primary scaling operators on which it is assumed to act.

Dynamic time-space symmetries with a generic dynamical exponent ($z \neq 1$ possible) often arise as "non-relativistic limits" of the conformal algebra. The two best-known examples are (1) the *Schrödinger algebra* $\mathfrak{sch}(d)$ which in (2) corresponds to $\ell = \frac{1}{2}$ (discovered in 1842/1843 by Jacobi and in 1881 by Lie) and (2) the *conformal Galilean algebra* $\mathrm{CGA}(d)$ [18] which corresponds in (2) to $\ell = 1$. These two important special cases can also be obtained by two distinct, complementary approaches

1. The non-relativistic limit of time-space conformal transformations such that a fixed value of the dynamical exponent z is assumed, reproduces the Schrödinger and conformal Galilean algebras from the restriction to flat time-like and light-like geodesics, along with $z = 2$ and $z = 1$ [13].
2. If one tries to include the transformations of scaling operators into the representation (3) of the infinite-dimensional algebra (3) only the cases with $\ell = \frac{1}{2}, 1$ close as a Lie algebra. Besides the conformal algebra, this reproduces the Schrödinger and conformal Galilean algebras [21].

Including the terms which describe the transformation of quasi-primary scaling operators often leads to central extensions of the above algebras. For $\ell = \frac{1}{2}$, one has instead of $\mathfrak{spi}(1, \frac{1}{2})$ the *Schrödinger-Virasoro algebra* $\mathfrak{sv} = \langle X_n, Y_m, M_n \rangle_{n\in\mathbb{Z}, m\in\mathbb{Z}+\frac{1}{2}}$ [19, 49] spanned by the generators

$$X_n = -t^{n+1}\partial_t - \frac{n+1}{2}t^n r\partial_r - \frac{n+1}{2}xt^n - \frac{n(n+1)}{4}\mathcal{M}t^{n-1}r^2$$
$$Y_m = -t^{m+\frac{1}{2}}\partial_r - \left(m+\frac{1}{2}\right)\mathcal{M}t^{m-\frac{1}{2}}r \quad , \quad M_n = -t^n\mathcal{M} \tag{4}$$

(x is the scaling dimension and $\mathcal{M}$ the mass) and the non-vanishing commutators

$$[X_n, X_{n'}] = (n-n')X_{n+n'} \quad , \quad [X_n, Y_m] = \left(\frac{n}{2} - m\right) Y_{n+m}$$
$$[X_n, M_{n'}] = -n' M_{n+n'} \quad , \quad [Y_m, Y_{m'}] = \left(m - m'\right) M_{m+m'} \tag{5}$$

Its maximal finite-dimensional sub-algebra is the Schrödinger algebra $\mathfrak{sch}(1) = \langle X_{\pm1,0}, Y_{\pm\frac{1}{2}}, M_0\rangle$, which centrally extends $\mathfrak{spi}(1, \frac{1}{2})$. It is the maximal dynamical symmetry of the free Schrödinger equation $\mathcal{S}\phi = 0$ with $\mathcal{S} = 2M_0X_{-1} - Y_{-\frac{1}{2}}^2 = 2\mathcal{M}\partial_t - \partial_r^2$, in the sense that solutions of $\mathcal{S}\phi = 0$ with scaling dimension $x = x_\phi = \frac{1}{2}$ are mapped onto solutions, a fact already known to Jacobi and to Lie. More generally, unitarity implies the bound $x \geq \frac{1}{2}$ [36].

On the other hand, for $\ell = 1$ one obtains the *altern-Virasoro algebra* $\mathfrak{av} := \langle X_n, Y_n\rangle_{n\in\mathbb{Z}}$ (also called "full CGA"), with an explicit representation spanned by [1, 3–5, 11, 12, 15, 18, 20, 21, 28, 31–33, 37, 40, 43]

$$X_n = -t^{n+1}\partial_t - (n+1)t^n r\partial_r - (n+1)xt^n - n(n+1)\gamma t^{n-1}r$$
$$Y_n = -t^{n+1}\partial_r - (n+1)\gamma t^n \tag{6}$$

which obeys (3) and has $\text{CGA}(1) = \langle X_{\pm1,0}, Y_{\pm1,0}\rangle$ as maximal finite-dimensional sub-algebra.[1] The representation (6) is spanned by the two scalars x and γ.

The relationship between $\mathfrak{sch}(1)$ and CGA(1) can in be understood in a different way by considering the imbedding $\mathfrak{sch}(1) \subset B_2$ into the complex Lie algebra B_2. This can be visualised in terms of a root diagram, see Fig. 1a, where the generators of $\mathfrak{sch}(1)$ are indicated by full black circles and the remaining ones by the grey circles. As it is well-known [34], a *standard parabolic sub-algebra* $\mathfrak{p}$, of a semi-simple Lie algebra $\mathfrak{g}$ consists of the Cartan sub-algebra $\mathfrak{h} \subset \mathfrak{g}$ and of all "positive" generators in $\mathfrak{g}$. The meaning of "positive" can be simply illustrated in Fig. 1 for the special case $\mathfrak{g} = B_2$: one draws a straight line through the centre of the root diagram and all generators on that line or to the right of it are "positive". From Fig. 1a, one also sees that the Schrödinger algebra can be extended to a parabolic sub-algebra $\widetilde{\mathfrak{sch}}(1) = \mathfrak{sch}(1) \oplus \mathbb{C}N$ by adding an extra generator N, which is indicated by the red double circle in the centre. Since the Weyl reflections and rotations can be used

[1] In the context of asymptotically flat $3D$ gravity, an isomorphic Lie algebra is known as BMS algebra, $\mathfrak{bms}_3 \equiv \text{CGA}(1)$ [6–9].

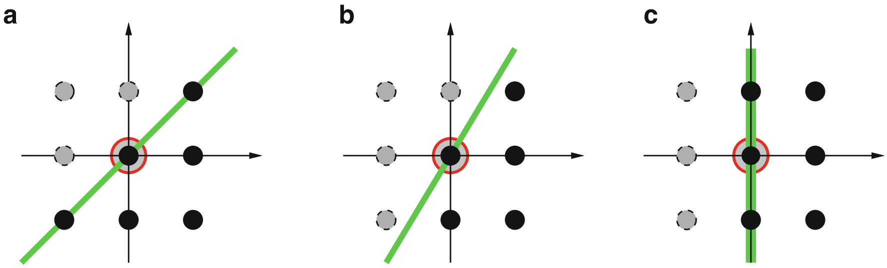

Fig. 1 Root diagrams of (**a**) $\mathfrak{sch}(1)$, (**b**) $\mathfrak{age}(1)$ and (**c**) $\mathfrak{alt}(1) = \text{CGA}(1)$ as sub-algebras of the complex Lie algebra B_2. If the second generator in the centre is included (*double red circle*) one obtains the maximal parabolic sub-algebras of B_2

to map isomorphic sub-algebras onto each other, the classification of the maximal parabolic sub-algebras of B_2 can now be illustrated simply through the value of the slope p of the straight line in Fig. 1 [25]:

1. if $p = 1$, one has $\widetilde{\mathfrak{sch}}(1) = \mathfrak{sch}(1) + \mathbb{C}N$, the parabolic extension of the Schrödinger algebra, see Fig. 1a. See below for explicit forms for N.
2. if $1 < p < \infty$, one has $\widetilde{\mathfrak{age}}(1) = \mathfrak{age}(1) + \mathbb{C}N$, the parabolic extension of the *ageing algebra*, see Fig. 1b.
3. if $p = \infty$, one has $\widetilde{\text{CGA}}(1) = \text{CGA}(1) + \mathbb{C}N$, the parabolic extension of the conformal Galilean algebra, see Fig. 1c.

While CGA(1) does not have a central extension, this is different in $d = 2$ space dimensions, where a so-called "exotic" central extension exists. This gives the *exotic conformal Galilean algebra* $\text{ECGA} = \langle X_{\pm 1,0}, Y_{\pm 1,0}, \theta, R_0 \rangle$ [38, 39] with an explicit representation (where $j, k, \ell = 1, 2$ and summation over repeated indices is implied)

$$\begin{aligned} X_n &= -t^{n+1}\partial_t - (n+1)t^n r_j \partial_j - x(n+1)t^n \\ &\quad -n(n+1)t^{n-1}\gamma_j r_j - n(n+1)h_j r_j \\ Y_n^{(j)} &= -t^{n+1}\partial_j - (n+1)t^n\gamma_j - (n+1)t^n h_j - n(n+1)\theta\varepsilon_{jk} r_k \\ J &= -\varepsilon_{k\ell}\, r_k \partial_\ell - \varepsilon_{k\ell}\,\gamma_k \frac{\partial}{\partial\gamma_\ell} - \frac{1}{2\theta} h_j h_j \end{aligned} \tag{7}$$

characterised by a scalar scaling dimension x and a vector $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$ of *"rapidities"* [12, 28, 40]. The components of the vector $\mathbf{h} = (h_1, h_2)$ satisfy $[h_i, h_j] = \varepsilon_{ij}\theta$, where θ is central. ε is the totally antisymmetric 2×2 tensor and $\varepsilon_{12} = 1$. The non-vanishing commutators of the ECGA read

$$\begin{aligned} [X_n, X_m] &= (n-m)X_{n+m} \quad , \quad \left[X_n, Y_m^{(i)}\right] = (n-m)Y_{n+m}^{(i)} \\ \left[Y_n^{(i)}, Y_m^{(j)}\right] &= \varepsilon_{ij}\,\delta_{n+m,0}\,(3\delta_{n,0} - 2)\,\theta \quad , \quad \left[J, Y_n^{(i)}\right] = \varepsilon_{ij}\,Y_n^{(j)} \end{aligned} \tag{8}$$

and the ECGA-invariant Schrödinger operator is

$$\mathcal{S} = -\theta X_{-1} + \varepsilon_{ij} Y_0^{(i)} Y_{-1}^{(j)} = \theta \partial_t + \varepsilon_{ij} \left(\gamma_i + h_i\right) \partial_j \tag{9}$$

with $x = x_\phi = 1$. The unitary bound gives $x \geq 1$ [40].

The common sub-algebra of $\mathfrak{sch}(1)$ and CGA(1) is called the *ageing algebra* $\mathfrak{age}(1) := \langle X_{0,1}, Y_{\pm\frac{1}{2}}, M_0 \rangle$ and does not include time-translations. Starting from the representation (4), only the generators X_n assume a more general form [26]

$$X_n = -t^{n+1}\partial_t - \frac{n+1}{2} t^n r \partial_r - \frac{n+1}{2} x t^n - n(n+1)\xi t^n - \frac{n(n+1)}{4} \mathcal{M} t^{n-1} r^2 \tag{10}$$

which also admits a more general invariant Schrödinger operator $\mathcal{S} = 2\mathcal{M}\partial_t - \partial_r^2 + 2\mathcal{M}t^{-1}\left(x + \xi - \frac{1}{2}\right)$, but without any constraint on neither x nor ξ [48]. This representation of $\mathfrak{age}(1)$ is characterised by the scalars $(x, \xi, \mathcal{M})$. The name of this algebra comes from its use as dynamical symmetry in *physical ageing*, which can be observed in strongly interacting many-body systems quenched from a disordered initial state to the co-existence regime below the critical temperature $T_c > 0$ where several equivalent equilibrium states exist. For example, for quenched Ising spins in $d \geq 2$ dimensions without disorder, nor frustrations, and with a purely relaxational dynamics without any conservation law, it can be shown that the dynamical exponent $z = 2$ [10]. Assuming $\mathfrak{age}(d)$ as a dynamical symmetry predicts the form of the two-time linear response function of the average order parameter $\langle \phi(t, \mathbf{r}) \rangle$ with respect to its canonically conjugate magnetic field $h(s, \mathbf{r}')$ [19, 25, 26]

$$R(t, s; \mathbf{r}) = \left. \frac{\delta \langle \phi(t, \mathbf{r}) \rangle}{\delta h(s, \mathbf{0})} \right|_{h=0} = \left\langle \phi(t, \mathbf{r}) \tilde{\phi}(s, \mathbf{0}) \right\rangle = s^{-1-a} F_R\left(\frac{t}{s}, \frac{\mathbf{r}^2}{t-s} \right) \tag{11}$$

$$F_R(y, u) = F_0\, \delta(\mathcal{M} - \tilde{\mathcal{M}})\, \Theta(y-1)\, y^{1+a'-\lambda_R/z} (y-1)^{-1-a'} \exp\left[-\frac{1}{2}\mathcal{M}u \right]$$

where the standard Janssen-de Dominicis formalism (see e.g. [23]) was used to re-write $R = \langle \phi\tilde{\phi} \rangle$ as a correlator of the order parameter scaling operator ϕ and the conjugate response operator $\tilde{\phi}$. Both of these are assumed to be quasi-primary under $\mathfrak{age}(d)$. The ageing exponents $a, a', \lambda_R/z$ are related to x, ξ and $\tilde{x}, \tilde{\xi}$ in a known way, e.g. $a' - a = \frac{2}{z}(\xi + \tilde{\xi})$. F_0 is a normalisation constant and the Θ-function expresses the causality condition $y = t/s > 1$, of which we shall say more in Sect. 3 below. Spatial translation-invariance was assumed.

The case of a Schrödinger-invariance response is obtained if one sets $\xi = \tilde{\xi} = 0$, hence $a = a'$ in (11).

Equation (11) has been confirmed in numerous spin systems (e.g. Ising, Potts, XY, spherical, Hilhorst-van Leeuven, Edwards-Wilkinson,…models) which undergo simple ageing with $z = 2$, both for the time- and space-dependence; either from a known exact solution or using simulational data. For a detailed review, see [23]. Current empirical evidence suggests that for quenches to low temperatures $T < T_c$, one should have for the second scaling dimensions $\xi + \tilde{\xi} = 0$, hence $a = a'$. However, the full representation (10) of $\mathfrak{age}(1)$ is needed in the $d = 1$ Glauber-Ising model, where the exact solution reproduces (11) with $a = 0$, $a' - a = -\frac{1}{2}$ and $\lambda_R = 1$ [26]. One might anticipate that $a' - a \neq 0$ for quenches to the critical point $T = T_c$.

For critical quenches, one has in general $z \neq 2$, such that (11) does no longer apply. However, the form of the auto-response $R(t,s) = R(t,s;\mathbf{0})$ does not contain the precise spatial form so that at least that part of (11) can be used for preliminary tests of dynamical symmetries for generic values of z.

In Sect. 2, various logarithmic representations of these algebras and some of their properties are reviewed. Known applications to physical ageing will be briefly discussed. In Sect. 3, the requirement of a physically sensible limit in the case of large spatial separation $|\mathbf{r}_1 - \mathbf{r}_2| \to \infty$ leads to the derivation of causality conditions. These inform on the interpretation in terms of either responses or correlators.

2 Logarithmic Representations

Logarithmic conformal field-theories arise from indecomposable but reducible representations of the Virasoro algebra [17, 41, 45, 46], see [14] for a collection of recent reviews. Formally, in the most simple case, this can be implemented [17, 44] by replacing the order parameter ϕ by a vector $\Phi = \begin{pmatrix} \psi \\ \phi \end{pmatrix}$ such that the scaling dimension x in the Lie algebra generators becomes a Jordan matrix $\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$. Anticipating the notation used for the algebras we are going to consider, we introduce, instead of a single two-point function $\langle \phi_1 \phi_2 \rangle$, the three two-point functions

$$
\begin{aligned}
&F := \langle \phi_1(t_1, \mathbf{r}_1) \phi_2^*(t_2, \mathbf{r}_2) \rangle \ , \ G := \langle \phi_1(t_1, \mathbf{r}_1) \psi_2^*(t_2, \mathbf{r}_2) \rangle \ , \\
&H := \langle \psi_1(t_1, \mathbf{r}_1) \psi_2^*(t_2, \mathbf{r}_2) \rangle
\end{aligned} \tag{12}
$$

Temporal and spatial translation-invariance imply that $F = F(t, \mathbf{r})$, $G = G(t, \mathbf{r})$ and $H = H(t, \mathbf{r})$ with $t = t_1 - t_2$ and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. The shape of these functions is derived from the algebras introduced in Sect. 1, as we now review.

2.1 Schrödinger Algebra

For the Schrödinger and the ageing algebras, the "complex conjugate" ϕ^* in (12) refers to the mapping $\mathcal{M} \mapsto -\tilde{\mathcal{M}}$ when in a response function such as (11) one goes from the order parameter ϕ to its conjugate response operator $\tilde{\phi}$ [23].

This is necessary in applications to physical ageing. In particular, extending $\mathfrak{sch}(1) \to \widetilde{\mathfrak{sch}}(1)$ and using the physical convention $\mathcal{M} \geq 0$ of non-negative masses, implies causality $t_1 - t_2 > 0$, as we shall see in Sect. 3. While common in statistical physics applications in models described by stochastic Langevin equations [26], this was recently re-discovered in string-theory contexts [42].

Replacing in the generators (4) the scaling dimension x by a 2×2 Jordan matrix, the Schrödinger Ward identities (or co-variance conditions) can be written down for the three two-point functions (12). The result is, in $d \geq 1$ dimensions [30]

$$\begin{aligned}
F(t,\mathbf{r}) &= \langle \phi_1(t,\mathbf{r})\phi_2^*(0,\mathbf{0})\rangle = 0, \\
G(t,\mathbf{r}) &= \langle \phi(t,\mathbf{r})\psi_2^*(0,\mathbf{0})\rangle = a\, t^{-2x_1} \exp\left[-\frac{\mathcal{M}_1\mathbf{r}^2}{2t}\right] \delta_{x_1,x_2}\delta_{\mathcal{M}_1,\mathcal{M}_2}, \qquad (13)\\
H(t,\mathbf{r}) &= \langle \psi_1(t,\mathbf{r})\psi_2^*(0,\mathbf{0})\rangle = t^{-2x_1}\, (b-2a\ln t) \exp\left[-\frac{\mathcal{M}_1\mathbf{r}^2}{2t}\right] \delta_{x_1,x_2}\delta_{\mathcal{M}_1,\mathcal{M}_2}.
\end{aligned}$$

where a,b are scalar normalisation constants. Time- and space-translation-invariance and also rotation-invariance for scalar ϕ, ψ were used.

2.2 Conformal Galilean Algebra

The representation (6) of CGA(1) depends on both the scaling dimension x [29] as well as the rapidity γ. Now, either of them may become a Jordan matrix and it turns out that the CGA(1)-commutators imply that *simultaneously* [28]

$$x \mapsto \begin{pmatrix} x & x' \\ 0 & x \end{pmatrix} \quad , \quad \gamma \mapsto \begin{pmatrix} \gamma & \gamma' \\ 0 & \gamma \end{pmatrix} \qquad (14)$$

In contrast to $\mathfrak{sch}(d)$, the "complex conjugate" is not needed here. The CGA-Ward identities lead to [28], immediately written down for $d \geq 1$ spatial dimensions

$$\begin{aligned}
F &= \langle \phi_1\phi_2\rangle(t,\mathbf{r}) = 0 \\
G &= \langle \phi_1\psi_2\rangle(t,\mathbf{r}) = a|t|^{-2x_1} e^{-2\boldsymbol{\gamma}_1\cdot\mathbf{r}/t}\, \delta_{x_1,x_2}\delta_{\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2}\, \delta_{x_1',x_2'}\delta_{\boldsymbol{\gamma'}_1,\boldsymbol{\gamma'}_2} \qquad (15)\\
H &= \langle \psi_1\psi_2\rangle(t,\mathbf{r}) = |t|^{-2x_1} e^{-2\boldsymbol{\gamma}_1\cdot\mathbf{r}/t}\left[b - 2a\frac{\mathbf{r}}{t}\cdot\boldsymbol{\gamma'}_1 - 2ax_1'\ln|t|\right] \\
&\qquad \times \delta_{x_1,x_2}\delta_{\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_2}\delta_{x_1',x_2'}\delta_{\boldsymbol{\gamma'}_1,\boldsymbol{\gamma'}_2}
\end{aligned}$$

where the normalisation $a = a(\boldsymbol{\gamma}_1^2, \boldsymbol{\gamma'}_1^2, \boldsymbol{\gamma}_1\cdot\boldsymbol{\gamma}_1')$ as follows from rotation-invariance for $d > 1$ and an analogous form holds for b.

2.3 Exotic Conformal Galilean Algebra

Again, both the scalar x as well as the vector $\boldsymbol{\gamma}$ may become simultaneously Jordan matrices, according to (14). One then needs four distinct two-point functions

$$F = \langle\phi_1\phi_2\rangle \;\;,\;\; G_{12} = \langle\phi_1\psi_2\rangle \;\;,\;\; G_{21} = \langle\psi_1\phi_2\rangle \;\;,\;\; H = \langle\psi_1\psi_2\rangle \tag{16}$$

which all depend merely on $t = t_1 - t_2$ and $\mathbf{r} = \mathbf{r_1} - \mathbf{r}_2$. The operators $\mathbf{h}, \theta$ are realised in terms of auxiliary variables $\boldsymbol{\nu}$ such that $h_i = \partial_{\nu_i} - \frac{1}{2}\epsilon_{ij}\nu_j\theta$ with $i, j = 1, 2$. Remarkably, it turns out that two cases must be distinguished [28]:

Case 1: *defined by* $x_1' \neq 0$ *or* $x_2' \neq 0$ *and* $F = 0$.
In what follows, the indices always refer to the identity of the two primary operators $\Phi_{1,2} = \begin{pmatrix}\psi_{1,2}\\ \phi_{1,2}\end{pmatrix}$. We also use the two-dimensional vector product (with a scalar value) $\mathbf{a}\wedge\mathbf{b} := \epsilon_{ij}a_ib_j$. Then $G_{12} = G(t,\mathbf{r}) = G(-t,-\mathbf{r}) = G_{21} =: G$ such that one has the constraints $x_1 = x_2$, $x_1' = x_2'$, $\theta_1 + \theta_2 = 0$ and (recall $d = 2$)

$$\begin{aligned}
G &= |t|^{-2x_1} e^{-(\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2)\cdot\mathbf{u}-\frac{1}{2}(\boldsymbol{\gamma}_1-\boldsymbol{\gamma}_2)\cdot(\boldsymbol{\nu}_1-\boldsymbol{\nu}_2)}\, e^{\theta_1\mathbf{u}\wedge(\boldsymbol{\nu}_1-\boldsymbol{\nu}_2)+\frac{1}{2}\theta_1\boldsymbol{\nu}_1\wedge\boldsymbol{\nu}_2}\, g_0(\mathbf{w}) \\
H &= |t|^{-2x_1} e^{-(\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2)\cdot\mathbf{u}-\frac{1}{2}(\boldsymbol{\gamma}_1-\boldsymbol{\gamma}_2)\cdot(\boldsymbol{\nu}_1-\boldsymbol{\nu}_2)}\, e^{\theta_1\mathbf{u}\wedge(\boldsymbol{\nu}_1-\boldsymbol{\nu}_2)+\frac{1}{2}\theta_1\boldsymbol{\nu}_1\wedge\boldsymbol{\nu}_2}\, h(\mathbf{u},\boldsymbol{\nu}_1,\boldsymbol{\nu}_2) \\
h &= h_0(\mathbf{w}) - g_0(\mathbf{w})\left(2x_1'\ln|t| + \mathbf{u}\cdot\left(\boldsymbol{\gamma}_1'+\boldsymbol{\gamma}_2'\right) + \frac{1}{2}\left(\boldsymbol{\nu}_1-\boldsymbol{\nu}_2\right)\cdot\left(\boldsymbol{\gamma}_1'-\boldsymbol{\gamma}_2'\right)\right)
\end{aligned} \tag{17}$$

together with the abbreviations $\mathbf{u} = \mathbf{r}/t$ and $\mathbf{w} := \mathbf{u} - \frac{1}{2}(\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2)$. The functions $g_0(\mathbf{w})$ and $h_0(\mathbf{w})$ remain undetermined.
Case 2: *defined by* $x_1' = x_2' = 0$, hence only the vector $\boldsymbol{\gamma}$ has a Jordan form.
One has the constraints $x_1 = x_2$, $\theta_1 + \theta_2 = 0$ and

$$\begin{aligned}
F &= |t|^{-2x_1} e^{-(\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2)\cdot\mathbf{u}-\frac{1}{2}(\boldsymbol{\gamma}_1-\boldsymbol{\gamma}_2)\cdot(\boldsymbol{\nu}_1-\boldsymbol{\nu}_2)}\, e^{\theta_1\mathbf{u}\wedge(\boldsymbol{\nu}_1-\boldsymbol{\nu}_2)+\frac{1}{2}\theta_1\boldsymbol{\nu}_1\wedge\boldsymbol{\nu}_2}\, f_0(\mathbf{w}) \\
G_{12} &= |t|^{-2x_1} e^{-(\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2)\cdot\mathbf{u}-\frac{1}{2}(\boldsymbol{\gamma}_1-\boldsymbol{\gamma}_2)\cdot(\boldsymbol{\nu}_1-\boldsymbol{\nu}_2)}\, e^{\theta_1\mathbf{u}\wedge(\boldsymbol{\nu}_1-\boldsymbol{\nu}_2)+\frac{1}{2}\theta_1\boldsymbol{\nu}_1\wedge\boldsymbol{\nu}_2}\, g_{12}(\mathbf{u},\boldsymbol{\nu}_1,\boldsymbol{\nu}_2) \\
G_{21} &= |t|^{-2x_1} e^{-(\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2)\cdot\mathbf{u}-\frac{1}{2}(\boldsymbol{\gamma}_1-\boldsymbol{\gamma}_2)\cdot(\boldsymbol{\nu}_1-\boldsymbol{\nu}_2)}\, e^{\theta_1\mathbf{u}\wedge(\boldsymbol{\nu}_1-\boldsymbol{\nu}_2)+\frac{1}{2}\theta_1\boldsymbol{\nu}_1\wedge\boldsymbol{\nu}_2}\, g_{21}(\mathbf{u},\boldsymbol{\nu}_1,\boldsymbol{\nu}_2) \\
H &= |t|^{-2x_1} e^{-(\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2)\cdot\mathbf{u}-\frac{1}{2}(\boldsymbol{\gamma}_1-\boldsymbol{\gamma}_2)\cdot(\boldsymbol{\nu}_1-\boldsymbol{\nu}_2)}\, e^{\xi_1\mathbf{u}\wedge(\boldsymbol{\nu}_1-\boldsymbol{\nu}_2)+\frac{1}{2}\theta_1\boldsymbol{\nu}_1\wedge\boldsymbol{\nu}_2}\, h(\mathbf{u},\boldsymbol{\nu_1},\boldsymbol{\nu}_2)
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
g_{12} &= g_0(\mathbf{w}) - f_0(\mathbf{w})\left(\mathbf{u} - \frac{1}{2}(\boldsymbol{\nu}_1 - \boldsymbol{\nu}_2)\right)\cdot\boldsymbol{\gamma}_2' \\
g_{21} &= g_0(\mathbf{w}) - f_0(\mathbf{w})\left(\mathbf{u} + \frac{1}{2}(\boldsymbol{\nu}_1 - \boldsymbol{\nu}_2)\right)\cdot\boldsymbol{\gamma}_1'
\end{aligned} \tag{19}$$

$$h = h_0(\mathbf{w}) - g_0(\mathbf{w}) \left(\mathbf{u} \cdot (\boldsymbol{\gamma}_1' + \boldsymbol{\gamma}_2') + \frac{1}{2}(\boldsymbol{\nu}_1 - \boldsymbol{\nu}_2) \cdot (\boldsymbol{\gamma}_1' - \boldsymbol{\gamma}_2') \right)$$
$$+ \frac{1}{2} f_0(\mathbf{w}) \left(\mathbf{u} + \frac{1}{2}(\boldsymbol{\nu}_1 - \boldsymbol{\nu}_2) \right) \cdot \boldsymbol{\gamma}_1' \left(\mathbf{u} - \frac{1}{2}(\boldsymbol{\nu}_1 - \boldsymbol{\nu}_2) \right) \cdot \boldsymbol{\gamma}_2'$$

The functions $f_0(\mathbf{w})$, $g_0(\mathbf{w})$ and $h_0(\mathbf{w})$ remain undetermined.

Finally, two distinct choices for the rotation generator have been considered in the littérature, namely (single-particle form)

$$J = -\mathbf{r} \wedge \partial_{\mathbf{r}} - \boldsymbol{\gamma} \wedge \partial_{\boldsymbol{\gamma}} - \frac{1}{2\theta} \mathbf{h} \cdot \mathbf{h} \quad \text{and} \quad R = -\mathbf{r} \wedge \partial_{\mathbf{r}} - \boldsymbol{\gamma} \wedge \partial_{\boldsymbol{\gamma}} - \boldsymbol{\nu} \wedge \partial_{\boldsymbol{\nu}} \tag{20}$$

The generator J arises naturally when one derives the generators of the ECGA from a contraction of a pair conformal algebras with non-vanishing spin [40], whereas the choice R has a fairly natural form, especially in the auxiliary variables $\boldsymbol{\nu}$. Both generators obey the same commutators (8) with the other generators of ECGA and commute with the Schrödinger operator (9). One speaks of *"J-invariance"* if the generator J is used and of *"R-invariance"*, if the generator R is used. The consequences of both cases are different [28]:

(A) If one uses R-invariance, in both cases the functions $f_0(\mathbf{w})$, $g_0(\mathbf{w})$ and $h_0(\mathbf{w})$ are short-hand notations for undetermined functions of nine rotation-invariant combinations of $\mathbf{w}$, $\boldsymbol{\gamma}_{1,2}$ and $\boldsymbol{\gamma}_{1,2}'$, for example

$$f_0 = f_0 \left(\mathbf{w}^2, \boldsymbol{\gamma}_1^2, \boldsymbol{\gamma}_2^2, \boldsymbol{\gamma}_1'^2, \boldsymbol{\gamma}_2'^2, \mathbf{w} \cdot \boldsymbol{\gamma}_1, \mathbf{w} \cdot \boldsymbol{\gamma}_2, \mathbf{w} \cdot \boldsymbol{\gamma}_1', \mathbf{w} \cdot \boldsymbol{\gamma}_2' \right) \tag{21}$$

and analogously for g_0 and h_0.

(B) For J-invariance, the $\boldsymbol{\gamma}$-matrices become diagonal, viz. $\boldsymbol{\gamma}_1' = \boldsymbol{\gamma}_2' = \mathbf{0}$. Then only case 1 retains a logarithmic (i.e. indecomposable) structure and, with $\hat{\epsilon} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$g_0 = g_0 \left(\boldsymbol{\gamma}_1^2, \boldsymbol{\gamma}_2^2, \boldsymbol{\gamma}_1 \cdot \boldsymbol{\gamma}_2, \mathbf{w} + \hat{\epsilon}\, (\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2)\, (2\theta_1)^{-1} \right)$$
$$h_0 = h_0 \left(\boldsymbol{\gamma}_1^2, \boldsymbol{\gamma}_2^2, \boldsymbol{\gamma}_1 \cdot \boldsymbol{\gamma}_2, \mathbf{w} + \hat{\epsilon}\, (\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2)\, (2\theta_1)^{-1} \right) \tag{22}$$

In all these ECGA-covariant two-point functions, there never is a constraint on the $\boldsymbol{\gamma}_i$, and on the $\boldsymbol{\gamma}_i'$ only in the case of J-invariance.

2.4 Ageing Algebra

Since the representation (10) of $\mathfrak{age}(1)$ contains the two independent scaling dimensions x, ξ, either of those may take a matrix form. From the commutators, it can be shown that both are simultaneously of Jordan form [22]

$$x \mapsto \begin{pmatrix} x & x' \\ 0 & x \end{pmatrix} \quad , \quad \xi \mapsto \begin{pmatrix} \xi & \xi' \\ 0 & \xi \end{pmatrix} \tag{23}$$

We use again the definitions (16). Since the space-dependent part of the two-point functions has the same form as already derived above, in the case of Schrödinger-invariance, we set $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 = \mathbf{0}$ and consider only the time-dependent part where only the values of the exponents change when $z \neq 2$ is admitted. The requirement of co-variance under this logarithmic representation of $\mathfrak{age}(1)$ leads to [22]

$$\begin{aligned}
F(t,s) &= s^{-(x_1+x_2)/2}\, \mathcal{F}\left(\frac{t}{s}\right) f_0 \\
G_{12}(t,s) &= s^{-(x_1+x_2)/2}\, \mathcal{F}\left(\frac{t}{s}\right) \left(g_{12}\left(\frac{t}{s}\right) + \gamma_{12}\left(\frac{t}{s}\right) \ln s \right) \\
G_{21}(t,s) &= s^{-(x_1+x_2)/2}\, \mathcal{F}\left(\frac{t}{s}\right) \left(g_{21}\left(\frac{t}{s}\right) + \gamma_{21}\left(\frac{t}{s}\right) \ln s \right) \\
H(t,s) &= s^{-(x_1+x_2)/2}\, \mathcal{F}\left(\frac{t}{s}\right) \left(h_0\left(\frac{t}{s}\right) + h_1\left(\frac{t}{s}\right) \ln s + h_2\left(\frac{t}{s}\right) \ln^2 s \right)
\end{aligned} \tag{24}$$

with the abbreviation $\mathcal{F}(y) = y^{2\xi_2/z+(x_2-x_1)/z}(y-1)^{-(x_1+x_2)/z-2(\xi_1+\xi_2)/z}$. Herein the scaling functions, depending only on $y = t/s$, are given by

$$\begin{aligned}
\gamma_{12}(y) &= -\frac{1}{2}x_2' f_0 \quad , \quad \gamma_{21}(y) = -\frac{1}{2}x_1' f_0 \\
h_1(y) &= -\frac{1}{2}\left(x_1' g_{12}(y) + x_2' g_{21}(y)\right) \quad , \quad h_2(y) = \frac{1}{4}x_1' x_2' f_0
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
g_{12}(y) &= g_{12,0} + \left(\frac{x_2'}{2} + \xi_2'\right) f_0 \ln\left|\frac{y}{y-1}\right| \\
g_{21}(y) &= g_{21,0} - \left(\frac{x_1'}{2} + \xi_1'\right) f_0 \ln|y-1| - \frac{x_1'}{2} f_0 \ln|y| \\
h_0(y) &= h_0 - \left[\left(\frac{x_1'}{2} + \xi_1'\right) g_{21,0} + \left(\frac{x_2'}{2} + \xi_2'\right) g_{12,0}\right] \ln|y-1|
\end{aligned}$$

Table 1 Constraints of co-variant two-point functions in logarithmic representations of some algebras of local scale-transformations

Algebra	Eq.	Constraints					
$\mathfrak{sch}$	(13)	$x_1 = x_2$	$x_1' = x_2' = 1$			$\mathcal{M} + \mathcal{M}^* = 0$	
$\mathfrak{age}$	(24)					$\mathcal{M}_1 + \mathcal{M}_2 = 0$	
CGA	(15)	$x_1 = x_2$	$x_1' = x_2'$	$\boldsymbol{\gamma}_1 = \boldsymbol{\gamma}_2$	$\boldsymbol{\gamma}_1' = \boldsymbol{\gamma}_2'$		
ECGA	(17)	$x_1 = x_2$	$x_1' = x_2'$			$\theta_1 + \theta_2 = 0$	R1
ECGA	(18)	$x_1 = x_2$	$x_1' = x_2' = 0$			$\theta_1 + \theta_2 = 0$	R2
ECGA	(17)	$x_1 = x_2$	$x_1' = x_2'$		$\boldsymbol{\gamma}_1' = \boldsymbol{\gamma}_2' = \mathbf{0}$	$\theta_1 + \theta_2 = 0$	J1

The equation labels refer to the explicit form of the two-point function. The constraints apply to scaling dimensions x, ξ, rapidities $\boldsymbol{\gamma}$ or the Bargman super-selection rules on the "masses" θ or $\mathcal{M}$. For the ECGA, the last three lines refer either to R-invariance with the two distinct cases labelled R1 and R2 and normalisations given by (21) or else J-invariance, labelled by J1, where the normalisations are given by (22)

$$
\begin{aligned}
&- \left[\frac{x_1'}{2} g_{21,0} - \left(\frac{x_2'}{2} + \xi_2' \right) g_{12,0} \right] \ln|y| \\
&+ \frac{1}{2} f_0 \left[\left(\left(\frac{x_1'}{2} + \xi_1' \right) \ln|y-1| + \frac{x_1'}{2} \ln|y| \right)^2 \right. \\
&\left. - \left(\frac{x_2'}{2} + \xi_2' \right)^2 \ln^2 \left| \frac{y}{y-1} \right| \right]
\end{aligned}
\tag{26}
$$

and $f_0, g_{12,0}, g_{21,0}, h_0$ are normalisation constants. There are no constraints on any of the x_i, x_i', ξ_i, ξ_i'. However, for $z = 2$ the Bargman superselection rule $\mathcal{M}_1 + \mathcal{M}_2 = 0$ holds true.

2.5 *Discussion*

Table 1 summarises some features of the various logarithmic representations considered here. First, logarithmic Schrödinger-invariance, with a single scaling dimension x elevated to a Jordan matrix, is the straightforward extension of analogous results of logarithmic conformal invariance. In the other algebra, a more rich structure arises since there are at least two distinct quantities (x and $\boldsymbol{\gamma}$ for CGA and ECGA and x and ξ for $\mathfrak{age}$, respectively) which simultaneously become Jordan matrices.

In this respect the results form the CGA is the next natural step of generalisations, in that all naturally expected constraints between the labels of the representation are realised. In addition, from the explicit for of $H = \langle \psi_1 \psi_2 \rangle$ in (15) one sees that for $x_1' = x_2' = 0$, no explicitly logarithmic terms remain, although the representation is still indecomposable. The possibility of finding such explicit examples of this kind was pointed out long ago [35]. More examples of this kind, including several ones of direct physical relevance, will be mentioned shortly. Next, when going over to the ECGA, we notice that the extra non-commutative structure with its non-trivial central charge has suppressed some of the constraints we had found before for the CGA. In addition, two quite distinct forms of the two-point functions are found. In the first case, see (17), the structure of the scaling function is quite analogous to the examples treated before, including an explicitly logarithmic contribution $\sim x_1' \ln|t|$. However, the second case gives the first surprise[2] that $F = \langle\phi_1\phi_2\rangle \neq 0$! Again, since now $x_1' = x_2' = 0$, no explicitly logarithmic term remains for this indecomposable representation. It is hoped that these explicit forms might be helpful in identifying physical examples with these representations.

Finally, the case of $\mathfrak{age}$ is again different, since the breaking of time-translation-invariance which was present in all other algebras studied here gives rise to new possibilities. Especially, besides the ubiquitous Bargman superselection rule, no further constraints remain. On the other hand, two explicitly logarithmic contributions $\sim \ln s$ and $\sim \ln^2 s$ are obtained.

The scaling function (24) has been used as a phenomenological device to describe numerical data for the auto-response function $R(t,s) = R(t,s;\mathbf{0})$ for the slow non-equilibrium relaxation in several model systems. For maximal flexibility, one interprets the measured response function as the correlator $R(t,s) = H(t,s) = \langle\psi(t)\tilde{\psi}(s)\rangle$ in the Janssen-de Dominicis formulation of non-equilibrium field-theory, where ψ is the logarithmic partner and $\tilde{\psi}$ is the corresponding response operator. Because of the excellent quality of the data collapse for several values of the waiting time s, one concludes that the logarithmic corrections to scaling which occur in (24) should be vanishing, which implies that $x' = x_\psi' = x_1' = 0$ and $\tilde{x}' = x_{\tilde{\psi}}' = x_2' = 0$. Hence, empirically, only the second scaling dimensions ξ and $\tilde{\xi}$ carry the indecomposable structure and the shape of $R(t,s)$ will be given by the scaling function h_0 in (26). Although there are no logarithmic corrections to time-dependent scaling, there are logarithmic modification in the shape of the scaling functions. Clearly, one can always arrange for the scaling $\xi' = \xi_\psi' = \xi_1' = 0, 1$ and $\tilde{\xi}' = \xi_{\tilde{\psi}}' = \xi_2' = 1$ such that four free parameters remain to be fitted to the data. Excellent fits have been obtained for (1) the Kardar-Parisi-Zhang equation of interface growth [27], (2) the directed percolation universality class [22] and (3) the critical $2D$ voter model on a triangular lattice [47]. These comparisons also clearly show that a non-logarithmic representation of $\mathfrak{age}$ with $\xi' = \tilde{\xi}' = 0$ would not nearly reproduce the data as satisfactorially. For a recent review and a detailed list of references, see [24].

[2]The only previously known example of this had been obtained for the ageing algebra, where time-translations are excluded, see (24).

3 Large-Distance Behaviour and Causality

In Sect. 2, the two-point functions were seen to be of the form $F(t,s;\mathbf{r}) = \langle\phi\phi^*\rangle \sim \exp\left(-\frac{\mathcal{M}}{2}\frac{r^2}{t-s}\right)$ for $\mathfrak{sch}(d)$ and $\sim \exp(-2\boldsymbol{\gamma}\cdot\mathbf{r}/t)$ for $\mathrm{CGA}(d)$, respectively, where the purely time-dependent parts are suppressed. *Can one show from an algebraic argument that* $|F(t,s;\mathbf{r})| \to 0$ *for large distances* $|\mathbf{r}| \to \infty$ *?*

As we shall see, the $F(t,s;\mathbf{r})$ cannot be considered as differentiable functions, but must rather seen as singular distributions, whose form may become more simple in convenient "dual" variables. It will be necessary to identify these first before trying to reconstruct F. For notational simplicity, we restrict to the scalar case.

3.1 Schrödinger Algebra

One introduces first a new coordinate ζ dual to $\mathcal{M}$ (consider as a "$(-1)^{\mathrm{st}}$" coordinate) by the transformation [16]

$$\hat{\phi}(\zeta,t,\mathbf{r}) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{d}\mathcal{M}\, e^{\mathrm{i}\mathcal{M}\zeta}\phi_{\mathcal{M}}(t,\mathbf{r}) \tag{27}$$

Next, one extends $\mathfrak{sch}(1)$ to the parabolic sub-algebra $\widetilde{\mathfrak{sch}}(1) \subset B_2$ by adding the extra generator N [25]. When acting on $\hat{\phi}$, the generators take the form

$$\begin{aligned}
X_n &= \frac{\mathrm{i}}{2}(n+1)nt^{n-1}r^2\partial_\zeta - t^{n+1}\partial_t - \frac{n+1}{2}t^n r\partial_r - \frac{n+1}{2}xt^n\\
Y_m &= \mathrm{i}\left(m+\frac{1}{2}\right)t^{m-1/2}r\partial_\zeta - t^{m+1/2}\partial_j\\
M_n &= \mathrm{i}t^n\partial_\zeta\\
N &= \zeta\partial_\zeta - t\partial_t + \xi\,.
\end{aligned} \tag{28}$$

Herein, the constant ξ is identical to the second scaling dimension which arises in the representation (10) of the ageing algebra $\mathfrak{age}(1)$. The Schrödinger-Ward identities of the generators $M_0, X_{-1}, Y_{-\frac{1}{2}}$ readily imply translation-invariance in ζ, t, r. Co-variance under the generators $X_{0,1}$ and $Y_{\frac{1}{2}}$ leads to $\hat{F}(\zeta,t,u) = |t|^{-x}f(u|t|^{-1})$, with $x := x_1 = x_2$ and where $u = 2\zeta t + r^2$ in an otherwise natural notation; f remains an undetermined function. This form is still to general to solve the question raised above. Co-variance under N restricts its form further, to a simple power law:

$$\hat{F}(\zeta,t,r) = \langle\hat{\phi}(\zeta,t,r)\hat{\phi}^*(0,0,0)\rangle = \hat{f}_0\,|t|^{-x}\left(\frac{2\zeta t + \mathrm{i}r^2}{|t|}\right)^{-x-\xi_1-\xi_2} \tag{29}$$

with a normalisation constant f_0. Now, one imposes the physical convention that the mass $\mathcal{M} > 0$ of the scaling operator ϕ should be positive. If $\frac{1}{2}(x_1+x_2)+\xi_1+\xi_2 > 0$, then a standard calculation of the inverse of the transformation (27) applied to both scaling operators in (29) leads to, already extended to dimensions $d \geq 1$

$$F(t,\mathbf{r}) = \delta(\mathcal{M}+\mathcal{M}^*)\,\delta_{x_1,x_2}\,\Theta(t)\,t^{-x_1}\,\exp\left[-\frac{\mathcal{M}}{2}\frac{\mathbf{r}^2}{t}\right]F_0(\mathcal{M}) \tag{30}$$

where the Θ-function expresses the causality condition $t > 0$ [25].

The same argument goes through for logarithmic representations of $\widetilde{\mathfrak{sch}}(d)$ [25].

3.2 Conformal Galilean Algebra

The dual coordinate ζ is now introduced via

$$\hat{\phi}(\zeta,t,\mathbf{r}) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\mathrm{d}\gamma\; e^{\mathrm{i}\gamma\zeta}\phi_\gamma(t,\mathbf{r}) \tag{31}$$

The generators of the parabolic sub-algebra $\widetilde{\mathrm{CGA}}(1) \subset B_2$ (see Fig.1c), including the new generator N, acting on $\hat{\phi}$, read

$$\begin{aligned} X_n &= -t^{n+1}\partial_t - (n+1)t^n r\partial_r + \mathrm{i}(n+1)nt^{n-1}r\partial_\zeta - (n+1)xt^n \\ Y_n &= -t^{n+1}\partial_r + \mathrm{i}(n+1)t^n\partial_\zeta \\ N &= -\zeta\partial_\zeta - r\partial_r - \nu \end{aligned} \tag{32}$$

Letting $\hat{F} = \langle\hat{\phi}_1\hat{\phi}_2\rangle$, time- and space-translation-invariance imply $\hat{F} = \hat{F}(\zeta_+,\zeta_-,t,r)$, with $\zeta_\pm := \frac{1}{2}(\zeta_1 \pm \zeta_2)$. There is no translation-invariance in the ζ_j; rather, combination of the generators $Y_{0,1}$ leads to $\partial_{\zeta_-}\hat{F} = 0$. As usual, combination of $X_{0,1}$ gives the constraint $x_1 = x_2$ and the two remaining generators of $\mathrm{CGA}(1)$ give $\hat{F} = |t|^{-2x_1}\hat{f}(\zeta_+ + \mathrm{i}r/t)$, with a yet un-determined function $\hat{f}$. As for $\mathfrak{sch}(1)$, this form is still too general to answer the question raised above. However, co-variance under N gives $\hat{f}(u) = \hat{f}_0 u^{-2\nu}$, with $2\nu := \nu_1 + \nu_2$ and a normalisation constant $\hat{f}_0$.

To proceed, we require the following fact [2, ch. 11].

Definition. *A function $g : \mathbb{H}_+ \to \mathbb{C}$, where $\mathbb{H}_+$ is the upper complex half-plane of all $w = u + \mathrm{i}\mathrm{v}$ with $\mathrm{v} > 0$, is in the* Hardy class H_2^+, *if $g(w)$ is holomorphic in $\mathbb{H}_+$ and if*

$$M^2 = \sup_{\mathrm{v}>0}\int_{\mathbb{R}}\mathrm{d}u\;|g(u+\mathrm{i}\mathrm{v})|^2 < \infty \tag{33}$$

We shall also need the Hardy class H_2^-, where $\mathbb{H}_+$ is replaced by the lower complex half-plane $\mathbb{H}_-$ and the supremum in (33) is taken over $\mathrm{v} < 0$.

Lemma ([2]). *If $g \in H_2^{\pm}$, then there are functions $\mathcal{G}_\pm \in L^2(0,\infty)$ such that for* $\mathrm{v} > 0$

$$g(w) = g(u \pm \mathrm{iv}) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \mathrm{d}\gamma\; e^{\pm \mathrm{i}\gamma w}\, \mathcal{G}_\pm(\gamma) \tag{34}$$

Next, we fix $\lambda := r/t$ and re-write the function $\hat{f}$ which determines the structure of the two-point function $\hat{F}$, as

$$\hat{f}(\zeta_+ + \mathrm{i}\lambda) = f_\lambda(\zeta_+) \tag{35}$$

Proposition. *If $\nu > \frac{1}{4}$ and if $\lambda > 0$, then $f_\lambda \in H_2^+$.*

Proof. The analyticity in $\mathbb{H}_+$ is obvious from the definition of f_λ. For the bound (33), observe that $|f_\lambda(u+\mathrm{iv})| = \left|\hat{f}_0(u + \mathrm{i}(\mathrm{v}+\lambda))^{-2\nu}\right| =$
$= \bar{f}_0 \left(u^2 + (\mathrm{v}+\lambda)^2\right)^{-\nu}$. Hence

$$M^2 = \sup_{\mathrm{v}>0} \int_{\mathbb{R}} \mathrm{d}u\; |f_\lambda(u+\mathrm{iv})|^2 = \bar{f}_0^2 \frac{\sqrt{\pi}\; \Gamma(2\nu - \frac{1}{2})}{\Gamma(2\nu)} \sup_{\mathrm{v}>0} (\mathrm{v}+\lambda)^{1-4\nu} < \infty$$

since the integral converges for $\nu > \frac{1}{4}$. □

Similarly, for $\nu > \frac{1}{4}$ and $\lambda < 0$, we have $f_\lambda \in H_2^-$.

For $\lambda > 0$, we use Eq. (34) from the lemma to re-write $\hat{f}$ as follows

$$\sqrt{2\pi}\,\hat{f}(\zeta_+ + \mathrm{i}\lambda) = \int_0^\infty \mathrm{d}\gamma_+ e^{\mathrm{i}(\zeta_+ + \mathrm{i}\lambda)\gamma_+} \hat{\mathcal{F}}_+(\gamma_+) = \int_{\mathbb{R}} \mathrm{d}\gamma_+ \Theta(\gamma_+)\, e^{\mathrm{i}(\zeta_+ + \mathrm{i}\lambda)\gamma_+} \hat{\mathcal{F}}_+(\gamma_+) \tag{36}$$

such that by inverting (31), the two-point function F finally becomes, with $x_1 = x_2$

$$\begin{aligned} F &= \frac{|t|^{-2x_1}}{\pi\sqrt{2\pi}} \int_{\mathbb{R}^2} \mathrm{d}\zeta_+ \mathrm{d}\zeta_-\; e^{-\mathrm{i}(\gamma_1+\gamma_2)\zeta_+} e^{-\mathrm{i}(\gamma_1-\gamma_2)\zeta_-} \\ &\quad \times \int_{\mathbb{R}} \mathrm{d}\gamma_+\; \Theta(\gamma_+)\hat{\mathcal{F}}_+(\gamma_+) e^{-\gamma_+\lambda} e^{\mathrm{i}\gamma_+\zeta_+} \\ &= \frac{|t|^{-2x_1}}{\pi\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}\gamma_+\; \Theta(\gamma_+)\hat{\mathcal{F}}_+(\gamma_+) e^{-\gamma_+\lambda} \int_{\mathbb{R}} \mathrm{d}\zeta_-\; e^{-\mathrm{i}(\gamma_1-\gamma_2)\zeta_-} \\ &\quad \times \int_{\mathbb{R}} \mathrm{d}\zeta_+\; e^{\mathrm{i}(\gamma_+ - \gamma_1 - \gamma_2)\zeta_+} \\ &= \delta(\gamma_1-\gamma_2)\Theta(\gamma_1) F_{0,+}(\gamma_1) e^{-2\gamma_1\lambda} |t|^{-2x_1} \end{aligned} \tag{37}$$

where in the last line, two δ-functions were used and $F_{0,+}$ contains the unspecified dependence on the positive constant γ_1.

For $\lambda < 0$, we can use again the second form of the lemma, with $f_\lambda \in H_2^-$, and find $F = \delta(\gamma_1 - \gamma_2)\Theta(-\gamma_1)F_{0,-}(\gamma_1)e^{2\gamma_1|\lambda|}|t|^{-2x_1}$. These two forms can be combined into a single one, immediately generalised to $d \geq 1$ dimensions, and assumed continuous in $\mathbf{r}$ and rotation-invariant as well

$$F(t, \mathbf{r}) = \delta_{x_1,x_2}\delta(\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2)\,|t|^{-2x_1} \exp\left[-2\left|\frac{\boldsymbol{\gamma}_1\cdot\mathbf{r}}{t}\right|\right] F_0(\boldsymbol{\gamma}_1^2) \tag{38}$$

3.3 Discussion

Surprisingly, our attempts to establish sufficient criteria that the two-point function $F(t, s; \mathbf{r})$ vanishes in the limit $|\mathbf{r}| \to 0$, led to qualitatively different types of results.

(A) For the Schrödinger algebra with the representation (4), the extension to the corresponding maximal parabolic sub-algebra and the dualisation of the mass $\mathcal{M}$ has led to the form (30). It is maximally asymmetric under permutation of its two scaling operators and obeys a causality condition $t_1 - t_2 > 0$. In applications, it should predict the form of *response functions*. Indeed, we quoted in Sect. 2 several examples where response functions of non-equilibrium many-body systems undergoing physical ageing are described by (30), or logarithmic extensions thereof.

(B) For the conformal Galilean algebra with the representation (6), there is no central extension which would produce a Bargman superselection rule for the rapidities $\boldsymbol{\gamma}$. An analogous extension to the maximal parabolic sub-algebra and the dualisation of the rapidities rather produced the form (38). It is fully symmetric under the permutation of its scaling operators. This is a characteristic of *correlation functions*. Our result therefore suggests that searches for physical applications of the conformal Galilean algebra should concentrate on studying co-variant correlators, rather than response functions.

Also, these examples indicate that a deeper analytic structure might be found upon investigating the dual two-point functions $\hat{F}$, rather than keeping masses $\mathcal{M}$ or rapidities $\boldsymbol{\gamma}$ fixed.

Another possibility concerns the extension of these lines to non-local representations of these algebras [48], see also elsewhere in this volume.

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New Type of $\mathcal{N} = 4$ Supersymmetric Mechanics

Evgeny Ivanov and Stepan Sidorov

Abstract We give a short account of the superfield approach based on deformed analogs of the standard $\mathcal{N}=4, d=1$ superspace and present a few models of supersymmetric quantum mechanics constructed within this new framework. The relevant superspaces are the proper cosets of the supergroup $SU(2|1)$. As instructive examples we consider the models associated with the worldline $SU(2|1)$ supermultiplets (**1,4,3**) and (**2,4,2**). An essential ingredient of these models is the mass parameter m which deforms the standard $\mathcal{N}=4, d=1$ supersymmetry to $SU(2|1)$ supersymmetry.

1 Introduction

Recently, there was a substantial interest in rigid supersymmetric theories based on curved analogs of the Poincaré supergroup in diverse dimensions [1]. One can hope that their study will lead to a further progress in understanding the generic gauge/gravity correspondence. Motivated by this interest, in [2] we defined the simplest analogous deformation of the one-dimensional $\mathcal{N} = 4, d = 1$ supersymmetry. The present talk is an overview of the relevant new class of models of supersymmetric quantum mechanics (SQM) and the appropriate superfield approach.

The symmetry group of the standard SQM models [3] is $\mathcal{N}$ extended $d = 1$ analog of the higher-dimensional super Poincaré groups:

$$\{Q^A, Q^B\} = 2\delta^{AB} H\,, \quad [H, Q^A] = 0\,, \quad A, B = 1 \ldots \mathcal{N}\,, \tag{1}$$

where Q^A are $\mathcal{N}$ real supercharges and H is the time-translation generator. The general automorphism group of this superalgebra is $O(\mathcal{N})$. A possible way of generalizing the relevant SQM models is suggested by the following form of the $\mathcal{N} = 2, d = 1$ superalgebra extended by the $U(1)$ automorphism generator J

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$$Q = \frac{1}{\sqrt{2}}\left(Q^1 + iQ^2\right), \quad \bar{Q} = \frac{1}{\sqrt{2}}\left(Q^1 - iQ^2\right), \tag{2}$$

$$\{Q, \bar{Q}\} = 2H,\ Q^2 = \bar{Q}^2 = 0,\ [H, Q] = [H, \bar{Q}] = 0,$$
$$[J, Q] = Q\,, \quad [J, \bar{Q}] = -\bar{Q}\,, \quad [H, J] = 0\,. \tag{3}$$

These relations are recognized as defining the superalgebra $u(1|1)$, H being the central charge generator.

The interpretation of $\mathcal{N} = 2, d = 1$ superalgebra as $u(1|1)$ suggests a new type of its extensions to the higher-rank $d = 1$ supersymmetries. It corresponds to the following sequence of embeddings:

$$(\mathcal{N} = 2\,,\ d = 1) \equiv u(1|1)\ \subset\ su(2|1)\ \subset\ su(2|2)\ \subset\ \ldots\,. \tag{4}$$

In the relevant superalgebras, the closure of supercharges contains, besides an analog of the Hamiltonian H, also internal symmetry generators. They commute with the Hamiltonian H, but not with the supercharges.

The SQM models to be reviewed here correspond to the simplest $su(2|1)$ case. Our basic aim is to construct the worldline superfield approach to $SU(2|1)$ and to demonstrate that most of the off-shell multiplets of $\mathcal{N} = 4, d = 1$ supersymmetry admit the well-defined $SU(2|1)$ analogs. In particular, the so-called "weak supersymmetry models" [4] prove to be associated with the $SU(2|1)$ multiplet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$. The second multiplet that we consider is the chiral multiplet $(\mathbf{2}, \mathbf{4}, \mathbf{2})$. An interesting feature of the relevant component SQM actions is the presence of the bosonic $d = 1$ Wess-Zumino (WZ) terms in parallel with the second-order kinetic terms.

2 $SU(2|1)$ Superspace

We consider the (centrally-extended) superalgebra $su(2|1)$:

$$\{Q^i, \bar{Q}_j\} = 2m\left(I^i_j - \delta^i_j F\right) + 2\delta^i_j H\,, \qquad \left[I^i_j, I^k_l\right] = \delta^k_j I^i_l - \delta^i_l I^k_j\,,$$
$$\left[I^i_j, \bar{Q}_l\right] = \frac{1}{2}\delta^i_j \bar{Q}_l - \delta^i_l \bar{Q}_j\,, \qquad \left[I^i_j, Q^k\right] = \delta^k_j Q^i - \frac{1}{2}\delta^i_j Q^k,$$
$$\left[F, \bar{Q}_l\right] = -\frac{1}{2}\bar{Q}_l\,, \qquad \left[F, Q^k\right] = \frac{1}{2}Q^k. \tag{5}$$

The bosonic subalgebra of (5) consists of the $U(2)$ symmetry generators (I^i_j, F) and the central charge generator H. In the quantum-mechanical realization of $SU(2|1)$ the central charge H is interpreted as the canonical Hamiltonian, while in the superspace realization it is the time-translation generator. The $SU(2|1)$ supersymmetry is a deformation of the standard $\mathcal{N} = 4, d = 1$ supersymmetry which is recovered in the limit $m = 0$.

2.1 Deformed Superspace

The superspace coordinates $\{t, \theta_i, \bar{\theta}^j\}$ are identified with the parameters of the following coset of the supergroup $SU(2|1)$:

$$\frac{SU(2|1)}{SU(2) \times U(1)} \sim \frac{\{Q^i, \bar{Q}_j, H, I^i_j, F\}}{\{I^i_j, F\}}. \tag{6}$$

The relevant coset element can be conveniently parametrized as

$$g = \exp\Big(itH + i\tilde{\theta}_i Q^i - i\bar{\tilde{\theta}}^j \bar{Q}_j\Big), \qquad \tilde{\theta}_i = \left[1 - \frac{2m}{3}\left(\bar{\theta}\cdot\theta\right)\right]\theta_i\,. \tag{7}$$

The superspace realization of the $SU(2|1)$ generators is given by[1]

$$\begin{aligned}
Q^i &= -i\frac{\partial}{\partial\theta_i} + 2im\bar{\theta}^i\bar{\theta}^j\frac{\partial}{\partial\bar{\theta}^j} + \bar{\theta}^i\frac{\partial}{\partial t}, \quad \bar{Q}_j = i\frac{\partial}{\partial\bar{\theta}^j} + 2im\theta_j\theta_k\frac{\partial}{\partial\theta_k} - \theta_j\frac{\partial}{\partial t},\\
I^i_j &= \left(\bar{\theta}^i\frac{\partial}{\partial\bar{\theta}^j} - \theta_j\frac{\partial}{\partial\theta_i}\right) - \frac{\delta^i_j}{2}\left(\bar{\theta}^k\frac{\partial}{\partial\bar{\theta}^k} - \theta_k\frac{\partial}{\partial\theta_k}\right),\\
F &= \frac{1}{2}\left(\bar{\theta}^k\frac{\partial}{\partial\bar{\theta}^k} - \theta_k\frac{\partial}{\partial\theta_k}\right), \quad H = i\partial_t.
\end{aligned} \tag{8}$$

The supercharges $Q^i, \bar{Q}_j$ generate the following coordinate transformations:

$$\begin{aligned}
\delta t &= i\left[\left(\epsilon\cdot\bar{\theta}\right) + \left(\bar{\epsilon}\cdot\theta\right)\right],\\
\delta\theta_i &= \epsilon_i + 2m\left(\bar{\epsilon}\cdot\theta\right)\theta_i\,, \quad \delta\bar{\theta}^j = \bar{\epsilon}^i - 2m\left(\epsilon\cdot\bar{\theta}\right)\bar{\theta}^i.
\end{aligned} \tag{9}$$

The invariant integration measure μ is defined as

$$\mu = dtd^2\theta d^2\bar{\theta}(1 + 2m\,\bar{\theta}\cdot\theta)\,, \quad \delta\mu = 0\,. \tag{10}$$

2.2 Covariant Derivatives

In order to construct the covariant derivatives of the superspace coordinates, we should calculate the left-covariant Cartan one-forms. They are defined by the relation

[1] We use the convention $\left(\bar{\theta}\cdot\theta\right) = \bar{\theta}^i\theta_i$.

$$\Omega := g^{-1}dg = i\Delta\theta_i\, Q^i - i\Delta\bar\theta^j\, \bar Q_j + i\Delta h_i^j\, I_j^i + i\Delta\hat h\, F + i\Delta t\, H\,, \tag{11}$$

and have the following explicit expressions:

$$\begin{aligned}
\Delta\theta_i &= d\theta_i + m\left(d\theta_l\bar\theta^l\theta_i - d\theta_i\bar\theta^k\theta_k\right) + \frac{m^2}{4}d\theta_i\left(\bar\theta\cdot\theta\right)^2,\\
\Delta\bar\theta^j &= d\bar\theta^j - m\left(d\bar\theta^l\theta_l\bar\theta^j - d\bar\theta^j\theta_k\bar\theta^k\right) + \frac{m^2}{4}d\bar\theta^j\left(\bar\theta\cdot\theta\right)^2,\\
\Delta t &= dt + i\left(d\theta_i\bar\theta^i + d\bar\theta^i\theta_i\right)\left[1 - 2m\left(\bar\theta\cdot\theta\right)\right],\\
\Delta\hat h &= -im\left(d\theta_i\bar\theta^i + d\bar\theta^i\theta_i\right)\left[1 - 2m\left(\bar\theta\cdot\theta\right)\right],\\
\Delta h_i^j &= im\left(d\theta_i\bar\theta^j + d\bar\theta^j\theta_i - \frac{\delta_i^j}{2}\left(d\theta_l\bar\theta^l + d\bar\theta^l\theta_l\right)\right)\left(1 - \frac{3m}{2}\left(\bar\theta\cdot\theta\right)\right)\\
&\quad - \frac{im^2}{2}\left(d\theta_l\bar\theta^l + d\bar\theta^l\theta_l\right)\left(\bar\theta^j\theta_i - \frac{\delta_i^j}{2}\left(\bar\theta\cdot\theta\right)\right).
\end{aligned} \tag{12}$$

The covariant derivatives can now be figured out from the covariant differential

$$\mathcal{D}\Phi_A := d\Phi_A + \left[i\Delta h_i^j\,\tilde I_j^i + i\Delta\hat h\,\tilde F\right]_A^B\Phi_B \equiv \left[\Delta\theta_i\mathcal{D}^i - \Delta\bar\theta^j\bar{\mathcal{D}}_j + \Delta t\,\mathcal{D}_t\right]\Phi_A\,, \tag{13}$$

where $\Phi_B(t,\theta_i,\bar\theta^j)$ is some superfield and B is the index of some $U(2)$ representation. They are

$$\begin{aligned}
\mathcal{D}^i &= \left[1 + m\left(\bar\theta\cdot\theta\right) - \frac{3m^2}{4}\left(\bar\theta\cdot\theta\right)^2\right]\frac{\partial}{\partial\theta_i} - m\bar\theta^i\theta_j\frac{\partial}{\partial\theta_j} - i\bar\theta^i\frac{\partial}{\partial t}\\
&\quad + m\bar\theta^i\tilde F - m\bar\theta^j\tilde I_j^i + \frac{m^2}{2}\left(\bar\theta\cdot\theta\right)\bar\theta^j\tilde I_j^i - \frac{m^2}{2}\bar\theta^i\bar\theta^j\theta_k\tilde I_j^k\,,\\
\bar{\mathcal{D}}_j &= -\left[1 + m\left(\bar\theta\cdot\theta\right) - \frac{3m^2}{4}\left(\bar\theta\cdot\theta\right)^2\right]\frac{\partial}{\partial\bar\theta^j} + m\bar\theta^k\theta_j\frac{\partial}{\partial\bar\theta^k} + i\theta_j\frac{\partial}{\partial t}\\
&\quad - m\theta_j\tilde F + m\theta_k\tilde I_j^k - \frac{m^2}{2}\left(\bar\theta\cdot\theta\right)\theta_k\tilde I_j^k + \frac{m^2}{2}\theta_j\bar\theta^l\theta_k\tilde I_l^k\,,\\
\mathcal{D}_t &= \partial_t\,.
\end{aligned} \tag{14}$$

3 The Supermultiplet (1, 4, 3)

The $SU(2|1)$ superfields are the appropriate analogs of the superfields defined on the standard superspace. For example, the multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ [5, 6] is described by the real neutral superfield $G(t,\theta,\bar\theta)$ satisfying the constraints

$$\varepsilon^{lj}\bar{\mathcal{D}}_l\,\bar{\mathcal{D}}_j G = \varepsilon_{lj}\mathcal{D}^l\,\mathcal{D}^j G = 0. \tag{15}$$

Their solution is

$$G = x - mx\left(\bar{\theta}\cdot\theta\right)\left[1 - 2m\left(\bar{\theta}\cdot\theta\right)\right] + \frac{\ddot{x}}{2}\left(\bar{\theta}\cdot\theta\right)^2 - i\left(\bar{\theta}\cdot\theta\right)\left(\theta_i\,\dot{\psi}^i + \bar{\theta}^j\,\dot{\bar{\psi}}_j\right)$$
$$+\left[1 - 2m\left(\bar{\theta}\cdot\theta\right)\right]\left(\theta_i\,\psi^i - \bar{\theta}^j\,\bar{\psi}_j\right) + \bar{\theta}^j\theta_i\,B^i_j\,, \quad B^k_k = 0\,. \tag{16}$$

The irreducible set of off-shell fields is $x(t)$, $\psi^i(t)$, $\bar{\psi}_i(t)$, $B^i_j(t)$, that just amounts to the $(\mathbf{1},\mathbf{4},\mathbf{3})$ content. In the limit $m = 0$, the real superfield G becomes the ordinary $(\mathbf{1},\mathbf{4},\mathbf{3})$ superfield. The ϵ transformation law of G,

$$\delta G = -\left(i\epsilon_i\,Q^i - i\bar{\epsilon}^j\,\bar{Q}_j\right)G, \tag{17}$$

implies

$$\delta x = \left(\bar{\epsilon}\cdot\bar{\psi}\right) - \left(\epsilon\cdot\psi\right), \qquad \delta\psi^i = i\bar{\epsilon}^i\dot{x} - m\bar{\epsilon}^i x + \bar{\epsilon}^k B^i_k\,,$$
$$\delta B^i_j = -2i\left(\epsilon_j\dot{\psi}^i + \bar{\epsilon}^i\dot{\bar{\psi}}_j - \frac{\delta^i_j}{2}\left[\epsilon_k\dot{\psi}^k + \bar{\epsilon}^k\dot{\bar{\psi}}_k\right]\right)$$
$$+2m\left(\bar{\epsilon}^i\bar{\psi}_j - \epsilon_j\psi^i + \frac{\delta^i_j}{2}\left[\bar{\epsilon}^k\bar{\psi}_k - \epsilon_k\psi^k\right]\right). \tag{18}$$

We construct the general Lagrangian and action as

$$\mathcal{L} = -\int d^2\theta\,d^2\bar{\theta}\left(1 + 2m\,\bar{\theta}\cdot\theta\right)\,f(G)\,, \quad S = \int dt\,\mathcal{L}\,. \tag{19}$$

After doing θ integral and eliminating the auxiliary field by its equation of motion,

$$B^i_j = \frac{g'(x)}{g(x)}\left(\frac{\delta^i_j}{2}\bar{\psi}_k\psi^k - \bar{\psi}_j\psi^i\right), \qquad g(x) = f''(x), \tag{20}$$

we obtain the on-shell Lagrangian

$$\mathcal{L} = \dot{x}^2 g(x) + i\left(\bar{\psi}_i\dot{\psi}^i - \dot{\bar{\psi}}_i\psi^i\right)g(x) - \frac{1}{2}\left(\bar{\psi}_i\psi^i\right)^2\left[g''(x) - \frac{3\left(g'(x)\right)^2}{2g(x)}\right]$$
$$-m^2x^2 g(x) + 2m\bar{\psi}_i\psi^i g(x) + mx\bar{\psi}_i\psi^i g'(x). \tag{21}$$

It can be simplified by passing to the new variables $y(x)$,

$$\dot{x}^2 g(x) = \frac{1}{2}\dot{y}^2\,, \quad \Rightarrow\ y'(x) = \sqrt{2g(x)}\,, \tag{22}$$

and $\zeta^i = \psi^i y'(x)$. In terms of the new variables, the Lagrangian is rewritten as

$$\mathcal{L} = \frac{\dot{y}^2}{2} + \frac{i}{2}\left(\bar{\zeta}_i\dot{\zeta}^i - \dot{\bar{\zeta}}_i\zeta^i\right) - \frac{m^2}{2}V^2(y) + m\bar{\zeta}_i\zeta^i V'(y)$$
$$-\frac{1}{2}\left(\bar{\zeta}_i\zeta^i\right)^2\partial_y\left(\frac{V'(y)-1}{V(y)}\right). \tag{23}$$

Here, $V(y) := xy'(x) = x(y)/x'(y)$. Thus we have obtained the Lagrangian involving an arbitrary function $V(y)$. The on-shell supersymmetry transformations read

$$\delta y = \bar{\epsilon}^k\bar{\zeta}_k - \epsilon_k\zeta^k,$$
$$\delta\zeta^i = i\bar{\epsilon}^i\dot{y} - m\bar{\epsilon}^i V(y) - \left(\epsilon_k\zeta^k\zeta^i + \bar{\epsilon}^k\bar{\zeta}_k\zeta^i - \bar{\epsilon}^i\bar{\zeta}_k\zeta^k\right)\frac{V'(y)-1}{V(y)}. \tag{24}$$

These Lagrangian and the transformation laws are recognized as those defining the general SQM model with the "weak" $\mathcal{N}=4$ supersymmetry [4].

4 The $(1,4,3)$ Oscillator Model

We consider the simplest Lagrangian

$$\mathcal{L} = \frac{\dot{x}^2}{2} - \frac{m^2x^2}{2} + \frac{i}{2}\left(\bar{\psi}_i\dot{\psi}^i - \dot{\bar{\psi}}_i\psi^i\right) + m\bar{\psi}_i\psi^i, \tag{25}$$

which corresponds to the Lagrangian (21) with $f(x) = x^2/4$. The action is invariant under the transformations

$$\delta x = \left(\bar{\epsilon}\cdot\bar{\psi}\right) - (\epsilon\cdot\psi)\,, \qquad \delta\psi^i = i\bar{\epsilon}^i\dot{x} - m\bar{\epsilon}^i x\,. \tag{26}$$

The conserved Noether charges read:

$$Q^i = \psi^i\,(p - imx)\,, \qquad \bar{Q}_i = \bar{\psi}_i\,(p + imx)\,,$$
$$F = \frac{1}{2}\psi^k\bar{\psi}_k\,, \qquad I^i_j = \psi^i\bar{\psi}_j - \frac{1}{2}\delta^i_j\psi^k\bar{\psi}_k\,. \tag{27}$$

The canonical Hamiltonian is easily calculated to be

$$H=\frac{p^2}{2}+\frac{m^2x^2}{2}+m\psi^i\bar{\psi}_i\,. \tag{28}$$

It provides an $SU(2|1)$ extension of the harmonic oscillator Hamiltonian.

The Poisson (Dirac) brackets are imposed as

$$\{x,p\}=1,\qquad \{\psi^i,\bar{\psi}_j\}=-i\delta^i_j\,. \tag{29}$$

We quantize them in the standard way

$$[\hat{x},\hat{p}]=i\,,\quad \{\hat{\psi}^i,\hat{\bar{\psi}}_j\}=\delta^i_j\,,\quad \hat{p}=-i\partial_x\,,\ \hat{\bar{\psi}}_j=\partial/\partial\hat{\psi}^j\,, \tag{30}$$

and write the quantum Hamiltonian as

$$\hat{H}=\frac{1}{2}\,(\hat{p}+im\hat{x})\,(\hat{p}-im\hat{x})+m\hat{\psi}^i\hat{\bar{\psi}}_i\,. \tag{31}$$

The Hamiltonian $\hat{H}$ and the remaining quantum generators

$$\hat{Q}^i=\hat{\psi}^i\,(\hat{p}-im\hat{x})\,,\qquad \hat{\bar{Q}}_i=\hat{\bar{\psi}}_i\,(\hat{p}+im\hat{x})\,, \tag{32}$$

$$\hat{F}=\frac{1}{2}\hat{\psi}^k\hat{\bar{\psi}}_k,\qquad \hat{I}^i_j=\hat{\psi}^i\hat{\bar{\psi}}_j-\frac{1}{2}\delta^i_j\hat{\psi}^k\hat{\bar{\psi}}_k\,. \tag{33}$$

constitute the $su(2|1)$ superalgebra (5).

4.1 Wave Functions

We construct the super-wave functions of the model in terms of the harmonic oscillator wave functions. The super wave-function $\Omega^{(\ell)}$ at the energy level ℓ reveals the four-fold degeneracy

$$\Omega^{(\ell)}=a^{(\ell)}|\ell\rangle+b^{(\ell)}_i\,\psi^i|\ell-1\rangle+\frac{1}{2}\,c^{(\ell)}\,\varepsilon_{ij}\psi^i\psi^j|\ell-2\rangle,\qquad \ell\geq 2\,, \tag{34}$$

where $|\ell\rangle, |\ell-1\rangle, |\ell-2\rangle$ are the harmonic oscillator functions at the relevant levels. The operators $\hat{p}\pm imx$ in (31) and (33) are treated as the creation and annihilation operators. We impose the standard conditions

$$\hat{\bar{\psi}}_k|\ell\rangle=0,\qquad (\hat{p}-im\hat{x})\,|0\rangle=0,\qquad (\hat{p}+im\hat{x})\,|\ell\rangle=|\ell+1\rangle. \tag{35}$$

The ground state ($\ell=0$) and the first excited states ($\ell=1$) are special, they encompass non-equal numbers of bosonic and fermionic states:

$$\Omega^{(0)} = a^{(0)}\,|0\rangle\,, \qquad \Omega^{(1)} = a^{(1)}\,|1\rangle + b_i^{(1)}\,\psi^i\,|0\rangle\,. \tag{36}$$

The ground state is annihilated by all $SU(2|1)$ generators including Q^i and $\bar{Q}_i$, so it is $SU(2|1)$ singlet. The states with $\ell = 1$ form the fundamental $(\mathbf{2}|\mathbf{1})$ representation of $SU(2|1)$. The action of the supercharges on them is given by

$$\begin{aligned} &Q^i\,\psi^k|0\rangle = 0\,, \qquad \bar{Q}_i\,\psi^k|0\rangle = \delta_i^k\,|1\rangle\,, \\ &Q^i\,|1\rangle = 2m\,\psi^i|0\rangle\,, \qquad \bar{Q}_i\,|1\rangle = 0\,. \end{aligned} \tag{37}$$

The states with $\ell \geq 2$ form the representations $(\mathbf{2}|\mathbf{2})$, with equal numbers of bosonic and fermionic states.

4.2 Spectrum and $SU(2|1)$ Representations

For all states $\ell \geq 0$, the spectrum of the Hamiltonian (33) is

$$\hat{H}\,\Omega^{(\ell)} = m\,\ell\,\Omega^{(\ell)}, \qquad m > 0. \tag{38}$$

It is instructive to see which values two $SU(2|1)$ Casimir operators C_2, C_3 take on all these states. The explicit form of these operators in terms of the $SU(2|1)$ generators is as follows

$$4m^2 C_2 = C_i^i\,, \quad 12\,m^3 C_3 = 6m^3 F'(1 + 2C_2) + m\,I_k^i C_i^k\,, \tag{39}$$

where

$$F' = F - \frac{1}{m}\,H\,, \quad C_j^i = 2m^2\,\delta_j^i\,\tilde{F}^2 - m^2\left\{I_l^i, I_j^l\right\} + m\left[Q^i, \bar{Q}_j\right]. \tag{40}$$

These expressions are valid irrespective of the particular realization of the $SU(2|1)$ generators.

For our quantum-mechanical "hat" realization Casimirs are reduced to the following nice form

$$m^2 C_2 = \hat{H}\left(\hat{H} - m\right), \quad m^3 C_3 = \hat{H}\left(\hat{H} - m\right)\left(\hat{H} - \frac{m}{2}\right). \tag{41}$$

Thus they are fully specified by the number ℓ:

$$C_2(\ell) = (\ell - 1)\,\ell, \quad C_3(\ell) = (\ell - 1/2)\,(\ell - 1)\,\ell. \tag{42}$$

These values of Casimirs characterize the finite-dimensional $SU(2|1)$ representations. The eigenvalues (42) can be written in the following generic form [7]:

$$C_2 = (\beta^2 - \lambda^2), \quad C_3 = \beta(\beta^2 - \lambda^2) = \beta C_2. \tag{43}$$

The positive number λ ("highest weight") can be half-integer or integer, while the number β is an arbitrary real number. Comparing this with the above values of Casimirs in terms of ℓ, we find that $\lambda = 1/2$ for any $\Omega^{(\ell)}$ and $\beta(\ell) = (\ell - 1/2)$.

The states with $\ell = 0, 1$ are atypical $SU(2|1)$ representations, because Casimir operators take zero values on them. Indeed, $Q^i\,\Omega^{(0)} = \bar{Q}_i\,\Omega^{(0)} = 0$, i.e. $\Omega^{(0)}$ is a singlet of $SU(2|1)$. The wave functions for $\ell = 1$ form the fundamental representation of $SU(2|1)$ (one bosonic and two fermionic states). The fact that the fundamental representation of $SU(2|1)$ is atypical is well known. On the other hand, on the $\ell \geq 2$ states both Casimirs are non-zero. Correspondingly, these states form typical $(\mathbf{2}|\mathbf{2})$ representations, with two bosonic and two fermionic states.

5 The Supermultiplet (2, 4, 2)

5.1 Chiral Subspaces

One can also define $SU(2|1)$ counterpart of the $\mathcal{N} = 4, d = 1$ chiral multiplet $(\mathbf{2}, \mathbf{4}, \mathbf{2})$. This is due to the existence of the invariant chiral coset $SU(2|1)$ superspaces

$$\frac{\{Q^i, \bar{Q}_j, H, I^i_k, F\}}{\{\bar{Q}_j, I^i_k, F\}} \sim (t_L, \theta_i), \qquad \frac{\{Q^i, \bar{Q}_j, H, I^i_k, F\}}{\{Q^i, I^i_k, F\}} \sim (t_R, \theta_i), \tag{44}$$

where

$$t_L = t + \frac{i}{2m}\ln\left[1 + 2m\left(\bar{\theta}\cdot\theta\right)\right], \quad t_R = t - \frac{i}{2m}\ln\left[1 + 2m\left(\bar{\theta}\cdot\theta\right)\right]. \tag{45}$$

The chiral subspaces are closed under the supersymmetry transformations

$$\delta\theta_i = \epsilon_i + 2m\,(\bar{\epsilon}\cdot\theta)\theta_i\,, \quad \delta t_L = 2i\,(\bar{\epsilon}\cdot\theta)\,, \qquad \text{c.c.}. \tag{46}$$

The multiplet $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ is described by the chiral superfield Φ satisfying the constraints

$$\bar{\mathcal{D}}_j\Phi = 0, \qquad \tilde{I}^i_j\Phi = 0, \qquad \tilde{F}\Phi = 2\kappa\Phi, \tag{47}$$

where κ is a fixed external $U(1)$ charge. The constraints fix the structure of Φ as

$$\begin{aligned} &\Phi(t, \theta, \bar{\theta}\,) = e^{2i\kappa m(t_L - t)}\Phi_L(t_L, \theta) = \left[1 + 2m\left(\bar{\theta}\cdot\theta\right)\right]^{-\kappa}\Phi_L(t_L, \theta), \\ &\Phi_L(t_L, \theta) = z + \sqrt{2}\,\theta_i\xi^i + \varepsilon^{ij}\theta_i\theta_j B, \end{aligned} \tag{48}$$

In the central basis $\{t, \theta_i, \bar{\theta}^k\}$, the same superfield is written as

$$\Phi(t, \theta, \bar{\theta}) = z + \sqrt{2}\,\theta_i \xi^i + \varepsilon^{ij}\theta_i\theta_j B + i\left(\bar{\theta}\cdot\theta\right)\nabla_t z + \sqrt{2}\,i\left(\bar{\theta}\cdot\theta\right)\theta_i\nabla_t\xi^i$$
$$-\frac{1}{2}\left(\bar{\theta}\cdot\theta\right)^2\left[2im\nabla_t z + \nabla_t^2 z\right], \quad \nabla_t = \partial_t + 2i\kappa m. \tag{49}$$

The superfields Φ and Φ_L transform according to

$$\delta\Phi = 2\kappa m\,(\epsilon_i\bar{\theta}^i + \bar{\epsilon}^j\theta_j)\,\Phi\,, \;\Rightarrow\; \delta\Phi_L = 4\kappa m\,\bar{\epsilon}^j\theta_j\,\Phi_L\,. \tag{50}$$

For the component fields, this implies

$$\delta z = -\sqrt{2}\,\epsilon_i\xi^i, \qquad \delta\xi^i = \sqrt{2}\,i\bar{\epsilon}^i\nabla_t z - \sqrt{2}\,\varepsilon^{ik}\epsilon_k B,$$
$$\delta B = -\sqrt{2}\,\varepsilon_{ik}\bar{\epsilon}^k\left[m\xi^i + i\nabla_t\xi^i\right]. \tag{51}$$

5.2 $SU(2|1)$ *Invariant Lagrangian*

General superfield Lagrangian is constructed as

$$\mathcal{L}_k = \frac{1}{4}\int d^2\theta\, d^2\bar{\theta}\,(1 + 2m\,\bar{\theta}\cdot\theta)\, f(\Phi, \Phi^\dagger), \tag{52}$$

where $f\left(\Phi, \Phi^\dagger\right)$ is the Kähler potential. This is a direct analog of the kinetic term defined in the standard $\mathcal{N} = 4$ mechanics based on the multiplet $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ [8]. After eliminating the complex auxiliary field B by its equation of motion,

$$B = -\frac{1}{2g}\varepsilon_{kl}\xi^k\xi^l g_z\,, \tag{53}$$

the Lagrangian takes the following on-shell form[2]

$$\mathcal{L} = g\dot{\bar{z}}\dot{z} + 2i\kappa m\left(\dot{\bar{z}}z - \dot{z}\bar{z}\right)g - \frac{im}{2}\left(\dot{\bar{z}}f_{\bar{z}} - \dot{z}f_z\right) - \frac{i}{2}\left(\bar{\xi}\cdot\xi\right)\left(\dot{\bar{z}}g_{\bar{z}} - \dot{z}g_z\right)$$
$$+\frac{i}{2}\left(\bar{\xi}_i\dot{\xi}^i - \dot{\bar{\xi}}_i\xi^i\right)g - m^2 V - m\left(\bar{\xi}\cdot\xi\right)U + \frac{1}{2}\left(\bar{\xi}\cdot\xi\right)^2 R\,, \tag{54}$$

where

$$V = \kappa\left(\bar{z}\partial_{\bar{z}} + z\partial_z\right)f - \kappa^2\left(\bar{z}\partial_{\bar{z}} + z\partial_z\right)^2 f,$$
$$U = \kappa\left(\bar{z}\partial_{\bar{z}} + z\partial_z\right)g - (1 - 2\kappa)\,g,$$

[2] Here, the lower indices mean the differentiation in $z, \bar{z}$: $f_z = \partial_z f$, $f_{\bar{z}} = \partial_{\bar{z}} f$, etc.

$$R = g_{z\bar{z}} - \frac{g_z g_{\bar{z}}}{g}\,, \tag{55}$$
$$f = f(z,\bar{z}), \qquad g = g(z,\bar{z}) = \partial_z\partial_{\bar{z}} f(z,\bar{z})\,.$$

The physical fields $(z, \bar{z}, \eta^i, \bar{\eta}_j)$ transform as

$$\delta z = -\sqrt{2}\,\epsilon_i \xi^i\,, \qquad \delta\xi^i = \sqrt{2}\,i\bar{\epsilon}^i\,(\partial_t + 2i\kappa m)\,z + \sqrt{2}\,\epsilon_k\xi^k\xi^i\,\frac{g_z}{g}. \tag{56}$$

The bosonic Lagrangian has the form

$$\mathcal{L} = g\dot{\bar{z}}\dot{z} + 2i\kappa m\left(\dot{\bar{z}}z - \dot{z}\bar{z}\right)g - \frac{im}{2}\left(\dot{\bar{z}}f_{\bar{z}} - \dot{z}f_z\right) - m^2 V\,. \tag{57}$$

Thus the standard $\mathcal{N} = 4, d = 1$ kinetic term is deformed to non-trivial Lagrangian with WZ-term, and potential term. The latter vanishes for $\kappa = 0$, while the WZ term vanishes only in the limit $m = 0$. So, the basic novel point compared to the standard $\mathcal{N} = 4$ Kähler sigma model for the multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ is the necessary presence of the WZ term with the strength m, alongside with the standard Kähler kinetic term.

5.3 *Quantum Generators*

To construct the quantum supercharges, we resort to the procedure worked out in [9]. Its basic point is the Weyl-ordering of the bosonic and fermionic operators in the classical Noether supercharges: the quantum Hamiltonian is then defined as the anticommutator of these ordered supercharges. As the first step, we impose Poisson (Dirac) brackets

$$\{z, p_z\} = 1, \qquad \{\xi^i, \bar{\xi}_j\} = -i\,\delta^i_j\, g^{-1}\,. \tag{58}$$

It is convenient to make the substitution

$$\left(z, \xi^i\right) \longrightarrow \left(z, \eta^i\right), \qquad \eta^i = g^{\frac{1}{2}}\xi^i\,,$$
$$\{z, p_z\} = 1, \qquad \{\eta^i, \bar{\eta}_j\} = -i\,\delta^i_j\,, \qquad \{p_z, \eta^i\} = \{p_z, \bar{\eta}_j\} = 0\,. \tag{59}$$

The standard way of quantization then implies

$$[\hat{z}, \hat{p}_z] = i\,, \quad \{\hat{\eta}^i, \hat{\bar{\eta}}_j\} = \delta^i_j\,, \quad [\hat{p}_z, \hat{\eta}^i] = [\hat{p}_z, \hat{\bar{\eta}}_j] = 0\,,$$
$$\hat{p}_z = -i\,\partial_z\,, \qquad \hat{\bar{\eta}}_j = \frac{\partial}{\partial\hat{\eta}^j}. \tag{60}$$

Using these relations, we obtain

$$\hat{Q}^i = \sqrt{2}\,\hat{\eta}^i\, g^{-\frac{1}{2}}\nabla_z\,, \qquad \hat{\bar{Q}}_j = \sqrt{2}\,\hat{\bar{\eta}}_j\, g^{-\frac{1}{2}}\bar{\nabla}_{\bar{z}}\,,$$
$$\hat{F} = -2\kappa\left(\hat{z}\partial_z - \hat{\bar{z}}\partial_{\bar{z}}\right) - \left(2\kappa - \frac{1}{2}\right)\hat{\eta}^k\,\hat{\bar{\eta}}_k\,, \qquad \hat{I}^i_j = \hat{\eta}^i\,\hat{\bar{\eta}}_j - \frac{1}{2}\delta^i_j\,\hat{\eta}^k\,\hat{\bar{\eta}}_k\,. \quad (61)$$

These operators satisfy the $su(2|1)$ superalgebra with the following quantum Hamiltonian

$$\hat{H} = g^{-1}\bar{\nabla}_{\bar{z}}\,\nabla_z - 2\kappa m\left(\hat{z}\partial_z - \hat{\bar{z}}\partial_{\bar{z}}\right) + m\,(1-2\kappa)\,\hat{\eta}^k\,\hat{\bar{\eta}}_k - \frac{1}{4}\,g^{-2}R\,\hat{\eta}_k\,\hat{\eta}^k\,\hat{\bar{\eta}}^i\,\hat{\bar{\eta}}_i\,. \quad (62)$$

Here, we used the relation

$$\left[\nabla_z, \bar{\nabla}_{\bar{z}}\right] = mg - \frac{1}{2}\,g^{-1}R\left(\hat{\eta}^k\,\hat{\bar{\eta}}_k - \hat{\bar{\eta}}_k\,\hat{\eta}^k\right), \quad (63)$$

where

$$\nabla_z = -i\,\partial_z - \frac{i}{2}m\partial_z f + \frac{i}{4}g^{-1}\partial_z g\left[\hat{\eta}^k, \hat{\bar{\eta}}_k\right],$$
$$\bar{\nabla}_{\bar{z}} = -i\,\partial_{\bar{z}} + \frac{i}{2}m\partial_{\bar{z}} f - \frac{i}{4}g^{-1}\partial_{\bar{z}} g\left[\hat{\eta}^k, \hat{\bar{\eta}}_k\right]. \quad (64)$$

6 Simplified Model on a Complex Plane

The model on a plane corresponds to the simplest choice $f\left(\Phi, \Phi^\dagger\right) = \Phi\Phi^\dagger$ in (54). It leads to the Lagrangian

$$\mathcal{L} = \dot{\bar{z}}\dot{z} + im\left(2\kappa - \frac{1}{2}\right)\left(\dot{\bar{z}}z - \dot{z}\bar{z}\right) + \frac{i}{2}\left(\bar{\xi}_i\dot{\xi}^i - \dot{\bar{\xi}}_i\xi^i\right)$$
$$+ 2\kappa\,(2\kappa - 1)\,m^2\bar{z}z + (1-2\kappa)\,m\left(\bar{\xi}\cdot\xi\right). \quad (65)$$

This Lagrangian is invariant under the transformations

$$\delta z = -\sqrt{2}\,\epsilon_i\,\xi^i, \qquad \delta\xi^i = \sqrt{2}\,i\bar{\epsilon}^i\,\dot{z} - 2\sqrt{2}\,\kappa m\bar{\epsilon}^i\,z. \quad (66)$$

The quantum Hamiltonian reads

$$\hat{H} = \bar{\nabla}_{\bar{z}}\,\nabla_z - 2\kappa m\left(\hat{z}\partial_z - \hat{\bar{z}}\partial_{\bar{z}}\right) + m\,(1-2\kappa)\,\hat{\eta}^k\,\hat{\bar{\eta}}_k \quad (67)$$

and forms, together with the quantum operators

$$\hat{Q}^i = \sqrt{2}\,\hat{\eta}^i \nabla_z\,, \qquad \hat{\bar{Q}}_j = \sqrt{2}\,\hat{\bar{\eta}}_j \bar{\nabla}_{\bar{z}}\,,$$
$$\hat{F} = -2\kappa\left(\hat{z}\partial_z - \hat{\bar{z}}\partial_{\bar{z}}\right) - \left(2\kappa - \frac{1}{2}\right)\hat{\eta}^k \hat{\bar{\eta}}_k\,, \qquad \hat{I}^i_j = \hat{\eta}^i \hat{\bar{\eta}}_j - \frac{1}{2}\delta^i_j \hat{\eta}^k \hat{\bar{\eta}}_k\,, \quad (68)$$

the $su(2|1)$ superalgebra (5). Here,

$$\nabla_z = -i\,\partial_z - \frac{i}{2}m\bar{z}, \quad \bar{\nabla}_{\bar{z}} = -i\,\partial_{\bar{z}} + \frac{i}{2}mz, \qquad \left[\nabla_z, \bar{\nabla}_{\bar{z}}\right] = m\,. \quad (69)$$

Note that the sum $\hat{H} - m\hat{F}$ does not depend on κ. This is a consequence of the fact that the supercharges do not depend on κ. Since the anticommutator $\{\hat{Q}, \hat{\bar{Q}}\}$ yields only the combination $\hat{H} - m\hat{F}$, without any specific splitting, the super wave functions do not contain κ-dependent terms in their η-expansions too.

6.1 Wave Functions and Spectrum

We will make use of the fact that there exists an extra $U(1)$ charge generator,

$$\hat{E} = -\left(\hat{z}\partial_z - \hat{\bar{z}}\partial_{\bar{z}}\right) - \hat{\eta}^k \hat{\bar{\eta}}_k\,, \quad (70)$$

which commutes with all $SU(2|1)$ generators, including $\hat{H}$. Hence we can construct the relevant wave functions in terms of the set of bosonic eigenfunctions of this external generator

$$\hat{E}\,\Omega = n\,\Omega, \qquad \hat{H}\,\Omega = \mathcal{E}\,\Omega = mq\,\Omega. \quad (71)$$

The first equation yields

$$\Omega = \bar{z}^n A\,(w)\,, \qquad w \equiv z\bar{z}\,. \quad (72)$$

Then the second equation amounts to the following one for $A\,(w)$:

$$\left[-w\partial_w^2 - (1+n)\partial_w + \frac{m^2}{4}w - \frac{m}{2}\right]A\,(w) = m\left(q - 2\kappa n + \frac{n}{2}\right)\,A\,(w)\,. \quad (73)$$

Solving the latter and combining it with the first solution (72), the eigenvalue problem for $\hat{H}$ is rewritten as

$$\hat{H}\,\Omega^{(\ell;n)} = \mathcal{E}^{(\ell;n)}\,\Omega^{(\ell;n)}\,, \quad (74)$$

with $\mathcal{E}^{(\ell;n)} = m(\ell + 2\kappa n)$. It is solved by

$$\Omega^{(\ell;n)} = \bar{z}^n e^{-\frac{mz\bar{z}}{2}} L_\ell^{(n)}(mz\bar{z}) = \frac{z^{-n}}{\ell!} e^{\frac{mz\bar{z}}{2}} \frac{d^\ell}{dw^\ell} \left(e^{-mw} w^{n+\ell}\right)\Big|_{w=z\bar{z}}. \tag{75}$$

The bosonic functions $L_\ell^{(n)}(mz\bar{z})$ are the generalized Laguerre polynomials, with ℓ a non-negative integer, $\ell \geq 0$. The number n is also integer and it takes the values $n \geq -\ell$, due to the orthogonality of the eigenfunctions $\Omega^{(\ell;n)}$. This means that the energies $\mathcal{E}^{(\ell;n)}$ are positive only under the following restriction on the parameter κ:

$$0 \leq \kappa \leq 1/2\,. \tag{76}$$

Acting by the supercharges on $\Omega^{(\ell;n)}$ and imposing the obvious physical condition,

$$\bar{\eta}_j\, \Omega^{(\ell;n)} = 0 \;\Rightarrow\; \bar{Q}_j\, \Omega^{(\ell;n)} = 0\,, \tag{77}$$

we obtain other eigenstates of $\hat{H}$ and $\hat{E}$. The full set of eigenfunctions obtained in this way reads:

$$\begin{aligned}
\Psi^{(\ell;n)} &= a^{(\ell;n)}\Omega^{(\ell;n)} + b_i^{(\ell;n)}\eta^i\, \Omega^{(\ell-1;n+1)} \\
&\qquad + \frac{1}{2}\, c^{(\ell;n)}\, \varepsilon_{ij}\eta^i\eta^j\, \Omega^{(\ell-2;n+2)}, \quad \ell \geq 2, \\
\Psi^{(1;n)} &= a^{(1;n)}\, \Omega^{(1;n)} + b_i^{(1;n)}\eta^i\, \Omega^{(0;n+1)}, \\
\Psi^{(0;n)} &= a^{(0;n)}\, \Omega^{(0;n)}\,.
\end{aligned} \tag{78}$$

These super wave functions span the full Hilbert space of quantum states of the model. The eigenvalues of $\hat{E}$ and $\hat{H}$ are given by

$$\hat{E}\,\Psi^{(\ell;n)} = n\,\Psi^{(\ell;n)}, \quad \hat{H}\,\Psi^{(\ell;n)} = \mathcal{E}^{(\ell;n)}\,\Psi^{(\ell;n)}\,, \quad \mathcal{E}^{(\ell;n)} = m\,(2\kappa n + \ell) \tag{79}$$

We observe that the ground state ($\ell = 0$) and the first excited states ($\ell = 1$) are special, in the sense that they comprise non-equal numbers of bosonic and fermionic states. Indeed,

$$Q^i\, \Omega^{(0;n)} = \bar{Q}_i\, \Omega^{(0;n)} = 0, \tag{80}$$

i.e. $\Omega^{(0;n)}$ is a singlet of $SU(2|1)$ for any n. The wave functions for $\ell = 1$ form the fundamental representation of $SU(2|1)$ (one bosonic and two fermionic states), while those for $\ell \geq 2$ form the typical (2|2) representations.

Casimir operators (39)–(40) for the considered model can be expressed through the operators $\hat{H}$ and $\hat{E}$:

$$m^2 C_2 = \left(\hat{H} - 2\kappa m \hat{E}\right)\left(\hat{H} - 2\kappa m \hat{E} - m\right), \tag{81}$$

$$m^3 C_3 = \left(\hat{H} - 2\kappa m \hat{E}\right)\left(\hat{H} - 2\kappa m \hat{E} - m\right)\left(\hat{H} - 2\kappa m \hat{E} - \frac{m}{2}\right). \tag{82}$$

For the quantum states they do not depend on the additional parameter κ and in fact have the same form (42) as for the $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ model

$$C_2(\ell) = (\ell - 1)\,\ell, \quad C_3(\ell) = (\ell - 1/2)\,(\ell - 1)\,\ell, \quad \beta(\ell) = (\ell - 1/2). \tag{83}$$

Thus they are vanishing for the wave functions with $\ell = 0, 1$, confirming the interpretation of the corresponding representations as atypical, and are non-vanishing on the wave functions with $\ell \geq 2$, implying them to form typical representations of $SU(2|1)$.

7 Summary and Outlook

We constructed the new type of $\mathcal{N} = 4$ supersymmetric mechanics which is based on the supergroup $SU(2|1)$. It is a deformation of the standard $\mathcal{N} = 4$ mechanics by a mass parameter m. These models are expected to be related to the rigid supersymmetric models in higher-dimensional curved superspaces.

We constructed the superfield formalism for two different coset manifolds of $SU(2|1)$ treated as the real and chiral $SU(2|1), d = 1$ superspaces. The corresponding SQM models are built on the off-shell multiplets $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ and $(\mathbf{2}, \mathbf{4}, \mathbf{2})$. The relevant Lagrangians were presented and the quantization was explicitly performed for some particular cases. The SQM models with $\mathcal{N} = 4$ "weak supersymmetry" [4] are easily reproduced from our superfield approach as those associated with the $SU(2|1)$ multiplet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$.

The models constructed reveal surprising features. For the $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ multiplet, the kinetic term of the physical bosonic field is inevitably accompanied by the generalized oscillator-type mass term with m playing the role of mass. For the $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ models, the kinetic term is accompanied by the $d = 1$ WZ term with the strength $\sim m$. In both cases the spaces of the quantum states reveal deviations from the standard rule of equality of the bosonic and fermionic states, in accordance with the existence of atypical $SU(2|1)$ representations.

There are some further lines of development which will be pursued in the future:

- Multi-particle extensions: to take a few superfields of one or different types, to construct the relevant off- and on-shell actions, to quantize, to identify the relevant target bosonic geometries (m-deformed?), etc.
- To inquire whether other $\mathcal{N} = 4, d = 1$ multiplets (e.g. the multiplet $(\mathbf{3}, \mathbf{4}, \mathbf{1})$) have their $SU(2|1)$ counterparts and to construct the corresponding

SQM models. In this connection, it would be useful to define some other coset $SU(2|1)$ superspaces. For instance, there exists the coset

$$\frac{\{Q^i, \bar{Q}_j, H, I^i_j, F\}}{\{Q^1, \bar{Q}_2, F, I^1_2, I^1_1 = -I^2_2\}} \sim \{Q^2, \bar{Q}_1, H, I^2_1\}, \tag{84}$$

which is none other than $SU(2|1)$ analog of the analytic harmonic $\mathcal{N} = 4, d = 1$ superspace [10]. The latter superspace is the carrier of the “root” $\mathcal{N} = 4, d = 1$ multiplet **(4, 4, 0)** from which all other $\mathcal{N} = 4, d = 1$ multiplets can be deduced, following the well defined procedure [11]. Thus the similar root multiplet can be defined in the $SU(2|1)$ case too.

- To generalize all this to the next in complexity case of the supergroup $SU(2|2)$. It involves eight supercharges and so can be treated as a deformation of $\mathcal{N} = 8, d = 1$ supersymmetry (and of $\mathcal{N} = (4, 4), d = 2$ supersymmetry, in fact).

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Vector-Valued Covariant Differential Operators for the Möbius Transformation

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Abstract We obtain a family of functional identities satisfied by vector-valued functions of two variables and their geometric inversions. For this we introduce particular differential operators of arbitrary order attached to Gegenbauer polynomials. These differential operators are symmetry breaking for the pair of Lie groups $(SL(2,\mathbb{C}), SL(2,\mathbb{R}))$ that arise from conformal geometry.

Keywords Branching law • Reductive Lie group • Symmetry breaking • Conformal geometry • Verma module • F-method • Time reversal operator

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1 A Family of Vector-Valued Functional Identities

Given a pair of functions f, g on $\mathbb{R}^2\setminus\{(0,0)\}$, we consider a $\mathbb{C}^2$-valued function $\mathbf{F} := \begin{pmatrix} f \\ g \end{pmatrix}$. Define its "twisted inversion" $\mathbf{F}_\lambda$ with parameter $\lambda \in \mathbb{C}$ by

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$$\mathbf{F}_\lambda^\vee(r\cos\theta, r\sin\theta) := r^{-2\lambda}\begin{pmatrix}\cos 2\theta & -\sin 2\theta\\ \sin 2\theta & \cos 2\theta\end{pmatrix}\mathbf{F}\left(\frac{-\cos\theta}{r}, \frac{\sin\theta}{r}\right). \tag{1}$$

Clearly, $\mathbf{F} \mapsto \mathbf{F}_\lambda^\vee$ is involutive, namely, $(\mathbf{F}_\lambda^\vee)_\lambda^\vee = \mathbf{F}$.

A pair of differential operators $\mathcal{D}_1, \mathcal{D}_2$ on $\mathbb{R}^2$ yields a linear map

$$\mathcal{D} : C^\infty(\mathbb{R}^2) \oplus C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}), \quad (f, g) \mapsto (\mathcal{D}_1 f)(x, 0) + (\mathcal{D}_2 g)(x, 0).$$

We write

$$\mathcal{D} := \mathrm{Rest}_{y=0} \circ (\mathcal{D}_1, \mathcal{D}_2).$$

Our main concern in this article is the following:

Question A *(1) For which parameters $\lambda, \nu \in \mathbb{C}$, do there exist differential operators $\mathcal{D}_1$ and $\mathcal{D}_2$ on $\mathbb{R}^2$ with the following properties?*

- *$\mathcal{D}_1$ and $\mathcal{D}_2$ have constant coefficients.*
- *For any $\mathbf{F} \in C^\infty(\mathbb{R}^2) \oplus C^\infty(\mathbb{R}^2)$, the functional identity*

$$(\mathcal{D}\mathbf{F}_\lambda^\vee)(x) = |x|^{-2\nu}(\mathcal{D}\mathbf{F})\left(-\frac{1}{x}\right), \quad \textit{for } x \in \mathbb{R}^\times \tag{$\mathcal{M}_{\lambda,\nu}$}$$

holds, where $\mathcal{D} = \mathrm{Rest}_{y=0} \circ (\mathcal{D}_1, \mathcal{D}_2)$.

(2) Find an explicit formula of such $\mathcal{D} \equiv \mathcal{D}_{\lambda,\nu}$ if exists.

Our motivation will be explained in Sect. 2 by giving three equivalent formulations of Question A. Here are some examples of the operators $\mathcal{D}_{\lambda,\nu}$ satisfying ($\mathcal{M}_{\lambda,\nu}$).

Example 1. (0) $\nu = \lambda$:

$$\mathcal{D}_{\lambda,\nu} := \mathrm{Rest}_{y=0} \circ (\mathrm{id}, 0),$$

namely,

$$\mathcal{D}_{\lambda,\nu}\begin{pmatrix} f\\ g\end{pmatrix}(x) = f(x, 0)$$

satisfies ($\mathcal{M}_{\lambda,\nu}$) for $\nu = \lambda$.

(1) $\nu = \lambda + 1$:

$$\mathcal{D}_{\lambda,\nu} := \mathrm{Rest}_{y=0} \circ \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right),$$

namely,

$$\mathcal{D}_{\lambda,\nu}\begin{pmatrix} f\\ g\end{pmatrix}(x) = \frac{\partial f}{\partial x}(x, 0) + \lambda\frac{\partial g}{\partial y}(x, 0)$$

satisfies ($\mathcal{M}_{\lambda,\nu}$) for $\nu = \lambda + 1$.

(2) $\nu = \lambda + 2$:

$$\mathcal{D}_{\lambda,\nu} := \mathrm{Rest}_{y=0} \circ \left(2(2\lambda+1)\frac{\partial^2}{\partial x \partial y}, (\lambda-1)\frac{\partial^2}{\partial x^2} + (\lambda+1)(2\lambda+1)\frac{\partial^2}{\partial y^2}\right),$$

namely,

$$\mathcal{D}_{\lambda,\nu}\begin{pmatrix} f \\ g \end{pmatrix}(x) = 2(2\lambda+1)\frac{\partial^2 f}{\partial x \partial y}(x,0) + (\lambda-1)\frac{\partial^2 g}{\partial x^2}(x,0) + (\lambda+1)(2\lambda+1)\frac{\partial^2 g}{\partial y^2}(x,0)$$

satisfies ($\mathcal{M}_{\lambda,\nu}$) for $\nu = \lambda + 2$.

Given $\mathcal{D} = \mathrm{Rest}_{y=0} \circ (\mathcal{D}_1, \mathcal{D}_2)$, define

$$\mathcal{D}^\vee := \mathrm{Rest}_{y=0} \circ (-\mathcal{D}_2, \mathcal{D}_1). \tag{2}$$

Clearly, $\mathcal{D}^\vee$ is determined only by $\mathcal{D}$, and is independent of the choice of $\mathcal{D}_1$ and $\mathcal{D}_2$. Proposition 1 below shows that the map $\mathcal{D} \mapsto \mathcal{D}^\vee$ is an automorphism of the set of the operators $\mathcal{D}$ such that ($\mathcal{M}_{\lambda,\nu}$) is satisfied.

Proposition 1. *If $\mathcal{D}$ satisfies ($\mathcal{M}_{\lambda,\nu}$) for all* $\mathbf{F}$*, so does $\mathcal{D}^\vee$.*

Proof. For $\mathbf{F} = \begin{pmatrix} f \\ g \end{pmatrix}$, we set ${}^\vee\mathbf{F} := \begin{pmatrix} g \\ -f \end{pmatrix}$. Then we have

$$^{\vee\vee}\mathbf{F} = -\mathbf{F}, \quad \mathcal{D}\left({}^\vee\mathbf{F}\right) = (\mathcal{D}^\vee)\mathbf{F}, \quad ({}^\vee\mathbf{F})^\vee_\lambda = {}^\vee(\mathbf{F}^\vee_\lambda). \tag{3}$$

To see this we note that $w := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ commutes with $\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$ and that $\mathcal{D}^\vee$ and ${}^\vee\mathbf{F}$ are expressed as $\mathcal{D}^\vee = \mathcal{D}w^{-1}$ and ${}^\vee\mathbf{F} = w^{-1}\mathbf{F}$, respectively. Therefore,

$$\begin{aligned} \left(\mathcal{D}^\vee \mathbf{F}^\vee_\lambda\right)(x) &= \mathcal{D}\left({}^\vee\mathbf{F}\right)^\vee_\lambda(x) \\ &= |x|^{-2\nu}\left(\mathcal{D}\,{}^\vee\mathbf{F}\right)\left(-\frac{1}{x}\right) \\ &= |x|^{-2\nu}\left(\mathcal{D}^\vee\,\mathbf{F}\right)\left(-\frac{1}{x}\right), \end{aligned}$$

where the passage from the first line to the second one is justified by the fact that ${}^\vee\mathbf{F}$ satisfies ($\mathcal{M}_{\lambda,\nu}$).

In order to answer Question A for general (λ, ν), we recall that the Gegenbauer polynomial or ultraspherical polynomial $C^\alpha_\ell(t)$ is a polynomial in one variable t of degree ℓ given by

$$C^\alpha_\ell(t) = \sum_{k=0}^{[\frac{\ell}{2}]}(-1)^k \frac{\Gamma(\ell-k+\alpha)}{\Gamma(\alpha)\Gamma(\ell-2k+1)k!}(2t)^{\ell-2k},$$

where $[s]$ denotes the greatest integer that does not exceed s. Following [9], we inflate $C_\ell^\alpha(t)$ to a polynomial of two variables by

$$C_\ell^\alpha(s,t) := s^{\frac{\ell}{2}} C_\ell^\alpha\left(\frac{t}{\sqrt{s}}\right). \tag{4}$$

By formally substituting $-\frac{\partial^2}{\partial x^2}$ and $\frac{\partial}{\partial y}$ to s and t in $C_\ell^\alpha(s,t)$, respectively, we obtain a homogeneous differential operator $\mathcal{C}_\ell^\alpha := C_\ell^\alpha\left(-\frac{\partial^2}{\partial x^2}, \frac{\partial}{\partial y}\right)$ of order ℓ on $\mathbb{R}^2$. Here are the first four operators:

$$\begin{aligned}
\mathcal{C}_0^\alpha &= \mathrm{id}, \\
\mathcal{C}_1^\alpha &= 2\alpha \tfrac{\partial}{\partial y}, \\
\mathcal{C}_2^\alpha &= \alpha\left(\tfrac{\partial^2}{\partial x^2} + 2(\alpha+1)\tfrac{\partial^2}{\partial y^2}\right), \\
\mathcal{C}_3^\alpha &= \tfrac{2}{3}\alpha(\alpha+1)\left(3\tfrac{\partial^3}{\partial x^2 \partial y} + 2(\alpha+2)\tfrac{\partial^3}{\partial y^3}\right).
\end{aligned}$$

Theorem A *Suppose that $a := \nu - \lambda$ is a non-negative integer. For $a > 0$, we define the following pair of homogeneous differential operators of order a on $\mathbb{R}^2$ by*

$$\mathcal{D}_1 := a(2\lambda + a - 1)\frac{\partial}{\partial x} \circ \mathcal{C}_{a-1}^{\lambda+\frac{1}{2}}$$

$$\mathcal{D}_2 := \left(2\lambda^2 + 2(a-1)\lambda + a(a-1)\right)\frac{\partial}{\partial y} \circ \mathcal{C}_{a-1}^{\lambda+\frac{1}{2}} + (\lambda-1)(2\lambda+1)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \circ \mathcal{C}_{a-2}^{\lambda+\frac{3}{2}}.$$

For $a = 0$, we set

$$\mathcal{D}_1 := \mathrm{id}, \quad \mathcal{D}_2 := 0.$$

Then $\mathcal{D} := \mathrm{Rest}_{y=0} \circ (\mathcal{D}_1, \mathcal{D}_2)$ and $\mathcal{D}^\vee := \mathrm{Rest}_{y=0} \circ (-\mathcal{D}_2, \mathcal{D}_1)$ satisfy the functional identity $(\mathcal{M}_{\lambda,\nu})$. Moreover, when $2\lambda \notin \{0, -1, -2, \cdots\}$, there exists a non-trivial solution to $(\mathcal{M}_{\lambda,\nu})$ only if $\nu - \lambda$ is a non-negative integer and any differential operator satisfying $(\mathcal{M}_{\lambda,\nu})$ is a linear combination of $\mathcal{D}$ and $\mathcal{D}^\vee$.

Notation: $\mathbb{N} := \{0, 1, 2, \ldots\}$,
$\mathbb{N}_+ := \{1, 2, \ldots\}$.

2 Three Equivalent Formulations

Question A arises from various disciplines of mathematics. In this section we describe it in three equivalent ways.

2.1 Covariance of $SL(2,\mathbb{R})$ for Vector-Valued Functions

For $\lambda \in \mathbb{C}$, we define a group homomorphism

$$\psi_\lambda : \mathbb{C}^\times \to GL(2,\mathbb{R}), \quad z = re^{i\theta} \mapsto r^\lambda \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{5}$$

For a $\mathbb{C}^2$-valued function $\mathbf{F}$ on $\mathbb{C} \simeq \mathbb{R}^2$, we set

$$\mathbf{F}_\lambda^h(z) := \psi_\lambda\left((cz+d)^{-2}\right)\mathbf{F}\left(\frac{az+b}{cz+d}\right)$$

for $\lambda \in \mathbb{C}$, $h^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$, and $z \in \mathbb{C}$ such that $cz+d \neq 0$.

Question A$'$ *(1) Determine complex parameters $\lambda, \nu \in \mathbb{C}$ for which there exist differential operators $\mathcal{D}_1$ and $\mathcal{D}_2$ on $\mathbb{R}^2$ with the following property: $\mathcal{D} = \mathrm{Rest}_{y=0} \circ (\mathcal{D}_1, \mathcal{D}_2)$ satisfies*

$$(\mathcal{D}\mathbf{F}_\lambda^h)(x) = |cx+d|^{-2\nu}(\mathcal{D}\mathbf{F})\left(\frac{ax+b}{cx+d}\right) \tag{6}$$

for all $\mathbf{F} \in C^\infty(\mathbb{C}) \oplus C^\infty(\mathbb{C})$, $h^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$, and $x \in \mathbb{R} \setminus \{-\frac{d}{c}\}$.

(2) Find an explicit formula of such $\mathcal{D} \equiv \mathcal{D}_{\lambda,\nu}$.

The equivalence between Questions A and A$'$ follows from the following three observations:

- The functional identity (6) for $h = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ $(t \in \mathbb{R})$ implies that $\mathcal{D} = \mathrm{Rest}_{y=0} \circ (\mathcal{D}_1, \mathcal{D}_2)$ is a translation invariant operator. Therefore, we can take $\mathcal{D}_1$ and $\mathcal{D}_2$ to have constant coefficients.
- $\mathbf{F}_\lambda^\vee = \mathbf{F}_\lambda^w$.
- The group $SL(2,\mathbb{R})$ is generated by w and $\left\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R}\right\}$.

2.2 Conformally Covariant Differential Operators

Let X be a smooth manifold equipped with a Riemannian metric g. Suppose that a group G acts on X by the map $G \times X \to X$, $(h,x) \mapsto h \cdot x$. This action is called *conformal* if there is a positive-valued smooth function (*conformal factor*) Ω on $G \times X$ such that

$$h^*(g_{h\cdot x}) = \Omega(h,x)^2 g_x \quad \text{for any } h \in G \text{ and } x \in X.$$

Given $\lambda \in \mathbb{C}$, we define a G-equivariant line bundle $\mathcal{L}_\lambda \equiv \mathcal{L}_\lambda^{\mathrm{conf}}$ over X by letting G act on the direct product $X \times \mathbb{C}$ by $(x, u) \mapsto (h \cdot x, \Omega(h, x)^{-\lambda} u)$ for $h \in G$. Then we have a natural action of G on the vector space $\mathcal{E}_\lambda(X) := C^\infty(X, \mathcal{L}_\lambda)$ consisting of smooth sections for $\mathcal{L}_\lambda$. Since $\mathcal{L}_\lambda \to X$ is topologically a trivial bundle, we may identify $\mathcal{E}_\lambda(X)$ with $C^\infty(X)$, and corresponding G-action on $C^\infty(X)$ is given as the multiplier representation $\varpi_\lambda \equiv \varpi_\lambda^X$:

$$(\varpi_\lambda(h) f)(x) = \Omega(h^{-1}, x)^\lambda f(h^{-1} \cdot x) \quad \text{for } h \in G \text{ and } f \in C^\infty(X).$$

See [7] for the basic properties of the representation $(\varpi_\lambda, C^\infty(X))$.

Example 2. We endow $\mathbb{P}^1\mathbb{C} \simeq \mathbb{C} \cup \{\infty\}$ with a Riemannian metric g via the stereographic projection of the unit sphere S^2:

$$\mathbb{R}^3 \supset S^2 \xrightarrow{\sim} \mathbb{C} \cup \{\infty\}, \quad (p, q, r) \mapsto \frac{p + \sqrt{-1} q}{1 + r}.$$

Then $g(u, v) = \frac{4}{(1+|z|^2)^2} (u, v)_{\mathbb{R}^2}$ for $u, v \in T_z\mathbb{C} \simeq \mathbb{R}^2$, and the Möbius transformation, defined by

$$\mathbb{P}^1\mathbb{C} \to \mathbb{P}^1\mathbb{C}, \quad z \mapsto g \cdot z = \frac{az + b}{cz + d} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}),$$

is conformal with conformal factor

$$\Omega(g, z) = |cz + d|^{-2}. \tag{7}$$

Therefore,

$$(\varpi_\lambda(h) f)(z) = |cz + d|^{-2\lambda} f\left(\frac{az + b}{cz + d}\right) \quad \text{for } h^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This is a (non-unitary) spherical principal series representation $\mathrm{Ind}_{B_\mathbb{C}}^{G_\mathbb{C}}(1 \otimes \lambda\alpha \otimes 1)$ of $G_\mathbb{C} = SL(2, \mathbb{C})$, where α is the unique positive restricted root which defines a Borel subgroup $B_\mathbb{C}$.

Let $\wedge^i T^*X$ be the ith exterior power of the cotangent bundle T^*X for $0 \leq i \leq n$, where n is the dimension of X. Then sections ω for $\wedge^i T^*X$ are ith differential forms on X, and G acts on $\mathcal{E}^i(X) = C^\infty(X, \wedge^i T^*X)$ as the pull-back of differential forms:

$$\varpi(h)\omega = (h^{-1})^* \omega \quad \text{for } \omega \in \mathcal{E}^i(X).$$

More generally, the tensor bundle $\mathcal{L}_\lambda \otimes \wedge^i T^*X$ is also a G-equivariant vector bundle over X, and we denote by $\varpi_{\lambda,i}^X$ the regular representation of G on the space of sections

$$\mathcal{E}^i_\lambda(X) := C^\infty(X, \mathcal{L}_\lambda \otimes \wedge^i T^* X).$$

By definition $\mathcal{E}^0_\lambda(X) = \mathcal{E}_\lambda(X)$. In our normalization we have a natural G-isomorphism

$$\mathcal{E}^0_n(X) \simeq \mathcal{E}^n_0(X),$$

if X admits a G-invariant orientation.

Denote by $\mathrm{Conf}(X)$ the full group of conformal transformations of the Riemannian manifold (X, g). Given a submanifold Y of X, we define a subgroup by

$$\mathrm{Conf}(X; Y) := \{\varphi \in \mathrm{Conf}(X) : \varphi(Y) = Y\}.$$

Then the induced action of $\mathrm{Conf}(X; Y)$ on the Riemannian manifold $(Y, g|_Y)$ is again conformal. We then consider the following problem.

Problem 1. (1) Given $0 \leq i \leq \dim X$ and $0 \leq j \leq \dim Y$, classify $(\lambda, \nu) \in \mathbb{C}^2$ such that there exists a non-zero local/non-local operator

$$T : \mathcal{E}^i_\lambda(X) \to \mathcal{E}^j_\nu(Y)$$

satisfying

$$\varpi^Y_{\nu,j}(h) \circ T = T \circ \varpi^X_{\lambda,i}(h) \quad \text{for all } h \in \mathrm{Conf}(X; Y).$$

(2) Find explicit formulæ of the operators $T \equiv T^{i,j}_{\lambda,\nu}$.

The case $i = j = 0$ is a question that was raised in [6, Problem 4.2] as a geometric aspect of the branching problem for representations with respect to the pair of groups $\mathrm{Conf}(X) \supset \mathrm{Conf}(X; Y)$.

As a special case, one may ask:

Question A″ *Solve Problem 1 for covariant differential operators in the setting that* $(X, Y) = (S^2, S^1)$ *and* $(i, j) = (1, 0)$.

We note that, for $(X, Y) = (S^2, S^1)$, there are natural homomorphisms

$$\begin{array}{ccc} G_{\mathbb{C}} := SL(2, \mathbb{C}) & \to & \mathrm{Conf}(X) \\ \cup & & \cup \\ G_{\mathbb{R}} := SL(2, \mathbb{R}) & \to & \mathrm{Conf}(X; Y), \end{array}$$

and the images of $SL(2, \mathbb{C})$ and $SL(2, \mathbb{R})$ coincide with the identity component groups of $\mathrm{Conf}(X) \simeq O(3, 1)$ and $\mathrm{Conf}(X; Y)$, respectively. Question A is equivalent to Question A$''$ with $\mathrm{Conf}(X; Y)$ replaced by its identity component $SO_0(2, 1) \simeq SL(2, \mathbb{R})/\{\pm I\}$. In fact, the differential operator $\mathcal{D} = \mathrm{Rest}_{y=0} \circ (\mathcal{D}_1, \mathcal{D}_2)$ in Question A gives a $G_{\mathbb{R}}$-equivariant differential operator

$$\mathcal{E}^1_{\lambda-1}(S^2) \to \mathcal{E}^0_\nu(S^1) \equiv \mathcal{E}_\nu(S^1)$$

in our normalization, which takes the form

$$\mathcal{E}^1(\mathbb{R}^2) \to C^\infty(\mathbb{R}), \quad f dx + g dy \mapsto (\mathcal{D}_1 f)(x, 0) + (\mathcal{D}_2 g)(x, 0)$$

in the flat coordinates via the stereographic projection.

2.3 Branching Laws of Verma Modules

Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, and $\mathfrak{b}$ a Borel subalgebra consisting of lower triangular matrices in $\mathfrak{g}$. For $\lambda \in \mathbb{C}$, we define a character of $\mathfrak{b}$, to be denoted by $\mathbb{C}_\lambda$, as

$$\mathfrak{b} \to \mathbb{C}, \quad \begin{pmatrix} -x & 0 \\ y & x \end{pmatrix} \mapsto \lambda x.$$

If $\lambda \in \mathbb{Z}$ then $\mathbb{C}_\lambda$ is the differential of the holomorphic character $\chi_{\lambda,\lambda}$ of the Borel subgroup $B_{\mathbb{C}}$, which will be defined in (9) in Sect. 3.1.

We consider a $\mathfrak{g}$-module, referred to as a Verma module, defined by

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda.$$

Then $\mathbf{1}_\lambda := 1 \otimes 1 \in M(\lambda)$ is a highest weight vector with weight $\lambda \in \mathbb{C}$, and it generates $M(\lambda)$ as a $\mathfrak{g}$-module. The $\mathfrak{g}$-module $M(\lambda)$ is irreducible if and only if $\lambda \notin \mathbb{N}$.

We consider the following algebraic question:

Question A$'''$ *(1) Classify* $(\mu, \lambda_1, \lambda_2) \in \mathbb{C}^3$ *such that*

$$\mathrm{Hom}_{\mathfrak{g}}\left(M(\mu), M(\lambda_1) \otimes M(\lambda_2)\right) \neq \{0\}.$$

(2) Find an explicit expression of $\varphi(\mathbf{1}_\mu)$ *in* $M(\lambda_1) \otimes M(\lambda_2)$ *for any* $\varphi \in \mathrm{Hom}_{\mathfrak{g}}\left(M(\mu), M(\lambda_1) \otimes M(\lambda_2)\right)$.

An answer to Question A$'''$ is given as follows:

Proposition 2. *If $\lambda_1+\lambda_2 \notin \mathbb{N}$ then the tensor product $M(\lambda_1)\otimes M(\lambda_2)$ decomposes into the direct sum of Verma modules as follows:*

$$M(\lambda_1) \otimes M(\lambda_2) \simeq \bigoplus_{a=0}^{\infty} M(\lambda_1+\lambda_2-2a).$$

For the proof, consult [4] for instance. In fact, in [4], one finds the (abstract) branching laws of (parabolic) Verma modules in the general setting of the restriction with respect to symmetric pairs. By the duality theorem ([8], [9, Theorem 2.7]) between differential *symmetry breaking operators* (covariant differential operators to submanifolds) and (discretely decomposable) branching laws of Verma modules, we have the following one-to-one correspondence

$$\begin{aligned}&\{\text{The differential operators } \mathcal{D} \text{ yielding the functional identity } (\mathcal{M}_{\lambda,\nu})\}\\ &\leftrightarrow \operatorname{Hom}_{\mathfrak{g}}(M(-2\nu), M(-\lambda-1)\otimes M(-\lambda+1)) \\ &\qquad\qquad \oplus \operatorname{Hom}_{\mathfrak{g}}(M(-2\nu), M(-\lambda+1)\otimes M(-\lambda-1)),\end{aligned} \tag{8}$$

because $T_o(G_{\mathbb{C}}/B_{\mathbb{C}}) \otimes \mathbb{C} \simeq \mathbb{C}_{-2} \boxtimes \mathbb{C} + \mathbb{C} \boxtimes \mathbb{C}_{-2}$ as $\mathfrak{b} \otimes \mathbb{C} \simeq \mathfrak{b} \oplus \mathfrak{b}$-modules. Combining this with Proposition 2, we obtain

Proposition 3. *If $2\lambda \notin -\mathbb{N}$ then a non-zero differential operator $\mathcal{D}$ satisfying $(\mathcal{M}_{\lambda,\nu})$ exists if and only if $\nu - \lambda \in \mathbb{N}$, and the set of such differential operators forms a two-dimensional vector space.*

Owing to Proposition 1, we get the two-dimensional solution space as the linear span of $\mathcal{D}$ and $\mathcal{D}^{\vee}$, once we find a generic solution $\mathcal{D}$.

3 Rankin–Cohen Brackets

As a preparation for the proof of Theorem A, we briefly review the Rankin–Cohen brackets, which originated in number theory [1, 2, 11].

3.1 Homogeneous Line Bundles over $\mathbb{P}^1\mathbb{C}$

First, we shall fix a normalization of three homogeneous line bundles over $X = \mathbb{P}^1\mathbb{C}$, namely, $\mathcal{L}_{\lambda}^{\mathrm{conf}}$ (Sect. 2), $\mathcal{L}_{\lambda}^{\mathrm{hol}}$, and $\mathcal{L}_{n,\lambda}$.

We define a Borel subgroup of $G_{\mathbb{C}} = SL(2,\mathbb{C})$ by

$$B_{\mathbb{C}} := \left\{ \begin{pmatrix} a & 0 \\ c & \frac{1}{a} \end{pmatrix} : a \in \mathbb{C}^{\times}, c \in \mathbb{C} \right\},$$

and identify $G_{\mathbb{C}}/B_{\mathbb{C}}$ with $X = \mathbb{P}^1\mathbb{C}$ by $hB_{\mathbb{C}} \mapsto h \cdot 0$.

Given $n \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$, we define a one-dimensional representation of $B_{\mathbb{C}}$ by

$$\chi_{n,\lambda} : B_{\mathbb{C}} \to \mathbb{C}^{\times}, \quad \begin{pmatrix} \frac{1}{re^{i\theta}} & 0 \\ c & re^{i\theta} \end{pmatrix} \mapsto e^{in\theta} r^{\lambda}, \tag{9}$$

and a $G_{\mathbb{C}}$-equivariant line bundle $\mathcal{L}_{n,\lambda} = G_{\mathbb{C}} \times_{B_{\mathbb{C}}} \chi_{n,\lambda}$ as the set of equivalence classes of $G_{\mathbb{C}} \times \mathbb{C}$ given by

$$(g, u) \sim (gb^{-1}, \chi_{n,\lambda}(b)u) \quad \text{for some } b \in B_{\mathbb{C}}.$$

The conformal line bundle $\mathcal{L}_{\lambda}^{\mathrm{conf}}$ defined in Sect. 2.2 amounts to $\mathcal{L}_{0,2\lambda}$ by the formula (7).

On the other hand, if $\lambda = n \in \mathbb{Z}$ then $\chi_{\lambda,\lambda}$ is a holomorphic character of $B_{\mathbb{C}}$, and consequently, $\mathcal{L}_{\lambda,\lambda} \to X$ becomes a holomorphic line bundle, which we denote by $\mathcal{L}_{\lambda}^{\mathrm{hol}}$. The complexified cotangent bundle $(T^*X) \otimes \mathbb{C}$ splits into a Whitney sum of the holomorphic and anti-holomorphic cotangent bundle $(T^*X)^{1,0} \oplus (T^*X)^{0,1}$, which amounts to $\mathcal{L}_{2,2} \oplus \mathcal{L}_{-2,2}$. In summary, we have:

Lemma 1. *We have the following isomorphisms of $G_{\mathbb{C}}$-equivariant line bundles over $X \simeq \mathbb{P}^1\mathbb{C}$.*

$$\begin{aligned} \mathcal{L}_{\lambda}^{\mathrm{hol}} &\simeq \mathcal{L}_{\lambda,\lambda} \quad && \text{for } \lambda \in \mathbb{Z}, \\ \mathcal{L}_{\lambda}^{\mathrm{conf}} &\simeq \mathcal{L}_{0,2\lambda} && \text{for } \lambda \in \mathbb{C}, \\ (T^*X)^{1,0} &\simeq \mathcal{L}_{2,2}, \\ (T^*X)^{0,1} &\simeq \mathcal{L}_{-2,2}. \end{aligned}$$

The line bundle $\mathcal{L}_{n,\lambda} \to X$ is $G_{\mathbb{C}}$-equivariant; thus, there is the regular representation $\pi_{n,\lambda}$ of $G_{\mathbb{C}}$ on $C^{\infty}(X, \mathcal{L}_{n,\lambda})$. This is called the (unnormalized, non-unitary) principal series representation of $G_{\mathbb{C}}$. The restriction to the open Bruhat cell $\mathbb{C} \hookrightarrow X = \mathbb{C} \cup \{\infty\}$ yields an injection $C^{\infty}(X, \mathcal{L}_{n,\lambda}) \hookrightarrow C^{\infty}(\mathbb{C})$, on which $\pi_{n,\lambda}$ is given as a multiplier representation:

$$(\pi_{n,\lambda}(h)F)(z) = \left(\frac{cz+d}{|cz+d|}\right)^{-n} |cz+d|^{-\lambda} F\left(\frac{az+b}{cz+d}\right) \quad \text{for } h^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Comparing this with the conformal construction of the representation ϖ_{λ} in Example 2, we have $\varpi_{\lambda} \simeq \pi_{0,2\lambda}$.

Similarly to the smooth line bundle $\mathcal{L}_{n,\lambda}$, we consider holomorphic sections for the holomorphic line bundle $\mathcal{L}_{\lambda}^{\mathrm{hol}}$. For this, let D be a domain of $\mathbb{C}$ and G a subgroup of $G_{\mathbb{C}}$, which leaves D invariant. Then we can define a representation, to be denoted by $\pi_{\lambda}^{\mathrm{hol}}$, of G on the space $\mathcal{O}(D) \equiv \mathcal{O}(D, \mathcal{L}_{\lambda}^{\mathrm{hol}})$ of holomorphic sections, which is identified with a multiplier representation

$$\left(\pi_\lambda^{\mathrm{hol}}(h)F\right)(z) = (cz+d)^{-\lambda} F\left(\frac{az+b}{cz+d}\right) \quad \text{for } F \in \mathcal{O}(D).$$

Example 3.

(1) $D = \{z \in \mathbb{C} : |z| < 1\}$, $G = SU(1,1)$.
(2) $D = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$, $G = SL(2,\mathbb{R})$.

(For the application below we shall use the unit disc model.)

3.2 Rankin–Cohen Bidifferential Operator

Let D be a domain in $\mathbb{C}$. For $a \in \mathbb{N}$ and $\lambda_1, \lambda_2 \in \mathbb{C}$, the bidifferential operator $\mathcal{RC}^a_{\lambda_1,\lambda_2} : \mathcal{O}(D) \otimes \mathcal{O}(D) \to \mathcal{O}(D)$, referred to as the *Rankin–Cohen bracket* [1, 11], is defined by

$$\mathcal{RC}^a_{\lambda_1,\lambda_2}(f_1 \otimes f_2)(z) := \sum_{\ell=0}^{a} (-1)^\ell \binom{\lambda_1+a-1}{\ell}\binom{\lambda_2+a-1}{a-\ell} \frac{\partial^{a-\ell} f_1}{\partial z^{a-\ell}}(z) \frac{\partial^\ell f_2}{\partial z^\ell}(z).$$

In the theory of automorphic forms, $\mathcal{RC}^a_{\lambda_1,\lambda_2}$ yields a new holomorphic modular form of weight $\lambda_1 + \lambda_2 + 2a$ out of two holomorphic modular forms f_1 and f_2 of weights λ_1 and λ_2, respectively.

From the viewpoint of representation theory, $\mathcal{RC}^a_{\lambda_1,\lambda_2}$ is an intertwining operator:

$$\pi^{\mathrm{hol}}_{\lambda_1+\lambda_2+2a}(h) \circ \mathcal{RC}^a_{\lambda_1,\lambda_2} = \mathcal{RC}^a_{\lambda_1,\lambda_2} \circ \left(\pi^{\mathrm{hol}}_{\lambda_1}(h) \otimes \pi^{\mathrm{hol}}_{\lambda_2}(h)\right) \tag{10}$$

for all $h \in G$.

The coefficients of the Rankin–Cohen brackets look somewhat complicated. Eicheler–Zagier [2, Chapter 3] found that they are related to those of a classical orthogonal polynomial. A short proof for this fact is given by the *F-method* in [9].

To see the relation, we define a polynomial $\mathrm{RC}^a_{\lambda_1,\lambda_2}(x,y)$ of two variables x and y by

$$\mathrm{RC}^a_{\lambda_1,\lambda_2}(x,y) := \sum_{\ell=0}^{a} (-1)^\ell \binom{\lambda_1+a-1}{\ell}\binom{\lambda_2+a-1}{a-\ell} x^{a-\ell} y^\ell, \tag{11}$$

so that the Rankin–Cohen bidifferential operator $\mathcal{RC}^a_{\lambda_1,\lambda_2}$ is given by

$$\mathcal{RC}^a_{\lambda_1,\lambda_2} = \mathrm{Rest}_{z_1=z_2=z} \circ \mathrm{RC}^a_{\lambda_1,\lambda_2}\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right).$$

The polynomial $\mathrm{RC}^a_{\lambda_1,\lambda_2}(x, y)$ is of homogeneous degree a. Clearly we have:

Lemma 2. $\mathrm{RC}^a_{\lambda_1,\lambda_2}(x, y) = (-1)^a \mathrm{RC}^a_{\lambda_2,\lambda_1}(y, x)$.

Second we recall that the Jacobi polynomial $P_\ell^{\alpha,\beta}(t)$ is a polynomial of one variable t of degree ℓ given by

$$P_\ell^{\alpha,\beta}(t) = \frac{\Gamma(\alpha+\ell+1)}{\Gamma(\alpha+\beta+\ell+1)} \sum_{m=0}^{\ell} \frac{\Gamma(\alpha+\beta+\ell+m+1)}{(\ell-m)!m!\Gamma(\alpha+m+1)} \left(\frac{t-1}{2}\right)^m.$$

We inflate it to a homogeneous polynomial of two variables x and y of degree ℓ by

$$P_\ell^{\alpha,\beta}(x, y) := y^\ell P_\ell^{\alpha,\beta}\left(2\frac{x}{y}+1\right).$$

For instance, $P_0^{\alpha,\beta}(x, y) = 1$ and $P_1^{\alpha,\beta}(x, y) = (2+\alpha+\beta)x + (\alpha+1)y$. It turns out that

$$\mathrm{RC}^a_{\lambda_1,\lambda_2}(x, y) = (-1)^a \mathrm{RC}^a_{\lambda_2,\lambda_1}(y, x).$$

In particular, the following holds.

Lemma 3. *We have*

$$\mathcal{RC}^a_{\lambda_1,\lambda_2} = (-1)^a \mathrm{Rest}_{z_1=z_2=z} \circ P_a^{\lambda_1-1,-\lambda_1-\lambda_2-2a+1}\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right).$$

4 Holomorphic Trick

In this section we give a proof for Theorem A by using the results of the previous sections

4.1 Restriction to a Totally Real Submanifold

Consider a totally real embedding of $X = \mathbb{P}^1\mathbb{C} \simeq \mathbb{C} \cup \{\infty\}$ defined by

$$\iota : \mathbb{P}^1\mathbb{C} \to \mathbb{P}^1\mathbb{C} \times \mathbb{P}^1\mathbb{C}, \quad z \mapsto (z, \bar{z}). \tag{12}$$

The map ι respects the action of $G_\mathbb{C}$ via the following group homomorphism (we regard $G_\mathbb{C}$ as a real group), denoted by the same letter,

$$\iota : G_\mathbb{C} \to G_\mathbb{C} \times G_\mathbb{C}, \quad g \mapsto (g, \bar{g}).$$

This is because $G_{\mathbb{C}}/B_{\mathbb{C}} \simeq \mathbb{P}^1\mathbb{C}$ and because the Borel subgroup $B_{\mathbb{C}}$ is stable by the complex conjugation $g \mapsto \bar{g}$. Then the following lemma is immediate from Lemma 1.

Lemma 4. *We have an isomorphism of $G_{\mathbb{C}}$-equivariant line bundles:*

$$\iota^* \left(\mathcal{L}^{\mathrm{hol}}_{\lambda_1} \boxtimes \mathcal{L}^{\mathrm{hol}}_{\lambda_2}\right) \simeq \mathcal{L}_{\lambda_1-\lambda_2,\lambda_1+\lambda_2}.$$

In particular,

$$\iota^*(\mathcal{L}^{\mathrm{hol}}_{\lambda+1} \boxtimes \mathcal{L}^{\mathrm{hol}}_{\lambda-1}) \simeq \mathcal{L}^{\mathrm{conf}}_{\lambda-1} \otimes (T^*X)^{1,0}, \tag{13}$$

$$\iota^*(\mathcal{L}^{\mathrm{hol}}_{\lambda-1} \boxtimes \mathcal{L}^{\mathrm{hol}}_{\lambda+1}) \simeq \mathcal{L}^{\mathrm{conf}}_{\lambda-1} \otimes (T^*X)^{0,1}. \tag{14}$$

Proposition 4. *The isomorphisms (13) and (14) induce injective $G_{\mathbb{C}}$-equivariant homomorphisms between equivariant sheaves:*

$$(\iota^*)^{1,0} : \mathcal{O}(\mathcal{L}^{\mathrm{hol}}_{\lambda+1}) \otimes \mathcal{O}(\mathcal{L}^{\mathrm{hol}}_{\lambda-1}) \to \mathcal{E}^{1,0}_{\lambda-1}, \quad f_1(z_1) \otimes f_2(z_2) \mapsto f_1(z) f_2(\bar{z}) dz,$$

$$(\iota^*)^{0,1} : \mathcal{O}(\mathcal{L}^{\mathrm{hol}}_{\lambda-1}) \otimes \mathcal{O}(\mathcal{L}^{\mathrm{hol}}_{\lambda+1}) \to \mathcal{E}^{0,1}_{\lambda-1}, \quad f_1(z_1) \otimes f_2(z_2) \mapsto f_1(z) f_2(\bar{z}) d\bar{z},$$

that is, $(\iota^*)^{1,0}$ and $(\iota^*)^{0,1}$ are injective on every open set D in $\mathbb{P}^1\mathbb{C}$, and

$$(\iota^*)^{1,0} \circ \left(\pi^{\mathrm{hol}}_{\lambda+1}(g) \otimes \pi^{\mathrm{hol}}_{\lambda-1}(\bar{g})\right) = \varpi_{\lambda-1,1}(g) \circ (\iota^*)^{1,0}$$

$$(\iota^*)^{0,1} \circ \left(\pi^{\mathrm{hol}}_{\lambda-1}(g) \otimes \pi^{\mathrm{hol}}_{\lambda+1}(\bar{g})\right) = \varpi_{\lambda-1,1}(g) \circ (\iota^*)^{0,1}$$

hold for any g whenever they make sense.

Proof. The injectivity follows from the identity theorem of holomorphic functions because $\iota : \mathbb{P}^1\mathbb{C} \to \mathbb{P}^1\mathbb{C} \times \mathbb{P}^1\mathbb{C}$ is a totally real embedding. The covariance property is derived from (13) and (14).

Fix $\lambda \in \mathbb{Z}$ and $a \in \mathbb{N}$. We set $\nu = \lambda + a$. We want to relate the Rankin–Cohen brackets $\mathcal{RC}^a_{\lambda\pm1,\lambda\mp1}$ to our differential operator $\mathcal{D}$ (see Question A) in the sense that both of the following diagrams commute:

$$\begin{array}{ccccc}
\mathcal{O}(\mathcal{L}^{\mathrm{hol}}_{\lambda+1}) \otimes \mathcal{O}(\mathcal{L}^{\mathrm{hol}}_{\lambda-1}) & \xrightarrow{(\iota^*)^{1,0}} & \mathcal{E}^{1,0}_{\lambda-1}(\mathbb{C}) \subset \mathcal{E}^1_{\lambda-1}(\mathbb{R}^2) & \simeq & C^\infty(\mathbb{R}^2) \oplus C^\infty(\mathbb{R}^2) \\
\mathcal{RC}^a_{\lambda+1,\lambda-1}\downarrow & & & & \downarrow\mathcal{D} \\
\mathcal{O}(\mathcal{L}^{\mathrm{hol}}_{2\lambda+2a}) & \xrightarrow{\iota^*} & \mathcal{E}_\nu(\mathbb{R}) & \simeq & C^\infty(\mathbb{R}),
\end{array}$$

and

$$\begin{array}{ccc} \mathcal{O}(\mathcal{L}^{\rm hol}_{\lambda-1}) \otimes \mathcal{O}(\mathcal{L}^{\rm hol}_{\lambda+1}) & \xhookrightarrow{(\iota^*)^{0,1}} & \mathcal{E}^{0,1}_{\lambda-1}(\mathbb{C}) \subset \mathcal{E}^{1}_{\lambda-1}(\mathbb{R}^2) \simeq C^\infty(\mathbb{R}^2) \oplus C^\infty(\mathbb{R}^2) \\ {\scriptstyle (-1)^a \mathcal{RC}^a_{\lambda-1,\lambda+1}} \downarrow & & \downarrow {\scriptstyle \mathcal{D}} \\ \mathcal{O}(\mathcal{L}^{\rm hol}_{2\lambda+2a}) & \xhookrightarrow{\iota^*} & \mathcal{E}_\nu(\mathbb{R}) \simeq C^\infty(\mathbb{R}). \end{array}$$

Here we have used the following identification:

$$\mathcal{E}^1_{\lambda-1}(\mathbb{R}^2) \simeq C^\infty(\mathbb{R}^2) \oplus C^\infty(\mathbb{R}^2), \quad f dx + g dy \mapsto (f, g).$$

We define homogeneous polynomials D_1, D_2 with real coefficients so that

$$D_1(x, y) + \sqrt{-1} D_2(x, y) = 2^{-a} \mathrm{RC}^a_{\lambda+1,\lambda-1}(x - \sqrt{-1} y, x + \sqrt{-1} y),$$

where $\mathrm{RC}^a_{\lambda_1,\lambda_2}(x, y)$ is a polynomial defined in (11). We set

$$\mathcal{D}_1 := D_1 \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right), \quad \mathcal{D}_2 := D_2 \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right), \quad \mathcal{D} := \mathrm{Rest}_{y=0} \circ (\mathcal{D}_1, \mathcal{D}_2). \tag{15}$$

Lemma 5. *For any holomorphic functions f_1 and f_2,*

$$\mathcal{D}\left((\iota^*)^{1,0}(f_1 \otimes f_2)\right) = \iota^* \mathcal{RC}^a_{\lambda+1,\lambda-1}(f_1 \otimes f_2),$$
$$\mathcal{D}\left((\iota^*)^{0,1}(f_1 \otimes f_2)\right) = (-1)^a \iota^* \mathcal{RC}^a_{\lambda-1,\lambda+1}(f_1 \otimes f_2).$$

Proof. Let $\omega := (\iota^*)^{1,0}(f_1 \otimes f_2) = f_1(z) f_2(\bar{z}) dz$. If we write $\omega = f dx + g dy$ then $f(z) = f_1(z) f_2(\bar{z})$ and $g = \sqrt{-1} f$. Therefore,

$$\mathcal{D}\omega = \mathrm{Rest}_{y=0} \circ (\mathcal{D}_1, \mathcal{D}_2) \begin{pmatrix} f \\ g \end{pmatrix},$$

$$(\mathcal{D}_1, \mathcal{D}_2) \begin{pmatrix} f \\ g \end{pmatrix} = \left(\mathcal{D}_1 + \sqrt{-1} \mathcal{D}_2)(f_1(z) f_2(\bar{z})\right).$$

If we write $\mathrm{RC}^a_{\lambda+1,\lambda-1}(x, y) = \sum_{\ell=0}^a r_\ell x^{a-\ell} y^\ell$ then

$$\begin{aligned} (\mathcal{D}_1 + \sqrt{-1} \mathcal{D}_2)\,(f_1(z) f_2(\bar{z})) &= \mathrm{RC}^a_{\lambda+1,\lambda-1}\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right)(f_1(z) f_2(\bar{z})) \\ &= \sum_{\ell=0}^a r_\ell \frac{\partial^{a-\ell} f_1}{\partial z^{a-\ell}}(z) \frac{\partial^\ell f_2}{\partial \bar{z}^\ell}(\bar{z}), \end{aligned}$$

because f_1 and f_2 are holomorphic. Taking the restriction to $y = 0$, we get

$$\mathcal{D}(\omega) = \sum_{\ell=0}^{a} r_\ell \frac{\partial^{a-\ell} f_1}{\partial x^{a-\ell}}(x) \frac{\partial^\ell f_2}{\partial x^\ell}(x) = \iota^* \mathcal{RC}^a_{\lambda+1,\lambda-1}(f_1 \otimes f_2).$$

Hence we have proved the first identity. The second identity follows from Lemma 2.

Remark 1. If we multiply the bidifferential operator $\mathcal{RC}^a_{\lambda+1,\lambda-1}$ by $\sqrt{-1}$ then obviously (10) holds, where the role of $(\mathcal{D}_1, \mathcal{D}_2)$ is changed into $(-\mathcal{D}_2, \mathcal{D}_1)$ because

$$\sqrt{-1}(\mathcal{D}_1 + \sqrt{-1}\mathcal{D}_2) = -\mathcal{D}_2 + \sqrt{-1}\mathcal{D}_1.$$

This explains Proposition 1 from the "holomorphic trick."

4.2 Identities of Jacobi Polynomials

For $a \in \mathbb{N}_+$, we define the following three meromorphic functions of λ by

$$A_a(\lambda) := \frac{2\lambda^2 + 2(a-1)\lambda + a(a-1)}{a(2\lambda + a - 1)},$$

$$B_a(\lambda) := \frac{(\lambda - 1)(2\lambda + 1)}{a(2\lambda + a - 1)},$$

$$U_a(\lambda) := \frac{2\left(\lambda + [\frac{a}{2}]\right)_{[\frac{a-1}{2}]}}{\left(\lambda + \frac{1}{2}\right)_{[\frac{a-1}{2}]}},$$

where $(\mu)_k := \mu(\mu+1)\cdots(\mu+k-1) = \frac{\Gamma(\mu+k)}{\Gamma(\mu)}$ is the Pochhammer symbol.

Proposition 5. *We have*

$$(1-z)^a P_a^{\lambda,-2\lambda-2a+1}\left(\frac{3+z}{1-z}\right)$$
$$= (-1)^{a-1} U_a(\lambda)\left((1 - A_a(\lambda)z) C_{a-1}^{\lambda+\frac{1}{2}}(z) + B_a(\lambda)(1-z^2) C_{a-2}^{\lambda+\frac{3}{2}}(z)\right).$$

Equivalently,

$$P_a^{\lambda,-2\lambda-2a+1}(x - \sqrt{-1}y, x + \sqrt{-1}y)$$
$$= (\sqrt{-1})^{a-1} U_a(\lambda)\Bigg(x C_{a-1}^{\lambda+\frac{1}{2}}(-x^2, y) + \sqrt{-1}\Bigg(A_a(\lambda) y C_{a-1}^{\lambda+\frac{1}{2}}(-x^2, y)$$
$$+ B_a(\lambda)(x^2 + y^2) C_{a-2}^{\lambda+\frac{3}{2}}(-x^2, y)\Bigg)\Bigg).$$

Proposition 5 will be used in the proof of Theorem A in the next subsection. We want to note that we wondered if the first equation of Proposition 5 was already known; however, we could not find the identity in the literature.

One might give an alternative proof of Proposition 5 by applying the F-method to a vector bundle case. We will discuss this approach in a subsequent paper.

4.3 Proof of Theorem A

The relations in Lemma 5 and the covariance property (10) of the Rankin–Cohen brackets imply that the differential operator $\mathcal{D}$ defined in (15) satisfies the covariance relations (6) on the image

$$(\iota^*)^{1,0}\big(\mathcal{O}(\mathcal{L}^{\mathrm{hol}}_{\lambda+1}) \otimes \mathcal{O}(\mathcal{L}^{\mathrm{hol}}_{\lambda-1})\big) + (\iota^*)^{0,1}\big(\mathcal{O}(\mathcal{L}^{\mathrm{hol}}_{\lambda-1}) \otimes \mathcal{O}(\mathcal{L}^{\mathrm{hol}}_{\lambda+1})\big).$$

In order to prove (6), we need to show that the image is dense in $C^\infty(\mathbb{R}^2) \oplus C^\infty(\mathbb{R}^2)$ topologized by uniform convergence on compact sets. To see this we note that the image contains a linear span of the following 1-forms

$$z^m \bar{z}^n dz, \quad z^m \bar{z}^n d\bar{z}, \quad (m, n \in \mathbb{N}).$$

Since a linear span of $(x + \sqrt{-1}\, y)^m (x - \sqrt{-1}\, y)^n$ $(m, n \in \mathbb{N})$ is dense in $C^\infty(\mathbb{R}^2)$ by the Stone–Weierstrass theorem, we conclude that $\mathcal{D}$ satisfies (6). An explicit formula for the operators $(\mathcal{D}_1, \mathcal{D}_2)$ is derived from the Rankin–Cohen brackets by using Lemma 3 and Proposition 5 for $\lambda \in \mathbb{Z}$. Then the covariance relations (1) are satisfied for all $\lambda \in \mathbb{C}$ because $\mathbb{Z}$ is Zariski dense in $\mathbb{C}$.

If $2\lambda \notin -\mathbb{N}$ then the dimension of solutions is two by Proposition 2 and the one-to-one correspondence (8). Since $\mathcal{D}$ and $\mathcal{D}^\vee$ are linearly independent for our solution $\mathcal{D}$, the linear span of $\mathcal{D}$ and $\mathcal{D}^\vee$ exhausts all the solutions by Proposition 1. Hence Theorem A is proved.

4.4 Scalar-Valued Case

So far we have discussed a family of vector-valued differential operators that yield functional identities satisfied by vector-valued functions. We close this article with some comments on the scalar-valued case.

Let $\lambda \in \mathbb{C}$. Given $f \in C^\infty(\mathbb{R}^2 \setminus \{(0,0)\}) \simeq C^\infty(\mathbb{C} \setminus \{0\})$, we define its *twisted inversion* $f_\lambda^\vee$ by

$$f_\lambda^\vee(r\cos\theta, r\sin\theta) := r^{-2\lambda} f\left(\frac{-\cos\theta}{r}, \frac{\sin\theta}{r}\right)$$

as in (1), and more generally,

$$f_\lambda^h(z) := |cz+d|^{-2\lambda} f\left(\frac{az+b}{cz+d}\right) \quad \text{for } h^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$$

as in (5).

For a differential operator $\mathcal{D}$ on $\mathbb{R}^2$, we define a linear operator $\tilde{\mathcal{D}} : C^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R})$ by

$$\tilde{\mathcal{D}} := \mathrm{Rest}_{y=0} \circ \mathcal{D}.$$

Fix $\lambda, \nu \in \mathbb{C}$. As in Questions A, A′, A″, and A‴, we may consider the following equivalent questions:

Question B *Find $\tilde{\mathcal{D}}$ with constant coefficients such that*

$$\left(\tilde{\mathcal{D}} f_\lambda^\vee\right)(x) = |x|^{-2\nu} (\tilde{\mathcal{D}} f)\left(-\frac{1}{x}\right) \text{ for all } f \in C^\infty(\mathbb{C}), h \in SL(2,\mathbb{R}), \text{ and } x \in \mathbb{R}^\times.$$

Question B′ *Find $\tilde{\mathcal{D}}$ such that*

$$\left(\tilde{\mathcal{D}} f_\lambda^h\right)(x) = |cx+d|^{-2\nu} (\tilde{\mathcal{D}} f)\left(\frac{ax+b}{cx+d}\right)$$

for all $f \in C^\infty(\mathbb{C}), h \in SL(2,\mathbb{R})$, and $x \in \mathbb{R}^\times$.

Question B″ *Find an explicit formula of conformally covariant differential operator $\mathcal{E}_\lambda(S^2) \to \mathcal{E}_\nu(S^1)$.*

Question B‴ *Find an explicit expression of the element $\varphi(\mathbf{1}_{-\nu})$ for any $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(M(-\nu), M(-\lambda) \otimes M(-\lambda))$, where $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$.*

An answer to Question B″ (and also in the case $S^{n-1} \subset S^n$ for arbitrary $n \geq 2$) was first given by Juhl [3]. In the flat model (Questions B and B′), if $a := \nu - \lambda \in \mathbb{N}$ then

$$\widetilde{C_a^{\lambda-\frac{1}{2}}} \equiv \mathrm{Rest}_{y=0} \circ C_a^{\lambda-\frac{1}{2}}\left(-\frac{\partial^2}{\partial x^2}, \frac{\partial}{\partial y}\right) : \mathcal{E}_\lambda(\mathbb{R}^2) \to \mathcal{E}_\nu(\mathbb{R}^1)$$

intertwines the $SL(2,\mathbb{R})$-action. There have been several proofs for this (and also for more general cases) based on:

- Recurrence relations among coefficients of $\mathcal{D}$ [3],
- F-method [5, 8, 9], and
- Residue formulæ of a meromorphic family of non-local symmetry breaking operators [6, 10].

The holomorphic trick in Sect. 4 applied to this case gives yet another proof by using the Rankin–Cohen brackets and the following proposition analogous to (and much simpler than) Proposition 5.

Proposition 6. *For $a \in \mathbb{N}$, we have*

$$(1-z)^a P_a^{\lambda-1,-2\lambda-2a+1}\left(\frac{3+z}{1-z}\right) = (-1)^a \frac{\left(\lambda+[\frac{a}{2}]\right)_{[\frac{a+1}{2}]}}{\left(\lambda-\frac{1}{2}\right)_{[\frac{a+1}{2}]}} C_a^{\lambda-\frac{1}{2}}(z).$$

Equivalently,

$$P_a^{\lambda-1,-2\lambda-2a+1}(x-\sqrt{-1}y, x+\sqrt{-1}y) = (\sqrt{-1})^a \frac{\left(\lambda+[\frac{a}{2}]\right)_{[\frac{a+1}{2}]}}{\left(\lambda-\frac{1}{2}\right)_{[\frac{a+1}{2}]}} C_a^{\lambda-\frac{1}{2}}(-x^2, y).$$

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Semi-classical Scalar Products in the Generalised $SU(2)$ Model

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Abstract In these notes we review the field-theoretical approach to the computation of the scalar product of multi-magnon states in the Sutherland limit where the magnon rapidities condense into one or several macroscopic arrays. We formulate a systematic procedure for computing the $1/M$ expansion of the on-shell/off-shell scalar product of M-magnon states in the generalised integrable model with $SU(2)$-invariant rational R-matrix. The coefficients of the expansion are obtained as multiple contour integrals in the rapidity plane.

1 Introduction

In many cases the calculation of form factors and correlation functions within quantum integrable models solvable by the Bethe Ansatz reduces to the calculation of scalar products of Bethe vectors. The best studied case is that of the models based on the $SU(2)$-invariant R-matrix. A determinant formula for the norm-squared of an on-shell state has been conjectured by Gaudin [1], and then proved by Korepin in [2]. Sum formulas for the scalar product between two generic Bethe states were obtained by Izergin and Korepin [2–4]. Furthermore, the scalar product between an on-shell and off-shell Bethe vector was expressed in determinant form by Slavnov [5]. This representation proved to be very useful in the computation of correlation functions of the XXX and XXZ models [6]. Although the Slavnov determinant formula is, by all evidence, not generalisable for higher rank groups, compact and potentially useful expressions of the scalar products as multiple contour integrals of (products of) determinants were proposed in [7–10].

The above-mentioned sum and determinant formulas are efficient for states composed of few magnons. In order to evaluate scalar products of multi-magnon states, new semi-classical methods specific for the problem need to be developed.

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Of particular interest is the evaluation of the scalar product of Bethe wave functions describing the lowest excitations above the ferromagnetic vacuum composed of given (large) number of magnons. The magnon rapidities for such excitations organise themselves in a small number of macroscopically large bound complexes [11, 12]. It is common to refer this limit as a thermodynamical, or semi-classical, or Sutherland limit. In the last years the thermodynamical limit attracted much attention in the context of the integrability in AdS/CFT [13], where it describes "heavy" operators in the $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory, dual to classical strings embedded in the curved $AdS_5 \times S^5$ space-time [14, 15]. It has been realised that the computation of some three-point functions of such heavy operators boils down to the computation of the scalar product of the corresponding Bethe wave functions in the thermodynamical limit [16–20].

In this notes, based largely on the results obtained in [16, 17, 21, 22], we review the field-theoretical approach developed by E. Bettelheim and the author [22], which leads to a systematical semi-classical expansion of the on-shell/off-shell scalar product. The field-theoretical representation is not sensitive to the particular representation of the monodromy matrix and we put it in the context of the generalised integrable model with $SU(2)$ invariant rational R-matrix.

The text is organised as follows. In Sect. 2 we remind the basic facts and conventions concerning the Algebraic Bethe Ansatz for rational $SU(2)$-invariant R-matrix. In Sect. 3 we give an alternative determinant representation of the on-shell/off-shell scalar product of two M-magnon Bethe vectors in spin chains with rational $SU(2)$-invariant R-matrix. This representation, which has the form of an $2M \times 2M$ determinant, possesses an unexpected symmetry: it is invariant under the group S_{2M} of the permutations of the *union* of the magnon rapidities of the left and the right states, while the Korepin sum formulas and the Slavnov determinant have a smaller $S_M \times S_M$ symmetry. We refer to the symmetric expression in question as $\mathscr{A}$-functional to underline the relation with a similar quantity, previously studied in the papers [18, 23] and denoted there by the same letter. In the generalised $SU(2)$-invariant integrable model the $\mathscr{A}$-functional depends on the ratio of the eigenvalues of the diagonal elements of the monodromy matrix on the pseudo-vacuum, considered as a free functional variable. In Sect. 4 we write the $\mathscr{A}$-functional as an expectation value in the Fock space of free chiral fermions. The fermionic representation implies that the $\mathscr{A}$-functional is a KP τ-function, but we do not use this fact explicitly. By two-dimensional bosonization we obtain a formulation of the $\mathscr{A}$-functional in terms of a chiral bosonic field with exponential interaction. The bosonic field describes a Coulomb gas of dipole charges. The thermodynamical limit $M \gg 1$ is described by an effective (0+1)-dimensional field theory, obtained by integrating the fast-scale modes of the original bosonic field. In terms of the dipole gas the effective theory contains composite particles representing bound states of any number of dipoles. The Feynman diagram technique for the effective field theory for the slow-scale modes is expected to give the perturbative $1/M$ expansion of the scalar product. We evaluate explicitly the first two terms of this expansion. The leading term reproduces the known expression as a contour integral of a dilogarithm, obtained by different methods in [23] and [16, 17], while the subleading term, given by a double contour integral, is a new result reported recently in [22].

2 Algebraic Bethe Ansatz for Integrable Models with $su(2)$ R-Matrix

We remind some facts about the ABA for the $su(2)$-type models and introduce our notations. The monodromy matrix $M(u)$ is a 2×2 matrix [24, 25]

$$M(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{1}$$

The matrix elements A, B, C, D are operators in the Hilbert space of the model and depend on the complex spectral parameter u called rapidity. The monodromy matrix obeys the RTT-relation (Yang-Baxter equation)

$$R(u-v)(M(u) \otimes I)(I \otimes M(v)) = (I \otimes M(u))(M(v) \otimes I)R(u-v). \tag{2}$$

Here I denotes the 2×2 identity matrix and the 4×4 matrix $R(u)$ is the $SU(2)$ rational R-matrix whose entries are c-numbers. The latter is given, up to a numerical factor, by

$$R_{\alpha\beta}(u) = u\, I_{\alpha\beta} + i\varepsilon\, P_{\alpha\beta}, \tag{3}$$

with the operator $P_{\alpha\beta}$ acting as a permutation of the spins in the spaces α and β. In the standard normalization $\varepsilon = 1$.

The RTT relation determines the algebra of the monodromy matrix elements, which is the same for all $su(2)$-type models. In particular, $[B(u), B(v)] = [C(u), C(v)] = 0$ for all u and v.

The trace $T = A + D$ of the monodromy matrix is called transfer matrix. Sometimes it is useful to introduce a twist parameter κ (see, for example, [26]). The twist preserves the integrability: the twisted transfer matrix

$$T(u) = \mathrm{tr}\left[\left(\begin{smallmatrix} 1 & 0 \\ 0 & \kappa \end{smallmatrix}\right) M(u)\right] = A(u) + \kappa\, D(u) \tag{4}$$

satisfies $[T(u), T(v)] = 0$ for all u and v.

To define a quantum-mechanical system completely, one must determine the action of the elements of the monodromy matrix in the Hilbert space. In the framework of the ABA the Hilbert space is constructed as a Fock space associated with a cyclic vector $|\Omega\rangle$, called pseudovacuum, which is an eigenvector of the operators A and D and is annihilated by the operator C:

$$A(u)|\Omega\rangle = a(u)|\Omega\rangle, \;\; D(u)|\Omega\rangle = d(u)|\Omega\rangle, \;\; C(u)|\Omega\rangle = 0. \tag{5}$$

The dual pseudo-vacuum satisfies the relations

$$\langle \Omega|A(u) = a(u)\langle \Omega|,\ \langle \Omega|D(u) = d(u)\langle \Omega|,\ \langle \Omega|B(u) = 0. \tag{6}$$

Here $a(u)$ and $d(u)$ are complex-valued functions whose explicit form depends on the choice of the representation of the algebra (2). We will not need the specific form of these functions, except for some mild analyticity requirements. In other words, we will consider the generalized $SU(2)$ model in the sense of [2], in which the functions $a(u)$ and $d(u)$ are considered as free functional parameters.

The vectors obtained from the pseudo-vacuum $|\Omega\rangle$ by acting with the "raising operators" $B(u)$,

$$|\mathbf{u}\rangle = B(u_1)\dots B(u_M)|\Omega\rangle, \quad \mathbf{u} = \{u_1, \dots, u_M\} \tag{7}$$

are called *Bethe states*. Since the B-operators commute, the state $|\mathbf{u}\rangle$ is invariant under the permutations of the elements of the set $\mathbf{u}$.

The Bethe states that are eigenstates of the (twisted) transfer matrix are called "on-shell". Their rapidities obey the Bethe Ansatz equations

$$\frac{a(u_j)}{d(u_j)} + \kappa\, \frac{Q_{\mathbf{u}}(u_j + i\varepsilon)}{Q_{\mathbf{u}}(u_j - i\varepsilon)} = 1 \qquad (j = 1, \dots, M). \tag{8}$$

Here and in the following we will use the notation

$$Q_{\mathbf{u}}(v) = \prod_{i=1}^{M}(v - u_i), \quad \mathbf{u} = \{u_1, \dots, u_M\}. \tag{9}$$

The corresponding eigenvalue of the transfer matrix $T(x)$ is

$$t(v) = \frac{Q_{\mathbf{u}}(v - i\varepsilon)}{Q_{\mathbf{u}}(v)} + \kappa\, \frac{d(v)}{a(v)}\, \frac{Q_{\mathbf{u}}(v + i\varepsilon)}{Q_{\mathbf{u}}(v)}\ . \tag{10}$$

If the rapidities $\mathbf{u}$ are generic, the Bethe state is called "off-shell".

In the unitary representations of the RTT-algebra, like the $\mathrm{XXX}_{1/2}$ spin chain, the on-shell states form a complete set in the Hilbert space. The XXX spin chain of length L can be deformed by introducing inhomogeneities $\theta_1, \dots, \theta_L$ associated with the L sites of the spin chain. The eigenvalues of the operators $A(v)$ and $D(v)$ on the vacuum in the inhomogeneous XXX chain are given by

$$a(v) = Q_{\boldsymbol{\theta}}(v + \tfrac{1}{2} i\varepsilon)\ ,\ \ d(v) = Q_{\boldsymbol{\theta}}(v - \tfrac{1}{2} i\varepsilon), \tag{11}$$

where the polynomial $Q_{\boldsymbol{\theta}}(x)$ is defined as[1]

$$Q_{\boldsymbol{\theta}}(x) = \prod_{l=1}^{L}(x - \theta_l), \quad \boldsymbol{\theta} = \{\theta_1, \ldots, \theta_L\}. \tag{12}$$

Any Bethe state is completely characterised by its *pseudo-momentum*, known also under the name of *counting function* [28]

$$2ip(v) = \log \frac{Q_{\mathbf{u}}(v + i\varepsilon)}{Q_{\mathbf{u}}(v - i\varepsilon)} - \log \frac{a(v)}{d(v)} + \log \kappa. \tag{13}$$

The Bethe equations (8) imply that

$$p(u_j) = 2\pi n_j - \pi \quad (j = 1, \ldots, M) \tag{14}$$

where the integers n_j are called mode numbers.

3 Determinant Formulas for the Inner Product

In order to expand the states $|\mathbf{v}\rangle$ with given a set of rapidities in the basis of eigenvectors $|\mathbf{u}\rangle$ of the monodromy matrix,

$$|\mathbf{v}\rangle = \sum_{\mathbf{u} \text{ on shell}} \frac{\langle\, \mathbf{u}|\mathbf{v}\,\rangle}{\langle\, \mathbf{u}|\mathbf{u}\,\rangle} |\mathbf{u}\rangle, \tag{15}$$

we need to compute the scalar product $\langle\, \mathbf{v}|\mathbf{u}\,\rangle$ of an off-shell and an on-shell Bethe state. The scalar product is related to the bilinear form

$$(\mathbf{v}, \mathbf{u}) = \langle \Omega| \prod_{j=1}^{M} C(v_j) \prod_{j=1}^{M} B(u_j)|\Omega\rangle \tag{16}$$

by $(\mathbf{u}, \mathbf{v}) = (-1)^M \langle\, \mathbf{u}^*|\mathbf{v}\,\rangle$. This follows from the complex Hermitian convention $B(u)^\dagger = -C(u^*)$. The inner product can be computed by commuting the B-operators to the left and the A-operators to the right according to the algebra (2), and then applying the relations (5) and (6). The resulting sum formula written down by Korepin [2] works well for small number of magnons but for larger M becomes intractable.

[1] This is a particular case of the Drinfeld polynomial $P_1(u)$ [27] when all spins along the chain are equal to $1/2$.

An important observation was made by N. Slavnov [5], who realised that when one of the two states is on-shell, the Korepin sum formula gives the expansion of the determinant of a sum of two $M \times M$ matrices.[2] Although the Slavnov determinant formula does not give obvious advantages for taking the thermodynamical limit, is was used to elaborate alternative determinant formulas, which are better suited for this task [16, 17, 21, 22].

Up to a trivial factor, the inner product depends on the functional argument

$$f(v) \equiv \kappa \, \frac{d(v)}{a(v)} \tag{17}$$

and on two sets of rapidities, $\mathbf{u} = \{u_1, \ldots, u_M\}$ and $\mathbf{v} = \{v_1, \ldots, v_M\}$. Since the rapidities within each of the two sets are not ordered, the inner product has symmetry $S_M \times S_M$, where S_M is the group of permutations of M elements. It came then as a surprise that the inner product can be written [21][3] as a restriction on the mass shell (for one of the two sets of rapidities) of an expression completely symmetric with respect of the permutations of the union $\mathbf{w} \equiv \{w_1, \ldots, w_{2M}\} = \{u_1, \ldots, u_M, v_1, \ldots, v_M\}$ of the rapidities of the two states:

$$(\mathbf{v}|\mathbf{u}) \underset{\mathbf{u}\to \text{on shell}}{=} \prod_{j=1}^{M} a(v_j) d(u_j) \, \mathscr{A}_{\mathbf{w}}[f] \,, \qquad \mathbf{w} = \mathbf{u} \cup \mathbf{v}, \tag{18}$$

where the functional $\mathscr{A}_{\mathbf{w}}[f]$ is given by the following $N \times N$ determinant ($N = 2M$)

$$\mathscr{A}_{\mathbf{w}}[f] = \det_{jk} \left(w_j^{k-1} - f(w_j)\,(w_j + i\varepsilon)^{k-1} \right) / \det_{jk} \left(w_j^{k-1} \right). \tag{19}$$

In the $\text{XXX}_{1/2}$ spin chain, the r.h.s. of (18) is proportional to the inner product of an off-shell Bethe state $|\mathbf{w}\rangle$ and a state obtained from the left vacuum by a global $SU(2)$ rotation [21]. Such inner products can be given statistical interpretation as a partial domain-wall partition function (pDWPF) [30]. In this case the identity (18) can be explained with the global $su(2)$ symmetry [21].

Another determinant formula, which is particularly useful for taking the thermodynamical limit, is derived in [22]:

$$\mathscr{A}_{\mathbf{w}} = \det\,(1 - K)\,, \tag{20}$$

[2]This property is particular for the $SU(2)$ model. The the inner product in the $SU(n)$ model is a determinant only for a restricted class of states [29].

[3]The case considered in [21] was that of the periodic inhomogeneous $\text{XXX}_{1/2}$ spin chain of length L, but the proof given there is trivially extended to the generalised $SU(2)$ model.

where the $N \times N$ matrix K has matrix elements

$$K_{jk} = \frac{Q_j}{w_j - w_k + i\varepsilon} \qquad (j, k = 1, \dots, N), \tag{21}$$

and the weights Q_j are obtained as the residues of the same function at the roots w_j:

$$Q_j \equiv \operatorname*{Res}_{z \to w_j} \mathcal{Q}(z), \quad \mathcal{Q}(z) \equiv f(z)\, \frac{Q_{\mathbf{w}}(z + i\varepsilon)}{Q_{\mathbf{w}}(z)}. \tag{22}$$

Here $Q_{\mathbf{w}}$ is the Baxter polynomial for the set $\mathbf{w}$, c.f. (9). The determinant formula (20) has the advantage that it exponentiates in a simple way:

$$\log \mathscr{A}_{\mathbf{w}}[f] = -\sum_{n=1}^{\infty} \frac{1}{n} \sum_{j_1,\dots,j_n=1}^{N} \frac{Q_{j_1}}{w_{j_1} - w_{j_2} + i\varepsilon}\, \frac{Q_{j_2}}{w_{j_1} - w_{j_3} + i\varepsilon} \cdots \frac{Q_{j_n}}{w_{j_n} - w_{j_1} + i\varepsilon}. \tag{23}$$

The identity (20) is the basis for the field-theoretical approach to the computation of the scalar product in the thermodynamical limit.

4 Field Theory of the Inner Product

4.1 The $\mathscr{A}$-Functional in Terms of Free Fermions

This determinant on the RHS of (20) can be expressed as a Fock-space expectation value for a Neveu-Schwarz chiral fermion living in the rapidity plane with two-point function

$$\langle 0|\psi(z)\psi^*(u)|0\rangle = \langle 0|\psi^*(z)\psi(u)|0\rangle = \frac{1}{z-u}\,. \tag{24}$$

Representing the matrix K in (20) as

$$K_{jk} = \langle 0|\psi^*(w_j + i\varepsilon)\psi(w_k)|0\rangle \tag{25}$$

it is easy to see that the $\mathscr{A}$-functional is given by the expectation value

$$\mathscr{A}_{\mathbf{w}}[f] = \langle 0| \exp\left(\sum_{j=1}^{N} Q_j\, \psi^*(w_j)\psi(w_j + i\varepsilon) \right) |0\rangle. \tag{26}$$

In order to take the large N limit, we will need reformulate the problem entirely in terms of the meromorphic function $\mathcal{Q}(z)$. The discrete sum of fermion bilinears in the exponent on the RHS of (26) can be written as a contour integral using the fact that the quantities Q_j, defined by (22), are residues of the same function $\mathcal{Q}(z)$ at $z = w_j$. As a consequence, the Fock space representation (26) takes the form

$$\mathscr{A}_{\mathbf{w}}[f] = \langle 0| \exp\left(\oint_{\mathcal{C}_{\mathbf{w}}} \frac{dz}{2\pi i} \mathcal{Q}(z)\, \psi^*(z)\psi(z+i\varepsilon) \right) |0\rangle\,, \tag{27}$$

where the contour $\mathcal{C}_{\mathbf{w}}$ encircles the points $\mathbf{w}$ and leaves outside all other singularities of $\mathcal{Q}$, as shown in Fig. 1. Expanding the exponent and performing the gaussian contractions, one writes the $\mathscr{A}$-functional in the form of a Fredholm determinant

$$\mathscr{A}_{\mathbf{w}}[f] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \oint\limits_{\mathcal{C}_{\mathbf{w}}^{\times n}} \prod_{j=1}^{n} \frac{dz_j\ \mathcal{Q}(z_j)}{2\pi i} \ \det_{j,k=1}^{n} \frac{1}{z_j - z_k + i\varepsilon}. \tag{28}$$

Since the function $\mathcal{Q}$ has exactly N poles inside the contour $\mathcal{C}_{\mathbf{w}}$, only the first N terms of the series are non-zero. The series exponentiates to

$$\log \mathscr{A}_{\mathbf{w}}[f] = -\sum_{n=1}^{\infty} \frac{1}{n} \oint\limits_{\mathcal{C}_{\mathbf{w}}^{\times n}} \frac{dz_1 \ldots dz_n}{(2\pi i)^n} \frac{\mathcal{Q}(z_1)}{z_1 - z_2 + i\varepsilon} \cdots \frac{\mathcal{Q}(z_n)}{z_n - z_1 + i\varepsilon}. \tag{29}$$

This is the vacuum energy of the fermionic theory, given by the sum of all vacuum loops. The factor (-1) comes from the Fermi statistics and the factor $1/n$ accounts for the cyclic symmetry of the loops. The series (29) can be of course obtained directly from (23).

4.2 *Bosonic Theory and Coulomb Gas*

Alternatively, one can express the $\mathscr{A}$-function in term of a chiral boson $\phi(x)$ with two-point function

$$\langle 0|\phi(z)\phi(u)|0\rangle = \log(z-u). \tag{30}$$

After bosonization $\psi(z) \to e^{\phi(z)}$ and $\psi^*(z) \to e^{-\phi(z)}$, where we assumed that the exponents of the gaussian field are normally ordered, the fermion bilinear $\psi^*(z)\psi(z+i\varepsilon)$ becomes, up to a numerical factor, a chiral vertex operator of zero charge

$$\mathcal{V}(z) \equiv e^{\phi(z+i\varepsilon)-\phi(z)}. \tag{31}$$

The coefficient is obtained from the OPE

$$e^{-\phi(z)}\, e^{\phi(u)} \sim \frac{1}{z-u}\, e^{\phi(u)-\phi(z)} \tag{32}$$

with $u = z + i\varepsilon$:

$$\psi^*(z)\psi(z+i\varepsilon) \quad \to \quad e^{-\phi(z)} e^{\phi(z+i\varepsilon)} = -\frac{1}{i\varepsilon}\,\mathcal{V}(z). \tag{33}$$

The bosonized form of the operator representation (27) is therefore

$$\mathscr{A}_{\mathbf{w}}[f] = \langle 0|\exp\left(-\frac{1}{i\varepsilon}\oint_{\mathcal{C}_{\mathbf{w}}}\frac{dz}{2\pi i}\,\mathcal{Q}(z)\,\mathcal{V}(z)\right)|0\rangle, \tag{34}$$

where $|0\rangle$ is the bosonic vacuum state with zero charge. Expanding the exponential and applying the OPE (32) one writes the expectation value as the grand-canonical Coulomb-gas partition function

$$\mathscr{A}_{\mathbf{w}}[f] = \sum_{n=0}^{N}\frac{(-1)^n}{n!}\prod_{j=1}^{n}\oint_{\mathcal{C}_{\mathbf{w}}}\frac{dz_j}{2\pi i}\,\frac{\mathcal{Q}(z_j)}{i\varepsilon}\,\prod_{j<k}^{n}\frac{(z_j-z_k)^2}{(z_j-z_k)^2-i\varepsilon^2}\,. \tag{35}$$

After applying the Cauchy identity, we get back the Fredholm determinant (28).

4.3 The Thermodynamical Limit

Although the roots $\mathbf{w} = \{w_1, \dots, w_N\}$ are off-shell, typically they can be divided into two or three on-shell subsets $\mathbf{w}^{(k)}$, each representing a lowest energy solution of the Bethe equations for given (large) magnon number $N^{(k)}$. The Bethe roots for such solution are organised in one of several arrays with spacing $\sim \varepsilon$, called macroscopic Bethe strings, and the distribution of the roots along these arrays is approximated by continuous densities on a collection of contours in the complex rapidity plane [11, 12, 14, 15].

We choose an N-dependent normalisation of the rapidity such that $\varepsilon \sim 1/N$. Then the typical size of the contours and the densities remains finite in the limit $\varepsilon \to 0$.

In order to compute the $\mathscr{A}$-functional in the large N limit, we will follow the method developed on [22] and based on the field-theoretical formulation of the problem, Eq. (34). The method involves a coarse-graining procedure, as does the original computation of the quantity $\mathcal{A}$, carried out in [23].

Let us mention that there is a close analogy between the above semiclassical analysis and the computation of the instanton partition functions of four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories in the so-called Ω-background,

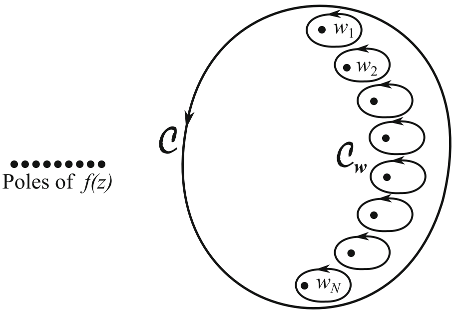


Fig. 1 Schematic representation of the contour $\mathcal{C}_{\mathbf{w}}$ and the deformed contour $\mathcal{C}$

characterised by two deformation parameters, ε_1 and ε_2 [31, 32], in the Nekrasov-Shatashvili limit $\varepsilon_2 \to 0$ [33]. In this limit the result is expressed in terms of the solution of a non-linear integral equation. The derivation, outlined in [33] and explained in great detail in the recent papers [34, 35], is based on the iterated Mayer expansion for a one-dimensional non-ideal gas. Our method is a field-theoretical alternative of the Mayer expansion of the gas of dipole charges created by the exponential operators $\mathcal{V}_n$. In our problem the saddle-point of the action (56) also lead to a non-linear integral equation, but the non-linearity disappears when $\varepsilon \to 0$.

Of crucial relevance to our approach is the possibility to deform the contour of integration. In order to take advantage of the contour-integral representation, the original integration contour $\mathcal{C}_{\mathbf{w}}$ surrounding the poles $\mathbf{w}$ of the integrand, should be deformed to a contour $\mathcal{C}$ which remains at finite distance from the singularities of the function $\mathcal{Q}$ when $\varepsilon \to 0$, as shown in Fig. 1. Along the contour $\mathcal{C}$ the function $\mathcal{Q}(z)$ changes slowly at distances $\sim \varepsilon$. In all nontrivial applications the weight function $\mathcal{Q}$ has additional poles, which are those of the function f. The contour $\mathcal{C}$ separates the roots $\mathbf{w}$ from the poles of f.

4.4 Coarse-Graining

We would like to compute the ε-expansion of the expectation value (34), with $\mathcal{C}_{\mathbf{w}}$ replaced by $\mathcal{C}$. This is a semi-classical expansion with Planck constant $\hbar = \varepsilon$. As any semi-classical expansion, the perturbative expansion in ε is an asymptotic expansion. Our strategy is to introduce a cutoff Λ, such that

$$\varepsilon \ll \Lambda \ll N\varepsilon \qquad (N\varepsilon \sim 1), \tag{36}$$

integrate the ultra-violet (fast-scale) part of the theory in order to obtain an effective infrared (slow-scale) theory. The splitting of the bosonic field into slow and fast pieces into slow and fast pieces is possible only in the thermodynamical limit $\varepsilon \to 0$. In this limit the dependence on Λ enters through exponentially small non-perturbative terms and the perturbative expansion in ε does not depend on Λ.

We thus cut the contour $\mathcal{C}$ into segments of length Λ and compute the effective action for the slow piece as the sum of the connected n-point correlators (cumulants) of the vertex operator $\mathcal{V}$. The nth cumulant $\Xi_n(z)$ is obtained by integrating the OPE of a product of n vertex operators

$$\mathcal{V}(z_1)\dots\mathcal{V}(z_n) = \prod_{j<k} \frac{(z_j - z_k)^2}{(z_j - z_k)^2 + \varepsilon^2} : \mathcal{V}(z_1)\dots\mathcal{V}(z_n) : \tag{37}$$

along a segment of the contour $\mathcal{C}$ of size Λ, containing the point z. Since we want to evaluate the effect of the short-distance interaction due to the poles, we can assume that the rest of the integrand is analytic everywhere. Then the integration can be performed by residues using the Cauchy identity. This computation has been done previously in [31] in a different context. The easiest way to compute the integral is to fix $z_1 = z$ and integrate with respect to $z_2, \dots, z_n$. We expand the numerical factor in (37) as a sum over permutations. The $(n-1)!$ permutations representing maximal cycles of length n give identical contributions to the residue. For the rest of the permutations the contour integral vanishes. We find ($z_{jk} \equiv z_j - z_k$)

$$\begin{aligned}\Xi_n &= \oint \frac{\mathcal{V}(z_1)\dots\mathcal{V}(z_n)}{(-i\varepsilon)^n\, n!} \prod_{k=2}^{n} \frac{dz_k}{2\pi i}\\ &\sim \frac{(n-1)!}{n!} \oint \frac{\prod_{k=2}^{n} \frac{dz_k}{2\pi i}\ : \mathcal{V}(z_1)\cdots\mathcal{V}(z_n) :}{(i\varepsilon - z_{12})\dots(i\varepsilon - z_{n-1,n})(i\varepsilon - z_{n,1})}\\ &= -\frac{1}{n^2 i\varepsilon}\, \mathcal{V}_n(z)\,,\end{aligned} \tag{38}$$

where

$$\mathcal{V}_n(z) \equiv\, : \mathcal{V}(z)\mathcal{V}(z+i\varepsilon)\dots\mathcal{V}(z+ni\varepsilon) : \,= e^{\phi(z+ni\varepsilon)-\phi(z)}\,. \tag{39}$$

The interaction potential of the effective coarse-grained theory therefore contains, besides the original vertex operator $\mathcal{V} \equiv \mathcal{V}_1$, all composite vertex operators $\mathcal{V}_n$ with $n \lesssim \Lambda$. If one repeats the computation (38) with the weights $\mathcal{Q}$, one obtains for the nth cumulant

$$\Xi_n(z) = -\frac{1}{i\varepsilon}\, \frac{\mathcal{Q}_n(z)\, \mathcal{V}_n(z)}{n^2}\,, \qquad \mathcal{Q}_n(z) = \mathcal{Q}(z)\mathcal{Q}(z+i\varepsilon)\dots\mathcal{Q}(z+in\varepsilon). \tag{40}$$

$$\begin{aligned}\Xi_n(z) = -\frac{1}{i\varepsilon}\, \frac{\mathcal{Q}_n(z)\, \mathcal{V}_n(z)}{n^2}\,, \qquad \mathcal{Q}_n(z) &= \mathcal{Q}(z)\mathcal{Q}(z+i\varepsilon)\dots\mathcal{Q}(z+in\varepsilon)\\ &= e^{-\Phi(x)+\Phi(x+ni\varepsilon)}\,.\end{aligned} \tag{41}$$

As the spacing $n\varepsilon$ should be smaller than the cut-off length Λ, from the perspective of the effective infrared theory all these particles are point-like. We thus obtained that in the semi-classical limit the $\mathscr{A}$-functional is given, up to non-perturbative terms, by the expectation value

$$\mathscr{A}_{\mathbf{u},\mathbf{z}} \approx \left\langle \exp\left(\frac{1}{\varepsilon} \sum_{n=1}^{\Lambda/\varepsilon} \frac{1}{n^2} \oint_{\mathcal{C}} \frac{dz}{2\pi}\, \mathcal{Q}_n(z)\, \mathcal{V}_n(z) \right) \right\rangle . \tag{42}$$

The effective potential can be given a nice operator form, which will be used to extract the perturbative series in ε. For that it is convenient to represent the function $f(z)$ as the ratio

$$f(z) = \frac{g(z)}{g(z+i\varepsilon)} = g(z)^{-1} \mathbb{D}\, g(z) , \tag{43}$$

where we introduced the shift operator

$$\mathbb{D} \equiv e^{i\varepsilon\partial} . \tag{44}$$

Then the weight factor $\mathcal{Q}_n$ takes the form

$$\mathcal{Q}_n = e^{-\Phi}\, \mathbb{D}^n\, e^{\Phi}, \qquad \Phi(z) = Q_{\mathbf{w}}(z)/g(z) , \tag{45}$$

and the series in the exponent in (42) can be summed up to

$$\mathscr{A}_{\mathbf{w}}[f] = \left\langle \exp\left(\frac{1}{\varepsilon} \oint_{\mathcal{C}} \frac{dz}{2\pi}\ : e^{-\Phi(z)-\phi(z)}\, \mathrm{Li}_2(\mathbb{D})\, e^{\Phi(z)+\phi(z)} : \right) \right\rangle , \tag{46}$$

with the operator $\mathrm{Li}_2(\mathbb{D})$ given by the dilogarithmic series

$$\mathrm{Li}_2(\mathbb{D}) = \sum_{n=1}^{\infty} \frac{\mathbb{D}^n}{n^2} . \tag{47}$$

Here we extended the sum over n to infinity, which can be done with exponential accuracy. The function $\Phi(z)$, which we will refer to as "classical potential", plays the role of classical expectation value for the bosonic field ϕ.

If we specify to the case of the (inhomogeneous, twisted) spin chain, considered in [22], then $f = \kappa\, d/a$ with a, d given by (11). In this case the classical potential is

$$\Phi(z) = \log Q_{\mathbf{w}}(z) - \log Q_{\boldsymbol{\theta}}(z - i\varepsilon/2). \tag{48}$$

Remark. Going back to the fermion representation, we write the result as a Fredholm determinant with different Fredholm kernel,

$$\mathscr{A}_{\mathbf{w}}[f] \approx \langle 0|\exp\left(\oint_{\mathcal{C}} \frac{dz}{2\pi i} e^{-\Phi(z)}\psi^*(z)\ \log(1-\mathbb{D})\ \psi(z)e^{\Phi(z)}\right)|0\rangle = \mathrm{Det}(1-\hat{\mathcal{K}}), \tag{49}$$

where the Fredholm operator $\hat{\mathcal{K}}$ acts in the space of functions analytic in the vicinity of the contour $\mathcal{C}$:

$$\hat{\mathcal{K}}\xi(z) = \oint_{\mathcal{C}} \frac{du}{2\pi i}\hat{\mathcal{K}}(z,u)\xi(u), \qquad \hat{\mathcal{K}}(z,u) = \sum_{n=1}^{\infty} \frac{e^{-\Phi(z)+\Phi(z+i\varepsilon n)}}{z-u+i\varepsilon n}. \tag{50}$$

The expression in terms of a Fredholm determinant can be obtained directly by performing the cumulant expansion for the expression of the $\mathscr{A}$-functional as a product of shift operators [17]

$$\mathscr{A}[f] = \frac{1}{\Psi_{\mathbf{w}}[g]} \prod_{j=1}^{N}(1-e^{i\varepsilon\partial/\partial w_j}) \prod_{j=1}^{N} \Psi_{\mathbf{w}}[g],$$

$$\Psi_{\mathbf{w}}[g] = \frac{\prod_{j<k}(w_j-w_k)}{\prod_{j=1}^{N} g(w_j)}, \quad f(z) = \frac{g(z)}{g(z+i\varepsilon)}. \tag{51}$$

4.5 The First Two Orders of the Semi-classical Expansion

The effective IR theory is compatible with the semi-classical expansion being of the form

$$\log \mathscr{A}_{\mathbf{w}} = \frac{F_0}{\varepsilon} + F_1 + \varepsilon F_2 + \cdots + \mathcal{O}(e^{-\Lambda/\varepsilon}). \tag{52}$$

Below we develop a diagram technique for computing the coefficients in the expansion. First we notice that the ε-expansion of the effective interaction in (46) depends on the field ϕ through the derivatives $\partial\phi, \partial^2\phi$, etc. We therefore consider the first derivative $\partial\phi$ as an independent field

$$\varphi(z) \equiv -\partial\phi(z) \tag{53}$$

with two-point function

$$G(z,u) = \partial_z\partial_u \log(z-u) = \frac{1}{(z-u)^2}. \tag{54}$$

In order to derive the diagram technique, we formulate the expectation value (46) as a path integral for the (0 + 1)-dimensional field $\varphi(x)$ defined on the contour $\mathcal{C}$. The two-point function (54) can be imposed in the standard way by introducing a second field $\rho(x)$ linearly coupled to φ. The path integral reads

$$\mathscr{A}_{\mathbf{w}}[f] = \int [D\varphi \, D\rho] \, e^{-\mathcal{Y}[\varphi,\rho]}, \tag{55}$$

with action functional

$$\mathcal{Y}[\varphi,\rho] = -\tfrac{1}{2} \int\limits_{\mathcal{C}\times\mathcal{C}} dz du \, \frac{\rho(z)\rho(u)}{(z-u)^2} + \oint_{\mathcal{C}} dx \, \rho(z)\varphi(z) + \oint_{\mathcal{C}} \frac{dz}{2\pi} \, W(\varphi,\varphi',\dots). \tag{56}$$

The dependence on ε is through the potential W, obtained by expanding the exponent in (46):

$$\begin{aligned} W(\varphi,\varphi',\dots) &= -\frac{1}{\varepsilon} \, e^{-\Phi(x)-\phi(x)} \, \mathrm{Li}_2(\mathbb{D}) \, e^{\Phi(x)+\phi(x)} \\ &= -\frac{1}{\varepsilon} \, \mathrm{Li}_2(\mathcal{Q}) + i \log(1-\mathcal{Q})\varphi - \frac{\varepsilon}{1-\mathcal{Q}}(\varphi^2 + \varphi') + \mathcal{O}(\varepsilon^2). \end{aligned} \tag{57}$$

The potential contains a constant term, which gives the leading contribution to the free energy, a tadpole of order 1 and higher vertices that disappears in the limit $\varepsilon \to 0$. The Feynman rules for the effective action $\mathcal{Y}[\varphi,\rho]$ are such that each given order in ε is obtained as a sum of finite number of Feynman graphs. For the first two orders one obtains

$$F_0 = \oint\limits_{\mathcal{C}} \frac{dx}{2\pi} \, \mathrm{Li}_2[\mathcal{Q}(x)], \tag{58}$$

$$F_1 = -\tfrac{1}{2} \oint\limits_{\mathcal{C}\times\mathcal{C}} \frac{dx\, du}{(2\pi)^2} \, \frac{\log[1-\mathcal{Q}(x)] \, \log[1-\mathcal{Q}(u)]}{(x-u)^2}. \tag{59}$$

where the double integral is understood as a principal value. The actual choice of the contour $\mathcal{C}$ is a subtle issue and depends on the analytic properties of the function $\mathcal{Q}(x)$. The contour should be placed in such away that it does not cross the cuts of the integrand (Figs. 2 and 3).

Returning to the scalar product and ignoring the trivial factors in (18), we find that the first two coefficients of the semi-classical expansion are given by Eqs. (58) and (59) with

$$\mathcal{Q} = e^{ip_{\mathbf{u}}+ip_{\mathbf{v}}}. \tag{60}$$

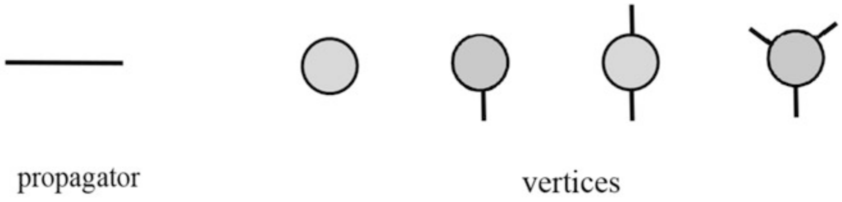
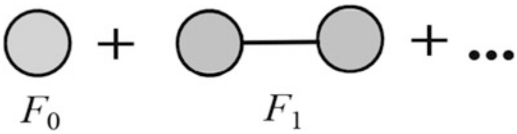


Fig. 2 Feynman rules for the effective field theory

Fig. 3 The leading and the subleading orders of the vacuum energy



5 Discussion

In these notes we reviewed the field-theoretical approach to the computation of scalar products of on-shell/off-shell Bethe vectors in the generalised model with $SU(2)$ rational R-matrix, which leads to a systematic procedure for computing the semi-classical expansion. The results reported here represent a slight generalisation if those already reported in [17, 21, 22]. We hope that the field-theoretical method could be used to compute scalar products in integrable models associated with higher rank groups, using the fact that the integrands in the multiple contour integrals of in [7–10] is expressed as products of $\mathscr{A}$-functionals.

The problem considered here is formally similar to the problem of computing the instanton partition functions in $\mathcal{N} = 1$ and $\mathcal{N} = 2$ SYM [31–33]. As a matter of fact, the scalar product in the form (35) is the grand-canonical version of the partition function of the $\mathcal{N} = 1$ SUSY in four dimensions, which was studied in a different large N limit in [36].

Our main motivation was the computation of the three-point function of heavy operators in $\mathcal{N} = 4$ four-dimensional SYM. Such operators are dual to classical strings in $AdS_5 \times S^5$ and can be compared with certain limit of the string-theory results. For a special class of three-point functions, the semi-classical expansion is readily obtained from that of the scalar product. The leading term F_0 should be obtained on the string theory side as the classical action of a minimal world sheet with three prescribed singularities. The comparison with the recent computation in [37] looks very encouraging. We expect that the meaning of the subleading term on the string theory side is that it takes account of the gaussian fluctuations around the minimal world sheet. In this context it would be interesting to obtain the subleading order of the heavy-heavy-light correlation function in the $su(2)$ sector in string theory [38–40]. In the near-plane-wave limit the subleading order was obtained in [41].

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Weak Poisson Structures on Infinite Dimensional Manifolds and Hamiltonian Actions

K.-H. Neeb, H. Sahlmann, and T. Thiemann

Abstract We introduce a notion of a weak Poisson structure on a manifold M modeled on a locally convex space. This is done by specifying a Poisson bracket on a subalgebra $\mathcal{A} \subseteq C^\infty(M)$ which has to satisfy a non-degeneracy condition (the differentials of elements of $\mathcal{A}$ separate tangent vectors) and we postulate the existence of smooth Hamiltonian vector fields. Motivated by applications to Hamiltonian actions, we focus on affine Poisson spaces which include in particular the linear and affine Poisson structures on duals of locally convex Lie algebras. As an interesting byproduct of our approach, we can associate to an invariant symmetric bilinear form κ on a Lie algebra $\mathfrak{g}$ and a κ-skew-symmetric derivation D a weak affine Poisson structure on $\mathfrak{g}$ itself. This leads naturally to a concept of a Hamiltonian G-action on a weak Poisson manifold with a $\mathfrak{g}$-valued momentum map and hence to a generalization of quasi-hamiltonian group actions.

1 Introduction

In geometric mechanics symplectic and Poisson manifolds form the basic underlying geometric structures on manifolds. In the finite dimensional context, this provides a perfect setting to model systems whose states depend on finitely many parameters [17]. In the context of symplectic geometry, resp., Hamiltonian flows, Banach manifolds were introduced by Marsden [16], and Weinstein obtained

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a Darboux Theorem for strong symplectic Banach manifolds [29].[1] Schmid's monograph [27] provides a nice introduction to infinite dimensional Hamiltonian systems. For more recent results on Banach–Lie–Poisson spaces we refer to the recent work of Ratiu, Odzijewicz and Beltita [2, 23–26] and in particular for [8] for certain classes of locally convex spaces.

In the present note we describe a possible approach to Poisson structures on infinite dimensional manifolds that works naturally for smooth manifolds modeled on locally convex spaces, such as spaces of test functions, smooth sections of bundles and distributions [9, 21]. Our requirements are minimal in the sense that any other concept of an infinite dimensional Poisson manifold should at least satisfy our requirements.

In the finite dimensional case, the main focus of the theory of Poisson manifolds lies on the Poisson tensor Λ which is a section of the vector bundle $\Lambda^2(T(M))$ and defines a skew-symmetric form on each cotangent space $T_m^*(M)$. This does not generalize naturally to infinite dimensional manifolds because continuous bilinear maps may be of infinite rank. Our main point is to define a weak Poisson structure on a smooth manifold M by a Poisson bracket $\{\cdot,\cdot\}$ on a unital subalgebra $\mathcal{A} \subseteq C^\infty(M)$ satisfying the Leibniz rule and the Jacobi identity. In addition to that, we require that $\mathcal{A}$ is large in the sense that, for every $m \in M$, the differentials $\mathtt{d}F(m)$, $F \in \mathcal{A}$, separate the points in the tangent space $T_m(M)$. We also require for each $H \in \mathcal{A}$ the existence of a smooth Hamiltonian vector field X_H determined by $\{F, H\} = X_H F$ for every $F \in \mathcal{A}$. The main difference to the traditional approaches is that we do not require the Poisson bracket to be defined on all smooth functions, instead we restrict the class of admissible differentials to define Poisson brackets. It turns out that this rather algebraic approach is strong enough to capture the main formal features of momentum maps and affine Poisson structures on locally convex space as well as their relations with Lie algebras and their duals. In the affine case $M = V$, the minimal choice of $\mathcal{A}$ is the subalgebra generated by a point separating subspace V_* of the topological dual space V'. In this context one can also enlarge the algebra $\mathcal{A}$ by adding certain exponential functions and extend the Poisson bracket appropriately; see [28] for such constructions.

Although our approach largely ignores geometric difficulties we hope that it provides a natural language for dealing with Poisson structures on rather general infinite dimensional manifolds and that this leads to precise specifications of the key difficulties arising for concrete examples. A discussion of similar structures is used in the context of hydrodynamics [13] and for free boundary problems [15].

One of our main objectives was to understand the nature of the affine Poisson structures arising implicitly on Lie algebras of smooth loops in the context of Hamiltonian actions of loop groups and quasihamiltonian actions [1] (Sect. 4).

Although the construction of the tangent bundle $T(M)$ of a locally convex manifold M and the Lie algebra $\mathcal{V}(M)$ of smooth vector fields on M follows pretty

[1] A symplectic form ω on M is called *strong* if, for every $p \in M$, every continuous linear functional on $T_p(M)$ is of the form $\omega_p(v, \cdot)$ for some $v \in T_p(M)$.

much the constructions from finite dimensional geometry (cf. [10, Chap. 8]), serious difficulties arise when one wants to put a smooth manifold structure on the cotangent bundle $T'(M) := \dot{\cup}_{p\in M} T_p(M)'$ whose elements are continuous linear functionals on the tangent spaces $T_p(M)$ of M. This works well for Banach manifolds when the dual spaces carry the norm topology, but if M is not modeled on a Banach space, there may not be any topology for which the natural chart changes for $T'(M)$ are smooth. Accordingly, cotangent bundles can be constructed naturally if M is an open subset of a locally convex space or if the tangent bundle $T(M)$ is trivial, in which case $T(M) \cong M \times V$ leads to $T'(M) \cong M \times V'$, so that any locally convex topology on V' leads to a manifold structure on $T'(M)$. This works in particular for Lie groups.

Since our main concern is with the algebraic framework for Poisson structures, we do not go into analytical aspects of symplectic leaves which are already subtle for Poisson manifolds not modeled on Hilbert spaces [2, 3, 26].

The structure of this paper is as follows. In Sect. 2 we introduce the notion of a weak Poisson manifold and discuss various types of examples, in particular affine ones and weak symplectic manifolds. We also take a brief look at Poisson maps arising from inclusions of submanifolds and from submersions. In Sect. 3 we then turn to momentum maps, which we consider as Poisson morphisms into affine Poisson spaces which arise naturally as subspaces of the dual of a Lie algebra $\mathfrak{g}$. If $\mathfrak{g}$ is the Lie algebra of a Lie group, we also have a global structure coming from the corresponding coadjoint action, but unfortunately there need not be any locally convex topology on $\mathfrak{g}'$ for which the coadjoint action is smooth.

As an interesting byproduct of our approach, one can use an invariant symmetric bilinear form κ and a κ-skew-symmetric derivation D on a Lie algebra $\mathfrak{g}$ to obtain a weak affine Poisson structure on $\mathfrak{g}$ itself. This leads naturally to a concept of a Hamiltonian G-action on a weak Poisson manifold with a $\mathfrak{g}$-valued momentum map. For the classical case where G is the loop group $\mathcal{L}(K) = C^\infty(\mathbb{S}^1, K)$ of a compact Lie group and the derivation is given by the derivative, we thus obtain the affine action on $\mathfrak{g} = \mathcal{L}(\mathfrak{k})$ which corresponds to the natural action of the gauge group $\mathcal{L}(K)$ on gauge potentials on the trivial K-bundle over $\mathbb{S}^1$. At this point we obtain a natural concept of a Hamiltonian $\mathcal{L}(K)$-space generalizing the one used in the context of quasi-hamiltonian K-spaces, where it is only defined for weak symplectic manifolds [1, 19].

2 Infinite Dimensional Poisson Manifolds

In this section we introduce the concept of a weak Poisson structure on a locally convex manifold. Our requirements are minimal in the sense that any other concept of an infinite dimensional Poisson manifold should at least satisfy our requirements. The concept discussed below is strong enough to capture the main algebraic features of momentum maps and the Poisson structure on the dual of a Lie algebra.

2.1 Locally Convex Manifolds

We first recall the basic concepts concerning infinite dimensional manifolds modeled on locally convex spaces. Throughout these notes all topological vector spaces are assumed to be Hausdorff.

Let E and F be locally convex spaces, $U \subseteq E$ open and $f\colon U \to F$ a map. Then the *derivative of* f *at* x *in the direction* h is defined as

$$\mathtt{d}f(x)(h) := (\partial_h f)(x) := \frac{d}{dt}\Big|_{t=0} f(x+th) = \lim_{t\to 0}\frac{1}{t}(f(x+th)-f(x))$$

whenever it exists. The function f is called *differentiable at* x if $\mathtt{d}f(x)(h)$ exists for all $h \in E$. It is called *continuously differentiable*, if it is differentiable at all points of U and

$$\mathtt{d}f\colon U \times E \to F, \quad (x,h) \mapsto \mathtt{d}f(x)(h)$$

is a continuous map. The map f is called a C^k*-map*, $k \in \mathbb{N}\cup\{\infty\}$, if it is continuous, the iterated directional derivatives

$$\mathtt{d}^j f(x)(h_1,\ldots,h_j) := (\partial_{h_j}\cdots\partial_{h_1} f)(x)$$

exist for all integers $1 \le j \le k$, $x \in U$ and $h_1,\ldots,h_j \in E$, and all maps $\mathtt{d}^j f\colon U \times E^j \to F$ are continuous. As usual, C^∞-maps are called *smooth*.

Once the concept of a smooth function between open subsets of locally convex spaces is established, it is clear how to define a locally convex smooth manifold. The tangent bundle $T(M)$ and the Lie algebra $\mathcal{V}(M)$ of smooth vector fields on M are now defined as in the finite dimensional case (cf. [10, Chap. 8]) and differential p-forms are defined as smooth functions on the p-fold Whitney sum $T(M)^{\oplus p}$. Although it is clear what the cotangent bundle is as a set, namely the disjoint union $T'(M) := \dot{\bigcup}_{p\in M} T_p(M)'$ of the topological dual spaces of the tangent spaces, in general it is not clear how to put a smooth manifold structure on $T'(M)$. This is due to the fact that the dual V' of the model space V need not carry a locally convex topology for which the chart changes for $T'(M)$ are smooth. For a Banach manifold this works with the natural Banach space structure on the dual, and it also works for manifolds with a single chart and the weak-$*$-topology on the dual, but for general locally convex manifolds M there seems to be no natural smooth structure on $T'(M)$ (see [9, 21] for more details).

2.2 Weak Poisson Manifolds

Definition 2.1. Let M be a smooth manifold modeled on a locally convex space. A *weak Poisson structure* on M is a unital subalgebra $\mathcal{A} \subseteq C^\infty(M,\mathbb{R})$, i.e., it contains the constant functions and is closed under pointwise multiplication, with the following properties:

(P1) $\mathcal{A}$ is endowed with a *Poisson bracket* $\{\cdot,\cdot\}$, this means that it is a Lie bracket, i.e.,

$$\{F,G\} = -\{G,F\}, \quad \{F,\{G,H\}\} = \{\{F,G\},H\} + \{G,\{F,H\}\}, \tag{J}$$

and it satisfies the Leibniz rule

$$\{F,GH\} = \{F,G\}H + G\{F,H\}. \tag{L}$$

(P2) For every $m \in M$ and $v \in T_m(M)$ satisfying $\mathrm{d}F(m)v = 0$ for every $F \in \mathcal{A}$ we have $v = 0$.

(P3) For every $F \in \mathcal{A}$, there exists a smooth vector field $X_H \in \mathcal{V}(M)$ with $X_H F = \{F,H\}$ for $F,H \in \mathcal{A}$. It is called the corresponding *Hamiltonian vector field*.

If (P1–3) are satisfied, then we call the triple $(M,\mathcal{A},\{\cdot,\cdot\})$ a *weak Poisson manifold*.

Remark 2.2. (a) (P2) implies that the vector field X_H in (P3) is uniquely determined by the relation $\{F,H\}(m) = (X_H F)(m) = \mathrm{d}F(m)X_H(m)$ for every $F \in \mathcal{A}$.

(b) For $F,G,H \in \mathcal{A}$,

$$[X_F,X_G]H = \{\{H,G\},F\} - \{\{H,F\},G\} = \{H,\{G,F\}\} = X_{\{G,F\}}H,$$

so that

$$[X_F,X_G] = X_{\{G,F\}} \quad \text{for} \quad F,G \in \mathcal{A}. \tag{1}$$

We also note that the Leibniz rule leads to

$$X_{FG} = FX_G + GX_F \quad \text{for} \quad F,G \in \mathcal{A}. \tag{2}$$

(c) If $\{\cdot,\cdot\}\colon \mathcal{A}\times\mathcal{A} \to \mathcal{A}$ is a skew-symmetric bracket satisfying the Leibniz rule, then the Jacobiator

$$\begin{aligned} J(F,G,H) &:= \{F,\{G,H\}\} + \{G,\{H,F\}\} + \{H,\{F,G\}\} \\ &= \{F,\{G,H\}\} - \{G,\{F,H\}\} - \{\{F,G\},H\} \end{aligned}$$

defines an alternating map $\mathcal{A}^3 \to \mathcal{A}$ which satisfies the Leibniz rule in every argument. It vanishes if and only if $\{\cdot,\cdot\}$ is a Lie bracket, i.e., if (P1) is satisfied. For a subset $\mathcal{S} \subseteq \mathcal{A}$ generating $\mathcal{A}$ as a unital algebra, this observation implies that J vanishes if it vanishes for $F,G,H \in \mathcal{S}$.

(d) If (P1) and (P2) are satisfied, then (2) implies that the subspace of all elements $X \in \mathcal{A}$ for which X_H as in (P3) exists is a subalgebra with respect to the pointwise product. Therefore it suffices to verify (P3) for a generating subset $\mathcal{S} \subseteq \mathcal{A}$.

Remark 2.3. From (P3) it follows that the value of the Poisson bracket

$$\{F,G\}(p) = \mathrm{d}F(p)X_G(p) = -\mathrm{d}G(p)X_F(p)$$

in $p \in M$ only depends on $\mathrm{d}F(p)$, resp., $\mathrm{d}G(p)$. On the separating subspace

$$T_p(M)_* := \{\mathrm{d}F(p)\colon F \in \mathcal{A}\} \subseteq T_p(M)'$$

we thus obtain a well-defined skew-symmetric bilinear map

$$\Lambda_p\colon T_p(M)_* \times T_p(M)_* \to \mathbb{R}, \quad \Lambda_p(\alpha,\beta) := \{F,G\}(p)$$

$$\text{for} \quad \alpha = \mathrm{d}F(p), \ \ \beta = \mathrm{d}G(p).$$

This suggests an extension of the Poisson bracket to the subalgebra $\mathcal{B} \subseteq C^\infty(M)$ of those functions F, for which $\mathrm{d}F(p) \in T_p(M)_*$ holds for every $p \in M$, by the formula

$$\{F,G\}(p) := \Lambda_p(\mathrm{d}F(p), \mathrm{d}G(p)).$$

At this point it is not clear that this results in a smooth function $\{F,G\}$ nor that, for $G \in \mathcal{B}$, there exists a smooth vector field X_G on M such that $\{F,G\} = X_G F$ holds for $F \in \mathcal{B}$ (cf. Example 2.13 below for criteria). If both these conditions are satisfied and, in addition, the Poisson bracket on $\mathcal{B}$ satisfies the Jacobi identity, then we can also work with the larger algebra $\mathcal{B}$ instead of $\mathcal{A}$.

Remark 2.4. Suppose that M is a Banach manifold. The notion of a Banach–Poisson manifold used in [25, 26] differs from our concept of a weak Poisson structure on M in the sense that it is required that $\mathcal{A} = C^\infty(M)$ and that every continuous linear functional on the dual space $T_p(M)'$ of the form $\alpha^\sharp := \Lambda_p(\alpha,\cdot) \in T_p(M)''$ can be represented by an element of $T_p(M)$.

Remark 2.5. Let $(M,\mathcal{A},\{\cdot,\cdot\})$ be a weak Poisson manifold. For $p \in M$, we call

$$C_p(M) := \{X_F(p)\colon F \in \mathcal{A}\} \subseteq T_p(M)$$

the *characteristic subspace in* p. Then

$$\omega_p\colon C_p(M) \times C_p(M) \to \mathbb{R},$$

$$\omega_p(X_F(p), X_G(p)) := \{F,G\}(p) = \mathrm{d}F(p)X_G(p) = -\mathrm{d}G(p)X_F(p)$$

is a well-defined skew-symmetric form. On the Lie algebra

$$\mathrm{ham}(M,\mathcal{A}) := \{X_F\colon F \in \mathcal{A}\} \subseteq \mathcal{V}(M)$$

of hamiltonian vector fields, every form ω_p defines a 2-cocycle

$$\tilde{\omega}_p(X,Y) := \omega_p(X(p), Y(p))$$

because

$$\tilde{\omega}_p([X_F, X_G], X_H) = \tilde{\omega}_p(X_{\{G,F\}}, X_H) = \{\{G,F\}, H\}(p)$$

and $\{\cdot,\cdot\}$ satisfies the Jacobi identity.

2.3 Examples of Weak Poisson Manifolds

We now turn to natural examples of weak Poisson manifolds.

Example 2.6 (Finite dimensional Poisson manifolds). Every finite dimensional (paracompact) Poisson manifold (M, Λ) carries a natural weak Poisson structure with $\mathcal{A} := C^\infty(M)$ and $\{F,G\}(m) := \Lambda_m(\mathtt{d}F(m), \mathtt{d}G(m))$. Then $T_m(M)^* = \{\mathtt{d}F(m) \colon F \in \mathcal{A}\}$ implies (P2) and the existence of $X_H \in \mathcal{V}(M)$ follows from the fact that every derivation of the algebra $C^\infty(M)$ is of the form $F \mapsto XF$ for some smooth vector field $X \in \mathcal{V}(M)$ [10, Thm. 8.4.18].

Remark 2.7. Let V be a real vector space. We call a linear subspace $V_* \subseteq V^*$ *separating* if $\alpha(v) = 0$ for every $\alpha \in V_*$ implies $v = 0$. This implies that, for every finite dimensional subspace $F \subseteq V$, the restriction map $V_* \to F^*$ is surjective, and this in turn implies that the natural map $S(V) \to \mathbb{R}^{V_*}$ of the symmetric algebra $S(V)$ over V to the algebra of functions on V_* is injective.

Theorem 2.8 (Affine Poisson Structures). *Let V be a locally convex space and $V_* \subseteq V'$ be a separating subspace. Further, let*

(a) $\Lambda \colon V_* \times V_* \to \mathbb{R}$ be a skew-symmetric bilinear map with the property that, for every $\alpha \in V_*$, there exists an element $\alpha^\sharp \in V$ with $\Lambda(\beta,\alpha) = \beta(\alpha^\sharp)$ for every $\beta \in V_*$, and

(b) let $[\cdot,\cdot]_0$ be a Lie bracket on V_* for which

(i) Λ is a 2-cocycle, i.e., $\Lambda([\alpha,\beta],\gamma) + \Lambda([\beta,\gamma],\alpha) + \Lambda([\gamma,\alpha],\beta) = 0$ for $\alpha,\beta,\gamma \in V_*$.

(ii) The linear maps $\mathrm{ad}_0\,\alpha \colon V_* \to V_*, \beta \mapsto [\alpha,\beta]_0$ have continuous adjoint maps $\mathrm{ad}_0^*\,\alpha \colon V \to V$ defined by $\beta(\mathrm{ad}_0^*\,\alpha v) = [\alpha,\beta]_0(v)$ for $\alpha,\beta \in V_*$ and $v \in V$.

This leads to a Lie algebra structure on the space $\hat{V}_* := \mathbb{R}1 \oplus V_*$ of affine functions on V by

$$[t+\alpha, s+\beta] := \Lambda(\alpha,\beta) + [\alpha,\beta]_0 \quad \text{for} \quad t,s \in \mathbb{R}, \alpha,\beta \in V_*.$$

Let $\mathcal{A} \cong S(V_*) \subseteq C^\infty(V)$ denote the unital subalgebra generated by V_*. Then $\mathtt{d}F(v) \in V_*$ for $F \in \mathcal{A}$ and $v \in V$, and

$$\{F, G\}(v) := \langle [dF(v), dG(v)], v\rangle \quad \textit{for} \quad v \in V, F, G \in \mathcal{A}$$

defines a weak Poisson structure on V.

This weak Poisson structure is *affine* in the sense that, for $\alpha, \beta \in V_*$, the function $\{\alpha, \beta\}$ on V is affine.

Proof. First we observe that, for every $F \in \mathcal{A}$ and $v \in V$, the Leibniz rule implies that the differential $\mathrm{d}F(v)$ is contained in V_*. Therefore $\{\cdot,\cdot\}$ defines a skew-symmetric bracket $\mathcal{A} \times \mathcal{A} \to \mathbb{R}^V$ satisfying the Leibniz rule. For $\alpha, \beta \in V_*$, the function $\{\alpha, \beta\}$ is contained in $\hat{V}_* \subseteq \mathcal{A}$, and this implies that $\{\mathcal{A}, \mathcal{A}\} \subseteq \mathcal{A}$. To verify the Jacobi identity, it suffices to do this on the generating subspace $V_* \subseteq \mathcal{A}$ (Remark 2.2(c)). For $\alpha, \beta, \gamma \in V_*$ we have $\{\alpha, \{\beta, \gamma\}\} = [\alpha, [\beta, \gamma]]$, so that (P1) follows from the Jacobi identity in the Lie algebra $\hat{V}_*$. Condition (P2) follows from the fact that $V_* \subseteq \mathcal{A}$ separates the points of V. To verify (P3), we first observe that, for $\alpha \in V_*$ and $F \in \mathcal{A}$, we have

$$\begin{aligned}\{F, \alpha\}(v) &= \langle [\mathrm{d}F(v), \alpha], v\rangle = \Lambda(\mathrm{d}F(v), \alpha) + [\mathrm{d}F(v), \alpha]_0(v) \\ &= \mathrm{d}F(v)(\alpha^\sharp) - \mathrm{d}F(v)(\mathrm{ad}_0\,\alpha)^* v.\end{aligned}$$

Therefore the affine vector field

$$X_\alpha(v) := \alpha^\sharp - (\mathrm{ad}_0\,\alpha)^* v \tag{3}$$

is a smooth vector field satisfying (P3). Now (P3) follows from an easy induction and (2) (cf. Remark 2.2(d)). This completes the proof. ☐

Specializing to the two particular cases $[\cdot,\cdot]_0 = 0$ and $\Lambda = 0$, we obtain constant, resp., linear Poisson structures as special cases.

Corollary 2.9 (Constant Poisson Structures). *Let V be a locally convex space, $V_* \subseteq V'$ be a separating subspace and $\Lambda\colon V_* \times V_* \to \mathbb{R}$ be a skew-symmetric bilinear map with the property that, for every $\alpha \in V_*$, there exists an element $\alpha^\sharp \in V$ with $\Lambda(\beta, \alpha) = \beta(\alpha^\sharp)$ for every $\beta \in V_*$. Let $\mathcal{A} \subseteq C^\infty(V)$ denote the unital subalgebra generated by the linear functions in V_*. Then*

$$\{F, G\}(v) := \Lambda(dF(v), dG(v)) \quad \textit{for} \quad v \in V, F, G \in \mathcal{A}$$

defines a weak Poisson structure on V.

Example 2.10 (Canonical Poisson Structures). Let V be a locally convex space and $V_* \subseteq V'$ be a separating subspace, endowed with a locally convex topology for which the pairing $V_* \times V \to \mathbb{R}$ is separately continuous. We consider the product space $W := V \times V_*$. Then $W_* := V_* \times V$ is a separating subspace of $W' \cong V' \times (V_*)'$,

$$\Lambda((\alpha, v), (\alpha', v')) := \alpha(v') - \alpha'(v)$$

is a skew-symmetric bilinear form on W_*, and for $(\alpha, v)^\sharp := (v, -\alpha) \in W$, we have

$$\Lambda((\alpha, v), (\alpha', v')) = \langle(\alpha, v), (v', -\alpha')\rangle = \langle(\alpha, v), (\alpha', v')^\sharp\rangle.$$

Therefore we obtain with Corollary 2.9 on W a constant weak Poisson structure with $\mathcal{A} \cong S(W_*)$ which is given on $W_* \times W_*$ by Λ.

Corollary 2.11 (Linear Poisson Structures). *Let V be a locally convex space, $V_* \subseteq V'$ be a separating subspace and $[\cdot, \cdot]$ be a Lie bracket on V_* for which the linear maps $\operatorname{ad}\alpha\colon V_* \to V_*$ have continuous adjoint maps $\operatorname{ad}^* \alpha\colon V \to V$. Let $\mathcal{A} \subseteq C^\infty(V)$ denote the unital subalgebra generated by V_*. Then*

$$\{F, G\}(v) := \langle[dF(v), dG(v)], v\rangle \quad \textit{for} \quad v \in V, F, G \in \mathcal{A}$$

defines a weak Poisson structure on V.

For a version of the preceding corollary for Banach spaces, we refer to [26, Thm. 3.2] and [23]. In this context V is a Banach space and $V_* := V'$ is the dual Banach space. Typical examples of Banach–Lie–Poisson space are the duals of C^*-algebras and preduals of W^*-algebras. Here the example of the space $V = \mathrm{Herm}_1(\mathcal{H})$ of hermitian trace class operators on a Hilbert space $\mathcal{H}$ is of particular importance in Quantum Mechanics. By the trace pairing, its dual can be identified with the Lie algebra of skew-hermitian compact operators.

Remark 2.12. In the context of Theorem 2.8 one can enlarge the algebra $\mathcal{A} \subseteq C^\infty(V)$ under the following topological assumptions. We assume that V_* carries a locally convex topology for which

(A1) the pairing $\langle\cdot, \cdot\rangle\colon V_* \times V \to \mathbb{R}$ is continuous,
(A2) the Lie bracket $[\cdot, \cdot]\colon V_* \times V_* \to \hat{V}_*$ is continuous,
(A3) the map $V_* \times V \to V, (\alpha, v) \mapsto (\operatorname{ad}_0 \alpha)^* v$ is continuous, and
(A4) the map $\sharp\colon V_* \to V$ is continuous.

Then

$$\mathcal{B} := \{F \in C^\infty(V) \colon \mathrm{d}F \in C^\infty(V, V_*)\}$$

is a subalgebra of $C^\infty(V)$ with respect to the pointwise multiplication. For $F, G \in \mathcal{B}$, the function

$$\{F, G\}(v) := [\mathrm{d}F(v), \mathrm{d}G(v)](v) = \langle[\mathrm{d}F(v), \mathrm{d}G(v)]_0, v\rangle + \Lambda(\mathrm{d}F(v), \mathrm{d}G(v))$$

is smooth and so is the vector field

$$X_G(v) = -(\operatorname{ad}_0 \mathrm{d}G(v))^* v + \mathrm{d}G(v)^\sharp$$

on V (cf. (3)) which satisfies

$$\{F, G\} = X_G F = \langle \mathtt{d}F, X_G\rangle \quad \text{and}$$

$$\langle \alpha, X_G(v)\rangle = \langle [\alpha, \mathtt{d}G(v)], v\rangle \quad \text{for} \quad \alpha \in V_*, v \in V.$$

For every $F \in \mathcal{B}$, we now identify $\mathtt{d}^2 F$ with a smooth function $\tilde{\mathtt{d}}^2 F \colon V \times V \to V_*$ which is linear in the second argument. The symmetry of the second derivative then leads to the relation

$$\mathtt{d}^2 F_v(w, u) = \langle \tilde{\mathtt{d}}^2 F_v(w), u\rangle = \langle \tilde{\mathtt{d}}^2 F_v(u), w\rangle.$$

We now show that $\{F, G\} \in \mathcal{B}$. The calculation

$$\begin{aligned} \mathtt{d}\{F, G\}(v)(h) &= [\mathtt{d}^2 F_v(h), \mathtt{d}G(v)](v) + [\mathtt{d}F(v), \mathtt{d}^2 G_v(h)](v) \\ &\quad + \langle [\mathtt{d}F(v), \mathtt{d}G(v)]_0, h\rangle \\ &= \mathtt{d}^2 F_v(h, X_G(v)) - \mathtt{d}^2 G_v(h, X_F(v)) + \langle [\mathtt{d}F(v), \mathtt{d}G(v)]_0, h\rangle \\ &= \langle \tilde{\mathtt{d}}^2 F_v(X_G(v)), h\rangle - \langle \tilde{\mathtt{d}}^2 G_v(X_F(v)), h\rangle + \langle [\mathtt{d}F(v), \mathtt{d}G(v)]_0, h\rangle \end{aligned}$$

shows that

$$\mathtt{d}\{F, G\}(v) = \tilde{\mathtt{d}}^2 F_v(X_G(v)) - \tilde{\mathtt{d}}^2 G_v(X_G(v)) + [\mathtt{d}F(v), \mathtt{d}G(v)]_0$$

is a smooth V_*-valued function. Therefore the Poisson bracket extends to $\mathcal{B}$. From

$$\begin{aligned} \langle [\mathtt{d}F(v), \mathtt{d}G(v)]_0, X_H(v)\rangle &= \langle [\mathtt{d}F(v), \mathtt{d}G(v)]_0, -(\mathrm{ad}_0\, \mathtt{d}H(v))^* v + \mathtt{d}H(v)^\sharp\rangle \\ &= \langle \big[[\mathtt{d}F(v), \mathtt{d}G(v)]_0, \mathtt{d}H(v)\big]_0, v\rangle \\ &\quad + \Lambda([\mathtt{d}F(v), \mathtt{d}G(v)]_0, \mathtt{d}H(v)) \\ &= \big[[\mathtt{d}F(v), \mathtt{d}G(v)]_0, \mathtt{d}H(v)\big](v) \\ &= \big[[\mathtt{d}F(v), \mathtt{d}G(v)], \mathtt{d}H(v)\big](v) \end{aligned}$$

we now derive

$$\begin{aligned} \{\{F, G\}, H\}(v) &= \mathtt{d}^2 F_v(X_H(v), X_G(v)) - \mathtt{d}^2 G_v(X_H(v), X_F(v)) \\ &\quad + \big[[\mathtt{d}F(v), \mathtt{d}G(v)], \mathtt{d}H(v)\big](v). \end{aligned}$$

Now the symmetry of the second derivative implies that the Poisson bracket on $\mathcal{B}$ satisfies the Jacobi identity, so that $(V, \mathcal{B}, \{\cdot, \cdot\})$ also is a weak Poisson structure on V.

If V is a Banach space with $V_* = V'$ (in particular if $\dim V < \infty$), then the preceding construction actually leads to all smooth functions $\mathcal{B} = C^\infty(V)$, so that we are in the context of Banach–Lie–Poisson spaces. However, one can do better:

Remark 2.13 (Glöckner's Locally Convex Poisson Vector Spaces). To obtain Poisson structures on V for the algebra $\mathcal{A} = C^\infty(V)$ of all smooth functions, one has to impose stronger assumptions on topologies on V and V_*. In [8, Def. 16.35] these are encoded in the concept of a *locally convex Poisson vector space*, which requires that the locally convex space V has the following properties:

(a) For the topology of uniform convergence on compact ($S = c$), resp., bounded ($S = b$) subsets of V (or even more general classes S of subsets) the linear injection $\eta_V: V \to (V'_S)'_S$, $\eta_V(v)(\alpha) = \alpha(v)$ is a topological embedding.
(b) The topology on every product space V^n is determined by its restriction to compact subsets (V is a k^∞ space).
(c) The dual space V'_S carries an S-hypocontinuous Lie bracket $[\cdot,\cdot]$, i.e., it is separately continuous and continuous on all subsets of the form $V'_S \times B$, $B \in S$.
(d) The Lie bracket on V'_S satisfies $\eta_V(v) \circ \mathrm{ad}_\alpha \in \eta_V(V)$ for $v \in V$ and $\alpha \in V'_S$.

If these conditions are satisfied, then [8, Thm. 16.40] asserts that, for two smooth functions $F, G \in C^\infty(V)$, their Poisson bracket

$$\{F, G\}(v) := \langle [\mathtt{d}F(v), \mathtt{d}G(v)], v\rangle$$

is smooth and that

$$X_F(v) := -\eta_V^{-1}(\eta_V(v) \circ \mathrm{ad}(\mathtt{d}F(v)))$$

is a smooth vector field satisfying $\{G, F\} = X_F G$. As in the preceding remark it now follows that $(V, C^\infty(V), \{\cdot,\cdot\})$ is a weak Poisson manifold. This is the special case of Corollary 2.11, where $V_* = V'_S$.

Example 2.14. (a) Let $\mathfrak{g}$ be a locally convex Lie algebra, i.e., a locally convex space with a continuous Lie bracket. We write $\mathfrak{g}'$ for its topological dual space, endowed with the weak-$*$-topology. Then Corollary 2.11 applies to $V := \mathfrak{g}'$ and $V_* := \mathfrak{g}$ because, for each $X \in \mathfrak{g}$, the bracket map $\mathrm{ad}\, X: \mathfrak{g} \to \mathfrak{g}$ has a continuous adjoint $\mathrm{ad}^* X: \mathfrak{g}' \to \mathfrak{g}'$. If $\mathfrak{g}$ is finite dimensional, we thus obtain the KKS (Kirillov–Kostant–Souriau) Poisson structure on $\mathfrak{g}^* = \mathfrak{g}'$.

(b) The preceding construction can be varied by changing the topology on $\mathfrak{g}'$ and by passing to a smaller subspace. Let $\mathfrak{g}_* \subseteq \mathfrak{g}'$ be a separating subspace on which the adjoint maps $\mathrm{ad}^* X\alpha := \alpha \circ \mathrm{ad}\, X$ induce for each $X \in \mathfrak{g}$ a continuous linear map. Then Corollary 2.11 applies with $V := \mathfrak{g}_*$ and $V_* := \mathfrak{g}$, and we thus obtain a weak Poisson structure on $\mathfrak{g}_*$ for which the Hamiltonian functions $H_X(\alpha) = \alpha(X)$ satisfy

$$\{H_X, H_Y\} = H_{[X,Y]} \quad \text{for} \quad X, Y \in g.$$

(c) Suppose that $\mathfrak{g}$ is a locally convex Lie algebra and $\kappa: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is a continuous non-degenerate symmetric bilinear form which is invariant under the adjoint representation, i.e.,

$$\kappa([x, y], z) + \kappa(y, [x, z]) = 0 \quad \text{for} \quad x, y, z \in \mathfrak{g}.$$

Then the natural map

$$\flat: \mathfrak{g} \to \mathfrak{g}', \quad X^\flat(Y) := \kappa(X, Y)$$

is injective and $\mathfrak{g}$-equivariant with respect to the adjoint and coadjoint representation, respectively. We may thus apply (b) with $\mathfrak{g}_* = \mathfrak{g}^\flat = \{X^\flat: X \in \mathfrak{g}\} \cong \mathfrak{g}$ to obtain a linear weak Poisson structure on $\mathfrak{g}$ with $\mathcal{A} \cong S(\mathfrak{g})$. The Hamiltonian functions $X^\flat(Y) = \kappa(X, Y)$ satisfy

$$\{X^\flat, Y^\flat\} = [X, Y]^\flat \quad \text{for} \quad X, Y \in \mathfrak{g}.$$

(d) Let $\mathfrak{g}$ be a locally convex Lie algebra and $\omega: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be a continuous 2-cocycle, i.e.,

$$\omega([X, Y], Z) + \omega([Y, Z], X) + \omega([Z, X], Y) = 0,$$

so that $\hat{\mathfrak{g}} = \mathbb{R} \oplus \mathfrak{g}$ is a locally convex Lie algebra with respect to the Lie bracket

$$[(t, X), (s, Y)] := (\omega(X, Y), [X, Y]).$$

We call it the *central extension defined by* ω. Identifying the element $(t, X) \in \hat{\mathfrak{g}}$ with the affine function $\alpha \mapsto t + \alpha(X)$ on $\mathfrak{g}'$, we obtain with Theorem 2.8 (for $V = \mathfrak{g}'$ and $V_* = \mathfrak{g}$) an affine weak Poisson structure on $\mathfrak{g}'$, for which the Hamiltonian functions $H_X(\alpha) = \alpha(X)$, $X \in \mathfrak{g}$, satisfy

$$\{H_X, H_Y\} = H_{[X,Y]} + \omega(X, Y) \quad \text{for} \quad X, Y \in \mathfrak{g}.$$

The assumptions of Theorem 2.8 are satisfied with $\Lambda = \omega$.

More generally, suppose that $\mathfrak{g}_* \subseteq \mathfrak{g}'$ is subspace separating the points of $\mathfrak{g}$ and on which the adjoint maps $\operatorname{ad}^* X$, $X \in \mathfrak{g}$, induce continuous endomorphisms. Assume further that it contains all functionals $i_X\omega$, $X \in \mathfrak{g}$. Then Theorem 2.8 yields an affine weak Poisson structure on $\mathfrak{g}_*$ with

$$\{H_X, H_Y\} = H_{[X,Y]} + \omega(X, Y) \quad \text{for} \quad X, Y \in \mathfrak{g}.$$

(e) To combine (c) and (d), we assume that, in addition to $\mathfrak{g}$ and κ as in (c), we are given a κ-skew symmetric continuous derivation $D: \mathfrak{g} \to \mathfrak{g}$, so that

$\omega(X, Y) = \kappa(DX, Y)$ is a 2-cocycle. Then we obtain an affine weak Poisson structure $(\mathcal{A}, \{\cdot, \cdot\}_{\kappa,D})$ on $\mathfrak{g}$ with $\mathcal{A} \cong S(\mathfrak{g})$. The Hamiltonian functions $X^\flat(Y) := \kappa(X, Y)$ satisfy

$$\{X^\flat, Y^\flat\}_{\kappa,D} = [X, Y]^\flat + \kappa(DX, Y) \quad \text{for} \quad X, Y \in \mathfrak{g}.$$

An important concrete class of examples to which the preceding constructions apply arise from loop algebras. We shall return to this example later, when we connect with Hamiltonian actions of loop groups (cf. Definition 4.3).

Example 2.15. Let $\mathfrak{k}$ be a Lie algebra which carries a non-degenerate invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$. Then the *loop algebra of* $\mathfrak{k}$ is the Lie algebra $\mathfrak{g} := \mathcal{L}(\mathfrak{k}) := C^\infty(\mathbb{S}^1, \mathfrak{k})$, endowed with the pointwise bracket. We identify the circle $\mathbb{S}^1$ with $\mathbb{R}/\mathbb{Z}$ and, accordingly, elements of $\mathfrak{g}$ with 1-periodic functions on $\mathbb{R}$. Then $\kappa(\xi, \eta) = \int_0^1 \langle \xi(t), \eta(t) \rangle \, dt$ is a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$ and $D\xi = \xi'$ is a skew-symmetric derivation. We thus obtain on $\mathfrak{g}$ with Example 2.14(e) an affine weak Poisson structure with

$$\{\xi^\flat, \eta^\flat\} = [\xi, \eta]^\flat + \int_0^1 \langle \xi'(t), \eta(t) \rangle \, dt.$$

Remark 2.16. Typical predual spaces $\mathfrak{g}_* \subseteq \mathfrak{g}'$ arise from geometric situations as follows (cf. [12]):

(a) If $\mathfrak{g} = C^\infty(M, \mathfrak{k})$, where $\mathfrak{k}$ is finite dimensional with a non-degenerate invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$ and μ is a measure on M which is equivalent to Lebesgue measure in charts, then we have an invariant pairing $\mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, (\xi, \eta) \mapsto \int_M \langle \xi, \eta \rangle \, d\mu$ which leads to $\mathfrak{g}_* \cong \mathfrak{g}$.
(b) If M is a compact smooth manifold and $\mathfrak{g} = \mathcal{V}(M)$, the Fréchet–Lie algebra of smooth vector fields on M, then the space $\mathfrak{g}_*$ of density-valued 1-forms α on M has a natural $\mathrm{Diff}(M)$-invariant pairing given by $(X, \alpha) \mapsto \int_M \alpha(X)$. Locally the elements of $\mathfrak{g}_*$ are represented by smooth 1-forms, so that $\mathfrak{g}_*$ is much smaller than the dual space $\mathfrak{g}'$ whose elements are locally represented by distributions.

In finite dimensions, symplectic manifolds provide the basic building blocks of Poisson manifolds because every Poisson manifold is naturally foliated by symplectic leaves. In the infinite dimensional context the situation becomes more complicated because a symplectic form $\omega: V \times V \to \mathbb{R}$ on a locally convex space needs not represent every continuous linear functional on V. If it does, ω is called *strong*, and *weak* otherwise. Accordingly, a 2-form ω on a smooth manifold M is called *strong* if all forms ω_p, $p \in M$, are strong, and *weak* otherwise.

Definition 2.17. A *weak symplectic manifold* is a pair (M,ω) of a smooth manifold M and a closed non-degenerate 2-form ω. For a weak symplectic manifold we write

$$\mathrm{ham}(M,\omega) := \{X \in \mathcal{V}(M) \colon (\exists H \in C^\infty(M))\, i_X\omega = \mathrm{d}H\}$$

for the Lie algebra of Hamiltonian vector fields on M and

$$\mathrm{sp}(M,\omega) := \{X \in \mathcal{V}(M) \colon \mathcal{L}_X\omega = \mathrm{d}(i_X\omega) = 0\}$$

for the larger Lie algebra of symplectic vector fields (cf. [22] for related constructions).

Proposition 2.18 (Poisson Structure on Weak Symplectic Manifolds). *Let (M,ω) be a weak symplectic manifold. Then*

$$\mathcal{A} := \{H \in C^\infty(M) \colon (\exists X_H \in \mathcal{V}(M))\ dH = i_{X_H}\omega\}$$

is a unital subalgebra of $C^\infty(M)$ and

$$\{F,G\} := \omega(X_F, X_G) = dF(X_G) = X_G F$$

defines on $\mathcal{A}$ a Poisson bracket satisfying (P1) and (P3).

If, in addition, for $v \in T_m(M)$, the condition $\omega(X(m),v) = 0$ for every $X \in$ $\mathrm{ham}(M,\omega)$, implies $v = 0$, then (P2) is also satisfied.[2]

Proof. Since ω is non-degenerate, the vector field X_H is uniquely determined by H. For $F, G \in \mathcal{A}$ we have

$$\mathrm{d}(FG) = F\mathrm{d}G + G\mathrm{d}F = i_{FX_G+GX_F}\omega,$$

which implies that $\mathcal{A}$ is a unital subalgebra of $C^\infty(M)$.

The closedness of the 1-forms $i_{X_H}\omega$ implies that $\mathcal{L}_{X_H}\omega = 0$. Further, $[\mathcal{L}_X, i_Y] = i_{[X,Y]}$ leads to

$$\begin{aligned} i_{[X_F,X_G]}\omega &= [\mathcal{L}_{X_F}, i_{X_G}]\omega = \mathcal{L}_{X_F}\big(i_{X_G}\omega\big) = \mathcal{L}_{X_F}\mathrm{d}G \\ &= \mathrm{d}\big(i_{X_F}\mathrm{d}G\big) + i_{X_F}\mathrm{d}(\mathrm{d}G) = \mathrm{d}\big(i_{X_F}\mathrm{d}G\big) = \mathrm{d}\{G,F\}. \end{aligned}$$

Since ω is non-degenerate, this implies $\{\mathcal{A},\mathcal{A}\} \subseteq \mathcal{A}$ with

$$[X_F, X_G] = X_{\{G,F\}} \quad \text{for} \quad F, G \in \mathcal{A}. \tag{4}$$

[2]This condition is satisfied for finite dimensional symplectic manifolds, for strongly symplectic smoothly paracompact Banach manifolds (cf. [14]) and for symplectic vector spaces.

It is clear that $\{\cdot,\cdot\}$ is bilinear and skew-symmetric, and from $\mathrm{d}(FG) = F\mathrm{d}G + G\mathrm{d}F$ we conclude that it satisfies the Leibniz rule. So it remains to check the Jacobi identity. This is an easy consequence of (4):

$$\begin{aligned}\{F,\{G,H\}\} &= X_{\{G,H\}}F = -[X_G, X_H]F \\ &= -X_G(X_H F) + X_H(X_G F) = \{G,\{F,H\}\} + \{\{F,G\},H\}.\end{aligned}$$

We have thus verified (P1) and (P3). For (P2) we further need that, for every $v \in T_m(M)$, the condition that $\omega(X(m), v) = 0$ for every $X \in \mathrm{ham}(M,\omega)$ implies $v = 0$. □

Example 2.19. If (V,ω) is a symplectic vector space, then a linear functional $\alpha\colon V \to \mathbb{R}$ is contained in the Poisson algebra $\mathcal{A}$ if and only if there exists a vector $v \in V$ with $i_v\omega = \alpha$. Then $H_v = \alpha = i_v\omega$ is the Hamiltonian function of the constant vector field v. Accordingly, the Poisson structure on V is determined by

$$\{H_v, H_w\} = \mathrm{d}H_v(w) = \omega(v,w) \quad \text{for} \quad v,w \in V. \tag{5}$$

Here (P2) follows from the non-degeneracy of ω.

2.4 Poisson Maps

It is now clear how to define the notion of a Poisson map between two weak Poisson manifolds. Here we take a closer look at Poisson maps arising from inclusions of submanifolds and from submersions which correspond to regular Poisson reduction. In the context of Hamiltonian actions, Poisson maps to weak affine Poisson space arise as momentum maps.

Definition 2.20. Let $(M_j, \mathcal{A}_j, \{\cdot,\cdot\}_j)$, $j = 1,2$, be weak Poisson manifolds. A smooth map $\varphi\colon M_1 \to M_2$ is called a *Poisson map*, or *morphism of Poisson manifolds*, if $\varphi^*\mathcal{A}_{M_2} \subseteq \mathcal{A}_{M_1}$ and $\varphi^*\{F,G\} = \{\varphi^*F, \varphi^*G\}$ for $F, G \in \mathcal{A}_{M_2}$.

Proposition 2.21 (Poisson Submanifolds). *Let $(M,\mathcal{A},\{\cdot,\cdot\})$ be a weak Poisson manifold and $N \subseteq M$ be a submanifold with the property that, for every $F \in \mathcal{A}$, the restriction of X_F to N is tangential to N. Then $\mathcal{I}_N := \{F \in \mathcal{A}\colon F|_N = 0\}$ is an ideal with respect to the Poisson bracket, i.e., $\{\mathcal{I}_N, \mathcal{A}\} \subseteq \mathcal{I}_N$, and the induced bracket on $\mathcal{A}_N := \mathcal{A}/\mathcal{I}_N \subseteq C^\infty(N)$ defines a weak Poisson structure on N such that the inclusion $N \hookrightarrow M$ is a morphism of weak Poisson manifolds.*

Proof. First we show that $\mathcal{I}_N$ is a Poisson ideal. So let $F \in \mathcal{I}_N$ and $G \in \mathcal{A}$. Then, for $n \in N$, $\{F,G\}(n) = \mathrm{d}F(n)X_G(n) = 0$ because F vanishes on N and $X_G(n) \in T_n(N)$. This implies that $\mathcal{A}_N$ inherits the structure of a Poisson algebra by

$$\{F|_N, G|_N\} := \{F,G\}|_N,$$

and that (P1) is satisfied.

If $v \in T_n(N)$, $n \in N$, satisfies $\mathrm{d}F(n)v = 0$ for every $F \in \mathcal{A}_N$, then the same holds for $F \in \mathcal{A}$, so that (P2) for $\mathcal{A}$ implies (P2) for $\mathcal{A}_N$.

To verify (P3), we simply observe that our assumption implies that

$$\{F|_N, G|_N\} = \{F, G\}|_N = (X_G F)|_N = (X_G|_N)F|_N.$$

□

Remark 2.17. (a) Let $\mathfrak{g}$ be a locally convex Lie algebra and endow $\mathfrak{g}'$ with the weak Poisson structure from Corollary 2.11 above. Let $C \in \mathfrak{z}(\mathfrak{g})$ be a central element. Then the hyperplane

$$N := \{\alpha \in \mathfrak{g}' : \alpha(C) = 1\}$$

is a submanifold of $\mathfrak{g}'$, and for every $F \in \mathcal{A}_{\mathfrak{g}'}$ and $\alpha \in N$, we have

$$0 = X_F(\alpha)H_C = \langle X_F(\alpha), C\rangle,$$

so that $X_F \in \mathcal{V}(N)$. Therefore the assumptions of Proposition 2.21 are satisfied, so that $\mathcal{A}_N := \mathcal{A}_{\mathfrak{g}'}|_N$ yields a weak Poisson structure on the hyperplane N.

(b) The preceding restriction is of particular importance if we are dealing with a central extension $\tilde{\mathfrak{g}} = \mathbb{R} \oplus_\omega \mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ with the bracket

$$(z, X), (z', X') = (\omega(X, X'), [X, X']),$$

where $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is a continuous 2-cocycle. Then $C := (1, 0)$ is a central element of $\tilde{\mathfrak{g}}$ and

$$H_C^{-1}(1) = \{1\} \times \mathfrak{g}' \subseteq \tilde{\mathfrak{g}}'$$

inherits a Poisson structure from $\mathcal{A}_{\tilde{\mathfrak{g}}'}$. Identifying the affine space $\mathfrak{g}'$ in the canonical fashion with the affine space $\{1\} \times \mathfrak{g}'$, we thus obtain a weak Poisson structure on $\mathfrak{g}'$, where $\mathcal{A} \subseteq C^\infty(\mathfrak{g}')$ is generated by the continuous affine functions, i.e., $\mathcal{A} \cong S(\mathfrak{g})$ as an associative algebra, and the Poisson bracket on $\mathcal{A}$ is determined by

$$\{H_X, H_Y\} = H_{[X,Y]} + \omega(X, Y) \quad \text{for} \quad H_X(\alpha) = \alpha(X), X \in \mathfrak{g}, \alpha \in \mathfrak{g}'$$

(cf. Example 2.14(d)).

Let $q : M \to N$ be a smooth *submersion*, i.e., q is surjective and has smooth local sections. This implies in particular that the subalgebra $q^*C^\infty(N)$ consists of those smooth functions on M which are constant along the fibers of q. The following proposition discusses the most regular form of Poisson quotients.

Proposition 2.23 (Smooth Poisson Quotients). *Let $(M, \mathcal{A}_M, \{\cdot,\cdot\})$ be a weak Poisson manifold and $q : M \to N$ be a submersion. Then a Poisson subalgebra $\mathcal{B} \subseteq q^*C^\infty(N) \cap \mathcal{A}_M$ is the image under q^* of a weak Poisson structure on N for which q is a Poisson map if and only if*

$$\ker T_m(q) = \{v \in T_m(M) : (\forall F \in \mathcal{B})\, \mathrm{d}F(m)v = 0\}. \tag{6}$$

Proof. Suppose first that q is a Poisson map w.r.t. the weak Poisson structure $(\mathcal{A}_N, \{\cdot,\cdot\})$ on N. Then $\mathcal{B} := q^*\mathcal{A}_N \subseteq \mathcal{A}_M$ is a Poisson subalgebra and property (P2) of $\mathcal{A}_N$ implies (6).

Suppose, conversely, that $\mathcal{B} \subseteq q^*C^\infty(N) \cap \mathcal{A}_M$ is a Poisson subalgebra satisfying (6). Let $\mathcal{A}_N \subseteq C^\infty(N)$ be the subalgebra with $q^*\mathcal{A}_N = \mathcal{B}$. Since q^* is injective, $\mathcal{A}_N$ inherits a natural Poisson algebra structure from $\mathcal{B}$. Hence $(N, \mathcal{A}_N, \{\cdot,\cdot\})$ satisfies (P1), and (P2) follows from (6). To see that (P3) also holds, let $f \in \mathcal{A}_N$ and $F = q^*f \in \mathcal{B}$. Then the corresponding Hamiltonian vector field $X_F \in \mathcal{V}(M)$ satisfies for every $G = q^*g \in \mathcal{B}$ the relation

$$\mathtt{d}g(q(m))T_m(q)X_F(m) = \mathtt{d}G(m)X_F(m) = \{G, F\}(m) = \{g, f\}(q(m)).$$

For $m' \in M$ with $q(m) = q(m')$, this leads to

$$\mathtt{d}g(q(m))T_m(q)X_F(m) = \mathtt{d}g(q(m))T_{m'}(q)X_F(m')$$

for every g, so that (P2) implies $T_m(q)X_F(m) = T_{m'}(q)X_F(m')$. Hence X_F is projectable to a vector field $Y \in \mathcal{V}(N)$ which is q-related to X_F. We then have for every $g \in \mathcal{A}_N$ the relation $\{g, f\} = Yg$, so that (P3) is also satisfied. □

Remark 2.24. If, in the context of Proposition 2.23, the subalgebra $\mathcal{B}$ is Poisson commutative, then (i) implies that the vector fields X_F, $F \in \mathcal{B}$, are tangential to the fibers of q, hence projectable to 0. We thus obtain the trivial Poisson structure on N for which all Poisson brackets vanish.

3 Momentum Maps

We now turn to momentum maps, which we consider as Poisson morphisms to affine Poisson spaces which arise naturally as subspaces of the duals of Lie algebras $\mathfrak{g}$. If $\mathfrak{g}$ is the Lie algebra of a Lie group, we also have a global structure coming from the corresponding coadjoint action, but unfortunately there need not be any locally convex topology on $\mathfrak{g}'$ for which the coadjoint action is smooth.

3.1 Momentum Maps as Poisson Morphisms

Since momentum maps are Poisson maps $\Phi\colon M \to V$, where V carries an affine weak Poisson structure (Theorem 2.8), we start with a characterization of such maps.

Proposition 3.1. *Let $(V, \mathcal{A}_V)$ be an affine Poisson manifold corresponding to a Lie algebra structure on the space $\hat{\mathcal{A}}_* = \mathcal{A}_* + \mathbb{R}1$ of affine functions on V, $(M, \mathcal{A}_M)$ a weak Poisson manifold and $\Phi\colon M \to V$ a smooth map such that $\varphi(\alpha) := \Phi^*\alpha = \alpha \circ \Phi \in \mathcal{A}_M$ for every $\alpha \in \hat{V}_*$. Then the following are equivalent*

(i) $\Phi^*: \mathcal{A}_M \to \mathcal{A}_V$ *is a homomorphism of Lie algebras, i.e.,* Φ *is a Poisson map.*
(ii) $\varphi: V_* \to \mathcal{A}_M$ *satisfies* $\varphi(\{\alpha, \beta\}) = \{\varphi(\alpha), \varphi(\beta)\}$ *for* $\alpha, \beta \in V_*$.
(iii) $\Phi: M \to V$ *satisfies the equivariance condition*

$$T_m(\Phi)X_{\varphi(\alpha)}(m) = X_\alpha(\Phi(m)) \quad \textit{for} \quad m \in M, \alpha \in V_*. \tag{7}$$

Proof. (i) ⇒ (ii) is trivial.

(ii) ⇒ (i): Clearly, $\Phi^*: \mathcal{A}_V \to \mathcal{A}_M$ is a homomorphism of commutative algebras because $\Phi^*(V_*) \subseteq \mathcal{A}_M$ and $\mathcal{A}_V$ is generated by V_*. Let $F, G \in \mathcal{A}_V$. For $p \in M$ we put $\alpha := \mathtt{d}F_{\Phi(p)}$ and $\beta := \mathtt{d}G_{\Phi(p)}$, which are elements of V_*. Then

$$\mathtt{d}(F \circ \Phi)_p = \alpha \circ T_p(\Phi) = \mathtt{d}(\Phi^*\alpha)_p = (\mathtt{d}\varphi(\alpha))_p,$$

and we thus obtain

$$\{\Phi^* F, \Phi^* G\}(p) = \mathtt{d}\varphi(\alpha)_p X_{\Phi^* G}(p) = \{\varphi(\alpha), \Phi^* G\}(p) = \{\varphi(\alpha), \varphi(\beta)\}(p)$$

and

$$\varphi([\alpha, \beta])(p) = \langle [\alpha, \beta], \Phi(p)\rangle = \{F, G\}(\Phi(p)).$$

This proves that (ii) implies (i).

(ii) ⇔ (iii): The equivariance relation (7) is an identity for elements of V. Hence it is satisfied if and only if it holds as an identity of real numbers when we apply elements of the separating subspace V_*. This means that

$$\mathtt{d}\varphi(\beta)_m X_{\varphi(\alpha)}(m) = \{\beta, \alpha\}(\Phi(m)) \quad \text{for} \quad m \in M, \alpha, \beta \in V_*.$$

Since the left hand side equals $\{\varphi(\beta), \varphi(\alpha)\}(m)$, this relation is equivalent to (ii).
□

The classical case of the preceding proposition is the one where $V = \mathfrak{g}'$ is the dual of locally convex Lie algebra, endowed with the weak-∗-topology.

Corollary 3.2. *Let* $\mathfrak{g}$ *be a locally convex Lie algebra, endow* $\mathfrak{g}'$ *with the canonical linear Poisson structure* $\mathcal{A}_{\mathfrak{g}'}$*, let* $(M, \mathcal{A}_M)$ *be a weak Poisson manifold and* $\Phi: M \to \mathfrak{g}'$ *be a map such that all functions* $\varphi_X(m) := \Phi(m)(X)$ *are contained in* $\mathcal{A}_M$*. Then the following are equivalent*

(i) $\Phi^*: \mathcal{A}_{\mathfrak{g}'} \to \mathcal{A}_M$ *is a homomorphism of Lie algebras, i.e.,* Φ *is a Poisson map.*
(ii) $\varphi: \mathfrak{g} \to \mathcal{A}_M$ *satisfies* $\varphi([X, Y]) = \{\varphi(X), \varphi(Y)\}$ *for* $X, Y \in \mathfrak{g}$.
(iii) $\Phi: M \to \mathfrak{g}'$ *satisfies the equivariance condition*

$$T_m(\Phi)X_{\varphi(X)}(m) = -\Phi(m) \circ \operatorname{ad} X \quad \textit{for} \quad m \in M, X \in \mathfrak{g}. \tag{8}$$

Remark 3.3. If we endow $\mathfrak{g}'$ with an affine Poisson structure corresponding to a Lie algebra cocycle ω, then the condition Corollary 3.2(ii) has to be modified to

$$\{\varphi(X), \varphi(Y)\} = \varphi([X, Y]) + \omega(X, Y) \quad \text{for} \quad X, Y \in \mathfrak{g}.$$

Definition 3.4. An infinitesimal action of the locally convex Lie algebra $\mathfrak{g}$ on the smooth manifold M is a Lie algebra homomorphism $\beta: \mathfrak{g} \to \mathcal{V}(M)$ for which all maps $\beta_m: \mathfrak{g} \to T_p(M), X \mapsto \beta(X)_m$ are continuous.

If $(M, \mathcal{A}_M, \{\cdot,\cdot\})$ is a weak Poisson manifold, then an infinitesimal action $\beta: \mathfrak{g} \to \mathcal{V}(M)$ of a locally convex Lie algebra on M is said to be *Hamiltonian* if there exists a homomorphism $\varphi: \mathfrak{g} \to \mathcal{A}_M$ of Lie algebras satisfying $X_{\varphi(Y)} = -\beta(Y)$ for every $Y \in \mathfrak{g}$. Then the map

$$\Phi: M \to \mathfrak{g}^*, \quad \Phi(m)(Y) := \varphi_Y(m)$$

is called the corresponding *momentum map*. Note that $\Phi(M) \subseteq \mathfrak{g}'$ is equivalent to the requirement that, for every $m \in M$, the linear functional $\mathfrak{g} \to \mathbb{R}, Y \mapsto \varphi_Y(m)$ is continuous.

Corollary 3.5. *If $\Phi: M \to \mathfrak{g}'$ is a momentum map for a Hamiltonian action of $\mathfrak{g}$ on the weak Poisson manifold $(M, \mathcal{A}_M, \{\cdot,\cdot\})$, then Φ is a Poisson map.*

Example 3.6. For a locally convex Lie algebra $\mathfrak{g}$, the infinitesimal coadjoint action $\beta: \mathfrak{g} \to \mathcal{V}(\mathfrak{g}')$ is given by the vector fields $\beta(X)(\alpha) := \alpha \circ \operatorname{ad} X = (\operatorname{ad} X)^*\alpha$. In view of Corollary 3.2, this action is Hamiltonian with momentum map $\Phi = \operatorname{id}_{\mathfrak{g}'}$.

Remark 3.7 (From Symplectic Actions to Hamiltonian Actions). Let (M, ω) be a connected weak symplectic manifold and $\mathcal{A}$ be as in Proposition 2.18. Further, let $\beta: \mathfrak{g} \to \operatorname{sp}(M, \omega)$ be an infinitesimal action by symplectic vector fields (cf. Definition 2.17). For β to be a Hamiltonian action requires a lift of this homomorphism to a Lie algebra homomorphism

$$\varphi: \mathfrak{g} \to (\mathcal{A}, \{\cdot,\cdot\}).$$

A necessary condition for such a lift to exist is that $\beta(\mathfrak{g}) \subseteq \operatorname{ham}(M, \omega)$. Even if this is the case, such a lift does not always exist. To understand the obstructions, we recall the short exact sequence

$$\mathbf{0} \to \mathbb{R} \to \mathcal{A} \to \operatorname{ham}(M, \omega) \to \mathbf{0},$$

which exhibits the Lie algebra $\mathcal{A}$ as a central extension of the Lie algebra $\operatorname{ham}(M, \omega)$ (cf. [22] for an in depth discussion of related central extensions).

Assuming that $\beta(\mathfrak{g}) \subseteq \operatorname{ham}(M, \omega)$, we consider the subspace

$$\hat{\mathfrak{g}} := \{(X, F) \in \mathfrak{g} \oplus \mathcal{A}: \beta(X) = -X_F\}$$

and observe that this is a Lie subalgebra of the direct sum $\mathfrak{g} \oplus \mathcal{A}$. Moreover, the projection $p(X, F) := X$ is a surjective homomorphism whose kernel consists of all pairs $(0, F)$, where F is a constant function. We thus obtain the central extension

$$\mathbb{R} \cong \mathbb{R}(0, 1) \to \hat{\mathfrak{g}} \xrightarrow{p} \mathfrak{g}.$$

The existence of a homomorphic lift $\varphi\colon \mathfrak{g} \to \mathcal{A}$ is equivalent to the existence of a splitting $\sigma\colon \mathfrak{g} \to \hat{\mathfrak{g}}$. Therefore the obstruction to the existence of φ is a central $\mathbb{R}$-extension of $\mathfrak{g}$, resp., a corresponding cohomology class in $H^2(\mathfrak{g}, \mathbb{R})$ (cf. [20]).

3.2 Infinite Dimensional Lie Groups

Before we turn to momentum maps and Hamiltonian actions, we briefly recall the basic concepts underlying the notion of an infinite dimensional Lie group. A *(locally convex) Lie group* G is a group equipped with a smooth manifold structure modeled on a locally convex space for which the group multiplication and the inversion are smooth maps. We write $\mathbf{1} \in G$ for the identity element. Then each $x \in T_{\mathbf{1}}(G)$ corresponds to a unique left invariant vector field x_l with $x_l(\mathbf{1}) = x$. The space of left invariant vector fields is closed under the Lie bracket of vector fields, hence inherits a Lie algebra structure. We thus obtain on $\mathfrak{g} := T_{\mathbf{1}}(G)$ a continuous Lie bracket which is uniquely determined by $[x, y] = [x_l, y_l](\mathbf{1})$ for $x, y \in \mathfrak{g}$. We shall also use the functorial notation $\mathbf{L}(G) := (\mathfrak{g}, [\cdot, \cdot])$ for the Lie algebra of G and, accordingly, $\mathbf{L}(\varphi) = T_{\mathbf{1}}(\varphi)\colon \mathbf{L}(G_1) \to \mathbf{L}(G_2)$ for the Lie algebra homomorphism associated to a smooth homomorphism $\varphi\colon G_1 \to G_2$ of Lie groups. Then $\mathbf{L}$ defines a functor from the category of locally convex Lie groups to the category of locally convex Lie algebras. If $\mathfrak{g}$ is a Fréchet, resp., a Banach space, then G is called a *Fréchet-*, resp., a *Banach–Lie group*.

A smooth map $\exp_G\colon \mathbf{L}(G) \to G$ is called an *exponential function* if each curve $\gamma_x(t) := \exp_G(tx)$ is a one-parameter group with $\gamma_x'(0) = x$. Not every infinite dimensional Lie group has an exponential function [21, Ex. II.5.5], but exponential functions are unique whenever they exist.

With the left and right multiplications $\lambda_g(h) := \rho_h(g) := gh$ we write $g.X = T_{\mathbf{1}}(\lambda_g)X$ and $X.g = T_{\mathbf{1}}(\rho_g)X$ for $g \in G$ and $X \in \mathfrak{g}$. Then the two maps

$$G \times \mathfrak{g} \to TG, \quad (g, X) \mapsto g.X \quad \text{and} \quad G \times \mathfrak{g} \to TG, \quad (g, X) \mapsto X.g \tag{9}$$

trivialize the tangent bundle TG.

3.3 Coadjoint Actions and Affine Variants

To add some global aspects to the Poisson structures on the dual $\mathfrak{g}'$ of a Lie algebra $\mathfrak{g}$, we assume that $\mathfrak{g} = \mathbf{L}(G)$ for a Lie group G. Then the *adjoint action* of G on $\mathfrak{g}$ is defined by $\mathrm{Ad}(g) := \mathbf{L}(c_g)$, where $c_g(x) = gxg^{-1}$ is the conjugation map.

The adjoint action is smooth in the sense that it defines a smooth map $G \times \mathfrak{g} \to \mathfrak{g}$. The *coadjoint action* on the topological dual space $\mathfrak{g}'$ is defined by

$$\mathrm{Ad}^*(g)\alpha := \alpha \circ \mathrm{Ad}(g)^{-1}.$$

The maps $\mathrm{Ad}^*(g)$ are continuous with respect to the weak-$*$-topology on $\mathfrak{g}'$ and all orbit maps for Ad^* are smooth because, for every $X \in \mathfrak{g}$ and $\alpha \in \mathfrak{g}'$, the map $g \mapsto \alpha(\mathrm{Ad}(g)^{-1}X)$ is smooth. If G is a Banach–Lie group, then the coadjoint action is smooth with respect to the norm topology on $\mathfrak{g}'$, but in general it is not continuous, as the following example shows.[3]

Example 3.8. Let V be a locally convex space and $\alpha_t(v) := e^t v$. Then the semidirect product

$$G := V \rtimes_\alpha \mathbb{R}, \qquad (v,t)(v',t') = (v + e^t v', t + t')$$

is a Lie group. From $c_{(v,t)}(w,s) = ((1-e^s)v + e^t w, s)$ we derive that

$$\mathrm{Ad}(v,t)(w,s) = (e^t w - sv, s).$$

Accordingly, we obtain

$$\mathrm{Ad}^*(v,t)(\alpha,u) = (e^{-t}\alpha, u + e^{-t}\alpha(v)).$$

If Ad^* is continuous, restriction to $t = 1$ implies that the evaluation map

$$V' \times V \to \mathbb{R}, \quad (\alpha, v) \mapsto \alpha(v)$$

is continuous, but w.r.t. the weak-$*$-topology on V', this happens if and only if V is finite dimensional. Therefore Ad^* is not continuous if $\dim V = \infty$.[4]

Remark 3.9. (a) If $\mathfrak{g}'$ is endowed with the affine Poisson structure corresponding to a 2-cocycle $\omega: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, then the corresponding infinitesimal action $\beta: \mathfrak{g} \to \mathcal{V}(\mathfrak{g}')$ of the Lie algebra $\mathfrak{g}$ by affine vector fields need not integrate to an action of a connected Lie group G with $\mathbf{L}(G) = \mathfrak{g}$, but if G is simply connected, then it does (cf. [20, Prop. 7.6]).

---

[3] By definition of the weak-$*$-topology on $\mathfrak{g}'$, which corresponds to the subspace topology with respect to the embedding $\mathfrak{g}' \hookrightarrow \mathbb{R}^{\mathfrak{g}}$, a map $\varphi: M \to \mathfrak{g}'$ is smooth with respect to this topology if and only if all functions $\varphi_X(m) := \varphi(m)(X)$ are smooth on M.

[4] One can ask more generally, for which locally convex spaces V and which topologies on V' the evaluation map $V \times V' \to \mathbb{R}$ is continuous. This happens if and only if the topology on V can be defined by a norm, and then the operator norm turns V' into a Banach space for which the evaluation map is continuous.

(b) The situation is much better for the Poisson structures on $\mathfrak{g}$ discussed in Example 2.14(e). Then the Hamiltonian vector field associated to $Y \in \mathfrak{g}$ is the affine vector field given by

$$X_{H_Y}(Z) = [Y, Z] - DY. \tag{10}$$

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\gamma_D \colon G \to \mathfrak{g}$ be a 1-cocycle for the adjoint action with $T_{\mathbf{1}}(\gamma_D) = D$. Here the cocycle condition is

$$\gamma_D(gh) = \gamma_D(g) + \mathrm{Ad}_g \gamma_D(h) \quad \text{for} \quad g, h \in G.$$

Since the adjoint action is smooth, such a cocycle exists if G is simply connected. Then we obtain an affine action of G on $\mathfrak{g}$ by

$$\mathrm{Ad}_g^D X := \mathrm{Ad}_g X - \gamma_D(g)$$

integrating the given infinitesimal action of $\mathfrak{g}$ determined by (10).

Definition 3.10. Let $(M, \mathcal{A})$ be a weak Poisson manifold, G a connected Lie group, and $\sigma \colon G \times M \to M$ a smooth (left) action. We also write $g.p := \sigma_g(p) := \sigma^p(g) := \sigma(g, p)$ and define the vector fields

$$X_\sigma(p) := T_{(\mathbf{1},p)}(\sigma)(X, 0) \quad \text{for} \quad X \in \mathfrak{g}.$$

Then we have a homomorphism

$$\mathbf{L}(\sigma) \colon \mathfrak{g} \to \mathcal{V}(M) \quad \text{with} \quad X \mapsto -X_\sigma$$

which defines an infinitesimal action of $\mathfrak{g}$ on M.

The action σ is called *Hamiltonian* if its derived action $\mathbf{L}(\sigma)$ is Hamiltonian, i.e., if there exists a homomorphism of Lie algebras $\varphi \colon \mathfrak{g} \to \mathcal{A}$ with $X_{\varphi(Y)} = Y_\sigma$ for $Y \in \mathfrak{g}$ such that, for every $m \in M$, the linear map $\Phi(m) \colon \mathfrak{g} \to \mathbb{R}, Y \mapsto \varphi(Y)(m)$ is continuous. Then $\Phi \colon M \to \mathfrak{g}'$ is called the corresponding *momentum map* (cf. Definition 3.4).

Remark 3.11. For any smooth left action $\sigma \colon G \times M \to M$ and $p \in M$, the right invariant vector field $X_r(g) = X.g$ on G and the corresponding vector field $X_\sigma \in \mathcal{V}(M)$ are σ^p-related. This follows from the relation $\sigma^p(hg) = h.\sigma^p(g)$ for $g, h \in G$. Combining this observation with the "Related Vector Field Lemma", one obtains a proof for $\mathbf{L}(\sigma) \colon \mathfrak{g} \to \mathcal{V}(M)$ being a homomorphism of Lie algebras.

Example 3.12. Let (V, ω) be a locally convex symplectic vector space and $G = (V, +)$ the translation group of V. Then the translation action $\sigma(v, w) := v + w$ of V on itself is symplectic and every constant vector field $v_\sigma(w) = v$ is Hamiltonian (cf. Example 2.19). The relation

$$\{v, w\} = \omega(v, w)$$

shows that there is no homomorphism $\varphi \colon \mathfrak{g} \to \mathcal{A}$ with $X_{\varphi(v)} = v_\sigma$ for every $v \in \mathfrak{g}$.

Remark 3.13. Of particular interest with respect to Poisson structures are Lie groups G whose Lie algebras $\mathfrak{g}$ can be approximated in a natural way by finite dimensional ones. This can be done by direct or projective limits.

(a) If $G = \varinjlim G_n$ is a Lie group whose Lie algebra $\mathfrak{g}$ is a directed union of a sequence of finite dimensional subalgebras $\mathfrak{g}_n = \mathbf{L}(G_n), n \in \mathbb{N}$, then $\mathfrak{g}$ carries the finest locally convex topology which actually coincides with the direct limit topology (see [6, 7] for direct limit manifolds and Lie groups). Then its topological dual $V := \mathfrak{g}'$, endowed with the topology of uniform convergence of bounded or compact subsets is a Fréchet space (isomorphic to a product $\mathbb{R}^{\mathbb{N}}$) and all assumptions (a)–(d) from Example 2.13 are satisfied [8, Rem. 16.34], so that we obtain a linear Poisson structure on $V = \mathfrak{g}'_c = \mathfrak{g}'_b$. In this case the coadjoint orbits of G are unions of finite dimensional manifolds, which can be used to obtain symplectic manifold structures on them (cf. [4, 6]).
(b) The opposite situation is obtained for Lie groups $G = \varprojlim G_n$ which are projective limits of finite dimensional Lie groups G_n (see [11]). Typical examples are groups of infinite jets of diffeomorphisms. Here $\mathfrak{g}$ is a Fréchet space (isomorphic to $\mathbb{R}^{\mathbb{N}}$) and the dual space $\mathfrak{g}'$ is the union of the dual spaces $\mathfrak{g}_n^*$. Endowed with the topology of uniform convergence of bounded or compact subsets the space $V = \mathfrak{g}'$ satisfies all assumptions (a)–(d) from Example 2.13 [8, Rem. 16.34]. In this case all coadjoint orbits are finite dimensional because they can be identified with coadjoint orbits of some G_n.

In both cases we obtain weak Poisson structures on $\mathfrak{g}'$ for which $\mathcal{A} = C^\infty(\mathfrak{g}')$ is the full algebra of smooth functions for a suitable topology which is the weak-$*$-topology in the first case and the finest locally convex topology in the second.

Example 3.14. Let G be a Lie group and $\mathfrak{g} = \mathbf{L}(G)$. Further, let $\mathfrak{g}_* \subseteq \mathfrak{g}'$ be an $\mathrm{Ad}^*(G)$-invariant separating subspace endowed with a locally convex topology for which the coadjoint action $\mathrm{Ad}_*(g) := \mathrm{Ad}^*(g)|_{\mathfrak{g}_*}$ on $\mathfrak{g}_*$ is smooth. Then $\mathfrak{g}_*$ carries a natural linear weak Poisson structure with $\mathcal{A} \cong S(\mathfrak{g})$ and

$$\{F, H\}(\alpha) = \langle \alpha, [\mathtt{d}F(\alpha), \mathtt{d}H(\alpha)]\rangle \quad \text{for} \quad \alpha \in \mathfrak{g}_*, F, H \in \mathcal{A}$$

(Example 2.14(b); see also [26, Sect. 4.2] for similar requirements in the context of Banach spaces).

For $X, Y \in \mathfrak{g}$, we have $\{H_X, H_Y\} = H_{[X,Y]}$ and the corresponding Hamiltonian vector fields are $X_{H_Y}(\alpha) = -\alpha \circ \mathrm{ad}\, Y$. Therefore the coadjoint action Ad_* on $\mathfrak{g}_*$ is Hamiltonian and its momentum map is the inclusion $\mathfrak{g}_* \hookrightarrow \mathfrak{g}'$.

For the coadjoint action Ad_* of G on $\mathfrak{g}_*$, the "tangent space" to the orbit of $\alpha \in \mathfrak{g}_*$ is the space $\{X_{\mathrm{Ad}_*}(\alpha) : X \in \mathfrak{g}\} = \alpha \circ \mathrm{ad}(\mathfrak{g})$. This is also the characteristic subspace of the Poisson structure (cf. Remark 2.5) and the corresponding skew-symmetric form is given by

$$\omega_\alpha(X_F(\alpha), X_H(\alpha)) = \{F, H\}(\alpha) = \mathtt{d}F(\alpha) X_H(\alpha),$$

resp.,

$$\omega_\alpha(\alpha \circ \operatorname{ad} X, \alpha \circ \operatorname{ad} Y) = \{H_X, H_Y\}(\alpha) = H_{[X,Y]}(\alpha) = \alpha([X,Y]).$$

Fix $\alpha \in \mathfrak{g}_*$. Then we obtain on G a 2-form by

$$\begin{aligned}\Omega_\alpha(X.g, Y.g) &:= \omega_{g.\alpha}(X_{\mathrm{Ad}_*}(g.\alpha), Y_{\mathrm{Ad}_*}(g.\alpha)) = \omega_{g.\alpha}([g.\alpha \circ \operatorname{ad} X, g.\beta \circ \operatorname{ad} Y]) \\ &= (g.\alpha)([X,Y]) = \alpha([\mathrm{Ad}_g^{-1} X, \mathrm{Ad}_g^{-1} Y]).\end{aligned}$$

This means that Ω is a left-invariant 2-form on G. Since $(\Omega_{\alpha,\mathbf{1}})(X,Y) = \alpha([X,Y])$ is a 2-cocycle, Ω is closed, the radical of $\Omega_{\mathbf{1}}$ coincides with the Lie algebra of the stabilizer subgroup G_α.

If $\mathcal{O}_\alpha := \mathrm{Ad}_*(G)\alpha$ carries a manifold structure for which the orbit map $G \to \mathcal{O}_\alpha$ is a submersion, we thus obtain on $\mathcal{O}_\alpha$ the structure of a weak symplectic manifold. However, if the Lie algebra $\mathfrak{g}$ is not a Hilbert space, then it is not clear how to obtain a manifold structure on $\mathcal{O}_\alpha$, resp., the homogeneous space G/G_α. In any case, we may consider the pair (G, Ω_α) as a non-reduced variant of the symplectic structure on the coadjoint orbit.

3.4 Cotangent Bundles of Lie Groups and Their Reduction

Let G be a Lie group, $\mathfrak{g} = \mathbf{L}(G)$ and $\mathfrak{g}_* \subseteq \mathfrak{g}'$ be as in Example 3.14, so that the coadjoint action Ad_* on $\mathfrak{g}_*$ is smooth. Then the "cotangent bundle"

$$T_*(G) := \bigcup_{g \in G} \{\alpha \in T_g'(G) \colon \alpha \circ T_{\mathbf{1}}(\rho_g) \in \mathfrak{g}_*\}$$

carries a natural Lie group structure for which it is isomorphic to the semidirect product $\mathfrak{g}_* \rtimes_{\mathrm{Ad}_*} G$. Here we identify (α, g) with the element $\alpha \circ T_{\mathbf{1}}(\rho_g)^{-1} \in T_g(G)'$, which leads to an injection $T_*(G) \hookrightarrow T'(G)$.

The lift of the left, resp., right multiplications to $T_*(G)$ is given by

$$\sigma_g^l(\alpha, h) = (\alpha \circ \mathrm{Ad}_g^{-1}, gh) \quad \text{and} \quad \sigma_g^r(\alpha, h) = (\alpha, hg). \tag{11}$$

The corresponding infinitesimal action is given by the vector fields

$$X_{\sigma^l}(\alpha, h) = (-\alpha \circ \operatorname{ad} X, X.h) \quad \text{and} \quad X_{\sigma^r}(\alpha, h) = (0, h.X).$$

The smooth 1-form defined by

$$\Theta(\alpha, g)(\beta, X.g) := \alpha(X)$$

is an analog of the Liouville 1-form. It follows from (11) that it is invariant under both actions σ^l and σ^r. Note that

$$\Theta(X_{\sigma^l})(\alpha, h) = \alpha(X) \quad \text{and} \quad \Theta(X_{\sigma^r})(\alpha, h) = \alpha(\mathrm{Ad}_h X).$$

Now

$$\Omega := -\mathtt{d}\Theta$$

is closed smooth 2-form on $T_*(G)$. To see that it is non-degenerate, we observe that its invariance under left and right translations and the Cartan formulas imply

$$(i_{X_{\sigma^l}}\Omega)_{(\alpha,g)}(\beta, Y.g) = \mathtt{d}(i_{X_{\sigma^l}}\Theta)_{(\alpha,g)}(\beta, Y.g) = \beta(X) \tag{12}$$

and, for the constant vertical vector field $Z(\alpha, g) = \gamma \in \mathfrak{g}_*$, the relation $\Theta(Z) = 0$ leads to

$$(i_Z\Omega)_{(\alpha,g)}(\beta, Y.g) = -(\mathcal{L}_Z\Theta)_{(\alpha,g)}(\beta, Y.g) = \gamma(Y). \tag{13}$$

We conclude that $(T_*(G), \Omega)$ is a weak symplectic manifold.

We thus obtain by Proposition 2.18 on $T_*(G)$ a weak Poisson structure on the subalgebra

$$\mathcal{A} := \{H \in C^\infty(T_*(G)) \colon (\exists X_H \in \mathcal{V}(T_*(G)))\ \mathtt{d}H = i_{X_H}\Omega\} \subseteq C^\infty(T_*(G)).$$

Let $C^\infty_*(G) \subseteq C^\infty(G)$ denote the subalgebra of smooth functions H whose differential $\mathtt{d}H$ defines a smooth section $G \to T_*(G)$, resp., a smooth function

$$\delta H \colon G \to \mathfrak{g}_*, \quad (\delta H)_g(X) := (\mathtt{d}H)_g(X.g).$$

Then (13) shows that, for $H \in C^\infty_*(G)$, the vertical vector field on $T_*(G)$ defined by $X_H(\alpha, g) := (\delta H(g), 0)$ satisfies

$$(i_{X_H}\Omega)_{(\alpha,g)}(\beta, Y.g) = (\delta H)_g(Y) = (\mathtt{d}H)_g(Y.g).$$

For the corresponding function $\tilde{H}$ on $T_*(G)$, we therefore have $\mathtt{d}\tilde{H} = i_{X_H}\Omega$, so that $H \in \mathcal{A}$. On the other hand, we have seen above that, for $X \in \mathfrak{g}$, the function $H_X(\alpha, g) = \alpha(X)$ on $T_*(G)$ satisfies $\mathtt{d}H_X = i_{X_{\sigma^l}}\Omega$. This shows that $\mathcal{A}$ contains the subalgebra $C^\infty_*(G)$ and the algebra $S(\mathfrak{g})$ of polynomial functions on the first factor $\mathfrak{g}_*$ generated by the functions H_X, $X \in \mathfrak{g}$. We therefore have $S(\mathfrak{g}) \otimes C^\infty_*(G) \subseteq \mathcal{A}$.

The Poisson bracket vanishes on $C^\infty_*(G)$, and, for $X \in \mathfrak{g}$ and $F \in C^\infty_*(G)$, we have

$$\{\tilde{F}, H_X\}(\alpha, g) = \mathtt{d}F_g(X.g) = (X_r F)(g) = \widetilde{X_r F}(\alpha, g).$$

We also note that, for $X, Y \in \mathfrak{g}$, we have by (12)

$$\{H_X, H_Y\}(\alpha, g) = \Omega(X_{\sigma^l}, Y_{\sigma^l})(\alpha, g) = (i_{X_{\sigma^l}}\Omega)_{(\alpha,g)}(-\alpha \circ \operatorname{ad} Y, Y.g)$$
$$= -(\alpha \circ \operatorname{ad} Y)(X) = \alpha([X, Y]) = H_{[X,Y]}(\alpha, g).$$

This implies that $S(\mathfrak{g})$ and $\mathcal{B} := S(\mathfrak{g}) \otimes C^\infty_*(G)$ are Poisson subalgebras of $\mathcal{A}$. In particular, $\mathcal{B}$ defines a weak Poisson structure on $T_*(G)$.

Consider the submersion $q: T_*(G) \to \mathfrak{g}_*, (\alpha, g) \mapsto \alpha$. Then $\mathcal{B} \cap q^*C^\infty(\mathfrak{g}_*) \cong S(\mathfrak{g})$, and since $H_X(\alpha, g) = \langle q(\alpha, g), X\rangle$, condition (6) in Proposition 2.23 is satisfied. Therefore q is a Poisson map if we endow $\mathfrak{g}_*$ with the Poisson structure determined on $\mathcal{A}_{\mathfrak{g}_*} = S(\mathfrak{g})$ by $\{H_X, H_Y\} = H_{[X,Y]}$ for $X, Y \in \mathfrak{g}$.

The fibers of q are the orbits of the right translation action σ^r which is a Hamiltonian action of G on $T_*(G)$ and $\mathcal{B}^{\sigma^r(G)} \cong S(\mathfrak{g})$ is the subalgebra of invariant functions in $\mathcal{B}$. On the other hand, q is a momentum map for the left action σ^l of G on $T_*(G)$. Therefore the passage to the orbit space $\mathfrak{g}_* \cong T_*(G)/\sigma_r(G)$ is an example of Poisson reduction from the Hamiltonian action σ^l to the coadjoint action Ad_* on $\mathfrak{g}_*$ (cf. [17, Thm. 13.1.1] for the finite dimensional case).

Remark 3.15 (Magnetic Cotangent Bundles). A natural variation of this construction is obtained by using a continuous 2-cocycle $b: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ to get a closed right invariant 2-form $B \in \Omega^2(G)$. If $\pi: T_*G \to G$ is the bundle projection, then

$$\Omega_b := \Omega + \pi^* B$$

is a closed right invariant 2-form on $T_*(G)$. Since its values in vertical directions are the same as for Ω, the form Ω_b is also non-degenerate. We thus obtain an infinite dimensional version of a magnetic cotangent bundle (cf. [17, Sect. 6.6], [18, Sect. 7.2]).

The Poisson bracket on $C^\infty_*(G)$ still vanishes, and, for $X \in \mathfrak{g}$ and $F \in C^\infty_*(G)$, we still have $\{\tilde{F}, H_X\} = \widetilde{X_r F}$. But for $X, Y \in \mathfrak{g}$ we obtain

$$\{H_X, H_Y\} = \Omega(X_{\sigma^l}, Y_{\sigma^l}) + B(X_r, Y_r) = H_{[X,Y]} + b(X, Y).$$

Therefore the quotient Poisson structure on $\mathfrak{g}_* \cong T_*(G)/\sigma^r(G)$ is the affine Poisson structure from Example 2.14(b) (see [5] for applications of these techniques).

4 Lie Algebra-Valued Momentum Maps

We have already seen in Example 2.14(e) how to obtain from an invariant symmetric bilinear form κ and a κ-skew-symmetric derivation D a weak affine Poisson structures on a Lie algebra $\mathfrak{g}$. This leads naturally to a concept of a Hamiltonian

G-action with a $\mathfrak{g}$-valued momentum map. For the classical case where G is the loop group $\mathcal{L}(K) = C^\infty(\mathbb{S}^1, K)$ of a compact Lie group and the derivation is given by the derivative, we thus obtain the affine action on $\mathfrak{g} = \mathcal{L}(\mathfrak{k})$ which corresponds to the action of $\mathcal{L}(K)$ on gauge potentials on the trivial K-bundle over $\mathbb{S}^1$.

4.1 Hamiltonian Actions for Affine Poisson Structures on Lie Algebras

Let G be a Lie group with Lie algebra $\mathfrak{g}$, $\kappa\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be a continuous $\mathrm{Ad}(G)$-invariant non-degenerate symmetric bilinear form and $D\colon \mathfrak{g} \to \mathfrak{g}$ be a continuous derivation for which we have a smooth Ad-cocycle $\gamma_D\colon G \to \mathfrak{g}$ with $\gamma_D{}'(\mathbf{1}) = D$. In Remark 3.9(b) we have seen that this leads to a smooth affine action of G on $\mathfrak{g}$ by

$$\mathrm{Ad}^D_g \xi := \mathrm{Ad}_g \xi - \gamma_D(g) \quad \text{for} \quad g \in G, \xi \in \mathfrak{g}.$$

We recall from Example 2.14(e) that $\mathfrak{g}$ carries a weak Poisson structure $\{\cdot,\cdot\} = \{\cdot,\cdot\}_{\kappa,D}$ with $\mathcal{A} \cong S(\mathfrak{g})$, generated by the functions $\xi^\flat := \kappa(\xi, \cdot)$. It is determined by

$$\{\xi^\flat, \eta^\flat\} = [\xi, \eta]^\flat + \kappa(D\xi, \eta) \quad \text{for} \quad \xi, \eta \in g.$$

For any $F \in \mathcal{A}$ and $\xi \in \mathfrak{g}$, the linear functional $\mathtt{d}F(\xi) \in \mathfrak{g}'$ is represented by κ, hence can be identified with an element $\nabla F(\xi) \in \mathfrak{g}$, the *$\kappa$-gradient of F in ξ*. In these terms, the Poisson structure on $\mathfrak{g}$ is given by

$$\{F, H\}(\xi) := \kappa(\xi, [\nabla F(\xi), \nabla H(\xi)]) + \kappa(D\nabla F(\xi), \nabla H(\xi)) \text{ for } F, H \in \mathcal{A}, \xi \in \mathfrak{g}.$$

The corresponding Hamiltonian vector fields are determined by

$$\begin{aligned}(X_H F)(\xi) = \{F, H\}(\xi) &= \kappa(\nabla F(\xi), [\nabla H(\xi), \xi]) + \kappa(D\nabla F(\xi), \nabla H(\xi)) \\ &= \mathtt{d}F(\xi)\big([\nabla H(\xi), \xi] - D\nabla H(\xi)\big),\end{aligned}$$

which leads to

$$X_H(\xi) = [\nabla H(\xi), \xi] - D\nabla H(\xi).$$

For $H = \eta^\flat$, $\eta \in \mathfrak{g}$, this specializes to

$$X_{\eta^\flat} = \mathrm{ad}\,\eta - D\eta = \eta_{\mathrm{Ad}^D} \quad \text{for} \quad \eta \in \mathfrak{g}. \tag{14}$$

4.2 Loop Groups and the Affine Action on Gauge Potentials

An important example arises for $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and the loop group $G = \mathcal{L}(K) := C^\infty(\mathbb{S}^1, K)$ where K is a Lie group for which $\mathfrak{k}$ carries a non-degenerate $\mathrm{Ad}(K)$-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$. We then put $D\xi = \xi'$ and $\kappa(\xi,\eta) = \int_0^1 \langle \xi(t), \eta(t)\rangle\, dt$ as in Example 2.15. Then $\gamma_D(g) = \delta^r(g) := g'g^{-1}$ is the right logarithmic derivative, so that

$$\mathrm{Ad}_g^D \xi = \mathrm{Ad}_g \xi - g'g^{-1} =: \xi^g \tag{15}$$

corresponds to the natural affine action on the space $\Omega^1(\mathbb{S}^1, \mathfrak{k}) \cong C^\infty(\mathbb{S}^1, \mathfrak{k})$ of gauge potentials of the trivial K-bundle $\mathbb{S}^1 \times K$ over $\mathbb{S}^1$.

For $\xi \in \mathcal{L}(\mathfrak{k})$, let $\gamma_\xi \colon \mathbb{R} \to K$ denote the unique solution of the initial value problem

$$\gamma(0) = \mathbf{1} \quad \text{and} \quad \delta^l(\gamma) := \gamma^{-1}\gamma' = \xi. \tag{16}$$

For each $s \in \mathbb{R}$ we write

$$\mathrm{Hol}_s \colon \mathcal{L}(\mathfrak{k}) \to K, \quad \xi \mapsto \gamma_\xi(s),$$

for the corresponding *holonomy map*. It satisfies the equivariance relation

$$\mathrm{Hom}_s(\xi^g) = g(0)\,\mathrm{Hol}_s(\xi) g(s)^{-1} \quad \text{for} \quad g \in \mathcal{L}(K). \tag{17}$$

In particular $\mathrm{Hol} := \mathrm{Hol}_1$ is equivariant with respect to the conjugation action of K on itself. This formula also implies that $\gamma_{\xi^g} = g(0)\gamma_\xi g^{-1}$, so that the affine $\mathcal{L}(K)$-action on $\mathfrak{g}$ corresponds on the level of curves to the multiplication with the pointwise inverse on the right.

Proposition 4.1. *For any Lie group K for which (16) is solvable,*[5] *the action (15) of the subgroup $\Omega(K) := \{g \in \mathcal{L}(K) \colon g(0) = \mathbf{1}\}$ on $\mathfrak{g}$ is free and its orbits coincide with the fibers of* Hol*, so that* Hol *induces a bijection*

$$\overline{\mathrm{Hol}} \colon \mathcal{L}(\mathfrak{k})/\Omega(K) \to K, \quad [\xi] \mapsto \mathrm{Hol}_1(\xi).$$

Proof. The relation $\xi^g = \xi$ implies $\gamma_{\xi^g} = \gamma_\xi$, so that $g(0)\gamma_\xi = \gamma_\xi g$. For $g(0) = \mathbf{1}$ this implies that $g = \mathbf{1}$ is constant. Therefore the action of the subgroup $\Omega(K)$ on $\mathfrak{g}$ is free and Hol is constant on the $\Omega(K)$-orbits.

Suppose, conversely, that $\mathrm{Hol}(\xi) = \mathrm{Hol}(\eta)$, i.e., $g_1 := \gamma_\xi(1) = \gamma_\eta(1)$. Since ξ and η are periodic, $\gamma_\xi(t+1) = g_1\gamma_\xi(t)$ and $\gamma_\eta(t+1) = g_1\gamma_\eta(t)$ holds for all $t \in \mathbb{R}$.

[5] This is the case for so-called *regular Lie groups* (cf. [21]). Banach–Lie groups and in particular finite dimensional Lie groups are regular.

Therefore $g(t) := \gamma_\eta(t)^{-1}\gamma_\xi(t)$ is a smooth periodic curve defining an element of $\Omega(K)$ with $\gamma_\eta = \gamma_\xi g^{-1}$. This in turn leads to the relation

$$\eta = \delta^l(\gamma_\eta) = \delta^l(g^{-1}) + \mathrm{Ad}_g \delta^l(\gamma_\xi) = \mathrm{Ad}_g \xi - \delta^r(g) = \xi^g.$$

□

Remark 4.2 (An Attempt on Poisson Reduction from $\mathcal{L}(\mathfrak{k})$ to K). For every connected Lie group K, the map $\mathrm{Hol}\colon \mathcal{L}(\mathfrak{k}) \to K$ is surjective and it is easy to see that it is a submersion. In view of Proposition 2.23, it makes sense to ask for a Poisson subalgebra $\mathcal{B} \subseteq \mathcal{A} \cong S(\mathcal{L}(\mathfrak{k}))$ that induces on K a Poisson structure for which q is a Poisson map. A natural candidate for $\mathcal{B}$ is the invariant subalgebra

$$\mathcal{B} := \mathcal{A}^{\Omega(K)}$$

consisting of Ω_K-invariant functions in $\mathcal{A}$, i.e., functions that are constant on the fibers of Hol.

If K is compact, then the exponential function $\exp\colon \mathfrak{k} \to K$ is surjective, and since $\mathrm{Hol}|_{\mathfrak{k}} = \exp$, it follows that $\mathrm{Hol}(\mathfrak{k}) = K$, which in turn means that every $\Omega(K)$-orbit meets the subspace $\mathfrak{k} \subseteq \mathcal{L}(\mathfrak{k})$ of constant functions. We conclude that the restriction map $R\colon \mathcal{B} \to \mathrm{Pol}(\mathfrak{k})$ is injective and that its image consists of polynomial functions on $\mathcal{L}(\mathfrak{k})$ that are constant on the fibers of the exponential function. Let $T \subseteq K$ be a maximal torus and $\mathfrak{t} = \mathbf{L}(T)$ be its Lie algebra. Then every $F \in \mathcal{B}$ restricts to a polynomial $F|_{\mathfrak{t}}$ which is constant on the cosets of the lattice $\ker(\exp|_{\mathfrak{t}})$, hence constant. Since every element $X \in \mathfrak{k}$ is contained in the Lie algebra of a maximal torus, it follows that F is constant on $\mathfrak{k}$, and therefore F is constant on $\mathcal{L}(\mathfrak{k})$. We conclude that $\mathcal{B} = \mathbb{R}\mathbf{1}$ contains only constant functions.

This shows that the algebra $\mathcal{A} \cong S(\mathfrak{g})$ of polynomial functions is too small to lead to a sufficiently large algebra of $\Omega(K)$-invariant functions. It is an interesting question whether there exists a suitable larger Poisson algebra $\tilde{\mathcal{A}} \supseteq \mathcal{A}$ for which $\tilde{\mathcal{A}}^{\Omega(K)}$ satisfies the assumptions of Proposition 2.23.

Definition 4.3. A *Hamiltonian $\mathcal{L}(K)$-space*[6] is a smooth weak Poisson manifold $(M, \mathcal{A}, \{\cdot,\cdot\})$, endowed with a smooth action $\sigma\colon \mathcal{L}(K) \times M \to M$ which has a smooth momentum map

$$\Phi\colon M \to \mathcal{L}(\mathfrak{k})$$

which is a Poisson map with respect to $(\mathcal{A}, \{\cdot,\cdot\})$.[7]

[6]This concept depends on the choice of the invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$ on the Lie algebra $\mathfrak{k}$. Changing this form leads to a different Poisson structure on $\mathcal{L}(\mathfrak{k})$.

[7]In [1] one finds this concept for the special case where (M, ω) is a weak symplectic manifold. In this case one requires the action σ to be symplectic and the existence of a smooth $\mathcal{L}(K)$-equivariant map $\Phi\colon M \to \mathcal{L}(\mathfrak{k})$ such that the functions

$$\varphi(\xi)(m) := \kappa(\Phi(m), \xi) \quad \text{satisfy} \quad i_{\xi_\sigma}\omega = \mathrm{d}(\varphi(\xi)).$$

Remark 4.4. Since the subgroup $\Omega(K) \subseteq \mathcal{L}(K)$ acts freely on $\mathfrak{g}$ and Φ is equivariant, it also acts freely on M, so that we can consider the holonomy space

$$\mathrm{Hol}(M) := M/\Omega(K),$$

and obtain a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\quad\Phi\quad} & \mathcal{L}(\mathfrak{k}) \\ \big\downarrow & & \big\downarrow \mathrm{Hol} \\ \mathrm{Hol}(M) & \xrightarrow{\quad\overline{\Phi}\quad} & K \end{array}$$

The geometric structure contained in the bottom row consists in an action of the Lie group $K \cong \mathcal{L}(K)/\Omega(K)$ on the orbit space $\mathrm{Hol}(M)$ and an equivariant map $\overline{\Phi}\colon \mathrm{Hol}(M) \to K$. If (M, ω) is weak symplectic, this is enriched by the data contained in natural differential forms on $\mathrm{Hol}(M)$ and K, which leads to the concept of a quasihamiltonian K-space for which $\overline{\Phi}\colon M \to K$ plays the role of a *group-valued momentum map*. If K is a compact Lie group and $\mathcal{L}(K)$ denotes a suitable Banach–Lie group of differentiable loops, such as H^1-loops, then the Equivalence Theorem in [1, Thm. 8.3] asserts that quasihamiltonian actions of K are in one-to-one correspondence with Hamiltonian $\mathcal{L}(K)$-actions on Banach manifolds M for which the momentum map $\Phi\colon M \to \mathcal{L}(\mathfrak{k})$ is proper.

Since our setup for Hamiltonian $\mathcal{L}(K)$-action uses only the invariant bilinear form on $\mathfrak{k}$, it is also valid for non-compact Lie groups K and even for infinite-dimensional ones, provided $\mathfrak{k}$ carries an invariant non-degenerate symmetric bilinear form.

In particular, the construction of a Lie group-valued momentum map $\mu = \exp\circ\Phi$ from a Lie algebra-valued momentum map $\Phi\colon M \to \mathfrak{g}$ with respect to a Poisson structure $\{\cdot,\cdot\}_{\kappa,D}$ on $\mathfrak{g}$ (cf. [1, Prop. 3.4]) works quite generally for any pair (κ, D) as in Sect. 4.1.

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These conditions are easily verified to be equivalent to ours (cf. Proposition 3.1).

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Bethe Vectors of gl(3)-Invariant Integrable Models, Their Scalar Products and Form Factors

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Abstract This short note corresponds to a talk given at *Lie Theory and Its Applications in Physics* (Varna, Bulgaria, June 2013) and is based on joint works with S. Belliard, S. Pakuliak and N. Slavnov, see arXiv:1206.4931, arXiv:1207.0956, arXiv:1210.0768, arXiv:1211.3968 and arXiv:1312.1488.

1 General Background

We first expose the general algebraic framework that will be needed for our calculation. This part is not new at all, it just recasts well-known facts from QISM approach, see e.g. [1–3] and references therein. We also use it to fix our notations.

1.1 R-Matrix

As usual in integrable systems, the basic tool is the so-called R-matrix $R(x, y) \in V \otimes V$, where $x, y \in \mathbb{C}$ are the spectral parameters and $V = \text{End}(\mathbb{C}^N)$ is a vector space. $R(x, y)$ obeys the Yang–Baxter equation, written in $V \otimes V \otimes V$:

$$R^{12}(x_1, x_2)\, R^{13}(x_1, x_3)\, R^{23}(x_2, x_3) = R^{23}(x_2, x_3)\, R^{13}(x_1, x_3)\, R^{12}(x_1, x_2).$$

Here and below, we will use the auxiliary space notation: the superscripts indicate in which copies of V spaces R acts non trivially. For instance, in $V \otimes V \otimes V$, we have:

$$R^{12}(x, y) = R(x, y) \otimes \mathbb{I} \quad \text{and} \quad R^{23}(x, y) = \mathbb{I} \otimes R(x, y)\,,$$

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while in $V^{\otimes N}$, we would have:

$$R^{k,k+1}(x,y) = \mathbb{I}^{\otimes(k-1)} \otimes R(x,y) \otimes \mathbb{I}^{N-k-1}.$$

1.2 Monodromy and Transfer Matrices

We define the monodromy matrix

$$T(x) = \sum_{i,j=1}^{N} e_{ij} \otimes T_{ij}(x) \in \text{End}(\mathbb{C}^N) \otimes \mathcal{A}[[x^{-1}]]\,,$$

where e_{ij} is the elementary $N \times N$ matrix with 1 at position (i,j). $T(x)$ obeys the commutation relations (or FRT relations)

$$R^{12}(x,y)\,T^1(x)\,T^2(y) = T^2(y)\,T^1(x)\,R^{12}(x,y). \quad (1)$$

Through these exchange relations, the monodromy matrix generates an algebra $\mathcal{A}$, defined by the choice of the R-matrix. Typically, $\mathcal{A}$ is the Yangian $Y(\mathfrak{gl}_N)$ or the quantum affine group $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$. The monodromy matrix leads to an integrable model through the transfer matrix

$$t(x) = tr_0 T^0(x) = \sum_{j=1}^{N} T_{jj}(x) \in \mathcal{A}[[x^{-1}]].$$

Integrability can be seen in the relation $[t(x)\,,\; t(y)] = 0$, that is valid at the algebraic level (i.e. in the $\mathcal{A}$ algebra), due to the relations (1).

In the following, we will deal with the Yangian $Y(\mathfrak{gl}_3)$, based on the $SU(3)$-invariant R-matrix

$$R(x,y) = \mathbf{I} + g(x,y)\mathbf{P} \in End(\mathbb{C}^3) \otimes End(\mathbb{C}^3) \quad \text{and} \quad g(x,y) = \frac{c}{x-y},$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{P}$ is the permutation matrix between two spaces $End(\mathbb{C}^3)$, and c is a constant. Note however that many properties will be also valid for the trigonometric R-matrix associated to the quantum group $\mathcal{U}_q(\widehat{\mathfrak{gl}}_3)$, and also for $Y(\mathfrak{gl}_N)$ or $\mathcal{U}_q(\widehat{\mathfrak{gl}}_N)$ algebras, see below.

1.3 Choice of a Physical Model

The choice of a representation for the algebra $\mathcal{A}$ leads to a physical model. For instance, taking for the monodromy and transfer matrices, the usual form

$$t(x) = tr_0 T^0(x) = tr_0 R^{01}(x,0)\, R^{02}(x,0) \cdots R^{0L}(x,0) \in (\text{End}(\mathbb{C}^N))^{\otimes L}\,,$$

we get an Hamiltonian acting on L copies of the fundamental representation of $\mathcal{A}$, $(\mathbb{C}^N)^{\otimes L}$: it is the generalized $\mathfrak{gl}_N$-XXX or $\mathfrak{gl}_N$-XXZ closed spin chain with L sites.

To summarize this algebraic part, we have a two step procedure for the determination of a physical model:

- The choice of an R-matrix, that fixes the algebra we are dealing with, that is to say the interaction in the bulk of the spin chain (leading to XXX, XXZ, ... models);
- The choice of the "spin content" of the chain, that is given by the choice of the representations of the algebra, in our context the form of the monodromy matrix.

Here, as already stated, we will deal with $\mathcal{A} = Y(\mathfrak{gl}_3)$. However, to be as general (and algebraic) as possible, we will not fix the representation we act on, and just assume that it is highest weight:

$$T_{jj}(w)|0\rangle = \lambda_j(w)|0\rangle,\ j = 1,2,3 \qquad T_{ij}(w)|0\rangle = 0, \qquad 1 \le i < j \le 3$$

for some arbitrary series $\lambda_j(w)$, $j = 1,2,3$. Up to a rescaling $T(w) \to \lambda_2^{-1}(w)$ $T(w)$, we will only need the ratios

$$r_1(w) = \frac{\lambda_1(w)}{\lambda_2(w)}, \qquad r_3(w) = \frac{\lambda_3(w)}{\lambda_2(w)}.$$

where r_1 and r_3 are free functional parameters.

1.4 Aim

The purpose in integrable systems is twofold:

1. Compute the Bethe vectors (BVs), eigenvectors of $t(x)$

$$t(x)\,\mathbb{B}^{a,b}(\bar{u},\bar{v}) = \tau(x|\bar{u},\bar{v})\,\mathbb{B}^{a,b}(\bar{u},\bar{v}).$$

   This part is well-understood and is done using the algebraic Bethe Ansatz method. It leads to the celebrated Bethe Ansatz eqs (BAE).
2. Compute correlation functions $< \mathcal{O}_1 \cdots \mathcal{O}_n >$ for some local operators $\mathcal{O}_j$. This calculation can be decomposed in four steps:

   (a) Express the operators $\mathcal{O}_j$ in terms of monodromy entries $T_{kl}(x)$;
   (b) Action of $T_{ij}(\bar{x})$ on $\mathbb{B}^{a,b}(\bar{u},\bar{v})$;
   (c) Scalar product of off-shell BVs (without BAE);
   (d) Form factors $\mathbb{C}^{a,b}(\bar{t},\bar{s})T_{ij}(\bar{x})\mathbb{B}^{a,b}(\bar{u},\bar{v})$.

In part 2, one needs to find *simple* (i.e. factorized) expressions in order to be able to take the thermodynamical limit and extract the asymptotic behavior of the correlation functions.

Here, we will present these two parts for the model based on $Y(\mathfrak{gl}_3)$. The calculations are rather technical, so that we will present here the results only, and refer to the original papers for the complete calculations. The presentation follows the plan explained above, and we will show how the techniques apply for other models in the conclusion.

2 Notation

Apart from the functions $g(x,y) = \frac{c}{x-y}$, $r_1(x)$ and $r_3(x)$ introduced above, we note

$$f(x,y) = \frac{x-y+c}{x-y}\,, \quad h(x,y) = \frac{f(x,y)}{g(x,y)}, \quad t(x,y) = \frac{g(x,y)}{h(x,y)}.$$

Clearly $f(x,y) = 1 + g(x,y)$ but this identification is not true for the q-analogues of these functions, so we keep this distinction.

To make presentation lighter, we will use the following conventions:

- "bar" always denote sets of variables: $\bar{w}$, $\bar{u}$, $\bar{v}$ etc.
- $|.|$ is the dimension of a set: $\bar{w} = \{w_1, w_2\} \;\Rightarrow\; |\bar{w}| = 2$, etc.
- Individual elements of the sets have Latin subscripts: w_j, u_k, etc.
- Subsets of variables are denoted by roman indices: $\bar{u}_{\rm I}$, $\bar{v}_{\rm iv}$, $\bar{w}_{\rm II}$, etc.
- Special case: $\bar{u}_j = \bar{u} \setminus \{u_j\}$, $\bar{w}_k = \bar{w} \setminus \{w_k\}$, etc.

We will also use shorthand notations for products of scalar functions:

$$f(\bar{u}_{\rm II}, \bar{u}_{\rm I}) = \prod_{u_j \in \bar{u}_{\rm II}} \prod_{u_k \in \bar{u}_{\rm I}} f(u_j, u_k), \quad r_1(\bar{u}_{\rm II}) = \prod_{u_j \in \bar{u}_{\rm II}} r_1(u_j),$$

$$g(v_k, \bar{w}) = \prod_{w_j \in \bar{w}} g(v_k, w_j), \quad etc.$$

3 Bethe Vectors

The framework for the construction of Bethe vectors is the Nested Bethe Ansatz as introduced in [4]. This technics is well-known, but the explicit expressions for these BVs are rather recent, so we briefly remind them here.

3.1 On-shell Bethe Vectors

The Bethe vectors $\mathbb{B}^{a,b}(\bar{u};\bar{v})$ depend on two sets of parameters $\bar{u} = \{u_1, \dots, u_a\}$ and $\bar{v} = \{v_1, \dots, v_b\}$. The superscripts a and b in $\mathbb{B}$ indicate the cardinalities of the sets, $|\bar{u}| = a$ and $|\bar{v}| = b$. They are eigenvectors of the transfer matrix

$$t(x)\,\mathbb{B}^{a,b}(\bar{u};\bar{v}) = \tau(x|\bar{u};\bar{v})\,\mathbb{B}^{a,b}(\bar{u};\bar{v}), \tag{2}$$

$$\tau(x|\bar{u};\bar{v}) = r_1(w)f(\bar{u},w) + f(w,\bar{u})f(\bar{v},w) + r_3(w)f(w,\bar{v}), \tag{3}$$

provided $\bar{u}$ and $\bar{v}$ obey the Bethe equations (BAEs):

$$r_1(\bar{u}_{\rm I}) = \frac{f(\bar{u}_{\rm I},\bar{u}_{\rm II})}{f(\bar{u}_{\rm II},\bar{u}_{\rm I})} f(\bar{v},\bar{u}_{\rm I}), \tag{4}$$

$$r_3(\bar{v}_{\rm I}) = \frac{f(\bar{v}_{\rm II},\bar{v}_{\rm I})}{f(\bar{v}_{\rm I},\bar{v}_{\rm II})} f(\bar{v}_{\rm I},\bar{u}). \tag{5}$$

that hold for arbitrary partitions of the sets $\bar{u}$ and $\bar{v}$ into subsets $\{\bar{u}_{\rm I},\ \bar{u}_{\rm II}\}$ and $\{\bar{v}_{\rm I},\ \bar{v}_{\rm II}\}$. In that case, the BVs will be called "on-shell", while they will be called "off-shell" is the BAEs are not obeyed. Of course, in that latter case, the BVs are not eigenvector of $t(x)$.

3.2 Dual Bethe Vectors $\mathbb{C}^{a,b}(\bar{u};\bar{v})$, $|\bar{u}| = a$, $|\bar{v}| = b$

Dual BVs are constructed as left eigenvectors of the transfer matrix:

$$\mathbb{C}^{a,b}(\bar{u};\bar{v})\,t(x) = \tau(x|\bar{u};\bar{v})\,\mathbb{C}^{a,b}(\bar{u};\bar{v}), \tag{6}$$

where the Bethe parameters $\bar{u}, \bar{v}$ obey the BAEs (4)–(5). Again, these dual BVs will be called on-shell when $\bar{u}$ and $\bar{v}$ obey the BAEs, while they will be called off-shell dual BVs when $\bar{u}, \bar{v}$ are left free.

3.3 Trace Formula

This is a known and quite general formula, given in [5] for $\mathfrak{gl}_N$ and $\mathcal{U}_q(\mathfrak{gl}_N)$ algebras, and generalized in [6] for superalgebras. It expresses $\mathbb{B}^{a,b}(\bar{u};\bar{v})$ as a trace in $a + b$ auxiliary spaces of products of monodromy matrices:

$$\mathbb{B}^{a,b}(\bar{u};\bar{v}) = tr\Big(\mathbb{T}(\bar{u};\bar{v})\,\mathbb{R}(\bar{u};\bar{v})\,e_{21}^{\otimes a} \otimes e_{32}^{\otimes b}\Big) \in Y(\mathfrak{gl}_3), \tag{7}$$

where $\mathbb{T}$ is some product of monodromy matrices $T(x)$ and $\mathbb{R}$ some product of R-matrices. Their explicit expression can be found in [5, 6].

3.4 Recursion Formulas

It can be shown that the Bethe vectors also obey the following recursion relations [7]:

$$\lambda_2(u_k)\,f(\bar v,u_k)\mathbb{B}^{a+1,b}(\bar u;\bar v) \;=\; T_{12}(u_k)\mathbb{B}^{a,b}(\bar u_k;\bar v) \tag{8}$$
$$+\sum_{i=1}^{b} g(v_i,u_k)\,f(\bar v_i,v_i)T_{13}(u_k)\mathbb{B}^{a,b-1}(\bar u_k;\bar v_i),$$

$$\lambda_2(v_k)\,f(v_k,\bar u)\mathbb{B}^{a,b+1}(\bar u;\bar v) = T_{23}(v_k)\mathbb{B}^{a,b}(\bar u;\bar v_k) \tag{9}$$
$$+\sum_{j=1}^{a} g(v_k,u_j)\,f(u_j,\bar u_j)T_{13}(v_k)\mathbb{B}^{a-1,b}(\bar u_j;\bar v_k).$$

Let us remark that (9) completely determines the Bethe vectors once $\mathbb{B}^{0,b}(\emptyset;\bar v)$ is known. In the same way, (10) completely determines the Bethe vectors once $\mathbb{B}^{a,0}(\bar u;\emptyset)$ is fixed.

3.5 Explicit Formulas

There is a third series of expressions for Bethe vectors, using partitions of $\bar u$ and $\bar v$ [7]:

$$\mathbb{B}^{a,b}(\bar u;\bar v) = \sum \frac{\mathsf{K}_k(\bar v_{\rm I}|\bar u_{\rm I})}{\lambda_2(\bar v_{\rm II})\lambda_2(\bar u)}\,\frac{f(\bar v_{\rm II},\bar v_{\rm I})\,f(\bar u_{\rm II},\bar u_{\rm I})}{f(\bar v_{\rm II},\bar u)\,f(\bar v_{\rm I},\bar u_{\rm I})}\;T_{12}(\bar u_{\rm II})T_{13}(\bar u_{\rm I})T_{23}(\bar v_{\rm II})\,|0\rangle, \tag{10}$$

$$\mathbb{B}^{a,b}(\bar u;\bar v) = \sum \frac{\mathsf{K}_k(\bar v_{\rm I}|\bar u_{\rm I})}{\lambda_2(\bar u_{\rm II})\lambda_2(\bar v)}\,\frac{f(\bar v_{\rm I},\bar v_{\rm II})\,f(\bar u_{\rm I},\bar u_{\rm II})}{f(\bar v_{\rm I},\bar u_{\rm I})\,f(\bar v,\bar u_{\rm II})}\;T_{23}(\bar v_{\rm II})T_{13}(\bar v_{\rm I})T_{12}(\bar u_{\rm II})\,|0\rangle, \tag{11}$$

$$\mathbb{B}^{a,b}(\bar u;\bar v) = \sum \frac{\mathsf{K}_k(\bar v_{\rm I}|\bar u_{\rm I})}{\lambda_2(\bar v_{\rm II})\lambda_2(\bar u)}\,\frac{f(\bar v_{\rm II},\bar v_{\rm I})\,f(\bar u_{\rm I},\bar u_{\rm II})}{f(\bar v,\bar u)}\;T_{13}(\bar u_{\rm I})T_{12}(\bar u_{\rm II})T_{23}(\bar v_{\rm II})\,|0\rangle, \tag{12}$$

$$\mathbb{B}^{a,b}(\bar u;\bar v) = \sum \frac{\mathsf{K}_k(\bar v_{\rm I}|\bar u_{\rm I})}{\lambda_2(\bar u_{\rm II})\lambda_2(\bar v)}\,\frac{f(\bar v_{\rm II},\bar v_{\rm I})\,f(\bar u_{\rm I},\bar u_{\rm II})}{f(\bar v,\bar u)}\;T_{13}(\bar v_{\rm I})T_{23}(\bar v_{\rm II})T_{12}(\bar u_{\rm II})\,|0\rangle. \tag{13}$$

The sums are taken over partitions of the sets $\bar u \Rightarrow \{\bar u_{\rm I},\bar u_{\rm II}\}$ and $\bar v \Rightarrow \{\bar v_{\rm I},\bar v_{\rm II}\}$ with the condition $0 \le |\bar u_{\rm I}| = |\bar v_{\rm I}| = k \le \min(a,b)$.

$\mathsf{K}_k(\bar v_{\rm I}|\bar u_{\rm I})$ is the Izergin–Korepin determinant [8, 9]

$$\mathsf{K}_k(\bar x|\bar y) = \prod_{\ell<m}^{k} g(x_\ell,x_m)g(y_m,y_\ell)\cdot h(\bar x,\bar y)\;\det_k\big[t(x_i,y_j)\big]. \tag{14}$$

3.6 All These Formulas are Related

Let us stress that all the above formulas define the same Bethe vectors, should they be on-shell or off-shell. For instance, one can show that

- The explicit expressions obey the recursion formulas;
- The trace formula obeys the recursion formulas too;
- Recursion formulas can be obtained starting from the trace formula.

Depending on the calculation, one can then freely choose any of these expression to prove a formula or a property of BVs.

4 Correlation Functions

We now turn to the second step of our program, that is, for a local operator $\mathcal{O}$, how to compute its mean value? As a first step, we are led with the following question:

How to compute $\mathcal{O}_{\mathbb{C},\mathbb{B}} = \langle\mathbb{C}|\mathcal{O}|\mathbb{B}\rangle$?

Assuming that $\{|\mathbb{B}\rangle\}$ forms a complete basis (of transfer matrix eigenspaces), we have

$$\mathcal{O}|\mathbb{B}\rangle = \sum_{\mathbb{B}'} \mathbb{O}_{\mathbb{B}\mathbb{B}'}|\mathbb{B}'\rangle, \tag{15}$$

so that we "only" need $\langle\mathbb{C}|\mathbb{B}'\rangle$ and of course the decomposition (15).

Now, for a spin chain of length L and based on $\mathfrak{gl}_N$-fundamental representations, local operators have a decomposition[1]

$$\mathcal{O} = \sum_{\ell=1}^{L}\sum_{i,j=1}^{N} \mathcal{O}_{ij}^{(\ell)}\, e_{ij}^{\ell}, \tag{16}$$

where e_{ij}^{ℓ} is the elementary matrix e_{ij} at site ℓ. Then, everything boils down to the calculation of $\langle\mathbb{C}|e_{ij}^{\ell}|\mathbb{B}\rangle$.

A further simplification occurs because of QISM. Indeed, the expression of e_{ij}^{ℓ}, $i, j = 1, 2, \dots, N$ and $\ell = 1, \dots L$ is known in terms of monodromy entries $T_{kl}(x)$, $k, l = 1, \dots, N$ [10]:

$$e_{ij}^{\ell} = (t(0))^{\ell-1}\, T_{ji}(0)\, (t(0))^{-\ell}. \tag{17}$$

Then, from (16) and (17), if we can compute $T_{kl}(x)\mathbb{B}^{a,b}(\bar{u};\bar{v})$ and $\mathbb{C}^{a,b}(\bar{w};\bar{z})\mathbb{B}^{a,b}(\bar{u};\bar{v})$, we are able to compute any correlation function. The following sections are devoted to the calculation of these two fundamental quantities in the case $N = 3$.

[1]The same ideas can be applied for a general spin chain, using an adapted basis.

5 Multiple Actions of $T_{ij}(\bar{x})$ on $\mathbb{B}^{a,b}(\bar{u};\bar{v})$

Using the explicit expressions of Sect. 3.5, we were able in [7] to compute explicitly the actions of $T_{ij}(\bar{x})$ on $\mathbb{B}^{a,b}(\bar{u};\bar{v})$. Denoting $\{\bar{u},\bar{x}\} = \bar{\eta}$, $\{\bar{v},\bar{x}\} = \bar{\xi}$ and the cardinalities by $|\bar{x}| = n$, $|\bar{\eta}| = a+n$ and $|\bar{\xi}| = b+n$, we have

$$T_{13}(\bar{x})\mathbb{B}^{a,b}(\bar{u};\bar{v}) = \lambda_2(\bar{x})\,\mathbb{B}^{a+n,b+n}(\bar{\eta};\bar{\xi}), \tag{18}$$

$$T_{12}(\bar{x})\mathbb{B}^{a,b}(\bar{u};\bar{v}) = (-1)^n\lambda_2(\bar{x})\sum f(\bar{\xi}_{\rm II},\bar{\xi}_{\rm I})\,\mathsf{K}_n(\bar{\xi}_{\rm I}|\bar{x}+c)\,\mathbb{B}^{a+n,b}(\bar{\eta};\bar{\xi}_{\rm II}), \tag{19}$$

$$T_{23}(\bar{x})\mathbb{B}^{a,b}(\bar{u};\bar{v}) = (-1)^n\lambda_2(\bar{x})\sum f(\bar{\eta}_{\rm I},\bar{\eta}_{\rm II})\,\mathsf{K}_n(\bar{x}|\bar{\eta}_{\rm I}+c)\,\mathbb{B}^{a,b+n}(\bar{\eta}_{\rm II};\bar{\xi}). \tag{20}$$

In (19), the sum is on partitions $\bar{\xi} = \{\bar{\xi}_{\rm I};\bar{\xi}_{\rm II}\}$ with $|\bar{\xi}_{\rm I}| = n$, while in (20), the sum is on partitions $\bar{\eta} = \{\bar{\eta}_{\rm I};\bar{\eta}_{\rm II}\}$ with $|\bar{\eta}_{\rm I}| = n$. Similar expressions can be obtained for any $T_{ij}(\bar{x})$ and for dual BVs, see [7].

Remark that the relations (19) and (20) imply recursion relations of Sect. 3.4 as a subcase (for $n = 1$).

Since the action of $T_{ij}(\bar{x})$ operators on BVs gives back BVs (that are a priori off-shell), it remains to compute scalar products of BVs to get the full form factor expression.

6 Scalar Products of BVs

In this section, we provide expression for the scalar product

$$\mathcal{S}_{a,b} \equiv \mathcal{S}_{a,b}(\bar{u}^C,\bar{u}^B|\bar{v}^C,\bar{v}^B) = \mathbb{C}^{a,b}(\bar{u}^C;\bar{v}^C)\,\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B), \tag{21}$$

where $\mathbb{C}^{a,b}(\bar{u}^C;\bar{v}^C)$ and $\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B)$ are general (dual) BVs. Let us stress that the superscripts ^{B} and ^{C} are used to denote *different* sets of (Bethe) parameters, completely independent one from each other.

6.1 Reshetikhin's Formula

There is a well-known formula, due to Reshetikhin [11], and valid for $\mathfrak{gl}_N$:

$$\mathcal{S}_{a,b} = \sum r_1(\bar{u}^B_{\rm I})r_1(\bar{u}^C_{\rm II})r_3(\bar{v}^B_{\rm I})r_3(\bar{v}^C_{\rm II})\,f(\bar{u}^C_{\rm I},\bar{u}^C_{\rm II})\,f(\bar{u}^B_{\rm II},\bar{u}^B_{\rm I})\,f(\bar{v}^C_{\rm II},\bar{v}^C_{\rm I})\,f(\bar{v}^B_{\rm I},\bar{v}^B_{\rm II})$$
$$\times\, f(\bar{v}^C_{\rm I},\bar{u}^C_{\rm I})\,f(\bar{v}^B_{\rm II},\bar{u}^B_{\rm II})\,Z_{a-k,n}(\bar{u}^C_{\rm II};\bar{u}^B_{\rm II}|\bar{v}^C_{\rm I};\bar{v}^B_{\rm I})Z_{k,b-n}(\bar{u}^B_{\rm I};\bar{u}^C_{\rm I}|\bar{v}^B_{\rm II};\bar{v}^C_{\rm II}), \tag{22}$$

where the sum is on partitions $\bar{u}^B = \{\bar{u}^B_{\rm I},\bar{u}^B_{\rm II}\}$, $\bar{u}^C = \{\bar{u}^C_{\rm I},\bar{u}^C_{\rm II}\}$ with $|\bar{u}^B_{\rm I}| = |\bar{u}^C_{\rm I}| = k$ for $k = 0,\dots,a$ $\bar{v}^B = \{\bar{v}^B_{\rm I},\bar{v}^B_{\rm II}\}$, $\bar{v}^C = \{\bar{v}^C_{\rm I},\bar{v}^C_{\rm II}\}$ with $|\bar{v}^B_{\rm I}| = |\bar{v}^C_{\rm I}| = n$ for $n = 0,\dots,b$.

$Z_{a,b}$ are the so-called highest coefficients

$$Z_{a,b}(\bar{t};\bar{x}|\bar{s};\bar{y}) = (-1)^b \sum K_b(\bar{s}-c|\bar{w}_{\rm I})K_a(\bar{w}_{\rm II}|\bar{t})K_b(\bar{y}|\bar{w}_{\rm I})f(\bar{w}_{\rm I},\bar{w}_{\rm II}), \tag{23}$$

where the sum is done over partitions of $\bar{w} = \{\bar{s}, \bar{x}\}$ into subsets $\bar{w}_{\rm I}$ and $\bar{w}_{\rm II}$ with $|\bar{w}_{\rm I}| = b$.

The formula is valid for a general scalar product, but as it stands, $\mathcal{S}_{a,b}$ is difficult to handle. To compute e.g. the thermodynamical limit of such formula, and to use it for the calculation of correlation functions, one needs to find a factorized form, containing only one determinant. It was done for the $\mathfrak{gl}_2$ case [12], but for $\mathfrak{gl}_3$ (and a fortiori for $\mathfrak{gl}_N$) no such formula is known yet. However, in some particular cases, there exists such a formula:

1. When computing the norm of a Bethe vector that is assumed to be on-shell, such an expression was obtained by Reshetikhin [11];
2. A nice factorized expression was obtained in [13], when some of the Bethe parameters tend to infinity;
3. When the BV is on-shell and the dual BV is "twisted on-shell" (see below), we were able to get a simplified expression [14];
4. In [15], we provided different expressions for the highest coefficients (23);
5. An interesting multiple integral expression for the scalar product of an on-shell and an off-shell BV was recently obtained in [16].

We present the points 3 and 4 in the two following sections.

6.2 *Highest Coefficients*

Highest coefficients were introduced by Reshetikhin [11] and play a central role in the expression of the scalar product of Bethe vectors. In fact, they can be viewed as partition functions of a statistical models with some particular boundary conditions. It is thus important to get different forms for them. We give here some examples of such formulas, a more complete list can be found in [15].

Sums on Partitions

There are different series of expressions for the highest coefficients. A first series is given by sums over partitions. The expression (23) is a first example of such formulas. Another example is given by

$$Z_{a,b}(\bar{t};\bar{x}|\bar{s};\bar{y}) = (-1)^a f(\bar{y},\bar{x})f(\bar{s},\bar{t}) \sum K_a(\bar{t}-c|\bar{\eta}_{\rm I})K_a(\bar{x}|\bar{\eta}_{\rm I}) \times K_b(\bar{\eta}_{\rm II}-c|\bar{s})f(\bar{\eta}_{\rm I},\bar{\eta}_{\rm II}), \tag{24}$$

where $\bar{\eta} = \{\bar{y}+c,\ \bar{t}\}$. The sum is taken with respect to partitions of the set $\bar{\eta}$ into subsets $\bar{\eta}_{\rm I}$ and $\bar{\eta}_{\rm II}$ with $\#\bar{\eta}_{\rm I} = a$.

Recursion Formulas

The most important property of the highest coefficient $Z_{a,b}$ is that its residues in its poles can be expressed in terms of $Z_{a-1,b}$ or $Z_{a,b-1}$. Since $Z_{a,b}$ is a rational function in all its variables, this property allows us to fix it unambiguously, provided we know some initial condition. It is easy to see that for $a = 0$ or $b = 0$ $Z_{a,b}$ coincides with K_n:

$$Z_{a,0}(\bar{t};\bar{x}|\emptyset;\emptyset) = K_a(\bar{x}|\bar{t}), \qquad Z_{0,b}(\emptyset;\emptyset|\bar{s};\bar{y}) = K_b(\bar{y}|\bar{s}). \tag{25}$$

Consider $Z_{a,b}$ as a function of s_b with all other variables fixed. Then it has simple poles at $s_b = y_m$, $m = 1,\dots,b$ and $s_b = t_\ell$, $\ell = 1,\dots,a$. Due to the symmetry of $Z_{a,b}$ over $\bar{y}$ and over $\bar{t}$ it is enough to find the residues at $s_b = y_b$ and $s_b = t_a$. These residues are given by:

$$\operatorname{Res} Z_{a,b}(\bar{t};\bar{x}|\bar{s};\bar{y})\Big|_{s_b=y_b} = -cf(y_b,\bar{s}_b)f(\bar{y}_b,y_b)f(y_b,\bar{x})Z_{a,b-1}(\bar{t};\bar{x}|\bar{s}_b;\bar{y}_b), \tag{26}$$

$$\begin{aligned}\operatorname{Res} Z_{a,b}(\bar{t};\bar{x}|\bar{s};\bar{y})\Big|_{s_b=t_a} &= cf(\bar{s}_b,t_a)f(t_a,\bar{t}_a)\sum_{p=1}^{a} g(x_p,t_a)f(\bar{x}_p,x_p)\\ &\quad\times Z_{a-1,b}(\bar{t}_a;\bar{x}_p|\{\bar{s}_b,\ x_p\};\bar{y}_b),\end{aligned} \tag{27}$$

where $\bar{s}_b = \bar{s} \setminus s_b$, $\bar{y}_b = \bar{y} \setminus y_b$, etc.

Contour Integral

There exists several representations for $Z_{a,b}$ in terms of multiple contour integrals of Cauchy type. Here, we give only one possible integral as example:

$$Z_{a,b}(\bar{t};\bar{x}|\bar{s};\bar{y}) = \frac{1}{(2\pi i c)^b b!}\oint_{\bar{w}} K_b(\bar{s}-c|\bar{z})K_b(\bar{y}|\bar{z})K_{a+b}(\bar{w}|\bar{t},\bar{z}+c)f(\bar{z},\bar{w})\mathcal{F}_b(\bar{z})\,d\bar{z}, \tag{28}$$

where we have a b-fold integral and

$$\mathcal{F}_b(\bar{z}) = \prod_{j=1}^{b} f^{-1}(z_j,\bar{z}_j).$$

Other expressions of the type (28), or implying a-fold integrals can be found in [15].

6.3 Scalar Product for Twisted Bethe Vectors

Here we consider an on-shell Bethe vector, eigenvector of the transfer matrix

$$t(x)\,\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B) = \tau(x|\bar{u}^B,\bar{v}^B)\,\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B), \tag{29}$$

where the Bethe parameters $\{\bar{u}^B;\bar{v}^B\}$ obey the Bethe equations (4)–(5). We also introduce, for any complex number κ, a twisted transfer matrix

$$t_\kappa(x) = T_{11}(x) + \kappa T_{22}(x) + T_{33}(x) = tr\big(M\,T(x)\big) \quad \text{with} \quad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{30}$$

and its *twisted* dual on-shell Bethe vector

$$\mathbb{C}_\kappa^{a,b}(\bar{u}^C;\bar{v}^C)\,t_\kappa(x) = \tau_\kappa(x|\bar{u}^C,\bar{v}^C)\,\mathbb{C}_\kappa^{a,b}(\bar{u}^C;\bar{v}^C). \tag{31}$$

It is an eigenvector of $t_\kappa(x)$ when the Bethe parameters $\bar{u}^C$, $\bar{v}^C$ obey the *twisted* BAEs

$$r_1(\bar{u}_{\rm I}) = \kappa\,\frac{f(\bar{u}_{\rm I},\bar{u}_{\rm II})}{f(\bar{u}_{\rm II},\bar{u}_{\rm I})}\,f(\bar{v},\bar{u}_{\rm I}), \tag{32}$$

$$r_3(\bar{v}_{\rm I}) = \kappa\,\frac{f(\bar{v}_{\rm II},\bar{v}_{\rm I})}{f(\bar{v}_{\rm I},\bar{v}_{\rm II})}\,f(\bar{v}_{\rm I},\bar{u}). \tag{33}$$

Let us stress that the superscripts ^{B} and ^{C} are there to distinguish the Bethe parameters of $\mathbb{B}^{ab}$ from those of $\mathbb{C}^{ab}$. In other words, the Bethe parameters $\{\bar{u}^B,\bar{v}^B\}$ are a priori not related to $\{\bar{u}^C,\bar{v}^C\}$.

In [14], we obtained an expression for the scalar product

$$\mathcal{S}_{a,b} \equiv \mathcal{S}_{a,b}(\bar{u}^C,\bar{u}^B|\bar{v}^C,\bar{v}^B) = \mathbb{C}_\kappa^{a,b}(\bar{u}^C;\bar{v}^C)\,\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B). \tag{34}$$

Indeed, the scalar product can be written as

$$\mathcal{S}_{a,b} = f(\bar{v}^C,\bar{u}^C)f(\bar{v}^B,\bar{u}^B)t(\bar{v}^C,\bar{u}^B)\Delta'_a(\bar{u}^C)\Delta_a(\bar{u}^B)\Delta'_b(\bar{v}^C)\Delta_b(\bar{v}^B)\,\det_{a+b}\mathcal{N}, \tag{35}$$

where

$$\Delta'_n(\bar{x}) = \prod_{j>k}^{n} g(x_j,x_k), \qquad \Delta_n(\bar{y}) = \prod_{j<k}^{n} g(y_j,y_k).$$

and $\mathcal{N}$ is a block-matrix of the size $(a+b)\times(a+b)$,

$$\mathcal{N} = \begin{pmatrix} \mathcal{N}^{(u)}(u_j^C, u_k^B) & \mathcal{N}^{(u)}(u_j^C, v_k^C) \\ \mathcal{N}^{(v)}(v_j^B, u_k^B) & \mathcal{N}^{(v)}(v_j^B, v_k^C) \end{pmatrix} = \left(\begin{array}{c|c} a\times a & a\times b \\ \hline b\times a & b\times b \end{array}\right),$$

whose full expression is given in Appendix 1. We show below how this expression can give rise to a factorized expression for form factors of the model.

Expression for a General Twist

A similar expression for $\mathcal{S}_{a,b}$ can be obtained when considering a general twist $\bar\kappa = (\kappa_1, \kappa_2, \kappa_3)$ of the transfer matrix

$$t_{\bar\kappa}(x) = \kappa_1 T_{11}(x) + \kappa_2 T_{22}(x) + \kappa_3 T_{33}(x).$$

However, in that case, the expression is valid only up to terms $(\kappa_i - 1)(\kappa_j - 1)$, $i, j = 1, 2, 3$, that are irrelevant for our purpose, as we shall see below. For further application it is useful to write the system of twisted Bethe equations in the logarithmic form. Let us define

$$\Phi_j = \log r_1(u_j^C) - \log\left(\frac{f(u_j^C, \bar u_j^C)}{f(\bar u_j^C, u_j^C)}\right) - \log f(\bar v^C, u_j^C), \qquad j = 1, \dots, a, \tag{36}$$

and

$$\Phi_{j+a} = \log r_3(v_j^C) - \log\left(\frac{f(\bar v_j^C, v_j^C)}{f(v_j^C, \bar v_j^C)}\right) - \log f(v_j^C, \bar u^C), \qquad j = 1, \dots, b. \tag{37}$$

Then the system of twisted Bethe equations for general $\bar\kappa$ takes the form

$$\begin{aligned} \Phi_j &= \log\kappa_2 - \log\kappa_1 + 2\pi i \ell_j, \qquad j = 1, \dots, a, \\ \Phi_{j+a} &= \log\kappa_2 - \log\kappa_3 + 2\pi i m_j, \qquad j = 1, \dots, b, \end{aligned} \tag{38}$$

where ℓ_j and m_j are some integers.

7 Form Factors

We present now the calculation [17] the form factor of the diagonal elements $T_{ss}(z)$

$$\mathcal{F}_{a,b}^{(s)}(z) \equiv \mathcal{F}_{a,b}^{(s)}(z|\bar u^C, \bar v^C; \bar u^B, \bar v^B) = \mathbb{C}^{a,b}(\bar u^C; \bar v^C) T_{ss}(z) \mathbb{B}^{a,b}(\bar u^B; \bar v^B), \tag{39}$$

where both $\mathbb{C}^{a,b}(\bar{u}^C;\bar{v}^C)$ and $\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B)$ are on-shell Bethe vectors. Form factors for off-diagonal elements $T_{j,j+1}(z)$ and $T_{j+1,j}(z)$ have been given in [18]. The form factors associated to $T_{13}(z)$ and $T_{31}(z)$ remain to be done. Of course, the ultimate goal would be to find a simple expression for the form factor when the Bethe vector $\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B)$ and/or the dual Bethe vector $\mathbb{C}^{a,b}(\bar{u}^C;\bar{v}^C)$ are off-shell. Up to now, such an expression is still missing.

A priori, from the knowledge of the actions (18)–(20) and the scalar products (22), we can deduce an expression of the form factor. However, the expression is rather complicated and difficult to handle. Fortunately, one can get another simpler form using the following trick.

Let us consider $\mathbb{C}^{a,b}_{\bar{\kappa}}(\bar{u}^C;\bar{v}^C)$ a twisted on-shell Bethe vector such that

$$\mathbb{C}^{a,b}_{\bar{\kappa}}(\bar{u}^C;\bar{v}^C)|_{\bar{\kappa}=1} = \mathbb{C}^{a,b}(\bar{u}^C;\bar{v}^C). \tag{40}$$

Then, the form factor (39) can be expressed as

$$\mathcal{F}^{(s)}(z|\bar{u}^C,\bar{v}^C;\bar{u}^B,\bar{v}^B) = \frac{dQ_{\bar{\kappa}}(z)}{d\kappa_s}\Big|_{\bar{\kappa}=1}, \qquad s=1,2,3$$

where $\bar{\kappa}=1$ means $\kappa_1=\kappa_2=\kappa_3=1$ and

$$\begin{aligned} Q_{\bar{\kappa}}(z) &= \mathbb{C}^{a,b}_{\bar{\kappa}}(\bar{u}^C;\bar{v}^C)\big(t_{\bar{\kappa}}(z)-t(z)\big)\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B) \\ &= \big(\tau_\kappa(z|\bar{u}^C,\bar{v}^C)-\tau(z|\bar{u}^B,\bar{v}^B)\big)\mathbb{C}^{a,b}_{\bar{\kappa}}(\bar{u}^C;\bar{v}^C)\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B). \end{aligned}$$

Then, it is clear that all depends on the expression of the scalar product $\mathbb{C}^{a,b}_{\bar{\kappa}}(\bar{u}^C;\bar{v}^C)\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B)$, and that we need to know this scalar product only up to terms $(\kappa_i-1)(\kappa_j-1)$, $i,j=1,2,3$. Depending on whether $\mathbb{C}^{a,b}(\bar{u}^C;\bar{v}^C)$ is $\big(\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B)\big)^\dagger$ or not, we get two different expressions:

When $\mathbb{C}^{a,b}(\bar{u}^C;\bar{v}^C)=\big(\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B)\big)^\dagger$

$$\begin{aligned} \mathcal{F}^{(s)}(z|\bar{u},\bar{v};\bar{u},\bar{v}) &= \|\mathbb{B}^{a,b}(\bar{u};\bar{v})\|^2\,\frac{d\tau_{\bar{\kappa}}(z|\bar{u}^C;\bar{v}^C)}{d\kappa_s}\Big|_{\bar{\kappa}=1} \\ &= (-1)^a c^{a+b} f(\bar{v},\bar{u})\prod_{j=1}^{a} f(u_j,\bar{u}_j)\prod_{k=1}^{b} f(v_k,\bar{v}_k)\;\det_{a+b+1}\Theta^{(s)}(z), \end{aligned} \tag{41}$$

where $\Theta^{(s)}(z)$ is an $(a+b+1)\times(a+b+1)$ matrix given in Appendix 2.

When $\mathbb{C}^{a,b}(\bar{u}^C;\bar{v}^C)\neq\big(\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B)\big)^\dagger$

$$\begin{aligned} \mathcal{F}^{(s)}_{a,b}(z|\bar{u}^C,\bar{v}^C;\bar{u}^B,\bar{v}^B) &= \Big(\tau(z|\bar{u}^C;\bar{v}^C)-\tau(z|\bar{u}^B;\bar{v}^B\Big) \\ &\quad\times\frac{d}{d\kappa_s}\Big(\mathbb{C}^{a,b}_{\bar{\kappa}}(\bar{u}^C;\bar{v}^C)\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B)\Big)\Big|_{\bar{\kappa}=1} \end{aligned}$$

$$= \frac{\tau(z|\bar{u}^C;\bar{v}^C) - \tau(z|\bar{u}^B;\bar{v}^B)}{\Omega_p}\, t(\bar{v}^C,\bar{u}^B)$$
$$\times\, \Delta'_a(\bar{u}^C)\Delta_a(\bar{u}^B)\Delta'_b(\bar{v}^C)\Delta_b(\bar{v}^B)\, \det_{a+b} \mathcal{N}^{(s,p)},$$

The integer p is such that $\Omega_p \neq 0$ where

$$\begin{aligned}
\Omega_k &= \prod_{\ell=1}^{a}(u_k^C - u_\ell^B)\prod_{\substack{\ell=1\\ \ell\neq k}}^{a}(u_k^C - u_\ell^C)^{-1}, \qquad k = 1,\dots,a,\\
\Omega_{a+k} &= \prod_{m=1}^{b}(v_k^B - v_m^C)\prod_{\substack{m=1\\ m\neq k}}^{b}(v_k^B - v_m^B)^{-1}, \qquad k = 1,\dots,b.
\end{aligned} \tag{42}$$

The matrix $\mathcal{N}^{(s,p)}$ has a special p^{th} row, but its determinant is independent of p. The form of $\mathcal{N}^{(s,p)}$ is given in Appendix 3.

Conclusion

For models with a $\mathfrak{gl}_3$ invariant R-matrix, we have presented several explicit expressions for (off-shell) Bethe vectors and dual BVs. We have also computed the multiple action of monodromy elements on these BVs. Both results are presented in term of Izergin-Korepin determinants and sums over partitions of sets of Bethe parameters.

In a second step, we calculated the scalar product of (twisted) on-shell BVs and the form factors of $T_{ss}(x)$, $s = 1, 2, 3$, of $T_{j,j+1}(x)$ and of $T_{j+1,j}(x)$, $j = 1, 2$. Both results were given in term of a single determinant (and product of scalar functions).

The ultimate goal is to obtain a single determinant expression for the correlation functions of the model, so as to study the thermodynamical limit and their asymptotics. Of course, to get to that point a lot remains to be done. For instance, it remains to compute the form factors of $T_{13}(x)$ and $T_{31}(x)$. The calculation of the scalar product of generic off-shell BVs (as a single determinant) is also lacking.

Certainly, a generalization to other integrable models is wanted. As a first step, we started to investigate the case of $\mathfrak{gl}_3$ XXZ spin chain (i.e. based on the R-matrix of $\mathcal{U}_q(\mathfrak{gl}_3)$):

1. The multiple action of $T_{ij}(x)$ generators on BVs was performed in [19];
2. The calculation of the highest coefficient was done in [20];
3. A Reshetikhin-like formula for scalar products of the $\mathcal{U}_q(\mathfrak{gl}_3)$ model is given in [21].

(continued)

Let us remark that to obtain these results, we used the current realization of $\mathcal{U}_q(\mathfrak{gl}_3)$ and the construction of Khoroshkin, Pakuliak and collaborators for BVs in this presentation [22]. This construction is valid for $\mathcal{U}_q(\mathfrak{gl}_N)$: a link between the current presentation of BVs and the explicit expression of BVs using the monodromy matrix for $\mathcal{U}_q(\mathfrak{gl}_N)$ is done in [23]. The use of a morphism between $\mathcal{U}_q(\mathfrak{gl}_N)$ and $\mathcal{U}_{q^{-1}}(\mathfrak{gl}_N)$ is essential in this construction.

Appendix 1: The Matrix $\mathcal{N}$

Diagonal blocks

$$\mathcal{N}^{(u)}(u_j^C, u_k^B) = h(\bar{v}^C, u_k^B)h(u_k^B, \bar{u}^C)\Big[\kappa t(u_k^B, u_j^C) + t(u_j^C, u_k^B)\frac{f(\bar{v}^B, u_k^B)}{f(\bar{v}^C, u_k^B)}\frac{h(\bar{u}^C, u_k^B)h(u_k^B, \bar{u}^B)}{h(u_k^B, \bar{u}^C)h(\bar{u}^B, u_k^B)}\Big] \qquad a \times a \text{ block}$$

$$\mathcal{N}^{(v)}(v_j^B, v_k^C) = h(v_k^C, \bar{u}^B)h(\bar{v}^B, v_k^C)\Big[t(v_j^B, v_k^C) + \kappa t(v_k^C, v_j^B)\frac{f(v_k^C, \bar{u}^C)}{f(v_k^C, \bar{u}^B)}\frac{h(\bar{v}^C, v_k^C)h(v_k^C, \bar{v}^B)}{h(v_k^C, \bar{v}^C)h(\bar{v}^B, v_k^C)}\Big] \qquad b \times b \text{ block}$$

Off-diagonal blocks

$$\mathcal{N}^{(u)}(u_j^C, v_k^C) = \kappa t(v_k^C, u_j^C)h(v_k^C, \bar{u}^C)h(\bar{v}^C, v_k^C) \qquad a \times b \text{ block}$$

$$\mathcal{N}^{(v)}(v_j^B, u_k^B) = t(v_j^B, u_k^B)h(\bar{v}^B, u_k^B)h(u_k^B, \bar{u}^B) \qquad b \times a \text{ block}$$

Appendix 2: The Matrix $\Theta^{(s)}$

First of all we define an $(a+b)\times(a+b)$ matrix θ with the entries

$$\theta_{j,k} = \frac{\partial \Phi_j}{\partial u_k^C}\bigg|_{\substack{\bar{u}^C=\bar{u}\\ \bar{v}^C=\bar{v}}}, \qquad k = 1, \dots, a,$$
$$\theta_{j,k+a} = \frac{\partial \Phi_j}{\partial v_k^C}\bigg|_{\substack{\bar{u}^C=\bar{u}\\ \bar{v}^C=\bar{v}}}, \qquad k = 1, \dots, b, \tag{43}$$

where the Φ_j are given by (36) and (37).

Then we extend the matrix θ to an $(a+b+1)\times(a+b+1)$ matrix $\Theta^{(s)}$ with $s=1,2,3$, by adding one row and one column

$$
\begin{aligned}
&\Theta^{(s)}_{j,k} = \theta_{j,k}, && j,k=1,\dots,a+b,\\
&\Theta^{(s)}_{a+b+1,k} = \frac{\partial\tau(z|\bar u,\bar v)}{\partial u_k}, && k=1,\dots,a,\\
&\Theta^{(s)}_{a+b+1,a+k} = \frac{\partial\tau(z|\bar u,\bar v)}{\partial v_k}, && k=1,\dots,b,\\
&\Theta^{(s)}_{j,a+b+1} = \delta_{s1}-\delta_{s2} && j=1,\dots,a,\\
&\Theta^{(s)}_{j+a,a+b+1} = \delta_{s3}-\delta_{s2} && j=1,\dots,b,\\
&\Theta^{(s)}_{a+b+1,a+b+1} = \left.\frac{\partial\tau_{\bar\kappa}(z|\bar u^C,\bar v^C)}{\partial\kappa_s}\right|_{\substack{\bar u^C=\bar u\\ \bar v^C=\bar v}} .
\end{aligned}
$$

Here the δ_{sk} are Kronecker deltas. Notice that $\Theta^{(s)}$ depends on s only in its last column.

Appendix 3: The Matrix $\mathcal{N}^{(s,p)}$

For $j\neq p$ we define the entries $\mathcal{N}^{(s,p)}_{j,k}$ of the $(a+b)\times(a+b)$ matrix $\mathcal{N}^{(s,p)}$ as

$$
\mathcal{N}^{(s)}_{j,k} = c\, g^{-1}(w_k,\bar u^C)\, g^{-1}(\bar v^C,w_k)\frac{\partial\tau(w_k|\bar u^C,\bar v^C)}{\partial u^C_j}, \tag{44}
$$

$$
j=1,\dots,a,\quad j\neq p,
$$

$$
\mathcal{N}^{(s)}_{a+j,k} = -c\, g^{-1}(\bar v^B,w_k)\, g^{-1}(w_k,\bar u^B)\frac{\partial\tau(w_k|\bar u^B,\bar v^B)}{\partial v^B_j}, \tag{45}
$$

$$
j=1,\dots,b,\quad j\neq p.
$$

In these formulas one should set $w_k=u^B_k$ for $k=1,\dots,a$ and $w_{k+a}=v^C_k$ for $k=1,\dots,b$.

The p-th row has the following elements

$$
\mathcal{N}^{(s)}_{p,k} = h(\bar v^C,w_k)h(w_k,\bar u^B)Y^{(s)}_k, \tag{46}
$$

where again $w_k = u_k^B$ for $k = 1, \dots, a$ and $w_{k+a} = v_k^C$ for $k = 1, \dots, b$, and

$$Y_k^{(s)} = c\,(\delta_{s1} - \delta_{s2}) + (\delta_{s1} - \delta_{s3}) u_k^B \left(1 - \frac{f(\bar{v}^B, u_k^B)}{f(\bar{v}^C, u_k^B)}\right), \qquad k = 1, \dots, a,$$

$$Y_{a+k}^{(s)} = c\,(\delta_{s3} - \delta_{s2}) + (\delta_{s1} - \delta_{s3})(v_k^C + c) \left(1 - \frac{f(v_k^C, \bar{u}^C)}{f(v_k^C, \bar{u}^B)}\right), \qquad (47)$$
$$k = 1, \dots, b.$$

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Polylogarithms and Multizeta Values in Massless Feynman Amplitudes

Ivan Todorov

Abstract The last two decades have seen a remarkable development of analytic methods in the study of Feynman amplitudes in perturbative quantum field theory. The present lecture offers a physicists' oriented survey of Francis Brown's work on singlevalued multiple polylogarithms, the associated multizeta periods and their application to Schnetz's graphical functions and to x-space renormalization. To keep the discussion concrete we restrict attention to explicit examples of primitively divergent graphs in a massless scalar QFT.

1 Introduction

It is refreshing for mathematically minded theorists that computer calculations in perturbative Quantum Field Theory (QFT) far from making analytic methods obsolete go in effect hand in hand with their developments. It took 9 years before an error in the first numerical calculation of the α^2 contribution to the anomalous magnetic moment $(g-2)$ of the electron was corrected by Petermann (and independently by Sommerfield) while computing the relevant seven Feynman diagrams analytically. The answer involves a $\zeta(3)$. (For a historical review—see [31]; for the expressions of the α^2 and α^3 contributions to $g-2$ in terms of zeta values of weight three and five, respectively, and for references to the original work of the late 1950s on the α^2 term and the mid 1990s on the α^3 graphs—see [28].) It was in the course of a calculation of the electron form-factors that multiple polylogarithms were used by Remiddi et al. and subsequently surveyed under the name of *harmonic polylogarithms* in [27]. (Later computer aided higher order calculations of $g-2$ took over—see the entertaining review of the field up to 2010 by Kinoshita [22].)

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Mathematicians were attracted to the beauty of the dilogarithm and the enigma of *multiple zeta values* (MZVs) since the work of Euler—see [10, 19, 34, 35] for reviews. The *singlevalued multiple polylogarithms* (SVMP), introduced and studied by Brown [4] were soon recognized to play a central role in Euclidean calculations of scattering amplitudes—see the systematic elaboration and application to the study of graphical functions in massless QFT in [30] as well as a choice of influential recent papers [15, 17, 18, 20, 21] and references to earlier work cited there.

The notion of a *Feynman period* [28], identified as *residue* of a primitively divergent graph, was used systematically in [25, 26] in the study of x-space renormalization of massless Feynman amplitudes. Such residues/periods appear in the perturbative expansion of the renormalization group beta function. They were studied by Broadhurst and Kreimer [3] back in 1996 up to nine loops in the φ^4 theory and found to be given in most cases by MZVs—i.e. by rational linear combinations of multiple (convergent) series

$$\zeta(n_1, \ldots, n_r) = \sum_{1 \le k_1 < \ldots < k_r} \frac{1}{k_1^{n_1} \ldots k_r^{n_r}} \qquad (n_i \in \mathbb{N}, \quad n_r > 1) . \tag{1}$$

The multiple polylogarithms were first encountered as multiple power series of a similar type, convergent in the unit circle. They admit an analytic continuation to the punctured projective plane $(z \in) \, \mathbb{CP}^1 \backslash \{0, 1, \infty\}$ given by multiple *iterated integrals* [5, 12] labeled by words in two letters $\{0, 1\}$. The MZVs appear as values of the multipolylogarithms at the boundary point $z = 1$. It is remarkable that this family of functions admits a double algebra structure: a *shuffle* and a *stuffle algebra* (both commutative) which incorporate a wide family of identities among them. Moreover, the SVMPs naturally form a shuffle subalgebra. Both algebraic structures pass to the MZV and allow to speak about the algebra of *singlevalued MZV* [8].

In general, the residue of a primitively (ultra-violet) divergent Feynman amplitude is defined by an integral over a compact projective space (see [25], Theorem 2.3). In many cases (for instance for amplitudes involving a conformally invariant integration) the same residue can be computed using integration over a (non-compact) unbounded domain. An example of this type, the wheel with n strokes was considered in [2] (and later surveyed in Appendix D to [25] and in [30, 32]). All φ^4 periods considered in [9, 28, 30] are of this type. This allows to compute such periods using recursive relations that involve integration over $\mathbb{R}^4$. Furthermore, it offers the possibility to treat graphs with internal vertices and thus to face the large x (infrared) behaviour.

The paper is organized as follows. We start in Sect. 2 with a basic example: integration over an internal vertex in the φ^4 theory yielding the Bloch–Wigner dilogarithm. The details of the calculation (using Gegenbauer polynomial technics [13]) are relegated to Appendix 1. Section 3 introduces the multipolylogarithms as iterated integrals L_w labeled by words w in two letters $\{0, 1\}$ obeying shuffle algebra relations. The (possibly regularized) value of $L_w(z)$ at $z = 1$ is identified with the

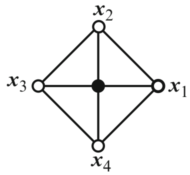

Fig. 1 Four-point graph; the *open circles* correspond to external vertices

(generalized) MZV ζ_w. The series MZV correspond to a passage from the two letter alphabet to one with an infinite number of letters:

$$\zeta(n_1, \ldots, n_r) = (-1)^r \zeta_{10^{n_1-1}\ldots 10^{n_r-1}}, \quad n_i = 1, 2, \ldots, \quad i = 1, \ldots, r \tag{2}$$

($n_r = 1$ corresponding to the generalized/regularized MZVs). It is for the MZVs that we also define (in Sect. 3.2) the stuffle relations (which reduce to easily derivable identities for the series (1)). The number of arguments r in the MZV (1) corresponding to the number of 1's in w is called *length* or *depth* while the number of all letters $\{0, 1\}$ of a word w is called its *weight*. We treat systematically the identities among MZV of weight up to five in Appendix 2. The study of the monodromy of multipolylogarithms (Sect. 3.3 and Appendix 3) is streamlined by the introduction of the generating series $L = L_{e_0 e_1}(z)$ and $Z = Z_{e_0 e_1}$ (29). It is a prerequisite for the study of the monodromy of L_w and hence for introducing SVMP by Brown's Theorem 3.1. Schnetz's notion of a *graphical function* is reviewed in Sect. 4. As an introduction to the generating series (46) for SVMP we work out in Sect. 4.1 the graphical function and the period for the wheel with n spokes which only involves the simpler SVMP of depth one. We return to our main example, the four loop amplitude G_4 (Fig. 1), in Sect. 4.2 (and Appendix 3). Its residue $I(G_4)$ is expressed as a sum of four pairs of SVMPs evaluated at $z = 1$: one of depth one, which reproduces the period of the wheel with four spokes

$$I(W_4) = \binom{6}{3} \zeta(5), \tag{3}$$

two of depth two with a negative contribution ($-20\,\zeta(5)$) to $I(G_4)$, and one of depth three whose contribution ($20\,\zeta(5)$) cancels that of the depth two terms. Thus we confirm the expected result $I(G_4) = I(W_4)$ (53) demonstrating that integration over internal vertices in a primitively divergent φ^4 graph commutes with taking the residue.

2 An Inspiring Example: The Bloch–Wigner Singlevalued Dilogarithm

The main example, on which we shall test the basic concepts and tools, reviewed in this lecture, is the massless four-point φ^4-amplitude in (Euclidean) position space

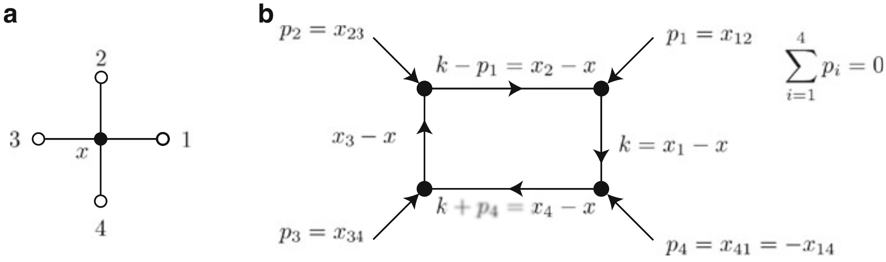

Fig. 2 Dual interpretation of the integral (5)

$$G_4(x_1, \ldots, x_4) = \frac{I(x_1, \ldots, x_4)}{x_{12}^2\, x_{23}^2\, x_{34}^2\, x_{14}^2}\,, \quad x_{ij} = x_i - x_j\,, \quad i, j = 1, \ldots, 4\,,$$

$$x_i = (x_i^\alpha,\ \alpha = 1, \ldots, 4)\,, \quad x_{ij}^2 = \sum_{\alpha=1}^{4} (x_{ij}^\alpha)^2\,, \tag{4}$$

where $I(x_1, \ldots, x_4)$ is the (conformally covariant) Feynman integral

$$I(x_1, \ldots, x_4) = \int \prod_{i=1}^{4} \frac{1}{(x_i - x)^2} \frac{d^4 x}{\pi^2} = \frac{\mathcal{F}(u, v)}{x_{13}^2\, x_{24}^2}\,, \tag{5}$$

u and v being the two independent cross ratios

$$u = \frac{x_{12}^2\, x_{34}^2}{x_{13}^2\, x_{24}^2}\,, \quad v = \frac{x_{14}^2\, x_{23}^2}{x_{13}^2\, x_{24}^2}\,. \tag{6}$$

The amplitude G_4 corresponds to the four-loop Feynman graph displayed on Fig. 1

The integral (5) can be interpreted both as a φ^4 integral in position space and as one corresponding to the box diagram of a φ^3 theory in momentum space (Fig. 2)

The second interpretation provides an elementary example of what came to be called a *dual conformal symmetry* [16]. It was for the momentum space box diagram (as the simplest example of a ladder graph) that the integral (5) was first computed [33] (back in 1993) using Melin transform. A modern computation using Gegenbauer polynomial technics [13] is sketched in Appendix 1. The result is expressed in terms of a dilogarithm function of a complex variable z and its conjugate $\bar{z}$ related to the conformal cross ratios (6) by

$$u = z\bar{z}\,, \qquad v = (1 - z)(1 - \bar{z})\,. \tag{7}$$

The derivation of Appendix 1 uses the fact that the four-dimensional hyperspherical Gegenbauer polynomial C_n^1 is expressed in terms of the Tchebyshev polynomial of the second kind:

$$|z|^n\, C_n^1\left(\frac{z+\bar{z}}{2\,|z|}\right) = \frac{z^{n+1}-\bar{z}^{n+1}}{z-\bar{z}} \quad (|z|^2 = z\,\bar{z})\,. \tag{8}$$

The result is a *singlevalued* real analytic function on $\mathbb{CP}^1\backslash\{0,1,\infty\}$ given by

$$\mathcal{F}(u,v) = x_{13}^2\, x_{24}^2\, I(x_1,\ldots,x_4) = \frac{4i\ D(z)}{z-\bar{z}} \tag{9}$$

where $D(z)$ is the singlevalued dilogarithm of David Wigner and Spencer Bloch—see [1, 35].

$$\begin{aligned} D(z) &= \mathrm{Im}\,(Li_2(z) + \ln|z|\ln(1-z)) \\ &= \frac{1}{4i}\left(2Li_2(z) - 2Li_2(\bar{z}) + \ln z\,\bar{z}\ln\frac{1-z}{1-\bar{z}}\right)\,. \end{aligned} \tag{10}$$

Here $Li_n(z)$ denotes the polylogarithm given for $|z| < 1$ by the power series

$$Li_n(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^n} \quad (Li_1(z) = -\ln(1-z))\,. \tag{11}$$

While $Li_2(z)$ has a *multivalued* analytic continuation to arbitrary complex z given by the integral

$$Li_2(z) = -\int_0^z \ln(1-t)\,\frac{dt}{t}\,, \tag{12}$$

that depends on the homotopy class of the path which joins 0 and z, the function (10) is singlevalued (and continuous) on the entire projective plane.

The symmetries of $D(z)$ can be best described by introducing a (real valued) function of four complex variables that behaves as a (scale invariant) local fermionic four-point amplitude in a two-dimensional conformal field theory:

$$\tilde{\mathcal{D}}(z_1,z_2,z_3,z_4) = D\left(\frac{z_{12}\,z_{34}}{z_{13}\,z_{24}}\right) \quad \text{where} \quad z_{ij} = z_i - z_j\,. \tag{13}$$

It is invariant under even permutations and changes sign under odd permutations of the variables $(z_1,\ldots,z_4)$. This implies

$$\begin{aligned} \mathcal{D}(z) &= \mathcal{D}\left(\frac{z-1}{z}\right) = \mathcal{D}\left(\frac{1}{1-z}\right) = -\mathcal{D}\left(\frac{1}{z}\right) \\ &= -\mathcal{D}(1-z) = -\mathcal{D}\left(\frac{z}{z-1}\right)(= -\mathcal{D}(\bar{z}))\,. \end{aligned} \tag{14}$$

The function $\tilde{\mathcal{D}}$ (13) gives the volume of the ideal (oriented) tetrahedron with vertices $z_1,\ldots,z_4$ on the absolute (also called horosphere) of Lobachevsky space and has already been studied by Lobachevsky himself (cf. [24]; for background on the Beltrami model of Lobachevsky space—see [23]).

The significance of this example stems from the fact that it displays properties common to low loop calculations in more structured quantum field theory models (such as the $N = 4$ super-Yang–Mills theory [17]) as well as in physically relevant calculations in quantum chromodynamics [15]. In particular, the singlevaluedness of Euclidean Feynman amplitudes is dictated by general considerations of the *symbol of iterated integrals* [18, 20, 21].

3 The Shuffle Algebra of Multipolylogarithms and of Multizeta Values

It is both fortunate and demanding for a newcomer in the field that the multipolylogarithms (as well as their values at $z = 1$—the MZV) appear with a rich algebraic structure.

3.1 The Algebra of Words in Two Letters: Recursive Definition of Polylogarithms

We start by introducing a family of iterated integrals.[1] Denote by $\{0,1\}^\times$ the set of words w in the two letters 0 and 1, including the empty word $\emptyset$. The multipolylogarithms of a single variable z are defined inductively by the differential equations

$$\frac{d}{dz} L_{wa}(z) = \frac{L_w(z)}{z-a}\,, \quad a \in \{0,1\}\,, \quad L_\emptyset = 1\,, \tag{15}$$

and the initial condition

$$L_w(0) = 0 \quad \text{for} \quad w \neq 0^n (= 0\ldots 0 - n \text{ times}), \quad L_{0^n}(z) = \frac{(\ln z)^n}{n!}\,. \tag{16}$$

In particular, for $n, n_i \geq 1$ we have

$$L_{1^n}(z) = \frac{[\ln(1-z)]^n}{n!}\,;\ (-1)^r L_{10^{n_1-1}\ldots 10^{n_r-1}}(z) = Li_{n_1\ldots n_r}(z)$$

$$\left(= \sum_{1\leq k_1<\ldots<k_r} \frac{z^{k_r}}{k_1^{n_1}\ldots k_r^{n_r}} \quad \text{for} \quad |z|<1\right). \tag{17}$$

[1]Iterated integrals were introduced in the mid 1950s and developed essentially single-handedly for over 20 years by Chen (1923–1987) [12] before gaining recognition in both mathematics and QFT—see [5].

For any ring R of numbers (which includes the ring $\mathbb{Z}$ of rational integers), we define the R-module $R(\{e_0, e_1\}^{\times})$ of formal linear combination of words in the alphabet $\{e_0, e_1\}$ and introduce the *shuffle algebra* $\mathrm{Sh}_R(e_0, e_1)$ equipping it with the (commutative) *shuffle product* $w \sqcup\!\sqcup\, w'$ defined recursively by

$$\emptyset \sqcup\!\sqcup\, w = w (= w \sqcup\!\sqcup\, \emptyset)\,, \quad au \sqcup\!\sqcup\, bv = a(u \sqcup\!\sqcup\, bv) + b(au \sqcup\!\sqcup\, v) \tag{18}$$

where u, v, w are (arbitrary) words while a, b are letters (note that the empty word is *not* a letter). Extending by (R-)linearity the correspondence $w \to L_w(z)$ one proves that the resulting map $\mathrm{Sh}_R(e_0, e_1) \to R(L_w)$ is a homomorphism of shuffle algebras:

$$L_{u \sqcup\!\sqcup v}(z) = L_u(z)\, L_v(z)\,. \tag{19}$$

In particular, it is easy to verify that the dilogarithm (12) disappears from the shuffle product:

$$L_{0 \sqcup\!\sqcup 1}(z) := L_{01}(z) + L_{10}(z) = L_0(z)\, L_1(z) = \ln z \ln(1-z)\,.$$

From the uniqueness of the solution of (15) under the condition (16) it is straightforward to prove that for a general word

$$w_{\mathbf{n}} = 0^{n_0}\, 10^{n_1 - 1} \ldots 10^{n_r - 1}\,, \quad n_0 = 0, 1, \ldots, n_i = 1, 2, \ldots\,, \tag{20}$$

we have

$$L_{w_{\mathbf{n}}}(z) = \sum_{\substack{k_0 \geq 0\, k_i \geq n_i, 1 \leq i \leq r \\ k_0 + k_1 + \ldots + k_r = n_0 + \ldots + n_r}} (-1)^{k_0 + n_0 + r} \prod_{i=1}^{r} \binom{k_i - 1}{n_i - 1} L_{0^{k_0}}(z)\, L_{k_1 - k_r}(z)\,. \tag{21}$$

3.2 Multiple Zeta Values (MZV)

For $n_r > 1$ in (17) (and in (21)) $L_{n_1 \ldots n_r}(z)$ is convergent at $z = 1$ and we define the MZVs as the values at 1 of the corresponding multipolylogarithms:

$$\zeta(n_1, \ldots, n_r) = Li_{n_1 \ldots n_r}(1)\,,$$
$$\zeta_{w_{\mathbf{n}}} = (-1)^{n_0 + r} \sum_{\substack{k_i \geq n_i \\ \sum_1^r k_i = n_0 + \sum_1^r n_i}} \prod_{i=1}^{r} \binom{k_i - 1}{n_i - 1} \zeta(k_1, \ldots, k_r)\,. \tag{22}$$

We extend this definition to all words by introducing the *regularized* MZV setting

$$\zeta_1 = -\zeta(1) = 0\, (= \zeta_0) \tag{23}$$

and postulating that ζ_w satisfy the shuffle relation

$$\zeta_{u ш v} = \zeta_u \, \zeta_v \,. \tag{24}$$

There is a second *stuffle product*, $\times$, defined on words in the infinite alphabet of positive integers which is suggested by identities for the series expansions of polylogs or MZV. Rather than reproducing the general definition (see [34]) we just give two simple examples: the *Nielsen reflection formula*

$$\zeta(m)\,\zeta(n) = \zeta(m,n) + \zeta(n,m) + \zeta(m+n) =: \zeta(m \times n) \tag{25}$$

and the relation

$$\begin{aligned}\zeta(\ell)\cdot\zeta(m,n) &= \zeta(\ell,m,n) + \zeta(m,\ell,n) + \zeta(m,n,\ell) \\ &\quad + \zeta(m+\ell,n) + \zeta(m,n+\ell) =: \zeta(\ell \times (m,n))\,,\end{aligned} \tag{26}$$

which suggests the general pattern. The stuffle identities that generalize (25), (26) prove that the product of MZV can be expanded as a linear combination of MZV with integer coefficients. They also allow to extend the notion of MZV to the case when the last entry is 1. The “regularized MZV” cancel in the difference of the two products yielding, in general, non-trivial identities as illustrated in the following example: subtracting the stuffle from the shuffle equation below,

$$\zeta(1)\,\zeta(2) = \zeta_{1 \Delta 10} = 2\zeta(1,2) + \zeta(2,1)$$

$$\zeta(1)\,\zeta(2) = \zeta(1 * 2) = \zeta(1,2) + \zeta(2,1) + \zeta(3)\,,$$

we obtain Euler’s identity

$$\zeta(1,2) = \zeta(3) \tag{27}$$

between two convergent series (see for a more systematic treatment of the resulting relations Appendix 2).

The number r of arguments in $\zeta(k_1,\ldots,k_r)$ corresponding to the number of 1’s in the word $w_{\mathbf{n}}$ (20) is called *length* (as in [34]) or *depth* (in [7, 30]) of $w_{\mathbf{n}}$. The number $|w|$ of all letters of the word w in the alphabet $\{0,1\}$ is called the *weight* of w.

For even $n\,(=\ 2,4,\ldots)$ the $\zeta(n)$ is a rational multiple of π^n (as established by the 27-year-old Euler in 1734—see detailed historical references in [14]; for a derivation à la Euler of the explicit formula (28) below in terms of the Bernoulli numbers B_{2k}—see [10]):

$$\zeta(2k) = 2^{2k-1}\frac{|B_{2k}|}{(2k)!}\,\pi^{2k}\,, \quad B_2 = \frac{1}{6}\,,\; B_4 = -\frac{1}{30}\,,\; B_6 = \frac{1}{42}\,,\ldots. \tag{28}$$

Calculating (by hand!) $\zeta(3)$ up to ten significant digits Euler verified that it is not given by π^3 times a rational number with a small denominator [14]. There is a far going (widely believed but completely unproven) conjecture that the numbers $\pi, \zeta(3), \zeta(5), \ldots$ are algebraically independent. All known relations among zeta values of odd weight involve MZVs of the same weight (like in (27)). One may call the relations coming from the shuffle and stuffle identities (see Appendix 2) *motivic*. (More precisely, starting from an abstract definition involving the fundamental group of $\mathbb{CP}^1 \backslash \{0, 1, \infty\}$—see [6] and the review [14]—one proves that these relations are indeed motivic—cf. also [34] and the explicit treatment of the special case of double zeta values in [11].) It is conjectured that all motivic zeta values are of this type. A further going conjecture (that would imply the above mentioned belief about the algebraic independence of odd zeta values and π) says that all relations among MZV are motivic.

3.3 Single Valued Multiple Polylogarithms (SVMP)

The monodromies $\mathcal{M}_0$ and $\mathcal{M}_1$ around the potential singularities 0 and 1 of the polylogarithms (11) and of L_{0^n} (16) are given by the unipotent operators

$$\mathcal{M}_0\, Li_n(z) = Li_n(z)\,, \quad \mathcal{M}_0\, L_{0^n}(z) = L_{0^n}(z) + 2\pi i\, L_{0^{n-1}}(z)\,,$$

$$\mathcal{M}_1\, Li_n(z) = Li_n(z) - 2\pi i\, L_{0^{n-1}}(z)\,.$$

More generally, introducing the generating function $L(z)(= L_{e_0\, e_1}(z) = 1 + \ln z\, e_0 + \ln(1-z)\, e_1 + \ldots)$ and its regularized limit $Z(= Z_{e_0\, e_1})$ at $z = 1$ (called the *Drinfeld associator*),

$$\begin{aligned} L(z) &= \sum_w L_w(z)\, w\,,\ Z = \sum_w \zeta_w\, w \\ &= 1 + \zeta(2)[e_0, e_1] + \zeta(3)[[e_0, e_1], e_0 + e_1] + \ldots \end{aligned} \tag{29}$$

(cf. Appendix 2) we can write (see Appendix 3)

$$\mathcal{M}_0\, L(z) = e^{2\pi i e_0}\, L(z)\,, \quad \mathcal{M}_1\, L(z) = Z\, e^{2\pi i e_1}\, Z^{-1}\, L(z)\,. \tag{30}$$

The first relation follows from the fact that $L(z)$ is the unique solution of the *Knizhnik–Zamolodchikov equation*

$$\frac{d}{dz} L(z) = L(z) \left(\frac{e_0}{z} + \frac{e_1}{z-1} \right) \tag{31}$$

obeying the asymptotic condition

$$L(z) = e^{e_0 \ln z} h_0(z)\,,$$
$$h_0(z) = 1 + e_1 \ln(1-z) + [e_0, e_1]\, Li_2(z) + e_1^2\, \frac{[\ln(1-z)]^2}{2} + \ldots \qquad (32)$$

(i.e. $h_0(0) = 1$, $h_0(z)$ being a formal power series in the words in $\{e_0, e_1\}^\times$ that is holomorphic in z in the neighbourhood of $z = 0$. The second relation (30) is implied by the fact that there exists a counterpart $h_1(z)$ of h_0, holomorphic around $z = 1$ and satisfying $h_1(1) = 1$ such that

$$L(z) = Z\, e^{e_1 \ln(1-z)}\, h_1(z) \qquad (33)$$

(see Appendix 3). The knowledge of the monodromy allows to construct singlevalued linear combinations of products of the type $L_{w'}(\bar z)\, L_w(z)$, the SVMP. A practitioner of two-dimensional (2D) conformal field theory will notice the analogy with constructing monodromy invariant 2D correlation functions out of (multivalued) chiral conformal blocks. It turns out that SVMP have simple characterization in terms of equations of type (15) (16) and form an interesting subalgebra of the shuffle algebra of multiple polylogarithms. The following result is due to Brown.

Theorem 3.1 ([4]). *(See also Theorem 2.5 of [30].) There exists a unique family of single valued functions* $\{P_w(z)\,,\ w \in \{0,1\}^\times\,,\ z \in \mathbb{C}\backslash\{0,1\}\}$ *each of which is a linear combination of* $L_u(\bar z)\, L_v(z)$ *of the same total weight,* $|u| + |v| = |w|$*, which satisfy the differential equations*

$$\partial\, P_{wa}(z) = \frac{P_w(z)}{z-a}\,, \quad \partial \equiv \frac{\partial}{\partial z}\,, \qquad (34)$$

such that

$$P_\emptyset = 1\,, \quad P_{0^n}(z) = \frac{(\ln z\,\bar z)^n}{n!}\,, \quad P_w(0) = 0 \ \textit{for}\ w \neq 0^n\ (w \neq \emptyset)\,. \qquad (35)$$

The functions P_w *satisfy the shuffle relations (19) and are linearly independent over the ring of polynomials* $\mathbb{C}\left[z, \frac{1}{z}, \frac{1}{1-z}; \bar z, \frac{1}{\bar z}, \frac{1}{1-\bar z}\right]$*. Every singlevalued linear combination of functions of the type* $L_{w'}(\bar z)\, L_{w''}(z)$ *can be written as a (unique) linear combination of* $P_w(z)$*.*

The functions P_w can be constructed explicitly in terms of the corresponding generating function (see [30], the text after Theorem 2.5; a special case of interest is reproduced in Sect. 4 below). The functions

$$P^0_w(z) = \sum_{uv=w} L_{\tilde u}(\bar z)\, L_v(z) \qquad (36)$$

(where $\tilde{u} = a_n \ldots a_1$ for $u = a_1 \ldots a_n$, $a_i \in \{0, 1\}$) can serve as a first step in the construction of P_w and actually coincide with P_w for words (of any weight but) of length/depth one as well as for all words of weight at most three. We find, in particular,

$$P_{01}(z) = L_{10}(\bar{z}) + L_{01}(z) + L_0(\bar{z})\, L_1(z) = Li_2(z) - Li_2(\bar{z}) + \ln z\,\bar{z} \ln(1-z)$$

$$P_{10}(z) = L_{01}(\bar{z}) + L_{10}(z) + L_1(\bar{z})\, L_0(z) = Li_2(\bar{z}) - Li_2(z) + \ln z\,\bar{z} \ln(1-\bar{z}) \,,$$

so that

$$P_{01} + P_{10} = \ln|z|^2 \, \ln|1-z|^2 = P_0\, P_1 \,,$$

in accord with the shuffle relation, while

$$P_{01}(z) - P_{10}(z) = 2(Li_2(z) - Li_2(\bar{z})) + \ln z\,\bar{z} \ln \frac{1-z}{1-\bar{z}} = 4i\; D(z) \tag{37}$$

reproduces the Bloch–Wigner function (10)—the only new SVMP of weight two.

The words w for which the SVMP P_w coincide with P_w^0 (33) include the wheel with n-spokes reviewed in Sect. 4.1 below.

As it is precisely the SVMP that appear in the calculation of Feynman amplitudes, it is natural to expect the Feynman periods (or residues) will also belong to the corresponding restricted shuffle subalgebra of "singlevalued MZV", generated by the values of SVMP at $z = 1$ (see [8]). This set turns out to be generated by the odd zeta values $\zeta(2n+1)$, $n = 1, 2, \ldots$. In particular, the Bloch–Wigner function (10) (34) vanishes for real z, hence so does the singlevalued counterpart of $\zeta(2)$:

$$\zeta^{SV}(2) = D(1) = 0\,. \tag{38}$$

4 Graphical Functions and Periods

4.1 *SVMP of Depth One and the Wheel: Generating Series for the General SVMP*

The computation of the integral (5) (or of its simplified version (56)) can be viewed as a first step in a recurrence in which $f_n(z) = F(z, W_n)$ are defined by

$$\partial\,\bar{\partial}\, f_2(z) = \frac{1}{z(1-\bar{z})} - \frac{1}{\bar{z}(1-z)} \Rightarrow f_2(z) = P_{01}(z) - P_{10}(z)$$

$$\partial\,\bar{\partial}\, f_{n+1}(z) = \frac{-1}{z\,\bar{z}}\, f_n(z) \quad \text{for} \quad n = 2, 3, \ldots,\; \partial = \frac{\partial}{\partial z}\,,\; \bar{\partial} = \frac{\partial}{\partial \bar{z}}\,, \tag{39}$$

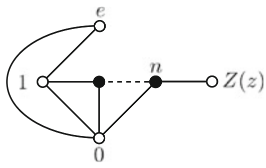

Fig. 3 Sequential graph for the wheel W_{n+1}

whose (unique) SVMP solution is

$$f_{n+1}(z) = (-1)^n \left(P_{0^{n-1}10^n}(z) - P_{0^n 10^{n-1}}(z)\right) . \tag{40}$$

Here $F(z, W_n)$ gives the Feynman amplitude corresponding to the sequential graph presented in Fig. 3

To prove this identification one uses the four-dimensional Laplace equation

$$-\frac{1}{4}\Delta_Z F(z, W_{n+1}) = \frac{1}{z\bar{z}} F(z, W_n) \tag{41}$$

and the expression for Δ_Z restricted on a function of z and $\bar{z}$:

$$\frac{1}{4}\Delta_Z F(z) = \frac{1}{z-\bar{z}} \partial\bar{\partial}\left[(z-\bar{z}) F(z)\right] \quad (Z^2 = z\bar{z},\ (Z-e)^2 = |z-1|^2). \tag{42}$$

The period $I(W_{n+1})$ of the wheel with $n+1$ spokes is now obtained as the limit of $F(z, W_{n+1})$ for $z \to 1$ $(Z \to e)$. (To see this, one should redraw Fig. 3 with the vertices $(e, 1, \ldots, n)$ on a circle and the vertex 0 in its centre.)

For a general word of weight $n_0 + n_1$ and depth one we can use (33) to write

$$\begin{aligned} P_{0^{n_0}10^{n_1-1}}(z) &= \sum_{k=0}^{n_0} (-1)^{k+1} \binom{n_1-1+k}{n_1-1} P_{0^{n_0-k}}(z)\, Li_{n_1+k}(z) \\ &+ \sum_{k=0}^{n_1-1} (-1)^{k+1} \binom{n_0+k}{n_0} P_{0^{n_1-1-k}}(z)\, Li_{n_0+k+1}(\bar{z}) \end{aligned} \tag{43}$$

(where P_{0^n} is given in (32)). Inserting this expression in (40) we obtain

$$F(z, W_{n+1}) = \frac{f_{n+1}(z)}{z-\bar{z}} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{n} P_{0^{n-k}}(z) \frac{Li_{n+k}(z) - Li_{n+k}(\bar{z})}{z-\bar{z}} . \tag{44}$$

In the limit $z \to 1$ only the term with $k = n$ contributes and we find

$$I(W_{n+1}) = F(1, W_{n+1}) = \binom{2n}{n} \zeta(2n-1) . \tag{45}$$

This result was first derived using a similar recursion by Broadhurst [2]. The above derivation follows Schnetz [30].

In general, the generating function of SVMP is given by (see [4, 30] and Appendix 3 below):

$$P_{e_0 e_1}(z) = \tilde{L}_{e_0 e_1'}(\bar{z})\, L_{e_0 e_1}(z) \tag{46}$$

where $\tilde{L} = \Sigma\, L_w\, \tilde{w}$ (cf. (36)) and e_1' is the unique solution of the equation

$$Z_{-e_0,-e_1'}\, e_1'\, Z^{-1}_{-e_0-e_1'} = Z_{e_0 e_1}\, e_1\, Z^{-1}_{e_0 e_1} \tag{47}$$

(see Appendix 3).

4.2 Single Valued MZV and the Period of G_4

The amplitude G_4 (4) and its period $I(G_4)$ corresponding to the graph on Fig. 1 is of interest as the first strongly connected (or "internally six connected" in the terminology of [30]) φ^4-graph that involves integration over an internal vertex. Albeit such an integral is known to be infrared convergent it may interfere with the causal factorization condition for the ultraviolet renormalization (the amplitude G_4 being primitively logarithmically divergent). A related question: the period of the amplitude belongs to the wheel series. If we can treat the vertex 0 (with four adjacent lines) as an external one then we should expect to have $I(G_4) = I(W_4) = \binom{6}{3}\zeta(5)$. If we treat it as an internal vertex—see Fig. 4 then we end up with a different graphical function. Indeed, the sequence of differential equations corresponding to the graph in Fig. 4 is

$$\begin{aligned}
g_2(z) &= f_2(z) = P_{01}(z) - P_{10}(z) \\
\partial\,\bar{\partial}\, g_3(z) &= \frac{-g_2(z)}{z\,\bar{z}(1-z)(1-\bar{z})} = \left(\frac{1}{z-1} - \frac{1}{z}\right)\left(\frac{1}{\bar{z}} - \frac{1}{\bar{z}-1}\right) g_2(z) \\
\partial\,\bar{\partial}\, g_4(z) &= \frac{-g_3(z)}{z\,\bar{z}}\,.
\end{aligned} \tag{48}$$

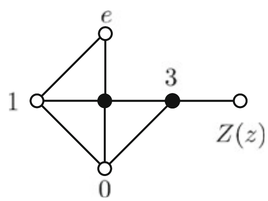

Fig. 4 Graph for the graphical function $g_4(z)$

The functions

$$\begin{aligned} g_3^0(z) &= P_{0100}(z) - P_{0010}(z) + P^0_{1010}(z) - P^0_{0101}(z) + P^0_{0011}(z) \\ &\quad - P^0_{1100}(z) + P^0_{1101}(z) - P^0_{1011}(z)\,, \\ g_4^0(z) &= P_{0^3 10^2}(z) - P_{0^2 10^3}(z) + P^0_{0^2 1010}(z) - P^0_{01010^2}(z) + P^0_{01^2 0^3}(z) \\ &\quad - P^0_{0^3 1^2 0}(z) + P^0_{01 01^2 0}(z) - P^0_{01^2 010}(z) \end{aligned} \tag{49}$$

where $P_w^0(z)$ are given by (36) provide a multivalued solution of the partial differential equations (48). The SVMP $g_3(z)$ is obtained from g_3^0 (49) by replacing $P_w^0(z)$ by

$$P_w(z) = P_w^0(z) + 2\,\zeta(3)\,\langle w, [[[e_0, e_1]\,e_1]\,e_0 + e_1]\rangle\, L_1(\bar{z}) \tag{50}$$

(see Appendix 3). The inner product in $\mathbb{Z}(\{e_0, e_1\}^{\times})$ is defined by setting $\langle u \mid v\rangle = 0$ if $u \neq v$ $\langle u \mid u\rangle = 1$ for any two words u and v. The period $I(G_4)$ is equal to the derivative $g_4'(z = 1)$ given by the limit

$$\begin{aligned} I(G_4) = \lim_{z \to 1} \frac{g_4(z)}{z - z'} &= P_{0^3 10}(1) - P_{0^2 10^2}(1) + P_{0^2 101}(1) - P_{01010}(1) \\ &\quad + P_{01^2 0^2}(1) - P_{0^3 1^2}(1) + P_{0101^2}(1) - P_{01^2 01}(1)\,. \end{aligned} \tag{51}$$

According to (40) and (45) the period of the wheel with four strokes is given by just the first two terms of (51):

$$I(W_4) = P_{0^3 10}(1) - P_{0^2 10^2}(1) = \binom{6}{3}\,\zeta(5) = 20\,\zeta(5)\,. \tag{52}$$

As verified in Appendix 3 the remaining six terms cancel against each other so that

$$I(G_4) = I(W_4) = 20\,\zeta(5) \tag{53}$$

which is a special case of the result of [9] concerning all zig-zag graphs. This calculation confirms the general argument of Sect. 2 of [28] demonstrating that periods in φ^4 theory are in fact associated to (completed by a "vertex at infinity") four-point graphs and do not depend on the choice of marked vertices $(\infty, 0, 1, z)$. Thus different (logarithmically divergent) Feynman amplitudes, in our example a four-point and a five-point one, may be renormalized by subtracting a pole term with the same residue (multiplied by a 12- and a 16-dimensional δ-function, respectively).

4.3 Concluding Remarks

In the early days of the development of the "dual resonance model" theoretists were joking about "physics of the red book"—meaning the volumes of the Bateman–Erdelyi classic "Higher Transcendental Functions". There is a marked difference between that old fad and the present day development of analytic methods in perturbative QFT calculations a basic ingredient of which is reviewed in this lecture. Quantum field theory is the language of the standard model of particle physics (which also gives room but is not reduced to speculative dreams that may serve a future theory). The family of multiple polylogarithms, omnipresent in perturbative calculations, far from being just another set of special functions, admits an interesting algebraic structure that passes to the physically relevant subfamily of SVMPs. Residues or periods typically expressed in terms of MZVs are central to our current understanding of ultraviolet renormalization. These developments have transformed QFT from a "reason for divorce between mathematics and physics" [14] half a century ago into a common playing ground for mathematicians and physicists, giving a new vigour to our field.

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Appendix 1: Computation of the Integral (5)

Using conformal invariance we can send the variable x_1 to infinity, x_4 to zero, x_2—to a unit 4-vector e and set

$$x_3 = Z\,, \quad \text{where} \quad Z^2 = z\bar{z}\,,\ 2Ze = z + \bar{z} \tag{54}$$

so that the cross ratios (6) assume the form

$$u = Z^2 = z\bar{z}\,, \quad v = (Z-e)^2 = (z-1)(\bar{z}-1) \tag{55}$$

in accord with (7). Then we can write, introducing spherical coordinates $x = r\omega$, $Z = |z|\,\omega_z$,

$$\begin{aligned}\mathcal{F}(u,v) = F(z) &= \frac{1}{\pi^2}\int \frac{d^4x}{x^2(x-e)^2(x-Z)^2} \\ &= \frac{1}{\pi^2}\int^{\infty} r\,dr \int_{\mathbb{S}^3} \frac{d^3\omega}{(r^2 - 2r\,e\cdot\omega + 1)(r^2 + |z|^2 - 2r\,|z|\,\omega\,\omega_z)}\,.\end{aligned} \tag{56}$$

Assuming $|z| < 1$ we can split the radial integral F into three terms $F = F_1 + F_2 + F_3$ corresponding to the domains $r < |z|$, $|z| < r < 1$ and $r > 1$, respectively. In the first one we can write

$$(r^2 - 2r\, e \cdot \omega + 1)^{-1} = \sum_{n=0}^{\infty} r^n\, C_n^1(\omega e)\,, \tag{57}$$

$$(r^2 + |z|^2 - 2r|z|\,\omega\,\omega_z)^{-1} = \frac{1}{|z|^2} \sum_{m=0}^{\infty} \left(\frac{r}{|z|}\right)^m C_m^1(\omega\,\omega_z)\ (\text{for } r < |z| < 1)$$

where the hyperspherical (Gegenbauer) polynomials C_n^1 can be written as

$$C_n^1(\cos\theta) = \frac{\sin(n+1)\,\theta}{\sin\theta}\,. \tag{58}$$

Using further the orthogonality relation

$$\int_{\mathbb{S}^3} C_n^1(\omega \cdot e)\, C_m^1(\omega\, w_z)\, \frac{d^3\omega}{\pi^2} = \frac{2\,\delta_{mn}}{n+1}\, C_n^1(\omega_z\, e) \tag{59}$$

where, according to (55)

$$\omega_z \cdot e = \frac{z + \bar{z}}{2\,|z|}\,. \tag{60}$$

Inserting in F_1 and using (58) (or (8)) and (11) we find

$$F_1(z) = \int_0^{|z|} \frac{r\,dr}{|z|^2} \sum_{n=0}^{\infty} \frac{2}{n+1} \frac{r^{2n}}{|z|^n}\, C_n^1\left(\frac{z+\bar{z}}{2|z|}\right) = \frac{Li_2(z) - Li_2(\bar{z})}{z - \bar{z}}\,. \tag{61}$$

The same result is obtained for $F_3(z)$:

$$F_3(z) = \int_1^{\infty} \frac{dr}{r^3} \sum_{n=0}^{\infty} \frac{2}{n+1} \frac{|z|^n}{r^{2n}}\, C_n^1\left(\frac{z+\bar{z}}{2|z|}\right) = \frac{Li_2(z) - Li_2(\bar{z})}{z - \bar{z}} = F_1(z)\,. \tag{62}$$

Finally,

$$F_2(z) = 2\int_{|z|}^{1} \frac{dr}{r}\, \frac{Li_1(z) - Li_1(\bar{z})}{z - \bar{z}} = \ln z\bar{z}\, \frac{\ln(1-z) - \ln(1-\bar{z})}{z - \bar{z}}\,; \tag{63}$$

this, together with (61), (62) completes the proof of (9) (10) for $|z| < 1$. The same expression can be obtained in a similar fashion for $|z| > 1$; alternatively, it can be deduced from the result for $|z| < 1$ using the symmetry of $F(z)$ implied by (14). The result can also be established by verifying that it is single valued and satisfies the first equation (39) (in view of the uniqueness of SVMP, Theorem 3.1; cf. [30]).

Appendix 2: Identities Among MZV

Equation (22) which relates the MZV ζ_w (labeled by words in the two letters $\{0, 1\}$) with $\zeta(n_1, \ldots, n_r)$, $n_i = 1, 2, \ldots$ becomes particularly simple for words of depth one,

$$\zeta_{0^{n_0}10^{n_1-1}} = (-1)^{n_0+1} \binom{n_0 + n_1 - 1}{n_1 - 1} \zeta(n_0 + n_1) \,. \tag{64}$$

This allows to write the depth one contribution to the generating function Z (29) in terms of multiple commutators:

$$\sum_{n=1}^{\infty} \zeta(n+1) \underbrace{[\ldots[}_{n} e_0, e_1], e_0], \ldots, e_0] = \sum_{n=1}^{\infty} \zeta(n+1) \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} e_0^k \, e_1 \, e_0^{n-k}, \tag{65}$$

which is another way to write down (64). It is more interesting—and more difficult— to deduce the relations among ζ_w for words of higher depth. We shall write down all such relations for depth two and weight $|w| \leq 5$. Note that the number d_n of linearly independent MZV of a given weight n can be read off the generating function conjectured by Don Zagier

$$\frac{1}{1 - t^2 - t^3} = \sum_{n=0}^{\infty} d_n \, t^n \quad d_2 = d_3 = d_4 = 1 \quad d_5 = d_6 = 2, \ldots \tag{66}$$

(and proven for the motivic analog of MZV by Brown [6]; in general, d_n provide an upper bound of the independent MZV).

The Euler's relation (27) is a special case of either of the following more general relations which only involve proper (convergent) zeta series:

$$\zeta(\underbrace{1, \ldots, 1}_{n-2}, 2) = \zeta(n) \,, \tag{67a}$$

$$\sum_{\substack{s_i \geq 1; s_k \geq 2 \\ \Sigma s_i = n}} \zeta(s_1, \ldots, s_k) = \zeta(n) \,. \tag{67b}$$

The "improper" (regularized) zeta value $\zeta(n, 1)$ is determined from the stuffle relation:

$$0 = \zeta(1) \, \zeta(n) = \zeta(1, n) + \zeta(n, 1) + \zeta(n+1) \,. \tag{68}$$

In particular, for $n = 2$, we find

$$\zeta(2, 1) = -\zeta(3) - \zeta(1, 2) = -2 \, \zeta(3) \,. \tag{69}$$

From Euler's formula

$$\zeta(2) = \frac{\pi^2}{6} \tag{70}$$

(a special case of (28)) and from the shuffle and stuffle relations one deduces that all zeta values of weight four are rational multiples of π^4 (in accord with the Zagier conjecture (66)). In particular, the relations for $10 ⧢ 10$, 10×10 and (67) for $n = 4$,

$$\zeta(2)^2 = \zeta_{10⧢10} = 2\,\zeta_{1010} + 4\,\zeta_{1100} = 2\,\zeta(2,2) + 4\,\zeta(1,3)\,,$$

$$\zeta(2)^2 = \zeta_{10\times 10} = 2\,\zeta(2,2) + \zeta(4)\,; \quad \zeta(1,3) + \zeta(2,2) = \zeta(4)\,,$$

allow to express all weight four words of length not exceeding two as integer multiples of $\zeta(1,3)$:

$$\zeta(4) = 4\,\zeta(1,3)(= \zeta(1,1,2))\,,\ \zeta(2,2) = 3\,\zeta(1,3)\,,\ \zeta(2)^2 = 10\,\zeta(1,3)$$

$$\Rightarrow \zeta(1,3) = \frac{\pi^4}{360}\,. \tag{71}$$

Proceeding in a similar fashion with the two products of the words 10 and 100 we find

$$\zeta(2)\,\zeta(3) = 3\,\zeta_{10100} + 6\,\zeta_{11000} + \zeta_{10010} = 3\,\zeta(2,3) + 6\,\zeta(1,4) + \zeta(3,2)\,,$$

$$\zeta(2)\,\zeta(3) = \zeta(2,3) + \zeta(3,2) + \zeta(5)\,; \quad \zeta(1,4) + \zeta(2,3) + \zeta(3,2) = \zeta(5)\,.$$

These three equations determine a two-dimensional space of zeta values of weight five (in accord with (66)). Selecting as a basis $\zeta(1,4)$ and $\zeta(2,3)$ we express the remaining convergent ζ-values of weight 5 in terms of this basis with positive integer coefficients

$$\zeta(1,1,3) = \zeta(1,4)\,,\ \zeta(1,2,2) = \zeta(2,3)\,,$$

$$\zeta(5) = 2\,\zeta(2,3) + 6\,\zeta(1,4)\,,\ \zeta(3,2) = \zeta(2,1,2) = \zeta(2,3) + 5\,\zeta(1,4)\,,$$

$$\zeta(2)\,\zeta(3) = 4\,\zeta(2,3) + 11\,\zeta(1,4) \tag{72}$$

(while $\zeta(4,1) = -\zeta(1,4) - \zeta(5) = -7\,\zeta(1,4) - 2\,\zeta(2,3)$).

For the study of single valued MZV it is more natural to use the basis $(\zeta(5), \zeta(2)\,\zeta(3))$ instead. Then we find

$$(\zeta(1,1,3) =)\,\zeta(1,4) = 2\,\zeta(5) - \zeta(2)\,\zeta(3)\,,$$

$$(\zeta(1,2,2) =)\, \zeta(2,3) = 3\,\zeta(2)\,\zeta(3) - \frac{11}{2}\,\zeta(5)$$

$$(\zeta(2,1,2) =)\, \zeta(3,2) = \frac{9}{2}\,\zeta(5) - 2\,\zeta(2)\,\zeta(3)\,;\ \zeta(4,1) = \zeta(2)\,\zeta(3) - 3\,\zeta(5)\,. \tag{73}$$

Brown [6] has demonstrated that a basis for "motivic" MZV for all weights is given by $\zeta(s_1, \ldots, s_k)$, with $s_i \in \{2, 3\}$.

From the iterated integral representation of MZV it follows that the generating function (29) satisfies:

$$Z^{-1}_{e_0 e_1} = Z_{e_1 e_0} = \tilde{Z}_{-e_0,-e_1}\,. \tag{74}$$

(The first equation incorporates, in particular, (67a).)

Appendix 3: Monodromy at $z = 1$: Single Valued MZV

The representation (33) can be obtained from (32) by noticing that the substitution $z \to 1 - z$ corresponds to the exchange $e_0 \leftrightarrow e_1$ and that the path from 0 to z can be viewed as a composition of two paths: from 0 to 1 and from 1 to z. For $0 < z < 1$ one should just set $h_1(z) = h_0(1 - z)$. Equation (30) follows from (32) (33) and the relations

$$\mathcal{M}_0 \ln z = \ln z + 2\pi i\,, \quad \mathcal{M}_1 \ln(1 - z) = \ln(1 - z) + 2\pi i\,. \tag{75}$$

Applying (30) one should take into account the relation (74)

$$Z^{-1}_{e_0,e_1} = Z_{e_1,e_0} = \tilde{Z}_{-e_0,-e_1} \tag{76}$$

where the tilde indicates that each word is replaced by its opposite. We leave it to the reader to verify that the first few terms in the expansion of (30) reproduce (75) and give

$$\mathcal{M}_1\, L_{01}(z) = L_{01}(z)(= \ln z \ln(1 - z) + Li_2(z))$$

$$\mathcal{M}_1\, L_{10}(z)(= \mathcal{M}_1(-Li_2(z))) = L_{10}(z) + 2\pi i \ln z\,. \tag{77}$$

We now proceed to the evaluation of the element e_1' defined by Eq. (47). To this end we introduce the Lie algebra valued function

$$F(e_0, e_1) = Z_{e_0 e_1} e_1 Z^{-1}_{e_0 e_1} - e_1 = \zeta(2)[[e_0, e_1], e_1] + \zeta(3)[[[e_0, e_1], e_1], e_0 + e_1] + \ldots \tag{78}$$

Equation (47) can then be solved recursively, writing $e_1' = \lim_{k\to\infty} e_1^{(k)}$ with

$$e_1^{(0)} = e_1\,,\; e_1^{(k+1)} = e_1 + F(e_0, e_1) + F_0(-e_0, -e_1^{(k)})\,. \tag{79}$$

The weight three term with $\zeta(2)$ cancels out and one finds

$$e_1' = e_1 + 2\,\zeta(3)\,[[[e_0, e_1], e_1], e_0 + e_1] + \zeta(5)(\ldots) + \ldots \tag{80}$$

where, according to Schnetz [30], the $\zeta(5)$ contribution consists of eight bracket words of weight six. (The $\zeta(3)$ contribution will be sufficient to the application that follows.)

The SVMPs in the right hand side of (51) are obtained from those in $g_3(z)$ by adding a letter 0 in front and at the end of each labeling word. Evaluating the regularized limit at $z = 1$ (and noting that for $L_{11}(z)$ it is zero) while $\bar{L}_{01}(1) = -\bar{L}_{10}(1) = \zeta(2)$ we find that for each (5-letter) word-label w in (51) we obtain the following counterpart of (50)

$$P_w(1) = P_w^0(1) + 2\,\zeta(2)\,\zeta(3)\,\langle w, w_{23}\rangle$$

$$w_{23} := [e_0, [[[e_0, e_1], e_1], e_0 + e_1]]\ . \tag{81}$$

We shall see that the role of the second term in the right hand side of (81) is to cancel the product $\zeta(2)\,\zeta(3)$ in $P_w^0(1)$, in accord with the observation that $\zeta^{\mathrm{SV}}(2) = 0$.

Indeed the depth one contributions are proportional to $\zeta(5)$:

$$P_{0^310}(1)(= P^0_{0^310}(1)) = L_{0^310}(1) + L_{010^3}(1) = 8\,\zeta(5)\,,$$

$$P_{0^210^2}(1) = 2\,L_{0^210^2}(1) = -12\,\zeta(5)$$

and their difference reproduces (52). For depth two we find (after cancelling the products $\zeta(2)\,\zeta(3)$) a negative multiple of $\zeta(5)$:

$$\begin{aligned} P^0_{0^2101}(1) &= \zeta_{0^2101} + \zeta_{1010^2} + \zeta_{100}\,\zeta_{01} \\ &= 3\,\zeta(4,1) + 2\,\zeta(3,2) + 2\,\zeta(2,3) - \zeta(2)\,\zeta(3) = 4\,\zeta(2)\,\zeta(3) - 11\,\zeta(5)\,, \end{aligned}$$

where in the last step we used (73), $\langle 0^2101, w_{23}\rangle = -2$ so that $P_{0^2101}(1) = P^0_{0^2101} + 2\,\zeta(2)\,\zeta(3)\langle 0^2101, w_{23}\rangle = -11\,\zeta(5)$; similarly $P_{01010}(1) = 4\,\zeta(5) = P_{0^31^2}(1)$, $P_{01^20^2}(1) = -\zeta(5)$, so that

$$P_{0^2101}(1) - P_{01010}(1) + P_{01^20^2}(1) - P_{0^31^2}(1) = -20\,\zeta(5)\,. \tag{82}$$

Finally, the depth three contribution is equal to that of depth one. Indeed we find, using [7],

$$P_{01012}(1) = \zeta_{01012} + \zeta_{12010} + \zeta_{10}\,\zeta_{012} + 6\,\zeta(2)\,\zeta(3) = 11\,\zeta(5)\,,\; P_{01201}(1) = -9\,\zeta(5)$$

$$\Rightarrow\; P_{01012}(1) - P_{01201}(1) = 20\,\zeta(5)\,. \tag{83}$$

This completes the proof of (53). (The expressions (82) and (83) can be also extracted from the polylog- and polyzeta-procedures of [29].)

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Reduction of Couplings in Quantum Field Theories with Applications in Finite Theories and the MSSM

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Abstract We apply the method of reduction of couplings in a Finite Unified Theory and in the MSSM. The method consists on searching for renormalization group invariant relations among couplings of a renormalizable theory holding to all orders in perturbation theory. It has a remarkable predictive power, since it leads to relations between gauge and Yukawa couplings in the dimensionless sectors and relations involving the trilinear terms and the Yukawa couplings, as well as a sum rule among scalar masses in the soft breaking sector, at the GUT scale. In both the MSSM and the FUT model we predict the masses of the top and bottom quarks and the light Higgs in remarkable agreement with the experiment. Furthermore we also predict the masses of the other Higgses, as well as the supersymmetric spectrum, the latter being in very comfortable agreement with the LHC bounds on supersymmetric particles.

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1 Introduction

The discovery of the Higgs boson [1–4] at LHC completes the search for the particles of the Standard Model (SM), and confirms the existence of a Higgs field and the spontaneous electroweak symmetry breaking mechanism as the way to explain the masses of the fundamental particles. The over twenty free parameters of the SM, the hierarchy problem, the existence of Dark Matter, the very small masses of the neutrinos, among others, point towards a more fundamental theory, whose goal among others should be to explain at least some of these facts.

The main achievement expected from a unified description of interactions is to understand the large number of free parameters of the Standard Model (SM) in terms of a few fundamental ones. In other words, to achieve *reduction of couplings* at a more fundamental level. To reduce the number of free parameters of a theory, and thus render it more predictive, one is usually led to introduce more symmetry. Supersymmetric Grand Unified Theories (GUTs) are very good examples of such a procedure [5–11].

For instance, in the case of minimal $SU(5)$, because of (approximate) gauge coupling unification, it was possible to reduce the gauge couplings by one and give a prediction for one of them. LEP data [12] seem to suggest that a further symmetry, namely $N = 1$ global supersymmetry [10, 11] should also be required to make the prediction viable. GUTs can also relate the Yukawa couplings among themselves, again $SU(5)$ provided an example of this by predicting the ratio M_τ/M_b [13] in the SM. Unfortunately, requiring more gauge symmetry does not seem to help, since additional complications are introduced due to new degrees of freedom and in the ways and channels of breaking the symmetry.

A natural extension of the GUT idea is to find a way to relate the gauge and Yukawa sectors of a theory, that is to achieve Gauge–Yukawa Unification (GYU) [14–16]. A symmetry which naturally relates the two sectors is supersymmetry, in particular $N = 2$ supersymmetry [17]. It turns out, however, that $N = 2$ supersymmetric theories have serious phenomenological problems due to light mirror fermions. Also in superstring theories and in composite models there exist relations among the gauge and Yukawa couplings, but both kind of theories have phenomenological problems, which we are not going to address here.

A complementary strategy in searching for a more fundamental theory, consists on looking for all-loop renormalization group invariant (RGI) relations holding below the Planck scale, which in turn are preserved down to the GUT scale [14, 15, 15, 16, 18–25]. Through the method of reduction of couplings it is possible to achieve Gauge–Yukawa Unification [14–16]. Even more remarkable is the fact that it is possible to find RGI relations among couplings that guarantee finiteness to all-orders in perturbation theory [26–28].

Although supersymmetry seems to be an essential feature for a successful realization of the above programme, its breaking has to be understood too, since it has the ambition to supply the SM with predictions for several of its free parameters.

Indeed, the search for RGI relations has been extended to the soft supersymmetry breaking sector (SSB) of these theories [23, 29], which involves parameters of dimension one and two.

2 The Method of Reduction of Couplings

In this section we will briefly outline the reduction of couplings method. Any RGI relation among couplings (i.e. which does not depend on the renormalization scale μ explicitly) can be expressed, in the implicit form $\Phi(g_1, \cdots, g_A) = \text{const.}$, which has to satisfy the partial differential equation (PDE)

$$\frac{d\Phi}{dt} = \sum_{a=1}^{A} \frac{\partial \Phi}{\partial g_a} \frac{dg_a}{dt} = \sum_{a=1}^{A} \frac{\partial \Phi}{\partial g_a} \beta_a = \nabla \Phi \cdot \beta = 0, \tag{1}$$

where $t = \ln \mu$ and β_a is the β-function of g_a. This PDE is equivalent to a set of ordinary differential equations, the so-called reduction equations (REs) [24, 25, 30],

$$\beta_g \frac{dg_a}{dg} = \beta_a \ , \ a = 1, \cdots, A \ , \tag{2}$$

where g and β_g are the primary coupling and its β-function, and the counting on a does not include g. Since maximally $(A - 1)$ independent RGI "constraints" in the A-dimensional space of couplings can be imposed by the Φ_a's, one could in principle express all the couplings in terms of a single coupling g. The strongest requirement is to demand power series solutions to the REs,

$$g_a = \sum_{n=0} \rho_a^{(n)} g^{2n+1} \ , \tag{3}$$

which formally preserve perturbative renormalizability. Remarkably, the uniqueness of such power series solutions can be decided already at the one-loop level [24, 25, 30].

Searching for a power series solution of the form (3) to the REs (2) is justified since various couplings in supersymmetric theories have the same asymptotic behaviour, thus one can rely that keeping only the first terms in the expansion is a good approximation in realistic applications.

3 Reduction of Couplings in Soft Breaking Terms

The method of reducing the dimensionless couplings was extended [23, 29], to the soft supersymmetry breaking (SSB) dimensionful parameters of $N = 1$ supersymmetric theories. In addition it was found [31, 32] that RGI SSB scalar masses in Gauge–Yukawa unified models satisfy a universal sum rule.

Consider the superpotential given by

$$W = \frac{1}{2}\,\mu^{ij}\,\Phi_i\,\Phi_j + \frac{1}{6}\,C^{ijk}\,\Phi_i\,\Phi_j\,\Phi_k\ , \tag{4}$$

where μ^{ij} (the mass terms) and C^{ijk} (the Yukawa couplings) are gauge invariant tensors and the matter field Φ_i transforms according to the irreducible representation R_i of the gauge group G. The Lagrangian for SSB terms is

$$-\mathcal{L}_{\rm SSB} = \frac{1}{6}\,h^{ijk}\,\phi_i\phi_j\phi_k + \frac{1}{2}\,b^{ij}\,\phi_i\phi_j + \frac{1}{2}\,(m^2)^j_i\,\phi^{*\,i}\phi_j + \frac{1}{2}\,M\,\lambda\lambda + \text{H.c.}, \tag{5}$$

where the ϕ_i are the scalar parts of the chiral superfields Φ_i, λ are the gauginos and M their unified mass, h^{ijk} and b^{ij} are the trilinear and bilinear dimensionful couplings respectively, and $(m^2)^j_i$ the soft scalars masses.

Let us recall that the one-loop β-function of the gauge coupling g is given by [33–37]

$$\beta^{(1)}_g = \frac{dg}{dt} = \frac{g^3}{16\pi^2}\left[\sum_i T(R_i) - 3\,C_2(G)\right]\ , \tag{6}$$

where $C_2(G)$ is the quadratic Casimir of the adjoint representation of the associated gauge group G. $T(R)$ is given by the relation $\text{Tr}[T^aT^b] = T(R)\delta^{ab}$ where T^a is the generators of the group in the appropriate representation. Similarly the β-functions of C_{ijk}, by virtue of the non-renormalization theorem, are related to the anomalous dimension matrix γ^i_j of the chiral superfields as:

$$\beta_C^{ijk} = \frac{dC_{ijk}}{dt}\ =\ C_{ijl}\,\gamma^l_k + C_{ikl}\,\gamma^l_j + C_{jkl}\,\gamma^l_i\ . \tag{7}$$

At one-loop level the anomalous dimension, $\gamma^{(1)}{}^i_j$ of the chiral superfield is [33–37]

$$\gamma^{(1)}{}^i_j = \frac{1}{32\pi^2}\,[\,C^{ikl}\,C_{jkl} - 2\,g^2\,C_2(R_i)\delta_{ij}\,], \tag{8}$$

where $C_2(R_i)$ is the quadratic Casimir of the representation R_i, and $C^{ijk} = C^*_{ijk}$. Then, the $N = 1$ non-renormalization theorem [38–40] ensures there are no extra mass and cubic-interaction-term renormalizations, implying that the β-functions of C_{ijk} can be expressed as linear combinations of the anomalous dimensions γ^i_j.

Here we assume that the reduction equations admit power series solutions of the form

$$C^{ijk} = g\sum_{n=0}\rho^{ijk}_{(n)}\,g^{2n}\ . \tag{9}$$

In order to obtain higher-loop results instead of knowledge of explicit β-functions, which anyway are known only up to two-loops, relations among β-functions are required.

The progress made using the spurion technique [40–44], leads to all-loop relations among SSB β-functions [45–50]. The assumption, following [46], that the relation among couplings

$$h^{ijk} = -M(C^{ijk})' \equiv -M \frac{dC^{ijk}(g)}{d \ln g} \, , \tag{10}$$

is RGI and furthermore, the use the all-loop gauge β-function of Novikov et al. [51, 52]

$$\beta_g^{\rm NSVZ} = \frac{g^3}{16\pi^2} \left[\frac{\sum_l T(R_l)(1-\gamma_l/2) - 3C_2(G)}{1 - g^2 C_2(G)/8\pi^2} \right] , \tag{11}$$

lead to the all-loop RGI sum rule [53] (assuming $(m^2)^i{}_j = m_j^2 \delta^i_j$),

$$m_i^2 + m_j^2 + m_k^2 = |M|^2 \left\{ \frac{1}{1 - g^2 C_2(G)/(8\pi^2)} \frac{d \ln C^{ijk}}{d \ln g} + \frac{1}{2} \frac{d^2 \ln C^{ijk}}{d(\ln g)^2} \right\} + \sum_l \frac{m_l^2 T(R_l)}{C_2(G) - 8\pi^2/g^2} \frac{d \ln C^{ijk}}{d \ln g} \, . \tag{12}$$

The all-loop results on the SSB β-functions lead to all-loop RGI relations (see e.g. [54]). If we assume:

(a) the existence of a RGI surfaces on which $C = C(g)$, or equivalently that

$$\frac{dC^{ijk}}{dg} = \frac{\beta_C^{ijk}}{\beta_g} \tag{13}$$

holds, i.e. reduction of couplings is possible, and

(b) the existence of a RGI surface on which

$$h^{ijk} = -M \frac{dC(g)^{ijk}}{d \ln g} \tag{14}$$

holds too in all-orders, then one can prove, [55, 56], that the following relations are RGI to all-loops (note that in both (a) and (b) assumptions above we do not rely on specific solutions of these equations)

$$M = M_0 \frac{\beta_g}{g}, \tag{15}$$

$$h^{ijk} = -M_0\, \beta_C^{ijk}, \tag{16}$$

$$b^{ij} = -M_0\, \beta_\mu^{ij}, \tag{17}$$

$$(m^2)^i{}_j = \frac{1}{2}\, |M_0|^2\, \mu \frac{d\gamma^i{}_j}{d\mu}, \tag{18}$$

where M_0 is an arbitrary reference mass scale to be specified shortly.

Finally we would like to emphasize that under the same assumptions (a) and (b) the sum rule given in Eq. (12) has been proven [53] to be all-loop RGI, which gives us a generalization of Eq. (18) to be applied in considerations of non-universal soft scalar masses, which are necessary in many cases including the MSSM.

As it was emphasized in [55] the set of the all-loop RGI relations (15)–(18) is the one obtained in the *Anomaly Mediated SB Scenario* [57, 58], by fixing the M_0 to be $m_{3/2}$, which is the natural scale in the supergravity framework. A final remark concerns the resolution of the fatal problem of the anomaly induced scenario in the supergravity framework, which is here solved thanks to the sum rule (12). Other solutions have been provided by introducing Fayet–Iliopoulos terms [59].

4 Applications of the Reduction of Couplings Method

In this section we show how to apply the reduction of couplings method in two scenarios, the MSSM and Finite Unified Theories. We will apply it only to the third generation of fermions and in the soft supersymmetry breaking terms. After the reduction of couplings takes place, we are left with relations at the unification scale for the Yukawa couplings of the quarks in terms of the gauge coupling according to Eq. (9), for the trilinear terms in terms of the Yukawa couplings and the unified gaugino mass Eq. (14), and a sum rule for the soft scalar masses also proportional to the unified gaugino mass Eq. (12), as applied in each model.

4.1 RE in the MSSM

We will examine here the reduction of couplings method applied to the MSSM, which is defined by the superpotential,

$$W = Y_t H_2 Q t^c + Y_b H_1 Q b^c + Y_\tau H_1 L \tau^c + \mu H_1 H_2, \tag{19}$$

with soft breaking terms,

$$
\begin{aligned}
-\mathcal{L}_{SSB} = \sum_{\phi} m_{\phi}^{2}\phi^{*}\phi + \left[m_3^2 H_1 H_2 + \sum_{i=1}^{3}\frac{1}{2}M_i\lambda_i\lambda_i + \text{h.c}\right] \\
+ \left[h_t H_2 Q t^c + h_b H_1 Q b^c + h_\tau H_1 L \tau^c + \text{h.c.}\right], \qquad (20)
\end{aligned}
$$

where the last line refers to the scalar components of the corresponding superfield. In general $Y_{t,b,\tau}$ and $h_{t,b,\tau}$ are 3×3 matrices, but we work throughout in the approximation that the matrices are diagonal, and neglect the couplings of the first two generations.

Assuming perturbative expansion of all three Yukawa couplings in favour of g_3 satisfying the reduction equations we find that the coefficients of the Y_τ coupling turn imaginary. Therefore, we take Y_τ at the GUT scale to be an independent variable. Thus, in the application of the reduction of couplings in the MSSM that we examine here, in the first stage we neglect the Yukawa couplings of the first two generations, while we keep Y_τ and the gauge couplings g_2 and g_1, which cannot be reduced consistently, as corrections. This "reduced" system holds at all scales, and thus serve as boundary conditions of the RGEs of the MSSM at the unification scale, where we assume that the gauge couplings meet [54].

In that case, the coefficients of the expansions (again at the GUT scale)

$$
\frac{Y_t^2}{4\pi} = c_1\frac{g_3^2}{4\pi} + c_2\left(\frac{g_3^2}{4\pi}\right)^2 \;; \quad \frac{Y_b^2}{4\pi} = p_1\frac{g_3^2}{4\pi} + p_2\left(\frac{g_3^2}{4\pi}\right)^2 \qquad (21)
$$

are given by

$$
\begin{aligned}
c_1 &= \frac{157}{175} + \frac{1}{35}K_\tau = 0.897 + 0.029K_\tau, \\
p_1 &= \frac{143}{175} - \frac{6}{35}K_\tau = 0.817 - 0.171K_\tau, \\
c_2 &= \frac{1}{4\pi}\,\frac{1457.55 - 84.491K_\tau - 9.66181K_\tau^2 - 0.174927K_\tau^3}{818.943 - 89.2143K_\tau - 2.14286K_\tau^2}, \\
p_2 &= \frac{1}{4\pi}\,\frac{1402.52 - 223.777K_\tau - 13.9475K_\tau^2 - 0.174927K_\tau^3}{818.943 - 89.2143K_\tau - 2.14286K_\tau^2},
\end{aligned} \qquad (22)
$$

where

$$
K_\tau = Y_\tau^2/g_3^2. \qquad (23)
$$

The couplings Y_t,Y_b and g_3 are not only reduced, but they provide predictions consistent with the observed experimental values. According to the analysis presented in Sect. 2 the RGI relations in the SSB sector hold, assuming the existence of RGI surfaces where Eqs. (13) and (14) are valid.

Since all gauge couplings in the MSSM meet at the unification point, we are led to the following boundary conditions at the GUT scale:

$$Y_t^2 = c_1 g_U^2 + c_2 g_U^4/(4\pi) \quad \text{and} \quad Y_b^2 = p_1 g_U^2 + p_2 g_U^4/(4\pi), \tag{24}$$

$$h_{t,b} = -M_U Y_{t,b}, \tag{25}$$

$$m_3^2 = -M_U \mu, \tag{26}$$

where M_U is the unification scale, $c_{1,2}$ and $p_{1,2}$ are the solutions of the algebraic system of the two reduction equations taken at the GUT scale (while keeping only the first term[1] of the perturbative expansion of the Yukawas in favour of g_3 for Eqs. (25) and (26)), and a set of equations resulting from the application of the sum rule

$$m_{H_2}^2 + m_Q^2 + m_{t^c}^2 = M_U^2, \quad m_{H_1}^2 + m_Q^2 + m_{b^c}^2 = M_U^2, \tag{27}$$

noting that the sum rule introduces four free parameters.

4.2 Predictions of the Reduced MSSM

With these boundary conditions we run the MSSM RGEs down to the SUSY scale, which we take to be the geometrical average of the stop masses, and then run the SM RGEs down to the electroweak scale (M_Z), where we compare with the experimental values of the third generation quark masses. The RGEs are taken at two-loops for the gauge and Yukawa couplings and at one-loop for the soft breaking parameters. We let M_U and $|\mu|$ at the unification scale to vary between $\sim$1 and $\sim$11 TeV, for the two possible signs of μ. In evaluating the τ and bottom masses we have taken into account the one-loop radiative corrections that come from the SUSY breaking [60, 61]; in particular for large $\tan\beta$ they can give sizeable contributions to the bottom quark mass.

Recall that Y_τ is not reduced and is a free parameter in this analysis. The parameter K_τ, related to Y_τ through Eq. (23) is further constrained by allowing only the values that are also compatible with the top and bottom quark masses simultaneously within 1 and 2σ of their central experimental value. In the case that $\text{sign}(\mu) = +$, there is no value for K_τ where both the top and the bottom quark masses agree simultaneously with their experimental value, therefore we only consider the negative sign of μ from now on. We use the experimental value of the top quark pole mass as

$$m_t^{\text{exp}} = (173.2 \pm 0.9)\ \text{GeV}\ . \tag{28}$$

[1]The second term can be determined once the first term is known.

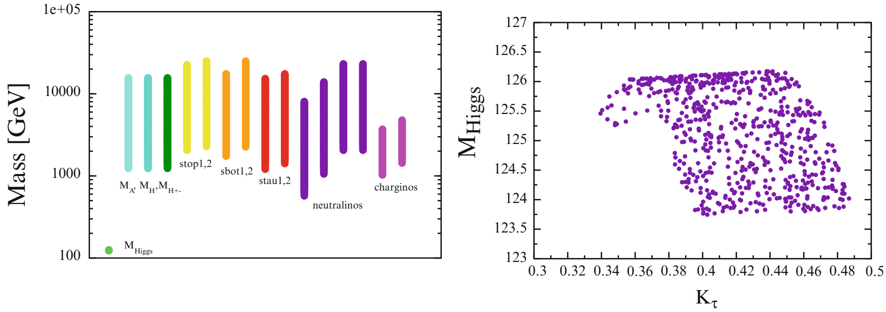

Fig. 1 The *left plot* shows the SUSY spectrum as function of the reduced MSSM. *From left to right* are shown: the lightest Higgs, the pseudoscalar one M_A, the heavy neutral one M^H, the two charged Higgses $M^{H\pm}$; then come the two stops, two bottoms and two status, the four neutralinos, and at the end the two charginos. The *right plot* shows the lightest Higgs boson mass as a function of $K_\tau = Y_\tau^2/g_3^2$

The bottom mass is calculated at M_Z to avoid uncertainties that come from running down to the pole mass and, as previously mentioned, the SUSY radiative corrections both to the tau and the bottom quark masses have been taken into account [62]

$$m_b(M_Z) = (2.83 \pm 0.10)\ \text{GeV}. \tag{29}$$

The variation of K_τ is in the range ∼0.33 to ∼0.5 if the agreement with both top and bottom masses is at the 2σ level.

Finally, assuming the validity of Eq. (14) for the corresponding couplings to those that have been reduced before, we calculate the Higgs mass as well as the whole Higgs and sparticle spectrum using Eqs. (24)–(27), and we present them in Fig. 1 The Higgs mass was calculated using a “mixed-scale” one-loop RG approach, which is known to approximate the leading two-loop corrections as evaluated by the full diagrammatic calculation [63, 64]. However, more refined Higgs mass calculations, and in particular the results of [65] are not (yet) included.

In the left plot of Fig. 1 we show the full mass spectrum of the model. We find that the masses of the heavier Higgses have relatively high values, above the TeV scale. In addition we find a generally heavy supersymmetric spectrum starting with a neutralino as LSP at ∼500 GeV and comfortable agreement with the LHC bounds due to the non-observation of coloured supersymmetric particles [66–68]. Finally note that although the $\mu < 0$ found in our analysis would disfavour the model in connection with the anomalous magnetic moment of the muon, such a heavy spectrum gives only a negligible correction to the SM prediction. We plan to extend our analysis by examining the restrictions that will be imposed in the spectrum by the B-physics and Cold Dark Matter (CDM) constraints.

In the right plot of Fig. 1 we show the results for the light Higgs boson mass as a function of K_τ. The results are in the range 123.7–126.3 GeV, where the uncertainty is due to the variation of K_τ, the gaugino mass M_U and the variation of the scalar soft masses, which are however constrained by the sum rules (27). The gaugino mass M_U is in the range ∼1.3 to ∼11 TeV, the lower values having been discarded since they do not allow for radiative electroweak symmetry breaking. To the lightest Higgs mass value one has to add at least ±2 GeV coming from unknown higher order corrections [69]. Therefore it is in excellent agreement with the experimental results of ATLAS and CMS [1–4].

4.3 Finiteness

Finiteness can be understood by considering a chiral, anomaly free, $N = 1$ globally supersymmetric gauge theory based on a group G with gauge coupling constant g. Consider the superpotential Eq. (4) together with the soft supersymmetry breaking Lagrangian Eq. (5). All the one-loop β-functions of the theory vanish if the β-function of the gauge coupling $\beta_g^{(1)}$, and the anomalous dimensions of the Yukawa couplings $\gamma_i^{j(1)}$, vanish, i.e.

$$\sum_i \ell(R_i) = 3C_2(G)\,, \ \frac{1}{2}C_{ipq}C^{jpq} = 2\delta_i^j g^2 C_2(R_i)\,, \tag{30}$$

where $\ell(R_i)$ is the Dynkin index of R_i, and $C_2(G)$ is the quadratic Casimir invariant of the adjoint representation of G. These conditions are also enough to guarantee two-loop finiteness [70]. A striking fact is the existence of a theorem [26–28], that guarantees the vanishing of the β-functions to all-orders in perturbation theory. This requires that, in addition to the one-loop finiteness conditions (30), the Yukawa couplings are reduced in favour of the gauge coupling to all-orders (see [71] for details). Alternatively, similar results can be obtained [72–74] using an analysis of the all-loop NSVZ gauge beta-function [51, 75].

Since we would like to consider only finite theories here, we assume that the gauge group is a simple group and the one-loop β-function of the gauge coupling g vanishes. We also assume that the reduction equations admit power series solutions of the form Eq. (9). According to the finiteness theorem of [26–28, 76], the theory is then finite to all orders in perturbation theory, if, among others, the one-loop anomalous dimensions $\gamma_i^{j(1)}$ vanish. The one- and two-loop finiteness for h^{ijk} can be achieved through the relation [77]

$$h^{ijk} = -MC^{ijk} + \cdots = -M\rho_{(0)}^{ijk}\, g + O(g^5)\,, \tag{31}$$

where ... stand for higher order terms.

In addition it was found that the RGI SSB scalar masses in Gauge–Yukawa unified models satisfy a universal sum rule at one-loop [31]. This result was generalized to two-loops for finite theories [32], and then to all-loops for general

Gauge–Yukawa and finite unified theories [53]. From these latter results, the following soft scalar-mass sum rule is found [32]

$$\frac{(\,m_i^2 + m_j^2 + m_k^2\,)}{MM^\dagger} = 1 + \frac{g^2}{16\pi^2}\,\Delta^{(2)} + O(g^4) \tag{32}$$

for i, j, k with $\rho^{ijk}_{(0)} \neq 0$, where $m^2_{i,j,k}$ are the scalar masses and $\Delta^{(2)}$ is the two-loop correction which vanishes for the universal choice, i.e. when all the soft scalar masses are the same at the unification point, as well as in the model considered here.

4.4 $SU(5)$ Finite Unified Theories

We examine an all-loop Finite Unified theory with $SU(5)$ as gauge group, where the reduction of couplings has been applied to the third generation of quarks and leptons. The particle content of the model we will study, which we denote **FUT** consists of the following supermultiplets: three ($\overline{\mathbf{5}} + \mathbf{10}$), needed for each of the three generations of quarks and leptons, four ($\overline{\mathbf{5}} + \mathbf{5}$) and one **24** considered as Higgs supermultiplets. When the gauge group of the finite GUT is broken the theory is no longer finite, and we will assume that we are left with the MSSM [15, 18–21].

A predictive Gauge–Yukawa unified $SU(5)$ model which is finite to all orders, in addition to the requirements mentioned already, should also have the following properties:

1. One-loop anomalous dimensions are diagonal, i.e., $\gamma_i^{(1)\,j} \propto \delta_i^j$.
2. Three fermion generations, in the irreducible representations $\overline{\mathbf{5}}_i, \mathbf{10}_i$ $(i = 1, 2, 3)$, which obviously should not couple to the adjoint **24**.
3. The two Higgs doublets of the MSSM should mostly be made out of a pair of Higgs quintet and anti-quintet, which couple to the third generation.

After the reduction of couplings the symmetry is enhanced, leading to the following superpotential [78]

$$W = \sum_{i=1}^{3}\Big[\,\frac{1}{2}g_i^u\,\mathbf{10}_i\mathbf{10}_i H_i + g_i^d\,\mathbf{10}_i\overline{\mathbf{5}}_i\,\overline{H}_i\,\Big] + g_{23}^u\,\mathbf{10}_2\mathbf{10}_3 H_4 \tag{33}$$

$$+ g_{23}^d\,\mathbf{10}_2\overline{\mathbf{5}}_3\,\overline{H}_4 + g_{32}^d\,\mathbf{10}_3\overline{\mathbf{5}}_2\,\overline{H}_4 + g_2^f\,H_2\,\mathbf{24}\,\overline{H}_2 + g_3^f\,H_3\,\mathbf{24}\,\overline{H}_3 + \frac{g^\lambda}{3}\,(\mathbf{24})^3\,.$$

The non-degenerate and isolated solutions to $\gamma_i^{(1)} = 0$ give us:

$$(g_1^u)^2 = \frac{8}{5}\,g^2\,,\ (g_1^d)^2 = \frac{6}{5}\,g^2\,,\ (g_2^u)^2 = (g_3^u)^2 = \frac{4}{5}\,g^2\,, \tag{34}$$

$$(g_2^d)^2 = (g_3^d)^2 = \frac{3}{5}\, g^2\ ,\ (g_{23}^u)^2 = \frac{4}{5}\, g^2\ ,\ (g_{23}^d)^2 = (g_{32}^d)^2 = \frac{3}{5}\, g^2\ ,$$
$$(g^\lambda)^2 = \frac{15}{7} g^2\ ,\ (g_2^f)^2 = (g_3^f)^2 = \frac{1}{2}\, g^2\ ,\ (g_1^f)^2 = 0\ ,\ (g_4^f)^2 = 0\ ,$$

and from the sum rule we obtain:

$$m_{H_u}^2 + 2m_{\mathbf{10}}^2 = M^2\ ,\ m_{H_d}^2 - 2m_{\mathbf{10}}^2 = -\frac{M^2}{3}\ ,\ m_{\overline{\mathbf{5}}}^2 + 3m_{\mathbf{10}}^2 = \frac{4M^2}{3}\ , \tag{35}$$

i.e., in this case we have only two free parameters $m_{\mathbf{10}}$ and M for the dimensionful sector.

As already mentioned, after the $SU(5)$ gauge symmetry breaking we assume we have the MSSM, i.e. only two Higgs doublets. This can be achieved by introducing appropriate mass terms that allow to perform a rotation of the Higgs sector [18–22, 79–81], in such a way that only one pair of Higgs doublets, coupled mostly to the third family, remains light and acquire vacuum expectation values. To avoid fast proton decay the usual fine tuning to achieve doublet-triplet splitting is performed, although the mechanism is not identical to minimal $SU(5)$, since we have an extended Higgs sector.

Thus, after the gauge symmetry of the GUT theory is broken we are left with the MSSM, with the boundary conditions for the third family given by the finiteness conditions, while the other two families are not restricted.

4.5 Predictions of the Finite Model

Since the gauge symmetry is spontaneously broken below $M_{\rm GUT}$, the finiteness conditions do not restrict the renormalization properties at low energies, and all it remains are boundary conditions on the gauge and Yukawa couplings (34), the $h = -MC$ (31) relation, and the soft scalar-mass sum rule at $M_{\rm GUT}$. The analysis follows along the same lines as in the MSSM case.

In Fig. 2 we show the **FUT** predictions for m_t and $m_b(M_Z)$ as a function of the unified gaugino mass M, for the two cases $\mu < 0$ and $\mu > 0$. The bounds on the $m_b(M_Z)$ and the m_t mass clearly single out $\mu < 0$, as the solution most compatible with these experimental constraints.

We now analyze the impact of further low-energy observables on the model **FUT** with $\mu < 0$. As additional constraints we consider the flavour observables BR($b \to s\gamma$) and BR($B_s \to \mu^+\mu^-$).

For the branching ratio BR($b \to s\gamma$), we take the value given by the Heavy Flavour Averaging Group (HFAG) is [82]

$$\mathrm{BR}(b \to s\gamma) = (3.55 \pm 0.24^{+0.09}_{-0.10} \pm 0.03) \times 10^{-4}. \tag{36}$$

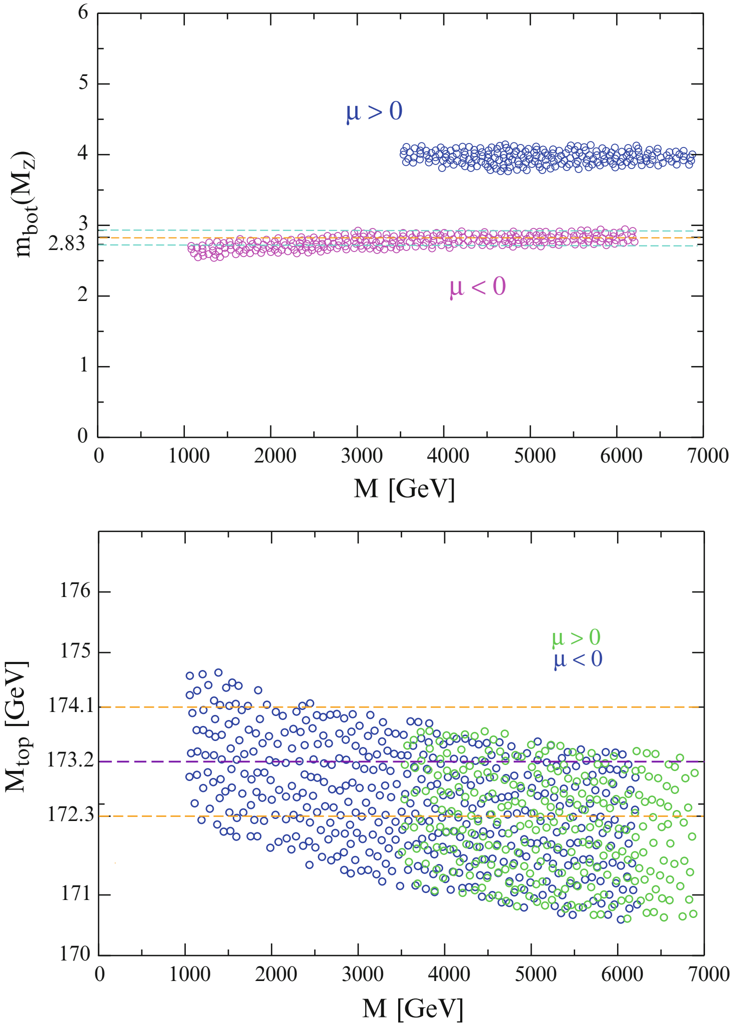

Fig. 2 The bottom quark mass at the Z boson scale (*left*) and top quark pole mass (*right*) are shown as function of M, the unified gaugino mass

For the branching ratio $\mathrm{BR}(B_s \to \mu^+\mu^-)$, the SM prediction is at the level of 10^{-9}, while we employ an upper limit of

$$\mathrm{BR}(B_s \to \mu^+\mu^-) \lesssim 4.5 \times 10^{-9} \tag{37}$$

at the 95% [83]. This is in relatively good agreement with the recent direct measurement of this quantity by CMS and LHCb Collaborations [84]. As we do not expect a sizable impact of the new measurement on our results, we stick for our analysis to the simple upper limit.

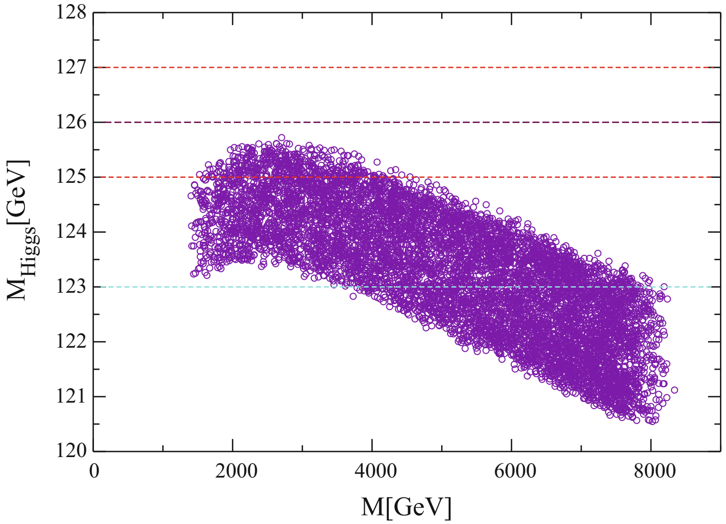

Fig. 3 The lightest Higgs mass, M_h, as function of M for the model **FUT** with $\mu < 0$

For the lightest Higgs mass prediction we used the code `FeynHiggs` [69,85–87]. The prediction for M_h of **FUT** with $\mu < 0$ is shown in Fig. 3, where the constraints from the two B physics observables are taken into account. The lightest Higgs mass ranges in

$$M_h \sim 121 - 126 \text{ GeV} , \tag{38}$$

where the uncertainty comes from variations of the soft scalar masses. To this value one has to add at least ± 2 GeV coming from unknown higher order corrections [69].[2] We have also included a small variation, due to threshold corrections at the GUT scale, of up to 5 % of the FUT boundary conditions. The masses of the heavier Higgs bosons are found at higher values in comparison with our previous analyses [65,89–91]. This is due to the more stringent bound on $\text{BR}(B_s \to \mu^+\mu^-)$, which pushes the heavy Higgs masses beyond $\sim$1 TeV, excluding their discovery at the LHC.

We impose now a further constraint on our results, which is the value of the Higgs mass

$$M_h \sim 126.0 \pm 1 \pm 2 \text{ GeV} , \tag{39}$$

[2] We have not yet taken into account the improved M_h prediction presented in [88] (and implemented into the most recent version of `FeynHiggs`), which will lead to an upward shift in the Higgs boson mass prediction.

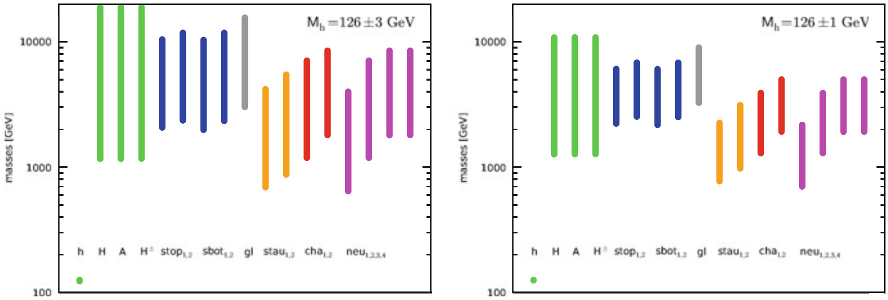

Fig. 4 The *left* (*right*) *plot* shows the spectrum after imposing the constraint $M_h = 126 \pm 3\,(1)$ GeV. The light (*green*) points are the various Higgs boson masses, the dark (*blue*) *points* following are the two scalar top and bottom masses, the *gray* ones are the gluino masses, then come the scalar tau masses in *orange* (*light gray*), the darker (*red*) points to the *right* are the two chargino masses followed by the *lighter shaded* (*pink*) points indicating the neutralino masses

where ± 3 GeV corresponds to the current theory and experimental uncertainty, and ± 1 GeV to a reduced theory uncertainty in the future.[3] We find that constraining the allowed values of the Higgs mass puts a limit on the allowed values of the unified gaugino mass, as can be seen from Fig. 3. The red lines correspond to the pure experimental uncertainty and restrict 2 TeV $\lesssim M \lesssim 5$ TeV. The blue line includes the additional theory uncertainty of ± 2 GeV. Taking this uncertainty into account no bound on M can be placed.

The full particle spectrum of model **FUT** with $\mu < 0$, compliant with quark mass constraints and the B-physics observables is shown in Fig. 4. It can be seen from the figure that the lightest observable SUSY particle (LOSP) is the light scalar tau. In the left (right) plot we impose $M_h = 126 \pm 3(1)$ GeV. Without any M_h restrictions the coloured SUSY particles have masses above ~1.8 TeV in agreement with the non-observation of those particles at the LHC [66–68]. Including the Higgs mass constraints in general favours the lower part of the SUSY particle mass spectra, but also cuts away the very low values. Going to the anticipated future theory uncertainty of M_h (as shown in the lower plot of Fig. 4) permits SUSY masses which would remain unobservable at the LHC, the ILC or CLIC. On the other hand, large parts of the allowed spectrum of the lighter scalar tau or the lighter neutralinos might be accessible at CLIC with $\sqrt{s} = 3$ TeV.

[3] In this analysis the new M_h evaluation [88] may have a relevant impact on the restrictions on the allowed SUSY parameter space shown below.

Conclusions

The serious problem of the appearance of many free parameters in the SM of Elementary Particle Physics, takes dramatic dimensions in the MSSM, where the free parameters are proliferated by at least hundred more, while it is considered as the best candidate for Physics Beyond the SM. The idea that the Theory of Particle Physics is more symmetric at high scales, which is broken but has remnant predictions in the much lower scales of the SM, found its best realisation in the framework of the MSSM assuming further a GUT beyond the scale of the unification of couplings. However, the unification idea, although successful, seems to have exhausted its potential to reduce further the free parameters of the SM.

A new interesting possibility towards reducing the free parameters of a theory has been put forward in [24, 25] which consists on a systematic search on the RGI relations among couplings. This method might lead to further symmetry, however its scope is much wider. After several trials it seems that the basic idea found very nice realisations in Finite Unified Theories and the MSSM. In the first case one is searching for RGI relations among couplings holding beyond the unification scale, which moreover guarantee finiteness to all-orders in perturbation theory. In the second, the search of RGI relations among couplings is concentrated within the MSSM itself and the assumption of GUT is not necessarily required. The results in both cases are indeed impressive as we have discussed. Certainly one can add some more comments on the Finite Unified Theories. These are related to some fundamental developments in Theoretical Particle Physics based on reconsiderations of the problem of divergencies and serious attempts to solve it. They include the motivation and construction of string and noncommutative theories, as well as $N = 4$ supersymmetric field theories [92, 93], $N = 8$ supergravity [94–98] and the AdS/CFT correspondence [99]. It is a thoroughly fascinating fact that many interesting ideas that have survived various theoretical and phenomenological tests, as well as the solution to the UV divergencies problem, find a common ground in the framework of $N = 1$ Finite Unified Theories, which have been discussed here. From the theoretical side they solve the problem of UV divergencies in a minimal way. On the phenomenological side in both cases of reduction of couplings discussed here the celebrated success of predicting the top-quark mass [18–20, 22, 23, 100] is now extended to the predictions of the Higgs masses and the supersymmetric spectrum of the MSSM, which so far have been confronted very successfully with the findings and bounds at the LHC.

The various scenarios will be refined/scrutinized in various ways in the upcoming years. Important improvements in the analysis are expected from progress on the theory side, in particular in an improved calculation of the

(continued)

light Higgs boson mass. The corrections introduced in [88] not only introduce a shift in M_h (which should to some extent be covered by the estimate of theory uncertainties). They will also reduce the theory uncertainties, see [88, 101], and in this way refine the selected model points, leading to a sharper prediction of the allowed spectrum. One can hope that with even more higher-order corrections included in the M_h calculation an uncertainty below the 0.5 GeV level can be reached.

The other important improvements in the future will be the continuing searches for SUSY particles at collider experiments. The LHC will re-start in 2015 with an increased center-of-mass energy of $\sqrt{s} \lesssim 14$ TeV, largely extending its SUSY search reach. The lower parts of the currently allowed/predicted colored SUSY spectra will be tested in this way. For the electroweak particles, on the other hand, e^+e^- colliders might be the better option. The ILC, operating at $\sqrt{s} \lesssim 1$ TeV, has only a limited potential for our model spectra. Going to higher energies, $\sqrt{s} \lesssim 3$ TeV, that might be realized at CLIC, large parts of the predicted electroweak model spectra can be covered.

All spectra, however, (at least with the current calculation of M_h and its corresponding uncertainty), contain parameter regions that will escape the searches at the LHC, the ILC and CLIC. In this case we would remain with a light Higgs boson in the decoupling limit, i.e. would be undistinguishable from a SM Higgs boson. The only hope to overcome this situation is that an improved M_h calculation would cut away such high spectra.

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Part II
String Theories and Gravity Theories

A SUSY Double-Well Matrix Model as 2D Type IIA Superstring

Fumihiko Sugino

Abstract We discuss correspondence between a simple supersymmetric matrix model with a double-well potential and two-dimensional type IIA superstrings on a nontrivial Ramond–Ramond background. In particular, we can see direct correspondence between single trace operators in the matrix model and vertex operators in the type IIA theory by computing scattering amplitudes and comparing the results in both sides.

1 Introduction

Nonperturbative aspects of noncritical bosonic string theory were vigorously investigated around 1990 by using solvable matrix models (for a review, see [1]), while little has been known for superstring theory, in particular which possesses target-space supersymmetry (SUSY). We would like to consider (solvable) matrix models describing superstring theory with target-space SUSY. We hope our analysis is helpful to understand nonperturbative dynamics of matrix models of super Yang–Mills type for critical superstring theory [2–4].

2 Double-Well SUSY Matrix Model

Kuroki and Sugino [5] discussed a following simple matrix model:

$$S = N\,\mathrm{tr}\left[\frac{1}{2}B^2 + iB(\phi^2 - \mu^2) + \bar{\psi}(\phi\psi + \psi\phi)\right], \tag{1}$$

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where B and ϕ are $N \times N$ hermitian matrices, and ψ and $\bar{\psi}$ are $N \times N$ Grassmann-odd matrices. The action S is invariant under SUSY transformations generated by Q and $\bar{Q}$:

$$Q\phi = \psi, \quad Q\psi = 0, \quad Q\bar{\psi} = -iB, \quad QB = 0, \tag{2}$$

$$\bar{Q}\phi = -\bar{\psi}, \quad \bar{Q}\bar{\psi} = 0, \quad \bar{Q}\psi = -iB, \quad \bar{Q}B = 0, \tag{3}$$

from which one can see that they are nilpotent: $Q^2 = \bar{Q}^2 = \{Q, \bar{Q}\} = 0$. After integrating out B, we have a scalar potential of a double-well shape: $\frac{1}{2}(\phi^2 - \mu^2)^2$. A large-$N$ saddle point solution for the eigenvalue distribution of the matrix ϕ: $\rho(x) \equiv \frac{1}{N}\text{tr}\,\delta(x - \phi)$ is given by

$$\rho(x) = \begin{cases} \frac{\nu_+}{\pi}\, x\, \sqrt{(x^2 - a^2)(b^2 - x^2)} & (a < x < b) \\ \frac{\nu_-}{\pi}\, |x|\, \sqrt{(x^2 - a^2)(b^2 - x^2)} & (-b < x < -a), \end{cases} \tag{4}$$

where $a = \sqrt{\mu^2 - 2}$ and $b = \sqrt{\mu^2 + 2}$. The filling fractions (ν_+, ν_-) satisfying $\nu_+ + \nu_- = 1$ indicate that $\nu_+ N$ ($\nu_- N$) eigenvalues are around the right (left) minimum of the double-well. The solution exists for $\mu^2 > 2$. The large-N free energy and the expectation values $\left\langle \frac{1}{N}\text{tr}\, B^n \right\rangle$ $(n = 1, 2, \cdots)$ evaluated at the solution turn out to all vanish [5]. This strongly suggests that the solution preserves SUSY. Thus, we conclude that the SUSY minima are infinitely degenerate and parameterized by (ν_+, ν_-) at large N.

On the other hand, for $\mu^2 < 2$, non SUSY saddle point solution is obtained [6]. The transition between the SUSY phase ($\mu^2 > 2$) and the SUSY broken phase ($\mu^2 < 2$) is of the third order.

In the next section, we will compute various correlation functions at the saddle point (4) and find new logarithmic critical behavior as $\mu^2 \to 2 + 0$. Based on the result, we will discuss correspondence between the matrix model and two-dimensional type IIA superstring theory on a nontrivial Ramond–Ramond (RR) background in Sects. 4 and 5.

Our matrix model is interpreted as the $O(n)$ model on a random surface [7] with $n = -2$, whose critical behavior is described by the $c = -2$ topological gravity [8]. The partition function after B, ψ and $\bar{\psi}$ are integrated out is expressed as a Gaussian one-matrix model by the Nicolai mapping $H = \phi^2$, where the H-integration is over the *positive definite* hermitian matrices, not over all the hermitian matrices. References [9, 10] discuss that the difference of the integration region has only effects which are nonperturbative in $1/N$, and the model can be regarded as the standard Gaussian matrix model at each order of genus expansion.

The Nicolai mapping changes the operators $\frac{1}{N}\text{tr}\,\phi^{2n}$ $(n = 1, 2, \cdots)$ to regular operators $\frac{1}{N}\text{tr}\, H^n$. Hence, the behavior of their correlators is expected to be described by the Gaussian one-matrix (the $c = -2$ topological gravity) at least perturbatively in $1/N$. However, the operators $\frac{1}{N}\text{tr}\,\phi^{2n+1}$ $(n = 0, 1, 2, \cdots)$ are mapped to $\pm\frac{1}{N}\text{tr}\, H^{n+1/2}$ that are singular at the origin. They are not observables in the $c = -2$ topological gravity, while they are natural observables as well as $\frac{1}{N}\mathrm{tr}\,\phi^{2n}$ in the original setting (1). In the next section, we will see that correlation functions among operators

$$\frac{1}{N}\mathrm{tr}\,\phi^{2n+1}, \qquad \frac{1}{N}\mathrm{tr}\,\psi^{2n+1}, \qquad \frac{1}{N}\mathrm{tr}\,\bar{\psi}^{2n+1} \qquad (n = 0, 1, 2, \cdots) \tag{5}$$

exhibit logarithmic singular behavior of powers of $\ln(\mu^2 - 2)$ at the planar topology.

3 Correlation Functions

The planar one-point function $\left\langle \frac{1}{N}\mathrm{tr}\,\phi^n \right\rangle_0$ $(n = 1, 2, \cdots)$ are computed as

$$\begin{aligned}\left\langle \frac{1}{N}\mathrm{tr}\,\phi^n \right\rangle_0 &= \int dx\, x^n \rho(x) \\ &= (\nu_+ + (-1)^n \nu_-)(2+\mu^2)^{n/2} F\left(-\frac{n}{2}, \frac{3}{2}, 3; \frac{4}{2+\mu^2}\right),\end{aligned} \tag{6}$$

where the suffix "0" in the left hand side indicates the planar contribution. For n even, the expression reduces to a polynomial of μ^2 giving nonsingular behavior as expected from the $c = -2$ topological gravity. On the other hand, when μ^2 is odd, it exhibits logarithmic singular behavior as $\mu^2 \to 2 + 0$:

$$\left\langle \frac{1}{N}\mathrm{tr}\,\phi^{2k+1} \right\rangle_0 \sim (\nu_+ - \nu_-) \frac{2^{k+2}}{\pi} \frac{(2k+1)!!}{(k+2)!}\, \omega^{k+2} \ln \omega \tag{7}$$

with $\omega \equiv \frac{1}{4}(\mu^2 - 2)$. The symbol "$\sim$" denotes equality up to additive less singular terms. Matrix models can be seen as a sort of "lattice models" for string theory. In the hypergeometric function $F\left(-\frac{n}{2}, \frac{3}{2}, 3; \frac{1}{1+\omega}\right)$ for n being odd, the logarithmic singular terms can be regarded as universal parts relevant to "continuum physics", whereas polynomials of ω as nonuniversal "lattice artifacts".

In [11], planar higher-point functions are obtained by introducing source terms $\sum_{p=1}^{\infty} j_p \mathrm{tr}\,\phi^p$ and considering a large-N saddle point equation in the presence of the source terms. Two-point functions are expressed as

$$\left\langle \Phi_{2k+1} \frac{1}{N}\mathrm{tr}\,\phi^{2\ell} \right\rangle_{C,0} \sim (\nu_+ - \nu_-)(\mathrm{const.})\omega^{k+1} \ln \omega, \tag{8}$$

$$\begin{aligned}\langle \Phi_{2k+1} \Phi_{2\ell+1} \rangle_{C,0} \sim &-(\nu_+ - \nu_-)^2 \frac{1}{2\pi^2} \frac{1}{k+\ell+1} \frac{(2k+1)!}{(k!)^2} \frac{(2\ell+1)!}{(\ell!)^2} \\ &\times \omega^{k+\ell+1} (\ln \omega)^2.\end{aligned} \tag{9}$$

Here, the suffix "C" means taking connected parts. In order to subtract nonuniversal contributions, we took a basis of the odd-power operators mixed with lower even-power operators:

$$\Phi_{2k+1} = \frac{1}{N}\mathrm{tr}\,\phi^{2k+1} + (\nu_+ - \nu_-)\sum_{i=1}^{k}\alpha_{2k+1,2i}(\omega)\frac{1}{N}\mathrm{tr}\,\phi^{2i} \tag{10}$$

with $\alpha_{2k+1,2i}(\omega)$ being a regular function at $\omega = 0$. The form of n-point functions of operators Φ_{2k+1} ($k = 0, 1, 2, \cdots$) is

$$\left\langle \prod_{i=1}^{n} \Phi_{2k_i+1} \right\rangle_{C,0} \sim (\nu_+ - \nu_-)^n \,(\mathrm{const.})\, \omega^{2-\gamma+\sum_{i=1}^{n}(k_i-1)}\,(\ln\omega)^n \tag{11}$$

with $\gamma = -1$. Besides the power of logarithm $(\ln\omega)^n$, it has the standard scaling behavior with the string susceptibility $\gamma = -1$ (the same as in the $c = -2$ topological gravity) and the gravitational scaling dimension k of Φ_{2k+1}, if we identify ω with "the cosmological constant" coupled to the lowest dimensional operator on a random surface [12–14].

Similarly to (10), we consider fermionic operators:

$$\begin{aligned} \Psi_1 &\equiv \tfrac{1}{N}\mathrm{tr}\,\psi, & \bar{\Psi}_1 &\equiv \frac{1}{N}\mathrm{tr}\,\bar{\psi}, \\ \Psi_3 &\equiv \tfrac{1}{N}\mathrm{tr}\,\psi^3 + (\mathrm{mixing}), & \bar{\Psi}_3 &\equiv \frac{1}{N}\mathrm{tr}\,\bar{\psi}^3 + (\mathrm{mixing}), \\ &\cdots, & &\cdots, \end{aligned} \tag{12}$$

where "(mixing)" means lower-power operators needed to subtract nonuniversal contributions. In [11], two-point correlators of fermions are also computed as

$$\left\langle \Psi_{2k+1}\bar{\Psi}_{2\ell+1}\right\rangle_{C,0} \sim \delta_{k,\ell}\, v_k\, (\nu_+ - \nu_-)^{2k+1}\omega^{2k+1}\ln\omega \tag{13}$$

with $v_0 = \frac{1}{\pi}$ and $v_1 = \frac{6}{\pi}$. The result tells us that Ψ_{2k+1} and $\bar{\Psi}_{2k+1}$ have the same gravitational scaling dimension k as Φ_{2k+1} besides the logarithmic factor.

4 2D Type IIA Superstring

The two-dimensional type II superstring theory discussed in [15–18] has the target space $(\varphi, x) \in$ (Liouville direction) $\times$ (S^1 with self-dual radius). The holomorphic energy-momentum tensor on the string world-sheet is

$$T = -\frac{1}{2}(\partial x)^2 - \frac{1}{2}\psi_x\partial\psi_x - \frac{1}{2}(\partial\varphi)^2 + \partial^2\varphi - \frac{1}{2}\psi_\ell\partial\psi_\ell \tag{14}$$

excluding ghosts' part. ψ_x and ψ_ℓ are superpartners of x and φ, respectively. Target-space supercurrents in the type IIA theory

$$q_+(z) = e^{-\frac{1}{2}\phi(z)-\frac{i}{2}H(z)-ix(z)}, \qquad \bar{q}_-(\bar{z}) = e^{-\frac{1}{2}\bar{\phi}(\bar{z})+\frac{i}{2}\bar{H}(\bar{z})+i\bar{x}(\bar{z})} \tag{15}$$

exist only for the S^1 target space of the self-dual radius. ϕ ($\bar{\phi}$) is the holomorphic (anti-holomorphic) bosonized superconformal ghost, and the fermions are bosonized as $\psi_\ell \pm i\psi_x = \sqrt{2}\,e^{\mp iH}$, $\bar{\psi}_\ell \pm i\bar{\psi}_x = \sqrt{2}\,e^{\mp i\bar{H}}$. In addition, we should care about cocycle factors in order to realize the anticommuting nature between q_+ and $\bar{q}_-$. Supercurrents with the cocycle factors are

$$\hat{q}_+(z) = e^{\pi\beta(\frac{1}{2}p_{\bar{\phi}}-i\frac{1}{2}p_{\bar{h}}-ip_{\bar{x}})}\,q_+(z), \qquad \hat{\bar{q}}_-(\bar{w}) = e^{-\pi\beta(\frac{1}{2}p_\phi+i\frac{1}{2}p_h+ip_x)}\,\bar{q}_-(\bar{w}), \tag{16}$$

where $\beta \in \mathbf{Z}+\frac{1}{2}$, and p_ϕ, p_h and p_x ($p_{\bar{\phi}}$, $p_{\bar{h}}$ and $p_{\bar{x}}$) are momentum modes of holomorphic part (anti-holomorphic part) of free bosons [19]. Then the supercharges

$$\hat{Q}_+ = \oint \frac{dz}{2\pi i}\,\hat{q}_+(z), \qquad \hat{\bar{Q}}_- = \oint \frac{d\bar{z}}{2\pi i}\,\hat{\bar{q}}_-(\bar{z}) \tag{17}$$

are nilpotent $\hat{Q}_+^2 = \hat{\bar{Q}}_-^2 = \{\hat{Q}_+, \hat{\bar{Q}}_-\} = 0$, which indeed matches the property of the supercharges Q and $\bar{Q}$ in the matrix model.

The spectrum except special massive states is represented by the NS "tachyon"[1] vertex operator (in (-1) picture):

$$T_k = e^{-\phi+ikx+p_\ell\varphi}, \qquad \bar{T}_{\bar{k}} = e^{-\bar{\phi}+i\bar{k}\bar{x}+p_\ell\bar{\varphi}}, \tag{18}$$

and by the R vertex operator (in $(-\frac{1}{2})$ picture):

$$V_{k,\epsilon} = e^{-\frac{1}{2}\phi+\frac{i}{2}\epsilon H+ikx+p_\ell\varphi}, \qquad \bar{V}_{\bar{k},\bar{\epsilon}} = e^{-\frac{1}{2}\bar{\phi}+\frac{i}{2}\bar{\epsilon}\bar{H}+i\bar{k}\bar{x}+p_\ell\bar{\varphi}} \tag{19}$$

with $\epsilon, \bar{\epsilon} = \pm 1$. Cocycle factors for the vertex operators are introduced as

$$\hat{T}_k(z) = e^{\pi\beta(p_{\bar{\phi}}+ikp_{\bar{x}})}\,T_k(z), \quad \hat{\bar{T}}_{\bar{k}}(\bar{z}) = e^{-\pi\beta(p_\phi+i\bar{k}p_x)}\,\bar{T}_{\bar{k}}(\bar{z}), \tag{20}$$
$$\hat{V}_{k,\epsilon}(z) = e^{\pi\beta(\frac{1}{2}p_{\bar{\phi}}+i\frac{\epsilon}{2}p_{\bar{h}}+ikp_{\bar{x}})}\,V_{k,\epsilon}(z), \quad \hat{\bar{V}}_{\bar{k},\bar{\epsilon}}(\bar{z}) = e^{-\pi\beta(\frac{1}{2}p_\phi+i\frac{\bar{\epsilon}}{2}p_h+i\bar{k}p_x)}\,\bar{V}_{\bar{k},\bar{\epsilon}}(\bar{z}).$$

Locality with the supercurrents, mutual locality, superconformal invariance (including the Dirac equation constraint) and the level matching condition determine physical vertex operators. As discussed in [17], there are two consistent sets of

[1] In two dimensions, "tachyon" turns out to be not truly tachyonic but massless.

physical vertex operators—"momentum background" and "winding background". Let us consider the "winding background".[2] The physical spectrum in the "winding background" is given by

$$
\begin{aligned}
&(\text{NS}, \text{NS}): && \hat{T}_k\, \hat{\bar{T}}_{-k} && (k \in \mathbf{Z} + \tfrac{1}{2}),\\
&(\text{R}+, \text{R}-): && \hat{V}_{k,+1}\, \hat{\bar{V}}_{-k,-1} && (k = \tfrac{1}{2}, \tfrac{3}{2}, \cdots),\\
&(\text{R}-, \text{R}+): && \hat{V}_{-k,-1}\, \hat{\bar{V}}_{k,+1} && (k = 0, 1, 2, \cdots),\\
&(\text{NS}, \text{R}-): && \hat{T}_{-k}\, \hat{\bar{V}}_{-k,-1} && (k = \tfrac{1}{2}, \tfrac{3}{2}, \cdots),\\
&(\text{R}+, \text{NS}): && \hat{V}_{k,+1}\, \hat{\bar{T}}_{k} && (k = \tfrac{1}{2}, \tfrac{3}{2}, \cdots),
\end{aligned}
\tag{21}
$$

where we take a branch of $p_\ell = 1 - |k|$ satisfying the locality bound $p_\ell \leq Q/2 = 1$ [20]. We can see that the vertex operators

$$
\hat{V}_{\frac{1}{2},+1}\, \hat{\bar{V}}_{-\frac{1}{2},-1}, \qquad \hat{T}_{-\frac{1}{2}}\, \hat{\bar{V}}_{-\frac{1}{2},-1}, \qquad \hat{V}_{\frac{1}{2},+1}\, \hat{\bar{T}}_{\frac{1}{2}}, \qquad \hat{T}_{-\frac{1}{2}}\, \hat{\bar{T}}_{\frac{1}{2}} \tag{22}
$$

form a quartet under $\hat{Q}_+$ and $\hat{\bar{Q}}_-$:

$$
\begin{aligned}
[\hat{Q}_+, \hat{V}_{\frac{1}{2},+1}\, \hat{\bar{V}}_{-\frac{1}{2},-1}] = \hat{T}_{-\frac{1}{2}}\, \hat{\bar{V}}_{-\frac{1}{2},-1}, \ \{\hat{Q}_+, \hat{T}_{-\frac{1}{2}}\, \hat{\bar{V}}_{-\frac{1}{2},-1}\} = 0,\\
\{\hat{Q}_+, \hat{V}_{\frac{1}{2},+1}\, \hat{\bar{T}}_{\frac{1}{2}}\} = \hat{T}_{-\frac{1}{2}}\, \hat{\bar{T}}_{\frac{1}{2}}, \qquad [\hat{Q}_+, \hat{T}_{-\frac{1}{2}}\, \hat{\bar{T}}_{\frac{1}{2}}] = 0,
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
[\hat{\bar{Q}}_-, \hat{V}_{\frac{1}{2},+1}\, \hat{\bar{V}}_{-\frac{1}{2},-1}] = -\hat{V}_{\frac{1}{2},+1}\, \hat{\bar{T}}_{\frac{1}{2}}, \ \{\hat{\bar{Q}}_-, \hat{V}_{\frac{1}{2},+1}\, \bar{T}_{\frac{1}{2}}\} = 0,\\
\{\hat{\bar{Q}}_-, \hat{T}_{-\frac{1}{2}}\, \hat{\bar{V}}_{-\frac{1}{2},-1}\} = \hat{T}_{-\frac{1}{2}}\, \hat{\bar{T}}_{\frac{1}{2}}, \qquad [\hat{\bar{Q}}_-, \hat{T}_{-\frac{1}{2}}\, \bar{T}_{\frac{1}{2}}] = 0.
\end{aligned}
\tag{24}
$$

Notice that (23) and (24) are isomorphic to (2) and (3), respectively. It leads to correspondence of single-trace operators in the matrix model to integrated vertex operators in the type IIA theory:

$$
\begin{aligned}
\Phi_1 &= \frac{1}{N} \mathrm{tr}\, \phi \Leftrightarrow \mathcal{V}_\phi(0) \equiv g_s^2 \int d^2z\, \hat{V}_{\frac{1}{2},+1}(z)\, \hat{\bar{V}}_{-\frac{1}{2},-1}(\bar{z}),\\
\Psi_1 &= \frac{1}{N} \mathrm{tr}\, \psi \Leftrightarrow \mathcal{V}_\psi(0) \equiv g_s^2 \int d^2z\, \hat{T}_{-\frac{1}{2}}(z)\, \hat{\bar{V}}_{-\frac{1}{2},-1}(\bar{z}),
\end{aligned}
$$

[2] We can repeat the parallel argument for "momentum background" in the type IIB theory, which is equivalent to the "winding background" in the type IIA theory through T-duality with respect to the S^1 direction.

$$\bar{\Psi}_1 = \frac{1}{N}\mathrm{tr}\,\bar{\psi} \Leftrightarrow \mathcal{V}_{\bar{\psi}}(0) \equiv g_s^2 \int d^2z\,\hat{V}_{\frac{1}{2},+1}(z)\,\hat{\bar{T}}_{\frac{1}{2}}(\bar{z}),$$

$$\frac{1}{N}\mathrm{tr}\,(-iB) \Leftrightarrow \mathcal{V}_B(0) \equiv g_s^2 \int d^2z\,\hat{T}_{-\frac{1}{2}}(z)\,\hat{\bar{T}}_{\frac{1}{2}}(\bar{z}), \tag{25}$$

where the bare string coupling g_s is put in the right hand sides to count the number of external lines of amplitudes in the IIA theory. Furthermore, it can be naturally extended as

$$\Phi_{2k+1} = \frac{1}{N}\mathrm{tr}\,\phi^{2k+1} + (\text{mixing}) \Leftrightarrow \mathcal{V}_\phi(k) \equiv g_s^2 \int d^2z\,\hat{V}_{k+\frac{1}{2},+1}(z)\,\hat{\bar{V}}_{-k-\frac{1}{2},-1}(\bar{z}),$$

$$\Psi_{2k+1} = \frac{1}{N}\mathrm{tr}\,\psi^{2k+1} + (\text{mixing}) \Leftrightarrow \mathcal{V}_\psi(k) \equiv g_s^2 \int d^2z\,\hat{T}_{-k-\frac{1}{2}}(z)\,\hat{\bar{V}}_{-k-\frac{1}{2},-1}(\bar{z}),$$

$$\bar{\Psi}_{2k+1} = \frac{1}{N}\mathrm{tr}\,\bar{\psi}^{2k+1} + (\text{mixing}) \Leftrightarrow \mathcal{V}_{\bar{\psi}}(k) \equiv g_s^2 \int d^2z\,\hat{V}_{k+\frac{1}{2},+1}(z)\,\hat{\bar{T}}_{k+\frac{1}{2}}(\bar{z}) \tag{26}$$

for higher $k(=1,2,\cdots)$. We see in (26) that the powers of matrices are interpreted as windings or momenta in the S^1 direction of the type IIA theory.

Note that (R−, R+) operators are singlets under the target-space SUSYs $\hat{Q}_+, \hat{\bar{Q}}_-$, and appear to have no counterpart in the matrix model side. Since the expectation value of operators measuring an RR charge $\langle \Phi_{2k+1}\rangle_0$ does not vanish as seen in (7), the matrix model is considered to correspond to the type IIA theory on a nontrivial background of the (R−, R+) fields. We may introduce the (R−, R+) background in the form of vertex operators, when the strength of the background $(\nu_+ - \nu_-)$ is small.

5 Correspondence Between the Matrix Model and the Type IIA Theory

Correlation functions among integrated vertex operators in the type IIA theory on the trivial background are given by

$$\left\langle \prod_i \mathcal{V}_i \right\rangle = \frac{1}{\mathrm{Vol.(CKV)}} \int \mathcal{D}(x,\varphi,H,\text{ghosts})\, e^{-S_{\mathrm{CFT}}} e^{-S_{\mathrm{int}}} \prod_i \mathcal{V}_i, \tag{27}$$

where

$$S_{\mathrm{CFT}} = \frac{1}{2\pi}\int d^2z \left[\partial x\bar{\partial}x + \partial\varphi\bar{\partial}\varphi + \frac{1}{2}\sqrt{\hat{g}}\hat{R}\varphi + \partial H\bar{\partial}H + (\text{ghosts})\right],$$

$$S_{\mathrm{int}} = \mu_1 \mathcal{V}_B^{(0,0)}(0) \equiv \mu_1 \int d^2z\,\hat{T}^{(0)}_{-\frac{1}{2}}(z)\hat{\bar{T}}^{(0)}_{\frac{1}{2}}(\bar{z}). \tag{28}$$

The 0-picture (NS, NS) "tachyon" is given by

$$\hat{T}^{(0)}_{-\frac{1}{2}}(z) = e^{\pi\beta(ip_{\bar{h}} - i\frac{1}{2}p_{\bar{x}})}\, \frac{i}{\sqrt{2}}\, e^{iH - i\frac{1}{2}x + \frac{1}{2}\varphi}(z),$$
$$\hat{\tilde{T}}^{(0)}_{\frac{1}{2}}(\bar{z}) = e^{-\pi\beta(-ip_h + i\frac{1}{2}p_x)}\, \frac{i}{\sqrt{2}}\, e^{-i\bar{H} + i\frac{1}{2}\bar{x} + \frac{1}{2}\bar{\varphi}}(\bar{z}). \quad (29)$$

We consider correlation functions in the IIA theory on a nontrivial (R−, R+) background as a form

$$\left\langle\left\langle \prod_i \mathcal{V}_i \right\rangle\right\rangle \equiv \left\langle \left(\prod_i \mathcal{V}_i \right) e^{W_{\rm RR}} \right\rangle. \quad (30)$$

The background $W_{\rm RR}$ is invariant under the target-space SUSYs:

$$W_{\rm RR} = (\nu_+ - \nu_-) \sum_{k\in\mathbf{Z}} a_k\, \mu_1^{k+1} \mathcal{V}^{\rm RR}_k,$$
$$\mathcal{V}^{\rm RR}_k \equiv \begin{cases} \int d^2z\, \hat{V}_{k,-1}(z) \hat{\tilde{V}}_{-k,+1}(\bar{z}) & (p_\ell = 1 - |k|,\ k \le 0) \\ \int d^2z\, \hat{V}^{(\rm nonlocal)}_{-k,-1}(z) \hat{\tilde{V}}^{(\rm nonlocal)}_{k,+1}(\bar{z}) & (p_\ell = 1 + |k|,\ k \ge 1) \end{cases} \quad (31)$$

with a_k being numerical constants. Although the nonlocal operators in (31) with $p_\ell > 1$ do not satisfy the Dirac equation constraint on the trivial background, these operators are necessary to make correspondence with the matrix model as we see later. Since the RR background possibly change the on-shell condition, it would be acceptable. We treat the RR background for $(\nu_+ - \nu_-)$ small as

$$\left\langle\left\langle \prod_i \mathcal{V}_i \right\rangle\right\rangle \equiv \left\langle \left(\prod_i \mathcal{V}_i \right) e^{W_{\rm RR}} \right\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \left(\prod_i \mathcal{V}_i \right) (W_{\rm RR})^n \right\rangle, \quad (32)$$

and the picture is adjusted by hand so that the total picture is equal to −2.

In computation of amplitudes in the type IIA theory, we consider the so-called $s = 0$ amplitude in the Liouville theory, which is interpreted as a bulk amplitude insensitive to details of the Liouville wall [21]. Computation in the Liouville theory [19] yields

$$\left\langle \mathcal{V}_B(0)\, \mathcal{V}_\phi(k)\, \mathcal{V}^{\rm RR}_\ell \right\rangle = -g_s^4\, \delta_{k,\ell}\, (2\ln\mu_1)\, e^{i2\pi\beta(-k^2 - \frac{1}{2}k + \frac{1}{4})}, \quad (33)$$

$$\left\langle \mathcal{V}_\phi(k_1), \mathcal{V}_\phi(k_2)\, \mathcal{V}^{\rm RR}_{\ell_1}\, \mathcal{V}^{\rm RR}_{\ell_2} \right\rangle = g_s^4\, (\delta_{\ell_1,k_1+k_2}\delta_{\ell_2,-1} + (\ell_1 \leftrightarrow \ell_2))\, c_L (2\ln\mu_1)^2$$

$$\times\frac{\pi}{2}\left(\frac{(k_1+k_2)!}{k_1!k_2!}\right)^2 e^{-i\pi\beta\{\sum_{i=1}^{2}(k_i+\frac{1}{2})^2+\sum_{i=1}^{2}\ell_i^2\}}. \quad (34)$$

Let us identify the coupling μ_1 of the Liouville interaction S_{int} in (28) with the "cosmological constant" ω by an appropriate shift of the Liouville coordinate. Then, it leads to the identification $N\,\text{tr}(-iB) \cong \frac{1}{4}\mathcal{V}_B^{(0,0)}(0)$, which is consistent to the last line in (25) (up to the choice of the picture) with $\frac{1}{N} \cong g_s$. Also, introducing coefficients c_k, d_k, $\bar{d}_k$, we precisely express the correspondence in (25) and (26) as

$$\Phi_{2k+1} \cong c_k\mathcal{V}_\phi(k), \qquad \Psi_{2k+1} \cong d_k\mathcal{V}_\psi(k), \qquad \bar{\Psi}_{2k+1} \cong \bar{d}_k\mathcal{V}_{\bar{\psi}}(k). \quad (35)$$

We put the overall normalization factor $\mathcal{N}$ in identifying the amplitudes in the matrix-model side and those in the IIA theory side:

$$\langle N\,\text{tr}(-iB)\,\Phi_{2k+1}\rangle_{C,0} \cong \mathcal{N}g_s^{-2}\left\langle\!\!\left\langle\frac{1}{4}\mathcal{V}_B^{(0,0)}(0)\,c_k\mathcal{V}_\phi(k)\right\rangle\!\!\right\rangle. \quad (36)$$

The left hand side is calculated by using (7):

$$\text{(LHS)} = -\frac{1}{4}\partial_\omega\,\langle\Phi_{2k+1}\rangle_0 \sim -(\nu_+-\nu_-)\frac{2^k}{\pi}\frac{(2k+1)!!}{(k+1)!}\,\omega^{k+1}\ln\omega. \quad (37)$$

On the other hand, under a suitable choice of the picture, leading nontrivial contribution for $(\nu_+-\nu_-)$ small to the right hand side is

$$\begin{aligned}
&\frac{1}{4}\mathcal{N}g_s^{-2}c_k\left\langle\mathcal{V}_B(0)\,\mathcal{V}_\phi(k)\,W_{\text{RR}}\right\rangle\\
&= \frac{1}{4}\mathcal{N}g_s^{-4}c_k(\nu_+-\nu_-)\sum_{\ell\in\mathbf{Z}}a_\ell\,\omega^{\ell+1}\left\langle\mathcal{V}_B(0)\,\mathcal{V}_\phi(k)\,\mathcal{V}_\ell^{\text{RR}}\right\rangle\\
&= -\frac{1}{2}(\nu_+-\nu_-)\mathcal{N}c_k\,a_k\,\omega^{k+1}(\ln\omega)\,e^{i2\pi\beta(-k^2-\frac{1}{2}k+\frac{1}{4})} \qquad (38)
\end{aligned}$$

where (33) was used. The identification (36) leads to

$$\mathcal{N}\,\hat{c}_k\hat{a}_k\,e^{i\pi\beta\frac{3}{4}} = \frac{2}{\pi}\frac{(2k+1)!}{k!(k+1)!} \quad (39)$$

with $\hat{c}_k \equiv c_k\,e^{-i\pi\beta(k+\frac{1}{2})^2}$ and $\hat{a}_k \equiv a_k\,e^{-i\pi\beta k^2}$.

Next, let us consider the correspondence

$$\langle\Phi_{2k_1+1}\Phi_{2k_2+1}\rangle_{C,0} \cong \mathcal{N}g_s^{-2}\left\langle\!\left\langle c_{k_1}\mathcal{V}_\phi(k_1)\,c_{k_2}\mathcal{V}_\phi(k_2)\right\rangle\!\right\rangle. \quad (40)$$

Leading nontrivial contribution to the right hand side is obtained from (34) as

$$\begin{aligned} &\mathcal{N} g_s^{-2} c_{k_1}\, c_{k_2} \left\langle \mathcal{V}_\phi(k_1)\, \mathcal{V}_\phi(k_2)\, \frac{1}{2!}(W_{\rm RR})^2 \right\rangle \\ &= \frac{1}{2}\mathcal{N} g_s^{-2} c_{k_1}\, c_{k_2}\, (\nu_+ - \nu_-)^2 \sum_{\ell_1,\ell_2\in\mathbf{Z}} a_{\ell_1}\, a_{\ell_2}\, \omega^{\ell_1+\ell_2+2} \left\langle \mathcal{V}_\phi(k_1)\, \mathcal{V}_\phi(k_2)\, \mathcal{V}^{\rm RR}_{\ell_1}\, \mathcal{V}^{\rm RR}_{\ell_2} \right\rangle \\ &= (\nu_+ - \nu_-)^2 \mathcal{N} g_s^2 c_L\, \hat{c}_{k_1}\, \hat{c}_{k_2}\, \hat{a}_{k_1+k_2}\, \hat{a}_{-1}\, 2\pi \left(\frac{(k_1+k_2)!}{k_1!k_2!}\right)^2 \omega^{k_1+k_2+1}\, (\ln\omega)^2, \end{aligned} \tag{41}$$

while the result of the left hand side is given by (9). Comparing these, we find the same dependence on $\nu_\pm$ and ω for any k_1 and k_2. In addition, we have an equation:

$$\begin{aligned} &\left(\frac{\hat{c}_{k_1}}{(2k_1+1)!}\right)\left(\frac{\hat{c}_{k_2}}{(2k_2+1)!}\right)(\hat{a}_{k_1+k_2}(k_1+k_2)!(k_1+k_2+1)!) \\ &= -\frac{1}{4\pi^3}\frac{1}{\mathcal{N} c_L \hat{a}_{-1}}, \end{aligned} \tag{42}$$

which is solved as

$$\hat{c}_k = \hat{c}_0\, e^{\gamma k}\, (2k+1)!, \qquad \hat{a}_k = \frac{\hat{a}_0\, e^{-\gamma k}}{k!(k+1)!} \qquad (k = 0, 1, 2, \cdots) \tag{43}$$

with γ being a numerical constant and $\hat{c}_0^2 \hat{a}_0 = -\frac{1}{4\pi^3}\frac{1}{\mathcal{N} c_L\, \hat{a}_{-1}}$. Remarkably, (39) is consistent to (43). It serves a quite nontrivial check of the correspondence.

So far, the correspondence seems consistent at the level of planar or tree amplitudes. Furthermore, the consistency is checked in amplitudes containing fermions and the torus partition function [19].

6 Summary and Discussion

We computed planar correlation functions in the double-well SUSY matrix model, and discussed its correspondence to two-dimensional type IIA superstring theory on (R−,R+) background by comparing amplitudes in both sides. This is an interesting example of matrix models for superstrings with target-space SUSY, in which various amplitudes are explicitly calculable.

Furthermore, instanton effects in the matrix model are calculated in [22]. Although such effects are of the order e^{-N} and vanish in the simple large N limit, they are nonvanishing in a double scaling limit

$$N \to \infty, \quad \omega \to 0 \quad \text{with } t \equiv N^{2/3}\omega \text{ fixed.} \tag{44}$$

The result shows that the supersymmetry is spontaneously broken by nonperturbative effects due to instantons. In particular, the instanton effects survive in the double scaling limit, which implies that supersymmetry breaking takes place by nonperturbative dynamics in the target space of the type IIA superstring theory. Corresponding Nambu-Goldstone fermions are identified with $\frac{1}{N}\mathrm{tr}\,\bar{\psi}$ and $\frac{1}{N}\mathrm{tr}\,\psi$ associated with the breaking of Q and $\bar{Q}$, respectively. It is interesting to investigate dynamics of D-branes in the type IIA theory and to reproduce the instanton contributions from the type IIA theory side.

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$f(R)$-Gravity: "Einstein Frame" Lagrangian Formulation, Non-standard Black Holes and QCD-Like Confinement/Deconfinement

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Abstract We consider $f(R) = R + R^2$ gravity interacting with a dilaton and a special non-standard form of nonlinear electrodynamics containing a square-root of ordinary Maxwell Lagrangian. In flat spacetime the latter arises due to a spontaneous breakdown of scale symmetry and produces an effective charge-confining potential. In the $R+R^2$ gravity case, upon deriving the explicit form of the equivalent *local* "Einstein frame" Lagrangian action, we find several physically relevant features due to the combined effect of the gauge field and gravity nonlinearities such as: appearance of dynamical effective gauge couplings and *confinement-deconfinement transition effect* as functions of the dilaton vacuum expectation value; new mechanism for dynamical generation of cosmological constant; deriving non-standard black hole solutions carrying additional constant vacuum radial electric field and with non-asymptotically flat "hedge-hog"-type spacetime asymptotics.

1 Introduction

$f(R)$-gravity models (where $f(R)$ is a nonlinear function of the scalar curvature R and, possibly, of other higher-order invariants of the Riemann curvature tensor $R^{\kappa}_{\lambda\mu\nu}$) are attracting a lot of interest as possible candidates to cure problems in the standard cosmological models related to dark matter and dark energy. For a recent review of $f(R)$-gravity see e.g. [1] and references therein.[1]

[1] The first R^2-model (within the second order formalism), which was proposed as the first inflationary model, appeared in [2].

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In the present contribution we consider $f(R)$-gravity coupled to scalar dilaton ϕ and most notably—to a *non-standard nonlinear gauge field system containing* $\sqrt{-F^2}$ (square-root of standard Maxwell kinetic term; see [3–5]), which is known to produce confining effective potential among quantized charged fermions in flat spacetime [4].

We describe in some detail the explicit derivation of the effective Lagrangian governing the $f(R)$-gravity dynamics in the so called "Einstein frame". The latter means that in terms of an appropriate *conformal rescaling* of the original spacetime metric $g_{\mu\nu} \to h_{\mu\nu} = f'_R\, g_{\mu\nu}$ (where $f'_R = df/dR$) the pertinent gravity part of the effective action assumes the standard form of Einstein–Hilbert action ($\sim R(h)$).

Our main goal is to derive a *local* "Einstein frame" effective Lagrangian *for the matter fields* as well—this is explicitly done for "$R + R^2$-gravity".

Namely, in the special case of $f(R) = R + \alpha R^2$ the passage to the "Einstein frame" entails non-trivial modifications in the effective matter Lagrangian, which *in combination with the special "square-root" gauge field nonlinearity* triggers various physically interesting effects:

- (i) appearance of dynamical effective gauge couplings and *confinement-deconfinement transition effect* as functions of the dilaton vacuum expectation value (v.e.v.);
- (ii) new mechanism for dynamical generation of cosmological constant;
- (iii) non-standard black hole solutions carrying a constant vacuum radial electric field (such electric fields do not exist in ordinary Maxwell electrodynamics) and exhibiting non-asymptotically flat "hedgehog"-type [6] spacetime asymptotics;
- (iv) the above non-standard black holes are shown to obey the first law of black hole thermodynamics;
- (v) obtaining new "tubelike universe" solutions of Levi–Civita–Bertotti–Robinson type $\mathcal{M}_2 \times S^2$ [7].

In addition, as shown in [8] coupling of the gravity/nonlinear gauge field system to *lightlike* branes produces "charge-"hiding" and charge-confining "thin-shell" wormhole solutions displaying QCD-like confinement.

The main motivation for including the nonlinear gauge field term $\sqrt{-F^2}$ comes from the works [9] of 't Hooft, who has shown that in any effective quantum gauge theory, which is able to describe linear confinement phenomena, the energy density of electrostatic field configurations should be a linear function of the electric displacement field in the infrared region (the latter appearing as an "infrared counterterm").

The simplest way to realize 't Hooft's ideas in flat spacetime has been worked out in [3–5] where the following nonlinear modification of Maxwell action has been proposed:

$$S = \int d^4x\, L(F^2) \quad , \quad L(F^2) = -\frac{1}{4}F^2 - \frac{f_0}{2}\sqrt{-F^2}\,, \tag{1}$$

$$F^2 \equiv F_{\mu\nu}F^{\mu\nu} \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ .$$

The square root of the Maxwell kinetic term naturally arises as a result of *spontaneous breakdown of scale symmetry* of the original scale-invariant Maxwell action with f_0 appearing as an integration constant responsible for the latter spontaneous breakdown.

For static field configurations the model (1) yields an electric displacement field $\mathbf{D} = \mathbf{E} - \frac{f_0}{\sqrt{2}}\frac{\mathbf{E}}{|\mathbf{E}|}$ and the corresponding energy density turns out to be $\frac{1}{2}\mathbf{E}^2 = \frac{1}{2}|\mathbf{D}|^2 + \frac{f_0}{\sqrt{2}}|\mathbf{D}| + \frac{1}{4}f_0^2$, so that it indeed contains a term linear w.r.t. $|\mathbf{D}|$ as predicted by the phenomenological theory of ‘t Hooft.

The *non-standard nonlinear* gauge field system (1) produces in flat spacetime [4], when coupled to quantized fermions, a confining effective potential $V(r) = -\frac{\beta}{r} + \gamma r$ (Coulomb plus linear one with $\gamma \sim f_0$) which is of the form of the well-known “Cornell” potential [10] in the phenomenological description of quarkonium systems in QCD.

2 $f(R)$-Gravity in the “Einstein Frame”

Consider $f(R) = R + \alpha R^2 + \ldots$ gravity (possibly with a bare cosmological constant Λ_0) coupled to a dilaton ϕ and a nonlinear gauge field system containing $\sqrt{-F^2}$:

$$S = \int d^4x \sqrt{-g}\Big[\frac{1}{16\pi}\Big(f\big(R(g,\Gamma)\big) - 2\Lambda_0\Big) + L(F^2(g)) + L_D(\phi, g)\Big] , \quad (2)$$

$$L(F^2(g)) = -\frac{1}{4e^2}F^2(g) - \frac{f_0}{2}\sqrt{-F^2(g)} , \quad (3)$$

$$F^2(g) \equiv F_{\kappa\lambda}F_{\mu\nu}g^{\kappa\mu}g^{\lambda\nu} \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (4)$$

$$L_D(\phi, g) = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) \ . \quad (5)$$

where $R(g,\Gamma) = R_{\mu\nu}(\Gamma)g^{\mu\nu}$ and $R_{\mu\nu}(\Gamma)$ is the Ricci curvature in the first order (Palatini) formalism, i.e., the spacetime metric $g_{\mu\nu}$ and the affine connection $\Gamma^\mu_{\nu\lambda}$ are *a priori* independent variables.

The equations of motion resulting from the action (2) read:

$$R_{\mu\nu}(\Gamma) = \frac{1}{f'_R}\left[8\pi T_{\mu\nu} + \frac{1}{2}f\big(R(g,\Gamma)\big)g_{\mu\nu}\right] , \quad (6)$$

$$f'_R \equiv \frac{df(R)}{dR} \quad , \quad \nabla_\lambda\left(\sqrt{-g}\, f'_R g^{\mu\nu}\right) = 0 , \quad (7)$$

$$\partial_\nu\Big(\sqrt{-g}\Big[1/e^2 - \frac{f_0}{\sqrt{-F^2(g)}}\Big]F_{\kappa\lambda}g^{\mu\kappa}g^{\nu\lambda}\Big) = 0 \ . \quad (8)$$

The total energy-momentum tensor is given by:

$$T_{\mu\nu} = \Big[L(F^2(g)) + L_D(\phi, g) - \frac{1}{8\pi}\Lambda_0\Big]g_{\mu\nu} \tag{9}$$
$$+\Big(1/e^2 - \frac{f_0}{\sqrt{-F^2(g)}}\Big)F_{\mu\kappa}F_{\nu\lambda}g^{\kappa\lambda} + \partial_\mu\phi\partial_\nu\phi\ .$$

Equation (7) leads to the relation $\nabla_\lambda\left(f'_R g_{\mu\nu}\right) = 0$ and thus it implies transition to the physical "Einstein frame" metrics $h_{\mu\nu}$ via conformal rescaling of the original metric $g_{\mu\nu}$ [11]:

$$g_{\mu\nu} = \frac{1}{f'_R}h_{\mu\nu} \quad , \quad \Gamma^\mu_{\nu\lambda} = \frac{1}{2}h^{\mu\kappa}\left(\partial_\nu h_{\lambda\kappa} + \partial_\lambda h_{\nu\kappa} - \partial_\kappa h_{\nu\lambda}\right)\ . \tag{10}$$

Using (10) the $f(R)$-gravity equations of motion (6) can be rewritten in the form of *standard* Einstein equations:

$$R_{\mu\nu} = 8\pi\left(T_{\mathrm{eff}\,\mu\nu} - \frac{1}{2}g_{\mu\nu}T_{\mathrm{eff}}\right) \tag{11}$$

where $T_{\mathrm{eff}} = g^{\mu\nu}T_{\mathrm{eff}\,\mu\nu}$ and with effective energy-momentum tensor $T_{\mathrm{eff}\,\mu\nu}$ of the following form:

$$T_{\mathrm{eff}\,\mu\nu} = \frac{1}{f'_R}\Big[T_{\mu\nu} - \frac{1}{4}g_{\mu\nu}T\Big] - \frac{1}{32\pi}g_{\mu\nu}R(T)\ . \tag{12}$$

Here $T \equiv g^{\mu\nu}T_{\mu\nu}$, $R(T)$ is the original scalar curvature determined as function of T from the trace of Eq. (6):

$$8\pi T = Rf'_R - 2f(R)\ , \tag{13}$$

and everywhere in (11)–(13) $g_{\mu\nu}$ and $\Gamma^\mu_{\nu\lambda}$ are understood as functions of the "Einstein frame" metric $h_{\mu\nu}$ (10).

3 Einstein-Frame Effective Action

We are now looking for an effective action $S_{\mathrm{eff}} = \int d^4x\sqrt{-h}\Big[\frac{1}{16\pi}R(h) + L_{\mathrm{eff}}\Big]$, where $R(h)$ is the standard Ricci scalar of the "Einstein frame" metric $h_{\mu\nu}$ and $L_{\mathrm{eff}} \equiv L_{\mathrm{eff}}(h_{\mu\nu}, A_\mu, \phi)$ is a *local function* of the corresponding (matter) fields and of their derivatives, such that it produces in the standard way the original $f(R)$-gravity equations of motion (6) (or equivalently (11)–(13)). L_{eff} will also include an *effective* cosmological constant term irrespective of the presence or absence of a bare cosmological constant Λ_0 in the original $f(R)$-gravity action (2).

L_{eff} must obey the following relation to the "Einstein frame" effective energy-momentum tensor (12):

$$T_{\mathrm{eff}\,\mu\nu} = h_{\mu\nu} L_{\mathrm{eff}} - 2\frac{\partial L_{\mathrm{eff}}}{\partial h^{\mu\nu}} \; . \tag{14}$$

Henceforth we will explicitly consider the simplest nonlinear $f(R)$-gravity case: $f(R) = R + \alpha R^2$ (so that $f'_R = 1 + 2\alpha R$).

The generic form of the matter Lagrangian reads:

$$L_m = L^{(0)} + L^{(1)}(g) + L^{(2)}(g) \; , \tag{15}$$

where the superscripts indicate homogeneity degree w.r.t. $g^{\mu\nu}$. Solving relation (14) by taking into account the conformal rescaling of $g_{\mu\nu}$ (10) and the homogeneity relation (15) we find the following *local* effective "Einstein frame" matter Lagrangian:

$$L_{\mathrm{eff}} = \frac{1}{1 - 64\pi\alpha L^{(0)}}\Big[L^{(0)} + L^{(1)}(h)\big(1 + 16\pi\alpha L^{(1)}(h)\big)\Big] + L^{(2)}(h) \; , \tag{16}$$

where now the superscripts indicate homogeneity degree w.r.t. $h^{\mu\nu}$.

Explicitly, in the case of $R + R^2$-gravity/nonlinear-gauge-field/dilaton system (2)–(5) we have (using shortcut notations $F^2(h) \equiv F_{\kappa\lambda}F_{\mu\nu}h^{\kappa\mu}h^{\lambda\nu}$ and $X(\phi, h) \equiv -\frac{1}{2}h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$):

$$L_{\mathrm{eff}} = -\frac{1}{4e^2_{\mathrm{eff}}(\phi)}F^2(h) - \frac{1}{2}f_{\mathrm{eff}}(\phi)\sqrt{-F^2(h)} \\ + \frac{X(\phi,h)\big(1 + 16\pi\alpha X(\phi,h)\big) - V(\phi) - \Lambda_0/8\pi}{1 + 8\alpha\,(8\pi V(\phi) + \Lambda_0)} \tag{17}$$

with the dynamically generated dilaton ϕ-dependent couplings:

$$\frac{1}{e^2_{\mathrm{eff}}(\phi)} = \frac{1}{e^2} + \frac{16\pi\alpha f_0^2}{1 + 8\alpha\,(8\pi V(\phi) + \Lambda_0)} \; , \tag{18}$$

$$f_{\mathrm{eff}}(\phi) = f_0 \frac{1 + 32\pi\alpha X(\phi,h)}{1 + 8\alpha\,(8\pi V(\phi) + \Lambda_0)} \; . \tag{19}$$

Here is an important observation about the effective action:

$$S_{\mathrm{eff}} = \int d^4x\sqrt{-h}\Big[\frac{R(h)}{16\pi} + L_{\mathrm{eff}}(h, A, \phi)\Big] \; . \tag{20}$$

Even if ordinary kinetic Maxwell term $-\frac{1}{4}F^2$ is absent in the original system ($e^2 \to \infty$ in (3)), such term is nevertheless *dynamically generated* in the "Einstein frame" action (17)–(20)—an explicit manifestation of the *combined effect* of gravitational and gauge field nonlinearities (αR^2 and $-\frac{f_0}{2}\sqrt{-F^2}$):

$$S_{\rm maxwell} = -4\pi\alpha f_0^2 \int d^4x \sqrt{-h} \frac{F_{\kappa\lambda} F_{\mu\nu} h^{\kappa\mu} h^{\lambda\nu}}{1 + 8\alpha\,(8\pi V(\phi) + \Lambda_0)} \; . \tag{21}$$

4 Confinement/Deconfinement Phases

In what follows we consider constant dilaton ϕ extremizing the effective Lagrangian (17) (i.e., the dilaton kinetic term $X(\phi, h)$ will be ignored in the sequel):

$$L_{\rm eff} = -\frac{1}{4e^2_{\rm eff}(\phi)} F^2(h) - \frac{1}{2} f_{\rm eff}(\phi) \sqrt{-F^2(h)} - V_{\rm eff}(\phi) \; , \tag{22}$$

$$V_{\rm eff}(\phi) = \frac{V(\phi) + \frac{\Lambda_0}{8\pi}}{1 + 8\alpha\,(8\pi V(\phi) + \Lambda_0)} \; , \tag{23}$$

$$f_{\rm eff}(\phi) = \frac{f_0}{1 + 8\alpha\,(8\pi V(\phi) + \Lambda_0)} \; , \tag{24}$$

$$\frac{1}{e^2_{\rm eff}(\phi)} = \frac{1}{e^2} + \frac{16\pi\alpha f_0^2}{1 + 8\alpha\,(8\pi V(\phi) + \Lambda_0)} \; . \tag{25}$$

Here we uncover the following important property: *the dynamical couplings and the effective potential are extremized simultaneously*, which is an explicit realization of the so called "least coupling principle" of Damour–Polyakov [12]:

$$\frac{\partial f_{\rm eff}}{\partial\phi} = -64\pi\alpha f_0 \frac{\partial V_{\rm eff}}{\partial\phi} \quad , \quad \frac{\partial}{\partial\phi} \frac{1}{e^2_{\rm eff}} = -(32\pi\alpha f_0)^2 \frac{\partial V_{\rm eff}}{\partial\phi} \quad \to \frac{\partial L_{\rm eff}}{\partial\phi} \sim \frac{\partial V_{\rm eff}}{\partial\phi} \; . \tag{26}$$

Therefore, at the extremum of $L_{\rm eff}$ (22) ϕ must satisfy:

$$\frac{\partial V_{\rm eff}}{\partial\phi} = \frac{V'(\phi)}{\left[1 + 8\alpha\,(\kappa^2 V(\phi) + \Lambda_0)\right]^2} = 0 \; . \tag{27}$$

There are two generic cases:

(A) *Confining phase*: Equation (27) is satisfied for some finite value ϕ_0 extremizing the original potential $V(\phi)$: $V'(\phi_0) = 0$.

(B) *Deconfinement phase*: For polynomial or exponentially growing original potential $V(\phi)$, so that $V(\phi) \to \infty$ when $\phi \to \infty$, we have:

$$\frac{\partial V_{\rm eff}}{\partial\phi} \to 0 \quad , \quad V_{\rm eff}(\phi) \to \frac{1}{64\pi\alpha} = {\rm const} \quad \text{when} \;\; \phi \to \infty \; , \tag{28}$$

i.e., for sufficiently large values of ϕ we find a *"flat region"* in the effective potential V_{eff}. This "flat region" triggers a *transition from confining to deconfinement dynamics*.

Namely, in the confining phase (A) (generic minimum ϕ_0 of the effective dilaton potential) we have shown in [13] that the following *confining potential* (linear w.r.t. r) acts on charged test point-particles:

$$\frac{\sqrt{2}\mathcal{E}|q_0|}{m_0^2} e_{\text{eff}}(\phi_0) f_{\text{eff}}(\phi_0)\, r \;, \tag{29}$$

where $\mathcal{E}, m_0, q_0$ are energy, mass and charge of the test particle.

In the deconfinement phase (B) ("flat-region" of the effective dilaton potential) we have:

$$f_{\text{eff}} \to 0 \quad , \quad e_{\text{eff}}^2 \to e^2 \tag{30}$$

and the effective gauge field Lagrangian (22) reduces to the ordinary *non-confining* one (the "square-root" term $\sqrt{-F^2}$ vanishes):

$$L_{\text{eff}}^{(0)} = -\frac{1}{4e^2} F^2(h) - \frac{1}{64\pi\alpha} \tag{31}$$

with an *induced* cosmological constant $\Lambda_{\text{eff}} = 1/8\alpha$, which is *completely independent* of the bare cosmological constant Λ_0.

5 Non-standard Black Holes and New "Tubelike" Solutions

From the effective Einstein-frame action (20) with L_{eff} as in (22) we find *non-standard* Reissner–Nordström-(anti-)de-Sitter-type black hole solutions in the confining phase (ϕ_0—generic minimum of the effective dilaton potential (23); $e_{\text{eff}}(\phi)$), $f_{\text{eff}}(\phi)$ as in (24)–(25)):

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) , \tag{32}$$

$$A(r) = 1 - \sqrt{8\pi}|Q| f_{\text{eff}}(\phi_0) e_{\text{eff}}(\phi_0) - \frac{2m}{r} + \frac{Q^2}{r^2} - \frac{\Lambda_{\text{eff}}(\phi_0)}{3} r^2 , \tag{33}$$

with *dynamically generated* cosmological constant:

$$\Lambda_{\text{eff}}(\phi_0) = \frac{\Lambda_0 + 8\pi V(\phi_0) + 2\pi e^2 f_0^2}{1 + 8\alpha\left(\Lambda_0 + 8\pi V(\phi_0) + 2\pi e^2 f_0^2\right)} . \tag{34}$$

The black hole's static spherically symmetric electric field contains apart from the Coulomb term an *additional constant radial "vacuum" piece* responsible for confinement (let us stress again that constant vacuum radial electric fields do not exist in ordinary Maxwell electrodynamics):

$$|F_{0r}| = |\mathbf{E}_{\text{vac}}| + \frac{|Q|}{\sqrt{4\pi}\, r^2}\Big(\frac{1}{e^2} + \frac{16\pi\alpha f_0^2}{1 + 8\alpha\,(8\pi V(\phi_0) + \Lambda_0)}\Big)^{-\frac{1}{2}} \tag{35}$$

$$|\mathbf{E}_{\text{vac}}| \equiv \Big(\frac{1}{e^2} + \frac{16\pi\alpha f_0^2}{1 + 8\alpha\,(8\pi V(\phi_0) + \Lambda_0)}\Big)^{-1} \frac{f_0/\sqrt{2}}{1 + 8\alpha\,(8\pi V(\phi_0) + \Lambda_0)}\ . \tag{36}$$

For the special value of ϕ_0 where $\Lambda_{\text{eff}}(\phi_0) = 0$ we obtain Reissner–Nordström-type black hole with a "hedgehog" [6] *non-flat-spacetime* asymptotics: $A(r) \to 1 - \sqrt{8\pi}|Q|\, f_{\text{eff}}(\phi_0)e_{\text{eff}}(\phi_0) \neq 1$ for $r \to \infty$.

Further we obtain Levi–Civitta–Bertotti–Robinson (LCBR) [7] type "tubelike" spacetime solutions with geometries $\mathcal{M}_2 \times S^2$ ($\mathcal{M}_2$—two-dimensional manifold) with metric of the form:

$$ds^2 = -A(\eta)dt^2 + \frac{d\eta^2}{A(\eta)} + r_0^2\big(d\theta^2 + \sin^2\theta d\varphi^2\big)\ \ ,\ \ -\infty < \eta < \infty\ , \tag{37}$$

and constant vacuum "radial" electric field $|F_{0\eta}| = |\mathbf{E}_{\text{vac}}|$, where the size of the S^2-factor is given by (using short-hand $\Lambda(\phi_0) \equiv 8\pi V(\phi_0) + \Lambda_0$):

$$\frac{1}{r_0^2} = \frac{4\pi}{1 + 8\alpha\Lambda(\phi_0)}\Big[\Big(1 + 8\alpha\,\big(\Lambda(\phi_0) + 2\pi f_0^2\big)\Big)\mathbf{E}_{\text{vac}}^2 + \frac{1}{4\pi}\Lambda(\phi_0)\Big]\ . \tag{38}$$

There are three distinct solutions for LBCR (37) where $\mathcal{M}_2 = AdS_2, Rind_2, dS_2$ (two-dimensional anti-de Sitter, Rindler and de Sitter spaces, respectively):

(i) LBCR type solution $AdS_2 \times S^2$ for strong $|\mathbf{E}_{\text{vac}}|$:

$$A(\eta) = 4\pi K(\mathbf{E}_{\text{vac}})\eta^2 \quad ,\quad K(\mathbf{E}_{\text{vac}}) > 0\ , \tag{39}$$

in the metric (37), η being the Poincare patch space-like coordinate of AdS_2, and

$$K(\mathbf{E}_{\text{vac}}) \equiv \Big(1{+}8\alpha\,\big(\Lambda(\phi_0) + 2\pi f_0^2\big)\Big)\mathbf{E}_{\text{vac}}^2 {-} \sqrt{2} f_0|\mathbf{E}_{\text{vac}}| {-} \frac{1}{4\pi}\Lambda(\phi_0)\ . \tag{40}$$

(ii) LBCR type solution $Rind_2 \times S^2$ when $K(\mathbf{E}_{\text{vac}}) = 0$:

$$A(\eta) = \eta \ \text{ for }\ 0 < \eta < \infty \quad \text{or} \quad A(\eta) = -\eta \ \text{ for }\ -\infty < \eta < 0 \tag{41}$$

(iii) LBCR type solution $dS_2 \times S^2$ for weak $|\mathbf{E}_{\text{vac}}|$:

$$A(\eta) = 1 - 4\pi|K(\mathbf{E}_{\text{vac}})|\,\eta^2 \quad ,\quad K(\mathbf{E}_{\text{vac}}) < 0\ . \tag{42}$$

6 Thermodynamics of Non-standard Black Holes

Consider static spherically symmetric metric $ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + r^2\big(d\theta^2 + \sin^2\theta d\varphi^2\big)$ with Schwarzschild-type horizon r_0, i.e., $A(r_0) = 0$, $\partial_r A\big|_{r_0} > 0$ and with $A(r)$ of the general "non-standard" form:

$$A(r) = 1 - c(Q_i) - 2m/r + A_1(r; Q_i) \,, \tag{43}$$

where Q_i are the rest of the black hole parameters apart from the mass m, and $c(Q_i)$ is generically a non-zero constant (as in (33)) responsible for a "hedgehog" [6] *non-flat spacetime asymptotics*.

The so called *surface gravity* κ proportional to Hawking temperature T_h is given by $\kappa = 2\pi T_h = \frac{1}{2}\partial_r A\big|_{r_0}$ (cf., e.g., [14]).

One can straightforwardly derive the first law of black hole thermodynamics for the non-standard black hole solutions with (43):

$$\delta m = \frac{1}{8\pi}\kappa\delta A_H + \tilde{\Phi}_i \delta Q_i \;,\; A_H = 4\pi r_0^2 \;,\; \tilde{\Phi}_i = \frac{r_0}{2}\frac{\partial}{\partial Q_i}\Big(A_1(r_0; Q_i) - c(Q_i)\Big) \,. \tag{44}$$

In the special case of non-standard Reissner–Nordström-(anti-)de-Sitter type black holes (32)–(34) with parameters (m, Q) the conjugate potential in (44):

$$\tilde{\Phi} = \frac{Q}{r_0} - \sqrt{2\pi} f_{\rm eff}(\phi_0) e_{\rm eff}(\phi_0) r_0 \equiv \frac{\sqrt{4\pi}}{e^2_{\rm eff}(\phi_0)} A_0 \,\big|_{r=r_0} \tag{45}$$

(with $e_{\rm eff}(\phi_0)$ and $f_{\rm eff}(\phi_0)$ as in (18)–(19)) is (up to a dilaton v.e.v.-dependent factor) the electric field potential A_0 ($F_{0r} = -\partial_r A_0$) of the nonlinear gauge system on the horizon.

Conclusions

In the present contribution we have uncovered a non-trivial interplay between a special gauge field non-linearity and $f(R)$-gravity. On one hand, the inclusion of the non-standard nonlinear "square-root" Maxwell term $\sqrt{-F^2}$ is the explicit realization of the old "classic" idea of 't Hooft [9] about the nature of low-energy confinement dynamics. On the other hand, coupling of the nonlinear gauge theory containing $\sqrt{-F^2}$ to $f(R) = R + \alpha R^2$ gravity plus scalar dilaton leads to a variety of remarkable effects:

- Dynamical effective gauge couplings and dynamical induced cosmological constant—functions of dilaton v.e.v..

(continued)

- New non-standard black hole solutions of Reissner–Nordström-(*anti*-)de-Sitter type carrying an additional constant vacuum radial electric field, in particular, non-standard Reissner–Nordström type black holes with asymptotically non-flat "hedgehog" behaviour.
- "Cornell"-type *confining* effective potential in charged test particle dynamics.
- Cumulative simultaneous effect of $\sqrt{-F^2}$ and R^2-terms—triggering *transition from confining to deconfinement phase*. Standard Maxwell kinetic term for the gauge field $-F^2$ is *dynamically generated* even when absent in the original "bare" theory.

Furthermore, as we have shown in [8]:

- Coupling to a charged lightlike brane produces a charge-"hiding" wormhole, where a genuinely charged matter source is detected as electrically neutral by an external observer.
- Coupling to two oppositely charged lightlike brane sources produces a two-"throat" wormhole displaying a genuine QCD-like charge confinement.

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The D-Brane Charges of $C_3/\mathbb{Z}_2$

Elaine Beltaos

Abstract The charges of WZW D-branes form a finite abelian group called the charge group. One approach to finding these groups is to use the conformal field theory description of D-branes, i.e. the charge equation. Using this approach, we work out the charge groups for the non-simply connected group $C_3/\mathbb{Z}_2$, which requires knowing the NIM-rep of the underlying conformal field theory.

1 Introduction

String theory remains a significant field of study in theoretical physics. Modern string theories contain both open and closed strings, where an open string can be topologically identified with the interval [0,1]. A major discovery by Polchinski et al. was the requirement that a consistent string theory contain higher dimensional objects (membranes), called Dirichlet-branes, or D-branes, where the endpoints of open strings reside (see e.g. [15]). These branes have physical properties, such as tension, and conserved quantities called *charges*. In this paper, we are interested in these charges for the WZW models. In particular, we determine the charges for the non-simply connected manifold corresponding to $C_3/\mathbb{Z}_2$.

The charges of a given WZW model form a finite abelian group, hence have the (unique) form

$$\mathbb{Z}_{M_1} \oplus \mathbb{Z}_{M_2} \oplus \cdots \oplus \mathbb{Z}_{M_s} \tag{1}$$

for some positive integers M_i such that each $M_i \mid M_{i-1}$. The determination of these groups for string theories on the simply connected Lie group SU(n) has been done in [13,14,16], and the groups were found to be $\mathbb{Z}_M$ where M is given in (3). The charge groups for the non-simply connected group SO$(3) = $ SU$(2)/\mathbb{Z}_2$ were determined in [4, 9] to be $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ if $4|k$ and $\mathbb{Z}_4$ if $4 \nmid k$, whereas [5] found different groups, corresponding to a different supersymmetric CFT. More generally, many of the charge groups for the non-simply connected quotients SU$(n)/\mathbb{Z}_d$, where $d|n$, were

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found in [9,10]. The comparison between the groups for SU(2) ($\mathbb{Z}_{k+2}$, where k is the level of the underlying affine algebra) and the quotient $SO(3) = SU(2)/\mathbb{Z}_2$ already shows how differently behaved the non-simply connected cases are. It would be very interesting to generalize the results of [9, 10] to all of the WZW models, and in this paper, we begin this work by determining the charge groups for $C_3/\mathbb{Z}_2$. There are two main approaches to determining D-brane charges: the (twisted) K-theory approach, and the conformal field theory approach; the latter involves solving Eq. (2) below, which is our focus. To solve this equation in the non-simply connected case requires knowing the NIM-reps (a NIM-rep is a nonnegative integer matrix representation of the fusion ring) of the associated conformal field theories. For the example in this paper, we use the NIM-reps for the C-series which appear in [2].

In a given model, the D-branes are indexed by *boundary states* (these will be discussed in more detail in Sect. 2), and satisfy the *charge equation*

$$\dim \lambda \;\; q_x = \sum_y \mathcal{N}_{\lambda,x}^{y} q_y, \tag{2}$$

where $\dim \lambda$ denotes the Weyl dimension of λ in the algebra $\overline{\mathfrak{g}}$, x, y are boundary states, q_a is the charge associated to the state a, $\mathcal{N}_{\lambda,x}^{y}$ are the NIM-rep coefficients, and the sum is over all boundary states. Unlike classical charges, D-brane charges are preserved only modulo some integer M, so Eq. (2) holds modulo M. In the simply connected case, the integer M has been determined for all algebras and levels [1,3,6]; it is given by the number

$$M(\mathfrak{g}; k) := \frac{k + h^{\vee}}{(k + h^{\vee}, L)}, \tag{3}$$

where $h^{\vee}$ is the dual Coxeter number, and L depends on $\overline{\mathfrak{g}}$. For example, for SU(2), $M(\mathfrak{g}; k) = k + 2$. By a *charge assignment*, we mean an assignment q_x to each boundary state x such that (2) is satisfied modulo M. The set of all charge assignments for a given k forms a group called the *charge group*. The charge group is a $\mathbb{Z}_{M(\mathfrak{g};k)}$-module, and so in particular, the integer M divides $M(\mathfrak{g}; k)$.

In the case of a compact, simple, simply connected Lie group G, such as SU(n), the boundary states are labelled by highest weight representations, and the charge equation becomes

$$\dim \lambda \, q_\mu = \sum_{\nu \in P_+^k} N_{\lambda\mu}^{\nu} q_\nu \;, \tag{4}$$

where $P_+^k(\mathfrak{g}) := \left\{(\lambda_0; \ldots, \lambda_r) \in \mathbb{N}^{r+1} \mid \sum_{\ell=0}^{r} a_\ell^{\vee} \lambda_\ell = k\right\}$ is the set of level k integrable highest weights for $\mathfrak{g}$ at level k, with horizontal subalgebra $\overline{\mathfrak{g}}$ of rank

r, and where $a_\ell^\vee$ are the dual Coxeter labels. Equation (4) has solutions q_λ = (dim λ) q_0[1] modulo $M(\mathfrak{g};k)$, yielding the aforementioned charge group $\mathbb{Z}_{M(\mathfrak{g};k)}$.

In Sect. 2, we review the NIM-rep, and in Sect. 3, we solve the charge equation for the case of $C_3/\mathbb{Z}_2$.

2 The NIM-Rep

In this section, we describe the NIM-rep for the rational conformal field theory with affine algebra $\mathfrak{g}$ at level k. The primaries are in one-to-one correspondence with the set $P_+^k(\mathfrak{g})$. We denote the fundamental weights by Λ_i and the vacuum by 0; this corresponds to the weight Λ_0.

2.1 The Fusion Ring

The S-matrix for the level k algebra $\mathfrak{g}$ is indexed by $P_+^k(\mathfrak{g})$ and is given by the Kac–Peterson formula [12]

$$S_{\lambda\mu} = \kappa^{-r/2} s \sum_{w \in \overline{W}} (\det w) \exp\left[-2\pi i \frac{w(\overline{\lambda + \rho}) \cdot (\overline{\mu + \rho})}{\kappa} \right], \tag{5}$$

where $\overline{W}$ is the $\overline{\mathfrak{g}}$ Weyl group, $\overline{\rho} = (1, \ldots, 1)$ is the $\overline{\mathfrak{g}}$ Weyl vector, $\overline{\lambda}$ denotes the weight $(\lambda_1, \ldots, \lambda_r)$, and κ and s are constants depending on r and k. We define *fusion coefficients* $N_{\lambda\mu}^\nu$ by Verlinde's formula

$$N_{\lambda\mu}^\nu = \sum_{\kappa \in P_+^k} \frac{S_{\lambda\kappa} S_{\mu\kappa} S_{\nu\kappa}^*}{S_{0\kappa}}, \tag{6}$$

where * denotes complex conjugate transpose. The fusion coefficients are non-negative integers, through which we define the *fusion ring*: that is, the unique commutative associative ring with basis P_+^k and ring operation $\lambda * \mu = \sum_\nu N_{\lambda\mu}^\nu \nu$. For example, the $A_1^{(1)}$ fusion coefficients are

$$N_{\lambda\mu}^\nu = \begin{cases} 1 & \text{if } \nu \equiv_2 \lambda + \mu \text{ and } |\lambda - \mu| \le \nu \le \min\{\lambda + \mu, 2k - \lambda - \mu\} \\ 0 & \text{else} \end{cases},$$

[1] We usually normalize this to $q_0 = 1$.

where $\equiv_2$ denotes congruence modulo 2. We also define *fusion matrices* N_λ by $(N_\lambda)_{\mu,\nu} = N_{\lambda\mu}^{\nu}$ for each $\lambda \in P_+^k$. These give a matrix representation of the fusion ring.

To develop the NIM-rep formulas that correspond to the non-simply connected groups, we must consider the fixed points of simple-currents. A *simple-current* is a weight $\nu \in P_+^k$ for which there exists a permutation J of P_+^k such that $N_{\nu,\lambda}^{\mu} = \delta_{\mu,J\lambda}$ with $\nu = J0$. This is equivalent to the set of ν such that $S_{0\nu} = S_{00}$. We identify the weight $J0$ with the permutation J and also call the latter a simple-current. The set $\mathcal{J}$ of all simple-currents of the model with Lie group G forms an abelian group, which in all cases except $E_8^{(1)}$ level 2, is isomorphic to the centre of the universal covering group of G and corresponds to a subset of automorphisms of the extended Dynkin diagram of the affine algebra [7]. For example, the simple-current group for SU(n) is isomorphic to $\mathbb{Z}_n$ and is generated by the order n rotational symmetry of the extended Dynkin diagram of the underlying affine algebra $A_{n-1}^{(1)}$. If J is a simple-current, we denote by $\langle J \rangle$ the subgroup of $\mathcal{J}$ generated by J. The fusion coefficients of (6) obey the symmetry

$$N_{J^a\lambda J^b\mu}^{J^{a+b}\nu} = N_{\lambda\mu}^{\nu} \tag{7}$$

with respect to the simple currents. If $\varphi \in \Phi$, then we get the useful special case $N_{\lambda\varphi}^{\nu} = N_{\lambda\varphi}^{J\nu}$.

2.2 Description of the NIM-Rep

A *NIM-rep* is a nonnegative integer representation of the fusion ring. We assign to each $\lambda \in P_+^k$ a nonnegative integer matrix $\mathcal{N}_\lambda$ such that $\mathcal{N}_\lambda \mathcal{N}_\mu = \sum_{\nu \in P_+^k} N_{\lambda\mu}^{\nu} \mathcal{N}_\nu$, where $\mathcal{N}_0 = I$ and $\mathcal{N}_{C\lambda} = \mathcal{N}_\lambda^t$, where I denotes the identity matrix, and ^{t} denotes transpose. Note that in the case of the C-series, charge-conjugation is trivial, and so the NIM-rep matrices are symmetric. Two NIM-reps $\mathcal{N}$ and $\mathcal{N}'$ are *equivalent* if there exists a permutation matrix P such that for all $\lambda \in P_+^k$, $\mathcal{N}_\lambda' = P^{-1}\mathcal{N}_\lambda P$. A NIM-rep is indexed by boundary states, which we will describe at the end of this section. The fusion matrix representation is a NIM-rep; in this case, the boundary states coincide with the set P_+^k.

As the matrices in a given NIM-rep are normal and commute, they are simultaneously diagonalized by a unitary matrix Ψ. Thus, they satisfy the Verlinde-like formula

$$\mathcal{N}_{\lambda x}^{y} = \sum_{\mu} \frac{\Psi_{x\mu} S_{\lambda\mu} \Psi_{y\mu}^*}{S_{0\mu}} , \tag{8}$$

where the sum is over all *exponents* of the NIM-rep (these will be described below), and x and y are boundary states. A NIM-rep is a homomorphic image of the fusion ring, which is itself a homomorphic image (i.e. quotient) of the representation ring of the underlying simple finite dimensional algebra, and is therefore completely determined by its values at the fundamental weights Λ_i. Thus it suffices to know the NIM-rep coefficients at the fundamental weights.

The S-matrix mentioned above, together with a diagonal matrix T, constitute the *modular data* of the theory.[2] They satisfy several properties. Among them: S is unitary and symmetric; T is of finite order; $S_{0\lambda} \geq S_{00} > 0$ for all $\lambda \in P_+^k$, and $(ST)^3 = S^2 =: C$, where C is an order two permutation matrix called *charge-conjugation*. A *modular invariant* is a nonnegative integer matrix M indexed by P_+^k such that $M_{00} = 1$ and M commutes with both S and T. For a given RCFT, the coefficient matrix $M_{\lambda\mu}$ of the *modular invariant partition function* $\mathcal{Z}(\tau) = \sum_{\lambda,\mu \in P_+^k} M_{\lambda\mu} \chi_\lambda(\tau) \chi_\mu^*(\tau)$, where τ is in the upper half plane and χ_λ, χ_μ are the RCFT characters, specialized to τ, is a modular invariant. For an introduction to modular data and modular invariants, see e.g. [11].

To each simple-current J of an affine algebra is associated the modular invariant

$$M[J]_{\lambda\mu} := \sum_{i=1}^{\mathrm{ord}(J)} \delta_{J^i\lambda,\mu} \delta^{\mathbb{Z}}(Q_J(\lambda) + i r_J) , \tag{9}$$

where $\delta^{\mathbb{Z}}(x) = 1$ if $x \in \mathbb{Z}$ and 0 else, and $Q_J(\lambda)$ and r_J are rational numbers that depend on $\mathfrak{g}$. This is a modular invariant partition function for a rational conformal field theory precisely when $T_{J0,J0} T_{00}^*$ is an nth root of unity (where n = ord (J)), corresponding to the model with group $G/\langle J \rangle$. The number of maximally symmetric, untwisted D-branes is equal to the trace of $M[J]$.

For example, the $\mathcal{D}$-series modular invariant for $A_1^{(1)}$, corresponding to the order-2 simple current is

$$\mathcal{D}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} .$$

Let J be a simple-current of order n. We denote by $[\lambda]$ the J-orbit $\{J^i\lambda \mid i = 0, \ldots, n-1\}$, and by ord λ the order of the stabilizer of λ in $\langle J \rangle$. The boundary states are then pairs $([\lambda], i)$, where $1 \leq i \leq$ ord λ. This Lie-theoretic interpretation was given by [8, 9]. The exponents of a modular invariant are members of the multi-set

[2]These matrices control the modularity of the RCFT characters and yield a representation of the modular group $\mathrm{SL}_2(\mathbb{Z})$ via the assignment

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto S \quad ; \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto T .$$

$\mathcal{E}(M)$ consisting of all λ with $M_{\lambda\lambda} \neq 0$, appearing with multiplicity $M_{\lambda\lambda}$. We will associate $\mathcal{E}(M)$ with the set $\{(\lambda, i) \mid 1 \leq i \leq M_{\lambda\lambda}\}$.

3 C_3 Charge Groups

In this section, we work out the charges of the maximally symmetric, untwisted D-branes for the algebra $C_3^{(1)}$, with the order two simple-current. We first give the $C_3^{(1)}$ data and NIM-rep.

3.1 The $C_3^{(1)}$ Data

The level k highest weights are labelled by the set $P_+^k(C_3^{(1)}) = \{(\lambda_0; \lambda_1, \lambda_2, \lambda_3) \mid \sum \lambda_i = k\}$. There is one simple-current, of order two, which has fixed points when k is even. These fixed points are in one-to-one correspondence with the set $\Phi :=$ $\{\varphi = (\varphi_0; \varphi_1, \varphi_1, \varphi_0) \mid \varphi_0 + \varphi_1 = k/2\}$, which has cardinality $k/2 + 1$.

Thus, in the present case, ord $\lambda = 1$ if $\lambda \notin \Phi$ and 2 if $\lambda \in \Phi$. If $\lambda \notin \Phi$, then we simply write $[\lambda]$ for the pair $([\lambda], 1)$. For example, when $k = 2$, there are two fixed points, namely (1; 0, 0, 1) and (0; 1, 1, 0), and there are eight boundary states: [2; 0, 0, 0], [1; 1, 0, 0], [1; 0, 1, 0], [0; 2, 0, 0], ([1; 0, 0, 1], 1), ([1; 0, 0, 1], 2), ([0; 1, 1, 0], 1), ([0; 1, 1, 0], 2).

Let $\varphi \in \Phi$. Then $\varphi = (\varphi_0; \varphi_1, \varphi_1, \varphi_0)$, where $\varphi_0 + \varphi_1 = k/2$. We define $\tilde{\varphi}$ to be the *truncated fixed point* $\tilde{\varphi} = (\varphi_0; \varphi_1)$, which lies in $P_+^{k/2}(C_1^{(1)})$. The $C_3^{(1)}$ NIM-rep is then given by the equations

$$\begin{aligned}
\mathcal{N}_{\lambda[\mu]}^{[\nu]} &= N_{\lambda\mu}^{\nu} + N_{\lambda\mu}^{J\nu} \\
\mathcal{N}_{\lambda([\varphi],i)}^{[\nu]} &= N_{\lambda\varphi}^{\nu} \\
\mathcal{N}_{\Lambda_i([\varphi],i)}^{([\psi],j)} &= \frac{1}{2} N_{\Lambda_i\varphi}^{\psi} \quad, i = 1,3 \\
\mathcal{N}_{\Lambda_2([\varphi],i)}^{([\psi],j)} &= \frac{1}{2}\left(N_{\Lambda_2\varphi}^{\psi} + (-1)^{i+j+1} \tilde{N}_{\tilde{\Lambda}_1\tilde{\varphi}}^{\tilde{\psi}}\right)
\end{aligned}$$

where tildes denote $C_1^{(1)}$ level $k/2$ quantities, and $\tilde{\varphi}$, $\tilde{\psi}$ are the truncated fixed points. The last two equations are given in [2].

3.2 Charge Equation for $C_3^{(1)}$

In this subsection, we sketch the solution to Eq. (2) for $C_3^{(1)}$, where k is even. It is sufficient that (2) is satisfied for the three fundamental weights, i.e., that the following three equations are satisfied:

$$6q_x = \sum_y \mathcal{N}_{\Lambda_1,x}^{y} q_y \tag{10}$$

$$14q_x = \sum_y \mathcal{N}_{\Lambda_2,x}^{y} q_y \tag{11}$$

$$14q_x = \sum_y \mathcal{N}_{\Lambda_3,x}^{y} q_y \ . \tag{12}$$

Throughout this section, we let $M_k := M(\mathfrak{g}; k)$.

For each weight, define $t(\lambda) = \lambda_1 + \lambda_3 \pmod 2$. The simple-current symmetry (7) gives a grading

$$N_{\lambda,\mu}^{\nu} \neq 0 \implies t(\lambda) + t(\mu) = t(\nu) \pmod 2$$

of the fusion coefficients.

Substituting $x = [0]$ and $\lambda \notin \Phi$ into (2) yields $q_{[\lambda]} = \dim \lambda \, q_{[0]}$. Letting $q_{[0]} \neq 0$ (the assignment $q_{[0]} = 0$ leads to the trivial group), we normalize this to $q_{[0]} = 1$, and so we have the assignment

$$q_{[\lambda]} = \dim \lambda \bmod M_k \,\forall\, \lambda \notin \Phi \ . \tag{13}$$

This gives us a copy of $\mathbb{Z}_{M_k}$ in (1). Now substituting $x = [0]$ and $\lambda = \varphi \in \Phi$, we have

$$q_{[\varphi,1]} + q_{[\varphi,2]} = \dim \varphi \mod M_k \ . \tag{14}$$

Finally, substituting $x = [\varphi, i]$ into Eqs. (10), (11), (12) gives a system of equations whose solution depends on the parity of M. If M_k is odd, then $q_{([\varphi],i)} = 0 \bmod M_k$, so the charge group is $\mathbb{Z}_{M_k}$, which agrees with the simply connected case. However, if M_k is even, then $q_{([\varphi],1)}$, $q_{([\varphi],2)}$ are both $M_k/2$ or 0 $(\mathrm{mod} M_k)$. Therefore, the charge group is $\mathbb{Z}_M \oplus \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$, where there are $|\Phi| = k/2 + 1$ copies of $\mathbb{Z}_2$.

4 Concluding Remarks

In this paper, we found the charge groups for the non-simply connected group $C_3/\mathbb{Z}_2$, via the conformal field theory description of D-branes, namely by solving Eq. (2) modulo the smallest M that we could find, which uses the NIM-rep formulas of [2]. It should be noted that C_r for $r > 3$ involve more number-theoretical subtleties, and a solution will be more challenging to obtain. As well, C_3 is an example of a non-pathological case (i.e. dim $J0 = 1 \pmod{M_k}$); already C_2 is pathological, and these pathological cases will require a different approach. In the case of the simply connected groups, the charge groups found via the conformal field theory description agreed with those found by the K-theory method. It would be interesting to work out the K-theory calculation for the non-simply connected groups and compare with the result in this paper.

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On Robertson Walker Solutions in Noncommutative Gauge Gravity

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Abstract Robertson–Walker solution is presented in terms of gauge fields in a de Sitter gauge theory of gravity (Chamseddine and Mukhanov, J High Energy Phys 3:033, 2010). For a vanishing torsion analogous (Zet et al., Int J Mod Phys C15(7):1031, 2004) we present the field strength tensor and the scalar analogous of the Ricci scalar. Following the noncommutative generalization (Chamseddine, Phys Lett B504:33, 2001) for the de Sitter gauge theory of gravity we study how the noncommutativity of space-time deform, through noncommutative parameters, the homogeneous isotropic solution of the commutative gauge theory of gravity. The study is realized with special conceived analytical procedures under GRTensorII for Maple that we designed for the specific quantities of the gauge theory of gravity (Babeti (Pretorian), Rom J Phys 57(5–6):785, 2012). Noncommutative deformations are obtained using a star product deformation of space time and the Seiberg–Witten map to express the deformed fields in terms of undeformed ones and noncommutative parameter. We analyze a space-time (Fabi et al., Phys Rev D78:065037, 2008) and a space-space noncommutativity. The gauge fields, the field strength tensor and the noncommutative analogue of the metric tensor, the noncommutative scalar analog to Ricci scalar are followed until second order in noncommutative parameter.

1 Introduction

We work with the model of gauge theory of gravitation that has the de-Sitter (DS) group SO(4,1) (10-dimensional) [3] as local symmetry and as base manifold, the commutative 4-dimensional Minkowski space-time, endowed with spherical symmetry:

$$ds^2 = -dt^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{1}$$

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The 10 infinitesimal generators of DS group $M_{AB} = -M_{BA}$, A,B=0,1,2,3,5, can be identified with the translations $P_a = -M_{a5}$ and the Lorentz rotations $M_{ab} = -M_{ba}$, a, b = 0,1,2,3. Therefore, we have 10 corresponding (non-deformed) gauge fields (or potentials) $\omega_\mu^{AB}(x) = -\omega_\mu^{BA}(x)$. The gauge fields are identified with the four tetrad fields (the gauge field of translational generator), $\omega_\mu^{a5}(x) = e_\mu^a(x)$, and the six antisymmetric spin connection $\omega_\mu^{AB}(x) = -\omega_\mu^{BA}(x)$. The field strength tensor [5], associated with the gauge fields $\omega_\mu^{AB}(x)$, which takes its values in the Lie algebra of the DS group (Lie algebra-valued tensor) can be separated into a tensor equivalent to the torsion and one equivalent to the curvature tensor:

$$F_{\mu\nu}^a = \partial_{[\mu} e_{\nu]}^a + \omega_{[\mu}^{ab} e_{\nu]}^c \eta_{bc}, \tag{2}$$

$$F_{\mu\nu}^{ab} = \partial_{[\mu} \omega_{\nu]}^{ab} + \omega_{[\mu}^{ac} \omega_{\nu]}^{db} \eta_{cd} + 4\lambda^2 e_{[\mu}^a e_{\nu]}^b, \tag{3}$$

with the brackets indicate antisymmetrization of indices and λ a real parameter.

The SO(4,1) group as the symmetry underlying the Universe give the appearance of a non-vanishing cosmological constant Λ, which is determined by the real parameter λ ($4\lambda^2 = -\Lambda/3$). When we consider the limit $\lambda \to 0$ i.e. the group contraction process, the de-Sitter group SO(4,1) reduces to the Poincaré group ISO(3,1), obtaining the commutative Poincaré gauge theory of gravitation.

In the gauge theory of gravitation the gauge invariant action $S = \frac{1}{16\pi G}\int d^4x\, e\, F$ is expressed in terms of gauge fields. The scalar $F = F_{\mu\nu}^{ab} \bar{e}_a^\mu \bar{e}_b^\nu$, with $e_\mu^a \bar{e}_b^\mu = \delta_b^a$, is corresponding to the Ricci scalar and we have $e = det(e_\mu^a)$. Corresponding to the metric tensor it can be defined the tensor $g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b$. Although the gauge invariant action appears to depend on the non-diagonal ω_μ^{AB} it is a function on $g_{\mu\nu}$ only.

2 Robertson–Walker Solution in the Commutative Theory

In order to apply the gauge theory formalism for gravity we choose a particular ansatz for gauge fields [6]:

$$\begin{aligned} e_\mu^0 &= (N(t), 0, 0, 0)\,, \qquad e_\mu^1 = \left(0, a(t)/\sqrt{1-kr^2}, 0, 0\right), \\ e_\mu^2 &= (0, 0, ra(t), 0)\,, \qquad e_\mu^3 = (0, 0, 0, ra(t)\sin\theta)\,, \end{aligned} \tag{4}$$

$$\begin{aligned} \omega_\mu^{01} &= (0, U(t,r), 0, 0)\,, \qquad \omega_\mu^{02} = (0, 0, V(t,r), 0)\,, \\ \omega_\mu^{03} &= (0, 0, 0, W(t,r)\sin\theta)\,, \qquad \omega_\mu^{12} = (0, 0, Y(t,r), 0)\,, \\ \omega_\mu^{13} &= (0, 0, 0, Z(r)\sin\theta)\,, \qquad \omega_\mu^{23} = (0, 0, 0, -\cos\theta)\,, \end{aligned} \tag{5}$$

with the constant k and the functions U, V, W, Y, Z of time t and 3D radius r. These gauge fields correspond to Robertson–Walker $g_{\mu\nu}$ and lead to the following non-null components of $F^a_{\mu\nu}$ and $F^{ab}_{\mu\nu}$:

$$F^1_{01} = \frac{\dot{a}}{\sqrt{1-kr^2}} - UN, \qquad F^2_{02} = r\dot{a} - VN, \qquad F^3_{03} = (r\dot{a} - WN)\sin\theta,$$
$$F^2_{12} = a\left(1 + \frac{Y}{\sqrt{1-kr^2}}\right), \qquad F^3_{13} = a\sin\theta\left(1 + \frac{Z}{\sqrt{1-kr^2}}\right), \tag{6}$$

respectively

$$F^{12}_{12} = \left(\frac{\partial Y}{\partial r} + UV - \frac{4\lambda^2 r a^2}{\sqrt{1-kr^2}}\right), \qquad F^{02}_{12} = \frac{\partial V}{\partial r} + UY,$$
$$F^{13}_{13} = \left(\frac{\partial Z}{\partial r} + UW - 4\lambda^2 r a^2\right)\sin\theta, \qquad F^{03}_{13} = \left(\frac{\partial W}{\partial r} + UZ\right)\sin\theta,$$
$$F^{01}_{01} = \frac{\partial U}{\partial t} - \frac{4\lambda^2 N a}{\sqrt{1-kr^2}}, \qquad F^{03}_{23} = (W - V)\cos\theta, \tag{7}$$
$$F^{23}_{23} = \left(1 - ZY - 4\lambda^2 r^2 a^2 + WV\right)\sin\theta, \qquad F^{13}_{23} = (Z - Y)\cos\theta,$$
$$F^{12}_{02} = \frac{\partial Y}{\partial t}, \qquad F^{02}_{02} = \frac{\partial V}{\partial t} - 4\lambda^2 N r a,$$
$$F^{03}_{03} = \left(\frac{\partial W}{\partial t} - 4\lambda^2 N r a\right)\sin\theta, \qquad F^{13}_{03} = \frac{\partial Z}{\partial t}\sin\theta,$$

where $\dot{a}$ is the derivative of $a(t)$ with respect to the variable t.

Following the case of null components $F^a_{\mu\nu}$ of the strength tensor we obtain some constraints on the arbitrary functions introduced in the spin connection components:

$$U(t,r) = \frac{\dot{a}(t)}{N(t)\sqrt{1-kr^2}}, \qquad V(t,r) = W(t,r) = \frac{r\dot{a}(t)}{N(t)}$$
$$Y(t,r) = Z(t,r) = -\sqrt{1-kr^2}. \tag{8}$$

Therefore, the spin connection components ω^{ab}_μ are determined by the tetrads e^a_μ in the case of null torsion. The scalar F, that define the action, depends on field strength tensor associated with the gauge fields and with the constraints (8) is

$$F = 6\frac{a\ddot{a}N - a\dot{a}\dot{N} + kN^3 + \dot{a}^2 N - 8\lambda^2 a^2 N^3}{a^2 N^3}. \tag{9}$$

With the supplementary condition N(t) = 1, for the case of null equivalent torsion, the spin connection components determined by the tetrads (4) are:

$$\omega_\mu^{01} = \left(0, \dot{a}(t)/\sqrt{1-kr^2}, 0, 0\right),\ \omega_\mu^{02} = (0, 0, r\dot{a}(t), 0)\,,$$
$$\omega_\mu^{03} = (0, 0, 0, r\dot{a}(t)\sin\theta)\,,\ \omega_\mu^{12} = \left(0, 0, -\sqrt{1-kr^2}, 0\right),$$
$$\omega_\mu^{13} = \left(0, 0, 0, -\sqrt{1-kr^2}\sin\theta\right),\ \omega_\mu^{23} = (0, 0, 0, -\cos\theta)\,, \tag{10}$$

and determine the following nonvanishing components $F_{\mu\nu}^{ab}$ of the field strength tensor

$$F_{12}^{12} = \frac{r}{\sqrt{1-kr^2}}\left(k - 4\lambda^2 a^2 + \dot{a}^2\right), \quad F_{13}^{13} = \frac{r\sin\theta}{\sqrt{1-kr^2}}\left(k - 4\lambda^2 a^2 + \dot{a}^2\right),$$
$$F_{01}^{01} = \frac{\ddot{a} - 4\lambda^2 a}{\sqrt{1-kr^2}}, \qquad F_{23}^{23} = r^2\sin\theta\left(k - 4\lambda^2 a^2 + \dot{a}^2\right), \tag{11}$$
$$F_{02}^{02} = r\left(\ddot{a} - 4\lambda^2 a\right), \qquad F_{03}^{03} = r\sin\theta\left(\ddot{a} - 4\lambda^2 a\right).$$

The resulting scalar F:

$$F = 6\left(\frac{\ddot{a}}{a} + \frac{k}{a^2} + \left(\frac{\dot{a}}{a}\right)^2 - 8\lambda^2\right), \tag{12}$$

for $\lambda \to 0$ is the known Ricci scalar for the Robertson–Walker metric.

3 Deformed Gauge Fields and Noncommutative Analogous Metric Tensor

In order to calculate the effect of the noncommutativity on the gauge fields we work with the canonical deformation of the Minkowski space-time based on $[x^\mu, x^\nu]_* = i\Theta^{\mu\nu}$ with real constant deformation parameter $\Theta^{\mu\nu} = -\Theta^{\nu\mu}$. As (star) $*$ product between the fields defined on this space-time we use the (associative) Moyal product, $* = e^{\frac{i}{2}\Theta^{\mu\nu}\overleftarrow{\partial_\mu}\overrightarrow{\partial_\nu}}$.

The noncommutative gauge theory (as the commutative one) is described in terms of gauge fields (or potentials), denoted here by $\hat{\omega}_\mu^{AB}(x, \Theta)$ and field strengths, denoted here by $\hat{F}_{\mu\nu}^{AB}$, that depend on deformation parameter of noncommutative coordinate algebra. Using the Seiberg–Witten map one expand the noncommutative gauge fields, that transform according to the noncommutative algebra, in terms of commutative gauge fields, that transform according to the commutative algebra. In powers of $\Theta^{\mu\nu}$, [2], (the (n) subscript indicates the n-th order in $\Theta^{\mu\nu}$) the tetrad fields, the spin connections and the field strength tensor are:

$$\hat{e}^a_\mu(x,\Theta) = e^a_\mu(x) + e_{(1)}{}^a_\mu(x) + e_{(2)}{}^a_\mu(x) + \dots$$
$$\hat{\omega}^{ab}_\mu(x,\Theta) = \omega^{ab}_\mu(x) + \omega_{(1)}{}^{ab}_\mu(x) + \omega_{(2)}{}^{ab}_\mu(x) + \dots \tag{13}$$
$$\hat{F}^{AB}_{\mu\nu}(x,\Theta) = F^{AB}_\mu(x) + F_{(1)}{}^{AB}_{\mu\nu}(x) + F_{(2)}{}^{AB}_{\mu\nu}(x) + \dots$$

The first order in noncommutative parameter of gauge fields is expressed in terms of zero order (from the commutative theory) gauge fields and zero order field strength tensor. For the case of $F^a_{\mu\nu} = 0$ in the zero order and using the usual brackets for the anticommutator we have:

$$e_{(1)}{}^a_\mu = -\frac{i}{4}\Theta^{\rho\sigma}\left(\omega^{ab}_\rho \partial_\sigma e^c_\mu + (\partial_\sigma \omega^{ab}_\mu + F^{ab}_{\sigma\mu})e^c_\rho\right)\eta_{bc}, \tag{14}$$

$$\omega_{(1)}{}^{ab}_\mu = -\frac{i}{4}\Theta^{\rho\sigma}\left\{\omega_\rho, \partial_\sigma\omega_\mu + F_{\sigma\mu}\right\}^{ab}. \tag{15}$$

The first order of field strength tensors depend on zero and first order gauge fields as:

$$F_{(1)}{}^a_{\mu\nu} = \partial_{[\mu} e_{(1)}{}^a_{\nu]} + \left(\omega_{(1)}{}^{ab}_{[\mu} e^c_{\nu]} + \omega^{ab}_{[\mu} e_{(1)}{}^c_{\nu]} + \omega^{ab}_{[\mu} *_{(1)} e^c_{\nu]}\right)\eta_{bc}, \tag{16}$$

$$F_{(1)}{}^{ab}_{\mu\nu} = \partial_{[\mu}\omega_{(1)}{}^{ab}_{\nu]} + \left[\omega_{(1)\mu}, \omega_\nu\right]^{ab} + \left[\omega_\mu, \omega_{(1)\nu}\right]^{ab} + \left[\omega_\mu, \omega_\nu\right]^{ab}_{*(1)}. \tag{17}$$

Even we have a vanishing $F^a_{\mu\nu}$ in the zero order at order one (16) is nonvanishing.

In order to be applied for the particular tetrad fields (4) and spin connection (10), all formulas are implemented in an analytical procedure conceived in GR Tensor II for Maple. Instead to present the second order terms for the gauge fields and field strength tensor as usually, they come in the particular form of analytical procedure that contain suggestive notations.

```
>grdef(`ev2{^a mu}:=-I/8*Tn{^rho^sig}*(om1{^a^c rho}*
ev{^d mu,sig}+om{^a^c rho}*(ev1{^d mu,sig}
+F1a{^d sig mu})+(I/2)*Tn{^lam^tau}*
om{^a^c rho,lam}*ev{^d mu,sig,tau}+
(om1{^a^c mu,sig}+F1ab{^a^c sig mu})*ev{^d rho}+
(om{^a^c mu,sig}+Fab{^a^c sig mu})*ev1{^d rho}+
(I/2)*Tn{^lam^tau}*((om{^a^c mu,sig,lam}+
Fab{^a^c sig mu,lam})*ev{^d rho,tau}))*eta1{c d}`);

>grdef(`om2{^a^b mu}:=(-I/8)*Tn{^rho^sig}*
(om1{^a^c rho}*(om{^b^d mu,sig}+Fab{^d^b sig mu})+
(om{^a^c mu,sig}+Fab{^a^c sig mu})*om1{^d^b rho}
+om{^a^c rho}*(om1{^d^b mu,sig}+F1ab{^d^b sig mu})
+(om1{^a^c mu,sig}+F1ab{^a^c sig mu})*om{^d^b rho}
+(I/2)*Tn{^lam^tau}*(om{^a^c rho,lam}*
(om{^d^b mu,sig,tau}+Fab{^d^b sig mu,tau})
+(om{^a^c mu,sig,lam}+Fab{^a^c sig mu,lam})*
omega{^d^b rho,tau}))*eta1{c d}`);
```

```
>grdef(`F2a{^a mu nu}:= ev2{^a nu,mu}-ev2{^a mu,nu}+
(om{^a^c mu}*ev2{^d nu}-om{^a^c nu}*ev2{^d mu}+
om{^a^c mu}*ev{^d nu}-om2{^a^c nu}*ev{^d mu}+
om{^a^c mu}*ev1{^d nu}-om1{^a^c nu}*ev1{^d mu}+
(I/2)*Tn{^rho^sig}*(om{^a^c mu,rho}*ev1{^d nu,sig}
-om{^a^c nu,rho}*ev1{^d mu,sig}+om1{^a^c mu,rho}*
ev{^d nu,sig}-om1{^a^c nu,rho}*ev{^d mu,sig})+
(-1/8)*Tn{^rho^sig}*Tn{^lam^tau}*
(om{^a^c mu,rho,lam}*ev{^d nu,sig,tau}
-om{^a^c nu,rho,lam}*ev{^d mu,sig,tau}))*eta1{c d}');

>grdef(`F2ab{^a^b mu nu}:=
om2{^a^b nu,mu}-om2{^a^b mu,nu}+
(om{^a^c mu}*om2{^d^b nu}-om2{^a^c nu}*om{^d^b mu}+
om2{^a^c mu}*om{^d^b nu}-om{^a^c nu}*om2{^d^b mu}+
om1{^a^c mu}*om1{^d^b nu}-om1{^a^c nu}*om1{^d^b mu}+
(I/2)*Tn{^rho^sig}*(om{^a^c mu,rho}*om1{^d^b nu,sig}
-om1{^a^c nu,rho}*om{^d^b mu,sig}+om1{^a^c mu,rho}*
om{^d^b nu,sig}-om{^a^c nu,rho}*om1{^d^b mu,sig})+
(-1/8)*Tn{^rho^sig}*Tnc{^lam^tau}*
(om{^a^c mu,rho,lam}*om{^d^b nu,sig,tau}
-om{^a^c nu,rho,lam}*om{^d^b mu,sig,tau}))*
eta1{c d}');
```

The noncommutative analogue of the metric tensor is defined using the hermitian conjugate of tetrads: $\hat{g}_{\mu\nu} = \frac{1}{2}\eta_{ab}\left(\hat{e}^a_\mu * \hat{e}^{b*}_\nu + \hat{e}^b_\nu * \hat{e}^{a*}_\mu\right)$. The noncommutative scalar analog to F is $\hat{F} = \hat{e}^\mu_a * \hat{F}^{ab}_{\mu\nu} * \hat{e}^\nu_b$, where $\hat{e}^\mu_a$ is the $*$inverse of $\hat{e}^a_\mu$. The part of analytical procedure for these quantities can be read in [1].

For arbitrary $\Theta^{\mu\nu}$, the deformed metric is not diagonal even if the commutative one has this property. We examine how the noncommutativity modifies the structure of the gravitational field for the particular case (4) in the situation of (10) for an time-space noncommutativity and an space-space noncommutativity.

For the time-space noncommutativity we choose the t-r noncommutativity ($\Theta^{tr} = -\Theta^{rt} = \Theta$) and applying the above formalism for the tetrad fields (4), spin connection (10) we obtain for the noncommutative analogue of the metric tensor

$$\hat{g}_{00} = -1 + \Theta^2 \frac{6\ddot{a}^2 + 5\dot{a}\dddot{a}}{16(1-kr^2)} + \lambda^2\Theta^2 \frac{8\lambda^2 a^2 + 3(\dot{a}^2 - 2a\ddot{a})}{4(1-kr^2)} + \mathcal{O}(\Theta^4)$$

$$\hat{g}_{11} = \frac{a^2}{1-kr^2} - \Theta^2 \frac{(1-kr^2)\left(\dot{a}^4 + 13a\dot{a}^2\ddot{a} + 12a^2\dot{a}\dddot{a} + 16a^2\dot{a}^2\right) + \left(3kr^2+4\right)k\dot{a}^2 + 4ka\ddot{a}(1+kr^2)}{16(1-kr^2)^3} - \\ - \lambda^2\Theta^2 \frac{a^2\left(8\lambda^2a^2 - 10\dot{a}^2 - 12a\ddot{a}\right)}{4(1-kr^2)^2} + \mathcal{O}(\Theta^4)$$

$$
\begin{aligned}
\hat{g}_{22} =& r^2a^2 + \Theta^2 \frac{1}{16}\left(4a\ddot{a} + 5\dot{a}^2 - ar^2\frac{8a\ddot{a}^2 + 9\dot{a}^2\ddot{a} + 4k\ddot{a} + 4a\dot{a}\dddot{a}}{1-kr^2}\right) - \\
&- \lambda^2\Theta^2\frac{a^2r^2}{4(1-kr^2)}\left(8\lambda^2a^2 - (2k + 7a\ddot{a} + 7\dot{a}^2)\right) + \mathcal{O}(\Theta^4) \\
&\hat{g}_{33} = \hat{g}_{22}\sin^2\theta \\
&\hat{g}_{01} = -\Theta^2\frac{kr\dot{a}\ddot{a}}{2(1-kr^2)^2} + \lambda^2\Theta^2\frac{9kr\dot{a}a}{8(1-kr^2)^2} + \mathcal{O}(\Theta^4).
\end{aligned} \tag{18}
$$

We have one nonzero off diagonal component in (18) and when we treat the analogue of the metric tensor as a standard metric tensor it is associated with an inhomogeneous isotropic space-time with respect to the worldline at r $= 0$, even if the parameter λ is nonzero. When the scale factor is a constant the noncommutative second order off diagonal components are null. For $\lambda \to 0$ the noncommutative analogue of the metric tensor

$$
\begin{aligned}
\hat{g}_{00} &= -1 + \Theta^2\tfrac{6\ddot{a}^2+5\dot{a}\dddot{a}}{16(1-kr^2)} + \mathcal{O}(\Theta^4) \\
\hat{g}_{11} &= \tfrac{a^2}{1-kr^2} - \Theta^2\tfrac{(1-kr^2)\left(\dot{a}^4+13a\dot{a}^2\ddot{a}+12a^2\dot{a}\dddot{a}+16a^2\dot{a}^2\right)+\left(3kr^2+4\right)k\dot{a}^2+4ka\ddot{a}(1+kr^2)}{16(1-kr^2)^3} \\
&\quad + \mathcal{O}(\Theta^4) \\
\hat{g}_{22} &= r^2a^2 + \Theta^2\tfrac{1}{16}\left(4a\ddot{a} + 5\dot{a}^2 - ar^2\tfrac{8a\ddot{a}^2+9\dot{a}^2\ddot{a}+4k\ddot{a}+4a\dot{a}\dddot{a}}{1-kr^2}\right) + \mathcal{O}(\Theta^4) \\
\hat{g}_{33} &= \hat{g}_{22}\sin^2\theta \\
\hat{g}_{01} &= -\Theta^2\tfrac{kr\dot{a}\ddot{a}}{2(1-kr^2)^2} + \mathcal{O}(\Theta^4).
\end{aligned} \tag{19}
$$

has no second order corrections for a constant scale factor. In the case of linear expansion for $\lambda \to 0$ small t can be defined using second order analysis of singular points of ordinary space time scalar curvature [4].

For the space-space noncommutativity we choose the r-θ noncommutativity ($\Theta^{r\theta} = -\Theta^{\theta r} = \Theta$) and, up to second order in Θ, we find, for $\lambda \to 0$, the following noncommutative tetrad fields after substituting into (13):

$$
\hat{e}^0_\mu = \left(1 - \Theta^2\tfrac{a\ddot{a}}{32(1-kr^2)}(3 - 7kr^2 - 4r^2\dot{a}^2),\ 0,\ \tfrac{i\Theta}{2}ra\dot{a},\ 0\right), \tag{20}
$$

$$
\begin{aligned}
\hat{e}^1_\mu = &\left(\Theta^2\tfrac{ra\dot{a}\ddot{a}(3kr^2-2)}{16(1-kr^2)^{3/2}},\ \tfrac{a}{\sqrt{1-kr^2}} - \Theta^2 a\tfrac{\dot{a}^2(5-kr^2-k^2r^4)+(r^2\dot{a}^4+4k)(1-kr^2)}{32(1-kr^2)^{3/2}},\right. \\
&\left. i\Theta\tfrac{a(3kr^2+\dot{a}^2r^2-1)}{4\sqrt{1-kr^2}},\ 0\right),
\end{aligned}
$$

$$
\hat{e}^2_\mu = \left(0,\ i\Theta\tfrac{ra\dot{a}^2}{4(1-kr^2)},\ ra - \Theta^2 ra(k + \dot{a}^2)\left(\tfrac{5}{8} + \tfrac{r^2\dot{a}^2}{32(1-kr^2)}\right),\ 0\right),
$$

$$\hat{e}^3_\mu = \Big(0,\ 0,\ 0,\ ra\sin\theta - \tfrac{i\Theta}{4}a\cos\theta + \Theta^2 a\sin\theta\,\tfrac{\dot{a}^2(20kr^2+4\dot{a}^2r^2-9)-8k(1-2kr^2)}{32(1-kr^2)}\Big),$$

and the deformed spin connection components:

$$\begin{aligned}
\hat{\omega}^{00}_\mu &= \left(0,\ -\Theta^2\frac{kr\dot{a}^2(1-3kr^2-r^2\dot{a}^2)}{16(1-kr^2)^2}, -i\Theta\frac{r\dot{a}^2}{2},\ 0\right), \qquad (21)\\
\hat{\omega}^{11}_\mu &= \left(0,\ -\Theta^2\frac{kr\dot{a}^2(1-3kr^2-r^2\dot{a}^2)}{16(1-kr^2)^2}, i\Theta r(k+\frac{\dot{a}^2}{2}),\ 0\right),\\
\hat{\omega}^{22}_\mu &= \left(0,\ 0, i\Theta r(k+\dot{a}^2),\ 0\right),\\
\hat{\omega}^{01}_\mu &= \Bigg(\Theta^2 r\ddot{a}\frac{4(k-\dot{a}^2)(1-kr^2)+kr^2\dot{a}^2}{32(1-kr^2)^{3/2}},\\
&\qquad \frac{\dot{a}}{\sqrt{1-kr^2}} - \Theta^2\dot{a}\frac{(5\dot{a}^2+r^2\dot{a}^4+4k)(1-kr^2)+3k^2r^4\dot{a}^2}{32(1-kr^2)^{5/2}},\\
&\qquad i\Theta\frac{\dot{a}(1-3kr^2-r^2\dot{a}^2)}{4\sqrt{1-kr^2}},\ 0\Bigg),\\
\hat{\omega}^{02}_\mu &= \Bigg(i\Theta\frac{\ddot{a}}{4},\ i\Theta\frac{r\dot{a}^3}{4(1-kr^2)},\\
&\qquad r\dot{a} + \Theta^2\frac{r\dot{a}(14k^2r^2-18k-16\dot{a}^2+13kr^2\dot{a}^2-r^2\dot{a}^4)}{32(1-kr^2)},\ 0\Bigg),\\
\hat{\omega}^{03}_\mu &= \Bigg(0,\ 0,\ 0,\ r\dot{a}\sin\theta + i\Theta\frac{\dot{a}\cos\theta}{4} +\\
&\qquad \Theta^2\frac{\dot{a}(10k^2r^3-6kr-7r\dot{a}^2+14kr^3\dot{a}^2+4r^3\dot{a}^4)}{32(1-kr^2)}\Bigg),\\
\hat{\omega}^{12}_\mu &= \Bigg(0,\ i\Theta\frac{kr^2\dot{a}^2}{4(1-kr^2)^{3/2}},\ -\sqrt{1-kr^2} + \Theta^2\Bigg(\frac{1}{32}\sqrt{1-kr^2}(16k+13\dot{a}^2)\\
&\qquad -r^2\frac{8k^2+9\dot{a}^4}{32\sqrt{1-kr^2}} + \frac{r^2\dot{a}^2(20kr^2-24k-3\dot{a}^2}{32(1-kr^2)^{5/2}}\Bigg),0\Bigg),\\
\hat{\omega}^{13}_\mu &= \Bigg(0,\ 0,\ 0,\ -\sqrt{1-kr^2}\sin\theta - i\Theta\cos\theta\frac{r\dot{a}^2}{4\sqrt{1-kr^2}}\\
&\qquad +\Theta^2\sin\theta\left(\frac{1}{16}\sqrt{1-kr^2}(2k+\dot{a}^2) + \frac{r^2\dot{a}^2(3\dot{a}^2-2k-2kr^2\dot{a}^2)}{32(1-kr^2)^{3/2}}\right)\Bigg),\\
\hat{\omega}^{23}_\mu &= \Bigg(0,\ 0,\ 0, -\cos\theta + i\Theta r\sin\theta\frac{k+\dot{a}^2}{2}\\
&\qquad + \Theta^2\sin\theta\frac{r^2(3\dot{a}^2+4k)(k+\dot{a}^2)-\dot{a}^2}{32(1-kr^2)}\Bigg).
\end{aligned}$$

Only one off diagonal component for the analogue of the metric tensor results in these coordinates

$$\begin{aligned}
\hat{g}_{00} &= -1 + \Theta^2 \tfrac{a\ddot{a}}{16(1-kr^2)}(3 - 7kr^2 - 4r^2\dot{a}^2) + \mathcal{O}(\Theta^4)\,,\\
\hat{g}_{11} &= \tfrac{a^2}{1-kr^2} - \Theta^2 \tfrac{a^2\left(\dot{a}^2(5-kr^2(1+kr^2))+4k(1-kr^2)\right)}{16(1-kr^2)^3} + \mathcal{O}(\Theta^4)\,,\\
\hat{g}_{22} &= r^2a^2 + \Theta^2 \tfrac{r^2a^2}{16}\left(\tfrac{3kr^2(\dot{a}^2+k)+1}{1-kr^2} - 26(\dot{a}^2+k)\right) + \mathcal{O}(\Theta^4)\,,\\
\hat{g}_{33} &= r^2a^2\sin^2\theta + \Theta^2\tfrac{a^2}{16}\left(\tfrac{7kr^2+1}{1-kr^2}\cos^2\theta\right.\\
&\quad \left. + \tfrac{r^2\dot{a}^2(20kr^2+4r^2\dot{a}^2-9)+4(4k^2r^4-3kr^2+1)}{1-kr^2}\sin^2\theta\right) + \mathcal{O}(\Theta^4)\,,\\
\hat{g}_{01} &= \Theta^2\tfrac{a^2\dot{a}\ddot{a}r}{16(1-kr^2)^2}(3kr^2-2) + \mathcal{O}(\Theta^4)\,.
\end{aligned} \tag{22}$$

Even in this simplest case of space-space noncommutativity the rotational invariance is broken even for $\lambda \to 0$ and worldline $r = 0$. We note that we receive second order correction for a constant scale factor but zero off diagonal components.

Conclusions
The corresponding deformed metric reveals the modified structure of gravitational field in the case of isotropic homogeneous Robertson–Walker space-time of the (commutative) gauge theory of gravitation. If we treat the noncommutative analogue of the metric tensor as a standard metric tensor we can examine the deformed space time for different scale factors in the case of constant noncommutative parameter.

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Some Power-Law Cosmological Solutions in Nonlocal Modified Gravity

Ivan Dimitrijevic, Branko Dragovich, Jelena Grujic, and Zoran Rakic

Abstract Modified gravity with nonlocal term $R^{-1}\mathcal{F}(\Box)R$, and without matter, is considered from the cosmological point of view. Equations of motion are derived. Cosmological solutions of the form $a(t) = a_0|t - t_0|^\alpha$, for the FLRW metric and $k = 0, \pm 1$, are found.

1 Introduction

Although very successful, Einstein theory of gravity is not a final theory. There are many its modifications, which are motivated by quantum gravity, string theory, astrophysics and cosmology (for a review, see [1]). One of very promising directions of research is *nonlocal modified gravity* and its applications to cosmology (as a review, see [2] and [3], see also contribution [4] in these proceedings). To solve cosmological Big Bang singularity, nonlocal gravity with replacement $R \to R + CR\mathcal{F}(\Box)R$ in the Einstein–Hilbert action was proposed in [5]. This nonlocal model is further elaborated is the series of papers [6–12].

In this brief paper we consider the action

$$S = \int d^4x \sqrt{-g}\Big(\frac{R}{16\pi G} + R^{-1}\mathcal{F}(\Box)R\Big), \tag{1}$$

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where R is scalar curvature, $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n \Box^n$ is an analytic function of the d'Alembert–Beltrami operator $\Box = \frac{1}{\sqrt{-g}}\partial_\mu \sqrt{-g} g^{\mu\nu}\partial_\nu$, $g = det(g_{\mu\nu})$. The action (1) was introduced in [13] as a new approach to nonlocal gravity. Its nonlocal term $R^{-1}\mathcal{F}(\Box)R = f_0 + R^{-1}\sum_{n=1}^{\infty} f_n \Box^n R$ contains f_0 which can be connected with the cosmological constant as $f_0 = -\frac{\Lambda}{8\pi G}$. This term is also invariant under transformation $R \to CR$, where C is a constant, i.e. this nonlocality does not depend on magnitude of the scalar curvature $R \neq 0$. Our intention is to present some cosmological solutions in this paper as a part of a systematic investigation of nonlocal gravity (1). In [13] similar power-law cosmological solution were obtained using some ansätze.

Note that there have been some investigations with $1/R$ modification of gravity, but they are not nonlocal and they have problems to be confirmed for the Solar System [14]. Let us mention that there are some other approaches to nonlocal gravity which contain $\Box^{-1}$ instead of $\Box$, see, e.g. [3, 15–17]. Nonlocality also improves renormalizability of gravity, see [18, 19] and references therein.

2 Equations of Motion

By variation of action (1) with respect to metric $g^{\mu\nu}$ one obtains the equations of motion for $g_{\mu\nu}$

$$\begin{aligned} R_{\mu\nu}V - &(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)V - \frac{1}{2}g_{\mu\nu}R^{-1}\mathcal{F}(\Box)R \\ &+\frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1}\Big(g_{\mu\nu}\big(\partial_\alpha\Box^l(R^{-1})\partial^\alpha\Box^{n-1-l}R + \Box^l(R^{-1})\Box^{n-l}R\big) \\ &-2\partial_\mu\Box^l(R^{-1})\partial_\nu\Box^{n-1-l}R\Big) = -\frac{G_{\mu\nu}}{16\pi G}, \\ V &= \mathcal{F}(\Box)R^{-1} - R^{-2}\mathcal{F}(\Box)R. \end{aligned} \tag{2}$$

In the case of the FLRW metric, Eq. (2) is equivalent to its trace and 00 component, respectively:

$$\begin{aligned} RV + 3\Box V + \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1}\big(\partial_\alpha\Box^l(R^{-1})\partial^\alpha\Box^{n-1-l}R + 2\Box^l(R^{-1})\Box^{n-l}R\big) \\ - 2R^{-1}\mathcal{F}(\Box)R = \frac{R}{16\pi G}, \end{aligned} \tag{3}$$

$$\begin{aligned} & R_{00}V - (\nabla_0\nabla_0 - g_{00}\Box)V - \frac{1}{2}g_{00}R^{-1}\mathcal{F}(\Box)R \\ & \quad + \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1}\Big(g_{00}\left(\partial_\alpha\Box^l(R^{-1})\partial^\alpha\Box^{n-1-l}R + \Box^l(R^{-1})\Box^{n-l}R\right) \\ & \quad - 2\partial_0\Box^l(R^{-1})\partial_0\Box^{n-1-l}R\Big) = -\frac{G_{00}}{16\pi G}. \end{aligned} \tag{4}$$

Equations (3) and (4) are more suitable for further investigation than (2).

In the FLRW metric $ds^2 = -dt^2 + a^2(t)\big(\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\big)$ one has $R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)$ and $\Box h(t) = -\ddot{h}(t) - 3H\dot{h}(t)$, where $H = \frac{\dot{a}}{a}$ is the Hubble parameter. In the sequel we solve equations of motions (3) and (4) for cosmological scale factor $a(t)$ and the corresponding R:

$$a(t) = a_0|t-t_0|^\alpha, \tag{5}$$

$$R(t) = 6\big(\alpha(2\alpha-1)(t-t_0)^{-2} + \frac{k}{a_0^2}(t-t_0)^{-2\alpha}\big). \tag{6}$$

3 Case $k = 0$, $\alpha \neq 0$ and $\alpha \neq \frac{1}{2}$

In this case, there is the following dependence on parameter α:

$$\begin{aligned} a &= a_0|t-t_0|^\alpha, & H &= \alpha(t-t_0)^{-1}, \\ R &= r(t-t_0)^{-2}, & r &= 6\alpha(2\alpha-1), \\ R_{00} &= 3\alpha(1-\alpha)(t-t_0)^{-2}, & G_{00} &= 3\alpha^2(t-t_0)^{-2}. \end{aligned} \tag{7}$$

Now expressions $\Box^n R$ and $\Box^n R^{-1}$ become

$$\begin{aligned} & \Box^n R = B(n,1)(t-t_0)^{-2n-2}, \quad \Box^n R^{-1} = B(n,-1)(t-t_0)^{2-2n}, \\ & B(n,1) = r(-2)^n n!\prod_{l=1}^{n}(1-3\alpha+2l),\ n \geq 1, \quad B(0,1) = r, \\ & B(n,-1) = (r)^{-1}2^n\prod_{l=1}^{n}(2-l)(-3-3\alpha+2l),\ n \geq 1, \quad B(0,-1) = r^{-1}. \end{aligned} \tag{8}$$

Note that $B(1,-1) = -2(3\alpha + 1)r^{-1}$ and $B(n,-1) = 0$ if $n \geq 2$. Also, we obtain

$$\begin{aligned} \mathcal{F}(\Box)R &= \sum_{n=0}^{\infty} f_n\, B(n,1)\,(t-t_0)^{-2n-2}, \\ \mathcal{F}(\Box)R^{-1} &= f_0\, B(0,-1)\,(t-t_0)^2 + f_1\, B(1,-1). \end{aligned} \tag{9}$$

Substituting these equations into trace and 00 component of the EOM one has

$$\begin{aligned} & r^{-1}\sum_{n=0}^{\infty} f_n B(n,1)\,(-3r + 6(1-n)(1-2n+3\alpha))\,(t-t_0)^{-2n} \\ & + r\sum_{n=0}^{1} f_n\,(rB(n,-1) + 3B(n+1,-1))\,(t-t_0)^{-2n} \\ & + 2r\sum_{n=1}^{\infty} f_n\gamma_n(t-t_0)^{-2n} = \frac{r^2}{16\pi G}(t-t_0)^{-2}, \end{aligned} \tag{10}$$

$$\begin{aligned} & \sum_{n=0}^{\infty} f_n r^{-1} B(n,1)\left(\frac{r}{2} - A_n\right)(t-t_0)^{-2n} + \sum_{n=0}^{1} f_n r B(n,-1)\, A_n\,(t-t_0)^{-2n} \\ & + \frac{r}{2}\sum_{n=1}^{\infty} f_n\delta_n(t-t_0)^{-2n} = \frac{-r^2}{32\pi G}\frac{\alpha}{2\alpha-1}(t-t_0)^{-2}, \end{aligned} \tag{11}$$

where

$$\gamma_n = \sum_{l=0}^{n-1} B(l,-1)(B(n-l,1) + 2(1-l)(n-l)B(n-l-1,1)), \tag{12}$$

$$\delta_n = \sum_{l=0}^{n-1} B(l,-1)(-B(n-l,1) + 4(1-l)(n-l)B(n-l-1,1)), \tag{13}$$

$$A_n = 6\alpha(1-n) - r\frac{\alpha-1}{2(2\alpha-1)} = \frac{r}{2}\frac{3-2n-\alpha}{2\alpha-1}. \tag{14}$$

Equations (10) and (11) can be split into system of pairs of equations with respect to each coefficient f_n. In the case $n > 1$, there are the following pairs:

$$\begin{aligned} & f_n\big(B(n,1)\,(-3r + 6(1-n)(1-2n+3\alpha)) + 2r^2\gamma_n\big) = 0, \\ & f_n\left(B(n,1)\left(\frac{r}{2} - A_n\right) + \frac{r^2}{2}\delta_n\right) = 0. \end{aligned} \tag{15}$$

Taking $\frac{3\alpha-1}{2}$ to be a natural number one obtains:

$$B(n,1) = r4^n n! \frac{(\frac{3}{2}(\alpha-1))!}{(\frac{3}{2}(\alpha-1)-n)!}, \quad n < \frac{3\alpha-1}{2}, \tag{16}$$

$$B(n,1) = 0, \quad n \geq \frac{3\alpha-1}{2}, \tag{17}$$

$$\gamma_n = 2B(0,-1)B(n-1,1)(3n\alpha - 2n^2 - 3\alpha - 1), \quad n \leq \frac{3\alpha-1}{2}, \tag{18}$$

$$\delta_n = 2B(0,-1)B(n-1,1)(2n^2 + 3n + 3\alpha - 3\alpha n + 1), \quad n \leq \frac{3\alpha-1}{2}, \tag{19}$$

$$\gamma_n = \delta_n = 0, \quad n > \frac{3\alpha-1}{2}. \tag{20}$$

If $n > \frac{3\alpha-1}{2}$, then $B(n,1) = \gamma_n = \delta_n = 0$ and hence the system is trivially satisfied for arbitrary value of coefficients f_n. On the other hand, for $2 \leq n \leq \frac{3\alpha-1}{2}$ the system has only trivial solution $f_n = 0$. When $n = 0$ the pair becomes

$$f_0\big(-2r + 6(1+3\alpha) + 3rB(1,-1)\big) = 0, \quad f_0 = 0 \tag{21}$$

and its solution is $f_0 = 0$. The remaining case $n = 1$ reads

$$f_1\big(-3r^{-1}B(1,1) + rB(1,-1) + 2\gamma_1\big) = \frac{r}{16\pi G},$$
$$f_1\Big(A_1(rB(1,-1) - r^{-1}B(1,1)) + \frac{1}{2}(B(1,1) + r\delta_1)\Big) = \frac{-r^2}{32\pi G}\,\frac{\alpha}{2\alpha-1}, \tag{22}$$

and it gives $f_1 = -\frac{3\alpha(2\alpha-1)}{32\pi G(3\alpha-2)}$.

Note that $f_1 \to \infty$ as $\alpha \to \frac{2}{3}$ and thus this solution cannot imitate the (dark) matter dominated universe.

4 Case $k = 0$, $\alpha \to 0$ (Minkowski Space)

Substituting (8) and (12) into trace Eq. (10) we obtain:

$$
\begin{aligned}
& f_0(-18\alpha(2\alpha-1)+6(1+3\alpha)) \\
& \quad + \sum_{n=1}^{\infty} f_n(-2)^n n! \prod_{l=1}^{n}(1-3\alpha+2l)\,(6(1-n)(1-2n+3\alpha) \\
& \quad - 18\alpha(2\alpha-1))\,(t-t_0)^{-2n} \\
& \quad +6\alpha(2\alpha-1)\Big(f_0(1-6(3\alpha+1)(6\alpha(2\alpha-1))^{-1}) + f_1(-6\alpha-2)(t-t_0)^{-2}\Big) \\
& \quad +2f_1(-12\alpha(2\alpha-1)(3-3\alpha)+12\alpha(2\alpha-1))(t-t_0)^{-2} \\
& \quad +12\alpha(2\alpha-1)\sum_{n=2}^{\infty} f_n(-2)^n(n-1)!\prod_{l=1}^{n-1}(1-3\alpha+2l) \\
& \quad \times\left(-3n\alpha+2n^2+1+3\alpha\right)(t-t_0)^{-2n} \; = \\
& \quad = \frac{(6\alpha(2\alpha-1))^2}{16\pi G}(t-t_0)^{-2}.
\end{aligned} \tag{23}
$$

Now, if $\alpha \to 0$ from the last equation we get

$$
\sum_{n=1}^{\infty} f_n(-1)^n(2n+1)!(1-n)(1-2n)(t-t_0)^{-2n} = 0. \tag{24}
$$

From this we conclude

$$
f_0, f_1 \in \mathbb{R}, \quad f_i = 0,\ i \geq 2. \tag{25}
$$

Substituting $f_i = 0,\ i \geq 2$ into Eq. (11) we obtain the following equation:

$$
\begin{aligned}
& f_0(3\alpha(3\alpha-4)) - 6f_1(1-\alpha)\frac{6\alpha(2\alpha-1)}{2}(1-\frac{1-\alpha}{2\alpha-1})(t-t_0)^{-2} \\
& \quad + f_0\frac{6\alpha(2\alpha-1)}{2}\frac{3-\alpha}{2\alpha-1} + 6f_1\alpha(-1-3\alpha)(1-\alpha)(t-t_0)^{-2} \\
& \quad + f_1(6\alpha(2\alpha-1)(3-3\alpha)+12\alpha(2\alpha-1))(t-t_0)^{-2} \\
& \quad = \frac{-(6\alpha(2\alpha-1))^2}{32\pi G}\frac{\alpha}{2\alpha-1}(t-t_0)^{-2}.
\end{aligned} \tag{26}
$$

When $\alpha \to 0$ we see that the last equation is also satisfied for any $f_0, f_1 \in \mathbb{R}$. This looks like Minkowski space solution, but this is not nonlocal gravity model (1), because all $f_n = 0,\ n \geq 2$. It follows that the above power-law cosmological solutions have not Minkowski space as their background, or in other words, they cannot be obtained as perturbations on Minkowski space.

Remark. To get the Minkowski space ($k = 0,\ a(t) = a_0 = const.$) for nonlocal gravity model (1) one can start from the de Sitter solution $a(t) = a_0 \exp(\lambda t)$ and to take limit $\lambda \to 0$. Really, it is easy to see that equations of motion (3) and (4) are satisfied for $a(t) = a_0 \exp(\lambda t)$ with $R = 12\lambda^2$ and $f_0 = -\frac{3}{8\pi G}\lambda^2$. Then $a(t) \to a_0$, $R \to 0$ and $f_0 \to 0$ as $\lambda \to 0$, and all f_i, $i \geq 1$ are arbitrary constants. This means that nonlocal gravity model (1) in its general form contains the de Sitter universe, which has Minkowski space as its background. This case will be further elaborated and presented elsewhere.

5 Case $k = 0$, $\alpha \to \frac{1}{2}$

Let $\alpha \to \frac{1}{2}$. Similarly as in the previous case, from (23) we obtain the equation

$$\sum_{n=1}^{\infty} f_n (-2)^n n! \prod_{l=1}^{n} (-\frac{1}{2} + 2l)(1-n)(\frac{5}{2} - 2n)(t-t_0)^{-2n} = 0. \tag{27}$$

From this, it follows

$$f_0, f_1 \in \mathbb{R}, \quad f_i = 0,\ i \geq 2. \tag{28}$$

Using $f_i = 0,\ i \geq 2$, from Eq. (26) we get

$$\frac{3}{2} f_1 (t-t_0)^{-2} = 0. \tag{29}$$

The corresponding solution is

$$f_0 \in \mathbb{R}, \quad f_i = 0,\ i \geq 1. \tag{30}$$

6 Case $k \neq 0$, $\alpha = 1$

In order to simplify expression (6) there are three possibilities: $\alpha = 0$, $\alpha = \frac{1}{2}$ and $\alpha = 1$. The first two of them do not yield solutions which satisfy the equations of motion. In the case $\alpha = 1$ we obtain

$$a = a_0|t-t_0|, \quad H = (t-t_0)^{-1}, \quad R = s(t-t_0)^{-2},\ s = 6\left(1 + \frac{k}{a_0^2}\right),$$
$$R_{00} = 0, \quad \Box R = 0, \quad \Box^n R^{-1} = D(n,-1)(t-t_0)^{2-2n},$$
$$D(0,-1) = s^{-1}, \quad D(1,-1) = -8s^{-1}, \quad D(n,-1) = 0,\ n \geq 2, \tag{31}$$

where $a_0 = c = 1$, because we work in natural system of units in which speed of light $c = 1$. When $k = -1$, then $s = R = 0$ and cosmological solution $a = |t - t_0|$ mimics the Milne universe, which is not a realistic cosmological model, but has been interesting as a pure kinematical model.

Using the above expressions, trace and 00 equations become respectively

$$
\begin{aligned}
&3f_0 + \sum_{n=0}^{1} f_n s D(n,-1)(t-t_0)^{-2n} + 4f_1(t-t_0)^{-2} = \frac{s}{16\pi G}(t-t_0)^{-2},\\
&-6f_0 s^{-1} + \frac{1}{2}f_0 + 6\sum_{n=0}^{1} f_n D(n,-1)(1-n)(t-t_0)^{-2n} + 2f_1(t-t_0)^{-2}\\
&= -\frac{s}{32\pi G}(t-t_0)^{-2}.
\end{aligned}
\tag{32}
$$

This system leads to conditions for f_0 and f_1 :

$$
\begin{aligned}
-2f_0 - 4f_1(t-t_0)^{-2} &= \frac{s}{16\pi G}(t-t_0)^{-2},\\
\frac{1}{2}f_0 + 2f_1(t-t_0)^{-2} &= -\frac{s}{32\pi G}(t-t_0)^{-2}.
\end{aligned}
\tag{33}
$$

The corresponding solution is

$$
f_0 = 0, \quad f_1 = \frac{-s}{64\pi G}, \quad f_n \in \mathbb{R}, \quad n \geq 2. \tag{34}
$$

Example. As an example, let us take

$$
\mathcal{F}(\Box) = -\frac{\Lambda}{8\pi G} + Ce^{-\beta\Box} = -\frac{\Lambda}{8\pi G} + C\sum_{n=0}^{\infty}\frac{(-\beta)^n}{n!}\Box^n. \tag{35}
$$

Thus, the coefficients f_n are given by

$$
f_0 = -\frac{\Lambda}{8\pi G} + C, \quad f_n = C\frac{(-\beta)^n}{n!},\ n \geq 1. \tag{36}
$$

To have power-law solution of $a(t)$ (5), we have to set $f_0 = 0$ and $f_1 = -\frac{3}{32\pi G}(1+k)$. Hence we have

$$
C = \frac{\Lambda}{8\pi G}, \quad \beta = \frac{3}{4\Lambda}(1+k), \tag{37}
$$

$$
\mathcal{F}(\Box) = \frac{\Lambda}{8\pi G}\left(e^{-\frac{3}{4\Lambda}(1+k)\Box} - 1\right), \tag{38}
$$

where $k = \pm 1, 0$. Note that (38) is valid also for $k = 0$. When $k = -1$, then $\mathcal{F}(\Box) = 0$ and $R = 0$, and there is Milne's expansion $a = |t - t_0|$.

7 Concluding Remarks

To summarize, in this paper we have presented some power-law cosmological solutions of the form $a(t) = a_0|t - t_0|^\alpha$, which are derived from modified gravity with nonlocal term $R^{-1}\mathcal{F}(\Box)R$. These solutions do not have appropriate Minkowski space background. However, in this nonlocal modified gravity model, there is the de Sitter bounce solution $a(t) = a_0 \exp(\lambda t)$, which in the limit $\lambda \to 0$ leads to the Minkowski space. There is also nonsingular bounce solution $a(t) = \frac{1}{\lambda}\cosh(\lambda t)$ for $k = +1$. Let us also mention solution $a(t) = \frac{1}{\lambda}\sinh(\lambda t)$, related to $k = -1$. In all these three cases scalar curvature is $R = 12\lambda^2$ and there is no restriction on coefficients f_n in $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n \Box^n$.

It is worth noting that there is solution $a(t) = |t - t_0|$ which corresponds to the Milne universe for $k = -1$. As an illustration we presented nonlocality by $\mathcal{F}(\Box) = \frac{\Lambda}{8\pi G}\left(e^{-\frac{3}{4\Lambda}(1+k)\Box} - 1\right)$.

Note also that all the above presented power-law solutions $a(t) = a_0|t - t_0|^\alpha$ have scalar curvature $R(t) = 6\big(\alpha(2\alpha - 1)(t - t_0)^{-2} + \frac{k}{a_0^2}(t - t_0)^{-2\alpha}\big)$ (6), which satisfies relation $\Box R = qR^2$, where parameter q depends on α. This quadratic relation $\Box R = qR^2$ was used in [13] as an Ansatz to solve equations of motion.

Finally, our nonlocality, having the form $R^{-1}\mathcal{F}(\Box)R$, does not depend on the magnitude of R, but has influence on the evolution of the universe, because time dependent operator $\Box = -\partial_t^2 - 3H(t)\partial_t$ acts on the time dependent scalar curvature $R(t) = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)$.

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On Nonlocal Modified Gravity and Cosmology

Branko Dragovich

Abstract Despite many nice properties and numerous achievements, general relativity is not a complete theory. One of actual approaches towards more complete theory of gravity is its nonlocal modification. We present here a brief review of nonlocal gravity with its cosmological solutions. In particular, we pay special attention to two nonlocal models and their nonsingular bounce solutions for the cosmic scale factor.

1 Introduction

Recall that General Relativity is the Einstein theory of gravity based on tensorial equation of motion for gravitational (metric) field $g_{\mu\nu}$: $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$, where $R_{\mu\nu}$ is the Ricci curvature tensor, R—the Ricci scalar, $T_{\mu\nu}$ is the energy-momentum tensor, and speed of light is $c = 1$. This Einstein equation follows from the Einstein–Hilbert action $S = \frac{1}{16\pi G}\int \sqrt{-g}\, R\, d^4x + \int \sqrt{-g}\mathcal{L}_m\, d^4x$, where $g = \det(g_{\mu\nu})$ and $\mathcal{L}_m$ is Lagrangian of matter.

Motivations for modification of general relativity are usually related to some problems in quantum gravity, string theory, astrophysics and cosmology (for a review, see [15, 42, 44]). We are here mainly interested in cosmological reasons to modify the Einstein theory of gravity. If general relativity is gravity theory for the universe as a whole and the universe has Friedmann–Lemaître–Robertson–Walker (FLRW) metric, then there is in the universe about 68 % of *dark energy*, 27 % of *dark matter*, and only 5 % of *visible matter* [1]. The visible matter is described by the Standard model of particle physics. However, existence of this 95 % of dark energy-matter content of the universe is still hypothetical, because it has been not verified in the laboratory ambient. Another cosmological problem is related to the Big Bang singularity. Namely, under rather general conditions, general relativity yields cosmological solutions with zero size of the universe at its beginning, what means an infinite matter density. Note that when physical theory contains singularity, it is not valid in the vicinity of singularity and must be appropriately modified.

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In this article, we briefly review nonlocal modification of general relativity in a way to point out cosmological solutions without Big Bang singularity. We consider two nonlocal models and present their nonsingular bounce cosmological solutions. To have more complete view of these models we also write down other exact solutions which are power-law singular ones of the form $a(t) = a_0 \, |t|^\alpha$.

In Sect. 2 we describe some general characteristics of nonlocal gravity which are useful for understanding what follows in the sequel. Section 3 contains a review of both nonsingular bounce and singular cosmological solutions for two nonlocal gravity models without matter. Last section is related to the discussion with some concluding remarks.

2 Nonlocal Gravity

The well founded modification of the Einstein theory of gravity has to contain general relativity and to be verified on the dynamics of the Solar system. Mathematically, it should be formulated within the pseudo-Riemannian geometry in terms of covariant quantities and equivalence of the inertial and gravitational mass. Consequently, the Ricci scalar R in gravity Lagrangian $\mathcal{L}_g$ of the Einstein–Hilbert action has to be replaced by a function which, in general, may contain not only R but also any covariant construction which is possible in the Riemannian geometry. Unfortunately, there are infinitely many such possibilities and so far without a profound theoretical principle which could make definite choice. The Einstein–Hilbert action can be viewed as a result of the principle of simplicity in construction of $\mathcal{L}_g$.

We consider here nonlocal modified gravity. In general, a nonlocal modified gravity model corresponds to an infinite number of spacetime derivatives in the form of some power expansions of the d'Alembert operator $\Box = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu$ or of its inverse $\Box^{-1}$, or some combination of both. We are mainly interested in nonlocality expressed in the form of an analytic function $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n \Box^n$. However, some models with $\Box^{-1} R$, have been also considered (see, e.g. [19, 20, 28, 29, 31–33, 41, 45, 46] and references therein). For nonlocal gravity with $\Box^{-1}$ see also [6, 39]. Many aspects of nonlocal gravity models have been considered, see e.g. [14, 17, 18, 25, 40] and references therein.

Motivation to modify gravity in a nonlocal way comes mainly from string theory. Namely, strings are one-dimensional extended objects and their field theory description contains spacetime nonlocality. We will discuss it in the framework of p-adic string theory in Sect. 4.

In order to better understand nonlocal modified gravity itself, we investigate it without matter. Models of nonlocal gravity which we mainly consider are given by the action

$$S = \int d^4x \sqrt{-g} \Big(\frac{R - 2\Lambda}{16\pi G} + R^q \mathcal{F}(\Box) R \Big), \quad q = +1, -1, \tag{1}$$

where Λ is cosmological constant, which is for the first time introduced by Einstein in 1917. Thus this nonlocality is given by the term $R^q\mathcal{F}(\Box)R$, where $q = \pm 1$ and $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n \Box^n$, i.e. we investigate two nonlocal gravity models: the first one with $q = +1$ and the second one with $q = -1$.

Before to proceed, it is worth mentioning that analytic function $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n \Box^n$, has to satisfy some conditions, in order to escape unphysical degrees of freedom like ghosts and tachyons, and to be asymptotically free in the ultraviolet region (see discussion in [10, 11]).

3 Models and Their Cosmological Solutions

In the sequel we shall consider the above mentioned two nonlocal models (1) separately for $q = +1$ and $q = -1$.

We use the FLRW metric $ds^2 = -dt^2 + a^2(t)\big(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\big)$ and investigate all three possibilities for curvature parameter $k = 0, \pm 1$. In the FLRW metric scalar curvature is $R = 6\left(\frac{\ddot a}{a} + \frac{\dot a^2}{a^2} + \frac{k}{a^2}\right)$ and $\Box = -\partial_t^2 - 3H\partial_t$, where $H = \frac{\dot a}{a}$ is the Hubble parameter. Note that we use natural system of units in which speed of light $c = 1$.

3.1 *Nonlocal Model Quadratic in R*

Nonlocal gravity model which is quadratic in R is given by the action [7, 8]

$$S = \int d^4x \sqrt{-g}\Big(\frac{R - 2\Lambda}{16\pi G} + R\mathcal{F}(\Box)R\Big). \tag{2}$$

This model is attractive because it is ghost free and has some nonsingular bounce solutions, which can solve the Big Bang cosmological singularity problem.

The corresponding equation of motion follows from the variation of the action (2) with respect to metric $g_{\mu\nu}$ and it is

$$\begin{aligned} &2R_{\mu\nu}\mathcal{F}(\Box)R - 2(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)(\mathcal{F}(\Box)R) - \frac{1}{2}g_{\mu\nu}R\mathcal{F}(\Box)R \\ &+\sum_{n=1}^{\infty}\frac{f_n}{2}\sum_{l=0}^{n-1}\Big(g_{\mu\nu}\left(g^{\alpha\beta}\partial_\alpha\Box^l R\partial_\beta\Box^{n-1-l}R + \Box^l R\Box^{n-l}R\right) \\ &-2\partial_\mu\Box^l R\partial_\nu\Box^{n-1-l}R\Big) = \frac{-1}{8\pi G}(G_{\mu\nu} + \Lambda g_{\mu\nu}). \end{aligned} \tag{3}$$

When metric is of the FLRW form in (3) then there are only two independent equations. It is practical to use the trace and 00 component of (3), and respectively they are:

$$6\Box(\mathcal{F}(\Box)R) + \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \left(\partial_\mu \Box^l R \partial^\mu \Box^{n-1-l} R + 2\Box^l R \Box^{n-l} R\right)$$
$$= \tfrac{1}{8\pi G} R - \tfrac{\Lambda}{2\pi G}, \tag{4}$$

$$2R_{00}\mathcal{F}(\Box)R - 2(\nabla_0\nabla_0 - g_{00}\Box)(\mathcal{F}(\Box)R) - \tfrac{1}{2} g_{00} R\mathcal{F}(\Box)R$$
$$+ \sum_{n=1}^{\infty} \tfrac{f_n}{2} \sum_{l=0}^{n-1} \left(g_{00}\left(g^{\alpha\beta}\partial_\alpha \Box^l R \partial_\beta \Box^{n-1-l} R + \Box^l R \Box^{n-l} R\right)\right.$$
$$\left. -2\partial_0 \Box^l R \partial_0 \Box^{n-1-l} R\right) = \tfrac{-1}{8\pi G}(G_{00} + \Lambda g_{00}). \tag{5}$$

We are interested in cosmological solutions for the universe with FLRW metric and even in such simplified case it is rather difficult to find solutions of the above equations. To evaluate the above equations, the following Ansätze were used:

- Linear Ansatz: $\Box R = rR + s$, where r and s are constants.
- Quadratic Ansatz: $\Box R = qR^2$, where q is a constant.
- Qubic Ansatz: $\Box R = qR^3$, where q is a constant.
- Ansatz $\Box^n R = c_n R^{n+1}, \ n \geq 1$, where c_n are constants.

In fact these Ansätze make some constraints on possible solutions, but on the other hand they simplify formalism to find a particular solution.

Linear Ansatz and Nonsingular Bounce Cosmological Solutions

Using Ansatz $\Box R = rR + s$ a few nonsingular bounce solutions for the scale factor are found: $a(t) = a_0 \cosh\left(\sqrt{\frac{\Lambda}{3}} t\right)$ (see [7, 8]), $a(t) = a_0 e^{\frac{1}{2}\sqrt{\frac{\Lambda}{3}} t^2}$ (see [37]) and $a(t) = a_0(\sigma e^{\lambda t} + \tau e^{-\lambda t})$ [22]. The first two consequences of this Ansatz are

$$\Box^n R = r^n (R + \frac{s}{r}), \ n \geq 1, \qquad \mathcal{F}(\Box)R = \mathcal{F}(r)R + \frac{s}{r}(\mathcal{F}(r) - f_0), \tag{6}$$

which considerably simplify nonlocal term.

Now we can search for a solution of the scale factor $a(t)$ in the form of a linear combination of $e^{\lambda t}$ and $e^{-\lambda t}$, i.e.

$$a(t) = a_0(\sigma e^{\lambda t} + \tau e^{-\lambda t}), \quad 0 < a_0, \lambda, \sigma, \tau \in \mathbb{R}. \tag{7}$$

Then the corresponding expressions for the Hubble parameter $H(t) = \frac{\dot{a}}{a}$, scalar curvature $R(t) = \frac{6}{a^2}(a\ddot{a} + \dot{a}^2 + k)$ and $\Box R$ are:

$$H(t) = \frac{\lambda(\sigma e^{\lambda t} - \tau e^{-\lambda t})}{\sigma e^{\lambda t} + \tau e^{-\lambda t}},$$
$$R(t) = \frac{6\left(2a_0^2\lambda^2\left(\sigma^2 e^{4t\lambda} + \tau^2\right) + k e^{2t\lambda}\right)}{a_0^2\left(\sigma e^{2t\lambda} + \tau\right)^2}, \tag{8}$$
$$\Box R = -\frac{12\lambda^2 e^{2t\lambda}\left(4a_0^2\lambda^2\sigma\tau - k\right)}{a_0^2\left(\sigma e^{2t\lambda} + \tau\right)^2}.$$

We can rewrite $\Box R$ as

$$\Box R = 2\lambda^2 R - 24\lambda^4, \qquad r = 2\lambda^2,\ s = -24\lambda^4. \tag{9}$$

Substituting parameters r and s from (9) into (6) one obtains

$$\Box^n R = (2\lambda^2)^n(R - 12\lambda^2),\ n \geq 1,$$
$$\mathcal{F}(\Box)R = \mathcal{F}(2\lambda^2)R - 12\lambda^2(\mathcal{F}(2\lambda^2) - f_0). \tag{10}$$

Using this in (4) and (5) we obtain

$$36\lambda^2\mathcal{F}(2\lambda^2)(R - 12\lambda^2) + \mathcal{F}'(2\lambda^2)\left(4\lambda^2(R - 12\lambda^2)^2 - \dot{R}^2\right)$$
$$- 24\lambda^2 f_0(R - 12\lambda^2) = \frac{R - 4\Lambda}{8\pi G}, \tag{11}$$

$$(2R_{00} + \frac{1}{2}R)\left(\mathcal{F}(2\lambda^2)R - 12\lambda^2(\mathcal{F}(2\lambda^2) - f_0)\right)$$
$$- \frac{1}{2}\mathcal{F}'(2\lambda^2)\left(\dot{R}^2 + 2\lambda^2(R - 12\lambda^2)^2\right) - 6\lambda^2(\mathcal{F}(2\lambda^2) - f_0)(R - 12\lambda^2)$$
$$+ 6H\mathcal{F}(2\lambda^2)\dot{R} = -\frac{1}{8\pi G}(G_{00} - \Lambda). \tag{12}$$

Substituting $a(t)$ from (7) into Eqs. (11) and (12) one obtains two equations as polynomials in $e^{2\lambda t}$. Taking coefficients of these polynomials to be zero one obtains a system of equations and their solution determines parameters $a_0, \lambda, \sigma, \tau$ and yields some conditions for function $\mathcal{F}(2\lambda^2)$. For details see [22].

Quadratic Ansatz and Power-Law Cosmological Solutions

New Ansätze $\Box R = rR$, $\Box R = qR^2$ and $\Box^n R = c_n R^{n+1}$, were introduced in [21] and they contain solution for $R = 0$ which satisfies also equations of motion. When $k = 0$ there is only static solution $a = constant$, and for $k = -1$ solution is $a(t) = |t|$.

In particular, Ansatz $\Box R = qR^2$ is very interesting. The corresponding differential equation for the Hubble parameter, if $k = 0$, is

$$\dddot{H} + 4\dot{H}^2 + 7H\ddot{H} + 12H^2\dot{H} + 6q(\dot{H}^2 + 4H^2\dot{H} + 4H^4) = 0 \tag{13}$$

with solutions

$$H_\eta(t) = \frac{2\eta + 1}{3}\frac{1}{t + C_1}, \quad q_\eta = \frac{6(\eta - 1)}{(2\eta + 1)(4\eta - 1)}, \quad \eta \in \mathbb{R} \tag{14}$$

and $H = \frac{1}{2}\frac{1}{t+C_1}$ with arbitrary coefficient q, what is equivalent to the ansatz $\Box R = rR$ with $R = 0$.

The corresponding scalar curvature is given by

$$R_\eta = \frac{2}{3}\frac{(2\eta + 1)(4\eta - 1)}{(t + C_1)^2}, \quad \eta \in \mathbb{R}. \tag{15}$$

By straightforward calculation one can show that $\Box^n R_n = 0$ when $n \in \mathbb{N}$. This simplifies the equations considerably. For this particular case of solutions operator $\mathcal{F}$ and trace Eq. (4) effectively become

$$\mathcal{F}(\Box) = \sum_{k=0}^{n-1} f_k \Box^k, \tag{16}$$

$$\sum_{k=1}^{n+1} f_k \sum_{l=0}^{k-1}(\partial_\mu \Box^l R \partial^\mu \Box^{k-1-l} R + 2\Box^l R \Box^{k-l} R) + 6\Box\mathcal{F}(\Box)R = \frac{R}{8\pi G}. \tag{17}$$

In particular case $n = 2$ the trace formula becomes

$$\frac{36}{35} f_0 R^2 + f_1(-\dot{R}^2 + \frac{12}{35}R^3) + f_2(-\frac{24}{35}R\dot{R}^2 + \frac{72}{1225}R^4) + f_3(-\frac{144}{1225}R^2\dot{R}^2) = \frac{R}{8\pi G}. \tag{18}$$

Some details on all the above three Ansätze can be found in [21].

3.2 Nonlocal Model with Term $R^{-1}\mathcal{F}(\Box)R$

This model was introduced recently [23] and its action may be written in the form

$$S = \int d^4x \sqrt{-g}\Big(\frac{R}{16\pi G} + R^{-1}\mathcal{F}(\Box)R\Big), \tag{19}$$

where $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n \Box^n$ and when $f_0 = -\frac{\Lambda}{8\pi G}$ it plays role of the cosmological constant. For example, $\mathcal{F}(\Box)$ can be of the form $\mathcal{F}(\Box) = -\frac{\Lambda}{8\pi G} e^{-\beta\Box}$.

The nonlocal term $R^{-1}\mathcal{F}(\Box)R$ in (19) is invariant under transformation $R \to CR$. It means that effect of nonlocality does not depend on the magnitude of scalar curvature R, but on its spacetime dependence, and in the FLRW case is sensitive only to dependence of R on time t. When $R = \textit{constant}$ there is no effect of nonlocality, but only of f_0 what corresponds to cosmological constant.

By variation of action (19) with respect to metric $g^{\mu\nu}$ one obtains the equations of motion for $g_{\mu\nu}$

$$\begin{aligned} R_{\mu\nu} V - (\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box) V - \tfrac{1}{2} g_{\mu\nu} R^{-1} \mathcal{F}(\Box) R \\ + \sum_{n=1}^{\infty} \tfrac{f_n}{2} \sum_{l=0}^{n-1} \Big(g_{\mu\nu} \big(\partial_\alpha \Box^l (R^{-1}) \partial^\alpha \Box^{n-1-l} R + \Box^l (R^{-1}) \Box^{n-l} R \big) \\ -2 \partial_\mu \Box^l (R^{-1}) \partial_\nu \Box^{n-1-l} R \Big) = -\tfrac{G_{\mu\nu}}{16\pi G}, \\ V = \mathcal{F}(\Box) R^{-1} - R^{-2} \mathcal{F}(\Box) R. \end{aligned} \tag{20}$$

Note that operator $\Box$ acts not only on R but also on R^{-1}. There are only two independent equations when metric is of the FLRW type.

The trace of Eq. (20) is

$$\begin{aligned} RV + 3\Box V + \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \big(\partial_\alpha \Box^l (R^{-1}) \partial^\alpha \Box^{n-1-l} R + 2\Box^l (R^{-1}) \Box^{n-l} R \big) \\ -2R^{-1} \mathcal{F}(\Box) R = \tfrac{R}{16\pi G}. \end{aligned} \tag{21}$$

The 00-component of (20) is

$$\begin{aligned} R_{00} V - (\nabla_0 \nabla_0 - g_{00}\Box) V - \tfrac{1}{2} g_{00} R^{-1} \mathcal{F}(\Box) R \\ + \sum_{n=1}^{\infty} \tfrac{f_n}{2} \sum_{l=0}^{n-1} \Big(g_{00} \big(\partial_\alpha \Box^l (R^{-1}) \partial^\alpha \Box^{n-1-l} R + \Box^l (R^{-1}) \Box^{n-l} R \big) \\ -2 \partial_0 \Box^l (R^{-1}) \partial_0 \Box^{n-1-l} R \Big) = -\tfrac{G_{00}}{16\pi G}. \end{aligned} \tag{22}$$

These trace and 00-component equations are equivalent for the FLRW universe in the equation of motion (20), but they are more suitable for usage.

Some Cosmological Solutions for Constant *R*

We are interested in some exact nonsingular cosmological solutions for the scale factor $a(t)$ in (20). The Ricci curvature R in the above equations of motion can be calculated by expression

$$R = 6 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right).$$

Case $k = 0$, $a(t) = a_0 e^{\lambda t}$**.**

We have $a(t) = a_0 e^{\lambda t}$, $\dot{a} = \lambda a$, $\ddot{a} = \lambda^2 a$, $H = \frac{\dot{a}}{a} = \lambda$ and $R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = 12\lambda^2$. Putting $a(t) = a_0 e^{\lambda t}$ in the above Eqs. (21) and (22), they are satisfied with $\lambda = \pm\sqrt{\frac{\Lambda}{3}}$, where $\Lambda = -8\pi G f_0$ with $f_0 < 0$.

Case $k = +1$, $a(t) = \frac{1}{\lambda}\cosh\lambda t$**.**

Starting with $a(t) = a_0 \cosh\lambda t$, we have $\dot{a} = \lambda a_0 \sinh\lambda t$, $H = \frac{\dot{a}}{a} = \lambda\tanh\lambda t$ and $R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{1}{a^2}\right) = 12\lambda^2$ if $a_0 = \frac{1}{\lambda}$. Hence Eqs. (21) and (22) are satisfied for cosmic scale factor $a(t) = \frac{1}{\lambda}\cosh\lambda t$.

In a similar way, one can obtain another solution:

Case $k = -1$, $a(t) = \frac{1}{\lambda}|\sinh\lambda t|$**.**

Thus we have the following three cosmological solutions for $R = 12\lambda^2$:

1. $k = 0$, $a(t) = a_0\, e^{\lambda t}$, nonsingular bounce solution.
2. $k = +1$, $a(t) = \frac{1}{\lambda}\cosh\lambda t$, nonsingular bounce solution.
3. $k = -1$, $a(t) = \frac{1}{\lambda}|\sinh\lambda t|$, singular cosmic solution.

All of this solution have exponential behavior for large value of time t.

Note that in all the above three cases the following two tensors have also the same expressions:

$$R_{\mu\nu} = \frac{1}{4}Rg_{\mu\nu}, \qquad G_{\mu\nu} = -\frac{1}{4}Rg_{\mu\nu}. \tag{23}$$

Minkowski background space follows from the de Sitter solution $k = 0$, $a(t) = a_0 e^{\lambda t}$. Namely, when $\lambda \to 0$ then $a(t) \to a_0$ and $H = R = 0$.

In all the above cases $\Box R = 0$ and thus coefficients f_n, $n \geq 1$ may be arbitrary. As a consequence, in these cases nonlocality does not play a role.

Some Power-Law Cosmological Solutions

Power-law solutions in the form $a(t) = a_0|t - t_0|^\alpha$, have been investigated by some Ansätze in [23] and without Ansätze [24]. The corresponding Ricci scalar and the Hubble parameter are:

$$R(t) = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{1}{a^2}\right) = 6\big(\alpha(2\alpha - 1)(t - t_0)^{-2} + \frac{k}{a_0^2}(t - t_0)^{-2\alpha}\big)$$

$$H(t) = \frac{\dot{a}}{a} = \frac{\alpha}{|t - t_0|}.$$

Now $\Box = -\partial_t^2 - \frac{3\alpha}{|t-t_0|}\partial_t$. An analysis has been performed for $\alpha \neq 0, \frac{1}{2}$, and also $\alpha \to 0$, $\alpha \to \frac{1}{2}$ for $k = +1, -1, 0$. For details, the reader refers to [23, 24].

4 Discussion and Concluding Remarks

To illustrate the form of the above nonlocality (2) it is worth to start from exact effective Lagrangian at the tree level for p-adic closed and open scalar strings. This Lagrangian is as follows (see, e.g. [13]):

$$L_p = -\frac{m^D}{2g^2}\frac{p^2}{p-1}\varphi p^{-\frac{\Box}{2m^2}}\varphi - \frac{m^D}{2h^2}\frac{p^4}{p^2-1}\phi p^{-\frac{\Box}{4m^2}}\phi + \frac{m^D}{h^2}\frac{p^4}{p^4-1}\phi^{p^2+1}$$
$$-\frac{m^D}{g^2}\frac{p^2}{p^2-1}\phi^{\frac{p(p-1)}{2}} + \frac{m^D}{g^2}\frac{p^2}{p^2-1}\varphi^{p+1}\phi^{\frac{p(p-1)}{2}}, \qquad (24)$$

where φ denotes open strings, D is spacetime dimensionality (in the sequel we shall take $D = 4$), and g and h are coupling constants for open and closed strings, respectively. Scalar field $\phi(x)$ corresponds to closed p-adic strings and could be related to gravity scalar curvature as $\phi = f(R)$, where f is an appropriate function. The corresponding equations of motion are:

$$p^{-\frac{\Box}{2m^2}}\varphi = \varphi^p\phi^{\frac{p(p-1)}{2}}, \quad p^{-\frac{\Box}{4m^2}}\phi = \phi^{p^2} + \frac{h^2}{2g^2}\frac{p-1}{p}\phi^{\frac{p(p-1)}{2}-1}\left(\varphi^{p+1}-1\right). \qquad (25)$$

There are the following constant vacuum solutions: (i) $\varphi = \phi = 0$, (ii) $\varphi = \phi = 1$ and (iii) $\varphi = \phi^{-\frac{p}{2}} = constant$.

In the case that the open string field $\varphi = 0$, one obtains equation of motion only for closed string ϕ. One can now construct a toy nonlocal gravity model supposing that closed scalar string is related to the Ricci scalar curvature as $\phi = -\frac{1}{m^2}R = -\frac{4}{3g^2}(16\pi G)R$. Taking $p = 2$, we obtain the following Lagrangian for gravity sector:

$$\mathcal{L}_g = \frac{1}{16\pi G}R - \frac{8}{3}\frac{C^2}{h^2}R\,e^{-\frac{\ln 2\,\Box}{4m^2}}R - \frac{1024}{405g^6h^2}(16\pi G)^3R^5. \qquad (26)$$

To compare third term to the first one in (26), let us note that $(16\pi G)^3R^5 = (16\pi GR)^4\frac{R}{16\pi G}$. It follows that $(GR)^4$ has to be dimensionless after rewriting it using constants c and $\hbar$. As Ricci scalar R has dimension $Time^{-2}$ it means that G has to be replaced by the Planck time as $t_P^2 = \frac{\hbar G}{c^5} \sim 10^{-88}s^2$. Hence $(GR)^4 \to (\frac{\hbar G}{c^5}R)^4 \sim 10^{-352}R^4$ and third term in (26) can be neglected with respect to the first one, except when $R \sim t_P^{-2}$. The nonlocal model with only first two terms corresponds to case considered above in this article. We shall consider this model including R^5 term elsewhere.

It is worth noting that the above two models with nonlocal terms $R\mathcal{F}(\Box)R$ and $R^{-1}\mathcal{F}(\Box)R$ are equivalent in the case when $R = constant$, because their equations of motion have the same solutions. These solutions do not depend on $\mathcal{F}(\Box) - f_0$. It would be useful to find cosmological solutions which have definite connection with the explicit form of nonlocal operator $\mathcal{F}(\Box)$.

Let us mention that many properties of (2) and its extended quadratic versions have been considered, see [9–11, 34, 35].

Nonlocal model (19) is a new one and was not considered before [23], it seems to be important and deserves further investigation. There are some gravity models modified by term R^{-1}, but they are neither nonlocal nor pass Solar system tests, see e.g. [30].

Note that nonlocal cosmology is related also to cosmological models in which matter sector contains nonlocality (see, e.g. [2–5, 16, 26, 27, 36]). String field theory and p-adic string theory models have played significant role in motivation and construction of such models.

Nonsingular bounce cosmological solutions are very important (as reviews on bouncing cosmology, see e.g. [12, 43]) and their progress in nonlocal gravity may be a further step towards cosmology of the cyclic universe [38].

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Part III
Integrable Systems

Vertex Operator Approach to Semi-infinite Spin Chain: Recent Progress

Takeo Kojima

Abstract Vertex operator approach is a powerful method to study exactly solvable models. We review recent progress of vertex operator approach to semi-infinite spin chain. (1) The first progress is a generalization of boundary condition. We study $U_q(\widehat{sl}(2))$ spin chain with a triangular boundary, which gives a generalization of diagonal boundary (Baseilhac and Belliard, Nucl Phys B873:550–583, 2013; Baseilhac and Kojima, Nucl Phys B880:378–413, 2014). We give a bosonization of the boundary vacuum state. As an application, we derive a summation formulae of boundary magnetization. (2) The second progress is a generalization of hidden symmetry. We study supersymmetry $U_q(\widehat{sl}(M|N))$ spin chain with a diagonal boundary (Kojima, J Math Phys 54(043507):40 pp., 2013). By now we have studied spin chain with a boundary, associated with symmetry $U_q(\widehat{sl}(N))$, $U_q(A_2^{(2)})$ and $U_{q,p}(\widehat{sl}(N))$ (Furutsu and Kojima, J Math Phys 41:4413–4436, 2000; Yang and Zhang, Nucl Phys B596:495–512, 2001; Kojima, Int J Mod Phys A26:1973–1989, 2011; Miwa and Weston, Nucl Phys B486:517–545, 1997; Kojima, J Math Phys 52(01351):26 pp., 2011), where bosonizations of vertex operators are realized by "monomial". However the vertex operator for $U_q(\widehat{sl}(M|N))$ is realized by "sum", a bosonization of boundary vacuum state is realized by "monomial".

1 Introduction

There have been many developments in exactly solvable lattice models. Various models were found to be solvable and various methods were invented to solve these models. Vertex operator approach is a powerful method to study exactly solvable lattice models. Solvability of lattice models is understood by means of commuting transfer matrix. The half transfer matrices are called "vertex operators" and are identified with the intertwiners of the irreducible highest weight representations of the quantum affine algebras $U_q(g)$. This identification is a basis of vertex operator

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approach. In [4], the vertex operator approach was extended to half-infinite XXZ spin chain with a diagonal boundary. In this paper we review recent progress of vertex operator approach to semi-infinite spin chain with a boundary. We start from solutions of the boundary Yang–Baxter equation, and introduce the transfer matrices in terms of a product of vertex operators. We diagonalize the transfer matrices by using bosonizations of the vertex operators, and study correlation functions.

The plan of the paper is as follows. In Sect. 2 we study $U_q(\widehat{sl}(2))$ spin chain with a triangular boundary, which is a generalization of diagonal boundary. We give a bosonization of the boundary vacuum state, and calculate boundary magnetization. In Sect. 3 we study supersymmetry $U_q(\widehat{sl}(M+1|N+1))$ spin chain with a diagonal boundary. We give bosonizations of boundary vacuum states. In section "Conclusion" we summarize a conclusion. Throughout this paper we use the following abbreviations.

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad (z;p)_\infty = \prod_{m=0}^{\infty}(1-p^m z), \quad \theta_m = \begin{cases} 1 & (m: \text{even}), \\ 0 & (m: \text{odd}). \end{cases} \tag{1}$$

2 XXZ Spin Chain with a Triangular Boundary

2.1 Transfer Matrix

The first progress is a generalization of boundary condition. We study XXZ spin chain with a triangular boundary [1, 2]. The Hamiltonian $H_B^{(\pm)}$ is given by

$$H_B^{(\pm)} = -\frac{1}{2}\sum_{k=1}^{\infty}(\sigma_{k+1}^x\sigma_k^x + \sigma_{k+1}^y\sigma_k^y + \Delta\sigma_{k+1}^z\sigma_k^z) - \frac{1-q^2}{4q}\frac{1+r}{1-r}\sigma_1^z - \frac{s}{1-r}\sigma_1^{\pm} \tag{2}$$

where $\sigma^x, \sigma^y, \sigma^z, \sigma^{\pm}$ are the standard Pauli matrices. In what follows we set $V = \mathbf{C}v_+ \oplus \mathbf{C}v_-$. Consider the infinite dimensional vector space $\cdots \otimes V_3 \otimes V_2 \otimes V_1$, where V_j are copies of V. Let us introduce the subspace $\mathcal{H}^{(i)}$ $(i = 0, 1)$ by

$$\mathcal{H}^{(i)} = Span\{\cdots \otimes v_{p(N)} \otimes \cdots \otimes v_{p(2)} \otimes v_{p(1)} |\, p(N) = (-1)^{N+i}\ (N \gg 1)\} \tag{3}$$

where $p : \mathbf{N} \to \{\pm\}$. The Hamiltonian $H_B^{(\pm)}$ acts on the subspace $\mathcal{H}^{(i)}$. Here we consider the model in the massive regime where $\Delta = \frac{q+q^{-1}}{2}$, $-1 < q < 0$, $-1 \le r \le 1$, $s \in \mathbf{R}$. In Sklyanin's framework [8], the transfer matrix $\hat{T}_B^{(\pm,i)}(\zeta; r, s)$ that was a generating function of the Hamiltonian $H_B^{(\pm)}$ was introduced. It is built from two objects: the R-matrix and the $K-$matrix. We introduces the R-matrix $R(\zeta)$ by

$$R(\zeta)=\frac{1}{\kappa(\zeta)}\begin{pmatrix}1&&&\\&\frac{(1-\zeta^2)q}{1-q^2\zeta^2}&\frac{(1-q^2)\zeta}{1-q^2\zeta^2}&\\&\frac{(1-q^2)\zeta}{1-q^2\zeta^2}&\frac{(1-\zeta^2)q}{1-q^2\zeta^2}&\\&&&1\end{pmatrix}. \tag{4}$$

Here we have set $\kappa(\zeta)=\zeta\frac{(q^4\zeta^2;q^4)_\infty(q^2/\zeta^2;q^4)_\infty}{(q^4/\zeta^2;q^4)_\infty(q^2\zeta^2;q^4)_\infty}$. The matrix elements of $R(\zeta)\in\mathrm{End}(V\otimes V)$ are given by $R(\zeta)v_{\epsilon_1}\otimes v_{\epsilon_2}=\sum_{\epsilon_1',\epsilon_2'=\pm}v_{\epsilon_1'}\otimes v_{\epsilon_2'}R(\zeta)_{\epsilon_1'\epsilon_2'}^{\epsilon_1\epsilon_2}$, where the ordering of the index is given by $v_+\otimes v_+, v_+\otimes v_-, v_-\otimes v_+, v_-\otimes v_-$. $R_{ij}(\zeta)$ acts as $R(\zeta)$ on the i-th and j-th components and as identity elsewhere. The R-matrix $R(\zeta)$ satisfies the Yang–Baxter equation.

$$R_{12}(\zeta_1/\zeta_2)R_{13}(\zeta_1/\zeta_3)R_{23}(\zeta_2/\zeta_3)=R_{23}(\zeta_2/\zeta_3)R_{13}(\zeta_1/\zeta_3)R_{12}(\zeta_1/\zeta_2). \tag{5}$$

The normalization factor $\kappa(\zeta)$ is determined by the following unitarity and crossing symmetry conditions: $R_{12}(\zeta)R_{21}(\zeta^{-1})=1,\quad R(\zeta)_{\epsilon_2\epsilon_1'}^{\epsilon_2'\epsilon_1}=R(-q^{-1}\zeta^{-1})_{-\epsilon_1\epsilon_2}^{-\epsilon_1'\epsilon_2'}$. Also, we introduce the triangular K-matrix $K^{(\pm)}(\zeta)=K^{(\pm)}(\zeta;r,s)$ by

$$K^{(+)}(\zeta;r,s)=\frac{\varphi(\zeta^2;r)}{\varphi(\zeta^{-2};r)}\begin{pmatrix}\frac{1-r\zeta^2}{\zeta^2-r}&\frac{s\zeta(\zeta^2-\zeta^{-2})}{\zeta^2-r}\\0&1\end{pmatrix}, \tag{6}$$

$$K^{(-)}(\zeta;r,s)=\frac{\varphi(\zeta^2;r)}{\varphi(\zeta^{-2};r)}\begin{pmatrix}\frac{1-r\zeta^2}{\zeta^2-r}&0\\\frac{s\zeta(\zeta^2-\zeta^{-2})}{\zeta^2-r}&1\end{pmatrix}, \tag{7}$$

where we have set $\varphi(z;r)=\frac{(q^4rz;q^4)_\infty(q^6z^2;q^8)_\infty}{(q^2rz;q^4)_\infty(q^8z^2;q^8)_\infty}$. The matrix elements of $K^{(\pm)}(\zeta)\in\mathrm{End}(V)$ are given by $K^{(\pm)}(\zeta)v_\epsilon=\sum_{\epsilon'=\pm}v_{\epsilon'}K^{(\pm)}(\zeta)_{\epsilon'}^{\epsilon}$, where the ordering of the index is given by v_+, v_-. The K-matrix $K^{(\pm)}(\zeta)$ satisfies the boundary Yang–Baxter equation:

$$\begin{aligned}&K_2^{(\pm)}(\zeta_2)R_{21}(\zeta_1\zeta_2)K_1^{(\pm)}(\zeta_1)R_{12}(\zeta_1/\zeta_2)=\\&=R_{21}(\zeta_1/\zeta_2)K_1^{(\pm)}(\zeta_1)R_{12}(\zeta_1\zeta_2)K_2^{(\pm)}(\zeta_2).\end{aligned} \tag{8}$$

The normalization factor $\varphi(z;r)$ is determined by the following boundary unitarity and boundary crossing symmetry : $K^{(\pm)}(\zeta)K^{(\pm)}(\zeta^{-1})=1$, $K^{(\pm)}(-q^{-1}\zeta^{-1})_{\epsilon_1}^{\epsilon_2}=\sum_{\epsilon_1',\epsilon_2'=\pm}R(-q\zeta^2)_{\epsilon_1'-\epsilon_2'}^{-\epsilon_1\epsilon_2}K^{(\pm)}(\zeta)_{\epsilon_2'}^{\epsilon_1'}$. We introduce the vertex operators $\hat{\Phi}_\epsilon^{(1-i,i)}(\zeta)$ $(\epsilon=\pm)$ which act on the space $\mathcal{H}^{(i)}$ $(i=0,1)$. Matrix elements are given by products of the R-matrix as follows:

$$(\hat{\Phi}_\epsilon^{(1-i,i)}(\zeta))_{\cdots p(N)\cdots p(2)\ p(1)}^{\cdots p(N)'\cdots p(2)' p(1)'} = \lim_{N\to\infty} \sum_{\mu(1),\mu(2),\cdots,\mu(N)=\pm} \prod_{j=1}^{N} R(\zeta)_{\mu(j-1)\ p(j)}^{\mu(j)\ p(j)'} \quad (9)$$

where $\mu(0) = \epsilon$ and $\mu(N) = (-1)^{N+1-i}$. We expect that the vertex operators $\hat{\Phi}_\epsilon^{(1-i,i)}(\zeta)$ give rise to well-defined operators. We set $\hat{\Phi}_\epsilon^{*(1-i,i)}(\zeta) = \hat{\Phi}_{-\epsilon}^{(1-i,i)}(-q^{-1}\zeta)$. Following the strategy proposed in [4], we introduce the transfer matrix $\hat{T}_B^{(\pm,i)}(\zeta;r,s)$ using the vertex operators.

$$\hat{T}_B^{(\pm,i)}(\zeta;r,s) = \sum_{\epsilon_1,\epsilon_2=\pm} \hat{\Phi}_{\epsilon_1}^{*(i,1-i)}(\zeta^{-1}) K^{(\pm)}(\zeta;r,s)_{\epsilon_1}^{\epsilon_2} \hat{\Phi}_{\epsilon_2}^{(1-i,i)}(\zeta). \quad (10)$$

Heuristic arguments suggest that the transfer matrix commutes:

$$[\hat{T}_B^{(\pm,i)}(\zeta_1;r,s), \hat{T}_B^{(\pm,i)}(\zeta_2;r,s)] = 0 \quad \text{for any } \zeta_1,\zeta_2. \quad (11)$$

The Hamiltonian $H_B^{(\pm)}$ (2) is obtained as

$$\left.\frac{d}{d\zeta}\hat{T}_B^{(\pm,i)}(\zeta;r,s)\right|_{\zeta=1} = \frac{4q}{1-q^2} H_B^{(\pm)} + \text{const.} \quad (12)$$

We are interested in diagonalization of the transfer matrix $\hat{T}_B^{(\pm,i)}(\zeta;r,s)$.

2.2 *Vertex Operator Approach*

We formulate the vertex operator approach to the half-infinite XXZ spin chain with a triangular boundary. Let V_ζ the evaluation representation of $U_q(\widehat{sl}(2))$. Let $V(\Lambda_i)$ the irreducible highest weight $U_q(\widehat{sl}(2))$ representation with the fundamental weights Λ_i $(i = 0,1)$. We introduce the vertex operators $\Phi_\epsilon^{(1-i,i)}(\zeta)$ as the intertwiner of $U_q(\widehat{sl}(2))$:

$$\Phi^{(1-i,i)}(\zeta) : V(\Lambda_i) \longrightarrow V(\Lambda_{1-i}) \otimes V_\zeta, \Phi^{(1-i,i)}(\zeta)\cdot x = \\ = \Delta(x)\cdot\Phi^{(1-i,i)}(\zeta), \quad (13)$$

for $x \in U_q(\widehat{sl}(2))$. We set the elements of the vertex operators : $\Phi^{(1-i,i)}(\zeta) = \sum_\epsilon \Phi_\epsilon^{(1-i,i)}(\zeta) \otimes v_\epsilon$. We set $\Phi_\epsilon^{*(1-i,i)}(\zeta) = \Phi_{-\epsilon}^{(1-i,i)}(-q^{-1}\zeta)$. Following the strategy of [4], as the generating function of the Hamiltonian $H_B^{(\pm)}$ we introduce the "renormalized" transfer matrix $T_B^{(\pm,i)}(\zeta;r,s)$:

$$T_B^{(\pm,i)}(\zeta;r,s)=g\sum_{\epsilon_1,\epsilon_2=\pm}\Phi_{\epsilon_1}^{*(i,1-i)}(\zeta^{-1})K^{(\pm)}(\zeta;r,s)_{\epsilon_1}^{\epsilon_2}\Phi_{\epsilon_2}^{(1-i,i)}(\zeta),\ g=\frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty}$$

Following strategy [4], we study our problem upon the following identification:

$$\begin{aligned}T_B^{(\pm,i)}(\zeta;r,s) = \hat{T}_B^{(\pm,i)}(\zeta;r,s), \Phi_\epsilon^{(1-i,i)}(\zeta) = \hat{\Phi}_\epsilon^{(1-i,i)}(\zeta), \Phi_\epsilon^{*(1-i,i)}(\zeta) \ =\\ = \hat{\Phi}_\epsilon^{*(1-i,i)}(\zeta).\end{aligned} \tag{14}$$

The point of using the vertex operators $\Phi_\epsilon^{(1-i,i)}(\zeta)$ associated with $U_q(\widehat{sl}(2))$ is that they are well-defined objects, free from the difficulty of divergence. It is convenient to diagonalize the "renormalized" transfer matrix $T_B^{(\pm,i)}(\zeta;r,s)$ instead of the Hamiltonian $H_B^{(\pm)}$.

2.3 Boundary Vacuum State

We are interested in bosonizations of the boundary vacuum states ${}_B\langle i;\pm|$ given by

$${}_B\langle i;\pm|T_B^{(\pm,i)}(\zeta;r,0) = \Lambda^{(i)}(\zeta;r){}_B\langle i;\pm|, \tag{15}$$

for $i=0,1$. Here we have set $\Lambda^{(0)}(\zeta;r)=1$ and $\Lambda^{(1)}(\zeta;r)=\frac{1}{\zeta^2}\frac{\Theta_{q^4}(r\zeta^2)\Theta_{q^4}(q^2r\zeta^{-2})}{\Theta_{q^4}(r\zeta^{-2})\Theta_{q^4}(q^2r\zeta^2)}$, where $\Theta_p(z)=(p;p)_\infty(z;p)_\infty(p/z;p)_\infty$. We introduce bosons a_m $(m\neq 0)$ and the zero-mode operator ∂,α by

$$[a_m,a_n]=\delta_{m+n,0}\frac{[2m]_q[m]_q}{m}\quad(m,n\neq 0),\quad[\partial,\alpha]=2. \tag{16}$$

The relation between the zero-mode and the fundamental weights are given by $[\partial,\Lambda_0]=0$ and $\Lambda_1=\Lambda_0+\frac{\alpha}{2}$. Using the bosonization of the vertex operators $\Phi_\epsilon^{(1-i,i)}(\zeta)$ we have a bosonization of the boundary vacuum state. The boundary vacuum states ${}_B\langle i;\pm|$ are realized by

$${}_B\langle 0;+| = {}_B\langle 0|\exp_q(-sf_0),\quad {}_B\langle 1;+| = {}_B\langle 1|\exp_{q^{-1}}\left(-\frac{s}{rq}e_1q^{-h_1}\right) \tag{17}$$

$${}_B\langle 0;-| = {}_B\langle 0|\exp_{q^{-1}}\left(\frac{s}{q}e_0q^{-h_0}\right),\quad {}_B\langle 1;-| = {}_B\langle 1|\exp_q\left(\frac{s}{r}f_1\right) \tag{18}$$

where we have used q-exponential $\exp_q(x)=\sum_{n=0}^\infty\frac{q^{\frac{n(n-1)}{2}}}{[n]_q!}x^n$. Here ${}_B\langle i|$ are given by

$$ {}_B\langle i| = \langle i|\exp(G_i), \quad G_i = -\frac{1}{2}\sum_{n=1}^{\infty}\frac{nq^{-2n}}{[2n]_q[n]_q}a_n^2 + \sum_{n=1}^{\infty}\delta_n^{(i)}a_n, $$
$$ \langle i| = 1 \otimes e^{-\Lambda_i}. \tag{19} $$

where we have set

$$ \delta_n^{(i)} = \theta_n \frac{q^{-3n/2}(1-q^n)}{[2n]_q} + \begin{cases} -\dfrac{q^{-5n/2}r^n}{[2n]_q} & (i=0), \\ +\dfrac{q^{-n/2}r^{-n}}{[2n]_q} & (i=1). \end{cases} \tag{20} $$

The boundary vacuum states $|\pm; i\rangle_B$ are realized similarly.

2.4 Boundary Magnetization

We study the boundary magnetization. Let $\mathcal{E}_{\epsilon,\epsilon'}$ be the matrix $E_{\epsilon,\epsilon'}$ at the first site of the space $\mathcal{H}^{(i)}$. We have a realization of this local operator

$$ \mathcal{E}_{\epsilon,\epsilon'} = g\, \Phi_{\epsilon}^{*(i,1-i)}(-q^{-1}\zeta)\Phi_{\epsilon'}^{(1-i,i)}(\zeta)\Big|_{\zeta=1}, \quad g = \frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty}. \tag{21} $$

Hence, using the bosonizations of the vertex operators, the Chevalley generators e_j, f_j, h_j $(j = 0, 1)$, and the boundary vacuum states, we calculate the following vacuum expectation values.

$$ \frac{{}_B\langle i;\pm|\mathcal{E}_{\epsilon,\epsilon'}|\pm; i\rangle_B}{{}_B\langle i;\pm|\pm; i\rangle_B}. \tag{22} $$

For instance, the boundary magnetizations are derived:

$$ \frac{{}_B\langle 0;-|\sigma_1^z|-;0\rangle_B}{{}_B\langle 0;-|-;0\rangle_B} = -1 - 2(1-r)^2\sum_{n=1}^{\infty}\frac{(-q^2)^n}{(1-rq^{2n})^2}, \tag{23} $$

$$ \frac{{}_B\langle 0;-|\sigma_1^+|-;0\rangle_B}{{}_B\langle 0;-|-;0\rangle_B} = s\left(2 + (1-r)\sum_{n=1}^{\infty}(-q^2)^n\frac{2q^{2n} - r(1+q^{4n})}{(1-rq^{2n})^2}\right), \tag{24} $$

$$ \frac{{}_B\langle 0;-|\sigma_1^-|-;0\rangle_B}{{}_B\langle 0;-|-;0\rangle_B} = 0. \tag{25} $$

This is **main result** of the paper [2].

3 $U_q(\widehat{sl}(M+1|N+1))$ Spin Chain with a Diagonal Boundary

3.1 Transfer Matrix

The second progress is a generalization of hidden symmetry [3,6,7,9]. We study U_q $(\widehat{sl}(M+1|N+1))$ spin chain with a diagonal boundary [6]. Let us set $-1 < q < 0$ and $r \in \mathbf{R}$. Let us set $M, N = 0, 1, 2, \cdots (M \neq N)$ and $L, K = 1, 2, \cdots, M+N+2$. For simplicity we assume the condition $L + K \leq M + 1$. (More general cases are studied in [6].) Let us introduce the signatures ν_i $(i = 1, 2, \cdots, M+N+2)$ by $\nu_1 = \cdots = \nu_{M+1} = +$, $\nu_{M+2} = \cdots = \nu_{M+N+2} = -$. Let us set the vector spaces $V_1 = \oplus_{j=1}^{M+1} \mathbf{C} v_j$ and $V_0 = \oplus_{j=1}^{N+1} \mathbf{C} v_{M+1+j}$. In this section we set $V = V_1 \oplus V_0$. The $\mathbf{Z}_2$-grading of the basis $\{v_j\}_{1 \leq j \leq M+N+2}$ of V is chosen to be $[v_j] = \frac{\nu_j + 1}{2}$ $(j = 1, 2, \cdots, M+N+2)$. A linear operator $S \in \mathrm{End}(V)$ is represented in the form of a $(M+N+2) \times (M+N+2)$ matrix : $S v_j = \sum_{i=1}^{M+N+2} v_i S_{i,j}$. The $\mathbf{Z}_2$-grading of $(M+N+2) \times (M+N+2)$ matrix $(S_{i,j})_{1 \leq i,j \leq M+N+2}$ is defined by $[S] = [v_i] + [v_j]$ $(mod.2)$ if RHS of the equation does not depend on i and j such that $S_{i,j} \neq 0$. We define the action of the operator $S_1 \otimes \cdots \otimes S_n$ where $S_j \in \mathrm{End}(V)$ have $\mathbf{Z}_2$-grading.

$$S_1 \otimes S_2 \otimes \cdots \otimes S_n \cdot v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_n}$$
$$= \exp\left(\pi\sqrt{-1} \sum_{k=1}^{n} [S_k] \sum_{l=1}^{k-1} [v_{j_l}]\right) S_1 v_{j_1} \otimes S_2 v_{j_2} \otimes \cdots \otimes S_n v_{j_n}. \qquad (26)$$

We set the R-matrix $R(z) \in \mathrm{End}(V \otimes V)$ for $U_q(\widehat{sl}(M+1|N+1))$ as follows.

$$R(z) = r(z)\bar{R}(z), \quad \bar{R}(z) v_{j_1} \otimes v_{j_2} = \sum_{k_1,k_2=1}^{M+N+2} v_{k_1} \otimes v_{k_2} \bar{R}(z)_{k_1,k_2}^{j_1,j_2}. \qquad (27)$$

Here we have set

$$\bar{R}(z)_{j,j}^{j,j} = \begin{cases} -1 & (1 \leq j \leq M+1), \\ -\dfrac{(q^2-z)}{(1-q^2 z)} & (M+2 \leq j \leq M+N+2), \end{cases} \qquad (28)$$

$$\bar{R}(z)_{i,j}^{i,j} = \frac{(1-z)q}{(1-q^2 z)} \quad (1 \leq i \neq j \leq M+N+2), \qquad (29)$$

$$\bar{R}(z)_{i,j}^{j,i} = \begin{cases} (-1)^{[v_i][v_j]} \dfrac{(1-q^2)}{(1-q^2z)} \ (1 \le i < j \le M+N+2), \\ \\ (-1)^{[v_i][v_j]} \dfrac{(1-q^2)z}{(1-q^2z)} \ (1 \le j < i \le M+N+2), \end{cases} \tag{30}$$

$$\bar{R}(z)_{i,j}^{i,j} = 0 \quad \text{otherwise.} \tag{31}$$

Here we have set

$$r(z) = z^{\frac{1-M+N}{M-N}} \exp\left(-\sum_{m=1}^{\infty} \frac{[(M-N-1)m]_q}{m[(M-N)m]_q} q^m (z^m - z^{-m})\right). \tag{32}$$

The R-matrix $R(z)$ satisfies the graded Yang–Baxter equation.

$$R_{12}(z_1/z_2)R_{13}(z_1/z_3)R_{23}(z_2/z_3) = R_{23}(z_2/z_3)R_{13}(z_1/z_3)R_{12}(z_1/z_2). \tag{33}$$

We set the diagonal K-matrix $K(z) \in \text{End}(V)$ for $U_q(\widehat{sl}(M+1|N+1))$ as follows.

$$K(z) = z^{-\frac{2M}{M-N}} \frac{\varphi(z)}{\varphi(z^{-1})} \bar{K}(z), \quad \bar{K}(z)v_j = \sum_{k=1}^{M+N+2} v_k \delta_{j,k} \bar{K}(z)_j^j, \tag{34}$$

where we have set

$$\bar{K}(z)_j^j = \begin{cases} 1 & (1 \le j \le L), \\ \dfrac{1-r/z}{1-rz} & (L+1 \le j \le L+K), \\ z^{-2} & (L+K+1 \le j \le M+N+2). \end{cases} \tag{35}$$

Here we have set

$$\begin{aligned} \varphi(z) = \exp\Bigg(& \sum_{m=1}^{\infty} \frac{[2(N+1)m]_q}{m[2(M-N)m]_q} z^{2m} + \\ & + \sum_{j=1}^{M} \sum_{m=1}^{\infty} \frac{[2(M-N-j)m]_q}{2m[2(M-N)m]_q} (1-q^{2m}) z^{2m} + \\ & + \sum_{j=M+2}^{M+N+1} \sum_{m=1}^{\infty} \frac{[2(-M-N-2-j)m]_q}{2m[2(M-N)m]_q} (1+q^{2m}) z^{2m} - \\ & - \sum_{m=1}^{\infty} \frac{[(M-N-1)m]_q}{2m[(M-N)m]_q} q^m z^{2m} + \end{aligned} \tag{36}$$

$$+\sum_{m=1}^{\infty}\left\{\frac{[(-M+N+L)m]_q}{m[(N-M)m]_q}(rq^{-L}z)^m+\right.$$
$$\left.+\frac{[(-M+N+L+K)m]_q}{m[(M-N)m]_q}(q^{L-K}z/r)^m\right\}\Bigg).$$

The K-matrix $K(z)\in \mathrm{End}(V)$ satisfies the graded boundary Yang–Baxter equation

$$K_2(z_2)R_{21}(z_1z_2)K_1(z_1)R_{12}(z_1/z_2)=R_{21}(z_1/z_2)K_1(z_1)R_{12}(z_1z_2)K_2(z_2) \quad (37)$$

We introduce the vertex operators $\hat{\Phi}_j(z)$ and the dual vertex operators $\hat{\Phi}_j^*(z)$ for $j=1,2,\cdots,M+N+2$. Matrix elements are given by products of the R-matrix

$$(\hat{\Phi}_j(z))_{\cdots p(N)\cdots p(2)\ p(1)}^{\cdots p(N)'\cdots p(2)' p(1)'}=\lim_{n\to\infty}\sum_{\mu(1),\mu(2),\cdots,\mu(n)=1}^{M+N+2}\prod_{j=1}^{n}R(z)_{\mu(j-1)\ p(j)}^{\mu(j)\ p(j)'}, \quad (38)$$

$$(\hat{\Phi}_j^*(z))_{\cdots p(N)\cdots p(2)\ p(1)}^{\cdots p(N)'\cdots p(2)' p(1)'}=\lim_{n\to\infty}\sum_{\mu(1),\mu(2),\cdots,\mu(n)=1}^{M+N+2}\prod_{j=1}^{n}R(z)_{p(j)\ \mu(j)}^{p(j)'\mu(j-1)}, \quad (39)$$

where $\mu(0)=j$. We expect that the vertex operators $\hat{\Phi}_j(z)$ and $\hat{\Phi}_j^*(z)$ give rise to well-defined operators. Let us set the transfer matrix $\hat{T}_B(z)$ by

$$\hat{T}_B(z)=\sum_{j=1}^{M+N+2}\hat{\Phi}_j^*(z^{-1})K(z)_j^j\hat{\Phi}_j(z)(-1)^{[\nu_j]}. \quad (40)$$

Heuristic arguments suggest that the transfer matrix commutes:

$$[\hat{T}_B(z_1),\hat{T}_B(z_2)]=0 \quad \text{for any } z_1,z_2. \quad (41)$$

The Hamiltonian of this model H_B is given by

$$H_B=\frac{d}{dz}T_B(z)|_{z=1}=\sum_{j=1}^{\infty}h_{j,j+1}+\frac{1}{2}\frac{d}{dz}K_1(z)|_{z=1}, \quad (42)$$

where $h_{j,j+1}=P_{j,j+1}\frac{d}{dz}R_{j,j+1}(z)|_{z=1}$.

3.2 Vertex Operator Approach

We formulate the vertex operator approach to $U_q(\widehat{sl}(M+1|N+1))$ spin chain with a diagonal boundary [6]. Let V_z the evaluation representation of $U_q(\widehat{sl}(M+1|N+1))$ and V_z^{*S} its dual. Let $L(\lambda)$ the irreducible highest representation with level-1 highest weight λ. We introduce the vertex operators $\Phi(z)$ and $\Phi^*(z)$ as the intertwiners of $U_q(\widehat{sl}(M+1|N+1))$:

$$\Phi(z) : L(\lambda) \to L(\mu) \otimes V_z, \quad \Phi(z) \cdot x = \Delta(x) \cdot \Phi(z), \tag{43}$$

$$\Phi^*(z) : L(\mu) \to L(\lambda) \otimes V_z^{*S}, \quad \Phi^*(z) \cdot x = \Delta(x) \cdot \Phi^*(z), \tag{44}$$

for $x \in U_q(\widehat{sl}(M+1|N+1))$. We expand the vertex operators $\Phi(z) = \sum_{j=1}^{M+N+2} \Phi_j(z) \otimes v_j$, $\Phi^*(z) = \sum_{j=1}^{M+N+2} \Phi_j^*(z) \otimes v_j^*$. We set the "normalized" transfer matrix $T_B(z)$ by

$$T_B(z) = g \sum_{j=1}^{M+N+2} \Phi_j^*(z^{-1}) K(z)_j^j \Phi_j(z) (-1)^{[v_j]}, \tag{45}$$

where we have used $g = e^{\frac{\pi\sqrt{-1}M}{2(M-N)}} \exp\left(-\sum_{m=1}^{\infty} \frac{[(M-N-1)m]_q}{m[(M-N)m]_q} q^m\right)$. Following the strategy proposed in [4], we consider our problem upon the following identification.

$$T_B(z) = \hat{T}_B(z), \quad \Phi_j(z) = \hat{\Phi}_j(z), \quad \Phi_j^*(z) = \hat{\Phi}_j^*(z). \tag{46}$$

The point of using the vertex operators $\Phi_j(z), \Phi_j^*(z)$ is that they are well-defined objects, free from the difficulty of divergence. It is convenient to diagonalize the "renormalized" transfer matrix $T_B(z)$ instead of the Hamiltonian H_B.

3.3 Boundary Vacuum State

In this section we give a bosonization of the boundary vacuum state $\langle B|$ given by

$$\langle B| T_B(z) = \langle B|. \tag{47}$$

Let us introduce the bosons and the zero-mode operator [5] by

$$a_n^k, b_n^l, c_n^l, Q_{a^k}, Q_{b^l}, Q_{c^l},$$
$$(n \in \mathbf{Z}, k = 1, 2, \cdots, M+1, l = 1, 2, \cdots, N+1), \tag{48}$$

satisfying the following commutation relations.

$$[a_m^i, a_n^j] = \delta_{i,j}\delta_{m+n,0}\frac{[m]_q^2}{m},\ [a_0^i, Q_{a^j}] = \delta_{i,j},\ [a_0^i, a_0^j] = 0, \tag{49}$$

$$[b_m^i, b_n^j] = -\delta_{i,j}\delta_{m+n,0}\frac{[m]_q^2}{m},\ [b_0^i, Q_{b^j}] = -\delta_{i,j},\ [b_0^i, b_0^j] = 0, \tag{50}$$

$$[c_m^i, c_n^j] = \delta_{i,j}\delta_{m+n,0}\frac{[m]_q^2}{m},\ [c_0^i, Q_{c^j}] = \delta_{i,j},\ [c_0^i, c_0^j] = 0. \tag{51}$$

Let us introduce the generating function $c^i(z) = -\sum_{n\neq 0}\frac{c_n^i}{[n]_q}z^{-n} + Q_{c^i} + c_0^i \log z$. We introduce the projection operators $\xi_0 = \prod_{j=1}^{N+1}\xi_0^j$ and $\eta_0 = \prod_{j=1}^{N+1}\eta_0^j$, where we have set $\xi^j(z) = \sum_{m\in\mathbf{Z}}\xi_m^j z^{-m} =: e^{-c^j(z)}:$ and $\eta^j(z) = \sum_{m\in\mathbf{Z}} \eta_m^j z^{-m-1} =: e^{c^j(z)}:$. Using the bosonizations of the vertex operators, we have a bosonization of the boundary vacuum state $\langle B|$. However the vertex operator for $U_q(\widehat{sl}(M+1|N+1))$ is realized by "sum", a bosonization of boundary vacuum state is realized by "monomial". Let us set the highest weight vector $v_{\Lambda_{M+1}}^* = \langle 0|e^{-\beta\sum_{i=1}^{M+1}Q_{a^i}+(1-\beta)\sum_{j=1}^{N+1}Q_{b^j}+\sum_{j=1}^{N+1}Q_{c^j}}$, where $\langle 0|$ satisfying $\langle 0|a_n^i = \langle 0|b_n^j = \langle 0|c_n^j = 0$ for $n\geq 0$ and $1\leq i\leq M+1$, $1\leq j\leq N+1$. Let us set

$$h_{i,m}^* = \sum_{j=1}^{M+N+1}\frac{[\alpha_{i,j}m]_q[\beta_{i,j}m]_q}{[(M-N)m]_q[m]_q}h_{j,m}, \tag{52}$$

where we have used $h_{i,m} = a_m^i q^{-|m|/2} - a_m^{i+1}q^{|m|/2}$, $h_{M+1,m} = a_m^{M+1} q^{-|m|/2} + b_m^1 q^{-|m|/2}$, and $h_{M+1+j,m} = -b_m^j q^{|m|/2} + b_m^{j+1}q^{-|m|/2}$. Here we have set

$$\alpha_{i,j} = \begin{cases} \mathrm{Min}(i,j) & (\mathrm{Min}(i,j)\leq M+1),\\ 2(M+1)-\mathrm{Min}(i,j) & (\mathrm{Min}(i,j)>M+1),\end{cases} \tag{53}$$

$$\beta_{i,j} = \begin{cases} M-N-\mathrm{Max}(i,j) & (\mathrm{Max}(i,j)\leq M+1),\\ -M-N-2+\mathrm{Max}(i,j) & (\mathrm{Max}(i,j)>M+1).\end{cases} \tag{54}$$

A bosonization of the boundary vacuum state $\langle B|$ is given by

$$\langle B| = v_{\Lambda_{M+1}}^* \exp(G)\cdot \eta_0\xi_0. \tag{55}$$

Here we have set the bosonic operator G by

$$G=-\frac{1}{2}\sum_{j=1}^{M+N+1}\sum_{m=1}^{\infty}\frac{mq^{-2m}}{[m]_q^2}h_{j,m}h_{j,m}^*-\frac{1}{2}\sum_{j=1}^{N+1}\sum_{m=1}^{\infty}\frac{mq^{-2m}}{[m]_q^2}c_m^jc_m^j$$
$$+\sum_{j=1}^{M+N+1}\sum_{m=1}^{\infty}\beta_{j,m}^{(3)}h_{j,m}^*+\sum_{j=1}^{N+1}\sum_{m=1}^{\infty}\gamma_{j,m}c_m^j, \tag{56}$$

where we have used

$$\gamma_{j,m}=-\frac{q^{-m}}{[m]_q}\theta_m, \tag{57}$$

$$\beta_{j,m}^{(3)}=\beta_{j,m}^{(1)}-\frac{r^mq^{(-L-3/2)m}}{[m]_q}\delta_{j,L}-\frac{q^{(L-K-3/2)m}/r^m}{[m]_q}\delta_{j,L+K}, \tag{58}$$

$$\beta_{j,m}^{(1)}=\begin{cases}\dfrac{q^{-3m/2}-q^{-m/2}}{[m]_q}\theta_m & (1\le j\le M),\\[2ex] \dfrac{-2q^{-3m/2}}{[m]_q}\theta_m & (j=M+1),\\[2ex] \dfrac{q^{-3m/2}+q^{-m/2}}{[m]_q}\theta_m & (M+2\le j\le M+N+1).\end{cases} \tag{59}$$

This is **main result** of the paper [6].

Conclusion

From the above progress, we suppose that the boundary vacuum state $\langle B|$ of semi-infinite $U_q(g)$ spin chain with a triangular boundary is realized as follows.

$$\langle B|=\langle vac|\exp(\mathcal{B})\exp_q(\mathcal{C}). \tag{60}$$

where $\mathcal{B}$ is a quadratic expression in the bosons and $\mathcal{C}$ is a simple expression in the Chevalley generators. We would like to check this conjecture in the future.

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Thermopower in the Coulomb Blockade Regime for Laughlin Quantum Dots

Lachezar S. Georgiev

Abstract Using the conformal field theory partition function of a Coulomb-blockaded quantum dot, constructed by two quantum point contacts in a Laughlin quantum Hall bar, we derive the finite-temperature thermodynamic expression for the thermopower in the linear-response regime. The low-temperature results for the thermopower are compared to those for the conductance and their capability to reveal the structure of the single-electron spectrum in the quantum dot is analyzed.

1 What are Quantum Dots and Why Study Them?

Quantum dots (QD) are mesoscopic conducting islands of two-dimensional (incompressible) electron gas constructed on the metal-oxide-semiconductor interface in a typical field-effect transistor [1,2]. The semiconductor bar contains a small number of bulk charge carriers (electrons or holes) which are pushed out to an overlaying oxide insulator layer by means of electric field perpendicular to the interface surface, creating in this way a two-dimensional film of strongly correlated electrons with a finite geometry realized by a confining potential. Under appropriate conditions (low temperature, high perpendicular magnetic fields in a high-mobility semiconductor samples) the strongly correlated electron gas can be found to be in the quantum Hall regime (integer or fractional) and for simplicity we will think of it as a two-dimensional droplet of quantum Hall liquid with disk shape whose dynamics is concentrated on the one-dimensional edge which is a circle.

The QDs have a number of interesting properties and are essential part of the so called single-electron transistors (SET) which explains why they have been the subject of intense research in recent years. Because of the small size of the QDs (typical circumference of several μm) and its isolation form the rest of the system (only small tunneling is considered), QDs are almost closed quantum systems with a discrete energy spectrum at very low temperatures, which make them similar to large artificial atoms in which one can investigate both fundamental concepts of quantum

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theory and important application aspects of nanoelectronics as well as transcend the cutting-edge research-and-development perspectives for the implementation of quantum computers and quantum information processing.

The incompressible fractional quantum Hall liquids have been successfully described by two-dimensional rational conformal field theories [3] (CFT) governing the dynamics of their edge excitations [4]. In this contribution we will show how one can use the CFT for QDs, realized inside of quantum Hall bar corresponding to the $\nu_H = 1/m$ Laughlin state, to calculate observable thermodynamic characteristics of the QDs, such as the tunneling conductance and thermopower.

2 Quantum Dots and Single-Electron Transistors

When a QD is equipped with drain and source gates, as shown on Fig. 1, by applying a drain-source voltage one could in principle transfer electrons from the left FQH liquid to the QD and then to the right FQH liquid. However, a tunneling electron from left to the QD must overcome the Coulomb charging energy $e^2/2C$, associated with adding one extra electron to the QD, where C is the total capacitance of the QD. When the QD is small so is C and this Coulomb charging energy could be large, so that at low temperature $k_B T \ll e^2/C$ and small bias the electron transfer is blocked. This is called the Coulomb blockade [1, 2, 5]. Because we are interested in the small-bias regime, which can be treated by linear response, one way to lift the Coulomb blockade at small bias is to add a third electrode called the Side gate, see Fig. 1. Then, by changing the gate voltage V_g one can shift the discrete energy levels of the QD, still in the linear response regime, to align them with the Fermi levels of the left and right FQH liquids and when this happens one electron can tunnel from left to the right through the QD. Since the electrons tunnel one-by-one with the variation of V_g this three-gate QD construction is called a Single-electron transistor, see Fig. 1 for its scheme.

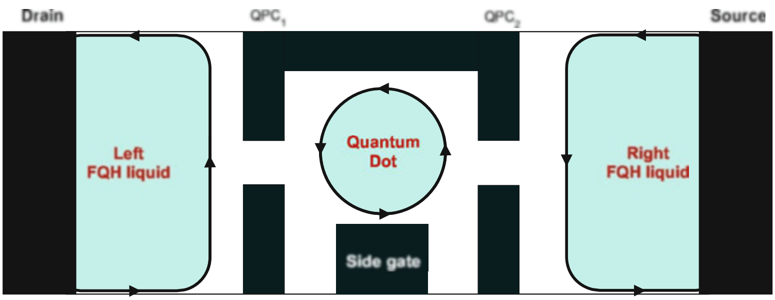

Fig. 1 Single-electron transistor realized by two quantum-point contacts (QPC_1 and QPC_2) inside of $\nu_H = 1/m$ Laughlin FQH state. The *arrows* show the direction of the propagation of the edge modes. Only electrons can tunnel between the left and right FQH liquids and the QD under appropriate conditions

The QD in the SET is an almost closed quantum system of size from 0.1 to 1 μ m with discrete single-electron energy levels of typical spacing $\Delta\varepsilon = \hbar 2\pi v_F/L$, where v_F is the Fermi velocity of the edge mode and L is the circumference of the edge circle. Only small tunneling is allowed between the leads and the QD, i.e., the tunneling conductances for QPC_1 and QPC_2 are much smaller than the conductance quantum: $G_{L/R} \ll e^2/h$, which guarantees that the single-particle energy levels in the QD remain discrete. At low temperature the number of electrons on the QD is quantized to be integer and can be computed as a derivative of the thermodynamic density of states with respect to the chemical potential—here we can use the RCFT partition function as a thermodynamical Grand potential. Thus the QDs are very similar to large artificial atoms—almost 1,000 times bigger than the average atoms, they are highly tunable, yet still purely quantum systems! For example, one magnetic flux quantum in an atom requires magnetic field of the order of 10^6 T, while for QDs the corresponding field is of order of 1 T [1]. This makes QDs very convenient for verification of fundamental concepts of quantum theory as well as for quantum computation and information processing.

For small QD and small bias the charging effects leading to the Coulomb blockade become important at low T such that $k_B T \ll e^2/C$. The variation of the side gate voltage V_g induces external electric charge on the QD and creates charge imbalance between the QD and the side gate which changes continuously the single-particle energies of the QD lifting in this way the CB [1,5].

Changing adiabatically the side gate voltage V_g at small-bias tunneling, between the left- and right- FQH liquids and the QD, results in a precise QD level spectroscopy which can be treated analytically in the linear response regime under the following conditions:

- low temperature $k_B T \ll e^2/C$
- low bias $V \ll e/C$
- low QPC conductances $G_{L,R} \ll e^2/h$

Under these conditions the sequential tunneling of electrons one-by-one is dominating the cotunneling, which is a higher-order process associated with almost simultaneous virtual tunneling of pairs of electrons [2], that will not be considered here.

3 QD Conductance–CFT Spectroscopy

The tunneling conductance of the QD in the linear response regime can be computed at low temperature from the Grand canonical partition function [6]

$$Z_{\text{disk}}(\tau,\zeta) = \text{tr}_{\mathcal{H}_{\text{edge}}}\, \mathrm{e}^{-\beta(H_{\text{CFT}}-\mu N_{\text{el}})} = \text{tr}_{\mathcal{H}_{\text{edge}}}\, \mathrm{e}^{2\pi i\tau(L_0-c/24)} e^{2\pi i\zeta J_0}, \tag{1}$$

which describes the dynamics of the edge in terms of CFT assuming that the bulk of the QD is inert. In Eq. (1) we have denoted by $H_{\mathrm{CFT}} = \hbar \frac{2\pi v_F}{L}\left(L_0 - \frac{c}{24}\right)$ the edge states' Hamiltonian, by $N_{\mathrm{el}} = -\sqrt{\nu_H} J_0$ the electron number operator on the edge, L_0 is the zero mode of the Virasoro stress-tensor [3], J_0 is the normalized zero mode of the $\widehat{u(1)}$ current algebra [3, 7] and ν_H denotes the FQH filling factor. The trace in Eq. (1) is taken over the edge-states' Hilbert space $\mathcal{H}_{\mathrm{edge}}$ whose structure might depend on the presence of quasiparticles in the bulk of the QD [7].

The modular parameters [3] of the rational CFT are related to the temperature T and chemical potential μ of the QD

$$\tau = i\pi \frac{T_0}{T}, \quad T_0 = \frac{\hbar v_F}{\pi k_B L}, \quad \zeta = i \frac{1}{2\pi k_B T}\mu. \tag{2}$$

The disk CFT partition function for the Grand canonical ensemble in presence of AB flux ϕ can be expressed in a compact way by shifting the chemical potential [8]

$$\zeta \to \zeta + \phi\tau, \quad Z^{\phi}_{\mathrm{disk}}(\tau,\zeta) \stackrel{\mathrm{def}}{=} \mathrm{tr}_{\mathcal{H}_{\mathrm{edge}}}\, \mathrm{e}^{-\beta(H_{\mathrm{CFT}}(\phi) - \mu N_{\mathrm{imb}}(\phi))} \equiv Z_{\mathrm{disk}}(\tau, \zeta + \phi\tau), \tag{3}$$

where $N_{\mathrm{imb}}(\phi) = N_{\mathrm{el}} - \nu_H \phi$ is the *particle imbalance due to the gate voltage*, see the explanations after Eq. (12) below; what we will need here is the last expression in Eq. (3). The thermodynamic Grand potential on the edge is expressed in terms of the partition function as usual

$$\Omega_{\phi}(T,\mu) = -k_B T \ln Z^{\phi}_{\mathrm{disk}}(\tau,\zeta). \tag{4}$$

The edge conductance has been shown to be proportional to the derivative of the thermodynamic density of states with respect to the chemical potential [6], i.e.

$$G_{\mathrm{is}}(\phi) = \frac{e^2}{h}\left(\nu_H + \frac{1}{2\pi^2}\left(\frac{T}{T_0}\right)\frac{\partial^2}{\partial\phi^2}\ln Z_{\phi}(T,0)\right). \tag{5}$$

The conductance for the $\nu = 1/3$ Laughlin QD, computed by Eq. (5) from the partition function (6) given in the next section with $l = 0$ at temperature $T = T_0$, shows vast regions in which it is zero (CB valleys) and sharp peaks at values $\phi_i = 3/2 + 3i$, $i = 0, \pm 1, \pm 2, \ldots$ as shown in Fig. 2.

4 The Laughlin QD Partition Function

The grand partition function for the edge of a QD in the $\nu_H = 1/m$ Laughlin FQH state can be written as

$$K_l(\tau,\zeta;m) = \frac{\mathrm{CZ}}{\eta(\tau)} \sum_{n=-\infty}^{\infty} q^{\frac{m}{2}\left(n+\frac{l}{m}\right)^2} \mathrm{e}^{2\pi i\zeta\left(n+\frac{l}{m}\right)}, \tag{6}$$

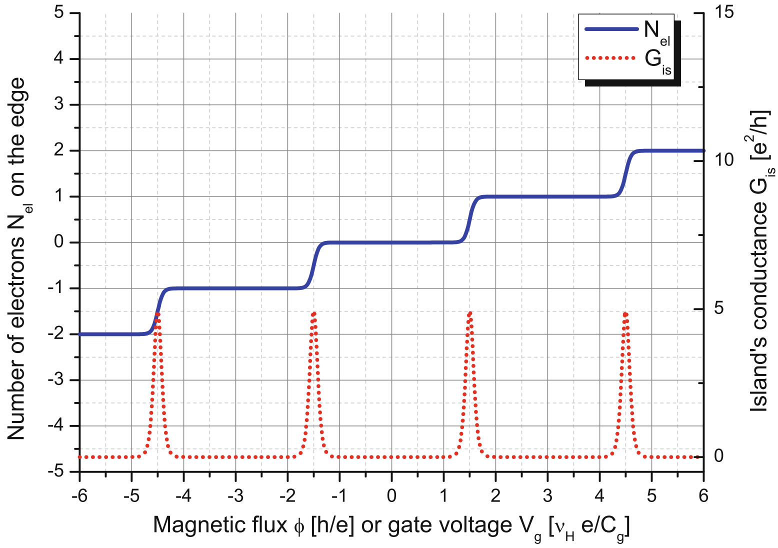

Fig. 2 Electron number average $N_{\rm el}$ on the edge and Coulomb blockade conductance $G_{\rm is}$ for the $\nu_H = 1/3$ Laughlin island without bulk quasiparticles as a function of the gate voltage at temperature $T = T_0$

where $q = \mathrm{e}^{-\beta\Delta\varepsilon} = \mathrm{e}^{2\pi i\tau}$ with $\beta = (k_B T)^{-1}$ and $\Delta\varepsilon = \hbar\frac{2\pi v_F}{L}$. The index of the K-function $l = -(m-1)/2, \ldots, (m-1)/2$ (m must be an odd integer) corresponds to a Hilbert space $\mathcal{H}_l$ with quasiparticles in the bulk [7] with electric charge l/m. The Dedekind function η and the Cappelli–Zemba factor [4] are

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \quad \mathrm{CZ} = \mathrm{e}^{-\pi\nu_H\frac{(\mathrm{Im}\,\zeta)^2}{\mathrm{Im}\,\tau}},$$

however, for our purposes they would be unimportant since we would set $\zeta = 0$ at the end [6, 8].

5 Thermopower: A Finer Spectroscopic Tool

The thermopower S, known also as the Seebeck coefficient, is the potential difference V between the leads of the SET when the two leads are at different temperature T_R and T_L, assuming that the difference is small $\Delta T = T_R - T_L \ll T_L$, under the condition that the current I between the leads is zero [2]. Usually thermopower is expressed as the ratio of the thermal conductance G_T and electric conductance G, i.e., $S = G_T/G$, however, this expression is not appropriate for SETs because $G = 0 = G_T$, while their ratio is finite, in vast intervals of flux (in the CB valleys), see Fig. 2. Fortunately, there is an alternative expression in terms of the average energy $\langle\varepsilon\rangle$ of the electrons tunneling through the QD [2]

$$S \equiv -\lim_{\Delta T\to 0} \frac{V}{\Delta T}\bigg|_{I=0} = -\frac{\langle\varepsilon\rangle}{eT}.$$

where $T = T_L + \Delta T/2$ is the temperature of the QD.

The average tunneling energy could be computed thermodynamically using as thermodynamical potential the rational CFT partition function for the FQH edge of the QD. To this end we notice that due to energy conservation in single-electron tunneling the average tunneling energy is simply the difference between the total thermodynamic average energy of the QD with $N+1$ and N electrons at the same temperature T and AB flux ϕ (respectively, gate voltage V_g) divided by the difference in the electron numbers of the QD as a function of ϕ

$$\langle\varepsilon\rangle^{\phi}_{\beta,\mu_N} = \frac{E^{\beta,\mu_{N+1}}_{\mathrm{QD}}(\phi) - E^{\beta,\mu_N}_{\mathrm{QD}}(\phi)}{N^{\beta,\mu_{N+1}}_{\mathrm{QD}}(\phi) - N^{\beta,\mu_N}_{\mathrm{QD}}(\phi)}. \tag{7}$$

Because we are working within the Grand canonical ensemble, the total energy of the QD with N electrons requires the chemical potential μ_N to be determined. It is defined as the chemical potential for which the average of the particle number operator is equal to the number N at zero gate voltage (AB flux)

$$\nu_H\left(\frac{\mu_N}{\Delta\epsilon} + \phi\right) - \frac{\partial\Omega_\phi(\beta,\mu_N)}{\partial\phi} = N. \tag{8}$$

The total energy of an N-electron QD within the *Constant Interaction model* [1] is

$$E^{\beta,\mu_N}_{\mathrm{QD}}(\phi) = \sum_{i=1}^{N_0} E_i(B) + \langle H_{\mathrm{CFT}}(\phi)\rangle_{\beta,\mu_N} + U(N), \tag{9}$$

where N_0 is the number of electrons in the bulk of the QD and $N - N_0 = N_{\mathrm{el}}$ is the number of electrons on the edge, $E_i(B)$, $i = 1,\ldots,N_0$, are the energies of the occupied single-electron states in the bulk of the QD, the expectation value $\langle\cdots\rangle_{\beta,\mu}$ is the Grand canonical average of the Hamiltonian H_{CFT} on the edge, and $U(N)$ is the (B-independent) electrostatic energy of the QD, including the contribution due to the gate voltage V_g is (see Eq. (1) in [1])

$$U(N) = \frac{\left[e(N-N_0) - C_g V_g\right]^2}{2C}, \tag{10}$$

where $N = N_0$ for $V_g = 0$. The total capacitance $C = C_g + C_1 + C_2$, where C_g is the capacitance of the side gate, C_1 and C_2 are the capacitances of the two QPCs, is assumed independent of N and this assumption a characteristic for the Constant Interaction model [1]. Within this model the energies E_i depend on the magnetic field B and on the gate voltage V_g, but not on N [5]. In the case of a FQH island we know that the variation of V_g modifies also the single-electron energies on the edge [6, 9–11] due to a variation of the CB island's area A, producing a variation of the AB flux ϕ. Because the variation of the gate voltage V_g induces (continuously varying) "external charge" $eN_g = C_g V_g$ on the edge, it is equivalent to the AB flux-induced variation of the particle number $N_\phi = \nu_H \phi$, so that we can take into account the subtler effects of the gate voltage on the edge energies $\langle H_{\mathrm{CFT}}(\phi)\rangle_{\beta,\mu_N}$ by introducing AB flux ϕ determined from[1]

$$\frac{C_g V_g}{e} \equiv \nu_H \phi, \quad \phi = \frac{e}{h}(A - A_0)\,B, \tag{11}$$

where A_0 is the area of the CB island at $V_g = 0$. Therefore, when we speak about Coulomb blockade caused by a variation of the AB flux ϕ we actually mean a variation of the gate voltage V_g determined from (11). It is worth stressing that the electron number N_{el} on the QD is quantized to be integer, while "particle number imbalance" $N_{\mathrm{imb}} = (N - N_0) - C_g V_g/e$, between the QD and the side gate, changes continuously when the gate voltage V_g is varied [1, 5]. It is also interesting to mention that according to (11) the AB flux distance between two neighboring CB peaks is $\Delta\phi = \nu_H^{-1}$ because then $\Delta N_\phi = 1$ so that an entire additional electron can be transferred through the QD. It corresponds to gate voltage periodicity between CB peaks equal to $e\Delta V_g = (1/\alpha_g)(e^2/C)$, where $\alpha_g = C_g/C$ is called the gate's lever arm [1].

Using the AB flux instead of the gate voltage like in Eq. (11) is convenient because the flux can be interpreted mathematically as a continuous twisting of the $\widehat{u(1)}$ charge of the underlying chiral algebra [3, 8], which is technically similar to the rational (orbifold) twisting of $\widehat{u(1)}$ current [12], i.e., its zero mode is modified by

$$J_0 \to \pi_\beta(J_0) = J_0 - \beta \quad \text{with} \quad \beta = -\sqrt{\nu_H}\phi. \tag{12}$$

Then the average of the twisted electric $\widehat{u(1)}$ current $\pi_\beta(J_0^{\mathrm{el}}) \equiv \sqrt{\nu_H}\pi_\beta(J_0)$ is proportional to the thermodynamic derivative of the Grand potential $\partial\Omega_\phi/\partial\phi = \langle\pi_\phi(J_0^{\mathrm{el}})\rangle$ whose physical meaning is the electrostatic charge imbalance between the CB island and the gate arising due to the gate voltage. The untwisted $\widehat{u(1)}$ charge, which is proportional to the electron number on the edge $J_0^{\mathrm{el}} = \sqrt{\nu_H} J_0 = -N_{\mathrm{el}}$, is according to (12) $\langle J_0^{\mathrm{el}}\rangle = \langle\pi_\phi(J_0^{\mathrm{el}})\rangle - \nu_H\phi$ and this is equivalent to the following Grand canonical thermal average of the electron particle number on the edge, which is illustrated in Fig. 2 for the $\nu_H = 1/3$ Laughlin state without quasiparticles in the bulk

[1]For a one-dimensional circular edge all thermodynamic quantities depend on the magnetic flux not on the magnetic filed itself. Thus, the flux of the constant B has the same effect on the partition function as the singular AB flux, which is however, easier to take into account analytically [8].

$$\langle N_{\rm el}(\phi)\rangle_{\beta,\mu_N} = -\frac{\partial \Omega_\phi(\beta,\mu_N)}{\partial\phi} + \nu_H\phi + \nu_H\left(\frac{\mu_N}{\Delta\varepsilon}\right)$$
$$= \nu_H\left(\phi + \frac{\mu_N}{\Delta\varepsilon}\right) + \frac{1}{2\pi^2}\left(\frac{T}{T_0}\right)\frac{\partial}{\partial\phi}\ln Z_\phi(T,\mu_N) \qquad (13)$$

6 Average Tunneling Energy

Taking into account Eqs. (7) and (9), and neglecting the electrostatic energy $U(N)$ for large CB islands as in [13], we can compute the thermodynamic average energy of a single electron tunneling to the QD with N electrons by

$$\langle\varepsilon\rangle^{\phi}_{\beta,\mu_N} = \frac{\langle H_{\rm CFT}(\phi)\rangle_{\beta,\mu_{N+1}} - \langle H_{\rm CFT}(\phi)\rangle_{\beta,\mu_N}}{\langle N_{\rm el}(\phi)\rangle_{\beta,\mu_{N+1}} - \langle N_{\rm el}(\phi)\rangle_{\beta,\mu_N}}. \qquad (14)$$

Notice that the first term in the r.h.s of Eq. (9) cancels, while the electrostatic energy $U(N)$ is subleading for large CB islands, which are of experimental interest [13, 14], and is omitted.

The average of the edge Hamiltonian is computed according to the standard formula for the Grand canonical ensemble [15]

$$\langle H_{\rm CFT}(\phi)\rangle_{\beta,\mu_N} = \Omega_\phi(T,\mu_N) - T\frac{\partial\Omega_\phi(T,\mu_N)}{\partial T} - \mu_N\frac{\partial\Omega_\phi(T,\mu_N)}{\partial\mu} \qquad (15)$$

where $\Omega_\phi(T,\mu_N)$ is the Grand potential in presence of AB flux ϕ defined in (4). Introducing the AB flux ϕ and chemical potential μ into the partition function (6) according to (3) and moving the ϕ and μ dependence into the index l of (6), see [6, 8], we obtain (a factor independent of μ and ϕ is omitted)

$$Z_\phi(T,\mu) = K_{\frac{\mu}{\Delta\varepsilon}+\phi}(\tau,0;m) \propto \sum_{n=-\infty}^{\infty} q^{\frac{m}{2}\left(n+\frac{\mu/\Delta\varepsilon+\phi}{m}\right)^2}. \qquad (16)$$

The partition function (16) has a remarkable symmetry—adding one electron to the ground state, which is equivalent to increasing the flux by m, does not change it, i.e., $Z_\phi(T,\mu^{\rm GS}_{N+1}) = Z_{\phi+m}(T,\mu^{\rm GS}_N) = Z_\phi(T,\mu^{\rm GS}_N)$, implying $\Omega_\phi(T,\mu^{\rm GS}_{N+1}) = \Omega_\phi(T,\mu^{\rm GS}_N)$ and

$$\frac{\partial\Omega_\phi(T,\mu^{\rm GS}_{N+1})}{\partial T} = \frac{\partial\Omega_\phi(T,\mu^{\rm GS}_N)}{\partial T}, \quad \frac{\partial\Omega_\phi(T,\mu^{\rm GS}_{N+1})}{\partial\mu} = \frac{\partial\Omega_\phi(T,\mu^{\rm GS}_N)}{\partial\mu}. \qquad (17)$$

Using the symmetry (17) we can find the difference between the ground-states chemical potentials of the QD with N and $N+1$ electrons. Indeed, writing Eq. (8) for N and $N+1$ electrons

$$\nu_H\left(\frac{\mu_N^{\mathrm{GS}}}{\Delta\varepsilon}+\phi\right)-\frac{\partial\Omega_\phi(\beta,\mu_N^{\mathrm{GS}})}{\partial\phi}=N$$

$$\nu_H\left(\frac{\mu_{N+1}^{\mathrm{GS}}}{\Delta\varepsilon}+\phi\right)-\frac{\partial\Omega_\phi(\beta,\mu_{N+1}^{\mathrm{GS}})}{\partial\phi}=N+1$$

and subtracting them we obtain $\mu_{N+1}^{\mathrm{GS}}-\mu_N^{\mathrm{GS}}=m\Delta\varepsilon$. This means that the chemical potentials μ_N^{GS} and μ_{N+1}^{GS} cannot be both set to 0. Adjusting the chemical potential for $\phi=0$ to be in the middle between μ_N^{GS} and μ_{N+1}^{GS} (center of the CB valley), i.e., assuming we obtain

$$\mu_N^{\mathrm{GS}}=-\frac{m}{2}\Delta\varepsilon,\quad \mu_{N+1}^{\mathrm{GS}}=\frac{m}{2}\Delta\varepsilon.$$

These values of the chemical potentials determine the ground-state energies of the CB island with N and $N+1$ electrons and their difference gives the addition energy characterizing the energy spacing of the CB conductance peaks. However, for the calculation of the average tunneling energy (14) we need to find the difference between the energies of the N-th occupied single-particle state in the QD and the next available one, which is not the ground state with $N+1$ electrons. Instead, the next available single-particle state can be obtained from the last occupied state by increasing adiabatically the AB flux threading the edge by exactly one flux quantum. This is equivalent to increasing $\mu/\Delta\varepsilon$ by 1 so that the difference between the two chemical potentials is $\mu_{N+1}-\mu_N=\Delta\varepsilon$. Therefore, choosing again a symmetric setup so that $\mu_N+\mu_{N+1}=0$, we obtain

$$\mu_N=-\frac{\Delta\varepsilon}{2},\quad \mu_{N+1}=\frac{\Delta\varepsilon}{2}. \tag{18}$$

Next, we can compute numerically the two edge energy averages (15) for a $\nu_H=1/3$ QD with N and $N+1$ electrons with chemical potentials (18). The plot of the thermopower for $T/T_0=1$ and $T/T_0=1.5$ and the conductance at $T/T_0=1$ are given in Fig. 3. The plot of the thermopower has a sawtooth shape like that in metallic CB islands [2]. Also it is interesting to note that thermopower vanishes at the conductance peaks position in the same way as it does for metallic islands, expressing the fact that the energy difference between the QD with N and $N+1$ electrons is zero at the maximum of the conductance peak. In the middle of the CB valleys the thermopower has sharp jumps (discontinuous at $T=0$), expressing the particle-hole symmetry in the centers of the valleys [2].

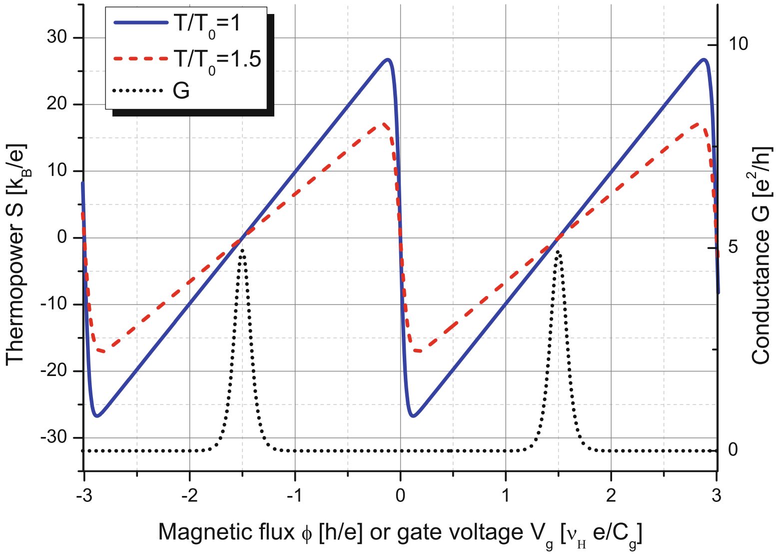

Fig. 3 Thermopower of the $\nu_H = 1/m$ Laughlin state with $m = 3$ at temperatures $T = T_0$ and $T = 1.5T_0$. The conductance at $T = T_0$ is also shown on the right vertical scale

7 Conclusion and Perspectives

We have shown that the Constant Interaction model works fine for the Laughlin CB islands. Thermopower is non-zero in the CB valleys while the electric and thermal conductances are both zero. The period of the thermopower is $\Delta\phi = m$ and its zeros correspond to the conductance peaks. Thermopower appears to be more sensitive to the neutral modes in the FQH liquid than the tunneling conductance which explains why it is considered a finer spectroscopic tool. This could make thermopower an appropriate observable, which could distinguish between different FQH states with similar CB conductance patterns [16], and therefore it would be interesting to apply this approach to FQH QDs with filling factors $\nu_H = n_H/d_H$ for $n_H \geq 2$, especially for non-Abelian FQH states. The sensitivity of the thermopower depends, however, on the relative sizes of the Coulomb charging energy and single-particle energies of the QD, which depend on the size and quality of the CB island. The experimental realization of CB islands in the fractional quantum Hall regime is challenging, however efforts have been made to measure the thermoelectric properties of such systems [13]. For example, in a recent experiment these properties have been investigated for the $\nu_H = 2/3$ FQH state [13, 14] which is similar to the $\nu_H = 1/3$ Laughlin state but is expected to have a more complicated structure related to neutral modes.

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On a Pair of Difference Equations for the ${}_4F_3$ Type Orthogonal Polynomials and Related Exactly-Solvable Quantum Systems

E.I. Jafarov, N.I. Stoilova, and J. Van der Jeugt

Abstract We introduce a pair of novel difference equations, whose solutions are expressed in terms of Racah or Wilson polynomials depending on the nature of the finite-difference step. A number of special cases and limit relations are also examined, which allow to introduce similar difference equations for the orthogonal polynomials of the ${}_3F_2$ and ${}_2F_1$ types. It is shown that the introduced equations allow to construct new models of exactly-solvable quantum dynamical systems, such as spin chains with a nearest-neighbour interaction and fermionic quantum oscillator models.

1 Introduction

The importance of orthogonal polynomials in the study of quantum dynamical systems is undisputable. Without the knowledge of basic properties of orthogonal polynomials, it is impossible to comprehend the existence of explicit solutions of quantum systems such as the quantum harmonic oscillator, the Coulomb problem or Heisenberg spin chains. A long time ago, different types of orthogonal polynomials were studied separately. Then the idea grew that some of them are special case of others, and that they can be generalized. Thus the discovered polynomials could be unified in a table, each having its own level and cell in that table. This table is called the Askey scheme of hypergeometric orthogonal polynomials. The importance of

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this table is that it gathers all polynomials, some of them satisfying an orthogonality relation in the continuous space and others in a discrete space, some with a finite support and others with an infinite support [1].

Hermite polynomials are the most attractive ones from the Askey scheme, because they have no free parameters and occupy the lowest level of the table, that is the level where there is no sign of the discreteness of the space. They are well known as the explicit solution of the 1D non-relativistic quantum harmonic oscillator in a canonical approach [2]. The dynamical symmetry of this quantum system is also well known and it is the Heisenberg–Weyl algebra. This algebra can be easily constructed by using the three-term recurrence relations of Hermite polynomials. If, as a next step, one drops the canonical commutation relation between position and momentum operator $[\hat{p}, \hat{x}] = -i$ [3], then one observes the very interesting behaviour of the solution of the 1D non-relativistic quantum harmonic oscillator. Now the solution is expressed in terms of the generalized Laguerre polynomials, and the dynamical symmetry of the system is the Lie superalgebra $osp\,(1|2)$. It is constructed by using two kind of three-term recurrence relations of the generalized Laguerre polynomials, which are intertwined. The existence of more than one recurrence relations for these polynomials has the following explanation. Laguerre polynomials occupy the next level in the Askey scheme: they generalize Hermite polynomials and have one free parameter. This parameter allows to separate the recurrence relations for even and odd polynomials, and thus obtain the new form of the recurrence relations for generalized Laguerre polynomials, which leads to the quite interesting so called non-canonical solution of the 1D non-relativistic quantum harmonic oscillator [4]. It is known that such a method can also be applied to polynomials from the next levels of the Askey scheme, and similar recurrence relations exist for continuous dual Hahn polynomials [5], generalizing both Meixner–Pollaczek and Laguerre polynomials. Their application allows to construct a new model of the quantum harmonic oscillator, whose algebra is the Lie algebra $su(1,1)$ deformed by a reflection operator [6]. A similar approach in finite-discrete configuration space leads to the new difference equations (or recurrence relations) for the Hahn or dual Hahn polynomials and they generalize the difference equation for Krawtchouk polynomials (due to duality of Krawtchouk polynomials, the difference equation can be transformed to the three-term recurrence relation). Application of such recurrence relations leads to two very interesting quantum mechanical solutions, one of which is a finite-discrete quantum oscillator model based on the Lie algebra $u(2)$ extended by a parity operator [7] and other one is the case of perfect state transfer over the spin chain of fermions with a nearest-neighbour interaction under absence of the external magnetic field [8].

In current work, we continue this procedure and report on the pairs of three-term difference equations and recurrence relations for the Racah and Wilson polynomials, which occupy the top level of the Askey scheme and generalize all discrete and continuous orthogonal polynomials from this table. We also discuss some special cases, when new three-term difference equations exist also for Hahn polynomials and they lead to a pair of difference equations for the continuous Hahn polynomials.

2 Racah Polynomials and New Three-Term Recurrence Relations

The Racah polynomial $R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$ of degree n $(n = 0,1,\ldots,m)$ in the variable x is defined by:

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta) = {}_4F_3\left(\begin{matrix} -n, n+\alpha+\beta+1, -x, x+\gamma+\delta+1 \\ \alpha+1, \beta+\delta+1, \gamma+1 \end{matrix}; 1\right), \tag{1}$$

where $\lambda(x) = x(x+\gamma+\delta+1)$ and $\alpha+1 = -m$ or $\beta+\delta+1 = -m$ or $\gamma+1 = -m$, with m being a nonnegative integer.

They satisfy a finite-discrete orthogonality relation of the following form:

$$\sum_{x=0}^{m} w(x)\, R_l(\lambda(x);\alpha,\beta,\gamma,\delta)\, R_n(\lambda(x);\alpha,\beta,\gamma,\delta) = h_n\delta_{ln}, \tag{2}$$

where

$$w(x) = \frac{(\alpha+1)_x(\beta+\delta+1)_x(\gamma+1)_x(\gamma+\delta+1)_x((\gamma+\delta+3)/2)_x}{(-\alpha+\gamma+\delta+1)_x(-\beta+\gamma+1)_x((\gamma+\delta+1)/2)_x(\delta+1)_x\, x!}, \tag{3}$$

$$h_n = M \cdot \frac{(n+\alpha+\beta+1)_n(\alpha+\beta-\gamma+1)_n(\beta+1)_n\, n!}{(\alpha+\beta+2)_{2n}(\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n}, \tag{4}$$

and with multiplier M being defined as

$$M = \begin{cases} \frac{(-\beta)_m(\gamma+\delta+2)_m}{(-\beta+\gamma+1)_m(\delta+1)_m} & \text{if} \quad \alpha+1=-m \\ \frac{(-\alpha+\delta)_m(\gamma+\delta+2)_m}{(-\alpha+\gamma+\delta+1)_m(\delta+1)_m} & \text{if} \quad \beta+\delta+1=-m \\ \frac{(\alpha+\beta+2)_m(-\delta)_m}{(\alpha-\delta+1)_m(\beta+1)_m} & \text{if} \quad \gamma+1=-m. \end{cases}$$

Then, one can introduce a pair of new difference equations for these polynomials in which Racah polynomials of the same degree n in variables x or $x+1$, and with parameters of type $(\alpha+1,\beta-1,\delta)$ and $(\alpha,\beta,\delta-1)$ are intertwined.

Proposition 1. *The Racah polynomials satisfy the following difference equations:*

$$\begin{aligned} &\frac{(x+\gamma+1)(x+\beta+\delta)}{2x+\gamma+\delta+1} R_n(\lambda(x+1);\alpha,\beta,\gamma,\delta-1) \\ &- \frac{(x-\beta+\gamma+1)(x+\delta)}{2x+\gamma+\delta+1} R_n(\lambda(x);\alpha,\beta,\gamma,\delta-1) \\ &= \frac{(n+\alpha+1)(n+\beta)}{\alpha+1} R_n(\lambda(x);\alpha+1,\beta-1,\gamma,\delta), \end{aligned} \tag{5}$$

$$\frac{(x+\alpha+2)(x+\gamma+\delta+1)}{2x+\gamma+\delta+2} R_n(\lambda(x+1);\alpha+1,\beta-1,\gamma,\delta)$$
$$-\frac{(x+1)(x-\alpha+\gamma+\delta)}{2x+\gamma+\delta+2} R_n(\lambda(x);\alpha+1,\beta-1,\gamma,\delta)$$
$$=(\alpha+1) R_n(\lambda(x+1);\alpha,\beta,\gamma,\delta-1). \tag{6}$$

Proof. We prove both equations by performing straightforward computations using known properties of hypergeometric functions and Pochhammer symbols. In the case of (5), one can rewrite the left-hand side in the following form:

$$(x+\gamma+1) R_n(\lambda(x+1);\alpha,\beta,\gamma,\delta-1)-(x-\beta+\gamma+1)$$
$$\times R_n(\lambda(x);\alpha,\beta,\gamma,\delta-1) +$$
$$+\frac{(x+\gamma+1)(x-\beta+\gamma+1)}{2x+\gamma+\delta+1}[R_n(\lambda(x);\alpha,\beta,\gamma,\delta-1)$$
$$-R_n(\lambda(x+1);\alpha,\beta,\gamma,\delta-1)]. \tag{7}$$

Then, a simple computations show that

$$(x+\gamma+1) R_n(\lambda(x+1);\alpha,\beta,\gamma,\delta-1)-(x-\beta+\gamma+1)$$
$$\times R_n(\lambda(x);\alpha,\beta,\gamma,\delta-1)$$
$$=-\sum_{k=0}^{n}\frac{(-n)_k(n+\alpha+\beta+1)_k(-x)_{k-1}(x+\gamma+\delta+1)_{k-1}}{(\alpha+1)_k(\beta+\delta)_k(\gamma+1)_k k!} \tag{8}$$
$$\times[(x+\gamma+1)(x+1)(x+\gamma+\delta+k) +$$
$$+(x-\beta+\gamma+1)(k-x-1)(x+\gamma+\delta)]$$

and

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta-1)-R_n(\lambda(x+1);\alpha,\beta,\gamma,\delta-1) \tag{9}$$
$$=\sum_{k=0}^{n}\frac{(-n)_k(n+\alpha+\beta+1)_k(-x)_{k-1}(x+\gamma+\delta+1)_{k-1}}{(\alpha+1)_k(\beta+\delta)_k(\gamma+1)_k k!}k(2x+\gamma+\delta+1).$$

Therefore, combining (8) and (9), we have the following expression for the left hand side of (5):

$$\sum_{k=0}^{n}\frac{(-n)_k(n+\alpha+\beta+1)_k(-x)_{k-1}(x+\gamma+\delta+1)_{k-1}}{(\alpha+1)_k(\beta+\delta)_k(\gamma+1)_k k!}$$
$$\times[\beta(k-x-1)(x+\gamma+\delta)-k(x+\gamma+1)(x+\beta+\delta)]. \tag{10}$$

Then, one can rewrite the right hand side of (5) as follows:

$$\frac{(n+\alpha+1)(n+\beta)}{\alpha+1}\; R_n(\lambda(x);\alpha+1,\beta-1,\gamma,\delta) \tag{11}$$

$$= \frac{(n+\alpha+1)(n+\beta)}{\alpha+1}\sum_{k=0}^{n}\frac{(-n)_k\,(n+\alpha+\beta+1)_k\,(-x)_k\,(x+\gamma+\delta+1)_k}{(\alpha+2)_k\,(\beta+\delta)_k\,(\gamma+1)_k\,k!}$$

$$= \sum_{k=0}^{n}\frac{(-n)_k\,(n+\alpha+\beta+1)_k\,(-x)_k\,(x+\gamma+\delta+1)_k}{(\alpha+1)_k\,(\beta+\delta)_k\,(\gamma+1)_k\,k!}\,\frac{(n+\alpha+1)(n+\beta)}{\alpha+k+1}$$

$$= \sum_{k=0}^{n}\frac{(-n)_k\,(n+\alpha+\beta+1)_k\,(-x)_k\,(x+\gamma+\delta+1)_k}{(\alpha+1)_k\,(\beta+\delta)_k\,(\gamma+1)_k\,k!}$$

$$\times\left[\frac{(n+\alpha+\beta+k+1)(n-k)}{\alpha+k+1}+(\beta+k)\right]$$

$$= \sum_{k=0}^{n}\frac{(-n)_k\,(n+\alpha+\beta+1)_k\,(-x)_{k-1}\,(x+\gamma+\delta+1)_{k-1}}{(\alpha+1)_k\,(\beta+\delta)_k\,(\gamma+1)_k\,k!}$$

$$\times\left[(\beta+k)(k-x-1)(x+\gamma+\delta+k)-k(\gamma+k)(\beta+\delta+k-1)\right].$$

Now, to prove (5), we just need to check that the following equality is correct:

$$\beta(k-x-1)(x+\gamma+\delta)-k(x+\gamma+1)(x+\beta+\delta) =$$

$$= (\beta+k)(k-x-1)(x+\gamma+\delta+k)-k(\gamma+k)(\beta+\delta+k-1), \tag{12}$$

which is obvious.

The proof of Eq. (6) is even simpler than that of Eq. (5). It is possible to rewrite (6) it as follows:

$$\sum_{k=0}^{n}\frac{(-n)_k\,(n+\alpha+\beta+1)_k\,(-x)_k\,(x+\gamma+\delta+2)_k}{(\alpha+1)_k\,(\beta+\delta)_k\,(\gamma+1)_k\,k!}\;\times \tag{13}$$

$$\times\;[(x-\alpha+\gamma+\delta)(k-x-1)+$$

$$+\;(x+\alpha+2)(x+\gamma+\delta+k+1)-(2x+\gamma+\delta+2)(\alpha+k+1)]=0.$$

Then (6) follows from the simple observation that

$$(x-\alpha+\gamma+\delta)(k-x-1)+(x+\alpha+2)(x+\gamma+\delta+k+1) =$$

$$= (2x+\gamma+\delta+2)(\alpha+k+1)\,. \tag{14}$$

□

There are three known cases, when the Racah polynomials $R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$ reduce to Hahn polynomials $Q_n(x;\alpha,\beta,m)$ [1, (9.2.15)–(9.2.17)], defined as

$$Q_n(x;\alpha,\beta,m) = {}_3F_2\left(\begin{array}{c} -n, n+\alpha+\beta+1, -x \\ \alpha+1, -m \end{array}; 1\right). \tag{15}$$

For the first two cases, $(\gamma+1=-m;\delta\to\infty)$ and $(\delta=-\beta-m-1;\gamma\to\infty)$, one recovers a pair of known difference equations for the Hahn polynomials $Q_n(x;\alpha+1,\beta-1,m)$ and $Q_n(x;\alpha,\beta,m)$ [8, (10)–(11)]. For the third case, $(\alpha+1=-m;\ \beta\to\beta+\gamma+m+1;\delta\to\infty)$ leads to a pair of new difference equations for Hahn polynomials, with a shift in m:

$$\begin{aligned}(x+1)\,Q_n(x;\alpha,\beta,m-1)-(x-m+1)\,Q_n(x+1;\alpha,\beta,m-1) &= \\ = m\cdot Q_n(x+1;\alpha,\beta,m), \qquad\qquad (16)\\ m(x-\beta-m)\,Q_n(x;\alpha,\beta,m)-m(x+\alpha+1)\,Q_n(x+1;\alpha,\beta,m) &= \\ = (n-m)(n+\alpha+\beta+m+1)\,Q_n(x;\alpha,\beta,m-1). \qquad\qquad (17)\end{aligned}$$

Under the limit $(\alpha=pt;\beta=(1-p)t;t\to\infty)$, these equations further reduce to a pair of difference equations for the Krawtchouk polynomials $K_n(x;p,m)$:

$$\begin{aligned}(x+1)\,K_n(x;p,m-1)+(m-x+1)\,K_n(x+1;p,m-1) &= \\ = m\,K_n(x+1;p,m), \qquad\qquad (18)\\ m(1-p)\,K_n(x;p,m)+m\cdot p\cdot K_n(x+1;p,m) = (m-n)\,K_n(x;p,m-1).\end{aligned}$$

Equations (5) and (6) can be useful for the construction of finite-discrete quantum oscillator models as well as exactly-solvable spin chains with nearest-neighbour interaction of $m+1$ fermions subject to a zero external magnetic field:

$$\hat{H} = \sum_{k=0}^{m-1} J_k\left(a_k^+ a_{k+1} + a_{k+1}^+ a_k\right), \tag{19}$$

where, J_k expresses the coupling strength between two neighbour fermions k and $k+1$ and has the following expression:

$$J_k = \begin{cases} \sqrt{(k+1)(m-k)\,f(\alpha,\beta,\delta)}; & \text{k - odd} \\ \sqrt{(k+2\alpha+2)(m-k+2\beta)\,g(\delta)}; & \text{k - even} \end{cases} \tag{20}$$

with $f(\alpha,\beta,\delta)$ and $g(\delta)$ defined as follows:

$$f(\alpha,\beta,\delta) = \frac{(k-2\alpha+2\delta-m)(k+2\beta+2\delta-1)}{(2k+2\delta-m-1)(2k+2\delta-m+1)}, \tag{21}$$

$$g(\delta) = \frac{(k - m + 2\delta - 1)(k + 2\delta)}{(2k + 2\delta - m - 1)(2k + 2\delta - m + 1)}. \tag{22}$$

3 Wilson Polynomials as Analytical Solutions of New Difference Equations

The Wilson polynomial $W_n\left(x^2; a, b, c, d\right)$ of degree n $(n = 0, 1, \ldots)$ in the variable x is defined by:

$$\frac{W_n\left(x^2; a, b, c, d\right)}{(a+b)_n (a+c)_n (a+d)_n} = {}_4F_3\left(\begin{matrix} -n, n+a+b+c+d-1, a+ix, a-ix \\ a+b, a+c, a+d \end{matrix}; 1\right) \tag{23}$$

The polynomial satisfies an orthogonality relation in the continuous space $[0, +\infty)$ under the condition $\mathrm{Re}\,(a, b, c, d) > 0$ [1, (9.1.2)].

By putting $\alpha = a + b - 1$, $\beta = c + d - 1$, $\gamma = a + d - 1$, $\delta = a - d$ and $x \to -a + ix$ in Eqs. (5) and (6) as well as taking into account the duality of Racah polynomials (1) in n and x, one can transfer them to Wilson polynomials and obtain the following three-term recurrence relations

$$W_n\left(x^2; a, b, c, d\right) = \frac{n+a+b+c+d-1}{2n+a+b+c+d-1} W_n\left(x^2; a, b, c, d+1\right) - \tag{24}$$
$$- \frac{n(n+a+b-1)(n+a+c-1)(n+b+c-1)}{2n+a+b+c+d-1} W_{n-1}\left(x^2; a, b, c, d+1\right),$$

$$\left(x^2 + d^2\right) W_n\left(x^2; a, b, c, d+1\right) = \tag{25}$$
$$= \frac{(n+a+d)(n+b+d)(n+c+d)}{2n+a+b+c+d} W_n\left(x^2; a, b, c, d\right) -$$
$$- \frac{1}{2n+a+b+c+d} W_{n+1}\left(x^2; a, b, c, d\right),$$

and the difference equations:

$$\left[\frac{(a+ix)(b+ix)}{2ix} e^{-\frac{i}{2}\partial_x} - \frac{(a-ix)(b-ix)}{2ix} e^{\frac{i}{2}\partial_x}\right] W_n\left(x^2; a+\tfrac{1}{2}, b+\tfrac{1}{2}, c, d\right)$$
$$= (n+a+b)\, W_n\left(x^2; a, b, c+\tfrac{1}{2}, d+\tfrac{1}{2}\right), \tag{26}$$

$$\left[\frac{(c+ix)(d+ix)}{2ix}e^{-\frac{i}{2}\partial_x}-\frac{(c-ix)(d-ix)}{2ix}e^{\frac{i}{2}\partial_x}\right]W_n\left(x^2;a,b,c+\tfrac{1}{2},d+\tfrac{1}{2}\right)$$
$$=(n+c+d)\,W_n\left(x^2;a+\tfrac{1}{2},b+\tfrac{1}{2},c,d\right). \tag{27}$$

Introducing orthonormalized Wilson polynomials, one can reformulate (26) and (27) in a more compact form:

$$\left[e^{-\frac{i}{2}\partial_x}-\frac{(a-ix)(b-ix)}{2ix}e^{\frac{i}{2}\partial_x}\frac{(c+ix)(d+ix)}{2ix}\right]\tilde{W}_n\left(x^2;\tfrac{1}{2},0\right)=$$
$$=\sqrt{(n+a+b)(n+c+d)}\tilde{W}_n\left(x^2;0,\tfrac{1}{2}\right),$$
$$\left[e^{-\frac{i}{2}\partial_x}-\frac{(c-ix)(d-ix)}{2ix}e^{\frac{i}{2}\partial_x}\frac{(a+ix)(b+ix)}{2ix}\right]\tilde{W}_n\left(x^2;0,\tfrac{1}{2}\right)=$$
$$=\sqrt{(n+a+b)(n+c+d)}\tilde{W}_n\left(x^2;\tfrac{1}{2},0\right), \tag{28}$$

where $\tilde{W}_n\left(x^2;\frac{1}{2},0\right)\equiv\tilde{W}_n\left(x^2;a+\frac{1}{2},b+\frac{1}{2},c,d\right)$ and $\tilde{W}_n\left(x^2;0,\frac{1}{2}\right)\equiv\tilde{W}_n\left(x^2;a,b,c+\frac{1}{2},d+\frac{1}{2}\right)$. As a special case, when $a=c$ and $b=d$, both (26) and (27) reduce to difference equations for the continuous dual Hahn polynomials $S_n\left(4x^2;2a,2b,\frac{1}{2}\right)$ [1, (9.3.6)]. Then, they can be considered as a fermionic extension of the quantum harmonic oscillator model, whose algebra is Lie algebra $su(1,1)$ deformed by a reflection operator [6]. Under another limit, reducing Wilson polynomials to continuous Hahn polynomials [1, (9.1.17)], one obtains from Eqs. (26) and (27) a pair of difference equations for continuous Hahn polynomials

$$\left[(ix+b)\,e^{-\frac{i}{2}\partial_x}-(ix-d)\,e^{\frac{i}{2}\partial_x}\right]p_n\left(x;0,\tfrac{1}{2}\right)=(n+b+d)\,p_n\left(x;\tfrac{1}{2},0\right), \tag{29}$$

$$\left[(ix+a)\,e^{-\frac{i}{2}\partial_x}-(ix-c)\,e^{\frac{i}{2}\partial_x}\right]p_n\left(x;\tfrac{1}{2},0\right)=(n+a+c)\,p_n\left(x;0,\tfrac{1}{2}\right), \tag{30}$$

where, $p_n\left(x;0,\frac{1}{2}\right)\equiv p_n\left(x;a,b+\frac{1}{2},c,d+\frac{1}{2}\right)$ and $p_n\left(x;\frac{1}{2},0\right)\equiv p_n(x;a+\frac{1}{2},b,c+\frac{1}{2},d)$.

Surprisingly, both Eqs. (29) and (30) generalize a difference equation, whose solution is the Meixner–Pollaczek polynomial [1, (9.7.5)]. Therefore, they can be considered as a fermionic extension of the $su(1,1)$ Meixner–Pollaczek oscillator [9].

Conclusion
Racah and Wilson polynomials, which occupy the top level in the Askey scheme of hypergeometric orthogonal polynomials, are defined through the ${}_4F_3$ type hypergeometric series. Under certain conditions, there is a well-known orthogonality relation for the Racah polynomials with respect to a discrete measure as well as for the Wilson polynomials with respect to a continuous measure. These polynomials are explicit analytical solutions of known difference equations with quadratic-like eigenvalues. In current work, we introduce a pair of novel difference equations or three-term recurrence relations, whose solutions are also expressed in terms of the Racah or Wilson polynomials depending on nature of the finite-difference step. The proof of these equations is presented for case of Racah polynomials. These equations may turn out to be good candidates for building some new fermionic oscillator models as well as exactly-solvable spin chains with a nearest-neighbour interaction. A number of special cases and limit relations are also examined, which allow to introduce similar difference equations for the orthogonal polynomials of the ${}_3F_2$ and ${}_2F_1$ types.

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Spin Chain Models of Free Fermions

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Abstract We consider the integrable open spin chain models formulated through the generators of the Hecke algebras which are realized in terms of free fermions.

1 *R*-Matrix of the Hecke Type

Let V_N be an N-dimensional complex vector space. Consider a linear invertible operator $\hat{R}$ which acts in $V_N \otimes V_N$. Using the operator $\hat{R}$ one can define the set of operators $\hat{R}_{k,k+1}$

$$\hat{R}_{k,k+1} = \underbrace{I_N \otimes \cdots \otimes I_N}_{k-1} \otimes \hat{R} \otimes \underbrace{I_N \otimes \cdots \otimes I_N}_{L-k-1}, \quad 1 \leq k \leq L-1, \tag{1}$$

which act in the vector space $V_N^{\otimes L}$. The invertible operator $\hat{R}$ is called the R-matrix if it satisfies the Yang–Baxter equation

$$\hat{R}_{k,k+1}\, \hat{R}_{k+1,k+2}\, \hat{R}_{k,k+1} = \hat{R}_{k+1,k+2}\, \hat{R}_{k,k+1}\, \hat{R}_{k+1,k+2}\,. \tag{2}$$

We say that the R-matrix is of the Hecke type if it satisfies the Hecke condition

$$\hat{R}^2 = (q - q^{-1})\, \hat{R} + I_N \otimes I_N\,, \tag{3}$$

where q is a parameter.

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The Hamiltonian for the open Hecke chain is defined as $H_L = \sum_{k=1}^{L-1} \hat{R}_{k,k+1}$. Now we are going to describe a rather general construction for the Hecke type R matrices.

Consider the R-matrix of the form

$$\hat{R} = \sum_{i=1}^{N} a_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} (a_{ij}\, e_{ij} \otimes e_{ji} + b_{ij}\, e_{ii} \otimes e_{jj}) \,, \tag{4}$$

where e_{ij} are the matrix units, i.e. $e_{ij} e_{rs} = \delta_{jr} e_{is}$ and a_{ij}, b_{ij} are the parameters.

It is shown in [1, 2] that the general expression (4) for the Hecke R-matrix is

$$\hat{R} = \sum_{i=1}^{N} a_i\, e_{ii} \otimes e_{ii} + \sum_{i \neq j} a_{ij} e_{ij} \otimes e_{ji} + (q - q^{-1}) \sum_{i<j} e_{ii} \otimes e_{jj} \,, \tag{5}$$

where for $i,\ j = \{1,\ 2,\ \ldots,\ N\}$

$$a_i^2 - (q - q^{-1}) a_i - 1 = 0 \,, \qquad a_{ij} a_{ji} = 1$$

and $i < j$ in the last term of (5) means that we choose any ordering on the set $i,\ j = \{1,\ \ldots,\ N\}$. For more details see, e.g., [1, 3, 4] and references therein.

2 Free Fermionic Realizations

The free fermionic integrable model were considered in several papers (see [5] and refs, therein). Here we propose another approach.

The aim of this section is to rewrite operators (1), where the R-matrix $\hat{R}$ is given by (5), in terms of free fermions.

Let $\mathcal{A}$ and $\mathcal{B}$ be two associative algebras over complex numbers and a_i ($i = 1, 2, 3, \ldots$) and b_α ($\alpha = 1, 2, 3, \ldots$) be the basis elements of $\mathcal{A}$ and $\mathcal{B}$, respectively. The standard direct product $\mathcal{A} \otimes \mathcal{B}$ of algebras $\mathcal{A}$ and $\mathcal{B}$ is defined as a vector space with the basis elements $a_i \otimes b_\alpha$ and multiplication rule

$$(a_i \otimes b_\alpha) \cdot (a_k \otimes b_\beta) = (a_i \cdot a_k \otimes b_\alpha \cdot b_\beta) \,. \tag{6}$$

Let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathbb{Z}_2$-graded algebras. We denote the grading ϵ of the basis elements by $\epsilon(a_i) = 0, 1 \bmod(2)$ and $\epsilon(b_\alpha) = 0, 1 \bmod(2)$. In this case, in addition to the usual direct product $\mathcal{A} \otimes \mathcal{B}$, one can define a new type algebra which is a graded direct product $\mathcal{A} \boxtimes \mathcal{B}$ of the algebras $\mathcal{A}$ and $\mathcal{B}$. As vector spaces the algebras $\mathcal{A} \boxtimes \mathcal{B}$ and $\mathcal{A} \otimes \mathcal{B}$ coincide with each other, but instead of the rule (6) for the algebra $\mathcal{A} \boxtimes \mathcal{B}$ we have the new graded multiplication

$$(a_i \boxtimes b_\alpha) \cdot (a_k \boxtimes b_\beta) = (-1)^{\epsilon(a_k)\epsilon(b_\alpha)} (a_i \cdot a_k \boxtimes b_\alpha \cdot b_\beta) \,. \tag{7}$$

The algebra $\mathcal{A} \boxtimes \mathcal{B}$ has the natural $\mathbb{Z}_2$-grading when the parity of the basis elements $(a_i \boxtimes b_\alpha)$ is postulated as $\epsilon(a_i \boxtimes b_\alpha) = \epsilon(a_i) + \epsilon(b_\alpha)$. In physical literature $\mathbb{Z}_2$-graded algebras are called super-algebras.

Let $\mathcal{A}$ be an associative algebra Mat_N of the $N \times N$ matrices, and we choose the matrix units $e_{ik} \in \mathrm{Mat}_N$, $(i,\ k = 1, \ldots, N)$ subject to the standard multiplication

$$e_{ij} \cdot e_{k\ell} = \delta_{jk} e_{i\ell}\,, \tag{8}$$

as the basis elements in Mat_N. Note that the unit element in Mat_N is $I_N = \sum_{i=1}^{N} e_{ii}$.

Let $\mathcal{B}$ be another associative algebra Mat_M with the basis elements $e_{\alpha\beta}$ $(\alpha,\ \beta = 1, 2, \ldots, M)$ and the standard multiplication.

By definition, the algebra $\mathrm{Mat}^{(0)}_{NM} = \mathrm{Mat}_N \otimes \mathrm{Mat}_M$ is the associative algebra with the $NM \times NM$ matrix units $e_{(i\alpha),(j\beta)} = e_{ij} \otimes e_{\alpha\beta}$, which obey the standard multiplication rules (cf. (8))

$$e_{(i\alpha),(j\beta)} \cdot e_{(k\rho),(\ell\sigma)} = \delta_{(j\beta),(k\rho)}\ e_{(i\alpha),(\ell\sigma)}\,, \tag{9}$$

where $\delta_{(j\beta),(k\rho)} \equiv \delta_{jk}\delta_{\beta\rho}$.

Now we suppose that Mat_N and Mat_M are associative $\mathbb{Z}_2$-graded algebras. In this case, it is convenient to denote the matrix units e_{ij} by E_{ij} and choose the so-called "along diagonal grading": $\epsilon\big(E_{ik}\big) = \epsilon(i+k)$, $\epsilon\big(E_{\alpha\beta}\big) = \epsilon(\alpha+\beta)$, where $\epsilon(i) = i \bmod 2$. Next we consider the associative algebra $\mathrm{Mat}^{(1)}_{NM} = \mathrm{Mat}_N \boxtimes \mathrm{Mat}_M$ generated by the elements $E_{ik} \boxtimes E_{\alpha\beta}$ and multiplication, which is defined by the relation (cf. (7))

$$\begin{aligned}\big(E_{ik} \cdot E_{\alpha\beta}\big)\big(E_{jm} \cdot E_{\gamma\xi}\big) &= (-1)^{\epsilon(j+m)\epsilon(\alpha+\beta)}\ \big(E_{ik}\,E_{jm}\big)\cdot\big(E_{\alpha\beta}\,E_{\gamma\xi}\big) = \\ &= (-1)^{\epsilon(j+m)\epsilon(\alpha+\beta)}\ \delta_{kj}\,\delta_{\beta\gamma}\,\big(E_{im}\cdot E_{\alpha\xi}\big)\,.\end{aligned}$$

Here and below we use the concise notation and omit the symbol $\boxtimes$ in formulas, i.e., we write $E_{ik} \cdot E_{\alpha\beta}$ instead of $E_{ik} \boxtimes E_{\alpha\beta}$.

Our aim is to find the elements $E_{(i\alpha),(k\beta)} \in \mathrm{Mat}^{(1)}_{NM}$, which form the algebra (9) of the $NM \times NM$ matrix units. The answer is given by the following proposition.

Proposition 1. *The elements*

$$E_{(i\alpha),(k\beta)} = (-1)^{\epsilon(k)\,\epsilon(\alpha+\beta)} E_{ik} \cdot E_{\alpha\beta} \quad \in \quad \mathrm{Mat}^{(1)}_{NM} \tag{10}$$

form the algebra of the $NM \times NM$ matrix units with multiplication rules (9). The parity of the elements $E_{(i\alpha),(k\beta)}$ is defined by means of the function $\epsilon(i+k+\alpha+\beta)$.

Proof. We search the elements $E_{(i\alpha),(k\beta)}$ in the form

$$E_{(i\alpha),(k\beta)} = S_{ik,\alpha\beta}\ E_{ik} \cdot E_{\alpha\beta}\,, \tag{11}$$

where $S_{ik,\alpha\beta}$ are the numbers such that elements (11) should satisfy (9). The conditions (9) lead to the relations

$$S_{(ik),(\alpha\beta)} = (-1)^{(\epsilon(j)+\epsilon(k))(\epsilon(\alpha)+\epsilon(\rho))} S_{(ij),(\alpha\rho)} S_{(jk),(\rho\beta)} .$$

It is easy to verify by direct calculation that one solution of these equations[1] is

$$S_{(ik),(\alpha\beta)} = (-1)^{\epsilon(k)(\epsilon(\alpha)+\epsilon(\beta))} ,$$

which gives (10). The parity of the element $E_{(i\alpha),(k\beta)}$ is defined by the parity of the product $E_{ik} \cdot E_{\alpha\beta}$ and, therefore, is equal to $\epsilon(i+k+\alpha+\beta)$. □

Proposition 2. *There is a linear isomorphic map* $\Phi : \mathrm{Mat}_N \otimes \mathrm{Mat}_M \to \mathrm{Mat}_N \boxtimes \mathrm{Mat}_M$ *of the associative algebras given by the formulae*

$$\Phi\big(e_{ik} \otimes e_{\alpha\beta}\big) = (-1)^{\epsilon(k)(\epsilon(\alpha)+\epsilon(\beta))} E_{ik} \cdot E_{\alpha\beta} = E_{(i\alpha),(k\beta)} . \tag{12}$$

Proof. Since the elements $e_{ik} \otimes e_{\alpha\beta}$ form the basis in $\mathrm{Mat}_N \otimes \mathrm{Mat}_M$ and $E_{(i\alpha),(k\beta)}$ (10) form the basis in $\mathrm{Mat}_N \boxtimes \mathrm{Mat}_M$, the linear mapping Φ: $\mathrm{Mat}^{(0)}_{NM} \to \mathrm{Mat}^{(1)}_{NM}$ is defined uniquely by formula (12). Proposition 1 then leads to the relation

$$\Phi\left((e_{ik} \otimes e_{\alpha\beta})(e_{jm} \otimes e_{\gamma\sigma})\right) = \Phi\left(e_{ik} \otimes e_{\alpha\beta}\right) \cdot \Phi\left(e_{jm} \otimes e_{\gamma\sigma}\right) ,$$

which means that the mapping (12) is a homomorphism. It is obvious that the mapping Φ defined in (12) is invertible and, therefore, an isomorphism. □

Remark. Let Mat_N, Mat_M and Mat_K be $\mathbb{Z}_2$-graded algebras of the $N \times N$, $M \times M$ and $K \times K$ matrices, respectively. Then one can check directly the associativity of the rule of the definition of the matrix units (10):

$$(-1)^{\epsilon(k+\beta)\epsilon(a+b)} E_{(i,\alpha),(k,\beta)} \cdot E_{ab} = (-1)^{\epsilon(k)\epsilon(\alpha+\beta)} E_{ik} \cdot E_{(\alpha,a),(\beta,b)} \equiv E_{(i,\alpha,a),(k,\beta,b)} .$$

Corollary. *The mapping (12) can be extended to the isomorphism*

$$\Phi : \ \mathrm{Mat}_{N_1} \otimes \cdots \otimes \mathrm{Mat}_{N_r} \to \mathrm{Mat}_{N_1} \boxtimes \cdots \boxtimes \mathrm{Mat}_{N_r} ,$$

by means of the recurrence relation

$$\begin{aligned} \Phi\big(e_{i_1k_1} \otimes e_{i_2k_2} \otimes \cdots \otimes e_{i_rk_r}\big) \ &= \ (-1)^{\epsilon(k_1)\epsilon(i_2+k_2+\cdots+i_r+k_r)} E_{i_1k_1} \cdot \\ &\cdot \Phi\big(e_{i_2k_2} \otimes \cdots \otimes e_{i_rk_r}\big) , \end{aligned} \tag{13}$$

where $i_m, k_m = \{1, \ldots, N_m\}$, *and the initial relation is* $\Phi\big(e_{ik}\big) = E_{ik}$.

[1]The other solution of the equations is, e.g., $S_{(ik),(\alpha\beta)} = (-1)^{\epsilon(\alpha)\left(\epsilon(i)+\epsilon(k)\right)}$ and the general solution is $S_{(ik),(\alpha\beta)} = (-1)^{\epsilon(k)\left(\epsilon(\alpha)+\epsilon(\beta)\right)} \hat{S}_{(ik),(rs)}$, where $\hat{S}_{(ik),(\alpha\beta)}$ is a solution of the equations $\hat{S}_{(ik),(\alpha\beta)} = \hat{S}_{(ij),(\alpha\rho)} \hat{S}_{(jk),(\rho\beta)}$.

Example. As an example of the above construction we deduce the matrix units for Mat_4 in terms of the algebra of two free fermions. Let us introduce the associative algebra $\mathcal{F}$ which is generated by two fermionic operators $\{\psi, \bar{\psi}\}$ with the standard commutation relations. The algebra $\mathcal{F}$ is called the algebra of one complex free fermion. For the 2×2 case we have the following free fermionic realization of the matrix units $e_{ik} = E_{ik}$:

$$E_2 = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} = \begin{pmatrix} \bar{\psi}\psi & \bar{\psi} \\ \psi & \psi\bar{\psi} \end{pmatrix}. \tag{14}$$

One can check directly that the elements e_{ij} of the matrix E_2 given in (14) satisfy relations (8). Thus, the elements e_{ij} (14) generate the $\mathbb{Z}_2$-graded algebra Mat_2.

Consider two $\mathbb{Z}_2$-graded algebras Mat_2, where the first algebra is generated by the free fermion $\{\psi, \bar{\psi}\}$ while the second one is generated by the free fermion $\{\xi, \bar{\xi}\}$.

For the direct product $\mathrm{Mat}_2 \otimes \mathrm{Mat}_2$ of two such algebras with the standard multiplication rule (6) it is easy to see that the elements $e_{(i\alpha),(k\beta)} = e_{ik} \otimes e_{\alpha\beta}$, where e_{ik} and $e_{\alpha\beta}$ are the matrix units (14), are the matrix units in $\mathrm{Mat}_2 \otimes \mathrm{Mat}_2$. Here the fermionic generators $\{\psi, \bar{\psi}\}$ of the first factor in $\mathrm{Mat}_2 \otimes \mathrm{Mat}_2$ commute with the generators $\{\xi, \bar{\xi}\}$ of the second factor in $\mathrm{Mat}_2 \otimes \mathrm{Mat}_2$. It means that the $\mathbb{Z}_2$-grading is not defined correctly for the algebra $\mathrm{Mat}_2 \otimes \mathrm{Mat}_2$.

On the other hand, one can consider the associative algebra $\mathrm{Mat}_2 \boxtimes \mathrm{Mat}_2$, where the algebras Mat_2 are generated by two independent free fermions $\{\psi, \bar{\psi}\}$ and $\{\xi, \bar{\xi}\}$, which satisfy the anti-commutation relations $[\psi, \bar{\xi}]_+ = [\xi, \bar{\psi}]_+ = [\psi, \xi]_+ = [\bar{\xi}, \bar{\psi}]_+ = 0$, i.e., the pairs of the fermions $\{\psi, \bar{\psi}\}$ and $\{\xi, \bar{\xi}\}$ anti-commute with each other and form the algebra of two free fermions $\mathcal{F}_2$.

In this case the elements $E^{(\psi)}_{ik} \cdot E^{(\xi)}_{rs} \in \mathrm{Mat}_2 \boxtimes \mathrm{Mat}_2$ do not represent the matrix units in $\mathrm{Mat}_2 \boxtimes \mathrm{Mat}_2$. Therefore, to construct such matrix units we need to use the linear isomorphism (12) of the associative algebras $\mathrm{Mat}_2 \otimes \mathrm{Mat}_2$ and $\mathrm{Mat}_2 \boxtimes \mathrm{Mat}_2$.

According to our choice of the parity $\epsilon(i) = i \bmod 2$and using (14) we obtain the following matrix units:

$$E_4 = \|E_{ab}\|_{a,b=1,\dots,4} = \begin{pmatrix} \bar{\psi}\psi\bar{\xi}\xi & -\bar{\psi}\psi\bar{\xi} & \bar{\psi}\bar{\xi} & \bar{\psi}\bar{\xi}\xi \\ -\bar{\psi}\psi\xi & \bar{\psi}\psi\xi\bar{\xi} & \bar{\psi}\xi\bar{\xi} & \bar{\psi}\xi \\ -\psi\xi & \psi\xi\bar{\xi} & \psi\bar{\psi}\xi\bar{\xi} & \psi\bar{\psi}\xi \\ \psi\bar{\xi}\xi & -\psi\bar{\xi} & \psi\bar{\psi}\bar{\xi} & \psi\bar{\psi}\bar{\xi}\xi \end{pmatrix}. \tag{15}$$

Here we represent the matrix $E_{(ij),(km)}$ as the 4×4 matrix (15) by ordering the pairs of indices (i, j) in (10) as follows: $(1, 1) \leftrightarrow 1$, $(1, 2) \leftrightarrow 2$, $(2, 1) \leftrightarrow 4$ and $(2, 2) \leftrightarrow 3$. Such ordering leads to the along diagonal grading $\epsilon(E_{ab}) = \epsilon(a + b)$ for the elements of the matrix $\|E_{ab}\|$.

3 Hecke R-Matrices in Terms of Free Fermions

The R-matrix (1), where $\hat{R}$ is given by (5), is the element of $\mathrm{Mat}_n^{\otimes L}$, where Mat_n is the associative algebra of the $n\times n$ matrix units e_{ij}. Using the relation $I_n = \sum_{r=1}^n e_{rr}$ and the rule (13), it is easy to find that

$$\Phi\Big(\underbrace{I_n\otimes\ldots\otimes I_n}_{(k-1)\text{ times}}\otimes e_{ii}\otimes e_{jj}\otimes\underbrace{I_n\otimes\ldots\otimes I_n}_{(L-k-1)\text{ times}}\Big) = E_{ii}^{(k)}\cdot E_{jj}^{(k+1)}\,,$$

$$\Phi\Big(\underbrace{I_n\otimes\ldots\otimes I_n}_{(k-1)\text{ times}}\otimes e_{ij}\otimes e_{ji}\otimes\underbrace{I_n\otimes\ldots\otimes I_n}_{(L-k-1)\text{ times}}\Big) = (-1)^{\epsilon(j)\left(\epsilon(i)+\epsilon(j)\right)}\,E_{ij}^{(k)}\cdot E_{ji}^{(k+1)}\,.$$

Here index k for the matrix units $E_{ij}^{(k)}$ indicates that they are constructed via free fermions, which are different for different indices k (sites of the chain). So, the image of the R-matrix in the associative $\mathbb{Z}_2$-graded algebra $\mathrm{Mat}_n^{\boxtimes L}$ is

$$\mathcal{R}_{k,k+1} = \Phi\big(\hat{R}_{k,k+1}\big) = \textstyle\sum_{i,j=1}^n\big(\hat{a}_{ij}\,E_{ij}^{(k)}\cdot E_{ji}^{(k+1)} + b_{ij}\,E_{ii}^{(k)}\cdot E_{jj}^{(k+1)}\big)\,, \tag{16}$$

where $\hat{a}_{ij} = (-1)^{\epsilon(j)(\epsilon(i)+\epsilon(j))}a_{ij}$ and $a_{ii} = a_i$.

We would like to realize this R-matrix in terms of free fermions. First, we realize the matrix units in the associative algebra $\mathrm{Mat}_n = \mathrm{Mat}_{2^N}$ by means of N free fermions, i.e. by the elements of the associative algebra $\mathcal{F}_N$.

For this aim it is more convenient to describe the elements of Mat_{2^N} by the multi-index. We introduce multi-indices $\mathbf{i} = (i_1, i_2, \ldots, i_N)$, where $i_r = 1,\ 2$. Parity of the element $E_{\mathbf{i},\mathbf{j}}$ will be given by $(-1)^{\epsilon(\mathbf{i})+\epsilon(\mathbf{j})}$, where $\epsilon(\mathbf{i}) = |\mathbf{i}| \bmod 2$ and $\mathbf{i}| \equiv \sum_{r=1}^n i_r$.

The matrix units $E_{\mathbf{i},\mathbf{j}}$ are the elements of the associative algebra $\mathcal{F}_N$

$$E_{\mathbf{i},\mathbf{j}} = \Phi\big(e_{i_1,j_1}\otimes e_{i_2,j_2}\otimes\ldots\otimes e_{i_N,j_N}\big) = (-1)^{\sum\limits_{k<m} j_k\,(i_m+j_m)}\,E_{i_1,j_1}^{(\psi_1)}\cdot\ldots\cdot E_{i_N,j_N}^{(\psi_N)}\,,$$

where $E_{i_m,j_m}^{(\psi_m)}$ $(m = 1,\ldots,N)$ are the 2×2 matrices given in (14). According to (16), the image of the R-matrix, as an element of the associative algebra $\mathcal{F}_{NL}$, is

$$\begin{aligned}\mathcal{R}_{k,k+1} &= \Phi\big(\hat{R}_{k,k+1}\big) = \textstyle\sum_{\mathbf{i}}a_{\mathbf{i}}E_{\mathbf{i},\mathbf{i}}^{(k)}E_{\mathbf{i},\mathbf{i}}^{(k+1)} + \sum_{\mathbf{i}\neq\mathbf{j}}a_{\mathbf{i},\mathbf{j}}E_{\mathbf{i},\mathbf{j}}^{(k)}E_{\mathbf{j},\mathbf{i}}^{(k+1)} + \\ &\quad + (q-q^{-1})\textstyle\sum_{\mathbf{i}<\mathbf{j}}E_{\mathbf{i},\mathbf{i}}^{(k)}E_{\mathbf{j},\mathbf{j}}^{(k+1)}\,,\end{aligned} \tag{17}$$

where $E_{\mathbf{i},\mathbf{j}}^{(k)} \in \mathcal{F}_N$ are the matrix units for $\mathrm{Mat}_{2^N}^{(k)}$ constructed by means of the free fermions $\psi_\alpha^{(k)}$ $(k = 1,\ldots,L;\ \alpha = 1,\ldots,N)$. For the parameters $a_{\mathbf{i}}$ and $a_{\mathbf{i},\mathbf{j}}$ we have

$$a_{\mathbf{i}}^2 - (q-q^{-1})a_{\mathbf{i}} - 1 = 0\,,\qquad a_{\mathbf{i},\mathbf{j}}a_{\mathbf{j},\mathbf{i}} = (-1)^{|\mathbf{i}|+|\mathbf{j}|},$$

and the relation $\mathbf{i} < \mathbf{j}$ is defined by any ordering of the set of multi-indices $\mathbf{i}$ which does not necessarily conserve "along diagonal grading".

4 Examples

The fundamental R-matrix for the $GL_q(n)$ quantum group is

$$\hat{R} = q \sum_{i=1}^{n} e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ij} \otimes e_{ji} + (q - q^{-1}) \sum_{i<j} e_{ii} \otimes e_{jj} \,, \tag{18}$$

where e_{ij} are the $n \times n$ matrix units. In particular, for the case of $GL_q(2)$, the Hamiltonian $\rho_R(\mathcal{H}_L)$ describes the XXZ open $\frac{1}{2}$-spin chain model.

Using our general construction we rewrite this model in terms of the free fermions. For $n = 2$ and ordering $1 < 2$, the R-matrix in the associative algebra generated by L free fermions is, according to (17),

$$\begin{aligned} \mathcal{R}_{k,k+1} = q\big(E_{11}^{(k)} E_{11}^{(k+1)} + E_{22}^{(k)} E_{22}^{(k+1)}\big) + E_{12}^{(k)} E_{21}^{(k+1)} - E_{21}^{(k)} E_{12}^{(k+1)} + \\ +(q - q^{-1}) E_{11}^{(k)} E_{22}^{(k+1)} \,. \end{aligned}$$

Using (14), we will get

$$\begin{aligned} \mathcal{R}_{k,k+1} = \bar{\psi}_{k+1}\, \psi_k + \bar{\psi}_k\, \psi_{k+1} - q^{-1} \bar{\psi}_k \psi_k - q \bar{\psi}_{k+1}\, \psi_{k+1} + \\ +(q + q^{-1}) \bar{\psi}_k\, \psi_k\, \bar{\psi}_{k+1}\, \psi_{k+1} + q \,. \end{aligned}$$

The generalization of the Hecke type R matrix (18) for $GL_q(n|m)$ has the form

$$\hat{R} = \sum_i (-1)^{[i]} q^{1-2[i]} e_{ii} \otimes e_{ii} + \sum_{i \neq j} (-1)^{[i][j]} e_{ij} \otimes e_{ji} + (q - q^{-1}) \sum_{i<j} e_{ii} \otimes e_{jj} \,, \tag{19}$$

where $i, j = 1, \ldots, n + m$, $[i] = 0$ for $i = 1, \ldots, n$ and $[i] = 1$ for $i = n + 1, \ldots, n + m$. In the case of $GL_q(1|1)$ $(n = m = 1)$ for (19), by using our construction, we have the fermionic image

$$\mathcal{R}_{k,k+1} = q^{-1}\big((\bar{\psi}_{k+1} + q\bar{\psi}_k)(\psi_{k+1} + q\psi_k) - 1\big) \,.$$

Now we consider the R-matrix (18) for the $GL_q(4)$ quantum group, where e_{ij} are the 4×4 matrix units. Our general construction gives the matrix units (15). In the last term of (18) we have to choose any ordering. We will consider the ordering

$$(11) \leftrightarrow 1 < (22) \leftrightarrow 2 < (12) \leftrightarrow 3 < (21) \leftrightarrow 4 \,.$$

In this ordering our construction for the fermionic image of the R-matrix (18) gives

$$\begin{aligned} \mathcal{R}_{k,k+1} \;=\; q \sum_{\mathbf{i}} E_{\mathbf{i},\mathbf{i}}^{(k)} E_{\mathbf{i},\mathbf{i}}^{(k+1)} + \sum_{\mathbf{i} \neq \mathbf{j}} (-1)^{|\mathbf{j}|(|\mathbf{i}|+|\mathbf{j}|)} E_{\mathbf{i},\mathbf{j}}^{(k)} E_{\mathbf{j},\mathbf{i}}^{(k+1)} \;+ \\ +\; (q - q^{-1}) \sum_{\mathbf{i}<\mathbf{j}} E_{\mathbf{i},\mathbf{i}}^{(k)} E_{\mathbf{j},\mathbf{j}}^{(k+1)} \,. \end{aligned}$$

If we substitute (15) into this formula we obtain the R-matrix for the two-fermionic model

$$\begin{aligned}\mathcal{R}_{k,k+1} = {} & q - q(\bar\psi_k\psi_k + \bar\xi_k\xi_k) - q^{-1}(\bar\psi_{k+1}\psi_{k+1} + \bar\xi_{k+1}\xi_{k+1}) + \\ & + (q+q^{-1})(\bar\psi_{k+1}\psi_{k+1}\bar\psi_k\psi_k + \bar\xi_{k+1}\xi_{k+1}\bar\xi_k\xi_k) + \\ & + (q-q^{-1})(\bar\psi_k\psi_k\bar\xi_k\xi_k - \bar\psi_{k+1}\psi_{k+1}\bar\xi_{k+1}\xi_{k+1}) + \\ & + (q\bar\psi_k\psi_k + q^{-1}\bar\psi_{k+1}\psi_{k+1})(\bar\xi_k\xi_k + \bar\xi_{k+1}\xi_{k+1}) + \\ & + (\bar\psi_k\psi_{k+1} - \bar\psi_{k+1}\psi_k)(\bar\xi_k\xi_{k+1} - \bar\xi_{k+1}\xi_k) - \\ & - (\bar\psi_k\psi_k + \bar\psi_{k+1}\psi_{k+1} - 1)(\bar\xi_k\xi_{k+1} + \bar\xi_{k+1}\xi_k) - \\ & - (\bar\psi_k\psi_{k+1} + \bar\psi_{k+1}\psi_k)(\bar\xi_k\xi_k + \bar\xi_{k+1}\xi_{k+1} - 1) - \\ & - (q+q^{-1})\bar\psi_k\psi_k\bar\psi_{k+1}\psi_{k+1}(\bar\xi_k\xi_k + \bar\xi_{k+1}\xi_{k+1}) - \\ & - 2\bar\xi_k\xi_k\bar\xi_{k+1}\xi_{k+1}(q\bar\psi_k\psi_k + q^{-1}\bar\psi_{k+1}\psi_{k+1}) + \\ & + 2(q+q^{-1})\bar\psi_k\psi_k\bar\psi_{k+1}\psi_{k+1}\bar\xi_k\xi_k\bar\xi_{k+1}\xi_{k+1}\,.\end{aligned}$$

Now we construct the two-fermionic model which, corresponds to the R-matrix (19) for $GL_q(2|2)$. We consider the ordering as above, we obtain the R-matrix which does not involve a term of an order 8 in fermions.

We obtain an interesting R-matrix for the two-fermionic system, when we choose $a_{(11)} = a_{(22)} = q$, $a_{(12)} = a_{(21)} = -q^{-1}$ and ordering $(12) < (11) < (21) < (22)$. In this case our construction gives the R-matrix

$$\begin{aligned}\mathcal{R}_{k,k+1} = {} & q - q^{-1}\bar\psi_k\psi_k - q^{-1}\bar\xi_k\xi_k - q\bar\psi_{k+1}\psi_{k+1} - q\bar\xi_{k+1}\xi_{k+1} + \\ & + (q\bar\xi_{k+1}\xi_{k+1} + q^{-1}\bar\xi_k\xi_k)(\bar\psi_{k+1}\psi_{k+1} + \bar\psi_k\psi_k) + \\ & + (\bar\psi_k\psi_{k+1} + \bar\psi_{k+1}\psi_k)(\bar\xi_k\xi_{k+1} + \bar\xi_{k+1}\xi_k) - \\ & - (\bar\psi_k\psi_k + \bar\psi_{k+1}\psi_{k+1} - 1)(\bar\xi_k\xi_{k+1} + \bar\xi_{k+1}\xi_k) - \\ & - (\bar\psi_k\psi_{k+1} + \bar\psi_{k+1}\psi_k)(\bar\xi_k\xi_k + \bar\xi_{k+1}\xi_{k+1} - 1)\,,\end{aligned}$$

which does not contain terms of an order of 6 and 8 in the fermions.

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Group Analysis of Generalized Fifth-Order Korteweg–de Vries Equations with Time-Dependent Coefficients

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Abstract We perform enhanced Lie symmetry analysis of generalized fifth-order Korteweg–de Vries equations with time-dependent coefficients. The corresponding similarity reductions are classified and some exact solutions are constructed.

1 Introduction

In this paper the class of generalized variable-coefficient fifth-order Korteweg–de Vries (fKdV) equations

$$u_t + u^n u_x + \alpha(t)u + \beta(t)u_{xxxxx} = 0 \tag{1}$$

is investigated from the Lie symmetry point of view. Here α and β are smooth nonvanishing functions of the variable t and n is a positive integer, $n \geqslant 2$. This work is a natural continuation of the study undertaken by ourselves in [7], where the group classification of Eq. (1) with $n = 1$ was carried out exhaustively. Lie symmetry analysis of the class (1) was initiated in [18]. We show that the results presented therein are incorrect. The case $n = 2$ was considered also in [17] but the complete group classification was not achieved.

Various generalizations of the Korteweg–de Vries equation appear in many physical models, including ones describing gravity waves, plasma waves and waves

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in lattices [3]. Equation (1) with $n = 1$, $\alpha = 0$ and $\beta =$ const models, for example, one-dimensional hydromagnetic waves in a cold quasi-neutral collision-free plasma propagating along the x-direction under the presence of a uniform magnetic field under some conditions, namely, when the propagation angle of the wave relative to the external magnetic field becomes special, critical angle [4]. More references on studies concerned with these equations can be found in [7].

The presence of variable coefficients in a differential equation that model certain real-world phenomenon often allows one to get better description of the phenomenon but, at the same time, makes the related studies of this equation, including group classification problems, more difficult. In recent works on Lie symmetry analysis it was shown that the usage of admissible transformations in many cases is a cornerstone that leads to exhaustive solution of group classification problems [1, 6, 12, 13]. That's why we firstly investigate admissible transformations in the class (1) in the next section and then proceed with the classification of Lie symmetries in Sect. 3. The corresponding reductions of Eq. (1) admitting extensions of Lie symmetry algebras are performed in Sect. 4, some exact solutions are constructed therein. We discuss the incorrectnesses of the results obtained in [17, 18] in the conclusion.

2 Admissible Transformations

An admissible transformation (called also form-preserving [5] or allowed [19] one) can be regarded as a triple consisting of two fixed equations from a class and a point transformation linking these equations [13]. The set of admissible transformations of a class of differential equations naturally possesses the groupoid structure with respect to the standard operation of transformations composition [12]. More details and examples on finding and usage of admissible transformations for generalized fKdV equations as well as definitions of different kinds of equivalence groups can be found in [6, 15].

We search for admissible transformations in class (1) using the direct method [5], i.e., we suppose that Eq. (1) is linked with an equation from the same class,

$$\tilde{u}_{\tilde{t}} + \tilde{u}^{\tilde{n}}\tilde{u}_{\tilde{x}} + \tilde{\alpha}(\tilde{t})\tilde{u} + \tilde{\beta}(\tilde{t})\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{x}} = 0, \tag{2}$$

by a nondegenerate point transformation of the form

$$\tilde{t} = T(t), \quad \tilde{x} = X^1(t)x + X^0(t), \quad \tilde{u} = U^1(t,x)u + U^0(t,x), \tag{3}$$

where T, X^1, X^0, U^1 and U^0 are arbitrary smooth functions of their variables with $T_t X^1 U^1 \neq 0$. We can restrict ourselves by consideration of point transformations of such a form instead of the most general form $\tilde{t} = T(t,x,u)$, $\tilde{x} = X(t,x,u)$, and $\tilde{u} = U(t,x,u)$, since the class (1) is a subclass (for $m = 5$) of the more general class of evolution equations,

$$u_t = F(t)u_m + G(t,x,u,u_1,\dots,u_{m-1}),$$

where $F \neq 0$, $G_{u_i u_{m-1}} = 0$, $i = 1, \ldots, m-1$, and $m \geqslant 2$, $u_m = \frac{\partial^m u}{\partial x^m}$, F and G are arbitrary smooth functions of their variables. It was proved in [15] that the latter class is normalized with respect to its equivalence group, where transformation components for independent and dependent variables are of the form (3).

Now we perform the change of variables (3) in Eq. (2). The partial derivatives involved in (1) are transformed as follows:

$$\tilde{u}_{\tilde{t}} = \frac{1}{T_t}\left(U^1_t u + U^1 u_t + U^0_t\right) - \frac{X^1_t x + X^0_t}{T_t X^1}\left(U^1_x u + U^1 u_x + U^0_x\right),$$

$$\tilde{u}_{\tilde{x}} = \frac{1}{X^1}\left(U^1_x u + U^1 u_x + U^0_x\right),$$

$$\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{x}} = \frac{1}{(X^1)^5}\big(U^1_{xxxxx} u + 5U^1_{xxxx} u_x + 10U^1_{xxx} u_{xx} + 10U^1_{xx} u_{xxx} + 5U^1_x u_{xxxx} + U^1 u_{xxxxx} + U^0_{xxxxx}\big).$$

We further substitute $u_t = -u^n u_x - \alpha(t)u - \beta(t)u_{xxxxx}$ to the obtained equation in order to confine it to the manifold defined by (1) in the fifth-order jet space with the independent variables (t, x) and the dependent variable u. Splitting the obtained identity with respect to the derivatives of u leads to the determining equations on the functions T, X^1, X^0, U^1 and U^0. Solving them we get, in particular, the conditions

$$\tilde{n} = n, \quad U^0 = U^1_x = 0, \quad \tilde{\beta} T_t - \beta (X^1)^5 = 0.$$

Then the rest of the determining equations result in

$$X^1_t = X^0_t = 0, \quad (U^1)^n T_t = X^1, \quad \tilde{\alpha} U^1 T_t = \alpha U^1 - U^1_t.$$

We solve these equations and get the following assertion.

Theorem 1. *The generalized equivalence group $G^{\sim}$ of the class (1) consists of the transformations*

$$\tilde{t} = T(t), \quad \tilde{x} = \delta_1 x + \delta_2, \quad \tilde{u} = \left(\frac{\delta_1}{T_t}\right)^{\frac{1}{n}} u,$$

$$\tilde{\alpha}(\tilde{t}) = \frac{\alpha}{T_t} + \frac{T_{tt}}{n T_t^2}, \quad \tilde{\beta}(\tilde{t}) = \frac{{\delta_1}^5}{T_t}\beta(t), \quad \tilde{n} = n,$$

where δ_j, $j = 1, 2$, are arbitrary constants, T is an arbitrary smooth function with $\delta_1 T_t > 0$.

The entire set of admissible transformations of the class (1) is generated by the transformations from the group $G^{\sim}$.

Remark 1. If we assume that the constant n varies in the class (1), then the equivalence group $G^{\sim}$ is generalized since n is involved explicitly in the transformation of the variable u. Since n is invariant under the action of transformations from the equivalence group, the class (1) can be considered as the union of its disjoint subclasses with fixed n. For each such subclass the equivalence group $G^{\sim}$ is usual one.

Using Theorem 1 we derive a criterion of reducibility of variable-coefficient Eq. (1) to constant coefficient equations from the same class.

Theorem 2. *A variable coefficient equation from the class (1) is reducible to the constant coefficient equation from the same class if and only if its coefficients* α *and* β *satisfy the equality*

$$n\,(\alpha/\beta)_t = (1/\beta)_{tt}\,. \tag{4}$$

Equivalence transformations from the group $G^{\sim}$ allow us to gauge one of the arbitrary element α or β to a simple constant value, for example, α can be set to zero or β to unity. The gauge $\alpha = 0$ leads to more essential simplification of the study than the gauge $\beta = 1$, therefore, the first one is preferable. Any equation from the class (1) can be mapped to an equation from the same class with $\tilde{\alpha} = 0$ by the equivalence transformation

$$\tilde{t} = \int e^{-n\int \alpha(t)\,\mathrm{d}t}\mathrm{d}t, \quad \tilde{x} = x, \quad \tilde{u} = e^{\int \alpha(t)\,\mathrm{d}t}u. \tag{5}$$

Then the single variable coefficient in the transformed equation will be expressed via α and β as $\tilde{\beta} = e^{n\int \alpha(t)\,dt}\beta$. (Here and in what follows an integral with respect to t should be interpreted as a fixed antiderivative.) Therefore, we can restrict ourselves to the study of the class

$$u_t + u^n u_x + \beta(t)u_{xxxxx} = 0. \tag{6}$$

This will not lead to a loss of generality as all results on symmetries, classical solutions and other related objects for Eq. (1) can be constructed using the similar results obtained for equations from the class (6) and equivalence transformation (5).

To derive the equivalence group for (6) we set $\tilde{\alpha} = \alpha = 0$ in the corresponding transformation presented in Theorem 1 and deduce that the function T is linear with respect to t. The following assertion is true.

Corollary 1. *The generalized equivalence group* $G_0^{\sim}$ *of the class (6) comprises the transformations*

$$\tilde{t} = \delta_3 t + \delta_4, \quad \tilde{x} = \delta_1 x + \delta_2, \quad \tilde{u} = \left(\frac{\delta_1}{\delta_3}\right)^{\frac{1}{n}} u, \quad \tilde{\beta}(\tilde{t}) = \frac{{\delta_1}^5}{\delta_3}\beta(t), \quad \tilde{n} = n, \tag{7}$$

where δ_j, $j = 1, 2, 3, 4$, *are arbitrary constants with* $\delta_1\delta_3 > 0$.

The entire set of admissible transformations of the class (6) is generated by the transformations from the group $G_0^{\sim}$.

Remark 1 is also true for the equivalence group $G_0^{\sim}$.

3 Lie Symmetries

In the previous section we have shown that the group classification problem for the class (1) reduces to the similar problem for its subclass (6). In order to carry out the group classification of (6) we use the classical algorithm [8]. Namely, we look for symmetry generators of the form $Q = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u$ and require that

$$Q^{(5)}\{u_t + u^n u_x + \beta(t)u_{xxxxx}\} = 0 \tag{8}$$

identically, modulo Eq. (6). Here $Q^{(5)}$ is the fifth prolongation of the operator Q [8, 9]. Note that the restriction on n to be integer is inessential for the group classification problem, so we can assume that n is a real nonzero constant.

The infinitesimal invariance criterion implies

$$\tau = \tau(t), \quad \xi = \xi(t,x), \quad \eta = \eta^1(t,x)u + \eta^0(t,x),$$

where τ, ξ, η^1 and η^0 are arbitrary smooth functions of their variables. The rest of the determining equations have the form

$$\begin{gathered}\tau\beta_t = (5\xi_x - \tau_t)\beta, \quad \eta^1_x = 2\xi_{xx}, \quad \eta^1_{xx} = \xi_{xxx}, \quad 2\eta^1_{xxx} = \xi_{xxxx},\\ \eta^1_x u^{n+1} + \eta^0_x u^n + (\eta^1_t + \eta^1_{xxxxx}\beta)u + \eta^0_t + \eta^0_{xxxxx}\beta = 0,\\ (\tau_t - \xi_x + n\eta^1)u^n + n\eta^0 u^{n-1} + (5\eta^1_{xxxx} - \xi_{xxxxx})\beta - \xi_t = 0.\end{gathered}$$

The derived determining equations were verified using GeM software package [2]. The latter two equations can be split with respect to different powers of u. Special cases of the splitting arise if $n = 0$ or $n = 1$. If $n = 0$ Eq. (6) are linear ones and, therefore, excluded from the consideration. The case $n = 1$ is thoroughly investigated in [7]. So, we concentrate our attention on the case $n \neq 0, 1$.

If $n \neq 0, 1$ the determining equations result in

$$\tau = (c_1 - c_2 n)t + c_3, \quad \xi = c_1 x + c_0, \quad \eta^1 = c_2, \quad \eta^0 = 0,$$

where c_i, $i = 0, \ldots, 3$, are arbitrary constants. Thus, the infinitesimal generator has the form

$$Q = ((c_1 - c_2 n)t + c_3)\partial_t + (c_1 x + c_0)\partial_x + c_2 u\partial_u.$$

The classifying equation on β is

$$((c_1 - c_2 n)t + c_3)\beta_t = (4c_1 + nc_2)\beta. \tag{9}$$

To derive the kernel A^{ker} of maximal Lie invariance algebras $A^{\max}$ of equations from the class (6) (i.e., the Lie invariance algebra admitted by any equation from (6)) we split in (9) with respect to β and β_t. Then $c_1 = c_2 = c_3 = 0$ and $Q = c_0\partial_x$. Thus, $A^{\mathrm{ker}} = \langle\partial_x\rangle$. To get possible extensions of A^{ker} we consider (9) not as an identity but as an equation on β, that has the form

$$(pt + q)\beta_t = r\beta. \tag{10}$$

The group classification of class (6) is equivalent to the integration of the latter equation up to the $G_0^{\sim}$-equivalence. The equivalence transformations (7) act on the coefficients $p,\ q$, and r of Eq. (10) as follows:

$$\tilde p = \kappa p, \quad \tilde q = \kappa(q\delta_3 - p\delta_4), \quad \tilde r = \kappa r,$$

where κ is a nonzero constant. Therefore, there are three inequivalent nonzero triples (p, q, r): $(1, 0, \rho)$, $(0, 1, 1)$ and $(0, 1, 0)$, where ρ is an arbitrary constant. We integrate (10) for these values of (p, q, r). Up to $G_0^{\sim}$-equivalence β takes the values from the set $\{\varepsilon t^{\rho},\ \varepsilon e^{t},\ \varepsilon\}$. Here ρ and ε are arbitrary constants with $\rho\varepsilon \neq 0$, $\varepsilon = \pm 1 \bmod G_0^{\sim}$. The last step is to substitute the obtained forms of β into Eq. (9) and to find the corresponding values of $c_i,\ i = 0, \ldots, 3$, that define the infinitesimal operator Q. We get that all $G_0^{\sim}$-inequivalent cases of Lie symmetry extension are exhausted by the following:

$$\beta = \varepsilon t^{\rho},\ \rho \neq 0\colon\ Q = \frac{5}{\rho + 1}c_1 t\partial_t + (c_1 x + c_0)\partial_x + \frac{\rho - 4}{n(\rho + 1)}c_1 u\partial_u,$$

$$\beta = \varepsilon e^{t}\colon\ Q = 5c_1\partial_t + (c_1 x + c_0)\partial_x + \frac{1}{n}c_1 u\partial_u,$$

$$\beta = \varepsilon\colon\ Q = (5c_1 t + c_3)\partial_t + (c_1 x + c_0)\partial_x - \frac{4}{n}c_1 u\partial_u,$$

where c_0, c_1 and c_3 are arbitrary constants. We have proved the following statement.

Theorem 3. *The kernel of the maximal Lie invariance algebras of nonlinear equations from the class (6) with* $n \neq 1$ *coincides with the one-dimensional algebra* $\langle\partial_x\rangle$. *All possible* $G_0^{\sim}$*-inequivalent cases of extension of the maximal Lie invariance algebras are exhausted by those presented in Cases 2–4 of Table 1.*

Proposition 1. *A group classification list for the class (1) up to* $G^{\sim}$*-equivalence coincides with the list presented in Table 1.*

Table 1 The group classification of the class (6) with $n \neq 0, 1$ up to $G_0^{\sim}$-equivalence

No.	$\beta(t)$	Basis of $A^{\max}$
1	$\forall$	∂_x
2	εt^ρ	$\partial_x,\ 5nt\partial_t + (\rho+1)nx\partial_x + (\rho-4)u\partial_u$
3	εe^t	$\partial_x,\ 5n\partial_t + nx\partial_x + u\partial_u$
4	ε	$\partial_x,\ \partial_t,\ 5nt\partial_t + nx\partial_x - 4u\partial_u$

Here ρ is an arbitrary nonzero constant; $\varepsilon = \pm 1 \bmod G_0^{\sim}$

Table 2 The group classification of the class (1) with $n \neq 0, 1$ using no equivalence

No.	$\beta(t)$	Basis of $A^{\max}$
1	$\forall$	∂_x
2	$\lambda T_t\big(T+\kappa\big)^\rho$	$\partial_x,\ 5n(T+\kappa)T_t^{-1}\partial_t + n(\rho+1)x\partial_x +$ $+ (\rho - 4 - 5n\alpha(t)(T+\kappa)T_t^{-1})u\partial_u$
3	$\lambda T_t e^{mT}$	$\partial_x,\ 5nT_t^{-1}\partial_t + mnx\partial_x + \big(m - 5n\alpha(t)T_t^{-1}\big)u\partial_u$
4	λT_t	$\partial_x,\ T_t^{-1}(\partial_t - \alpha(t)u\partial_u),\ 5nTT_t^{-1}\partial_t + nx\partial_x -$ $- (4 + 5n\alpha(t)TT_t^{-1})u\partial_u$

Here λ, κ, ρ, and m are arbitrary constants with $\lambda\rho m \neq 0$, $T = T(t) = \int e^{-n\int \alpha(t)\,\mathrm{d}t}\mathrm{d}t$, and the function $\alpha(t)$ is arbitrary in all cases

Proposition 2. *An equation of the form (1) admits a three-dimensional Lie symmetry algebra if and only if it is point-equivalent to the constant-coefficient fKdV equation* $u_t + u^n u_x + \varepsilon u_{xxxxx} = 0$ *from the same class.*

For convenience of further applications we present in Table 2 the complete list of Lie symmetry extensions for the initial class (1), where arbitrary elements are not simplified by equivalence transformations (the detailed procedure of deriving such a list from a simplified one is described in [14]).

The obtained group classification results give all Eq. (1) for which the classical method of Lie reduction can be applied.

4 Symmetry Reductions and Construction of Exact Solutions

One of the most efficient techniques for construction of solutions for nonlinear partial differential equations is the Lie reduction method, based on the usage of Lie symmetries that correspond to Lie groups of continuous point transformations [8,9]. Any (1+1)-dimensional partial differential equation admitting a one-parameter Lie symmetry group (acting regularly and transversally on a manifold defined by this equation) can be reduced to an ordinary differential equation. Lie reduction method is well known and algorithmic [8, 9]. In order to get an optimal system of group-invariant solutions reductions should be performed with respect to subalgebras from the optimal system [8, Section 3.3].

To find optimal systems of one-dimensional subalgebras for Lie algebras $A^{\max}$ presented in Table 1, we firstly consider their structure, using notations of [11]. In Cases 2 and 3 the maximal Lie-invariance algebras are two-dimensional. In Case 2 with $\rho = -1$ it is Abelian ($2A_1$). The algebras adduced in Case 2 with $\rho \neq -1$ and Case 3 are non-Abelian (A_2). The three-dimensional algebra with basis operators presented in Case 4 is of the type $A^a_{3.5}$, where $a = 1/5$.

Therefore, optimal systems of one-dimensional subalgebras of the maximal Lie invariance algebras $A^{\max}$ presented in Table 1 are the following:

$2_{\rho\neq-1}$: $\mathfrak{g}_0 = \langle\partial_x\rangle$, $\mathfrak{g}_{2.1} = \langle 5nt\partial_t + (\rho+1)nx\partial_x + (\rho-4)u\partial_u\rangle$;
$2_{\rho=-1}$: $\mathfrak{g}_0 = \langle\partial_x\rangle$, $\mathfrak{g}^a_{2.2} = \langle nt\partial_t + a\partial_x - u\partial_u\rangle$, where a is an arbitrary constant;
3: $\mathfrak{g}_0 = \langle\partial_x\rangle$, $\mathfrak{g}_3 = \langle 5n\partial_t + nx\partial_x + u\partial_u\rangle$;
4: $\mathfrak{g}_0 = \langle\partial_x\rangle$, $\mathfrak{g}^\sigma_{4.1} = \langle\partial_t + \sigma\partial_x\rangle$, $\mathfrak{g}_{4.2} = \langle 5nt\partial_t + nx\partial_x - 4u\partial_u\rangle$; $\sigma \in \{-1, 0, 1\}$.

We do not perform the reductions with respect to the subalgebra $\mathfrak{g}_0$ since they lead to constant solutions only. The reductions with respect to other one-dimensional subalgebras from the found optimal lists are presented in Table 3.

It is possible to consider also reductions of the generalized fKdV equations to algebraic equations using two-dimensional subalgebras of their Lie invariance algebras. There is only one such subalgebra that leads to a nonconstant solution, it is the subalgebra

$$\langle\partial_t,\ 5nt\partial_t + nx\partial_x - 4u\partial_u\rangle$$

of the algebra $A^{\max}$ presented in Case 4 of Table 1. The corresponding Ansatz $u = Cx^{-\frac{4}{n}}$ reduces the equation

$$u_t + u^n u_x + \varepsilon u_{xxxxx} = 0 \tag{11}$$

to an algebraic equation on the constant C. We solve it and get the stationary solution

$$u = (-8\varepsilon(n+1)(n+2)(n+4)(3n+4))^{\frac{1}{n}}(nx)^{-\frac{4}{n}}.$$

Table 3 Similarity reductions of the equations $u_t + u^n u_x + \beta(t)u_{xxxxx} = 0$

No.	$\beta(t)$	$\mathfrak{g}$	ω	Ansatz	Reduced ODE
1	$\varepsilon t^\rho,\ \rho \neq -1$	$\mathfrak{g}_{2.1}$	$xt^{-\frac{\rho+1}{5}}$	$u = t^{\frac{\rho-4}{5n}}\varphi(\omega)$	$\varepsilon\varphi''''' + \left(\varphi^n - \frac{\rho+1}{5}\omega\right)\varphi' +$ $+ \frac{\rho-4}{5n}\varphi = 0$
2	εt^{-1}	$\mathfrak{g}^a_{2.2}$	$x - \frac{a}{n}\ln t$	$u = t^{-\frac{1}{n}}\varphi(\omega)$	$\varepsilon\varphi''''' + \left(\varphi^n - \frac{a}{n}\right)\varphi' -$ $- \frac{1}{n}\varphi = 0$
3	εe^t	$\mathfrak{g}_3$	$xe^{-\frac{1}{5}t}$	$u = e^{\frac{1}{5n}t}\varphi(\omega)$	$\varepsilon\varphi''''' + \left(\varphi^n - \frac{1}{5}\omega\right)\varphi' +$ $+ \frac{1}{5n}\varphi = 0$
4	ε	$\mathfrak{g}^\sigma_{4.1}$	$x - \sigma t$	$u = \varphi(\omega)$	$\varepsilon\varphi''''' + (\varphi^n - \sigma)\varphi' = 0$
5	ε	$\mathfrak{g}_{4.2}$	$xt^{-\frac{1}{5}}$	$u = t^{-\frac{4}{5n}}\varphi(\omega)$	$\varepsilon\varphi''''' + \left(\varphi^n - \frac{\omega}{5}\right)\varphi' -$ $- \frac{4}{5n}\varphi = 0$

Here a is an arbitrary constant, $\sigma \in \{-1, 0, 1\}$, $\varepsilon = \pm 1 \bmod G_0^\sim$, $n \neq 0, 1$

of Eq. (11). Using this solution and equivalence transformation (5) we construct simple nonstationary exact solution,

$$u = (-8\varepsilon(n+1)(n+2)(n+4)(3n+4))^{\frac{1}{n}}(nx)^{-\frac{4}{n}}e^{-\int\alpha(t)\,\mathrm{d}t},$$

for the fKdV equation with time-dependent coefficients

$$u_t + u^n u_x + \alpha(t)u + \varepsilon e^{-n\int\alpha(t)\,dt}u_{xxxxx} = 0, \tag{12}$$

where α is an arbitrary nonvanishing smooth function.

If $n = 2$ the travelling wave solution

$$u = \pm 2\sqrt{-10\varepsilon}\left(3\tanh(x + 24\varepsilon t)^2 - 2\right)$$

of Eq. (11) is known [10]. Using (5) we get the exact solution of Eq. (12) with $n = 2$,

$$u = \pm 2\sqrt{-10\varepsilon}\left(3\tanh\left(x + 24\varepsilon\textstyle\int e^{-2\int\alpha(t)\,\mathrm{d}t}\mathrm{d}t\right)^2 - 2\right)e^{-\int\alpha(t)\,\mathrm{d}t}.$$

It is worthy to note that the obtained reductions to ODEs can be used for construction of numerical solutions of the generalized fKdV equations, see [6, 16] for details.

5 Conclusion and Discussion

In this paper we present the exhaustive group classification of generalized fKdV equations with time dependent coefficients of the general form (1). The complete result is achieved due to the use of equivalence transformations. We show that up to point equivalence the group classification problem for the initial class can be reduced to a simpler problem for its subclass with $\alpha = 0$ (6). After the group classification for the subclass (6) is performed, the most general forms of Eq. (1) admitting Lie symmetry extensions can be easily recovered using equivalence transformations. The derived results together with ones obtained in [7] for the case $n = 1$ give the complete solution of the group classification problem for nonlinear equations of the form (1).

We mentioned in the introduction that Lie symmetry analysis of the class (1) was initiated in [18], and the case $n = 2$ was also treated separately in [17]. However, the results presented therein are either incorrect [18] or incomplete [17]. Here we discuss main lucks of the results obtained in those two papers.

In [17] only some cases of Lie symmetry extensions for equations of the form (1) with $n = 2$ were found, namely, the cases with $\alpha = \text{const}$ and $\alpha = 1/t$. If one performs the group classification up to the corresponding equivalence transformations it is enough to consider the case $\alpha = 0$. If one wants to get the classification, where all equations admitting Lie symmetry extensions are presented, not only their inequivalent representatives, then all such equations will have the coefficient α being arbitrary, so the cases $\alpha = \text{const}$ and $\alpha = 1/t$ can be considered as particular examples only. Moreover, even studying these particular cases the authors of [17] missed one case of Lie symmetry extension for each value of α considered by them. For example, for the case $\alpha = 0$ this is $\beta = \varepsilon(t + \delta)^\rho$, where ε, δ and ρ are arbitrary constants with $\varepsilon\rho \neq 0$. Nevertheless, at least dimensions and basis operators of the found Lie symmetry algebras for those particular cases derived in [17] are correct in contrast to the results presented in [18].

In [18] the authors state that they find three cases of Lie symmetry extensions for Eq. (1) and in each derived case the corresponding Lie symmetry algebra is four-dimensional. This is a false assertion. In this paper and in [7] we show that Eq. (1) admits four-dimensional Lie symmetry algebra if and only if $n = 1$ and, moreover, the equation is point-equivalent to the simplest constant-coefficient fKdV equation $u_t + uu_x + \mu u_{xxxxx} = 0$, where $\mu = \text{const}$. So, the results of [18] are principally incorrect.

In the modern group analysis of differential equations the solution of a group classification problem should be inseparably linked with the study of admissible transformations in the corresponding class of equations. Neglecting of this often leads to incomplete results as shown in the discussion. Moreover, the knowledge of such transformations can be used for solving other problems concerned with the study of classes of variable-coefficient differential equations or their systems. In particular, in the recent work [15] the application of admissible transformations to the study of integrability was analyzed.

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A Construction of Generalized Lotka–Volterra Systems Connected with $\mathfrak{sl}_n(\mathbb{C})$

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Abstract We construct a large family of Hamiltonian systems which are connected with root systems of complex simple Lie algebras. These systems are generalizations of the KM system. The Hamiltonian vector field is homogeneous cubic but in a number of cases a simple change of variables transforms such a system to a quadratic Lotka–Volterra system. We classify all possible Lotka–Volterra systems that arise via this algorithm in the A_n case.

1 Introduction

The Volterra model, also known as the KM system is a well-known integrable system defined by

$$\dot{x}_i = x_i(x_{i+1} - x_{i-1}) \qquad i = 1, 2, \ldots, n, \tag{1}$$

where $x_0 = x_{n+1} = 0$. It was studied by Lotka in [7] to model oscillating chemical reactions and by Volterra in [10] to describe population evolution in a hierarchical system of competing species. It was first solved by Kac and van-Moerbeke in [6], using a discrete version of inverse scattering due to Flaschka [5]. In [8] Moser gave a solution of the system using the method of continued fractions and in the process he constructed action-angle coordinates. Equation (1) can be considered as a finite-dimensional approximation of the Korteweg–de Vries (KdV) equation. The Poisson bracket for this system can be thought as a lattice generalization of the Virasoro algebra [4].

The Volterra system is associated with a simple Lie algebra of type A_n in the sense that it can be written in Lax pair form $\dot{L} = [B, L]$ where $L = \sum_{i=1}^{n} a_i \,(X_{\alpha_i} + X_{-\alpha_i})$ and $B = \sum_{i=1}^{n-1} a_i a_{i+1} \left(X_{\alpha_i+\alpha_{i+1}} - X_{-\alpha_i-\alpha_{i+1}}\right)$ with $\{\alpha_1, \ldots, \alpha_n\}$ being the simple roots of the root system of the Lie algebra of type A_n and X_{α_i} the corresponding root vectors. This Lax pair is due to Moser [8]; it gives a

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polynomial (in fact cubic) system of differential equations. The change of variables $x_i = 2a_i^2$, produces Eq. (1). The purpose of this paper is to generalize this Lax pair and produce a larger class of Hamiltonian systems which we call generalized Volterra systems since in some cases by a simple change of variables we produce Lotka–Volterra systems.

In this paper we restrict our attention in the A_n case. However, this algorithm applies, more generally for each complex simple Lie algebra. In dimension 3 this procedure produces only two systems, the KM system and the periodic KM system. In dimensions 4 and 5 (i.e. the cases of A_3 and A_4) and by allowing the use of complex coefficients this method works in all possible cases and in fact we have verified using Maple that all the resulting systems are Liouville integrable. To establish integrability we have used standard techniques of Lax pairs and Poisson geometry and also a particular technique of Moser which uses the square of the Lax matrix. After the definition of Lotka–Volterra systems in Sect. 2, we describe our algorithm in Sect. 3. Finally in Sect. 4 we give a classification of all cases which give rise to Lotka–Volterra systems via the transformation $a_i \to 2a_i^2$. We also explicitly present the corresponding Lotka–Volterra systems.

2 Lotka–Volterra Systems

The KM-system belongs to a large class of the so called Lotka–Volterra systems. The most general form of the Lotka–Volterra equations is

$$\dot{x}_i = \varepsilon_i x_i + \sum_{j=1}^{n} a_{ij} x_i x_j, \quad i = 1, 2, \ldots, n.$$

We may assume that there are no linear terms ($\varepsilon_i = 0$). We also assume that the matrix $A = (a_{ij})$ is skew-symmetric. All these systems can be written in Hamiltonian form using the Hamiltonian function

$$H = x_1 + x_2 + \cdots + x_n .$$

Hamilton's equations take the form $\dot{x}_i = \{x_i, H\} = \sum_{j=1}^{n} \pi_{ij}$ with quadratic functions

$$\pi_{i,j} = \{x_i, x_j\} = a_{ij} x_i x_j, \quad i, j = 1, 2, \ldots, n. \tag{2}$$

From the skew symmetry of the matrix $A = (a_{ij})$ it follows that the Jacobi identity is satisfied.

The Poisson tensor (2) is Poisson isomorphic to the constant Poisson structure defined by the constant matrix A, see [1]. If $\mathbf{k} = (k_1, k_2 \cdots, k_n)$ is a vector in the kernel of A then the function

$$f = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$$

is a Casimir. Indeed for an arbitrary function g the Poisson bracket $\{f, g\}$ is

$$\{f, g\} = \sum_{i,j=1}^{n} \{x_i, x_j\} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij} k_i \right) x_j f \frac{\partial g}{\partial x_j} = 0 .$$

If the matrix A has rank r then there are $n - r$ functionally independent Casimirs. This type of integral can be traced back to Volterra [10]; see also [1,2,9].

3 Generalized Volterra Systems

We recall the following procedure from [3]. Let $\mathfrak{g}$ be any simple Lie algebra equipped with its Killing form $\langle \cdot \mid \cdot \rangle$. One chooses a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and a basis Π of simple roots for the root system Δ of $\mathfrak{h}$ in $\mathfrak{g}$. The corresponding set of positive roots is denoted by Δ^+. To each positive root α one can associate a triple $(X_\alpha, X_{-\alpha}, H_\alpha)$ of vectors in $\mathfrak{g}$ which generate a Lie subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. The set $(X_\alpha, X_{-\alpha})_{\alpha \in \Delta^+} \cup (H_\alpha)_{\alpha \in \Pi}$ is a basis of $\mathfrak{g}$, called a root basis. Let $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ and let $X_{\alpha_1}, \ldots, X_{\alpha_\ell}$ be the corresponding root vectors in $\mathfrak{g}$. Define

$$L = \sum_{\alpha_i \in \Pi} a_i (X_{\alpha_i} + X_{-\alpha_i}).$$

To find the matrix B we use the following procedure. For each i, j form the vectors $\left[X_{\alpha_i}, X_{\alpha_j}\right]$. If $\alpha_i + \alpha_j$ is a root then include a term of the form $a_i a_j \left[X_{\alpha_i}, X_{\alpha_j}\right]$ in B. We make B skew-symmetric by including the corresponding negative root vectors $a_i a_j [X_{-\alpha_i}, X_{-\alpha_j}]$. Finally, we define the system using the Lax equation $\dot{L} = [L, B]$. For a root system of type A_n we obtain the KM system.

In this paper we generalize this algorithm as follows. Consider a subset Φ of Δ^+ such that $\Pi \subset \Phi \subset \Delta^+$. The Lax matrix is easy to construct

$$L = \sum_{\alpha_i \in \Phi} a_i (X_{\alpha_i} + X_{-\alpha_i}) .$$

Here we use the following enumeration of Φ which we assume to have m elements. The variables a_j correspond to the simple roots α_j for $j = 1, 2, \ldots, \ell$. We assign the variables a_j for $j = \ell + 1, \ell + 2, \ldots, m$ to the remaining roots in Φ. To construct the matrix B we use the following algorithm. Consider the set $\Phi \cup \Phi^-$ which consists of all the roots in Φ together with their negatives and let $\Psi = \left\{\alpha + \beta \mid \alpha, \beta \in \Phi \cup \Phi^-, \ \alpha + \beta \in \Delta^+\right\}$. Define

$$B = \sum c_{ij} a_i a_j (X_{\alpha_i + \alpha_j} - X_{-\alpha_i - \alpha_j}) \tag{3}$$

where $c_{ij} = \pm 1$ if $\alpha_i + \alpha_j \in \Psi$ with $\alpha_i, \alpha_j \in \Phi \cup \Phi^-$ and 0 otherwise. In all eight cases in A_3 we are able to make the proper choices of the sign of the c_{ij} so that we can produce a Lax pair. This method produces a Lax pair in all but five out of sixty four cases in A_4. However, when we allow the c_{ij} to take the complex values $\pm i$ we are able to produce a Lax pair in all 64 cases. By using Maple we were able to check that all these examples in A_3 and A_4 are in fact Liouville integrable. We will not attempt to prove the integrability of these systems in general due to the complexity of their definition. In this paper we restrict our attention to some examples in the A_n case. Examples from other Lie algebras will be presented in a future publication.

This algorithm for certain subsets Φ recovers well known integrable systems. For example for $\Phi = \Pi$, the simple roots of the root system A_n, and $c_{i,i+1} = 1$ for $i = 1, 2, \ldots, n-1$ we obtain the KM system while for $\Phi = \Pi \cup \{\alpha_{n+1}\}$, the simple roots and the highest root, the choice of the signs $c_{i,i+1} = 1$ for $i = 1, 2, \ldots, n-1$ and $c_{1,n+1} = c_{n,n+1} = -1$ produce the periodic KM system. In the next proposition we present a sufficient (but not necessary) condition on the subset Φ which gives a consistent Lax pair.

Proposition 1. *Let $\Pi \subset \Phi \subset \Delta^+$ be a subset of the positive roots with the property that whenever $\alpha, \beta, \gamma \in \Phi \cup \Phi^-$ then $\alpha + \beta + \gamma \neq 0$ and if $\alpha + \beta + \gamma \in \Delta^+$ then $\alpha + \beta + \gamma \in \Phi$. Also let B be the matrix constructed using the algorithm described in (3). Then for any choice of the signs $c_{i,j}$ the pair L, B is a Lax pair.*

This condition is of course not necessary. For example the KM and the periodic KM systems do not fall in this class.

Example 1. Let Φ be the subset of the positive roots of the root system A_n containing all the roots of odd height. We immediately see that Φ satisfies the hypothesis of Proposition 1 and therefore for all possible choices of the signs $c_{i,j}$ we have a consistent Lax pair. For example when $n = 3$, $\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$ and this choice gives rise to the matrix

$$L = \sum_{i=1}^{3} a_i (X_{\alpha_i} + X_{-\alpha_i}) + a_4 (X_{\alpha_1+\alpha_2+\alpha_3} + X_{-\alpha_i-\alpha_2-\alpha_3}) \, .$$

The skew symmetric matrix B constructed using (3) has upper triangular part

$$(c_{1,2} a_1 a_2 + c_{3,4} a_3 a_4)\, X_{\alpha_1+\alpha_2} + (c_{1,4} a_1 a_4 + c_{2,3} a_2 a_3)\, X_{\alpha_2+\alpha_3}$$

Now we easily verify that all 16 possible choices of the signs $c_{i,j}$ give consistent Lax pairs. Of course only half of them give possibly non-isomorphic systems and only one of them gives a Lotka Volterra system (see Theorem 1), the well known periodic KM system.

Example 2. For the root system of type A_3 if we take $\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2\}$ then

$$\Psi = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}.$$

In this example the variables a_i for $i = 1, 2, 3$ correspond to the three simple roots $\alpha_1, \alpha_2, \alpha_3$ and the variable a_4 to the root $\alpha_1 + \alpha_2$. We obtain the following Lax pair:

$$L = \begin{pmatrix} 0 & a_1 & a_4 & 0 \\ a_1 & 0 & a_2 & 0 \\ a_4 & a_2 & 0 & a_3 \\ 0 & 0 & a_3 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 & -a_4a_2 & a_1a_2 & -a_4a_3 \\ a_4a_2 & 0 & -a_1a_4 & a_2a_3 \\ -a_1a_2 & a_1a_4 & 0 & 0 \\ a_4a_3 & -a_2a_3 & 0 & 0 \end{pmatrix}.$$

Using the substitution $x_i = a_i^2$ followed by scaling, the system defined by the Lax equation $\dot{L} = [L, B]$ is transformed to the following Lotka–Volterra system.

$$\dot{x_1} = x_1x_2 - x_1x_4, \qquad \dot{x_2} = -x_2x_1 + x_2x_3 + x_2x_4,$$
$$\dot{x_3} = -x_3x_2 + x_3x_4, \qquad \dot{x_4} = x_4x_1 - x_4x_2 - x_4x_3\,.$$

This system is integrable. There exist two functionally independent Casimir functions $F_1 = x_1x_3 = \det L$ and $F_2 = x_1x_2x_4$. The standard quadratic Poisson bracket (2) is defined by the relations $\{x_i, x_j\} = r_{i,j}x_ix_j$ where $r_{1,2} = r_{2,3} = r_{3,4} = r_{2,4} = -r_{1,4} = 1$ and $r_{1,3} = 0$. One can find the Casimirs by computing the kernel of the skew symmetric matrix $A = (r_{i,j})_{1 \le i,j \le 4}$. The additional integral is the Hamiltonian $H = x_1 + x_2 + x_3 + x_4 = \operatorname{tr} L^2$.

4 Subsets Φ Giving Rise to Lotka Volterra Systems

In this section we classify the subsets of the positive roots containing the simple roots which give rise to Lotka Volterra systems via the transformation $x_i = 2a_i^2$. We also explicitly describe each system associated with this subsets. We have the following theorem.

Theorem 1. *The only choices for the subset Φ of Δ^+ so that the corresponding generalized Volterra systems, under the substitution $x_i = 2a_i^2$, are transformed into Lotka–Volterra systems are the following five.*

1. $\Phi = \Pi$,
2. $\Phi = \Pi \cup \{\alpha_2 + \alpha_3 + \cdots + \alpha_{n-1}\}$,
3. $\Phi = \Pi \cup \{\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}\}$,
4. $\Phi = \Pi \cup \{\alpha_2 + \alpha_3 + \cdots + \alpha_n\}$,
5. $\Phi = \Pi \cup \{\alpha_1 + \alpha_2 + \cdots + \alpha_n\}$.

We outline the proof of this theorem. First we prove the theorem for the special case where Φ is the subset of the positive roots containing the simple roots and only one extra root. This is done by explicitly writing down the matrix $[B, L]$ and setting equal to zero the coefficients of the root vectors corresponding to roots not appearing in Φ. We end up with a linear system of the signs $c_{i,j}$, which in order to have a solution, the extra root $\alpha_{n+1} \in \Phi$ must be of the form $\alpha_{n+1} = \alpha_k + \alpha_{k+1} + \ldots + \alpha_m$ with $k \leq 2$ and $m \geq n-1$. Since subsystems of Lotka Volterra systems are also Lotka Volterra systems, the proof of Theorem 1 is a case by case verification of all of the 16 possible subsets Φ containing the simple roots and roots in

$$\{\alpha_k + \alpha_{k+1} + \ldots + \alpha_m : k \leq 2 \text{ and } m \geq n-1\}.$$

Below we describe the corresponding Lotka Volterra systems.

Case (1) gives rise to the KM system while case (5) gives rise to the periodic KM system. Case (2) corresponds to the Lax equation $\dot{L} = [L, B]$ with L matrix

$$L = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_1 & 0 & a_2 & 0 & & 0 & a_{n+1} & 0 \\ 0 & a_2 & 0 & a_3 & \ddots & & 0 & 0 \\ \vdots & 0 & a_3 & \ddots & \ddots & & & 0 \\ 0 & & \ddots & \ddots & 0 & a_{n-2} & 0 & \vdots \\ 0 & 0 & & & a_{n-2} & 0 & a_{n-1} & 0 \\ 0 & a_{n+1} & 0 & & 0 & a_{n-1} & 0 & a_n \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_n & 0 \end{pmatrix}.$$

The skew symmetric matrix B is defined using the method described in Sect. 3. More explicitly its upper triangular part is given by the formula

$$\sum_{i=1}^{n-1} a_i a_{i+1} X_{\alpha_i + \alpha_{i+1}} - a_{n-1} a_{n+1} X_{\alpha_{n+1} - \alpha_{n-1}} - a_2 a_{n+1} X_{\alpha_{n+1} - \alpha_2}$$
$$- a_1 a_{n+1} X_{\alpha_1 + \alpha_{n+1}} - a_n a_{n+1} X_{\alpha_{n+1} + \alpha_n}.$$

After substituting $x_i = 2a_i^2$ for $i = 1, \ldots, n+1$, the Lax pair L, B becomes equivalent to the following equations of motion:

$$\begin{aligned} \dot{x}_1 &= x_1(x_2 - x_{n+1}), \\ \dot{x}_2 &= x_2(x_3 - x_1 - x_{n+1}), \\ \dot{x}_i &= x_i(x_{i+1} - x_{i-1}), \qquad i = 3, 4, \ldots, n-2, n \\ \dot{x}_{n-1} &= x_{n-1}(x_n - x_{n-2} + x_{n+1}), \\ \dot{x}_{n+1} &= x_{n+1}(x_1 + x_2 - x_{n-1} - x_n). \end{aligned}$$

It is easily verified that for n even, the rank of the corresponding Poisson matrix is n and the function $f = x_2 x_3 \cdots x_{n-1} x_{n+1}$ is the Casimir of the system, while for n odd, the rank of the Poisson matrix is $n-1$ and the functions $f_1 = x_1 x_3 \cdots x_n = \sqrt{\det L}$ and $f_2 = x_2 x_3 \cdots x_{n-1} x_{n+1}$ are the Casimirs.

Case (3) corresponds to the Lax pair whose Lax matrix L is given by

$$L = \sum_{i=1}^{n+1} a_i \left(X_{\alpha_i} + X_{-\alpha_i}\right)$$

with $a_{n+1} = \alpha_1 + \ldots + \alpha_{n-1}$. The upper triangular part of the skewsymmetric matrix B is

$$\sum_{i=1}^{n-1} a_i a_{i+1} X_{\alpha_i+\alpha_{i+1}} - a_{n-1} a_{n+1} X_{\alpha_{n+1}-\alpha_{n-1}} - a_1 a_{n+1} X_{\alpha_{n+1}-\alpha_1} -$$

$$- \; a_n a_{n+1} X_{\alpha_{n+1}+\alpha_n} \, .$$

After substituting $x_i = 2a_i^2$ for $i = 1, \ldots, n+1$, we obtain the following equivalent equations of motion:

$$\begin{aligned}
\dot{x}_1 &= x_1(x_2 - x_{n+1}) \\
\dot{x}_i &= x_i(x_{i+1} - x_{i-1}), \; i = 2, 3, 4, \ldots, n-2, n \\
\dot{x}_{n-1} &= x_{n-1}(x_n - x_{n-2} + x_{n+1}) \\
\dot{x}_{n+1} &= x_{n+1}(x_1 - x_n - x_{n-1}).
\end{aligned}$$

For n even, the rank of the Poisson matrix is n and the function $f = x_1 x_2 \cdots x_{n-1} x_{n+1}$ is the Casimir, while for n odd, the rank of the Poisson matrix is $n-1$ and the functions $f_1 = x_1 x_3 x_5 \cdots x_n = \sqrt{\det L}$ and $f_2 = x_1 x_2 \cdots x_{n-1} x_{n+1}$ are Casimirs.

The system obtained in case (4) turns out to be isomorphic to the one in case (3). In fact, the change of variables $u_{n+1-i} = -x_i$ for $i = 1, 2, \ldots, n$ and $u_{n+1} = -x_{n+1}$ in case (3) gives the corresponding system of case (4).

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Systems of First-Order Ordinary Differential Equations Invariant with Respect to Linear Realizations of Two- and Three-Dimensional Lie Algebras

Oksana Kuriksha

Abstract The complete group classification of systems of two first-order ordinary differential equations with respect to point transformations linear in dependent variables is carried out.

1 Introduction

The group analysis of differential equations (DEs) has appeared in works by outstanding mathematician Lie in nineteenth century. He made a fundamental contribution to the problem of exact solvability of ordinary differential equations (ODEs) by quadratures. Lie has shown that special methods of integration of such equations (specific changes of variables, the method of integrating multiplier, etc.) can be derived in a regular way using the group theory [1,2].

Finding symmetries of DEs is an algorithmic procedure implemented in many computer algebra packages. However, these packages are effective mainly for equations without free parameters. Group classification of a class of DEs as a rule is a non-trivial problem. At the same time just such problems are very important since they allow to find a number of ODEs integrable in quadratures.

In this paper we classify systems of first-order ODEs,

$$\dot{u}_a = f_a(u_1, u_2), \tag{1}$$

where u_a are unknown functions of t, $\dot{u}_a = du_a/dt, a = 1, 2$.

Systems of Eq. (1) are widely used in mathematical biology [3,4] and diffusion theory [5].

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2 Symmetry of the System (1)

It is well known that systems (1) admit infinite-dimensional Lie symmetry algebras which, unfortunately, can not be described constructively [6]. However, it is possible to make a preliminary group classification for these equations if we impose some a priori restrictions on the class of symmetries.

In this paper we present a complete group classification of equations of (1) with respect to groups of point transformations linear in dependent variables u_a. Such classification is still a rather complicated problem. To solve it we use the ideas proposed and implemented in [7–9]. First we specify the generic form of basis elements of symmetry algebra, which can be realized on the set of solutions of Eq. (1). Then we use the invariance criterium to complete this specification.

Since Eq. (1) do not depend on t explicitly, they admit the obvious symmetry, with respect to shifts of independent variable t. The corresponding infinitesimal generator is

$$X_0 = \partial_t. \tag{2}$$

Other symmetry operators are supposed to be of the form

$$X = \eta\partial_t + \pi^a\partial_{u_a}, \tag{3}$$

where $\pi^a = \pi^{ab}u_b + \omega^a$ and η, π^{ab}, ω^a are functions of t.

First, we find all inequivalent two-dimensional symmetry algebras for the system (1), which include infinitesimal operators X_0 and X. They should satisfy the condition

$$[X_0, X] = \alpha X_0 + \beta X, \tag{4}$$

where α and β are real constants.

Equivalence transformations which keep the form of Eq. (1) are given by the following formula

$$u_a \to \Lambda^{ab}u_b + \varphi^a, \tag{5}$$

where Λ^{ab} and φ^a are arbitrary constants, and Λ^{ab} is an invertible matrix.

Substituting (3) into (4) we obtain the system of determining equations

$$\dot{\eta} = \alpha + \beta\eta, \quad \dot{\pi}^{ab} = \beta\pi^{ab}, \quad \dot{\omega}^a = \beta\omega^a.$$

Solving this system, we find functions η, π^{ab} and ω^a which determine infinitesimal operator (3). As a result we come to the following statement.

Theorem 1. *Up to transformations (5), there exist exactly six inequivalent infinitesimal operators X, satisfying condition (4):*

$$\begin{gathered}X_1 = \mu t\partial_t - u_1\partial_{u_1} - \nu u_2\partial_{u_2}, \quad X_2 = \mu t\partial_t - \partial_{u_1} - u_2\partial_{u_2}, \\ X_3 = \mu t\partial_t - \nu\partial_{u_1} - \partial_{u_2}, \quad X_4 = e^{\lambda t}\left(u_1\partial_{u_1} + \nu u_2\partial_{u_2}\right), \\ X_5 = e^{\lambda t}\left(\partial_{u_1} + u_2\partial_{u_2}\right), \quad X_6 = e^{\lambda t}(\mu\partial_{u_1} + \partial_{u_2}),\end{gathered} \tag{6}$$

where μ, ν and λ are arbitrary constants.

The next step is to construct three-dimensional Lie algebras, which include the infinitesimal operator X_0 and two infinitesimal operators of the generic form (3). They should satisfy the conditions

$$[X_0, X_a] = \alpha_a X_0 + \beta_{ab}X_b, \quad [X_1, X_2] = \alpha_0 X_0 + \beta_{0b}X_b. \tag{7}$$

Similarly, substituting two infinitesimal operators of the generic form (3) into (7), we find possible functions η, π^{ab} and ω^a. As a result we get the following statement.

Theorem 2. *There are 42 inequivalent realizations of three-dimensional Lie algebras for systems of ODEs of form (1) that include operator (2) and two operators of the form (3). These additional pairs of operators are enumerated in the following formula:*

$$\begin{gathered}
R_1: \quad \mu t\partial_t + u_1\partial_{u_1} + \nu t u_2\partial_{u_2}, \quad X_7 = u_2\partial_{u_2}; \\
R_2: \quad X_8 = \mu t\partial_t - u_1\partial_{u_1}, \quad X_9 = \nu t\partial_t - u_2\partial_{u_2}; \\
R_3: \quad F_1 u_1\partial_{u_1} + G_1 u_2\partial_{u_2}, \quad F_2 u_1\partial_{u_1} + G_2 u_2\partial_{u_2}; \\
R_4: \quad X_{10} = \mu t\partial_t + u_1\partial_{u_1} + \nu t\partial_{u_2}, \quad X_{11} = \partial_{u_2}; \\
R_5: \quad \mu t\partial_t + \nu t u_1\partial_{u_1} + \partial_{u_2}, \quad X_{12} = u_1\partial_{u_1}; \\
R_6: \quad (F_1 + G_1 u_1)\partial_{u_2}, \quad (F_2 + G_2 u_1)\partial_{u_2}; \\
R_7: \quad F_1 u_1\partial_{u_1} + G_1\partial_{u_2}, \quad F_2 u_1\partial_{u_1} + G_2\partial_{u_2}; \\
R_8: \quad X_{11}, \quad X_{13} = \mu t\partial_t + \partial_{u_1} + \nu t\partial_{u_2}; \\
R_9: \quad X_{14} = \partial_{u_1} + u_1\partial_{u_2}, \quad \mu t\partial_t + X_{11} + \nu t X_{14}; \\
R_{10}: \quad X_{11}, \quad X_{15} = \mu t\partial_t + (\nu t + u_1)\partial_{u_2}; \\
R_{11}: \quad F_1 X_{11} + G_1 X_{14}, \quad F_2 X_{11} + G_2 X_{14}; \\
R_{12}: \quad \mu t\partial_t + (u_1 + \nu t)\partial_{u_1} + \lambda u_2\partial_{u_2}, \quad X_{16} = \partial_{u_1}; \\
R_{13}: \quad X_{17} = (\mu u_1 - u_2)\partial_{u_1} + (u_1 + \mu u_2)\partial_{u_2}, \quad \lambda t\partial_t + X_{18} + \nu t X_{17}; \\
R_{14}: \quad X_{18} = u_1\partial_{u_1} + u_2\partial_{u_2}, \quad \lambda t\partial_t + X_{17} + \nu t X_{18}; \\
R_{15}: \quad X_{19} = u_1\partial_{u_2}, \quad \mu t\partial_t + X_{20} + \nu t X_{19};
\end{gathered}$$

$$R_{16}: \quad X_{19}, \quad \mu t\partial_t + u_1\partial_{u_1} + (\lambda u_2 + \nu t u_1)\partial_{u_2}, \quad \lambda \neq 1;$$

$$R_{17}: \quad \mu t\partial_t + X_{14} + \nu t(X_7 + X_{18}), \quad X_7 + X_{18};$$

$$R_{18}: \quad -X_{19}, \quad \mu t\partial_t + u_1\partial_{u_1} + (1 - \nu t u_1)\partial_{u_2};$$

$$R_{19}: \quad X_1|_{\nu=1} - X_{19}, \quad X_{21} = \nu t\partial_t - u_1\partial_{u_2};$$

$$R_{20}: \quad F_1X_{20} + G_1X_{19}, \quad F_2X_{20} + G_2X_{19};$$

$$R_{21}: \quad F_1X_{18} + G_1X_{17}, \quad F_2X_{18} + G_2X_{17};$$

$$R_{22}: \quad X_8, \quad X_{22} = \nu t\partial_t - \partial_{u_2};$$

$$R_{23}: \quad X_{22}, \quad X_{23} = \mu t\partial_t - \partial_{u_1};$$

$$R_{24}: \quad X_{21}, \quad X_{23};$$

$$R_{25}: \quad X_{11}, \quad X_{13} + X_{19};$$

$$R_{26}: \quad X_{22}, \quad X_{23} - X_{19};$$

$$R_{27}: \quad X_1, \quad X_{16};$$

$$R_{28}: \quad X_1|_{\nu=1}, \quad X_{24} = \lambda\partial_{u_1} + \partial_{u_2};$$

$$R_{29}: \quad X_{24}, \quad \mu t\partial_t + \nu t X_{24} + X_{18};$$

$$R_{30}: \quad X_2, \quad X_{11};$$

$$R_{31}: \quad X_1|_{\nu=1}, \quad \nu t\partial_t - X_{17};$$

$$R_{32}: \quad F_1\partial_{u_1} + G_1\partial_{u_2}, \quad F_2\partial_{u_1} + G_2\partial_{u_2};$$

$$R_{33}: \quad X_1|_{\nu\neq 1}, \quad X_{19};$$

$$R_{34}: \quad X_7 + X_{13}, \quad X_{11};$$

$$R_{35}: \quad X_8 - X_{11}, \quad -X_{19};$$

$$R_{36}: \quad X_9, \quad X_{19};$$

$$R_{37}: \quad X_{18}, \quad X_{23} - X_{19};$$

$$R_{38}: \quad X_{20} = u_1\partial_{u_1} + (u_1 + u_2)\partial_{u_2}, \quad \mu t\partial_t + \nu t X_{20} + X_{19};$$

$$R_{39}: \quad X_{19}, \quad \mu t\partial_t + (\nu t u_1 + u_2)\partial_{u_2};$$

$$R_{40}: \quad X_1|_{\nu=1} - X_{19}, \quad X_{11};$$

$$R_{41}: \quad X_{11}, \quad X_{15} + X_{18};$$

$$R_{42}: \quad X_{19}, \quad \mu t\partial_t + (1 + \nu t u_1)\partial_{u_2}.$$

Here (F_1, G_1) *and* (F_2, G_2) *are solutions of the system* $F_t = \lambda F + \nu G$, $G_t = \sigma F + \gamma G$, *where* λ, ν, σ, γ *are arbitrary constants.*

3 Construction of Invariant Equations

Let us rewrite the system of ODEs (1) in the form

$$F_a(u_1, u_2, \dot{u}_1, \dot{u}_2) = \dot{u}_a - f_a(u_1, u_2) = 0. \tag{8}$$

The infinitesimal operator X is a symmetry operator of Eq. (8), if [6]:

$$X^{(1)} F\big|_{[F]} = 0,$$

where $[F]$ is the manifold determined by Eq. (8) in the first-order jet-space over the space of variables t, u_1, u_2, and $X^{(1)} = X + \xi^a \partial_{\dot{u}_a}$ is the first prolongation of the infinitesimal operator X. The coefficients ξ^a are calculated by the formula $\xi^a = D_t(\pi^a) - \dot{u}_a D_t(\eta)$, where $D_t = \partial_t + \dot{u}_a \partial_{u_a} + \ddot{u}_a \partial_{\dot{u}_a} + \cdots$ is the operator of total differentiation with respect to t.

Acting by the prolonged infinitesimal operator $X^{(1)}$ on the function $F = (F_1, F_2)$ and equate the resulting expression to zero we obtain:

$$X^{(1)} F = 0, \quad \text{or} \quad \dot{\pi}^{ab} u_b + \pi^{ab} \dot{u}_b + \dot{\omega}^a - \dot{\eta} \dot{u}_a = \left(\pi^{cb} u_b + \omega^c\right) \frac{\partial f_a}{\partial u_c}.$$

The transition to the manifold defined by $[F]$ is made by substituting $\dot{u}_a = f_a$ into the latter equality. As a result we obtain the determining equations:

$$\dot{\pi}^{ab} u_b + \pi^{ab} f_b + \dot{\omega}^a - \dot{\eta} f_a = \left(\pi^{cb} u_b + \omega^c\right) \frac{\partial f_a}{\partial u_c}. \tag{9}$$

For each case presented in Theorems 1 and 2 we substitute the expressions for the coefficients π^{ab}, ω^a and η into (9). As a result, we obtain the system of determining equations for arbitrary elements f_a. Solving these equations we obtain the lists of Eq. (1) with non-equivalent symmetries, presented in the following theorems. These lists do not include linear and autonomic systems (1) which are integrable independently on their symmetries.

Theorem 3. *Inequivalent systems of the form (1) invariant with respect to the two-dimensional Lie algebras listed in Theorem 1, have the following forms:*

$$X_1: \quad \dot{u}_1 = u_1^{1+\mu} F_1\left(u_2 u_1^{-\nu}\right), \quad \dot{u}_2 = u_1^{\nu+\mu} F_2\left(u_2 u_1^{-\nu}\right);$$

$$X_2: \quad \dot{u}_1 = u_2^{\mu} F_1\left(u_2 e^{-u_1}\right), \quad \dot{u}_2 = u_2^{\mu+1} F_2\left(u_2 e^{-u_1}\right);$$

$$X_3: \quad \dot{u}_1 = e^{\mu u_2} F_1\left(\nu u_2 - u_1\right), \quad \dot{u}_2 = e^{\mu u_2} F_2\left(\nu u_2 - u_1\right);$$

$$X_4: \quad \dot{u}_1 = u_1\left(\lambda \ln|u_1| + F_1\left(u_2 u_1^{-\nu}\right)\right), \quad \dot{u}_2 = u_2\left(\lambda \ln|u_2| + F_2\left(u_2 u_1^{-\nu}\right)\right);$$

$$X_5: \quad \dot{u}_1 = \lambda u_1 + F_1\left(u_2 e^{-u_1}\right), \quad \dot{u}_2 = \lambda u_1 u_2 + u_2 F_2\left(u_2 e^{-u_1}\right);$$

$$X_6: \quad \dot{u}_1 = \lambda u_1 + F_1\left(\nu u_2 - u_1\right), \quad \dot{u}_2 = \lambda u_2 + F_2\left(\nu u_2 - u_1\right).$$

Theorem 4. *Inequivalent systems of the form (1) invariant with respect to the three-dimensional Lie algebras R_n, are presented by the following list:*

$$
\begin{gathered}
R_2: \quad \dot{u}_1 = C_1 u_1^{1+\mu} u_2^{\nu}, \quad \dot{u}_2 = C_2 u_1^{\mu} u_2^{1+\nu};\\
R_3: \quad \dot{u}_1 = u_1\left(\lambda \ln u_1 + \nu \ln u_2 + C_1\right), \quad \dot{u}_2 = u_2\left(\sigma \ln u_1 + \gamma \ln u_2 + C_2\right);\\
R_7: \quad \dot{u}_1 = u_1\left(\lambda \ln u_1 + \nu u_2 + C_1\right), \quad \dot{u}_2 = \sigma \ln u_1 + \gamma u_2 + C_2;\\
R_9: \quad if \quad \mu \neq 0: \quad \dot{u}_1 = \tfrac{\nu}{\mu} + C_1 e^{\frac{\mu}{2}u_1^2 - \mu u_2},\\
\dot{u}_2 = \tfrac{\nu}{\mu} u_1 + (C_1 u_1 + C_2) e^{\frac{\mu}{2}u_1^2 - \mu u_2};\\
if \quad \mu = 0: \quad \dot{u}_1 = \nu(2u_2 - u_1^2) + C_1,\\
\dot{u}_2 = 2\nu u_1(2u_2 - u_1^2) + u_1 + C_2;\\
R_{11}: \quad \dot{u}_1 = -\tfrac{\sigma}{2}u_1^2 + \gamma u_1 + \sigma u_2 + C_1,\\
\dot{u}_2 = -\tfrac{\sigma}{2}u_1^3 + \left(\gamma - \tfrac{\lambda}{2}\right)u_1^2 + (\nu + C_1 + \sigma u_2)u_1 + \lambda u_2 + C_2;\\
R_{13}, \quad \lambda = 0:\\
\dot{u}_1 = \tfrac{\nu}{2}(\mu u_1 - u_2)\ln\left(u_1^2 + u_2^2\right) + C_1 u_1 - \mu\nu(\mu u_1 - u_2)\arctan\tfrac{u_2}{u_1} + (C_2 + \nu)u_2,\\
\dot{u}_2 = \tfrac{\nu}{2}(\mu u_2 + u_1)\ln\left(u_1^2 + u_2^2\right) + C_1 u_2 - \mu\nu(\mu u_2 + u_1)\arctan\tfrac{u_2}{u_1} - (C_2 + \nu)u_1;\\
R_{14}, \quad \lambda = 0:\\
\dot{u}_1 = \nu u_1 \arctan\tfrac{u_2}{u_1} + C_1 u_1 + C_2 u_2, \quad \dot{u}_2 = \nu u_2 \arctan\tfrac{u_2}{u_1} + C_1 u_2 - C_2 u_1;\\
R_{17}, \quad \mu = \nu = 0: \qquad (10)\\
\dot{u}_1 = C_1\sqrt{u_1^2 - 2u_2}, \quad \dot{u}_2 = C_1 u_1\sqrt{u_1^2 - 2u_2} + C_2(u_1^2 - 2u_2);\\
R_{19}: \quad \dot{u}_1 = C_1 u_1^{\mu-\nu+1} e^{\nu\frac{u_2}{u_1}}, \quad \dot{u}_2 = \left(C_1 u_1^{\mu-\nu} u_2 + C_2 u_1^{\mu-\nu+1}\right) e^{\nu\frac{u_2}{u_1}};\\
R_{20}: \quad \dot{u}_1 = u_1\left((\lambda - \nu)\ln u_1 + C_1\right) + \nu u_2,\\
\dot{u}_2 = u_1\left((\lambda - \nu - \gamma + \sigma)\ln u_1 + C_2\right) + u_2\left(\nu + \gamma + (\lambda - \nu)\ln u_1 + C_1 + \nu\tfrac{u_2}{u_1}\right);\\
R_{21}: \quad \dot{u}_1 = u_1 g_1 + u_2 g_2, \quad \dot{u}_2 = u_2 g_1 - u_1 g_2,\\
where \quad g_1 = \tfrac{\mu\sigma + \lambda}{2}\ln\left(u_1^2 + u_2^2\right) + \left(\nu + \mu(\gamma - \lambda) - \sigma\mu^2\right)\arctan\tfrac{u_2}{u_1} + C_1,\\
g_2 = -\tfrac{\sigma}{2}\ln\left(u_1^2 + u_2^2\right) + (\mu\sigma - \gamma)\arctan\tfrac{u_2}{u_1} + (C_2 + \sigma);\\
R_{22}: \quad \dot{u}_1 = C_1 u_1^{1+\mu} e^{\nu u_2}, \quad \dot{u}_2 = C_2 u_1^{\mu} e^{\nu u_2};\\
R_{26}: \quad \dot{u}_1 = C_1 e^{\frac{u_1}{2}(2\nu - \mu u_1) + \mu u_2}, \quad \dot{u}_2 = (C_1 u_1 + C_2) e^{\frac{u_1}{2}(2\nu - \mu u_1) + \mu u_2};\\
R_{38}, \quad \mu \neq 0: \quad \dot{u}_1 = \tfrac{\nu}{\mu} u_1 + C_1 u_1^{\mu+1} e^{-\frac{\mu u_2}{u_1}},
\end{gathered}
$$

$$\dot{u}_2 = \tfrac{\nu}{\mu}(u_1 + u_2) + (C_1u_2 + C_2u_1)\,u_1^{\mu}e^{-\frac{\mu u_2}{u_1}};$$
$$\mu = 0: \quad \dot{u}_1 = \nu u_2 + u_1(C_1 - \nu\ln|u_1|),$$
$$\dot{u}_2 = C_2u_1 - \nu(u_1 + u_2)\ln|u_1| + (\nu + C_1 + \tfrac{\nu u_2}{u_1})u_2.$$

4 Equivalence Transformations

We have found a complete list of systems (1) invariant with respect to two- and three-dimensional Lie algebras. Some of these equations can be simplified using the equivalence transformations that preserve the differential structure of this class of systems. The continuous equivalence transformations are generated by the infinitesimal operators of the following general form

$$Q = \varphi(t)\partial_t + (\alpha_{ab}(t)u_b + \beta_a(t))\partial_{u_a}.$$

Operators Q generates an equivalence transformation for (1) iff the commutator of its first prolongation $Q^{(1)}$ with the first prolongations of found symmetry operators Y_s, $s = \overline{1,l}$ is a linear combination of these symmetry operators with functional coefficients,

$$[Q^{(1)}, Y_s] = \gamma_{ss'}(t, u_1, u_2)Y_{s'}, \quad s = \overline{1,l}. \tag{11}$$

As an example, consider the following transformation for the system invariant with respect to the realization R_{14} (10):

$$\dot{u}_1 = \nu u_1 \arctan\frac{u_2}{u_1} + C_1u_1 + C_2u_2, \quad \dot{u}_2 = \nu u_2 \arctan\frac{u_2}{u_1} + C_1u_2 - C_2u_1. \tag{12}$$

To exclude linear systems we suppose that $\nu \neq 0$. The system (12) admits the three-dimensional Lie algebra with infinitesimal operators

$$X_0 = \partial_t, \quad X_1 = u_1\partial_{u_1} + u_2\partial_{u_2}, \quad X_2 = \nu t X_1 - u_2\partial_{u_1} + u_1\partial_{u_2}. \tag{13}$$

Their first prolongations are

$$Y_0 = \partial_t, \quad Y_1 = X_1 + \dot{u}_1\partial_{\dot{u}_1} + \dot{u}_2\partial_{\dot{u}_2},$$
$$Y_2 = X_2 + (\nu u_1 + \nu t\dot{u}_1 - \dot{u}_2)\partial_{\dot{u}_1} + (\nu u_2 + \nu t\dot{u}_2 + \dot{u}_1)\partial_{\dot{u}_2}.$$

The first prolongation of the infinitesimal operator Q takes the form

$$Q^{(1)} = Q + (\dot{\alpha}_{ab}(t)u_b + \alpha_{ab}(t)\dot{u}_b + \dot{\beta}_a(t) - \dot{u}_a\dot{\varphi}(t))\partial_{\dot{u}_a}.$$

The equations obtained from (11) for $s = 1$ lead to the conditions

$$\alpha_{11} = \kappa_{11} + \gamma_2 t + \tfrac{\nu\gamma_3}{2} t^2, \quad \alpha_{12} = \kappa_{12} - \gamma_3 t, \quad \alpha_{21} = \kappa_{21} + \gamma_3 t,$$
$$\alpha_{22} = \kappa_{22} + \gamma_2 t + \tfrac{\nu\gamma_3}{2} t^2, \quad \varphi = \kappa_0 + \gamma_1 t, \quad \gamma_a = \text{const}, \quad \beta_a = \text{const}.$$

Equation (11) for $s = 2, 3$ result in the conditions

$$\beta_a = 0, \quad \kappa_{11} = \kappa_{22}, \quad \kappa_{12} = -\kappa_{21}.$$

Thus, Q is a linear combination of the infinitesimal operators

$$\partial_t, \quad u_1\partial_{u_1} + u_2\partial_{u_2}, \quad t(u_1\partial_{u_1} + u_2\partial_{u_2}) - u_2\partial_{u_1} + u_1\partial_{u_2},$$
$$t\partial_t, \quad -u_2\partial_{u_1} + u_1\partial_{u_2}, \quad \tfrac{\nu t^2}{2}(u_1\partial_{u_1} + u_2\partial_{u_2}) + t(-u_2\partial_{u_1} + u_1\partial_{u_2}).$$

The first three operators are Lie symmetry operators of the system (12). Therefore, the groups of transformations which correspond to these infinitesimal operators leave the system invariant. The last infinitesimal operator is not a Lie symmetry, but it generates the equivalence transformation. The operator $t\partial_t$ generates scaling transformations of the variable t. This group can be expanded by adding the discrete transformation $t \to -t$. Then system (12) takes the form

$$\dot{u}_1 = u_1 \arctan\frac{u_2}{u_1} + \tilde{C}_1 u_1 + \tilde{C}_2 u_2, \quad \dot{u}_2 = u_2 \arctan\frac{u_2}{u_1} + \tilde{C}_1 u_2 - \tilde{C}_2 u_1, \quad (14)$$

where $\tilde{C}_a = \nu^{-1} C_a$.

The operator $-u_2\partial_{u_1} + u_1\partial_{u_2}$ corresponds to the group of rotations of the dependent variables. The rotation $u_1 \to u_1 \cos\tilde{C}_1 + u_2 \sin\tilde{C}_1$, $u_2 \to u_2 \cos\tilde{C}_1 - u_1 \sin\tilde{C}_1$ maps the system (14) into the same system with $\tilde{C}_1 = 0$.

Consider now the operator $\frac{\nu t^2}{2}(u_1\partial_{u_1} + u_2\partial_{u_2}) + t(-u_2\partial_{u_1} + u_1\partial_{u_2})$. It corresponds to the one-parameter group of transformations of the dependent variables: $u_1 \to e^{\frac{\varepsilon t^2}{2}}(u_1 \cos\varepsilon t - u_2 \sin\varepsilon t)$, $u_2 \to e^{\frac{\varepsilon t^2}{2}}(u_2 \cos\varepsilon t + u_1 \sin\varepsilon t)$. The transformations of the arbitrary elements ν and C_a of the class (12) are given by the formulas $\nu \to \nu$, $C_1 \to C_1$, and $C_2 \to C_2 - \varepsilon$. If we select $\varepsilon = \tilde{C}_2$, then the system of Eq. (14) reduces to the form

$$\dot{u}_1 = u_1 \arctan\frac{u_2}{u_1}, \quad \dot{u}_2 = u_2 \arctan\frac{u_2}{u_1}. \quad (15)$$

Thus, up to the obtained equivalence transformations it is sufficient to consider (15) instead of (12). In the same way other systems listed in Sect. 3 can be reduced to simpler forms, but we prefer to present more general expressions (10).

5 Integration of Systems that Admit Group Transformations

If a system of ODEs admits a three-dimensional Lie symmetry algebra, it can be integrated by quadratures using the standard Lie algorithm. Systems that admit two-dimensional Lie symmetry algebras can be reduced to autonomous systems, which also are integrated in quadratures. The procedure of integration of ODEs that admit Lie symmetry algebra is known and described in the monographes [2, 6, 10]. We illustrate the procedure by the following example.

Consider the system (12) that admits Lie symmetry algebra (13). We introduce new variables $\tilde{u}_1 = \tilde{u}_1(u_1, u_2)$ and $\tilde{u}_2 = \tilde{u}_2(u_1, u_2)$ such that the operators X_1 and X_2 are transformed to the shift operators. Such variables are solutions of the following system of equations:

$$X_1\tilde{u}_1 = 1, \quad X_1\tilde{u}_2 = 0, \quad X_2\tilde{u}_1 = 0, \quad X_2\tilde{u}_2 = 1,$$

which are $\tilde{u}_1 = \frac{1}{2}\ln(u_1^2 + u_2^2) - \nu t \arctan\frac{u_2}{u_1}$, and $\tilde{u}_2 = \arctan\frac{u_2}{u_1}$. Then the system of Eq. (12) and the infinitesimal operators (13) take the form:

$$\dot{\tilde{u}}_1 = C_1 + C_2\nu t, \quad \dot{\tilde{u}}_2 = -C_2, \quad \tilde{X}_1 = \partial_{\tilde{u}_2}, \quad \tilde{X}_2 = \partial_{\tilde{u}_1}.$$

The obtained system of ODEs can be easily integrated:

$$\tilde{u}_1 = C_1 t + \frac{C_2\nu}{2}t^2 + C_3, \quad \tilde{u}_2 = -C_2 t + C_4.$$

Returning to the functions u_1 and u_2 we obtain the solution of (12):

$$u_1 = e^{-\frac{C_2\nu}{2}t^2 + (C_4\nu + C_1)t + C_3}\cos(C_2 t - C_3),$$
$$u_2 = -e^{-\frac{C_2\nu}{2}t^2 + (C_4\nu + C_1)t + C_3}\sin(C_2 t - C_3).$$

All systems from the list (10) can be integrated in an analogous way.

Conclusion

We carry out the complete group classification of systems of first-order ODEs of the form (1) with respect to Lie groups of transformations that are linear in the dependent variables u_1 and u_2. The found equations admit two- or three-dimensional algebras of symmetries. The importance of found lists is that they present all inequivalent systems (1) integrable by quadratures using their symmetries. An algorithm for constructing such solutions is well known and is implemented in computer algebra systems, e.g., Maple and Mathematica.

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Part IV
Supersymmetry and Quantum Groups

On Principal Finite W-Algebras for Certain Orthosymplectic Lie Superalgebras and $F(4)$

Elena Poletaeva

Abstract We study finite W-algebras associated to even regular (principal) nilpotent elements for basic classical Lie superalgebras. We describe the principal finite W-algebras for Lie superalgebras $\mathfrak{osp}(1|2)$, $\mathfrak{osp}(1|4)$, $\mathfrak{osp}(2|2)$, $\mathfrak{osp}(3|2)$, and obtain partial results for the exceptional classical Lie superalgebra $F(4)$.

1 Introduction

A finite W-algebra is a certain associative algebra attached to a pair $(\mathfrak{g}, e)$ where $\mathfrak{g}$ a complex semi-simple Lie algebra and $e \in \mathfrak{g}$ is a nilpotent element.

Finite W-algebras for semi-simple Lie algebras were introduced by A. Premet [9] (see also [5]). In the case of Lie superalgebras, finite W-algebras were studied by mathematicians and physicists in the following works [1, 2, 10, 11]. The principal finite W-algebras for $\mathfrak{gl}(m|n)$ associated to regular (principal) nilpotent elements were described as certain truncations of a shifted version of the super-Yangian of $\mathfrak{gl}(1|1)$ in [2].

In [6, 7] we obtained the precise description of the principal finite W-algebras for classical Lie superalgebras of Type I and defect one, and for the exceptional Lie superalgebra $D(2,1;\alpha)$. In [8] we studied in detail the case when $\mathfrak{g} = Q(n)$. In particular, we proved that the principal finite W-algebra for $Q(n)$ is isomorphic to a quotient of the super-Yangian of $Q(1)$.

In this paper we describe the principal finite W-algebras for certain orthosymplectic Lie superalgebras: $\mathfrak{osp}(1|2)$, $\mathfrak{osp}(1|4)$, $\mathfrak{osp}(2|2)$ and $\mathfrak{osp}(3|2)$. We also obtain partial results for the exceptional classical Lie superalgebra $F(4)$. This is a joint work with V. Serganova.

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2 Preliminaries

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a basic classical Lie superalgebra, i.e. $\mathfrak{g}$ is simple, $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra, and $\mathfrak{g}$ has an even non-degenerate invariant supersymmetric bilinear form $(\cdot|\cdot)$. Let $e \in \mathfrak{g}_{\bar{0}}$ be an even nilpotent element. By the Jacobson-Morozov theorem, a nonzero e can be included in $\mathfrak{sl}(2) =< e, h, f >$. As in the Lie algebra case, the linear operator ad$h$ defines a Dynkin $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$, where

$$\mathfrak{g}_j = \{x \in \mathfrak{g} \mid \mathrm{ad}h(x) = jx\}.$$

Let $\mathfrak{g}^e = \mathrm{Ker}(\mathrm{ad}e)$. Note that as in the Lie algebra case, $\dim \mathfrak{g}^e = \dim \mathfrak{g}_0 + \dim \mathfrak{g}_1$, and $\mathfrak{g}^e \subseteq \bigoplus_{j \geq 0} \mathfrak{g}_j$. Let $\chi \in \mathfrak{g}_{\bar{0}}^* \subset \mathfrak{g}^*$ be defined by $\chi(x) = (x|e)$ for all $x \in \mathfrak{g}$. Note that $\chi([X, Y])$ defines a non-degenerate skew-symmetric even bilinear form on $\mathfrak{g}_{-1}$. Let $\mathfrak{l}$ be a maximal isotropic subspace with respect to this form. We consider a nilpotent subalgebra $\mathfrak{m} = (\bigoplus_{j \leq -2} \mathfrak{g}_j) \bigoplus \mathfrak{l}$ of $\mathfrak{g}$. The restriction of χ to $\mathfrak{m}$, $\chi : \mathfrak{m} \longrightarrow \mathbb{C}$, defines a one-dimensional representation $\mathbb{C}_\chi =< v >$ of $\mathfrak{m}$. Let I_χ be the left ideal of $U(\mathfrak{g})$ generated by $a - \chi(a)$ for all $a \in \mathfrak{m}$.

Definition 1. The induced $\mathfrak{g}$-module

$$Q_\chi := U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi \cong U(\mathfrak{g})/I_\chi$$

is called *the generalized Whittaker module*.

Definition 2. *The finite W-algebra* associated to the nilpotent element e is

$$W_\chi := \mathrm{End}_{U(\mathfrak{g})}(Q_\chi)^{op}.$$

Note that by Frobenius reciprocity

$$\mathrm{End}_{U(\mathfrak{g})}(Q_\chi) = \mathrm{Hom}_{U(\mathfrak{m})}(\mathbb{C}_\chi, Q_\chi).$$

That defines an identification of W_χ with the subspace

$$Q_\chi^{\mathfrak{m}} = \{u \in Q_\chi \mid au = \chi(a)u \text{ for all } a \in \mathfrak{m}\}. \quad (1)$$

In what follows we denote by $\pi : U(\mathfrak{g}) \to U(\mathfrak{g})/I_\chi$ the natural projection. By above

$$W_\chi = \{\pi(y) \in U(\mathfrak{g})/I_\chi \mid (a - \chi(a))y \in I_\chi \text{ for all } a \in \mathfrak{m}\}, \quad (2)$$

or, equivalently,

$$W_\chi = \{\pi(y) \in U(\mathfrak{g})/I_\chi \mid \mathrm{ad}(a)y \in I_\chi \text{ for all } a \in \mathfrak{m}\}.$$

The algebra structure on W_χ is given by

$$\pi(y_1)\pi(y_2) = \pi(y_1 y_2)$$

for $y_i \in U(\mathfrak{g})$ such that $\mathrm{ad}(a)y_i \in I_\chi$ for all $a \in \mathfrak{m}$ and $i = 1, 2$.

Definition 3. A nilpotent element $e \in \mathfrak{g}_{\bar{0}}$ is called *regular* nilpotent, if $\mathfrak{g}_{\bar{0}}^e$ attains the minimal dimension, which is equal to $\mathrm{rank}(\mathfrak{g}_{\bar{0}})$.

Theorem 1 (B. Kostant, [4]). *For a reductive Lie algebra $\mathfrak{g}$ and a regular nilpotent element $e \in \mathfrak{g}$, the finite W-algebra W_χ is isomorphic to the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$.*

This theorem does not hold for Lie superalgebras, since W_χ must have a non-trivial odd part, and the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ is even.

Let $\mathfrak{l}'$ be some subspace in $\mathfrak{g}_{-1}$ satisfying the following two properties:

- $\mathfrak{g}_{-1} = \mathfrak{l} \oplus \mathfrak{l}'$,
- $\mathfrak{l}'$ contains a maximal isotropic subspace with respect to the form defined by $\chi([\cdot,\cdot])$ on $\mathfrak{g}_{-1}$.

If $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is even, then $\mathfrak{l}'$ is a maximal isotropic subspace. If $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is odd, then $\mathfrak{l}^\perp \cap \mathfrak{l}'$ is one-dimensional and we fix $\theta \in \mathfrak{l}^\perp \cap \mathfrak{l}'$ such that $\chi([\theta, \theta]) = 2$. It is clear that $\pi(\theta) \in W_\chi$ and $\pi(\theta)^2 = 1$.

Definition 4. Define a $\mathbb{Z}$-grading on $T(\mathfrak{g})$ by setting the degree of $g \in \mathfrak{g}_j$ to be $j + 2$. This induces a filtration on $U(\mathfrak{g})$ and therefore on $U(\mathfrak{g})/I_\chi$ which is called the *Kazhdan filtration*. We will denote by Gr_K the corresponding graded algebras. Since by (2) $W_\chi \subset U(\mathfrak{g})/I_\chi$, we have an induced filtration on W_χ.

Let $\mathfrak{p} = \bigoplus_{j \geq 0} \mathfrak{g}_j$. By the PBW theorem, $U(\mathfrak{g})/I_\chi \simeq S(\mathfrak{p} \oplus \mathfrak{l}')$ as a vector space. Therefore $\mathrm{Gr}_K(U(\mathfrak{g})/I_\chi)$ is isomorphic to $S(\mathfrak{p} \oplus \mathfrak{l}')$ as a vector space. The Dynkin $\mathbb{Z}$-grading of $\mathfrak{g}$ induces the grading on $S(\mathfrak{p} \oplus \mathfrak{l}')$. For any $X \in U(\mathfrak{g})/I_\chi$ let $Gr_K(X)$ denote the corresponding element in $\mathrm{Gr}_K(U(\mathfrak{g})/I_\chi)$, and $P(X)$ denote the highest weight component of $Gr_K(X)$ in the Dynkin $\mathbb{Z}$-grading. We denote by $\deg P(X)$ the Kazhdan degree of $Gr_K(X)$ and by $\mathrm{wt} P(X)$ the weight of the highest weight component of $Gr_K(X)$.

Theorem 2 ([8], Theorem 2.5). *Let $X \in W_\chi$. If $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is even, then $P(X) \in S(\mathfrak{g}^e)$, and if $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is odd, then $P(X) \in S(\mathfrak{g}^e \oplus \mathbb{C}\theta)$.*

Theorem 3 (A. Premet, [9]). *Let $\mathfrak{g}$ be a semi-simple Lie algebra. Then the associated graded algebra $Gr_K W_\chi$ is isomorphic to $S(\mathfrak{g}^e)$.*

Theorem 4 ([8], Proposition 2.7). *Let $y_1, \ldots, y_p$ be a basis in $\mathfrak{g}^e$ homogeneous in the good $\mathbb{Z}$-grading. Assume that there exist $Y_1, \ldots, Y_p \in W_\chi$ such that $P(Y_i) = y_i$ for all $i = 1, \ldots, p$. Then*

(a) *if* $\dim(\mathfrak{g}_{-1})_{\bar 1}$ *is even, then* $Y_1, \dots, Y_p$ *generate* W_χ*, and if* $\dim(\mathfrak{g}_{-1})_{\bar 1}$ *is odd, then* $Y_1, \dots, Y_p$ *and* $\pi(\theta)$ *generate* W_χ*;*
(b) *if* $\dim(\mathfrak{g}_{-1})_{\bar 1}$ *is even, then* $Gr_K W_\chi \simeq S(\mathfrak{g}^e)$*, and if* $\dim(\mathfrak{g}_{-1})_{\bar 1}$ *is odd, then* $Gr_K W_\chi \simeq S(\mathfrak{g}^e) \otimes \mathbb{C}[\xi]$*, where* $\mathbb{C}[\xi]$ *is the exterior algebra generated by one element* ξ*.*

3 Principal Finite *W*-Algebras for Orthosymplectic Lie Superalgebras

Recall that $\mathfrak{g} = \mathfrak{osp}(m|2n) \subset \mathfrak{gl}(m|2n)$ is the Lie superalgebra which preserves a non-degenerated supersymmetric even bilinear form on a superspace V with $\dim V = (m|2n)$. We will study the case when $e \in \mathfrak{g}_{\bar 0}$ is a regular nilpotent element. Recall that $\mathrm{def}(\mathfrak{osp}(2m+1|2n)) = \mathrm{def}(\mathfrak{osp}(2m|2n)) = \min(m,n)$, where def stands for *defect*. We observed in [7] that

$$\dim(\mathfrak{g}^e)_{\bar 1} = 2\mathrm{def}\mathfrak{g}, \text{ if } \mathfrak{g} = \mathfrak{osp}(2m+1|2n),\ m \geq n,\ \text{or } \mathfrak{osp}(2m|2n),\ m \leq n,$$
$$\dim(\mathfrak{g}^e)_{\bar 1} = 2\mathrm{def}\mathfrak{g} + 1,\ \text{if } \mathfrak{g} = \mathfrak{osp}(2m+1|2n),\ m < n,\ \text{or } \mathfrak{osp}(2m|2n),\ m > n.$$

3.1 *The Case of* $\mathfrak{osp}(1|2)$

Let $\mathfrak{g} = \mathfrak{osp}(1|2) =< X, Y, H \mid s, r >$, where

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$s = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad r = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Consider the even non-degenerate invariant supersymmetric bilinear form $(a|b) = \frac{1}{2}str(ab)$ on $\mathfrak{g}$: $(s|r) = 1, \quad (X|Y) = -\frac{1}{2}, \quad (H|H) = -1.$

Let $\mathfrak{sl}(2) =< e, h, f >$, where $e = X, h = H, f = Y$. Note that $\mathfrak{g}^e =< X \mid r >$. The element h defines a $\mathbb{Z}$-grading on $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2, \text{ where}$$
$$\mathfrak{g}_{-2} =< Y >, \quad \mathfrak{g}_{-1} =< s >, \quad \mathfrak{g}_0 =< H >, \quad \mathfrak{g}_1 =< r >, \quad \mathfrak{g}_2 =< X > .$$

Note that $\mathfrak{m} = \mathfrak{g}_{-2}$, and $\chi(Y) = -\frac{1}{2}$. Let $\theta = s$. Note that $\pi(\theta) \in W_\chi$, and $\pi(\theta)^2 = \frac{1}{2}$.

Let Ω be the Casimir element of $\mathfrak{g}$. Then

$$\pi(\Omega) = \pi(2X + H - H^2 + 2r\theta).$$

Let

$$R = \pi(r - H\theta).$$

Note that $\pi(\Omega)$ and R belong to W_χ, and $P(\pi(\Omega)) = 2X$, $P(R) = r$.

Proposition 1. *The principal finite W-algebra W_χ is generated by $\pi(\Omega)$ and two odd elements $\pi(\theta)$ and R. The defining relations are*

$$\begin{aligned}&[\pi(\Omega), R] = [\pi(\Omega), \pi(\theta)] = 0,\\ &[R, R] = \pi(\Omega), \quad [R, \pi(\theta)] = -\tfrac{1}{2}, \quad [\pi(\theta), \pi(\theta)] = 1.\end{aligned}$$

3.2 The Case of $\mathfrak{osp}(1|4)$

Let $\mathfrak{g} = \mathfrak{osp}(1|4)$, where

$$X = \left(\begin{array}{c|cc|cc} 0&0&0&0&0\\ \hline 0&0&1&0&0\\ 0&0&0&0&0\\ \hline 0&0&0&0&0\\ 0&0&0&-1&0 \end{array}\right), \quad Y = \left(\begin{array}{c|cc|cc} 0&0&0&0&0\\ \hline 0&0&0&0&0\\ 0&1&0&0&0\\ \hline 0&0&0&0&-1\\ 0&0&0&0&0 \end{array}\right),$$

$$H_1 = \left(\begin{array}{c|cc|cc} 0&0&0&0&0\\ \hline 0&1&0&0&0\\ 0&0&0&0&0\\ \hline 0&0&0&-1&0\\ 0&0&0&0&0 \end{array}\right), \quad H_2 = \left(\begin{array}{c|cc|cc} 0&0&0&0&0\\ \hline 0&0&0&0&0\\ 0&0&1&0&0\\ \hline 0&0&0&0&0\\ 0&0&0&0&-1 \end{array}\right),$$

$$P_1 = \left(\begin{array}{c|cc|cc} 0&0&0&0&0\\ \hline 0&0&0&1&0\\ 0&0&0&0&0\\ \hline 0&0&0&0&0\\ 0&0&0&0&0 \end{array}\right), \quad P_2 = \left(\begin{array}{c|cc|cc} 0&0&0&0&0\\ \hline 0&0&0&0&0\\ 0&0&0&0&1\\ \hline 0&0&0&0&0\\ 0&0&0&0&0 \end{array}\right), \quad P_3 = \left(\begin{array}{c|cc|cc} 0&0&0&0&0\\ \hline 0&0&0&0&1\\ 0&0&0&1&0\\ \hline 0&0&0&0&0\\ 0&0&0&0&0 \end{array}\right),$$

$$Q_1 = \left(\begin{array}{c|cc|cc} 0&0&0&0&0\\ \hline 0&0&0&0&0\\ 0&0&0&0&0\\ \hline 0&1&0&0&0\\ 0&0&0&0&0 \end{array}\right), \quad Q_2 = \left(\begin{array}{c|cc|cc} 0&0&0&0&0\\ \hline 0&0&0&0&0\\ 0&0&0&0&0\\ \hline 0&0&0&0&0\\ 0&0&1&0&0 \end{array}\right), \quad Q_3 = \left(\begin{array}{c|cc|cc} 0&0&0&0&0\\ \hline 0&0&0&0&0\\ 0&0&0&0&0\\ \hline 0&0&1&0&0\\ 0&1&0&0&0 \end{array}\right),$$

$$s_1 = \left(\begin{array}{c|cc|cc} 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad s_2 = \left(\begin{array}{c|cc|cc} 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$r_1 = \left(\begin{array}{c|cc|cc} 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad r_2 = \left(\begin{array}{c|cc|cc} 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Consider the even non-degenerate invariant supersymmetric bilinear form $(a|b) = -str(ab)$ on $\mathfrak{g}$:

$$(P_1|Q_1) = (P_2|Q_2) = 1, (P_3|Q_3) = 2, (X|Y) = 2, (H_1|H_1) = (H_2|H_2) = 2,$$
$$(r_1|s_1) = (r_2|s_2) = 2.$$

Let $\mathfrak{sl}(2) =< e, h, f >$, where $e = X + P_2, f = 3Y + 4Q_2$, $h = \text{diag}(0|3, 1, -3, -1)$. Note that $\mathfrak{g}^e =< X + P_2, P_1|r_1 >$. The element h defines a $\mathbb{Z}$-grading on $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_{-6} \oplus \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \oplus \mathfrak{g}_4 \oplus \mathfrak{g}_6, \text{ where}$$
$$\mathfrak{g}_{-6} =< Q_1 >, \quad \mathfrak{g}_{-4} =< Q_3 >, \quad \mathfrak{g}_{-3} =< s_1 >, \quad \mathfrak{g}_{-2} =< Y, Q_2 >,$$
$$\mathfrak{g}_{-1} =< s_2 >, \quad \mathfrak{g}_0 =< H_1, H_2 >, \quad \mathfrak{g}_1 =< r_2 >, \quad \mathfrak{g}_2 =< X, P_2 >,$$
$$\mathfrak{g}_3 =< r_1 >, \quad \mathfrak{g}_4 =< P_3 >, \quad \mathfrak{g}_6 =< P_1 > .$$

Note that $\mathfrak{m} = \mathfrak{g}_{-6} \oplus \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2}$, and $\chi(Y) = 2$, $\chi(Q_2) = 1$.

Let $\theta = s_2$. Note that $\pi(\theta) \in W_\chi$, and $\pi(\theta)^2 = -1$. Let Ω be the Casimir element of $\mathfrak{g}$. Then

$$\pi(\Omega) = \pi(2X + 2P_2 + \frac{1}{2}(H_1^2 + H_2^2 - 3H_1 - H_2) - r_2 s_2).$$

Let

$$R = \pi(r_1 - \frac{1}{2}H_1 r_2 + \frac{1}{2}r_2 - \frac{1}{2}P_2 s_2 + \frac{1}{2}\sum_{i=1}^{2} H_i s_2 - \frac{1}{8}\sum_{i,j=1}^{2} H_i H_j s_2).$$

Note that $\pi(\Omega)$ and R belong to W_χ and

$$P(\pi(\Omega)) = 2X + 2P_2, \quad P(R) = r_1.$$

Proposition 2. *The principal finite W-algebra W_χ is generated by even elements $\pi(\Omega)$ and C, where $P(C) = P_1$, and odd elements $\pi(\theta)$ and R, which satisfy the following relations:*

$$[\pi(\theta), R] = \frac{1}{2}\pi(\Omega) - \frac{3}{8}, \quad [\pi(\theta), \pi(\theta)] = -2.$$

Conjecture 1 ([6]). The principal finite W-algebra W_χ for $\mathfrak{osp}(1|2n)$ is generated by the first n Casimir elements of $\mathfrak{g}$ and odd elements $\pi(\theta)$ and R, so that:

$$[R, R] \in Z(\mathfrak{g}), \quad [R, \pi(\theta)] \in Z(\mathfrak{g}), \quad [\pi(\theta), \pi(\theta)] = -2.$$

3.3 The Case of $\mathfrak{osp}(2|2)$

Let $\mathfrak{g} = \mathfrak{osp}(2|2)$, where

$$X = \left(\begin{array}{cc|cc} 0&0&0&0\\ 0&0&0&0\\ \hline 0&0&0&1\\ 0&0&0&0 \end{array}\right), Y = \left(\begin{array}{cc|cc} 0&0&0&0\\ 0&0&0&0\\ \hline 0&0&0&0\\ 0&0&1&0 \end{array}\right), H = \left(\begin{array}{cc|cc} 0&0&0&0\\ 0&0&0&0\\ \hline 0&0&1&0\\ 0&0&0&-1 \end{array}\right), H_1 = \left(\begin{array}{cc|cc} 1&0&0&0\\ 0&-1&0&0\\ \hline 0&0&0&0\\ 0&0&0&0 \end{array}\right),$$

$$s_1 = \left(\begin{array}{cc|cc} 0&0&1&0\\ 0&0&0&0\\ \hline 0&0&0&0\\ 0&-1&0&0 \end{array}\right), s_2 = \left(\begin{array}{cc|cc} 0&0&0&0\\ 0&0&1&0\\ \hline 0&0&0&0\\ -1&0&0&0 \end{array}\right), r_1 = \left(\begin{array}{cc|cc} 0&0&0&1\\ 0&0&0&0\\ \hline 0&1&0&0\\ 0&0&0&0 \end{array}\right), r_2 = \left(\begin{array}{cc|cc} 0&0&0&0\\ 0&0&0&1\\ \hline 1&0&0&0\\ 0&0&0&0 \end{array}\right).$$

Note that $\mathfrak{g}$ is of type I, i.e. $\mathfrak{g}_{\bar{1}}$ is a direct sum of two simple $\mathfrak{g}_{\bar{0}}$-submodules. Then $\mathfrak{g}$ admits a $\mathbb{Z}$-grading:

$$\mathfrak{g} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{1}, \quad \text{where}$$
$$\mathfrak{g}^0 =< X, Y, H, H_1 >, \quad \mathfrak{g}^{-1} =< r_1, s_1 >, \quad \mathfrak{g}^1 =< r_2, s_2 > .$$

Consider the even non-degenerate invariant supersymmetric bilinear form $(a|b) = -str(ab)$ on $\mathfrak{g}$:

$$(X|Y) = 1, \quad (H|H) = 2, \quad (H_1|H_1) = -2,$$
$$(r_1|s_2) = 2, \quad (r_2|s_1) = 2.$$

Let $\mathfrak{sl}(2) =< e, h, f >$, where $e = X, h = H, f = Y$. Note that $\mathfrak{g}^e = < X, H_1 \mid r_1, r_2 >$. The element h defines a $\mathbb{Z}$-grading on $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2, \text{ where}$$

$$\mathfrak{g}_{-2} =< Y >, \quad \mathfrak{g}_{-1} =< s_1, s_2 >, \quad \mathfrak{g}_0 =< H_1, H >,$$
$$\mathfrak{g}_1 =< r_1, r_2 >, \quad \mathfrak{g}_2 =< X > .$$

Note that $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{l}$, where $\mathfrak{l} =< s_1 >$, and $\chi(Y) = 1$.

Let Ω be the Casimir element of $\mathfrak{g}$. W_χ has even generators $C, \pi(\Omega)$ and odd generators R_1, R_2:

$$C = \pi(H_1), \quad \pi(\Omega) = \pi(2X + \tfrac{1}{2}H^2 - \tfrac{1}{2}H_1^2 - r_1 s_2),$$
$$R_1 = \pi(r_1), \quad R_2 = \pi(r_2 + \tfrac{1}{2}(H + H_1)s_2),$$

Note that $P(C) = H_1$, $P(\pi(\Omega)) = 2X$, $P(R_1) = r_1$, $P(R_2) = r_2$.

Proposition 3. *The principal finite W-algebra W_χ is generated by even elements $\pi(\Omega)$, C, and odd elements R_1 and R_2. The defining relations are*

$$[\pi(\Omega), C] = [\pi(\Omega), R_i] = 0, \quad i = 1, 2,$$
$$[C, R_1] = R_1, \quad [C, R_2] = -R_2,$$
$$[R_i, R_i] = 0, \quad i = 1, 2, \quad [R_1, R_2] = \pi(\Omega).$$

Remark 1. Note that the superalgebra $\mathfrak{osp}(2|2)$ is of Type I and defect one. The general theorem for such superalgebras was stated in [7] (Theorem 2).

3.4 The Case of $\mathfrak{osp}(3|2)$

Let $\mathfrak{g} = \mathfrak{osp}(3|2)$, where

$$X_1 = \left(\begin{array}{ccc|cc} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad Y_1 = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad H_1 = \left(\begin{array}{ccc|cc} 2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$X_2 = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad Y_2 = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right), \quad H_2 = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right),$$

$$r_1 = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{array}\right), \quad r_2 = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$s_1 = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad s_2 = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{array}\right),$$

$$q_1 = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad q_2 = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{array}\right).$$

Consider the even non-degenerate invariant supersymmetric bilinear form $(a|b) = -\frac{1}{2}str(ab)$ on $\mathfrak{g}$:

$$(X_1|Y_1) = -2, \quad (H_1|H_1) = -4, \quad (X_2|Y_2) = \tfrac{1}{2}, \quad (H_2|H_2) = 1,$$
$$(q_1|q_2) = 1, \quad (r_1, s_1) = -1, \quad (r_2|s_2) = 1.$$

Let $\mathfrak{sl}(2) =< e, h, f >$, where $e = X_1 + X_2, h = H_1 + H_2, f = Y_1 + Y_2$. Note that $\mathfrak{g}^e =< X_1, X_2 \mid r_1 + r_2, q_1 >$. The element h defines a $\mathbb{Z}$-grading of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3, \text{ where}$$

$$\mathfrak{g}_{-3} =< q_2 >, \quad \mathfrak{g}_{-2} =< Y_1, Y_2 >, \quad \mathfrak{g}_{-1} =< s_1, s_2 >, \quad \mathfrak{g}_0 =< H_1, H_2 >,$$
$$\mathfrak{g}_1 =< r_1, r_2 >, \quad \mathfrak{g}_2 =< X_1, X_2 >, \quad \mathfrak{g}_3 =< q_1 > .$$

Note that $\mathfrak{m} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{l}$, where $\mathfrak{l} =< s_1 >$, and $\chi(Y_1) = -2$, $\chi(Y_2) = \frac{1}{2}$.

W_χ has even generators C_1, C_2 and odd generators R_1, R_2:

$$C_1 = \pi(2X_1 - \tfrac{1}{4}H_1^2 - \tfrac{1}{2}H_1 + H_2 - 2r_2 s_2), \quad C_2 = \pi(2X_2 + H_2^2 - 2H_2),$$
$$R_1 = \pi(r_1 + r_2 + (\tfrac{1}{2}H_1 + H_2)s_2),$$
$$R_2 = \pi(q_1 + 2s_2 X_2 + \tfrac{1}{2}H_1 r_2 + H_2 r_1 + (\tfrac{1}{2}H_1 + H_2)H_2 s_2).$$

Note that $P(C_1) = 2X_1$, $P(C_2) = 2X_2$, $P(R_1) = r_1 + r_2$, $P(R_2) = q_1$. Let Ω be the Casimir element of $\mathfrak{g}$. Then

$$\pi(\Omega) = \pi(2X_1 + 2X_2 - \frac{1}{4}H_1^2 + H_2^2 - \frac{1}{2}H_1 - H_2 - 2r_2 s_2).$$

Hence

$$\pi(\Omega) = C_1 + C_2.$$

Proposition 4. *The principal finite W-algebra W_χ is generated by even elements $\pi(\Omega)$, C_2 and odd R_1. The relations are*

$$\begin{aligned}
&[\pi(\Omega), C_2] = [\pi(\Omega), R_i] = 0, \text{ for } i = 1, 2,\\
&[C_2, R_1] = R_1 - 2R_2, \quad [C_2, R_2] = -3R_2 - 2C_2 R_1,\\
&[R_1, R_2] = \tfrac{1}{2}\pi(\Omega), \quad [R_2, R_2] = -C_2\pi(\Omega) - 2R_2 R_1, \quad [R_1, R_1] = \pi(\Omega).
\end{aligned}$$

4 Principal Finite *W*-Algebra for *F*(4)

Recall that $\mathfrak{g} = F(4) = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is the exceptional basic classical Lie superalgebra, where $\mathfrak{g}_{\bar{0}} = \mathfrak{sl}(2) \oplus \mathfrak{so}(7)$, $\mathfrak{g}_{\bar{1}} = U \otimes V$, U is the standard $\mathfrak{sl}(2)$-module, and $V = \Lambda(\xi_1, \xi_2, \xi_3)$ is the Grassmann algebra (see [3]).

Let $\{X, H, Y\}$ be the standard basis in $\mathfrak{sl}(2)$ and $\{x, y\}$ be the standard basis in U. Let $C(\xi_i, \eta_i)$ be the Clifford algebra with generators ξ_i, η_i, $i = 1, 2, 3$, and relations:

$$\xi_i \xi_j = -\xi_j \xi_i, \quad \eta_i \eta_j = -\eta_j \eta_i, \quad \eta_i \xi_j = \delta_{i,j} - \xi_j \eta_i, \quad i, j = 1, 2, 3.$$

Note that $\mathfrak{so}(7)$ can be realized inside $C(\xi_i, \eta_i)$ as follows:

$$\mathfrak{so}(7) = < \xi_i \xi_j,\ \eta_i \eta_j,\ \xi_i \eta_j - \frac{1}{2}\delta_{ij},\ \xi_i,\ \eta_j \mid i, j = 1, 2, 3 > . \tag{3}$$

The commutator $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}]$ is given by the natural action of $\mathfrak{sl}(2)$ on U and of $\mathfrak{so}(7)$ on V. Note that the action of ξ_i on V is the multiplication in the Grassmann algebra, and η_i acts by ∂_{ξ_i}.

Let $\mathcal{P} : U \times U \to \mathfrak{sl}(2)$ be the $\mathfrak{sl}(2)$-invariant bilinear mapping defined as follows:

$$\mathcal{P}(x, x) = 2X,\ \mathcal{P}(y, y) = -2Y,\ \mathcal{P}(x, y) = \mathcal{P}(y, x) = -H,$$

and let $< \cdot, \cdot >$ be the non-degenerate skew-symmetric form on U defined by

$$< x, y >= - < y, x >= 1.$$

To describe the commutator $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$, we consider the paring on $\mathfrak{so}(7)$ defined by a non-degenerate invariant symmetric bilinear form on $\mathfrak{so}(7)$. For example, the form $(a,b) = \frac{1}{2}tr(ab)$ defines the paring $u^i \leftrightarrow u_i$. Explicitly,

$$\begin{array}{l} \xi_i\xi_j \leftrightarrow -\eta_i\eta_j, \quad \xi_i\eta_j \leftrightarrow \xi_j\eta_i, \quad i \neq j, \\ \xi_i\eta_i - \frac{1}{2} \leftrightarrow \xi_i\eta_i - \frac{1}{2}, \quad \xi_i \leftrightarrow \frac{1}{2}\eta_i, \quad i = 1,2,3. \end{array}$$

Consider the symmetric $\mathfrak{so}(7)$-invariant bilinear form $\Psi(\cdot,\cdot)$ on V defined as follows:

$$\begin{array}{l} \Psi(v,w) = \partial_{\xi_1}\partial_{\xi_2}\partial_{\xi_3}(vw), \text{ if } v, w \neq 1, \\ \Psi(v,w) = -\partial_{\xi_1}\partial_{\xi_2}\partial_{\xi_3}(vw), \text{ if } v = 1 \text{ or } w = 1, \end{array}$$

where v, w are monomials in V. Let

$$\mathcal{Q}(v,w) = \sum_i \Psi(u^i v, w)u_i,$$

where u_i runs through the basis (3) of $\mathfrak{so}(7)$. The commutator $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ is defined as follows:

$$[p \otimes v, q \otimes w] = \mathcal{P}(p,q)\Psi(v,w) + \frac{4}{3} < p,q > \mathcal{Q}(v,w),$$

where $p, q \in U$, $v, w \in V$. Consider the following even non-degenerate invariant supersymmetric bilinear form on $\mathfrak{g}$:

$$\begin{array}{l} (X|Y) = -\frac{1}{2}, \quad (H|H) = -1, \quad (\xi_i\xi_j|\eta_i\eta_j) = -\frac{3}{4}, \quad (\xi_i\eta_j|\xi_j\eta_i) = \frac{3}{4}, \quad i \neq j, \\ (\xi_i|\eta_i) = \frac{3}{2}, \quad (\xi_i\eta_i - \frac{1}{2}|\xi_i\eta_i - \frac{1}{2}) = \frac{3}{4}, \quad i = 1,2,3, \\ (x \otimes \xi_1\xi_2\xi_3|y \otimes 1) = 1, (y \otimes \xi_1\xi_2\xi_3|x \otimes 1) = -1, \\ (y \otimes \xi_1\xi_2|x \otimes \xi_3) = 1, (x \otimes \xi_1\xi_2|y \otimes \xi_3) = -1, \\ (x \otimes \xi_1\xi_3|y \otimes \xi_2) = 1, (x \otimes \xi_1|y \otimes \xi_2\xi_3) = -1, \\ (x \otimes \xi_2|y \otimes \xi_1\xi_3) = 1, (y \otimes \xi_1|x \otimes \xi_2\xi_3) = 1. \end{array}$$

Let $\mathfrak{sl}(2) =< e, h, f >$, where $e = \xi_1\eta_2 + \xi_2\eta_3 + \xi_3 + X$, $f = \xi_2\eta_1 + 10\xi_3\eta_2 + 6\eta_3 + Y$, $h = 6\xi_1\eta_1 + 4\xi_2\eta_2 + 2\xi_3\eta_3 - 6 + H$. Note that e is a regular nilpotent element.

We have that

$$\mathfrak{g}^e_{\bar{0}} =< c_i \mid i = 1,\ldots,4 >, \quad \mathfrak{g}^e_{\bar{1}} =< r_i \mid i = 1,2,3 >, \tag{4}$$

where

$$c_1 = X,\ c_2 = \xi_1\eta_2 + \xi_2\eta_3 + \xi_3,\ c_3 = \xi_1 + 2\xi_2\xi_3,\ c_4 = \xi_1\xi_2\ ,$$
$$r_1 = x \otimes (\xi_1 + \xi_2\xi_3),\ r_2 = y \otimes \xi_1\xi_2\xi_3 - x \otimes \xi_1\xi_2,\ r_3 = x \otimes \xi_1\xi_2\xi_3.$$

Note that $\dim(\mathfrak{g}^e) = (4|3)$. In fact, according to [7], if $\mathfrak{g} = F(4)$, then $\dim(\mathfrak{g}^e_{\bar{1}}) = 2\mathrm{def}\mathfrak{g} + 1$, and $\mathrm{def}(F(4)) = 1$. Note also that the Lie algebra $\mathfrak{g}^e_{\bar{0}}$ is abelian, and the nonzero commutation relations between c_i and r_i are as follows:

$$[c_1, r_2] = r_3, \quad [c_2, r_2] = r_3, \quad [c_3, r_1] = 3r_3,$$
$$[r_1, r_1] = -4c_1, \quad [r_1, r_2] = \tfrac{2}{3}c_3, \quad [r_2, r_2] = -\tfrac{8}{3}c_4.$$

The element h defines a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{-10 \le j \le 10} \mathfrak{g}_j$, where

$$\mathfrak{g}_{-10} =< \eta_1\eta_2 >, \quad \mathfrak{g}_{\pm 9} = 0, \quad \mathfrak{g}_{-8} =< \eta_1\eta_3 >, \quad \mathfrak{g}_{-7} =< y \otimes 1 >,$$
$$\mathfrak{g}_{-6} =< \eta_2\eta_3, \eta_1 >, \quad \mathfrak{g}_{-5} =< x \otimes 1, y \otimes \xi_3 >,$$
$$\mathfrak{g}_{-4} =< \xi_3\eta_1, \eta_2 >, \quad \mathfrak{g}_{-3} =< x \otimes \xi_3, y \otimes \xi_2 >, \quad \mathfrak{g}_{-2} =< \xi_2\eta_1, \xi_3\eta_2, \eta_3, Y >,$$
$$\mathfrak{g}_{-1} =< x \otimes \xi_2, y \otimes \xi_1, y \otimes \xi_2\xi_3 >, \quad \mathfrak{g}_0 =< H, \xi_i\eta_i - \tfrac{1}{2} \mid i = 1, 2, 3 >,$$
$$\mathfrak{g}_1 =< x \otimes \xi_1, y \otimes \xi_1\xi_3, x \otimes \xi_2\xi_3 >, \quad \mathfrak{g}_2 =< \xi_1\eta_2, \xi_2\eta_3, \xi_3, X >,$$
$$\mathfrak{g}_3 =< y \otimes \xi_1\xi_2, x \otimes \xi_1\xi_3 >, \quad \mathfrak{g}_4 =< \xi_1\eta_3, \xi_2 >,$$
$$\mathfrak{g}_5 =< x \otimes \xi_1\xi_2, y \otimes \xi_1\xi_2\xi_3 >, \quad \mathfrak{g}_6 =< \xi_2\xi_3, \xi_1 >, \quad \mathfrak{g}_7 =< x \otimes \xi_1\xi_2\xi_3 >,$$
$$\mathfrak{g}_8 =< \xi_1\xi_3 >, \quad \mathfrak{g}_{10} =< \xi_1\xi_2 > .$$

Note that $\mathfrak{m} = (\bigoplus_{j \le -2} \mathfrak{g}_j) \bigoplus \mathfrak{l}$, where $\mathfrak{l} =< x \otimes \xi_2 >$, and $\chi(x \otimes \xi_2) = 0$, $\chi(\xi_2\eta_1) = \chi(\xi_3\eta_2) = \frac{3}{4}$, $\chi(\eta_3) = \frac{3}{2}$, $\chi(Y) = -\frac{1}{2}$. Let $\theta = y \otimes (\xi_1 + \xi_2\xi_3)$. Note that $\theta \in \mathfrak{g}_{-1}$, $\pi(\theta) \in W_\chi$ and $\pi(\theta)^2 = -1$. Note that the following elements C_1, C_2 and R_1 belong to W_χ:

$$C_1 = \pi(X + H - \tfrac{1}{2}H^2),$$
$$C_2 = \pi\Big(\xi_1\eta_2 + \xi_2\eta_3 + \xi_3 - H + (x \otimes \xi_1)(y \otimes \xi_2\xi_3) + (x \otimes \xi_2\xi_3)(y \otimes \xi_1) + \tfrac{2}{3}\textstyle\sum_{i=1}^{3}(\xi_i\eta_i - \tfrac{1}{2})^2 - \tfrac{4}{3}(\xi_1\eta_1 + \xi_2\eta_2 - 1)\Big),$$
$$R_1 = \pi(x \otimes (\xi_1 + \xi_2\xi_3) + H\theta).$$

We have that $P(C_1) = c_1,\ P(C_2) = c_2$, and $P(R_1) = r_1$.

Conjecture 2. There exists an element $R_2 \in W_\chi$ such that $P(R_2) = r_2$.

Idea of Proof. Recall that we identify W_χ with Q_χ^m (see (1)). The elements $x \otimes \xi_2, \xi_2\eta_1, \xi_3\eta_2, \eta_3, Y$ generate $\mathfrak{m}$. We can show that W_χ has an element $R_2 = u + w$, where $u, w \in Q_\chi$, such that

$$au = \chi(a)u, \quad aw = \chi(a)w \quad \text{for} \quad a = x \otimes \xi_2, \xi_2\eta_1, \eta_3, Y,$$

where

$$u = \pi\Big(y \otimes \xi_1\xi_2\xi_3 - x \otimes \xi_1\xi_2 - (H + \tfrac{2}{3}(\xi_3\eta_3 - \tfrac{1}{2}))(y \otimes \xi_1\xi_2) + \\ + \tfrac{2}{3}(\xi_2 + 2H(\xi_2\eta_3) + \tfrac{4}{3}(\xi_3\eta_3 - \tfrac{1}{2})(\xi_2\eta_3))(y \otimes \xi_2\xi_3) + \tfrac{4}{3}(\xi_2\eta_3)(x \otimes \xi_2\xi_3)\Big),$$

so that $P(u) = r_2$, $\deg P(u) = 7, \mathrm{wt} P(u) = 5$, and $\deg P(w) = 7$, $\mathrm{wt} P(w) < 5$. Then $P(R_2) = r_2$.

Proposition 5. *The principal finite W-algebra W_χ is generated by $\pi(\Omega)$, $C_1, \pi(\theta)$ and R_2, where Ω is the Casimir element of $\mathfrak{g}$.*

Proof. Note that

$$2C_1 + 2C_2 = \pi(\Omega), \quad [C_1, \pi(\theta)] = R_1 - \frac{1}{2}\pi(\theta). \tag{5}$$

Observe that if $X, Y \in W_\chi$, $P(X), P(Y) \in \mathfrak{g}^e$ and $[P(X), P(Y)] \neq 0$, then $P([X, Y]) = [P(X), P(Y)]$. Set

$$R_3 = [C_1, R_2], \quad C_3 = \frac{3}{2}[R_1, R_2], \quad C_4 = -\frac{3}{8}[R_2, R_2]. \tag{6}$$

Since $P(C_1) = c_1$, $P(R_2) = r_2$ and $[c_1, r_2] = r_3$, then $P(R_3) = r_3$. Since $P(R_1) = r_1$, $P(R_2) = r_2$ and $[r_1, r_2] = \frac{2}{3}c_3$, then $P(C_3) = c_3$. Finally, since $P(R_2) = r_2$ and $[r_2, r_2] = -\frac{8}{3}c_4$, then $P(C_4) = c_4$. Note that $P(C_i)$ for $i = 1, \ldots, 4$ and $P(R_j)$ for $j = 1, 2, 3$ form a homogeneous basis of $\mathfrak{g}^e$, see (4). Then by Theorem 4 (a) C_i, R_j and $\pi(\theta)$ generate W_χ. It follows from (5) and (6) that $\pi(\Omega), C_1, \pi(\theta)$ and R_2 generate W_χ. □

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Super-de Sitter and Alternative Super-Poincaré Symmetries

V.N. Tolstoy

Abstract It is well-known that de Sitter Lie algebra $\mathfrak{o}(1,4)$ contrary to anti-de Sitter one $\mathfrak{o}(2,3)$ does not have a standard $\mathbb{Z}_2$-graded superextension. We show here that the Lie algebra $\mathfrak{o}(1,4)$ has a superextension based on the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading. Using the standard contraction procedure for this superextension we obtain an *alternative* super-Poincaré algebra with the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading.

1 Introduction

In supergravity theory (SUGRA) already for more than 20 years there is the following unsolved (up to now) problem. All physical reasonable solutions of SUGRA models with cosmological constants Λ have been constructed for the case $\Lambda < 0$, i.e. for the anti-de Sitter metric

$$g_{ab}=\text{diag}\,(1,-1,-1,-1,1), \quad (a,b\ =\ 0,1,2,3,4) \tag{1}$$

with the space-time symmetry $\mathfrak{o}(2,3)$. In the case $\Lambda > 0$, i.e. for the Sitter metric

$$g_{ab}=\text{diag}\,(1,-1,-1,-1,-1), \quad (a,b\ =\ 0,1,2,3,4) \tag{2}$$

with the space-time symmetry $\mathfrak{o}(1,4)$ no reasonable solutions have been found. For example, in SUGRA it was obtained the following relation

$$\Lambda = -3m^2, \tag{3}$$

where m is the massive parameter of gravitinos. Thus if $\Lambda > 0$, then m is imaginary.

In my opinion these problems for the case $\Lambda > 0$ are connected with superextensions of anti-de Sitter $\mathfrak{o}(2,3)$ and de Sitter $\mathfrak{o}(1,4)$ symmetries. The $\mathfrak{o}(2,3)$ symmetry has the superextension—the superalgebra $\mathfrak{osp}(1|(2,3))$. This is the usual $\mathbb{Z}_2$-graded

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superalgebra. In the case of $\mathfrak{o}(1,4)$ such superextension does not exist. However the Lie algebra $\mathfrak{o}(1,4)$ has an *alternative* superextension that is based on the $\mathbb{Z}_2\times\mathbb{Z}_2$-grading and a preliminary analysis shows that we can construct the reasonable SUGRA models for the case $\Lambda > 0$. In this paper we shall consider certain $\mathbb{Z}_2\times\mathbb{Z}_2$-graded supersymmetries, but we will not discuss supergravity models based on such supersymmetries.

All standard relativistic SUSY (super-anti de Sitter, super-Poincaré, super-conformal, extended N-supersymmetry, etc) are based on usual ($\mathbb{Z}_2$-graded) Lie superalgebras ($osp(1|(2,3))$, $su(N|(2,2))$, $osp(N|(2,3))$ etc). It turns out that every standard relativistic SUSY has an alternative variant based on an alternative ($\mathbb{Z}_2\times\mathbb{Z}_2$-graded) Lie superalgebra:

$$\text{Standard relativistic SUSY} \rightleftarrows \text{Alternative relativistic SUSY}$$

Distinctive features of the standard and alternative relativistic symmetries (in the example of Poincaré SUSY) are connected with the relations between the four-momenta and the Q-charges and also between the space-time coordinates and the Grassmann variables. Namely, we have.

(I) *For the standard ($\mathbb{Z}_2$-graded) Poincaré SUSY:*

$$[P_\mu, Q_\alpha] = [P_\mu, \bar{Q}_{\dot\alpha}] \;=\; 0\,, \quad \{Q_\alpha, \bar{Q}_{\dot\beta}\} \;=\; 2\sigma^\mu_{\alpha\dot\beta}P_\mu, \tag{4}$$

$$[x_\mu, \theta_\alpha] = [x_\mu, \dot\theta_{\dot\alpha}] \;=\; \{\theta_\alpha, \bar\theta_{\dot\beta}\} \;=\; 0. \tag{5}$$

(II) *For the alternative ($\mathbb{Z}_2\times\mathbb{Z}_2$-graded) Poincaré SUSY:*

$$\{P_\mu, Q_\alpha\} = \{P_\mu, \bar{Q}_{\dot\alpha}\} \;=\; 0, \quad [Q_\alpha, \bar{Q}_{\dot\beta}] \;=\; 2\sigma^\mu_{\alpha\dot\beta}P_\mu, \tag{6}$$

$$\{x_\mu, \theta_\alpha\} = \{x_\mu, \dot\theta_{\dot\alpha}\} \;=\; [\theta_\alpha, \bar\theta_{\dot\beta}] \;=\; 0. \tag{7}$$

We wrote down only the relations which are changed in the $\mathbb{Z}_2$- and $\mathbb{Z}_2\times\mathbb{Z}_2$-cases.

The paper is organized as follows. Section 2 provides definitions and general structure of $\mathbb{Z}_2$- and $\mathbb{Z}_2\times\mathbb{Z}_2$-graded superalgebras and also some classification of such simple Lie superalgebras. In Sect. 3 we describe the orthosymplectic $\mathbb{Z}_2$- and $\mathbb{Z}_2\times\mathbb{Z}_2$-graded superalgebras $\mathfrak{osp}(1|4)$ and $\mathfrak{osp}(1|2,2)$ and their real forms. We show here that a real form of $\mathfrak{osp}(1|4)$ contains $\mathfrak{o}(2,3)$ and a real form of $\mathfrak{osp}(1|2,2)$ contains $\mathfrak{o}(1,4)$. In Sect. 4 using the standard contraction procedure for the superextension $\mathfrak{osp}(1|2,2)$ we obtain an *alternative* super-Poincaré algebra with the $\mathbb{Z}_2\times\mathbb{Z}_2$-grading.

2 $\mathbb{Z}_2$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$-Graded Lie Superalgebras

A $\mathbb{Z}_2$-Graded Superalgebra [1] A $\mathbb{Z}_2$-graded Lie superalgebra (LSA) $\mathfrak{g}$, as a linear space, is a direct sum of two graded components

$$\mathfrak{g} = \bigoplus_{a=0,1} \mathfrak{g}_a \;=\; \mathfrak{g}_0 \oplus \mathfrak{g}_1 \tag{8}$$

with a bilinear operation (the general Lie bracket), $[\![\cdot,\cdot]\!]$, satisfying the identities:

$$\deg([\![x_a, y_b]\!]) = \deg(x_a) + \deg(y_b) \;=\; a + b \pmod{2}, \tag{9}$$

$$[\![x_a, y_b]\!] = -(-1)^{ab}[\![y_b, x_a]\!], \tag{10}$$

$$[\![x_a, [\![y_b, z]\!]]\!] = [\![[\![x_a, y_b]\!], z]\!] + (-1)^{ab}[\![y_b, [\![x_a, z]\!]]\!], \tag{11}$$

where the elements x_a and y_b are homogeneous, $x_a \in \mathfrak{g}_a$, $y_b \in \mathfrak{g}_b$, and the element $z \in \mathfrak{g}$ is not necessarily homogeneous. The grading function $\deg(\cdot)$ is defined for homogeneous elements of the subspaces $\mathfrak{g}_0$ and $\mathfrak{g}_1$ modulo 2, $\deg(\mathfrak{g}_0) = 0$, $\deg(\mathfrak{g}_1) = 1$. The first identity (9) is called the grading condition, the second identity (10) is called the symmetry property and the condition (11) is the Jacobi identity. It follows from (9) that $\mathfrak{g}_0$ is a Lie subalgebra in $\mathfrak{g}$, and $\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module. It follows from (9) and (10) that the general Lie bracket $[\![\cdot,\cdot]\!]$ for homogeneous elements posses two values: commutator $[\cdot,\cdot]$ and anticommutator $\{\cdot,\cdot\}$.

A $\mathbb{Z}_2 \times \mathbb{Z}_2$-Graded Superalgebra [4] A $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded LSA $\tilde{\mathfrak{g}}$, as a linear space, is a direct sum of four graded components

$$\tilde{\mathfrak{g}} = \bigoplus_{\mathbf{a}=(a_1,a_2)} \tilde{\mathfrak{g}}_{\mathbf{a}} \;=\; \tilde{\mathfrak{g}}_{(0,0)} \oplus \tilde{\mathfrak{g}}_{(1,1)} \oplus \tilde{\mathfrak{g}}_{(1,0)} \oplus \tilde{\mathfrak{g}}_{(0,1)} \tag{12}$$

with a bilinear operation $[\![\cdot,\cdot]\!]$ satisfying the identities (grading, symmetry, Jacobi):

$$\deg([\![x_{\mathbf{a}}, y_{\mathbf{b}}]\!]) = \deg(x_{\mathbf{a}}) + \deg(y_{\mathbf{b}}) \;=\; \mathbf{a} + \mathbf{b} \;=\; (a_1 + b_1, a_2 + b_2), \tag{13}$$

$$[\![x_{\mathbf{a}}, y_{\mathbf{b}}]\!] = -(-1)^{\mathbf{ab}}[\![y_{\mathbf{b}}, x_{\mathbf{a}}]\!], \tag{14}$$

$$[\![x_{\mathbf{a}}, [\![y_{\mathbf{b}}, z]\!]]\!] = [\![[\![x_{\mathbf{a}}, y_{\mathbf{b}}]\!], z]\!] + (-1)^{\mathbf{ab}}[\![y_{\mathbf{b}}, [\![x_{\mathbf{a}}, z]\!]]\!], \tag{15}$$

where the vector $(a_1 + b_1, a_2 + b_2)$ is defined $\bmod{(2,2)}$ and $\mathbf{ab} = a_1b_1 + a_2b_2$. Here in (13)–(15) $x_{\mathbf{a}} \in \tilde{\mathfrak{g}}_{\mathbf{a}}$, $y_{\mathbf{b}} \in \tilde{\mathfrak{g}}_{\mathbf{b}}$, and the element $z \in \tilde{\mathfrak{g}}$ is not necessarily homogeneous. It follows from (13) that $\tilde{\mathfrak{g}}_{(0,0)}$ is a Lie subalgebra in $\tilde{\mathfrak{g}}$, and the subspaces $\tilde{\mathfrak{g}}_{(1,1)}$, $\tilde{\mathfrak{g}}_{(1,0)}$ and $\tilde{\mathfrak{g}}_{(0,1)}$ are $\tilde{\mathfrak{g}}_{(0,0)}$-modules. It should be noted that $\tilde{\mathfrak{g}}_{(0,0)} \oplus \tilde{\mathfrak{g}}_{(1,1)}$ is a Lie subalgebra in $\tilde{\mathfrak{g}}$ and the subspace $\tilde{\mathfrak{g}}_{(1,0)} \oplus \tilde{\mathfrak{g}}_{(0,1)}$ is a $\tilde{\mathfrak{g}}_{(0,0)} \oplus \tilde{\mathfrak{g}}_{(1,1)}$-module, and moreover $\{\tilde{\mathfrak{g}}_{(1,1)}, \tilde{\mathfrak{g}}_{(1,0)}\} \subset \tilde{\mathfrak{g}}_{(0,1)}$ and vice versa $\{\tilde{\mathfrak{g}}_{(1,1)}, \tilde{\mathfrak{g}}_{(0,1)}\} \subset \tilde{\mathfrak{g}}_{(1,0)}$. It follows from (13) and (14) that the general Lie bracket $[\![\cdot,\cdot]\!]$ for homogeneous elements posses two values: commutator $[\cdot,\cdot]$ and anticommutator $\{\cdot,\cdot\}$ as well as in the previous $\mathbb{Z}_2$-case.

Let us introduce a useful notation of parity of homogeneous elements: *the parity $p(x)$ of a homogeneous element x is a scalar square of its grading* $\deg(x)$ *modulo 2*. It is evident that for the $\mathbb{Z}_2$-graded superalgebra $\mathfrak{g}$ the parity coincides with the grading: $p(\mathfrak{g}_a) = \deg(\mathfrak{g}_a) = \bar{a}$ $(\bar{a} = \bar{0}, \bar{1})$.[1] In the case of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra $\tilde{\mathfrak{g}}$ we have

$$p(\tilde{\mathfrak{g}}_{\mathbf{a}}) := \mathbf{a}^2 \;=\; a_1^2 + a_2^2 \qquad (\text{mod } 2), \tag{16}$$

that is

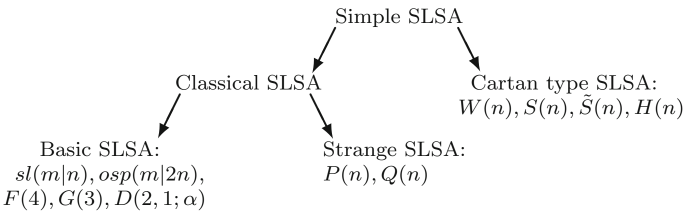
$$p(\tilde{\mathfrak{g}}_{(0,0)}) = p(\tilde{\mathfrak{g}}_{(1,1)}) \;=\; \bar{0}, \qquad p(\tilde{\mathfrak{g}}_{(1,0)}) \;=\; p(\tilde{\mathfrak{g}}_{(0,1)}) \;=\; \bar{1}. \tag{17}$$

Homogeneous elements with the parity $\bar{0}$ are called even and with parity $\bar{1}$ are odd. Thus,

$$\tilde{\mathfrak{g}}=\tilde{\mathfrak{g}}_{\bar{0}} \oplus \tilde{\mathfrak{g}}_{\bar{1}}, \qquad \tilde{\mathfrak{g}}_{\bar{0}} \;=\; \tilde{\mathfrak{g}}_{(0,0)} \oplus \tilde{\mathfrak{g}}_{(1,1)}, \qquad \tilde{\mathfrak{g}}_{\bar{1}} \;=\; \tilde{\mathfrak{g}}_{(1,0)} \oplus \tilde{\mathfrak{g}}_{(0,1)}. \tag{18}$$

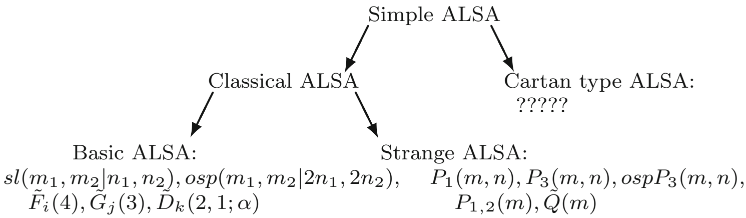
The even subspace $\tilde{\mathfrak{g}}_{\bar{0}}$ is a subalgebra and the odd one $\tilde{\mathfrak{g}}_{\bar{1}}$ is a $\tilde{\mathfrak{g}}_{\bar{0}}$-module. Thus the parity unifies "cousinly" the $\mathbb{Z}_2$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebras.

Classification of the $\mathbb{Z}_2$- and $\mathbb{Z}_2\times\mathbb{Z}_2$-Graded Simple Lie Superalgebras A complete list of simple $\mathbb{Z}_2$-graded (standard) Lie superalgebras was obtained by Kac [1]. The following scheme resumes the classification [2]:



[1]Integer value of the parity will be denoted with the bar.

There is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-analog (alternative superalgebras) of this scheme:



where $i = 1, 2, \ldots, 6$, $j = 1, 2, 3$, $k = 1, 2, 3$. It should be noted that the classification of the classical series $sl(m_1, m_2|n_1, n_2)$, $osp(m_1, m_2|2n_1, 2n_2)$ and all strange series was obtain by Rittenberg and Wyler in [4].

There are numerous references about the $\mathbb{Z}_2$-graded Lie superalgebras and their applications. Unfortunately, in the $\mathbb{Z}_2 \times \mathbb{Z}_2$-case the situation is somewhat poor. There are a few references where some $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras were studied and applied [3–8].

Analysis of matrix realizations of the basic $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras shows that these superalgebras (as well as the $\mathbb{Z}_2$-graded Lie superalgebras) have Cartan-Weyl and Chevalley bases, Weyl groups, Dynkin diagrams, etc. However these structures have a specific characteristics for the $\mathbb{Z}_2$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded cases. Let us consider, for example, the Dynkin diagrams. In the case of the $\mathbb{Z}_2$-graded superalgebras the nodes of the Dynkin diagram and corresponding simple roots occur at three types:

white ○, gray ⊗, dark ●.

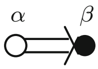
While in the case of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebras we have six types of nodes:

(00)-white ○, (11)-white ○, (10)-gray ⊗,
(01)-gray ⊗, (10)-dark ●, (01)-dark ●.

In the next section we consider in detail two basic superalgebras of rank 2: the orthosymplectic $\mathbb{Z}_2$-graded superalgebra $\mathfrak{osp}(1|4)$ and the orthosymplectic $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra $\mathfrak{osp}(1|2, 2) := \mathfrak{osp}(1, 0|2, 2)$. It will be shown that their real forms, which contain the Lorentz subalgebra $\mathfrak{o}(1, 3)$, give us the super-anti-de Sitter (in the $\mathbb{Z}_2$-graded case) and super-de Sitter (in the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded case) Lie superalgebras.

3 Anti-de Sitter and de Sitter Superalgebras

The Orthosymplectic $\mathbb{Z}_2$-Graded Superalgebra $\mathfrak{osp}(1|4)$ The Dynkin diagram:



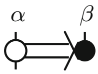
The Serre relations:

$$[e_{\pm\alpha}, [e_{\pm\alpha}, e_{\pm\beta}]] = 0\,, \qquad [\{[e_{\pm\alpha}, e_{\pm\beta}], e_{\pm\beta}\}, e_{\pm\beta}] = 0. \tag{19}$$

The root system Δ_+:

$$\underbrace{2\beta,\ \ 2\alpha+2\beta,\ \ \alpha,\ \ \alpha+2\beta}_{\deg(\cdot)=0},\ \ \underbrace{\beta,\ \ \alpha+\beta}_{\deg(\cdot)=1}. \tag{20}$$

The Orthosymplectic $\mathbb{Z}_2 \times \mathbb{Z}_2$-Graded Superalgebra $\mathfrak{osp}(1|2,2)$ The Dynkin diagram:



The Serre relations:

$$\{e_{\pm\alpha}, \{e_{\pm\alpha}, e_{\pm\beta}\}\} = 0, \qquad \{[\{e_{\pm\alpha}, e_{\pm\beta}\}, e_{\pm\beta}], e_{\pm\beta}\} = 0. \tag{21}$$

The root system Δ_+:

$$\underbrace{2\beta,\ \ 2\alpha+2\beta}_{\deg(\cdot)=(00)},\ \ \underbrace{\alpha,\ \ \alpha+2\beta}_{\deg(\cdot)=(11)},\ \ \underbrace{\ \ \beta\ \ }_{\deg(\cdot)=(10)},\ \ \underbrace{\alpha+\beta}_{\deg(\cdot)=(01)}. \tag{22}$$

Commutation relations, which contain Cartan elements, are the same for the $\mathfrak{osp}(1|4)$ and $\mathfrak{osp}(1|2,2)$ superalgebras and they are:

$$\begin{aligned} \llbracket e_{\gamma}, e_{-\gamma'} \rrbracket &= \delta_{\gamma,\gamma'} h_{\gamma}, \\ [h_{\gamma},\, e_{\gamma'}] &= (\gamma, \gamma') e_{\gamma'} \end{aligned} \tag{23}$$

for $\gamma, \gamma' \in \{\alpha, \beta\}$. These relations together with the Serre relations (19) and (21) correspondingly are called the defining relations of the superalgebras $\mathfrak{osp}(1|4)$

and $\mathfrak{osp}(1|2,2)$ correspondingly. It is easy to see that these defining relations are invariant with respect to the non-graded Cartan involution $(^{\dagger})$ $((x^{\dagger})^{\dagger} = x$, $[\![x, y]\!]^{\dagger} = [\![y^{\dagger}, x^{\dagger}]\!]$ for any homogenous elements x and y):

$$e^{\dagger}_{\pm\gamma} = e_{\mp\gamma}, \qquad h^{\dagger}_{\gamma} = h_{\gamma}. \tag{24}$$

The composite root vectors $e_{\pm\gamma}$ $(\gamma \in \Delta_{+})$ for $\mathfrak{osp}(1|4)$ and $\mathfrak{osp}(1|2,2)$ are defined as follows

$$\begin{aligned} e_{\alpha+\beta} &:= [\![e_{\alpha},\, e_{\beta}]\!], & e_{\alpha+2\beta} &:= [\![e_{\alpha+\beta},\, e_{\beta}]\!], \\ e_{2\alpha+2\beta} &:= \frac{1}{\sqrt{2}}\{e_{\alpha+\beta}, e_{\alpha+\beta}\}, & e_{2\beta} &:= \frac{1}{\sqrt{2}}\{e_{\beta},\, e_{\beta}\}, \\ e_{-\gamma} &:= e^{\dagger}_{\gamma}\,. \end{aligned} \tag{25}$$

These root vectors satisfy the non-vanishing relations:

$$\begin{aligned} [e_{\alpha}, e_{\alpha+2\beta}] &= (-1)^{\deg\alpha\cdot\deg\beta}\sqrt{2}e_{2\alpha+2\beta}, & [e_{\alpha}, e_{2\beta}] &= \sqrt{2}\,e_{\alpha+2\beta}, \\ [\![e_{\alpha+\beta}, e_{-\alpha}]\!] &= -(-1)^{\deg\alpha\cdot\deg\beta}e_{\beta}, & [e_{\alpha+2\beta}, e_{-\alpha}] &= -\sqrt{2}\,e_{2\beta}, \\ [e_{2\alpha+2\beta}, e_{-\alpha}] &= -(-1)^{\deg\alpha\cdot\deg\beta}\sqrt{2}e_{\alpha+2\beta}, & [e_{2\beta}, e_{-\beta}] &= -\sqrt{2}\,e_{\beta}, \\ [\![e_{\alpha+2\beta}, e_{-\alpha-\beta}]\!] &= -(-1)^{\deg\alpha\cdot\deg\beta}e_{\beta}, & [\![e_{\beta}, e_{-\alpha-\beta}]\!] &= e_{-\alpha}, \\ [\![e_{\beta}, e_{-\alpha-2\beta}]\!] &= -e_{-\alpha-\beta}, & [e_{2\alpha+2\beta}, e_{-\alpha-\beta}] &= -\sqrt{2}e_{\alpha+\beta}, \\ [e_{\alpha+2\beta}, e_{-2\alpha-2\beta}] &= -(-1)^{\deg\alpha\cdot\deg\beta}\sqrt{2}e_{-\alpha}, & [e_{2\beta},\, e_{-\alpha-2\beta}] &= -\sqrt{2}\,e_{-\alpha}, \\ \{e_{\alpha+\beta}, e_{-\alpha-\beta}\} &= h_{\alpha} + h_{\beta}, & [e_{\alpha+2\beta}, e_{-\alpha-2\beta}] &= -h_{\alpha} - 2h_{\beta}, \\ [e_{2\beta}, e_{-2\beta}] &= -2h_{\beta}, & [e_{2\alpha+2\beta}, e_{-2\alpha-2\beta}] &= -2h_{\alpha} - 2h_{\beta}. \end{aligned} \tag{26}$$

The rest of non-zero relations is obtained by applying the operation $(^{\dagger})$ to these relations.

Now we find real forms of $\mathfrak{osp}(1|4)$ and $\mathfrak{osp}(1|2,2)$, which contain the real Lorentz subalgebra $\mathfrak{o}(1,3)$. It is not difficult to check that the antilinear mapping $(^{*})$ $((x^{*})^{*} = x$, $[\![x, y]\!]^{*} = [\![y^{*}, x^{*}]\!]$ for any homogenous elements x and y) given by

$$\begin{aligned} e^{*}_{\pm\alpha} &= -(-1)^{\deg\alpha\cdot\deg\beta}e_{\mp\alpha}, & e^{*}_{\pm\beta} &= -i\,e_{\pm(\alpha+\beta)}, \\ e^{*}_{\pm2\beta} &= -e_{\pm(2\alpha+2\beta)}, & e^{*}_{\pm(\alpha+2\beta)} &= -e_{\pm(\alpha+2\beta)}, \\ h^{*}_{\alpha} &= h_{\alpha}, & h^{*}_{\beta} &= -h_{\alpha} - h_{\beta}. \end{aligned} \tag{27}$$

is an antiinvolution and the desired real form with respect to the antiinvolution is presented as follows.

The Lorentz algebra $\mathfrak{o}(1,3)$:

$$\begin{aligned}
L_{12} &= -\frac{1}{2}h_\alpha,\\
L_{13} &= -\frac{i}{2\sqrt{2}}\Big(e_{2\beta}+e_{2\alpha+2\beta}+e_{-2\beta}+e_{-2\alpha-2\beta}\Big),\\
L_{23} &= -\frac{1}{2\sqrt{2}}\Big(e_{2\beta}-e_{2\alpha+2\beta}-e_{-2\beta}+e_{-2\alpha-2\beta}\Big),\\
L_{01} &= \frac{i}{2\sqrt{2}}\Big(e_{2\beta}+e_{2\alpha+2\beta}-e_{-2\beta}-e_{-2\alpha-2\beta}\Big),\\
L_{02} &= \frac{1}{2\sqrt{2}}\Big(e_{2\beta}-e_{2\alpha+2\beta}+e_{-2\beta}-e_{-2\alpha-2\beta}\Big),\\
L_{03} &= -\frac{i}{2}(h_\alpha+2h_\beta).
\end{aligned} \tag{28}$$

The generators $L_{\mu 4}$:

$$\begin{aligned}
L_{04} &= -\frac{i}{2}\Big(e_{\alpha+2\beta}+(-1)^{\deg\alpha\cdot\deg\beta}e_{-\alpha-2\beta}\Big),\\
L_{14} &= -\frac{i}{2}\Big(e_{\alpha}+(-1)^{\deg\alpha\cdot\deg\beta}e_{-\alpha}\Big),\\
L_{24} &= \frac{1}{2}\Big(e_{\alpha}-(-1)^{\deg\alpha\cdot\deg\beta}e_{-\alpha}\Big),\\
L_{34} &= -\frac{i}{2}\Big(e_{\alpha+2\beta}-(-1)^{\deg\alpha\cdot\deg\beta}e_{-\alpha-2\beta}\Big).
\end{aligned} \tag{29}$$

Here are: $\deg\alpha = 0$, $\deg\beta = 1$, i.e. $(-1)^{\deg\alpha\cdot\deg\beta} = 1$, for the case of the $\mathbb{Z}_2$-grading; $\deg\alpha = (1,1)$, $\deg\beta = (1,0)$, i.e. $(-1)^{\deg\alpha\cdot\deg\beta} = -1$, for the case of the $\mathbb{Z}_2\times\mathbb{Z}_2$-grading.

The all elements L_{ab} $(a,b = 0,1,2,3,4)$ satisfy the relations

$$\begin{aligned}
[L_{ab}, L_{cd}] &= i\big(g_{bc}\,L_{ad}-g_{bd}\,L_{ac}+g_{ad}\,L_{bc}-g_{ac}\,L_{bd}\big),\\
L_{ab} &= -L_{ba}, \qquad L_{ab}^{*} = L_{ab},
\end{aligned} \tag{30}$$

where the metric tensor g_{ab} is given by

$$\begin{aligned}
g_{ab} &= \mathrm{diag}\,(1,-1,-1,-1,g_{44}^{(\alpha)}),\\
g_{44}^{(\alpha)} &= (-1)^{\deg\alpha\cdot\deg\beta}.
\end{aligned} \tag{31}$$

Thus we see that in the case of the $\mathbb{Z}_2$-grading, $(-1)^{\deg\alpha\cdot\deg\beta} = 1$, the generators (28) and (29) generate the anti-de-Sitter algebra $\mathfrak{o}(2,3)$, and in the case of the $\mathbb{Z}_2\times\mathbb{Z}_2$-grading, $(-1)^{\deg\alpha\cdot\deg\beta} = -1$, the generators (28) and (29) generate the de-Sitter algebra $\mathfrak{o}(1,4)$.

Finally we introduce the "supercharges":

$$\begin{aligned} Q_1 &:= \sqrt{2}\exp\Bigl(-\frac{i\pi}{4}\Bigr)\,e_{\alpha+\beta}, \qquad & Q_2 &:= \sqrt{2}\exp\Bigl(-\frac{i\pi}{4}\Bigr)\,e_{-\alpha-\beta},\\ \bar{Q}_{\dot{1}} &:= \sqrt{2}\exp\Bigl(-\frac{i\pi}{4}\Bigr)\,e_{\beta}\,, \qquad & \bar{Q}_{\dot{2}} &:= \sqrt{2}\exp\Bigl(-\frac{i\pi}{4}\Bigr)\,e_{-\beta}. \end{aligned} \tag{32}$$

They have the following commutation relations between themselves:

$$\begin{aligned} \{Q_1,Q_1\} &= -i2\sqrt{2}e_{2\alpha+2\beta} \;=\; 2(L_{13}-iL_{23}-L_{01}+iL_{02}),\\ \{Q_2,Q_2\} &= -i2\sqrt{2}e_{-2\alpha-2\beta} \;=\; 2(L_{13}+iL_{23}-L_{01}-iL_{02}),\\ \{Q_1,Q_2\} &= -i2(h_\alpha+h_\beta) \;=\; 2(L_{03}+iL_{12}),\\ \{\bar{Q}_{\dot\eta},\bar{Q}_{\dot\xi}\} &= \{Q_\zeta,Q_\eta\}^* \qquad (\bar{Q}_{\dot\eta} \;=\; Q_\eta^* \;\text{ for }\; \eta=1,2;\;\dot\eta=\dot{1},\dot{2}), \end{aligned} \tag{33}$$

$$\begin{aligned} [\![Q_1,\bar{Q}_{\dot{1}}]\!] &= -i2e_{\alpha+2\beta} \;=\; 2(L_{04}+L_{34}),\\ [\![Q_1,\bar{Q}_{\dot{2}}]\!] &= -i2e_{\alpha} \;=\; 2(L_{14}-iL_{24}),\\ [\![Q_2,\bar{Q}_{\dot{1}}]\!] &= -i2(-1)^{\deg\alpha\cdot\deg\beta}e_{-\alpha} \;=\; 2(L_{14}+iL_{24}),\\ [\![Q_2,\,\bar{Q}_{\dot{2}}]\!] &= -i2(-1)^{\deg\alpha\cdot\deg\beta}e_{-\alpha-2\beta} \;=\; 2(L_{04}-L_{34}). \end{aligned} \tag{34}$$

Here $[\![\cdot,\cdot]\!]\equiv\{\cdot,\cdot\}$ for the $\mathbb{Z}_2$-case and $[\![\cdot,\cdot]\!]\equiv[\cdot,\cdot]$ for the $\mathbb{Z}_2\times\mathbb{Z}_2$-case. Using the explicit formulas (28), (29), (32) and the commutation relations (26) we can also calculate commutation relations between the operators L_{ab} and the supercharges Q's and $\bar{Q}$'s .

4 $\mathbb{Z}_2$- and $\mathbb{Z}_2\times\mathbb{Z}_2$-Graded Poincaré Superalgebras

Using the standard contraction procedure: $L_{\mu 4} = R\,P_\mu$ $(\mu = 0,1,2,3)$, $Q_\alpha \to \sqrt{R}\;Q_\alpha$ and $\bar{Q}_{\dot\alpha} \to \sqrt{R}\;\bar{Q}_{\dot\alpha}$ $(\alpha = 1,2;\ \dot\alpha = \dot{1},\dot{2})$ for $R\to\infty$ we obtain the super-Poincaré algebra (standard and alternative) which is generated by $L_{\mu\nu}$, P_μ, Q_α, $\bar{Q}_{\dot\alpha}$ where $\mu,\nu = 0,1,2,3$; $\alpha = 1,2$; $\dot\alpha = \dot{1},\dot{2}$, with the relations (we write down only those which are distinguished in the $\mathbb{Z}_2$- and $\mathbb{Z}_2\times\mathbb{Z}_2$-cases).

(I) *For the $\mathbb{Z}_2$-graded Poincaré SUSY:*

$$[P_\mu, Q_\alpha] = [P_\mu, \bar{Q}_{\dot\alpha}] = 0\,, \quad \{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu. \tag{35}$$

(II) *For the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Poincaré SUSY:*

$$\{P_\mu, Q_\alpha\} = \{P_\mu, \bar{Q}_{\dot\alpha}\} = 0\,, \quad [Q_\alpha, \bar{Q}_{\dot\beta}] = 2\sigma^\mu_{\alpha\dot\beta} P_\mu, \tag{36}$$

Let us consider the supergroups associated to the $\mathbb{Z}_2$- and $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Poincaré superalgebras. A group element g is given by the exponential of the super-Poincaré generators, namely

$$g(x^\mu, \omega^{\mu\nu}, \theta^\alpha, \bar\theta^{\dot\alpha}) = \exp(x^\mu P_\mu + \omega^{\mu\nu} M_{\mu\nu} + \theta^\alpha Q_\alpha + \bar{Q}_{\dot\alpha}\bar\theta^{\dot\alpha}). \tag{37}$$

Because the grading of the exponent is zero ((0) or (00)) the result is as follows.

(1) $\mathbb{Z}_2$-*case*: $\deg P = \deg x = 0$, $\deg Q = \deg \bar{Q} = \deg \theta = \deg \bar\theta = 1$. This means that

$$[x_\mu, \theta_\alpha] = [x_\mu, \bar\theta_{\dot\alpha}] = \{\theta_\alpha, \bar\theta_{\dot\beta}\} = \{\theta_\alpha, \theta_\beta\} = \{\bar\theta_{\dot\alpha}, \bar\theta_{\dot\beta}\} = 0. \tag{38}$$

(2) $\mathbb{Z}_2 \times \mathbb{Z}_2$-*case*: $\deg P = \deg x = (11)$, $\deg Q = \deg \theta = (10)$, $\deg \bar{Q} = \deg \bar\theta = (01)$. This means that

$$\{x_\mu, \theta_\alpha\} = \{x_\mu, \bar\theta_{\dot\alpha}\} = [\theta_\alpha, \bar\theta_{\dot\beta}] = \{\theta_\alpha, \theta_\beta\} = \{\bar\theta_{\dot\alpha}, \bar\theta_{\dot\beta}\} = 0. \tag{39}$$

One defines the superspaces as the coset spaces of the standard and alternative super-Poincaré groups by the Lorentz subgroup, parameterized the coordinates x^μ, θ^α, $\bar\theta^{\dot\alpha}$, subject to the condition $\bar\theta^{\dot\alpha} = (\theta^\alpha)^*$. We can define a superfield $\mathcal{F}$ as a function of superspace.

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Localizations of $U_q(\mathfrak{sl}(2))$ and $U_q(\mathfrak{osp}(1|2))$ Associated with Euclidean and Super Euclidean Algebras

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Abstract We construct homomorphisms from the Euclidean and super Euclidean algebras, $\mathfrak{iso}(2)$ and $U(\widetilde{\mathfrak{iso}}(2))$, onto their images in localizations of $U_q(\mathfrak{sl}(2))$ and $U_q(\mathfrak{osp}(1|2))$, respectively, and, conversely, we describe homomorphisms of $U_q(\mathfrak{sl}(2))$ and $U_q(\mathfrak{osp}(1|2))$ into localizations of $U(\mathfrak{iso}(2))$ and $U(\widetilde{\mathfrak{iso}}(2))$. These homomorphisms give results on the relationship between the representation theory of the respective algebras, and, in particular, lead to new representations of $U_q(\mathfrak{osp}(1|2))$.

1 Introduction

This paper generalizes the ideas in [1, 2] and [3] to quantum super algebras. In those papers we described homomorphisms of Lie algebras and their q deformations into commutative algebraic extensions of quotient rings of enveloping algebras (localization). In this paper we describe supersymmetric analogs of those results. Here we show in complete analogy with the $U_q(\mathfrak{so}(2,1))$ case treated in [2] that it is possible to construct homomorphisms from $U_q(\mathfrak{osp}(1|2))$ into localizations of $U(\widetilde{\mathfrak{iso}}(2))$ and $U(\mathfrak{iso}(2))$ and, conversely, homomorphism of $\mathfrak{iso}(2)$ and $\widetilde{\mathfrak{iso}}(2)$ into localizations of $U_q(\mathfrak{osp}(1|2))$ and $U_q(\mathfrak{sl}(2))$. These homomorphism enable us to construct new representations of $U_q(\mathfrak{osp}(1|2))$ out of representations of $\mathfrak{iso}(2)$ and $U_q(\mathfrak{sl}(2))$. We believe that at least some of our results are capable of generalization to other q deformations of super algebras such as $U_q(\mathfrak{osp}(1|2n))$ [4].

Note on notation: except for elements of the Cartan subalgebras for which we always use plain faced letters, quantities made out of elements of $U(\mathfrak{iso}(2))$ and $U\left(\widetilde{\mathfrak{iso}}(2)\right)$ and of their localizations are usually denoted with bold faced letters and we use plain faced letters to denote elements of $\mathfrak{sl}(2)_q$ and $\mathfrak{osp}(1|2)_q$ and their localizations. Elements of super algebras are always denoted with tildes placed over the letters.

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2 The Algebras $U_q(\mathfrak{sl}(2))$ and $U(\mathfrak{iso}(2))$

We define the q-deformation $U_q(\mathfrak{sl}(2)) \simeq U_q(\mathfrak{so}(3,\mathbb{C}))$ of the simple Lie algebra $\mathfrak{sl}(2)$ as the unital associative algebra with generators $E,\ F,\ K,\ K^{-1}$ and relations [5]

$$KK^{-1} = K^{-1}K = I, KEK^{-1} = q^2 E, KFK^{-1} = q^{-2}F$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}} \ .$$

Equivalently, with $K = q^H$ we have generators $H,\ X^{\pm}$ with relations:

$$[H, X^{\pm}] = \pm 2X^{\pm}, \tag{1}$$

$$[X^+, X^-] = [H]_{q^2} \tag{2}$$

where $[x]_q = \frac{(q^{x/2} - q^{-x/2})}{(q^{1/2} - q^{-1/2})}$, $E = X^+$, $F = X^-$ and $[\cdot,\cdot]$ denote commutator. The Casimir operator is

$$\Delta_q = X^+X^- + \left(\left[\frac{H-1}{2}\right]_{q^2}\right)^2 - \frac{1}{4}\cdot I = X^-X^+ + \left(\left[\frac{H+1}{2}\right]_{q^2}\right)^2 - \frac{1}{4}\cdot I \ . \tag{3}$$

Define a real form $U_q(\mathfrak{so}(2,1))$ with generators L_{ij} $(i, j = 1.\,2,\ 3, i < j)$ specified by $X^{\pm} = L_{13} \mp iL_{23},\ L_{12} = -\frac{i}{2}\ H$. The L_{ij} are preserved under the following antilinear, anti-involution (star structure) [5]: $\omega(H) = H\ ,\ \omega(X^{\pm}) = -\ X^{\mp}$.

A basis for the Euclidean Lie algebra $\mathfrak{iso}(2)$ is $\mathbf{L}_{12}$ and $\mathbf{P}_i$ $(i = 1,\ 2)$. They satisfy the following commutation relations:

$$[\mathbf{L}_{12}\ ,\ \mathbf{P}_2] = -\,\mathbf{P}_1\ \ ,\ \ [\mathbf{L}_{12}\ ,\ \mathbf{P}_1] = \mathbf{P}_2\ \ ,$$

$$[\mathbf{P}_1\ ,\ \mathbf{P}_2] = 0\ \ .$$

Complexified translations generators are $\mathbf{P}^{\pm} = -\mathbf{P}_1 \pm i\mathbf{P}_2$. We also define as above $H = 2i\mathbf{L}_{12}$ and it generates an $SO(2)$ subgroup whose Lie algebra is $\mathfrak{so}(2)$. We have:

$$[H, \mathbf{P}^{\pm}] = \pm 2\mathbf{P}^{\pm}\ ,\ [\mathbf{P}^+, \mathbf{P}^-] = 0\,. \tag{4}$$

The enveloping algebra of $\mathfrak{iso}(2)$ is $U(\mathfrak{iso}(2))$. The center $Z(U(\mathfrak{iso}(2)))$ of $U(\mathfrak{iso}(2))$ is generated by

$$Y^2 = \mathbf{P}^+\mathbf{P}^- = \mathbf{P}^-\mathbf{P}^+. \tag{5}$$

3 The Algebras $U_{\tilde{q}}(\mathfrak{osp}(1|2))$ and $U\big(\widetilde{\mathfrak{iso}}(2)\big)$

The $\tilde{q}$-deformation $U_{\tilde{q}}(\mathfrak{osp}(1|2)) = \mathfrak{osp}(1|2)_{\tilde{q}}$ of the orthosymplectic Lie super algebra $\mathfrak{osp}(1|2)$ is defined as the unital associative algebra with generators e, f, k, k^{-1} and relations [5,6]:

$$kk^{-1} = k^{-1}k = I, kek^{-1} = \tilde{q}e, kfk^{-1} = \tilde{q}^{-1}f, ef + fe = \frac{k-k^{-1}}{\tilde{q}-\tilde{q}^{-1}} \ .$$

The $\mathbb{Z}_2$ grading on $U_{\tilde{q}}(\mathfrak{osp}(1|2))$ is $d(e) = d(f) = 1, d(k) = 0$ where $d(x)$ is the parity of x. Let $\{\cdot,\cdot\}$ denote anticommutator and let $k = \tilde{q}^{\tilde{H}}$, $e = \tilde{X}^+$, $f = \tilde{X}^-$ and we obtain generators $\tilde{H}$, $\tilde{X}^\pm$ with relations:

$$[\tilde{H}, \tilde{X}^\pm] = \pm\tilde{X}^\pm \ , \qquad (1^{bis})$$

$$\{\tilde{X}^+, \tilde{X}^-\} = [\tilde{H}]_{\tilde{q}^2} \ . \qquad (2^{bis})$$

The Casimir operator of $U_{\tilde{q}}(\mathfrak{osp}(1|2))$ is $\tilde{\Delta}_{\tilde{q}} = \tilde{S}_{\tilde{q}}^{\,2} + 2\cdot I$ with [6]

$$\tilde{S}_{\tilde{q}} = \frac{\tilde{q}^{1/2}k - \tilde{q}^{-1/2}k^{-1}}{\tilde{q}-\tilde{q}^{-1}} - (\tilde{q}^{1/2}+\tilde{q}^{-1/2})fe = [\tilde{H}+\frac{1}{2}]_{\tilde{q}^2} - [2]_{\tilde{q}}\tilde{X}^-\tilde{X}^+ =$$
$$-\frac{\tilde{q}^{-1/2}k - \tilde{q}^{1/2}k^{-1}}{(\tilde{q}-\tilde{q}^{-1})} + (\tilde{q}^{1/2}+\tilde{q}^{-1/2})ef = -[\tilde{H}-\frac{1}{2}]_{\tilde{q}^2} + [2]_{\tilde{q}}\tilde{X}^+\tilde{X}^-. \qquad (3^{bis})$$

It is straightforward to show that $\tilde{S}_{\tilde{q}}$ anticommutes with $\tilde{X}^+$ and $\tilde{X}^-$ and commutes with $\tilde{H}$. A star structure (or real form) is specified as follows. Let $\tilde{\omega}$ be such that $\tilde{\omega}(\tilde{H}) = \tilde{H}$, $\tilde{\omega}(\tilde{X}^\pm) = -\tilde{X}^\mp$. $\tilde{\omega}$ is, as in the $\mathfrak{sl}(2)$ case, an antilinear, anti-involution.

A basis for the three dimensional super Euclidean Lie algebra $\widetilde{\mathfrak{iso}}(2)$ is given by $\tilde{\mathbf{L}}_{12} = -\frac{i}{2}\tilde{H}$ and $\tilde{\mathbf{P}}_i$ $(i = 1,2)$ with commutation relations

$$[\tilde{H}, \tilde{\mathbf{P}}^\pm] = \pm\tilde{\mathbf{P}}^\pm \ , \ \{\tilde{\mathbf{P}}^+, \tilde{\mathbf{P}}^-\} = 0 \ . \qquad (4^{bis})$$

The universal enveloping algebra of $\widetilde{\mathfrak{iso}}(2)$ is $U\big(\widetilde{\mathfrak{iso}}(2)\big)$. Let $\tilde{Y}^2 \in Z(U\big(\widetilde{\mathrm{iso}}(2)\big))$ be given by

$$\tilde{Y}^2 = -i(-1)^{\tilde{H}}\,\tilde{\mathbf{P}}^+\tilde{\mathbf{P}}^- = i(-1)^{\tilde{H}}\tilde{\mathbf{P}}^-\tilde{\mathbf{P}}^+. \qquad (5^{bis})$$

4 Localizations of Algebras

R is a ring with unity and $S \neq \emptyset$ a multiplicatively closed subset of R such that $0 \notin S,\ 1_R \in S$ where I_R the identity in R. A nonzero element a in a ring R is said to be a left [resp. right] zero divisor if there exists a nonzero $b \in R$ such that $ab = 0$ [resp. $ba = 0$]. A zero divisor is an element of R which is both a left and a right zero divisor.

Definition 1. A ring Q is said to be a left quotient ring of R with respect to S if there exists a ring homomorphism $\varphi : R \longrightarrow Q$ such that the following conditions are satisfied:

(1) $\varphi(s)$ is a unit in Q for all $s \in S$ (This means that $a = \varphi(s)$ is both left and right invertible i.e. $\exists c \in Q$(resp. $b \in Q$) such that $ca = I_Q$(resp. $ab = I_Q$).);
(2) every element of Q is in the form $(\varphi(s))^{-1}\varphi(r)$, for some $r \in R,\ s \in S$;
(3) $\ker\varphi = \{r \in R :\ sr = 0, \text{for some s} \in S\}$.

The left (resp. right) quotient ring of R w.r.t. S, if it exists, is called the left (resp. right) localization of R at S and it is denoted by $S^{-1}R$ (resp. RS^{-1}). If $S = R$, the localization $S^{-1}R$ is the left skew field of fractions of R i.e.the left quotient field of R.

Note that if $rs = 0$, for some $r \in R,\ s \in S$, then $s'r = 0$, for some $s' \in S$. This is because $0 = \varphi(rs) = \varphi(r)\varphi(s)$ and thus $\varphi(r) = 0$, since, by condition 1), $\varphi(s)$ is a unit of Q. We need to multiply fractions like $(s^{-1}a)(s'^{-1}b)$ so we must be able to move s'^{-1} to the other side of a. This leads to the Ore condition: $Ra \cap Rs \neq \emptyset$ for $a \in R$ and $s \in S$. It is a necessary and sufficient condition for the existence of localizations [7].

Integral (no zero divisors) Noetherian rings satisfy the Ore condition [8], so that we can construct localizations. Examples of Noetherian rings include enveloping algebras of finite dimensional Lie algebras and, at least for semisimple ones, their q-deformations. For a proof that $U_q(\mathfrak{sl}(2))$ is Noetherian see [9]. It is easy to adapt the just mentioned proof in [9] for $U_q(\mathfrak{sl}(2))$ to the case of $U_q(\mathfrak{osp}(1|2)$ in order to show that the Ore condition also holds for $U_q(\mathfrak{osp}(1|2)$. In what follows we need to consider algebraic extensions (e.g. extensions of quotient rings obtain by adjoining square roots of operators) of localizations and we sometimes refer to these also as localizations.

It is important for us to know when a given representation of a ring R lifts to a representation of its localization. Suppose $f : R \longrightarrow R_1$ is a ring homomorphism and $Q = S^{-1}R\ \left(RS^{-1}\right)$ is a left (right) quotient ring of R with respect to S. If Q is a left quotient ring of R and φ is the map in Definition 2.1, then for all $r \in R$, $s \in S$ we define $g : Q \longrightarrow R_1$ by $f(s)g((\varphi(s))^{-1}\varphi(r)) := f(r)$ and similarly for right quotient rings.

Lemma 1. *If $f(s)$ is a unit in R_1 for every $s \in S$, then g is well-defined and is the unique ring homomorphism $g : Q \longrightarrow R_1$ which extends f.*

A proof of this Lemma can be found in [10].

5 Homomorphisms of $U_q(\mathfrak{sl}(2))$ and $U(\mathfrak{iso}(2))$ into Localizations of $U(\mathfrak{iso}(2))$ and $U_q(\mathfrak{sl}(2))$

We shall make frequent use of the following result and it is readily established by using Eq. (4) and the Maclaurin series formula: let f be any analytic function, then

$$\mathbf{P}^{\pm} f\left(\left[\frac{H}{2}\right]_q\right) \;=\; f\left(\left[\frac{H\mp 2}{2}\right]_q\right)\mathbf{P}^{\pm} \;. \tag{6}$$

We shall also make use of a similar equation for $U_q(\mathfrak{sl}(2))$; it is the same as Eq. (6) but with $\mathbf{P}^{\pm}$ replaced by $X^{\pm}$.

We define

$$\mathbf{X}^{\pm} = \mathbf{L}_{13} \mp i\mathbf{L}_{23} = \left\{\pm\frac{1}{Y}\left[\frac{H\mp 1}{2}\right]_{q^2} + I\right\}\mathbf{P}^{\pm} = \mathbf{P}^{\pm}\left\{\pm\frac{1}{Y}\left[\frac{H\pm 1}{2}\right]_{q^2} + I\right\} \tag{7}$$

where Y is a solution of the algebraic equation $Y^2 - \mathbf{P}^{+}\mathbf{P}^{-} = 0$ in $U(\mathfrak{iso}(2))$. Note that we used Eq. (6) to obtain the last equality. To make sense out of quantities like $\left[\frac{H\pm 1}{2}\right]_{q^2}$ in Eq. (7) as elements of $U(\mathfrak{iso}(2))$ we consider formal series expansions in H. This requires going to an extension of $U(\mathfrak{iso}(2))$ which allows for such arbitrary formal series (cf. [11]). To keep things simple we do not make a distinction between enveloping algebras and necessary such extensions for incorporating formal series expansions. (Observe that similar observations apply to the same quantities viewed as elements of $U_q(\mathfrak{sl}(2))$.) Let $\tau(X^{\pm}) = \mathbf{X}^{\pm}$ and $\tau(H) = H$. This defines a mapping τ from $U_q(\mathfrak{sl}(2))$ into an algebraic extension of the localization of $U(\mathfrak{iso}(2))$ with denominators consisting of powers of Y.

Proposition 1. *τ is a homomorphism from $U_q(\mathfrak{sl}(2))$ onto its image. In particular the $\mathbf{X}^{\pm}$ defined by Eq. (7) together with H satisfy the relations, Eqs. (1) and (2), of the generators of $U_q(\mathfrak{sl}(2))$. Furthermore, let Δ_q be defined by Eq. (3) but with $\mathbf{X}^{\pm}$ replacing $X^{\pm}$. Then $\Delta_q = Y^2 - \frac{1}{4}\cdot I$.*

The only difficult part of the proof of this proposition is to show that the $\mathbf{X}^{\pm}$ and H satisfy the defining relations, Eqs. (1) and (2), of $U_q(\mathfrak{sl}(2))$ and for this we refer the reader to [2].

Now let Y be such that it commutes with all elements of $U_q(\mathfrak{sl}(2))$ and satisfies the equation

$$Y^2 \;=\; \Delta_q \;+\; \frac{1}{4}\cdot I \;, \tag{8}$$

and let $P^{\pm} = (D_L^{\pm})^{-1}X^{\pm} = X^{\pm}(D_R^{\pm})^{-1}$ with $D_L^{\pm} = \left(\pm\frac{1}{Y}\left[\frac{H\mp 1}{2}\right]_{q^2} + I\right)$ and $D_R^{\pm} = \left(\pm\frac{1}{Y}\left[\frac{H\pm 1}{2}\right]_{q^2} + I\right)$ and define τ' by $\tau'(\mathbf{P}^{\pm}) = P^{\pm}$ and $\tau'(H) = H$.

Proposition 2. *τ' extends by linearity to a homomorphism from $\mathfrak{iso}(2)$ into a localization of $U_q(\mathfrak{sl}(2))$. In particular, $\tau'(\boldsymbol{P}^{\pm})$ and $\tau'(H)$ satisfy the commutation relations, Eqs. (4), of $\mathfrak{iso}(2)$ and, furthermore, $P^+P^- = Y^2$.*

For the proof we again refer to [2].

6 Homomorphisms of $U_{\tilde{q}}(\mathfrak{osp}(1|2))$ into Localizations of $U\left(\widetilde{\mathfrak{iso}}(2)\right)$ and $U\left(\mathfrak{iso}(2)\right)$

We start by establishing homomorphisms from $\widetilde{\mathfrak{iso}}(2)$ and $\mathfrak{iso}(2)$ onto their images in certain spaces which we now make precise. Define τ_0 by $\tau_0(H) = -2\tilde{H}$, $\tau_0(\mathbf{P}^+) = -\tilde{\mathbf{P}}^-$, $\tau_0(\mathbf{P}^-) = e^{-i\pi\tilde{H}}\tilde{\mathbf{P}}^+$ and $\tilde{\tau}_0$ by $\tilde{\tau}_0(\tilde{H}) = -\frac{1}{2}H$, $\tilde{\tau}_0(\tilde{\mathbf{P}}^-) = -\mathbf{P}^+$, $\tilde{\tau}_0(\tilde{\mathbf{P}}^+) = e^{-\frac{i\pi}{2}H}\mathbf{P}^-$. (We again refer the reader to [11] in order to give a precise meaning to expressions like $e^{-\frac{i\pi}{2}H}$ and $e^{-i\pi\tilde{H}}$.)

Proposition 3. *τ_0 and $\tilde{\tau}_0$ define Lie algebra and Lie super algebra homomorphisms from $\mathfrak{iso}(2)$ and $\widetilde{\mathfrak{iso}}(2)$ onto their images in $U\left(\widetilde{\mathfrak{iso}}(2)\right)$ and $U(\mathfrak{iso}(2))$, respectively.*

The proof of the proposition is easy. We extend τ_0 and $\tilde{\tau}_0$ by linearity to $\mathfrak{iso}(2)$ and $\widetilde{\mathfrak{iso}}(2)$, respectively, and verify the respective commutation relations.

We now describe homomorphisms of $U_{\tilde{q}}(\mathfrak{osp}(1|2))$ and $U\left(\widetilde{\mathfrak{iso}}(2)\right)$ into extensions of localizations of $U\left(\widetilde{\mathfrak{iso}}(2)\right)$ and $U_{\tilde{q}}(\mathfrak{osp}(1|2))$, respectively, i.e. the analogs of Propositions 1 and 2. Let $\tilde{\mathbf{X}}^{\pm} =$

$$\left(\frac{1}{\tilde{Y}}\sqrt{e^{i\pi\tilde{H}}\left[\tilde{H} \mp \frac{1}{2}\right]_{\tilde{q}^2}} + \sqrt{\pm I}\right)\frac{\tilde{\mathbf{P}}^{\pm}}{\sqrt{[2]_{\tilde{q}}}} =$$

$$= \frac{\tilde{\mathbf{P}}^{\pm}}{\sqrt{[2]_{\tilde{q}}}}\left(\frac{\pm i}{\tilde{Y}}\sqrt{e^{i\pi\tilde{H}}\left[\tilde{H} \pm \frac{1}{2}\right]_{\tilde{q}^2}} + \sqrt{\pm I}\right) \qquad (7^{bis})$$

where I is the identity in $U\left(\widetilde{\mathfrak{iso}}(2)\right)$. In obtaining the last term of this equation we used the equations $\sqrt{[\tilde{H} \mp \frac{1}{2}]_{\tilde{q}^2}}\tilde{\mathbf{P}}^{\pm} = \tilde{\mathbf{P}}^{\pm}\sqrt{[\tilde{H} \pm \frac{1}{2}]_{\tilde{q}^2}}$ and $e^{i\pi\frac{\tilde{H}}{2}}\tilde{\mathbf{P}}^{\pm} = \pm i\tilde{\mathbf{P}}^{\pm}e^{i\pi\frac{\tilde{H}}{2}}$ which equations follow easily from Eq. (4^{bis}).

Proposition 4. *If $\tilde{Y}$ is such that it commutes with all elements of $U\left(\widetilde{\mathfrak{iso}}(2)\right)$ and satisfies Eq. (5^{bis}), then Eq. (7^{bis}) define a homomorphism $\tilde{\tau}$ from $U_{\tilde{q}}(\mathfrak{osp}(1|2))$ into a localization of $U\left(\widetilde{\mathfrak{iso}}(2)\right)$, with $\tilde{\tau}(\tilde{X}^{\pm}) = \tilde{\mathbf{X}}^{\pm}$ and $\tilde{\tau}(\tilde{H}) = \tilde{H}$. Furthermore, let $\tilde{\mathbf{S}}_{\tilde{q}}$ be defined by Eq. (3^{bis}) but with $\tilde{\mathbf{X}}^{\pm}$ replacing $\tilde{X}^{\pm}$, then $\tilde{\mathbf{S}}_{\tilde{q}} = -e^{-i\pi\tilde{H}}\tilde{Y}^2$.*

The proof of this proposition is straightforward and is very similar to the proof of Proposition 1. The main part of the proof is to show that the $\tilde{\mathbf{X}}^{\pm}$ defined by Eq. (7^{bis}) together with $\tilde{H}$ satisfy the relations, Eqs. (1^{bis}) and (2^{bis}), of the generators of $U_{\tilde{q}}(\mathfrak{osp}(1|2))$.

Now let $\tilde{P}^{\pm} = (\tilde{D}_L^{\pm})^{-1}\tilde{X}^{\pm} = \tilde{X}^{\pm}(\tilde{D}_R^{\pm})^{-1}$ with

$$\sqrt{[2]_{\tilde{q}}}\,\tilde{D}_L^{\pm} = \frac{1}{\tilde{Y}}\sqrt{e^{i\pi\tilde{H}}\left[\tilde{H} \mp \frac{1}{2}\right]_{\tilde{q}^2}} + \sqrt{\pm I} \quad \text{and}$$

$$\sqrt{[2]_{\tilde{q}}}\,\tilde{D}_R^{\pm} = \frac{\pm i}{\tilde{Y}}\sqrt{e^{i\pi\tilde{H}}\left[\tilde{H} \pm \frac{1}{2}\right]_{\tilde{q}^2}} + \sqrt{\pm I},$$

where now I is the identity in $U_{\tilde{q}}(\mathfrak{osp}(1|2))$.

Proposition 5. *Let $\tilde{\tau}'(\tilde{\mathbf{P}}^{\pm}) = \tilde{P}^{\pm}$ and $\tilde{\tau}'(\tilde{H}) = \tilde{H}$. If $\tilde{Y}^2$ is such that it commutes with all elements of $U_{\tilde{q}}(\mathfrak{osp}(1|2))$ and satisfies*

$$\tilde{Y}^2 + e^{i\pi\tilde{H}}\tilde{S}_{\tilde{q}} = 0, \tag{8^{bis}}$$

then $\tilde{\tau}'$ extends to a homomorphism of $\widetilde{\mathfrak{iso}}(2)$ onto its image in a localization of $U_{\tilde{q}}(\mathfrak{osp}(1|2))$. In particular $\tilde{P}^{\pm}$ and $\tilde{H}$ satisfy the commutation relations (4^{bis}) of $U\big(\widetilde{\mathrm{iso}}(2)\big)$ and, furthermore, $\tilde{P}^+\tilde{P}^- = i e^{-i\pi\tilde{H}}\tilde{Y}^2$.

Proof. We shall establish that $\{\tilde{P}^+, \tilde{P}^-\} = 0$, since the rest of the proof is straightforward. From the definitions of $\tilde{P}^+$ and $\tilde{P}^-$ we have:

$$
\begin{aligned}
&[2]_{\tilde{q}}(\tilde{P}^+\tilde{P}^- + \tilde{P}^-\tilde{P}^+) = \qquad (9)\\
&= \frac{1}{\left(\frac{1}{\tilde{Y}}\sqrt{(-1)^{\tilde{H}}\left[\tilde{H}-\frac{1}{2}\right]_{\tilde{q}^2}} + I\right)}\tilde{X}^+\tilde{X}^-\frac{1}{\left(\frac{-i}{\tilde{Y}}\sqrt{(-1)^{\tilde{H}}\left[\tilde{H}-\frac{1}{2}\right]_{\tilde{q}^2}} + i\,I\right)} + \\
&+ \frac{1}{\left(\frac{1}{\tilde{Y}}\sqrt{(-1)^{\tilde{H}}\left[\tilde{H}+\frac{1}{2}\right]_{\tilde{q}^2}} + i\,I\right)}\tilde{X}^-\tilde{X}^+\frac{1}{\left(\frac{i}{\tilde{Y}}\sqrt{(-1)^{\tilde{H}}\left[\tilde{H}+\frac{1}{2}\right]_{\tilde{q}^2}} + I\right)} = \\
&= i\,\tilde{Y}^2\tilde{X}^+\tilde{X}^-\frac{1}{\left(\sqrt{(-1)^{\tilde{H}}\left[\tilde{H}-\frac{1}{2}\right]_{\tilde{q}^2}} + \tilde{Y}\right)\left(\sqrt{(-1)^{\tilde{H}}\left[\tilde{H}-\frac{1}{2}\right]_{\tilde{q}^2}} - \tilde{Y}\right)} - \\
&- i\,\tilde{Y}^2\tilde{X}^-\tilde{X}^+\frac{1}{\left(\sqrt{(-1)^{\tilde{H}}\left[\tilde{H}+\frac{1}{2}\right]_{\tilde{q}^2}} + i\,\tilde{Y}\right)\left(\sqrt{(-1)^{\tilde{H}}\left[\tilde{H}+\frac{1}{2}\right]_{\tilde{q}^2}} - i\,\tilde{Y}\right)} = \\
&= i\,\tilde{Y}^2\left\{\frac{\tilde{X}^+\tilde{X}^-}{\left((-1)^{\tilde{H}}\left[\tilde{H}-\frac{1}{2}\right]_{\tilde{q}^2} - \tilde{Y}^2\right)} - \frac{\tilde{X}^-\tilde{X}^+}{\left((-1)^{\tilde{H}}\left[\tilde{H}+\frac{1}{2}\right]_{\tilde{q}^2} + \tilde{Y}^2\right)}\right\} =
\end{aligned}
$$

$$= i\ \tilde{Y}^2 \left\{ \frac{\tilde{X}^+\tilde{X}^-}{\left((-1)^{\tilde{H}}\left[\tilde{H}-\frac{1}{2}\right]_{\tilde{q}^2}+(-1)^{\tilde{H}}\tilde{S}_q\right)} - \right.$$

$$\left. - \frac{\tilde{X}^-\tilde{X}^+}{\left((-1)^{\tilde{H}}\left[\tilde{H}+\frac{1}{2}\right]_{\tilde{q}^2}-(-1)^{\tilde{H}}\tilde{S}_q\right)} \right\} =$$

$$= \frac{i\ \tilde{Y}^2}{(-1)^{\tilde{H}}} \left\{ \frac{\tilde{X}^+\tilde{X}^-}{\left(\left[\tilde{H}-\frac{1}{2}\right]_{\tilde{q}^2}+\tilde{S}_q\right)} - \frac{\tilde{X}^-\tilde{X}^+}{\left(\left[\tilde{H}+\frac{1}{2}\right]_{\tilde{q}^2}-\tilde{S}_q\right)} \right\}$$

where in obtaining the second to last line we used Eq. (8^{bis}). Now use Eq. (3^{bis}) twice to obtain the desired result.

7 Representations

Let R be $U_q(\mathfrak{sl}(2))$ and $\tilde{R}$ be $U_{\tilde{q}}(\mathfrak{osp}(1|2))$. Consider an R ($\tilde{R}$) module, V ($\tilde{V}$). For $\lambda \in \mathbb{C}$ define the λ-weight space of V ($\tilde{V}$) to be $V_\lambda := \{e_n \in V | He_n = \lambda e_n\}$ ($\tilde{V}_\lambda := \{e_n \in \tilde{V} | \tilde{H}e_n = \lambda e_n\}$). We call V ($\tilde{V}$) a *highest weight module* if there exists a λ_0 such that (1) dim $V_{\lambda_0} = 1$ (dim $\tilde{V}_{\lambda_0} = 1$), (2) $V = RV_{\lambda_0}$ ($\tilde{V} = \tilde{R}\tilde{V}_{\lambda_0}$) and (3) if $V_\mu \neq 0$ ($\tilde{V}_\mu \neq 0$), then $\lambda_0 - \mu \in \mathbb{N} \cup \{0\}$. The λ_0 which satisfies these conditions is unique and is called the *highest weight* of V ($\tilde{V}$).

Set $\mathfrak{b} := \mathbb{C}H + \mathbb{C}X^+$. This is the analog of a Borel subalgebra of $\mathfrak{sl}(2)$. If $\lambda \in \mathbb{C}$ write $\mathbb{C}_\lambda$ for the one dimensional $\mathfrak{b}$-module killed by $H - \lambda I$ and X^+. The *Verma module* of highest weight $\lambda_0 \in \mathbb{C}$ is $V(\lambda_0) := R \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda_0}$. It follows that $V(\lambda_0) \simeq \mathbb{C}[X^-]$ as a left $\mathbb{C}[X^-]$-module (cf.[9]). Set $\mathbb{C}e_{\lambda_0} := I \otimes \mathbb{C}_\lambda$. Each $(X^-)^k e_{\lambda_0}$ is of weight $\lambda_0 - k$ and $V(\lambda_0) = \sum_{k\in\mathbb{N}\cup\{0\}}^{\oplus} V_{\lambda_0-k}$ where $V_{\lambda_0-k} = \mathbb{C}e_{\lambda_0-k}$. $V(\lambda_0)$ is a highest weight module with highest weight λ_0 and $\dim_{\mathbb{C}} V_{\lambda-k} = 1$ for each $k \in \mathbb{N} \cup \{0\}$. The action of X^+ on a weight vector increases its weight by 1. For the analogous construction of the Verma module of highest weight λ_0 for $\tilde{R}$ simply replace everywhere H, X^+ and X^- by $\tilde{H}$, $\tilde{X}^+$ and $\tilde{X}^-$. Call it $\tilde{V}(\lambda_0)$.

It is well-known that, at least for q not a root of unity, the following statements are equivalent (compare [5]): (1) The R-module, $V(\lambda_0)$ ($\tilde{R}$-module, $\tilde{V}(\lambda_0)$), is reducible; (2) $V(\lambda_0)$ $\left(\tilde{V}(\lambda_0)\right)$ admits a singular vector $(X^-)^\ell e_{\lambda_0}$ $\left((\tilde{X}^-)^\ell e_{\lambda_0}\right)$ for $\ell \in \mathbb{N}$ ($\ell > 0$); (3) The unique irreducible quotient module of $V(\lambda_0)$ $\left(\tilde{V}(\lambda_0)\right)$, which we denote by $W(\lambda_0)$ $\left(\tilde{W}(\lambda_0)\right)$, is finite dimensional.

We now prove that we cannot get representations of $\widetilde{\mathfrak{iso}}(2)$ from highest weight representations of $U_{\tilde{q}}(\mathfrak{osp}(1|2))$ by using Proposition 5. Using the fact that every highest weight module is a quotient module of a Verma module, it suffices to show that the action of $\tilde{D}_R^+$ in $\tilde{V}(\lambda_0)$ vanishes on e_{λ_0} for any $\lambda_0 \in \mathbb{C}$, since then $\tilde{D}_R^+$

has no well-defined inverse in $\tilde{V}(\lambda_0)$ and we cannot use Proposition 5 to define an action of $\tilde{P}^+$ in $\tilde{V}(\lambda_0)$. Using the definition of $\tilde{D}_R^+$ we have:

$$\begin{aligned}\tilde{D}_R^+ e_{\lambda_0} &= \frac{i}{\sqrt{[2]_{\tilde{q}}}\,\tilde{Y}}\left(\sqrt{e^{i\pi\tilde{H}}\left[\tilde{H}+\frac{1}{2}\right]_{\tilde{q}^2}}-i\tilde{Y}\right)e_{\lambda_0} = \\ &= \frac{i}{\sqrt{[2]_{\tilde{q}}}\,\tilde{Y}}\left(\sqrt{e^{i\pi\tilde{H}}\left[\tilde{H}+\frac{1}{2}\right]_{\tilde{q}^2}}-\sqrt{e^{i\pi\tilde{H}}\tilde{S}_{\tilde{q}}}\right)e_{\lambda_0} = \\ &= \frac{i}{\sqrt{[2]_{\tilde{q}}}\,\tilde{Y}}\left(\sqrt{e^{i\pi\tilde{H}}\left[\tilde{H}+\frac{1}{2}\right]_{\tilde{q}^2}}-\sqrt{e^{i\pi\tilde{H}}\left[\tilde{H}+\frac{1}{2}\right]_{\tilde{q}^2}}\right)e_{\lambda_0} = 0.\end{aligned}$$

A similar computation establishes the analogous result that we cannot get representations of $\mathfrak{iso}(2)$ out of highest weight representations of $U_q(\mathfrak{sl}(2))$ by using Proposition 2.

We shall now make use of the Propositions to construct new representations of $U_{\tilde{q}}(\mathfrak{osp}(1|2))$ out of representations of $\mathfrak{iso}(2)$ and also representations of $\mathfrak{iso}(2)$ out of representations of $U_q(\mathfrak{sl}(2))$. Combining these two results we can thus obtain new representations of $U_{\tilde{q}}(\mathfrak{osp}(1|2))$ out of representations of $U_q(\mathfrak{sl}(2))$.

We first construct representations of $\mathfrak{iso}(2)$ from representations of $U_q(\mathfrak{sl}(2))$. Let $\mathcal{H}_{(m,\epsilon)}$ be the one dimensional vector space $\mathbb{C}e_m$ with $m = n+\epsilon, n = 0, \pm1, \pm2, \ldots$ for fixed $\epsilon = 0$ or $\frac{1}{2}$. For $\sigma \in \mathbb{C}$ and for $q \in \mathbb{C}$, $q \neq 0$ and not a root of unity, the following formulae define a representation $d\pi^{\sigma,\epsilon}$ of $U_q(\mathfrak{sl}(2))$ on the space $\oplus_{n\in\mathbb{Z}}\mathcal{H}_{(m,\epsilon)}$:

$$d\pi^{\sigma,\epsilon}(H)e_m = 2m\, e_m\ ,\ d\pi^{\sigma,\epsilon}(X^{\pm})e_m = [-\sigma \pm m]_{q^2} e_{m\pm1}\ . \tag{10}$$

Recall the real form $U_q(\mathfrak{so}(2,1))$ of $U_q(\mathfrak{sl}(2))$ introduced in Sect. 2. For $|q| = 1$ and q not a root of unity we have the following [3]: (1) for $\sigma = i\rho - \frac{1}{2}$ ($\rho \in \mathbb{R}$), the representation space is $\mathcal{H}^{(\sigma,\epsilon)} = \sum_m{}^{\oplus}\mathcal{H}_{(m,\epsilon)}$ and $d\pi^{\sigma,\epsilon}$ is the (infinitesimally unitarizable) principal series of $U_q(\mathfrak{so}(2,1))$; (2) for $\sigma = \epsilon$ mod(2) and $\sigma = \ell$ with $\ell < -\frac{1}{2}$, (a) the representation space $X_+^{-\ell,\epsilon}$ is the linear span of the above e_m with $m > -\ell$, (b) the representation space $X_-^{-\ell,\epsilon}$ is the linear span of the e_m with $m < \ell$. $d\pi^{\sigma,\epsilon}$ acts irreducibly on $X_\pm^{-\ell,\epsilon}$. These give q deformed discrete series of $U_q(\mathfrak{so}(2,1))$.

We now construct representations of $\mathfrak{iso}(2)$ out of the (infinitesimally unitarizable) principal series representations of $U_q(\mathfrak{so}(2,1))$ using Proposition 2. We start with a given such $d\pi^{\sigma,\epsilon}$. A simple calculation using Eqs. (3), (8) and (9) and Eq. (5) of [12] shows $Y^2 = ([i\rho]_{q^2})^2\ I$ on $\mathcal{H}^{(\sigma,\epsilon)}$ (I is the identity on $\mathcal{H}^{(\sigma,\epsilon)}$). It follows that the actions of $(D_R^{\pm})^{-1}$ in the representation exist as operators on the representation space. This is seen as follows: $d\pi^{\sigma,\epsilon}([\frac{H\pm1}{2}]_{q^2})\ |m\ >= \frac{\sin((m\pm\frac{1}{2})\alpha)}{\sin\alpha})\ |m\ >\ \ (q = e^{i\alpha})$, and $Y = i\frac{\sinh\alpha\rho}{\sin\alpha}I$ (taking the positive square root of Y^2). It follows that the eigenvalues of $d\pi^{\sigma,\epsilon}(\mp\frac{1}{Y}[\frac{H\pm 1}{2}]_{q^2}+I)$ are never zero and Y^{-1} is well-defined on $\mathcal{H}^{(\sigma,\epsilon)}$ provided $\rho \neq 0$. Thus, $d\pi^{\sigma,\epsilon}(\mp\frac{1}{Y}[\frac{H\pm 1}{2}]_{q^2}+I)$ has no nonzero eigenvalues and $d\pi^{\sigma,\epsilon}(D)$ is invertible. Using Proposition 2 and Eq. (9) , we can easily write down the action of the $\mathbf{P}^{\pm}$ on the e_m of the representation space $\mathcal{H}^{(\sigma,\epsilon)}$. We find:

$$d\tilde{\pi}_q^{\sigma,\epsilon}(\mathbf{P}^{\pm})\, e_m = -\left(\frac{[i\rho]_{q^2}[\sigma\mp m]_{q^2}}{\left[\sigma+\frac{1}{2}\right]_{q^2}\mp\left[m\pm\frac{1}{2}\right]_{q^2}}\right) e_{m\pm 1}\ . \tag{11}$$

Finally, using Propositions 3 and 4 we construct representations of$U_{\tilde{q}}(\mathfrak{osp}(1|2))$ out of representations of $U(\mathfrak{iso}(2))$ and $U\left(\widetilde{\mathfrak{iso}}(2)\right)$. We start with the positive mass representations of the Euclidean Lie algebra. They are characterized by a real number ρ ($\rho \neq 0$)and an integer ϵ which is either 0 or $\frac{1}{2}$. They are described as follows [13]. The representation space is $\mathcal{H}^{(i\rho,\epsilon)} = \sum_m{}^{\oplus}\mathcal{H}_{(m,\epsilon)}$ where the $\mathcal{H}_{(m,\epsilon)}$ are the same one dimensional vector spaces introduced above. The actions of the generators of $U(\mathfrak{iso}(2))$ on $\mathcal{H}^{(i\rho,\epsilon)}$ are given by

$$d\pi^{\rho,\epsilon}(\mathbf{P}^{\pm})\, e_m = -\ (i\ \rho)\, e_{m\pm 1} \tag{12}$$

$$d\pi^{\rho,\epsilon}(H)\, e_m = 2m\, e_m. \tag{13}$$

Using Proposition 3 we obtain the following representation of $U\left(\widetilde{\mathfrak{iso}}(2)\right)$ on $\mathcal{H}^{(i\rho,\epsilon)}$:

$$d\tilde{\pi}^{\rho,\epsilon}(\tilde{\mathbf{P}}^{\pm})\, e_m = \mp\ (i\ \rho)\, e^{-i\frac{\pi}{2}\{(m-1)\pm(m-1)\}}\, e_{m\mp 1} \tag{14}$$

$$d\tilde{\pi}^{\rho,\epsilon}(\tilde{H})\, e_m = -m\, e_m\ . \tag{15}$$

Now use Proposition 4 together with the Lemma to obtain the representation of $U_{\tilde{q}}(\mathfrak{osp}(1|2))$ on $\mathcal{H}^{(i\rho,\epsilon)}$. We claim that the conditions of the Lemma are satisfied provided zero is in the resolvent set of $d\tilde{\pi}^{\rho,\epsilon}(\tilde{Y})$. It is easy to see that this is always the case for any nonzero ρ and any ϵ, since from Eq. (5^{bis}) we have $d\tilde{\pi}^{\rho,\epsilon}(\tilde{Y}^2)e_m = -i(-1)^{d\tilde{\pi}^{\rho,\epsilon}(\tilde{H})}d\tilde{\pi}^{\rho,\epsilon}(\tilde{\mathbf{P}}^{+})d\tilde{\pi}^{\rho,\epsilon}(\tilde{\mathbf{P}}^{-})e_m$, and using Eqs. (13) and (14) we easily obtain

$$d\tilde{\pi}^{\rho,\epsilon}(\tilde{Y}^2) = -i(-1)^{2\epsilon}\rho^2 I \tag{16}$$

where I is the identity operator on $\mathcal{H}^{(i\rho,\epsilon)}$. Hence, since $\rho \neq 0$, $d\tilde{\pi}^{\rho,\epsilon}(\tilde{Y}^2)$ is invertible and so also its square root

$$d\tilde{\pi}^{\rho,\epsilon}(\tilde{Y}) = i\sqrt{i}(-1)^{\epsilon}\rho\, I. \tag{17}$$

Clearly the image of $\tilde{Y}^2$ and its square root are units in the localization of the algebraic extension of $U\left(\widetilde{\mathfrak{iso}}(2)\right)$ obtained by adjoining the square root of $\tilde{Y}^2$ and from Eq. (16) we see that the conditions of the Lemma are satisfied. Using (7^{bis})

together with Eqs. (13), (14) and (16) we can explicitly construct the representation of $U_{\tilde{q}}(\mathfrak{osp}(1|2))$ on the above representation space. We obtain:

$$\tilde{X}^+ e_m = \frac{1}{\sqrt{[2]_{\tilde{q}}}}(-i\rho)\left\{\frac{(-1)^{m/2}\sqrt{[m-\frac{1}{2}]_{q^2}}}{i\sqrt{i}(-1)^{\epsilon}\rho} - 1\right\}(-1)^m e_{m-1} \tag{18}$$

$$\tilde{X}^- e_m = \frac{1}{\sqrt{[2]_{\tilde{q}}}}(i\rho)\left\{\frac{(-1)^{-m/2}\sqrt{[m+\frac{1}{2}]_{q^2}}}{i\sqrt{i}(-1)^{\epsilon}\rho} + i\right\} e_{m+1} \tag{19}$$

with the action of $\tilde{H}$ on the representation space being given by Eq. (14). These representations of $U_{\tilde{q}}(\mathfrak{osp}(1|2))$ seem to be new.

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On the 2D Zero Modes' Algebra of the SU(n) WZNW Model

Ludmil Hadjiivanov and Paolo Furlan

Abstract A quantum group covariant extension of the chiral parts of the Wess-Zumino-Novikov-Witten (WZNW) model on a compact Lie group G gives rise to two matrix algebras with non-commutative entries. These are generated by "chiral zero modes" a^i_α , $\bar{a}^\beta_j$ which combine, in the $2D$ model, into $Q^i_j = a^i_\alpha \otimes \bar{a}^\alpha_j$. The Q-operators provide important information about the internal symmetry and the fusion ring. Here we review earlier results about the $SU(n)$ WZNW Q-algebra and its Fock representation for $n = 2$ and make the first steps towards their generalization to $n \geq 3$.

1 Introduction

The object of our study, the "zero modes", appear naturally in the splitting of the (single valued) $2D$ WZNW field $G(x, \bar{x}) = (G^A_B(x, \bar{x}))$ into left and right quantum group covariant chiral components $g^A_\alpha(x)$ and $\bar{g}^\alpha_B(\bar{x})$. The latter are necessarily quasiperiodic, i.e. have monodromies: for example, $g^A_\alpha(x + 2\pi) = g^A_\beta(x)\, M^\beta_\alpha$. The *chiral zero modes* $a = (a^i_\alpha)$ and $\bar{a} = (\bar{a}^\alpha_j)$ are assumed to diagonalize the left and right monodromy matrices, respectively, so that

$$G^A_B(x, \bar{x}) = g^A_\alpha(x) \otimes \bar{g}^\alpha_B(\bar{x}) = u^A_i(x) \otimes Q^i_j \otimes \bar{u}^j_B(\bar{x})\,, \quad Q^i_j := a^i_\alpha \otimes \bar{a}^\alpha_j \tag{1}$$

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(summation over repeated upper and lower indices is implicitly understood), where the chiral fields $u(x)$ and $\bar{u}(\bar{x})$ have diagonal monodromies. We call hereafter $Q := (Q^i_j)$ the matrix of $2D$ *WZNW zero modes*.

The concept of chiral WZNW zero modes, classical or quantum, appeared in [1, 2, 6], and has been developed further in [12, 13, 15]. The Q-algebra has been studied, in the $SU(2)$ case, in [11]. It can be shown that a finite dimensional quotient of it, and the Fock representation thereof, provide a link to the internal symmetry and the fusion of the unitary WZNW model. We will describe below the first steps in attempt to extend this framework to $SU(n)$, $n \geq 3$.

2 Chiral WZNW Zero Modes

The (left sector) chiral zero modes' algebra $\mathcal{M}_q$ for the $SU(n)$ WZNW model at level k has been introduced in [15]. It is generated by the n mutually commuting operators q^{p_j} whose product is equal to the unit operator,

$$q^{p_i} q^{p_j} = q^{p_j} q^{p_i} \ , \qquad \prod_{j=1}^{n} q^{p_j} = 1 \ , \qquad j = 1, \ldots, n \ , \tag{2}$$

and by the entries of the $n \times n$ zero modes' quantum matrix $a = (a^i_\alpha)$ satisfying quadratic exchange relations,

$$\begin{aligned} & a^j_\beta a^i_\alpha \, [p_{ij} - 1] = a^i_\alpha a^j_\beta \, [p_{ij}] - a^i_\beta \, a^j_\alpha \, q^{\epsilon_{\alpha\beta} p_{ij}} \qquad (\text{for} \quad i \neq j \quad \text{and} \quad \alpha \neq \beta\,) \, , \\ & [a^j_\alpha , a^i_\alpha] = 0 \, , \qquad a^i_\alpha a^i_\beta = q^{\epsilon_{\alpha\beta}} \, a^i_\beta a^i_\alpha \, , \qquad i, j \, , \alpha \, , \beta = 1, \ldots, n \\ & (\, \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha} \, , \quad \epsilon_{\alpha\beta} = 1 \quad \text{for} \quad \alpha > \beta \, , \qquad [p] := \frac{q^p - q^{-p}}{q - q^{-1}}\,) \end{aligned} \tag{3}$$

the following mixed relations with q^{p_j},

$$q^{p_j} \, a^i_\alpha = q^{\delta^i_j - \frac{1}{n}} \, a^i_\alpha \, q^{p_j} \qquad \Rightarrow \qquad q^{p_{j\ell}} a^i_\alpha = q^{\delta^i_j - \delta^i_\ell} a^i_\alpha \, q^{p_{j\ell}} \quad (\, p_{j\ell} := p_j - p_\ell\,) \tag{4}$$

and the (n-linear in the zero modes) inhomogeneous *determinant condition*

$$\frac{1}{[n]!} \, \epsilon_{i_1 \ldots i_n} \, a^{i_1}_{\alpha_1} \ldots a^{i_n}_{\alpha_n} \, \varepsilon^{\alpha_1 \ldots \alpha_n} =: \det(a) = \mathcal{D}_q(p) := \prod_{i<j} [p_{ij}] \, . \tag{5}$$

The ε-tensor in (5) is totally q-antisymmetric,

$$\varepsilon^{\alpha_1 \ldots \alpha_i \alpha_{i+1} \ldots \alpha_n} = -q^{-\epsilon_{\alpha_i \alpha_{i+1}}} \, \varepsilon^{\alpha_1 \ldots \alpha_{i+1} \alpha_i \ldots \alpha_n} \ , \quad i = 1, \ldots, n-1 \tag{6}$$

its non-zero components being given by

$$\varepsilon^{\alpha_1 \dots \alpha_n} = q^{-\frac{n(n-1)}{4}} (-q)^{\ell(\sigma)} \qquad \text{for} \qquad \sigma = \begin{pmatrix} n & \dots & 1 \\ \alpha_1 & \dots & \alpha_n \end{pmatrix} \in \mathcal{S}_n \tag{7}$$

(the length $\ell(\sigma)$ of an element σ of the symmetric group $\mathcal{S}_n$ is equal to the number of *inversions* which, in our notation, are the pairs (α_i, α_j) such that $\alpha_i < \alpha_j$ for $i < j$) while $\epsilon_{i_1 \dots i_n} = (-1)^{\ell(\eta)}$ for $\eta = \left(\begin{smallmatrix} n & \dots & 1 \\ i_1 & \dots & i_n \end{smallmatrix} \right) \in \mathcal{S}_n$.

The exchange relations (3) originate from

$$\hat{R}_{12}(p)\, a_1\, a_2 = a_1\, a_2\, \hat{R}_{12} \qquad \Leftrightarrow \qquad \hat{R}^{ij}_{\ i'j'}(p)\, a^{i'}_\alpha a^{j'}_\beta = a^i_{\alpha'} a^j_{\beta'} \hat{R}^{\alpha'\beta'}_{\ \alpha\beta} \tag{8}$$

where $\hat{R}_{12} = P_{12} R_{12}$, $\hat{R}_{12}(p) = P_{12} R_{12}(p)$, P_{12} is the permutation matrix, R_{12} the Drinfeld-Jimbo quantum R-matrix for $U_q(s\ell(n))$ [4, 17] and $R_{12}(p)$ the corresponding *dynamical* quantum R-matrix [5, 15, 16]. Explicitly,

$$q^{-\frac{1}{n}}\, \hat{R}^{\alpha\beta}_{\alpha'\beta'} = \delta^\alpha_{\beta'} \delta^\beta_{\alpha'} + (q^{-1} - q^{-\epsilon_{\alpha\beta}})\, \delta^\alpha_{\alpha'} \delta^\beta_{\beta'}\,, \quad \epsilon_{\alpha\beta} = \begin{cases} 1\,, & \alpha > \beta \\ 0\,, & \alpha = \beta \\ -1\,, & \alpha < \beta \end{cases} \tag{9}$$

(our deformation parameter is $q = e^{-i\frac{\pi}{h}}$ where the *height* $h = k + n$) and

$$\begin{aligned} & q^{-\frac{1}{n}}\, \hat{R}^{ij}_{\ i'j'}(p) \ = \ a_{ij}(p)\, \delta^i_{j'} \delta^j_{i'} + b_{ij}(p)\, \delta^i_{i'} \delta^j_{j'}\,, \\ & a_{ii}(p) = q^{-1}\,, \quad a_{ij}(p) = \alpha(p_{ij})\, \frac{[p_{ij} - 1]}{[p_{ij}]}\,, \quad i \neq j \qquad (\alpha(p_{ji}) = \frac{1}{\alpha(p_{ij})})\,, \\ & b_{ii}(p) = 0\,, \quad b_{ij}(p) = \frac{q^{-p_{ij}}}{[p_{ij}]}\,, \quad i \neq j\,, \end{aligned} \tag{10}$$

respectively. Indeed, getting rid of the denominators in (10) and using the identity $[p-1] - q^{\pm 1}[p] = -q^{\pm p}$, we obtain (3) for $\alpha(p_{ij}) = 1$.

The right zero modes' algebra $\bar{\mathcal{M}}_q$ is generated by $\bar{a} = (\bar{a}^\alpha_i)$ and $q^{\bar{p}_j}$. The relevant relations follow from the left sector's ones according to the rules

$$q \ \to \ q^{-1}\,, \qquad (a^{-1})^\alpha_i \ \to \ \bar{a}^\alpha_i\,, \qquad q^{p_j} \ \to \ q^{\bar{p}_j} \tag{11}$$

which can be justified e.g. by examining the classical chiral symplectic forms and the subsequent canonical quantization procedure [12]. Thus $q^{\bar{p}_j}$ satisfy relations identical to (2) as well as mixed exchange relations

$$q^{\bar{p}_j}\, \bar{a}^\alpha_i = q^{\delta_{ij} - \frac{1}{n}}\, \bar{a}^\alpha_i\, q^{\bar{p}_{ij}} \qquad \Rightarrow \qquad q^{\bar{p}_{j\ell}}\, \bar{a}^\alpha_i = q^{\delta_{ij} - \delta_{i\ell}}\, \bar{a}^\alpha_i\, q^{\bar{p}_{j\ell}}\,. \tag{12}$$

The right sector counterpart of (8) has the form

$$\hat{R}_{12}\,\bar{a}_1\,\bar{a}_2 \;=\; \bar{a}_1\,\bar{a}_2\,\hat{\bar{R}}_{12}(\bar{p})\;. \tag{13}$$

The fact that the *constant* R-matrices in (8) and (13) are the same ensures the local commutativity of the $2D$ field (1); there is no such requirement however for the *dynamical* ones. Inserting explicitly the α-dependence in the notations of the dynamical R-matrices (so that e.g. $\hat{R}_{12}(p) \equiv \hat{R}^{(1)}_{12}(p)$ for $\alpha(p_{ij}) = 1$ in (10)) we observe that (13) becomes *identical* to (8) if we choose $\hat{\bar{R}}_{12}(\bar{p}) = {}^t\hat{R}_{12}(\bar{p}) \equiv \hat{R}^{(\alpha)}_{12}(\bar{p})$ for

$$\alpha(\bar{p}_{ij}) = \frac{[\bar{p}_{ij}+1]}{[\bar{p}_{ij}-1]}\ \ (= \frac{1}{\alpha(\bar{p}_{ji})})\,. \tag{14}$$

To this end we note that the constant R-matrix (9) is symmetric ($\hat{R}^{\alpha\beta}_{\ \ \alpha'\beta'} = \hat{R}^{\alpha'\beta'}_{\ \ \alpha\beta}$, i.e. $\hat{R}_{12} = {}^t\hat{R}_{12}$) and that $q^{\bar{p}_{ij}}$ commutes with $\bar{a}^\alpha_i\bar{a}^\beta_j$, cf. (12), so there is no change in the argument of $\hat{\bar{R}}_{12}(\bar{p})$ when it is moved to the left of $\bar{a}_1\bar{a}_2$ in (13). Getting rid of the denominators, we obtain

$$\begin{aligned}&\bar{a}^\beta_j\bar{a}^\alpha_i\,[\hat{p}_{ij}-1] = \bar{a}^\alpha_i\bar{a}^\beta_j\,[\hat{p}_{ij}] - \bar{a}^\beta_i\,\bar{a}^\alpha_j\,q^{\epsilon_{\alpha\beta}\hat{\bar{p}}_{ij}} \qquad (\text{for}\quad i\neq j\quad \text{and}\quad \alpha\neq\beta)\,,\\ &[\bar{a}^\alpha_j,\bar{a}^\alpha_i] = 0\,,\qquad \bar{a}^\alpha_i\bar{a}^\beta_i = q^{\epsilon_{\alpha\beta}}\,\bar{a}^\beta_i\bar{a}^\alpha_i\,,\qquad \alpha,\beta,i,j = 1,\dots,n\,.\end{aligned} \tag{15}$$

That (3) and (15) coincide is a desirable result, as the left and the right sector quantities appear in (1) on equal footing. It also suggests that the definition of $\det(\bar{a})$ and the condition it satisfies are identical to (5), up to exchanging upper and lower indices; note that (7) implies

$$\varepsilon_{\alpha_1\dots\alpha_n} = \varepsilon^{\alpha_1\dots\alpha_n} \quad\Rightarrow\quad \varepsilon^{\alpha_1\dots\alpha_n}\,\varepsilon_{\alpha_1\dots\alpha_n} = [n]!\ := [n][n-1]\dots[1]\,. \tag{16}$$

The chiral matrix algebras generate Fock spaces $\mathcal{F}_q = \mathcal{M}_q\mid 0\rangle$ and $\bar{\mathcal{F}}_q = \bar{\mathcal{M}}_q\,|\,0\rangle$ with vacuum vector $|\,0\rangle$ satisfying

$$p_{ij}\ |0\rangle = (j-i)\ |0\rangle = \bar{p}_{ij}\ |0\rangle\,,\qquad a^i_\alpha\ |0\rangle = 0 = \bar{a}^\alpha_i\ |0\rangle\quad \text{for}\quad i\neq 1\,. \tag{17}$$

Justification of (17) can be found in [10, 12]; we will only note here that the eigenvalues of $p_{i\,i+1}$ and $\bar{p}_{i\,i+1}$, $i = 1,\dots,n$ play the role of *shifted integral* $s\ell(n)$ *weights*.

For $q^h = -1$, the condition $[p_{ij}]\,v = 0\ \ (i\neq j)$ for some vector $v\in\mathcal{F}_q\otimes\bar{\mathcal{F}}_q$ implies that $p_{ij}\,v = N h\,v$ for some integer N. One infers from (3) (and similarly, from(15)) that

$$[p_{ij}]\,v = 0\ \Rightarrow\ a^i_\alpha a^j_\beta\,v = a^j_\alpha a^i_\beta\,v\,,\quad [\bar{p}_{ij}]\,v = 0\ \Rightarrow\ \bar{a}^\alpha_i\bar{a}^\beta_j\,v = \bar{a}^\alpha_j\bar{a}^\beta_i\,v\,. \tag{18}$$

Let e.g. $\mathcal{J}_q^{(h)}$ be the two-sided ideal of $\mathcal{M}_q$ generated by the h-th powers of all a_α^i and the $2h$-th powers of $q^{p_{ij}}$. It is easy to see that the quotient $\mathcal{M}_q^{(h)} := \mathcal{M}_q/\mathcal{J}_q^{(h)}$ is non-trivial, due to the relation (valid for $i \neq j$, $\alpha \neq \beta$)

$$[p_{ij}-1](a_\beta^j)^m a_\alpha^i = a_\alpha^i (a_\beta^j)^m [p_{ij}] - [m](a_\beta^j)^{m-1} a_\beta^i \, a_\alpha^j \, q^{\epsilon_{\alpha\beta} p_{ij}} \tag{19}$$

generalizing the first Eq. (3) for any positive integer m. (Equation (19) is easily be proved by induction, using the q-number relation $[p+m] = [p][m+1] - [p-1][m]$.) By a similar construction we obtain the quotient right sector zero modes' algebra $\bar{\mathcal{M}}_q^{(h)}$. We further define restricted Fock spaces and their tensor product

$$\mathcal{F}_q^{(h)} \otimes \bar{\mathcal{F}}_q^{(h)} = \mathcal{M}_q^{(h)} \otimes \bar{\mathcal{M}}_q^{(h)} \, |0\rangle \tag{20}$$

on which the algebra of Q-operators will act.

3 Q-Algebra—The $n = 2$ Case

A great simplification in the $n = 2$ case comes from the fact that the exchange relations combine with the determinant condition (5), which in this case is also bilinear in the zero modes, to form powerful operator identities.

For $n = 2$ and $q = e^{\pm i \frac{\pi}{h}}$ the chiral Fock space $\mathcal{F}_q$ carries a representation of the $2h^3$-dimensional *restricted* quantum group $\overline{U}_q = \overline{U}_q(s\ell(2))$ generated by E, F, K such that $E^h = 0 = F^h$, $K^{2h} = 1$ [14]. The restricted Fock space $\mathcal{F}_q^{(h)}$ is h^2-dimensional. The entries of the $2D$ zero modes' matrix

$$Q = (Q^i_j) = \begin{pmatrix} Q^1_1 & Q^1_2 \\ Q^2_1 & Q^2_2 \end{pmatrix} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{21}$$

have the following properties [11].

- If $(a_\alpha^i)^h = 0 = (\bar{a}_j^\alpha)^h \ \ \forall\, \alpha \in \{1, 2\}$, then $(Q^i_j)^h = 0$.
- Diagonal and off-diagonal elements of Q commute:

$$AB = BA\,,\ CA = AC\,,\ BD = DB\,,\ CD = DC\,. \tag{22}$$

- The triples A, D, L and B, C, N, generate two commuting $\overline{U}_q$ algebras:

$$\begin{aligned} &[A, D] = [L]\,,\ \ LA = q^2 AL\,,\ \ LD = q^{-2} DL\,,\ \ L^{\pm 1} := -q^{\pm p} \otimes q^{\pm \bar{p}} \\ &[B, C] = [N]\,,\ \ NB = q^2 BN\,,\ \ NC = q^{-2} CN\,,\ \ N^{\pm 1} := -q^{\pm p} \otimes q^{\mp \bar{p}} \\ &A^h = D^h = 0 = B^h = C^h\,,\quad L^{2h} = 1 = N^{2h} \quad (p = p_{12}\,,\ \bar{p} = \bar{p}_{12})\,. \end{aligned} \tag{23}$$

- The vacuum representation of the off-diagonal Q-algebra is one-dimensional:

$$B\,|0\rangle = 0 = C\,|0\rangle\,,\quad N\,|0\rangle = -\,|0\rangle\qquad (\Rightarrow\ [N]\,|0\rangle = 0\,)\,. \tag{24}$$

- The diagonal Q-algebra generates an indecomposable representation of $\overline{U}_q$ (a *Verma module* [7,8] $\mathcal{V}_1^+$, $\dim \mathcal{V}_1^+ = h$, with a 1-dimensional submodule),

$$A\mid m\rangle = [m+1]\mid m+1\rangle\,,\quad D\mid m\rangle = [m+1]\mid m-1\rangle\quad (\,D\,|0\rangle = 0\,)\,,$$
$$(L + q^{2(m+1)})\mid m\rangle = 0\quad \text{for}\quad \mid m\rangle := \frac{A^m}{[m]!}\,|0\rangle\,,\quad m = 0,\dots,h-1\,. \tag{25}$$

- The invariant hermitean scalar product on (25) (s.t. $A^\dagger = D\,,\ L^\dagger = L^{-1}$) is semidefinite, the $\overline{U}_q$-invariant subspace $\mathbb{C}\,|\,h-1\rangle \subset \mathcal{V}_1^+$ being isotropic:

$$(m' \mid m) = [m+1]\,\delta_{mm'} \equiv \frac{\sin\,(m+1)\frac{\pi}{h}}{\sin\frac{\pi}{h}}\,\delta_{mm'}\,,\quad m = 0,\dots,h-1\,. \tag{26}$$

Note that the dimension of the quotient space $\mathcal{V}_1^+/\{\mathbb{C}\mid h-1\rangle\}$ coincides with the number $h-1 = k+1$ of (integrable) sectors in the unitary $\widehat{su}(2)_k$ WZNW model [3]. This is a manifestation of a much deeper result providing an interpretation analogous to covariant quantization of gauge theories [11]. Without going into details, we would like to call special attention to the fact that only the *diagonal* entries of the matrix Q (21) are represented non-trivially. It guarantees, together with (4) and (12), that the eigenvalues of p and $\bar{p}$ on the diagonal Q-vectors $\mid m\rangle$ (25) coincide.

4 Q-Algebra—The General n Case

The general n case is much harder to explore, partly because the n-linear determinant conditions for the chiral zero modes should be considered for $n \geq 3$ separately from the quadratic exchange relations. For this reason we will only comment below the extensions to higher n of the first two points listed in Sect. 3 for $n = 2$, leaving the rest for a future work.

It turns out that the generalization of the first one is straightforward.

Proposition. *If* $(a^i_\alpha)^h = 0 = (\bar{a}^\alpha_j)^h\quad \forall\,\alpha \in \{1,\dots,n\}$, *then* $(Q^i_j)^h = 0$.

Proof. The indices i and j play no role here; introducing the "α-components" $Q_\alpha := a^i_\alpha \otimes \bar{a}^\alpha_j$ (no summation in α is assumed) of $Q^i_j = \sum_{\alpha=1}^n Q_\alpha$, we have

$$(Q_\alpha)^h = (a^i_\alpha)^h \otimes (\bar{a}^\alpha_j)^h = 0\,,\qquad Q_\alpha\,Q_\beta = a^i_\alpha a^i_\beta \otimes \bar{a}^\alpha_j \bar{a}^\beta_j = q^{2\epsilon_{\alpha\beta}}\,Q_\beta\,Q_\alpha\,. \tag{27}$$

We will perform the proof by induction in n, observing that

$$Q_\alpha\,(Q_1+\dots+Q_{\alpha-1}) = q^2 (Q_1+\dots+Q_{\alpha-1})\,Q_\alpha\ , \quad \alpha = 2,\dots,n\ . \tag{28}$$

The calculation is based on the q-binomial identity (in fact, the case $n=2$)

$$Q_2 Q_1 = q^2 Q_1 Q_2 \quad \Rightarrow \quad (Q_1+Q_2)^m = \sum_{r=0}^{m} \binom{m}{r}_+ Q_1^r Q_2^{m-r} \tag{29}$$

where $\binom{m}{r}_+ = \frac{(m)_+!}{(r)_+!(m-r)_+!}\ ,\ (r)_+! = (r)_+ \dots (1)_+\ ,\ (r)_+ = \frac{q^{2r}-1}{q^2-1}$, implying

$$(Q_1+Q_2)^h = (Q_1)^h + \sum_{r=1}^{h-1} \binom{h}{r}_+ Q_1^r Q_2^{h-r} + (Q_2)^h = 0 \tag{30}$$

(Eq. (29) can be proved by induction in m). Equations (28)–(30) imply

$$(Q_1+\dots+Q_\alpha)^h = (Q_1+\dots+Q_{\alpha-1})^h + (Q_\alpha)^h = (Q_1+\dots+Q_{\alpha-1})^h\ , \tag{31}$$

etc. The following general formula can be proved by induction as well:

$$\left(\sum_{\alpha=1}^{n} Q_\alpha\right)^h = \sum_{\alpha=1}^{n} (Q_\alpha)^h + (h)_+! \times \\ \times \sum_{\substack{m_1+m_2+\dots+m_n=h \\ 0\le m_i \le h-1}} \frac{(Q_1)^{m_1}}{(m_1)_+!}\frac{(Q_2)^{m_2}}{(m_2)_+!}\dots\frac{(Q_n)^{m_n}}{(m_n)_+!} = 0.\ \blacksquare \tag{32}$$

In compliance with the final remark of Sect. 3, we will make the following

Conjecture. Any Q-monomial containing off-diagonal entries of Q annihilates the vacuum vector.

Recall that in the $n=2$ case this property is valid, due to the general fact (following from (17)) that $Q^i_j \mid 0\rangle = 0$ for $i \neq j$ and the commutativity of the diagonal and off-diagonal entries of Q (22) which however doesn't hold in general but is replaced by the following corollaries of (3) and (15).

Lemma 1. *The entries of Q belonging to the same row or column commute:*

$$[Q^j_i, Q^\ell_i] = 0 = [Q^i_j, Q^i_\ell]\ . \tag{33}$$

Proof. It is sufficient to explore the case in (33) when the different indices (j and ℓ) are carried by the left sector variables since the bar quantities satisfy identical relations. We obtain (assuming implicitly that equal upper and lower *greek* i.e.

quantum group, indices are summed over all admissible values from 1 to n, if no restrictions are indicated under a summation symbol)

$$
\begin{aligned}
&[p_{\ell j}-1]\, Q^j_i\, Q^\ell_i = [p_{\ell j}-1]\,(a^j_\beta\otimes\bar a^\beta_i)(a^\ell_\alpha\otimes\bar a^\alpha_i) = [p_{\ell j}-1]\, a^j_\beta a^\ell_\alpha\otimes\bar a^\beta_i\bar a^\alpha_i = \\
&= [p_{\ell j}-1]\sum_\alpha a^j_\alpha a^\ell_\alpha\otimes\bar a^\alpha_i\bar a^\alpha_i + \sum_{\alpha\neq\beta}[p_{\ell j}-1]\, a^j_\beta\, a^\ell_\alpha\otimes\bar a^\beta_i\bar a^\alpha_i = \\
&= [p_{\ell j}-1]\sum_\alpha a^\ell_\alpha a^j_\alpha\otimes\bar a^\alpha_i\bar a^\alpha_i + \sum_{\alpha\neq\beta}\left(a^\ell_\alpha a^j_\beta\,[p_{\ell j}] - a^\ell_\beta\, a^j_\alpha\, q^{\epsilon_{\alpha\beta}p_{\ell j}}\right)\otimes\bar a^\beta_i\bar a^\alpha_i = \\
&= [p_{\ell j}-1]\sum_\alpha a^\ell_\alpha a^j_\alpha\otimes\bar a^\alpha_i\bar a^\alpha_i + \sum_{\alpha\neq\beta} a^\ell_\beta\, a^j_\alpha\,\left(q^{\epsilon_{\alpha\beta}}[p_{\ell j}] - q^{\epsilon_{\alpha\beta}p_{\ell j}}\right)\otimes\bar a^\beta_i\bar a^\alpha_i = \\
&= [p_{\ell j}-1]\, a^\ell_\beta\, a^j_\alpha\otimes\bar a^\beta_i\bar a^\alpha_i = [p_{\ell j}-1]\, Q^\ell_i\, Q^j_i \quad \text{i.e.,} \quad [p_{\ell j}-1]\,[Q^j_i\,,Q^\ell_i] = 0
\end{aligned} \tag{34}
$$

(we have applied (3), exchanged the dummy indices α and β in a term on the fourth line and then used the identity $q^\epsilon[p]-q^{\epsilon p} = [p-1]$ for $\epsilon = \pm 1$). The first relation (33) $[Q^j_i, Q^\ell_i] = 0$ follows since, by exchanging the upper (left sector) indices j and ℓ, we can also derive that

$$
[p_{j\ell}-1]\,[Q^\ell_i\,,Q^j_i] = [p_{\ell j}+1]\,[Q^j_i\,,Q^\ell_i] = 0\,, \tag{35}
$$

and there is no vector on which the operators $[p_{\ell j}+1]$ and $[p_{\ell j}-1]$ vanish simultaneously. In a similar way one obtains from (15) that $[Q^i_j\,,Q^i_\ell] = 0$. ■

Lemma 2. *The entries of Q belonging to different rows and columns satisfy*

$$
\begin{aligned}
&([p_{ij}-1]\otimes[\bar p_{\ell m}] - [p_{ij}]\otimes[\bar p_{\ell m}-1])\, Q^i_\ell\, Q^j_m \quad (\equiv [p_{ij}-\bar p_{\ell m}]\, Q^i_\ell\, Q^j_m\,) = \\
&= [p_{ij}-1]\otimes[\bar p_{\ell m}]\, Q^j_\ell\, Q^i_m - [p_{ij}]\otimes[\bar p_{\ell m}-1]\, Q^i_m\, Q^j_\ell \quad (\,i\neq j\,,\ \ell\neq m\,).
\end{aligned} \tag{36}
$$

Remark. Below we will make use of the following q-identities:

$$
\begin{aligned}
&[p\pm 1]\otimes[\bar p\,] - [p\,]\otimes[\bar p\pm 1] = \mp[p-\bar p\,] := \mp\,\frac{q^p\otimes q^{-\bar p} - q^{-p}\otimes q^{\bar p}}{q-q^{-1}}\,,\\
&[p\pm 1]\otimes[\bar p\,] - [p\,]\otimes[\bar p\mp 1] = \pm[p+\bar p\,] := \pm\,\frac{q^p\otimes q^{\bar p} - q^{-p}\otimes q^{-\bar p}}{q-q^{-1}}\,,\\
&[p\,]\otimes q^{\epsilon\bar p} - q^{\epsilon p}\otimes[\bar p\,] =: [p-\bar p\,]\,,\quad \epsilon = \pm 1\,.
\end{aligned} \tag{37}
$$

Proof. Equation (36) is suggested by (8) and (13), (14) implying

$$
\hat R^{ij}_{\ i'j'}(p)\, Q^{i'}_\ell\, Q^{j'}_m = Q^i_{\ell'}\, Q^j_{m'}\,(\hat R^{(\alpha)})^{\ell' m'}_{\ \ell m}(\bar p) \tag{38}
$$

but can be also verified directly with the help of (3), (15) and (37):

$$
\begin{aligned}
&[p_{ij}-1]\otimes[\bar{p}_{\ell m}]\, Q^j_\ell\, Q^i_m - [p_{ij}]\otimes[\bar{p}_{\ell m}-1]\, Q^i_m\, Q^j_\ell = \\
&= [p_{ij}-1]\otimes[\bar{p}_{\ell m}] \sum_\alpha a^j_\alpha\, a^i_\alpha \otimes \bar{a}^\alpha_\ell\, \bar{a}^\alpha_m \ + \\
&+ \sum_{\alpha\neq\beta}([p_{ij}]\, a^i_\alpha\, a^j_\beta - q^{\epsilon_{\alpha\beta} p_{ij}}\, a^i_\beta\, a^j_\alpha)\otimes[\bar{p}_{\ell m}]\,\bar{a}^\beta_\ell\,\bar{a}^\alpha_m - \\
&-[p_{ij}]\otimes[\bar{p}_{\ell m}-1] \sum_\alpha a^i_\alpha\, a^j_\alpha \otimes \bar{a}^\alpha_m\, \bar{a}^\alpha_\ell \ - \\
&- \sum_{\alpha\neq\beta}[p_{ij}]\, a^i_\beta\, a^j_\alpha \otimes (\,[\bar{p}_{\ell m}]\,\bar{a}^\alpha_\ell\,\bar{a}^\beta_m - q^{\epsilon_{\alpha\beta}\bar{p}_{\ell m}}\bar{a}^\beta_\ell\,\bar{a}^\alpha_m\,) = \\
&= [p_{ij}-\bar{p}_{\ell m}] \sum_\alpha a^i_\alpha\, a^j_\alpha \otimes \bar{a}^\alpha_\ell\, \bar{a}^\alpha_m \ + \\
&+ \sum_{\alpha\neq\beta}(\,[p_{ij}]\otimes q^{\epsilon_{\alpha\beta}\bar{p}_{\ell m}} - q^{\epsilon_{\alpha\beta} p_{ij}}\otimes[\bar{p}_{\ell m}]\,)\, a^i_\beta\, a^j_\alpha \otimes \bar{a}^\beta_\ell\,\bar{a}^\alpha_m = \\
&= [p_{ij}-\bar{p}_{\ell m}]\, Q^i_\ell\, Q^j_m \qquad (\, i\neq j\ ,\ \ell\neq m\,)\ . \ \blacksquare
\end{aligned}
\tag{39}
$$

Let us see what the above two Lemmas tell us in the cases involving diagonal entries of Q. Equation (33) implies that

$$
[Q^j_i, Q^i_i] = 0 = [Q^i_j, Q^i_i]\ , \tag{40}
$$

while Eq. (36) gives rise to the following relations valid for $i\neq j\neq \ell\neq i$ (which is only possible if $n\geq 3$):

$$
\begin{aligned}
&[p_{ij}-1]\otimes[\bar{p}_{i\ell}]\, Q^j_\ell\, Q^i_i = [p_{ij}]\otimes[\bar{p}_{i\ell}+1]\, Q^i_i\, Q^j_\ell - [p_{ij}+\bar{p}_{i\ell}]\, Q^i_\ell\, Q^j_i\ , \\
&[p_{ij}]\otimes[\bar{p}_{i\ell}-1]\, Q^j_\ell\, Q^i_i = [p_{ij}+1]\otimes[\bar{p}_{i\ell}]\, Q^i_i\, Q^j_\ell - [p_{ij}+\bar{p}_{i\ell}]\, Q^j_i\, Q^i_\ell\ .
\end{aligned}
\tag{41}
$$

So an off-diagonal Q-operator can jump over a diagonal one, except in cases when the p-dependent coefficients in the left-hand sides of the two identities (41) vanish simultaneously (note that the last terms of (41) only contain off-diagonal Q-operators). Moreover, if $[p_{ij}]\, v=0$ or $[\bar{p}_{i\ell}]\, v=0$, then

$$
[p_{ij}]\, v = 0 \ \Rightarrow\ Q^j_\ell\, Q^i_i\, v = Q^i_\ell\, Q^j_i\, v\ , \quad [\bar{p}_{i\ell}]\, v=0\ \Rightarrow\ Q^j_\ell\, Q^i_i\, v = Q^j_i\, Q^i_\ell\, v \tag{42}
$$

by (18), so the only obstacle arises when we apply (41) to vectors v satisfying

$$
[p_{ij}-1]\, v = 0 = [\bar{p}_{i\ell}-1]\, v \qquad \text{for} \qquad i\neq j\neq \ell\neq i\ . \tag{43}
$$

The above facts open the possibility to prove the Conjecture by induction in the number of diagonal Q-operators applied to the vacuum, starting with $v_0 = \mid 0\rangle$ and $v_1 = Q^1_1 \mid 0\rangle$. To find out if and when (43) occurs, we need to explore the space of diagonal Q-vectors $\mathcal{F}^{diag} = \{v \mid v = P(Q^n_n, \dots, Q^1_1) \mid 0\rangle\}$ and its subspace $\mathcal{F}' \subset \mathcal{F}^{diag}$ that is annihilated by all off-diagonal elements, $Q^r_s\, \mathcal{F}' = 0\,,\ r \neq s$. (In these terms our conjecture is equivalent to $\mathcal{F}' \overset{(?)}{=} \mathcal{F}^{diag}$.) The exchange relations for diagonal elements that follow from (36)

$$[p_{ij}] \otimes [\bar{p}_{ij} + 1]\, Q^i_i\, Q^j_j - [p_{ij} - 1] \otimes [\bar{p}_{ij}]\, Q^j_j\, Q^i_i = [p_{ij} + \bar{p}_{ij}]\, Q^i_j\, Q^j_i\ , \tag{44}$$

$$[p_{ij} + 1] \otimes [\bar{p}_{ij}]\, Q^i_i\, Q^j_j - [p_{ij}] \otimes [\bar{p}_{ij} - 1]\, Q^j_j\, Q^i_i = [p_{ij} + \bar{p}_{ij}]\, Q^j_i\, Q^i_j \qquad (i \neq j)\ .$$

imply (as the eigenvalues of p_{ij} and $\bar{p}_{ij}$ on $\mathcal{F}^{diag}$ are equal)

$$[p_{ij} + 1]\, Q^i_i\, Q^j_j \ \approx\ [p_{ij} - 1]\, Q^j_j\, Q^i_i \tag{45}$$

where the "weak equality" sign refers to an identity that holds on $\mathcal{F}'$.

As already mentioned, these are just the first steps in our study of the Q-algebra and its vacuum representation for $n \geq 3$. The obvious immediate tasks are the completion of the proof of the diagonality conjecture and the description of $\mathcal{F}^{diag}$. To this end, one should take next into account (besides the bilinear exchange relations) the n-linear determinant condition (which suggests a basis in $\mathcal{F}^{diag}$ labelled by $su(n)$ Young diagrams [9]) and also some trilinear relations following from the chiral structure of the Q-operators. Together with $(Q^i_i)^h = 0$ (32) and (45), the latter seem to imply the finite dimensionality of $\mathcal{F}^{diag}$.

5 Discussion and Outlook

It would be intriguing to look for a possible connection of the diagonal Q-algebra with the algebra of the (phase model) *"hopping operators"* $\{Q_1, \dots, Q_n\}$ on a circle (also called *"affine local plactic algebra"*). The latter is characterized by the relations

$$[Q_i, Q_j] = 0\,, \quad \text{if } i \neq j \pm 1 \bmod n$$

$$Q_i\, Q^2_j = Q_j\, Q_i\, Q_j\ , \quad Q^2_i\, Q_j = Q_i\, Q_j\, Q_i\ , \quad \text{if } i = j + 1 \bmod n \tag{46}$$

and provides a description of the (unitary) $\widehat{su}(n)_k$ affine fusion ring [18, 19]. In contrast to our (diagonal) Q-algebra, it *does not* depend explicitly on the level k which only labels its representations. Although it is clear from the outset that the two algebras are not isomorphic, relations (46) can suggest the correct procedure needed to obtain the physical subquotient space for general n.

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Part V
Conformal Field Theories

Breaking $so(4)$ Symmetry Without Degeneracy Lift

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Abstract We consider on S_R^3 the quantum motion of a scalar particle of mass m, perturbed by the trigonometric Scarf potential (Scarf I) with one internal quantized dimensionless parameter, ℓ, the 3D orbital angular momentum, and another, an external scale introducing continuous parameter, B. We show that a loss of the geometric hyper-spherical $so(4)$ symmetry of the free motion can occur that leaves intact the unperturbed $\mathcal{N}^2$-fold degeneracy patterns, with $\mathcal{N} = (\ell + n + 1)$ and n denoting the nodes of the wave function. Our point is that although the number of degenerate states for any $\mathcal{N}$ matches dimensionality of an irreducible $so(4)$ representation space, the corresponding set of wave functions do not transform irreducibly under any $so(4)$. Indeed, in expanding the Scarf I wave functions in the basis of properly identified $so(4)$ representation functions, we find power series in the perturbation parameter, B, where 4D angular momenta $K \in [\ell, \mathcal{N} - 1]$ contribute up to the order $\mathcal{O}\left(\frac{2mR^2B}{\hbar^2}\right)^{\mathcal{N}-1-K}$. In this fashion, we work out an explicit example on a symmetry breakdown by external scales that retains the degeneracy. The scheme extends to $so(d + 2)$ for any d.

1 Introduction

The theory of Lie algebras provides, in terms of its invariants, a powerful tool for the description of observed constants of motion both in free and interacting systems and enables in this manner uncovering of universal physical laws. In spectral problems, symmetry as a rule is signaled by energy values degenerate with respect to certain sets of quantum numbers, an indication that a Lie algebra might exist whose irreducible representations have dimensionalities that match the number of states in the levels. In this fashion, a relationship between symmetry and degeneracy can be established. Any N-fold degenerate system is $gl(N, R)$ symmetric in so far as by virtue of Sturm-Liouville's theory of differential equations, any linear superposition

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of solutions characterized by a common eigenvalue is again a solution to the same eigenvalue. The case of our interest here is the one in which the degeneracy patterns can be mapped on the irreducible representations of a Lie algebra distinct from $gl(N, R)$. Popular examples are the spectra of the Harmonic-Oscillator–, and the Coulomb problems, whose Hamilton operators can be cast in their turn as $su(3)$, and $so(4)$ invariants, respectively. Especially in the latter case, the $\mathcal{N}^2$-fold degeneracies of the states in a level ($\mathcal{N}$ being the principal quantum number, $\mathcal{N} = n + \ell + 1 \in [1, \infty)$, and ℓ and n denoting the orbital angular momentum value, and the number of nodes, respectively) has been explained in terms of $so(4)$ irreducible representations of dimensionalities, $\mathcal{N}^2$. It has been realized already in the early days of quantum mechanics that a Hamiltonian with Coulomb interaction can be cast in the form of a Casimir invariant of the isometry algebra $so(4)$ of the three-dimensional (3D) sphere, S^3_R with R being the hyper-radius [4]. This example shows that a relationship between symmetry and degeneracy can be at the very root of spectroscopic studies, a reason for which it is important to understand as to what extent Lie-algebraic degeneracy patterns are at par with the correct transformation properties of the wave functions under the algebra in question. Our point is that degeneracy alone is not sufficient to claim a particular Lie algebraic symmetry of the Hamiltonian. On the example of the quantum motion of a scalar particle on S^3_R, perturbed by the trigonometric Scarf potential (Scarf I), we show that the perturbation completely retains the $so(4)$ degeneracies of the free motion without that the "perturbed" wave functions would behave as eigenfunctions of an $so(4)$ Casimir operator.

The contribution is structured as follows. In the next section we study the $so(4)$ symmetry properties of the hyper-geometric differential equation for the Gegenbauer polynomials, $\mathcal{G}^\lambda_n(x)$, for $\lambda = (\ell + 1)$ with ℓ non-negative integer. First we observe that in subjecting the eigenvalue problem of the canonical $so(4)$ Casimir operator to a similarity transformation by $(1 - \sin^2\chi)^{\frac{\lambda}{2} - \frac{1}{4}}$, the square-root of the weight function of the Gegenbauer polynomials, and setting $x = \sin\chi$, with χ standing for the second polar angle in E_4, amounts to the Gegenbauer equation, thus making the $so(4)$ symmetry of the latter manifest. As long as free quantum motion on S^3_R can be cast as the eigenvalue problem of the Casimir operator of the transformed $so(4)$, whose wave functions are the Gegenbauer polynomials, $so(4)$ has been proved to be the relevant symmetry both of the spectrum and the wave functions. This contrasts the case of the Jacobi polynomials, $P^{\alpha_\ell\beta_\ell}_n(x)$, considered in Sect. 3 for the following parameter values, $\alpha_\ell = \ell + \frac{1}{2} - b$, and $\beta_\ell = \ell + \frac{1}{2} + b$, which present themselves as linear combinations of Gegenbauer polynomials of equal $\lambda = (\ell + 1)$ parameters but different degrees, n, and do not behave as $so(4)$ representation functions. Nonetheless, because of the above specific choice of the parameters, the Jacobi polynomial equation can be transformed to a motion on S^3_R perturbed by the trigonometric Scarf potential, whose spectrum carries by chance same $so(4)$ degeneracy patterns as the free motion, without that this symmetry is shared by the wave functions. In this manner, we explicitly work out an example of breaking $so(4)$ by a perturbation without degeneracy lift. The paper closes with brief conclusions.

2 The Gegenbauer Polynomial Equation as Eigenvalue Problem of An $so(4)$ Casimir Operator

The Gegenbauer polynomial equation [1] for the special choice of the parameter, $\lambda = \ell + 1$, with ℓ non-negative integer, is given by

$$\left(1 - x^2\right) \frac{\mathrm{d}^2 \mathcal{G}_n^{\ell+1}(x)}{\mathrm{d}x^2} - (2\ell + 3)\, x \frac{\mathrm{d}\mathcal{G}_n^{\ell+1}(x)}{\mathrm{d}x} + n(n + 2\ell + 2)\mathcal{G}_n^{\ell+1}(x) = 0. \quad (1)$$

At the same time, the eigenvalue problem of the well known Casimir operator, $\mathcal{K}^2$, of the $so(4)$ isometry algebra of the three dimensional (3D) unit sphere, to be denoted by S^3, reads

$$\left[\mathcal{K}^2 - K(K+2)\right] Y_{K\ell m}(\chi, \theta, \varphi) = 0,\ \mathcal{K}^2 = \frac{(-1)}{\cos^2 \chi} \frac{\partial}{\partial \chi} \cos^2 \chi \frac{\partial}{\partial \chi} + \frac{\mathbf{L}^2(\theta, \varphi)}{\cos^2 \chi},$$
$$Y_{K\ell m}(\chi, \theta, \varphi) = \cos^\ell \chi \mathcal{G}_{n=K-\ell}^{\ell+1}(\sin \chi) Y_{\ell m}(\theta, \varphi), \qquad K = n + \ell,$$
$$\mathbf{L}^2(\theta, \varphi) Y_{\ell m}(\theta, \varphi) = \ell(\ell+1) Y_{lm}(\theta, \varphi). \quad (2)$$

Here, $\mathbf{L}(\theta, \varphi)$ is the 3D angular momentum operator, K, ℓ and m are in turn the 4D-, 3D, and 2D angular momentum values, $Y_{K\ell m}(\chi, \theta, \varphi)$ are the 4D spherical harmonics, with $\chi \in \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$, and $\theta \in [0, \pi]$ standing for the two polar angles parameterizing S^3, and $\varphi \in [0, 2\pi]$ denoting the ordinary azimuthal angle. In the so called quasi-radial variable [6], χ, Eq. (2) reduces to

$$\left[-\frac{1}{\cos^2 \chi} \frac{\partial}{\partial \chi} \cos^2 \chi \frac{\partial}{\partial \chi} + \frac{\ell(\ell+1)}{\cos^2 \chi} - K(K+2)\right] \cos^\ell \chi \mathcal{G}_{K-\ell}^{\ell+1}(\sin \chi) = 0, \quad (3)$$

and it is straightforward to check that (3) is equivalent to

$$\left[\tilde{\mathcal{K}}^2 - (n+l)(n+l+2))\right] \mathcal{G}_{K-\ell}^{\ell+1}(\sin \chi) = 0,$$
$$\text{with} \qquad \tilde{\mathcal{K}}^2 = \cos^{-\ell} \chi \mathcal{K}^2 \cos^\ell \chi, \quad (4)$$

because of

$$\tilde{\mathcal{K}}^2 = \cos^{-\ell} \chi \mathcal{K}^2 \cos^\ell \chi = -\frac{\mathrm{d}^2}{\mathrm{d}\chi^2} + (2\ell + 2) \tan \chi \frac{\mathrm{d}}{\mathrm{d}\chi} + \ell(\ell + 2). \quad (5)$$

The $\cos^\ell \chi$ function relates to the square-root of the weight function, $\omega^\lambda(x)$, of the Gegenbauer polynomials, $\mathcal{G}_n^\lambda(x)$, as,

$$\omega^\lambda(x) = (1 - x^2)^{\lambda - \frac{1}{2}}, \qquad x = \sin \chi, \quad \lambda = (\ell + 1),$$
$$\cos^\ell \chi = \sqrt{\frac{\omega^{\ell+1}(\sin \chi)}{\frac{\mathrm{d}x}{\mathrm{d}\chi}}}. \quad (6)$$

Therefore, upon changing variable in (5) to $x = \sin\chi$, and back-substituting in (3), one obtains the claimed equality between the $so(4)$ Null operator,

$$\left[\tilde{\mathcal{K}}^2 - K(K+2)\right], \quad K = n + \ell, \tag{7}$$

and the Gegenbauer polynomial equation as,

$$\left[\tilde{\mathcal{K}}^2 - (n+l)(n+l+2)\right] \mathcal{G}^{\ell+1}_{K-\ell}(x) = \left(1 - x^2\right) \frac{\mathrm{d}^2 \mathcal{G}^{\ell+1}_{K-\ell}(x)}{\mathrm{d}x^2} - \\ - (2\ell+3) \frac{\mathrm{d}\mathcal{G}^{\ell+1}_{K-\ell}(x)}{\mathrm{d}x} + n(n+2\ell+2)\mathcal{G}^{\ell+1}_{K-\ell}(x) = 0. \tag{8}$$

The latter equation means that the Gegenbauer polynomials, occasionally termed to as ultra-spherical polynomials, are representation functions to an $so(4)$ algebra obtained from the canonical one according to (6) through a similarity transformation by the square-root of their weight function and upon accounting for a change of variable. An interesting connection between the latter equation and the 1D Schrödinger equation with the $\sec^2\chi$ potential can be established upon substituting,

$$\cos^{\ell}\chi \mathcal{G}^{\ell+1}_{K-\ell}(\sin\chi) = \frac{\mathcal{U}^{\ell}_n(\chi)}{\cos\chi}, \quad n = K - \ell. \tag{9}$$

In so doing, one finds that $\mathcal{U}^{\ell}_n(\chi)$ satisfies the 1D Schrödinger equation with the $\sec^2\chi$ potential according to,

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}\chi^2} + \frac{\ell(\ell+1)}{\cos^2\chi}\right]\mathcal{U}^{\ell}_n(\chi) = (K+1)^2\mathcal{U}^{\ell}_n(\chi), \tag{10}$$

whose spectrum is characterized by $(K+1)^2$-fold degeneracy of the levels, just as the H atom, due to $\sum_{\ell=0}^{\ell=K}(2\ell+1) = (K+1)^2$. Therefore, the $so(4)$ symmetry of the Gegenbauer polynomials shows up as $so(4)$ degeneracy patterns in the spectrum of the corresponding 1D Schrödinger equation with the $\sec^2\chi$ interaction. More general, there are several two-parameter potentials, $v(z;\alpha,\beta)$ for which the Schrödinger equation,

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}z^2} + v(z;\alpha,\beta))\right] R^{\alpha\beta}_n(z) = \epsilon R^{\alpha\beta}_n(z), \tag{11}$$

can be exactly solved by reducing it to a hyper-geometric differential equation by means of a point-canonical transformation of the type [8],

$$R^{\alpha\beta}_n(z) = R^{\alpha\beta}_n\left(z = f(x)\right) \stackrel{\mathrm{def}}{:=} g^{\alpha\beta}_n(x) = \sqrt{\omega^{\alpha\beta}(x)} J^{\alpha\beta}_n(x)\frac{1}{\sqrt{\frac{\mathrm{d}f(x)}{\mathrm{d}x}}}, \quad x \in [a,b], \tag{12}$$

where $J_n^{\alpha\beta}(x)$ are polynomials of degree n and orthogonal with respect to their weight-function $\omega^{\alpha\beta}(x)$ according to

$$\int_0^\infty R_n^{\alpha\beta}(z)R_{n'}^{\alpha\beta}(z)\mathrm{d}z=\int_a^b g_n^{\alpha\beta}(x)g_{n'}^{\alpha\beta}(x)\mathrm{d}f(x)=\int_a^b \omega^{\alpha\beta}(x)J_n^{\alpha\beta}(x)J_{n'}^{\alpha\beta}(x)\mathrm{d}x \tag{13}$$

And vice verse, any hyper-geometric differential equation can be brought back to an 1D Schrödinger equation in (11) by inverting the transformation in (12).

The above procedure establishes an interesting link between the symmetry properties of orthogonal polynomials and the degeneracies in the corresponding potential spectra. In the next subsection we shall see that a Lie algebraic degeneracy in the Schrödinger spectrum can appear by chance and without it being shared by the polynomial equation.

3 A Jacobi Polynomial Equation as Eigenvalue Problem of a "Frustrated" $so(4)$ Casimir Operator

The hyper-geometric differential equation solved by the Jacobi polynomial reads [1],

$$\begin{aligned}&\left(1-x^2\right)\frac{\mathrm{d}^2P_n^{\alpha\beta}(x)}{\mathrm{d}x^2}+\left[(\beta-\alpha)-(\alpha+\beta+2)\,x\right]\frac{\mathrm{d}P_n^{\alpha\beta}(x)}{\mathrm{d}x}\;+\\&+n(n+\alpha+\beta+1)P_n^{\alpha\beta}(x)\;=\;0,\end{aligned} \tag{14}$$

and acquires a shape pretty close to (1) for the following special choice of the parameters,

$$\alpha_\ell=\ell-b+\frac{1}{2},\quad \beta_\ell=\ell+b+\frac{1}{2}, \tag{15}$$

namely,

$$\begin{aligned}&\left(\tilde{\mathcal{K}}^2-(n+\ell)(n+\ell+2)+2b\frac{\mathrm{d}}{\mathrm{d}x}\right)P_n^{\ell-b+\frac12,\ell+b+\frac12}(x)\;=\\&=\;\left(1-x^2\right)\frac{\mathrm{d}^2P_n^{\ell-b+\frac12,\ell+b\frac12}(x)}{\mathrm{d}x^2}+\left[2b-(2\ell+3)\,x\right]\frac{\mathrm{d}P_n^{\ell-b+\frac12,\ell+b+\frac12}(x)}{\mathrm{d}x}\;+\\&+n(n+2\ell+2)P_n^{\ell-b+\frac12,\ell+b+\frac12}(x)=0.\end{aligned} \tag{16}$$

The latter relation reveals the Jacobi equation as the $so(4)$ Null-operator in (7), "frustrated" by the gradient term $\left[-2b\frac{\mathrm{d}}{\mathrm{d}x}\right]$. In consequence, the Jacobi polynomials

Table 1 Decompositions of some of the Jacobi polynomials $P_n^{\ell-b+\frac{1}{2},\ell+b+\frac{1}{2}}(\sin\chi)$ in (15) for fixed ℓ in $so(4)$ representation functions of 4D angular momenta $K\in[\ell,\kappa]$ with $\kappa=n+\ell=\mathcal{N}-1$

ℓ	K	$P_{n=\kappa-\ell}^{(\alpha_\ell,\beta_\ell)}(\sin\chi)$	=	$\sum_{K=\ell}^{\kappa} c_{K\kappa}(b)\,\mathcal{G}_{K-\ell}^{\ell+1}(\sin\chi)$
κ	$K=\kappa$	$P_0^{(\alpha_k,\beta_k)}(\sin\chi)$	=	$\mathcal{G}_0^{k+1}(\sin\chi)$
$\kappa-1$	$K\in[\kappa-1,\kappa]$	$P_1^{(\alpha_{k-1},\beta_{k-1})}(\sin\chi)$	=	$\frac{(2k+1)}{4k}\,\mathcal{G}_1^{k}(\sin\chi)-$ $-\,b\,\mathcal{G}_0^{k}(\sin\chi)$
$\kappa-2$	$K\in[\kappa-2,\kappa]$	$P_2^{(\alpha_{k-2},\beta_{k-2})}(\sin\chi)$	=	$\frac{1}{8}\frac{(2k+1)}{(k-1)}\,\mathcal{G}_2^{k-1}(\sin\chi)-$ $-\,\frac{b}{2}\frac{k}{(k-1)}\,\mathcal{G}_1^{k-1}(\cot\chi)+$ $+\,\frac{b^2}{2}\,\mathcal{G}_0^{k-1}(\sin\chi)$
$\kappa-3$	$K\in[\kappa-3,\kappa]$	$P_3^{(\alpha_{k-3},\beta_{k-3})}(\sin\chi)$	=	$\frac{1}{32}\frac{(4\kappa^2-1)}{(\kappa^2-3\kappa+2)}\mathcal{G}_3^{(k-2)}(\sin\chi)$ $-\frac{b}{8}\frac{(2\kappa^2-\kappa)}{(\kappa^2-3\kappa+2)}\mathcal{G}_2^{(k-2)}(\sin\chi)$ $+\,\frac{b^2}{8}\frac{(2k-1)}{(k-2)}\,\mathcal{G}_1^{(k-2)}(\sin\chi)$ $-\,\frac{b}{24}\frac{(4b^2k-4b^2+2k+1)}{(k-1)}\times$ $\times\,\mathcal{G}_0^{(k-2)}(\sin\chi)$

The K labeled Gegenbauer polynomials contribute to the order $\mathcal{O}\left(b^{\kappa-K}\right)$ to the expansion and give the order to which the $so(4)$ symmetry fades away, with b defined in (15) and (18)

do not behave as $so(4)$ representation functions. This is best illustrated through the finite series decomposition of a Jacobi polynomial of degree n in Gegenbauer polynomials of degrees running from 0 to n, shown in Table 1. In recalling that the degrees of the Gegenbauer polynomials under considerations express in terms of the 4D angular momentum values, K, as $n=(K-\ell)$, the decompositions present themselves as mixtures of $so(4)$ representation functions of different 4D angular momentum values, $K\in[\ell,\ell+n]$.

Despite the absence of $so(4)$ symmetry of the Jacobi polynomials, a curiosity occurs insofar as the associated 1D Schrödinger equation (in units of $\hbar^2/(2mR^2)$),

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}\chi^2}+v_{\mathrm{ScI}}\left(\chi;\alpha_\ell,\beta_\ell\right)\right]R_n^{\ell-b+\frac{1}{2},\ell+b+\frac{1}{2}}(\chi)=\epsilon\,R_n^{\ell-b+\frac{1}{2},\ell+b+\frac{1}{2}}(\chi)\,, \tag{17}$$

$$v_{\mathrm{ScI}}\left(\chi;\alpha_\ell,\beta_\ell\right)=\frac{b^2+\ell(\ell+1)}{\cos^2\chi}-\frac{b(2\ell+1)\tan\chi}{\cos\chi},\quad b=\frac{B(2mR^2)}{\hbar^2}, \tag{18}$$

$$R_n^{\ell-b+\frac{1}{2},\ell+b+\frac{1}{2}}(\chi)=e^{-b\tanh^{-1}\sin\chi}\cos^{\ell+1}\chi P_n^{\ell-b+\frac{1}{2},\ell+b+\frac{1}{2}}(\sin\chi)\,, \tag{19}$$

$$\mathcal{N} = n + \ell + 1, \quad \epsilon = \mathcal{N}^2, \quad \epsilon = \frac{E(2mR^2)}{\hbar^2}, [E], [B] = \text{MeV}, \tag{20}$$

reduced to the hyper-geometric differential equation along the line of the above Eqs. (13)–(12) for $\chi = f(x) = \sin^{-1} x$, and

$$\omega^{\ell-b+\frac{1}{2},\ell+b+\frac{1}{2}}(x) = e^{-b\tanh^{-1}x}(1-x^2)^{\ell+\frac{1}{2}}, \tag{21}$$

exhibits same degeneracy patterns as the fully $so(4)$ symmetric problem in (10) and the underlying (8). In (18), the $v_{\mathrm{ScI}}(\chi; \alpha_\ell, \beta_\ell)$ potential is known under the name of the trigonometric Scarf potential, abbreviated, Scarf I ([8] and references therein). Under the substitution,

$$R_n^{\ell-b+\frac{1}{2},\ell+b+\frac{1}{2}}(\chi) = \frac{U_n^{\ell-b+\frac{1}{2},\ell+b+\frac{1}{2}}(\chi)}{\cos\chi}, \tag{22}$$

Eq. (18) is transformed to motion on S^3 perturbed by Scarf I. The expansions in Table 1 apply equally well to the wave functions $U_n^{\ell-b+\frac{1}{2},\ell+b+\frac{1}{2}}(\chi)$ which can not transform as $so(4)$ representation functions despite the $so(4)$ degeneracy patterns in the spectrum. In this fashion, we worked out an example that a Lie algebraic symmetry in a spectral problem does not necessarily imply same symmetry of the Hamiltonian. Figure 1 is illustrative of this type of $so(4)$ breaking.

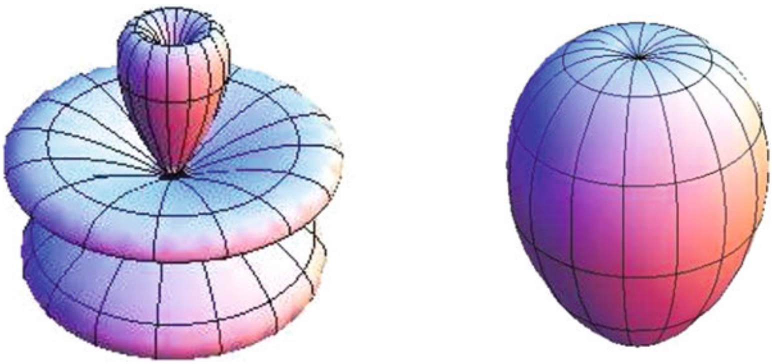

Fig. 1 The breaking of the $so(4)$ symmetry of the free motion of a scalar particle on S^3 in (3)–(5) and (10), through the external scale $B = \frac{\hbar^2 b}{2mR^2}$, due to a perturbation by the trigonometric Scarf potential (17)–(20). The wave function $U_3^{\frac{3}{2}-b,\frac{3}{2}+b}(\chi)$ in (22) (*right*) in comparison to its counterpart, $\cos^\ell \chi \mathcal{G}_3^2(\sin\chi)$ in (4) (*left*) describing the unperturbed $so(4)$ symmetric motion. These functions describe equal energies in the respective potential problems in Eqs. (10) and (20)

Conclusions

In this work we constructed an explicit example for the possibility to remove a Lie algebraic symmetry of a Hamiltonian by perturbation and without lifting the unperturbed degeneracy patterns in the spectrum. The clue of this observation is that Lie algebraic degeneracy patterns can throughout be tolerant towards external scales, such as masses, temperatures, lengths etc. Such a type of $so(4)$ symmetry lift could reconcile the experimentally detected conformal symmetry patterns in the spectra of the high-lying light flavored hadrons, both baryons and mesons, with the conformal symmetry removal through the dilation mass. The relevance of the conformal symmetry for QCD is predicted by the AdS_5/CFT_4 duality and is compatible with spectroscopic data on the light-flavored hadron spectra (see Fig. 2) due to the walking of the strong coupling constant in the infrared towards a fixed value [3], sketched in Fig. 3. The relevance of the hyper-spherical geometry in conformal field theories is derived from the possibility of mapping a flat space-time QFT on Einstein's closed universe, $\mathcal{R}^1 \otimes S^3_R$, whose isometry

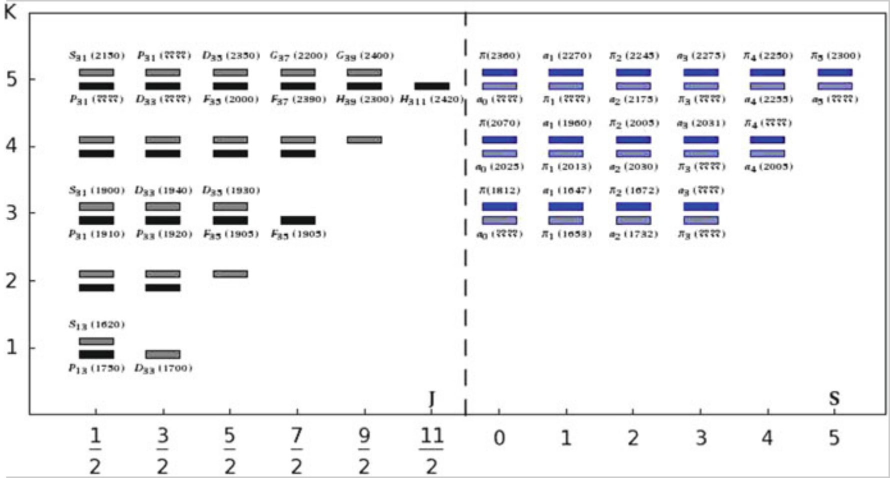

Fig. 2 Hydrogen like (conformal type) degeneracy in the reported spectra of the excited $L_{3(2J)}$, i.e. Δ baryons (*left*) and the high-lying light flavored mesons (*right*) (for details on the notations and more references see [2, 7]). *Full* and *shadowed bricks* denote degenerate hadron states of opposite parities. The *numbers inside of the parenthesis* give the masses (in MeV) while the *question marks* denote "missing" states. The meson sector is close to parity doubled, a possible hint on chiral symmetry restoration from the spontaneously broken Nambu-Goldstone—to the manifest Wigner-Wyle mode. Notice the pronounced supersymmetric baryon-meson degeneracy

(continued)

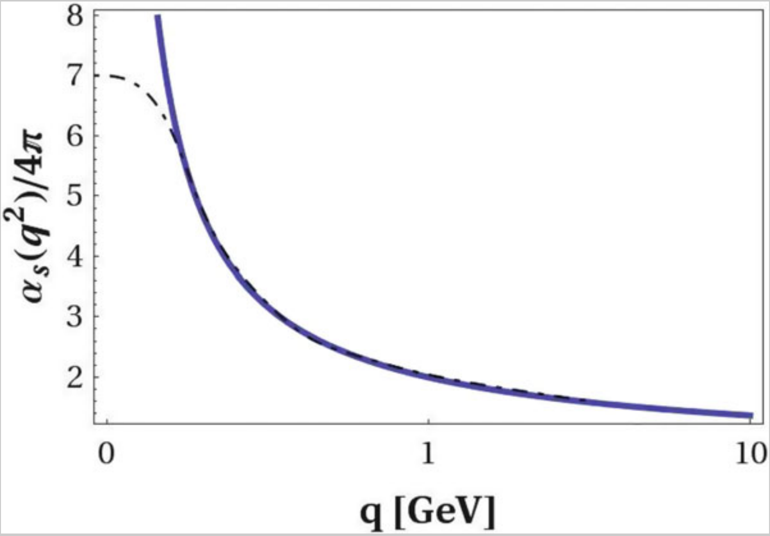

Fig. 3 Schematic presentation of the walking (*dashed line*) of the strong coupling constant in the infrared according to [3]

algebra is the covering of the conformal one, a result due to [9]. The so called compactified Minkowski space time, in being of finite 3D volume, provides a natural scenario for the QCD confinement phenomenon [10] and the inverse of the S^3_R radius provides a natural scale that can be interpreted as the temperature [5].

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On the Relation Between an $\mathcal{N} = 1$ Supersymmetric Liouville Field Theory and a Pair of Non-SUSY Liouville Fields

Leszek Hadasz and Zbigniew Jaskólski

Abstract We discuss a relation between the tensor product of the $\mathcal{N} = 1$ super-Liouville field theory with the imaginary free fermion and a certain projected tensor product of the real and the imaginary Liouville field theories. Using techniques of two dimensional, conformal field theory we give a complete proof of their equivalence in the NS sector.

1 Introduction

Several years ago the so called AGT relation between partition functions of $\mathcal{N} = 2$ superconformal $\mathrm{SU}(N)$ gauge theories in four dimensions and correlation functions in the two-dimensional Liouville/Toda field theories was established [1,15]. One of its essential generalizations, first formulated in [2], was the proposal that $\mathcal{N} = 2$ $\mathrm{SU}(N)$ gauge theories on $\mathbb{R}^4/\mathbb{Z}_p$ should be related to certain coset conformal fields theories. Some further checks of the AGT relation for $N = p = 2$, corresponding to the $\mathcal{N} = 1$ super-Liouville theory, were done in the NS sector in [4,6,7] and in the R sector in [3,11]. It was in particular observed in [6,7] that the blow-up formula for the Nekrasov partition function suggests a precise relation between $\mathcal{N} = 1$ super-Liouville and Liouville conformal blocks. An explanation of this phenomenon on the CFT side was given in [16]. It was motivated by old results [8,9,12,13] relating various rational models realized as quotients,

$$V(p,m) \sim \frac{\widehat{\mathrm{SU}}(2)_p \times \widehat{\mathrm{SU}}(2)_m}{\widehat{\mathrm{SU}}(2)_{p+m}}.$$

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The case relevant for the present discussion is the relation between the Virasoro minimal models $V(m) = V(1,m)$ and the $\mathcal{N} = 1$ superconformal models $SV(m) = V(2,m)$:

$$V(1) \otimes SV(m) \sim V(m) \otimes_P V(m+1), \quad m = 1, 2, \ldots,$$

where the symbol $\otimes_P$ denotes a projected tensor product in which only selected pairs of conformal families are present. The nonrational counterpart of this relation proposed in [16] takes the schematic form

$$\text{free fermion} \otimes \mathcal{N} = 1 \text{ super-Liouville} \quad \sim \quad \text{Liouville} \otimes_P \text{Liouville}. \tag{1}$$

In the NS sector this relation has been made much more precise in [5] where it was used as an essential element of the proof of the AGT correspondence in the case of $N = p = 2$. The extension of (1) to the Ramond sector along with some nontrivial checks were presented in [14].

Although most of the ingredients and constructions were already discussed in [5] and [14] a precise content of (1) as an exact equivalence of CFT models was an open problem. The aim of this letter is to show how the gaps present in [5, 14] can be filled. For technical details the reader may consult [10].

2 Liouville Field Theory

The Liouville field theory on a flat, two-dimensional space is described by an action

$$S_{\rm L}[\varphi] = \int d^2z \left[|\partial\varphi|^2 + \mu \mathrm{e}^{2b\varphi} \right]$$

where φ is a real, bosonic field. It possesses a holomorphic current (energy-momentum tensor):

$$T(z) = -\frac{1}{2}\left(\partial\varphi\right)^2 + Q\partial^2\varphi, \qquad Q = b + b^{-1},$$

of conformal weights $(2, 0)$, whose modes form the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m \left(m^2 - 1\right) \delta_{m+n}$$

with the central charge $c = 1 + 6Q^2$.

For $c > 0$ (what corresponds to $Q \in \mathbb{R}$) the solution of Liouville field theory is known:

- its Hilbert space is

$$\mathcal{H} = \int_{\oplus} \pi_{\mathsf{Vir}}(p) \otimes \bar{\pi}_{\mathsf{Vir}}(p)\, dp, \qquad p \in \mathbb{R}_+$$

  where $\pi_{\mathsf{Vir}(p)}$ is the Virasoro Verma module build on the highest weight state $|p\rangle$ with the highest weight (eigenvalue of L_0) equal to $\Delta_p = \frac{1}{4}Q^2 + p^2$,
- three-point function of primary fields

$$V_p(z,\bar{z}) \cong \mathrm{e}^{2\alpha\varphi}, \qquad \alpha = \frac{Q}{2} + ip,$$

  is expressible through the special function

$$\Upsilon_b(x) = \int_0^\infty \frac{dt}{t}\left[\left(\frac{Q}{2} - x\right)^2 \mathrm{e}^{-t} - \frac{\sinh^2\left(\frac{Q}{2} - x\right)}{\sinh\frac{bt}{2}\sinh\frac{t}{2b}}\right]$$

  and reads

$$C_{\mathrm{L}}(\alpha_1,\alpha_2,\alpha_3) \sim \frac{\Upsilon_b(2\alpha_1)\Upsilon_b(2\alpha_2)\Upsilon_b(2\alpha_3)}{\Upsilon_b(\alpha_{1+2+3} - Q)\Upsilon_b(\alpha_{1+2-3})\Upsilon_b(\alpha_{2+3-1})\Upsilon_b(\alpha_{3+1-2})}$$

  with $\alpha_{1+2-3} \equiv \alpha_1 + \alpha_2 - \alpha_3$ etc.

Three-point correlation function $C_\Upsilon(\alpha_3,\alpha_2,\alpha_1)$ of Liouville field theory with $c < 1$ was obtained by analytically continuing difference equation satisfied by $C_{\mathrm{L}}(\alpha_3,\alpha_2,\alpha_1)$ to the region $c < 1$ [17]. This model is still to some extend mysterious: $C_\Upsilon(\alpha_3,\alpha_2,0)$ does not vanish for $\alpha_3 \neq \alpha_2$, some degenerate fields do not decouple, there exist extra operator (beside identity) of dimension 0 and the spectrum, on which one should factorize correlation functions, is not known.

2.1 $\mathcal{N} = 1$ *Superconformal Liouville Field Theory*

The action of the model reads

$$S_{\mathrm{SL}}[\varphi,\psi] = \int d^2z\left[|\partial\varphi|^2 + \psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi} + \mu\bar{\psi}\psi\mathrm{e}^{b\varphi}\right]$$

where $\psi, \bar{\psi}$ are two-dimensional fermions with conformal weights $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, respectively. This model possesses two holomorphic currents

$$T(z) = -\tfrac{1}{2}(\partial\varphi)^2 - \tfrac{1}{2}\psi\partial\psi + Q\partial^2\varphi, \qquad G(z) = -i\psi\partial\varphi + iQ\partial\psi.$$

Their modes satisfy the Neveu-Schwarz–Ramond (or NSR) algebra

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \tfrac{c}{12}m\left(m^2-1\right)\delta_{m+n},\\
[L_m, G_k] &= \left(\tfrac{1}{2}m-k\right)G_{m+k},\\
\{G_k, G_l\} &= 2L_{k+l} + \tfrac{c}{3}\left(k^2-\tfrac{1}{4}\right)\delta_{k+l},
\end{aligned}
$$

with $c = \frac{3}{2} + 3Q^2$.

We shall only discuss the Neveu-Schwarz (or NS) sector with the half-integer k.

- The super-primary fields in this sector are $\Phi_\alpha \cong \mathrm{e}^{\alpha\varphi}$, with equal left and right conformal dimensions $\Delta_p = \frac{1}{8}Q^2 + \frac{1}{2}p^2,\ \alpha = \frac{1}{2} + ip$.
- The basic three-point coupling constants

$$C_{\mathrm{SL}}(\alpha_3, \alpha_2\alpha_1) \sim \langle \Phi_{\alpha_3}\Phi_{\alpha_2}\Phi_{\alpha_1}\rangle$$

  and

$$\tilde{C}_{\mathrm{SL}}(\alpha_3, \alpha_2\alpha_1) \sim \left\langle \Phi_{\alpha_3}\left(\bar{G}_{-\frac{1}{2}}G_{-\frac{1}{2}}\cdot\Phi_{\alpha_2}\right)\Phi_{\alpha_1}\right\rangle,$$

  are known and expressible through the functions

$$\Upsilon_0(x) = \Upsilon_b\left(\tfrac{x+b}{2}\right)\Upsilon_b\left(\tfrac{x+b^{-1}}{2}\right),\ \ \Upsilon_1(x) = \Upsilon_b\left(\tfrac{x}{2}\right)\Upsilon_b\left(\tfrac{x+Q}{2}\right).$$

  Let us denote by $\tilde{\mathcal{F}}_{\mathrm{NS}}$ the algebra of fermionic modes

$$\{f_r, f_s\} = \delta_{r+s,0},\ \ r,s\in\mathbb{Z}+\tfrac{1}{2},\qquad \{G_k, f_r\} = [L_m, f_r] = 0.$$

Using them together with the modes of NS algebra one can construct two sets of generators:

$$
\begin{aligned}
L_n^{\mathrm{L}} &= \frac{1}{1-b^2}L_n - \frac{1+2b^2}{1-b^2}L_n^{\mathfrak{f}} + \frac{b}{1-b^2}\sum_r f_{n-r}G_r,\\
L_n^{\Gamma} &= \frac{1}{1-b^{-2}}L_n - \frac{1+2b^{-2}}{1-b^{-2}}L_n^{\mathfrak{f}} + \frac{b^{-1}}{1-b^{-2}}\sum_r f_{n-r}G_r.
\end{aligned}
$$

They form two mutually commuting Virasoro algebras with central charges

$$
\begin{aligned}
c^{\mathrm{L}} &= 1 + 6\left(Q^{\mathrm{L}}\right)^2,\ Q^{\mathrm{L}} = b^{\mathrm{L}} + \frac{1}{b^{\mathrm{L}}},\ b^{\mathrm{L}} = \frac{2b}{\sqrt{2-2b^2}},\\
c^{\mathrm{L}} &= 1 - 6\left(Q^{\Gamma}\right)^2,\ Q^{\Gamma} = \frac{1}{b^{\Gamma}} - b^{\Gamma},\ \frac{1}{b^{\Gamma}} = \frac{2}{\sqrt{2-2b^2}}.
\end{aligned}
$$

For generic values of the momentum p the highest weight representation of $\mathsf{NSR} \oplus \tilde{\mathcal{F}}_{\mathrm{NS}}$ is irreducible and its character is given by

$$\chi_{\mathsf{NSR}\oplus\tilde{\mathcal{F}}_{\mathrm{NS}}}(q) = \chi_{\mathcal{F}_{\mathrm{NS}}}(q)^2 \chi_{\mathcal{B}}(q)$$

where

$$\chi_{\mathcal{F}_{\mathrm{NS}}}(q) = \prod_{n>0}\left(1+q^{n-\frac{1}{2}}\right), \quad \chi_{\mathcal{B}}(q) = \prod_{n>0}\frac{1}{1-q^n}.$$

The Jacobi triple product identity

$$\prod_{n>0}\left(1+q^{n-\frac{1}{2}}\right)^2 (1-q^n) = \sum_{k\in\mathbb{Z}} q^{\frac{k^2}{2}}$$

then gives a decomposition of the highest weight module of $\mathsf{NSR} \oplus \tilde{\mathcal{F}}_{\mathrm{NS}}$ into the direct sum

$$\pi_{\mathsf{NSR}\oplus\tilde{\mathcal{F}}_{\mathrm{NS}}} = \bigoplus_{j\in\mathbb{Z}} \pi^j_{\mathsf{Vir}\oplus\mathsf{Vir}}.$$

Here $\pi^j_{\mathsf{Vir}\oplus\mathsf{Vir}}$ is the Verma module of the algebra $\mathsf{Vir} \oplus \mathsf{Vir}$ with the highest weight $|p, j\rangle$,

$$L_0^{\mathrm{L},\Gamma}|p,j\rangle = \Delta^{\mathrm{L},\Gamma}(p,j)|p,j\rangle, \qquad L_n^{\mathrm{L},\Gamma}|p,j\rangle = 0, \quad n>0$$

where

$$\Delta^{\mathrm{L}}(p,j) = \frac{1}{1-b^2}\left(\frac{Q^2}{8} + \frac{(p+ij\,b)^2}{2}\right),$$

$$\Delta^{\Gamma}(p,j) = \frac{1}{1-b^{-2}}\left(\frac{Q^2}{8} + \frac{\left(p+ij\,b^{-1}\right)^2}{2}\right).$$

3 The Relation

To precisely formulate the relation between double Liouville and super-Liouville field theory we have to:

- construct the states $|p, j\rangle$ in $\pi_{\mathsf{NS}\oplus\tilde{\mathcal{F}}_{\mathrm{NS}}}$,
- construct the corresponding operators (they are primary fields with respect to $\mathsf{Vir} \oplus \mathsf{Vir}$, but descendants of NS),

- compute normalization factors and check the equality of three-point correlation functions on both sides,
- prove the equality of higher-point correlation functions.

There exist a well known family of "free field" (or Feigin-Fuchs) representations of the NSR generators in terms od the bosonic and fermionic oscillators,

$$L_0(p) = \sum_{m\geqslant 1} c_{-m}c_m + \sum_{k\geqslant \frac12} k\,\psi_{-k}\psi_k + \tfrac18 Q^2 + \tfrac12 p^2,$$

$$L_n(p) = \tfrac12 \sum_{m\neq 0,n} c_{n-m}c_m + \tfrac12 \sum_{k\in\mathbb{Z}+\frac12} k\,\psi_{n-k}\psi_k + \left(\tfrac{inQ}{2} + p\right)c_n, \qquad n\neq 0,$$

$$G_k(p) = \sum_{m\neq 0} c_m\psi_{k-m} + (iQk+p)\psi_k,$$

where

$$[c_m, c_n] = m\delta_{m+n}, \qquad \{\psi_r, \psi_s\} = \delta_{r+s}.$$

If we denote by $|\omega\rangle$ the Fock vacuum of the c_m, ψ_k, f_r algebra and define

$$\chi_k = f_k - i\psi_k$$

then the states

$$|p, j\rangle_F = \Omega(p,j)\chi_{-\frac{2j-1}{2}}\ldots\chi_{-\frac12}|\omega\rangle$$

do satisfy defining equations of $|p, j\rangle$.

In order to view them as states in $\pi_{\mathsf{NS}\oplus\tilde{\mathcal{F}}_{\mathrm{NS}}}$ we need to express — using the form of the Feigin-Fuchs representation of NSR — the state

$$\psi_{-\frac{2j-1}{2}}\ldots\psi_{-\frac12}|\omega\rangle$$

in the basis of the NS Verma module.

Let J, K, K' and M, N, N' denote multiindices. We have

$$c_{-M}\psi_{-J}|\omega\rangle = \sum_{N,N',K,K'} \langle\omega|G^{\dagger}_{-K'}(p)L^{\dagger}_{-N'}(p)c_{-M}\psi_{-J}|\omega\rangle \times$$
$$\times B^{N'K',NK}\, L_{-N}G_{-K}|\Delta_p\rangle$$

where $B^{N'K',NK}$ is the inverse to the Gramm-Shapovalov matrix on π_{NS}. The coefficients of this expansion are rational functions of p with poles at some subset of zeroes of the Kac determinant.

Let us define

$$\Omega(p,j) = \prod_{\langle r,s\rangle\in\mathcal{Z}} \left(2ip + rb + sb^{-1}\right)$$

where $\mathcal{Z}$ is chosen such that the coefficients $c_{\frac{1}{2}\frac{1}{2}\cdots\frac{1}{2}}, c_{\frac{1}{2}\frac{1}{2}\cdots\frac{3}{2}}, \ldots$ of the expansion

$$\Omega(p,j)\psi_{-\frac{2j-1}{2}}\ldots\psi_{-\frac{1}{2}}|\omega\rangle = \left(c_{\frac{1}{2}\frac{1}{2}\cdots\frac{1}{2}}(p)G_{-\frac{1}{2}}^{j^2} + c_{\frac{1}{2}\frac{1}{2}\cdots\frac{3}{2}}G_{-\frac{1}{2}}^{j^2-3}G_{-\frac{3}{2}} + \ldots\right)|\omega\rangle$$

are polynomials in p with no common factor. One can then show that

$$\mathcal{Z} = \{\langle r,s\rangle : \quad r,s\in\mathbb{Z}_{>0},\ r+s\equiv 0 \bmod 2,\ r+s\leqslant 2j\}.$$

We have already demonstrated that (as the vector spaces)

$$\bigoplus_{j\in\mathbb{Z}} \pi_{\mathsf{Vir}}(\Delta^{\mathrm{L}}(p,j))\otimes\pi_{\mathsf{Vir}}(\Delta^{\Gamma}(p,j)) = \pi_{\mathrm{NS}}(\Delta_p)\otimes\pi_{\tilde{\mathcal{F}}_{\mathrm{NS}}}$$

where

$$L^{\mathrm{L}}_{-M}|\,\nu^{\mathrm{L}}_{p,j}\rangle\otimes L^{\Gamma}_{-N}|\,\nu^{\Gamma}_{p,j}\rangle \quad\longrightarrow\quad L^{\mathrm{L}}_{-M}L^{\Gamma}_{-N}\,|\,p,j\,\rangle$$

The equivalence above is a unitary isomorphism if we assume on the l.h.s. the scalar product such that

$$\langle\nu^{\mathrm{L}}_{p,j}\otimes\nu^{\Gamma}_{p,j}\,|\,\nu^{\mathrm{L}}_{p,j'}\otimes\nu^{\Gamma}_{p,j'}\rangle = \langle\,p,j\ \,|p,-j\,\rangle\,\delta_{j+j',0}.$$

The skew form of this product is the only one consistent with the complex weights $\Delta^{\mathrm{L}}(p,j), \Delta^{\Gamma}(p,j), j\neq 0$ and the hermiticity of $L^{\mathrm{L}}_0, L^{\Gamma}_0$.

A counterpart of the map between vector spaces

$$L^{\mathrm{L}}_{-M}L^{\Gamma}_{-N}\,|\,p,j\,\rangle_n \quad\longrightarrow\quad L^{\mathrm{L}}_{-M}|\,\nu^{\mathrm{L}}_{p,j}\rangle\otimes L^{\Gamma}_{-N}|\,\nu^{\Gamma}_{p,j}\rangle.$$

is a map between chiral vector operators

$$\mathcal{L}^{\mathrm{L}}_{-M}\mathcal{L}^{\Gamma}_{-N}V_{p,j} \quad\longrightarrow\quad \mathcal{L}^{\mathrm{L}}_{-M}V^{\mathrm{L}}_{p,j}\otimes\mathcal{L}^{\Gamma}_{-N}V^{\Gamma}_{p,j}.$$

In particular

$$V_{p,j} \quad\longrightarrow\quad V^{\mathrm{L}}_{p,j}\otimes V^{\Gamma}_{p,j}.$$

Let us denote $\mathsf{NS}\oplus\tilde{\mathcal{F}}_{\mathrm{NS}}\equiv\mathsf{A}_{\mathrm{NS}}$. The matrix element of the chiral field $V_{p_2,j_2}(z)$ between the states $|\,p_1,j_1\,\rangle\equiv\xi_{p_1,j_1}$ and $|\,p_3,j_3\,\rangle$ can be written as

$$\langle\, p_3, j_3 \,|\, V_{p_2,j_2}(1)\,|\, p_1, j_1 \,\rangle = \mathsf{C}^{\text{NS}}_b(\alpha_3,\alpha_2,\alpha_1)\; \rho^{\text{A}}_{\text{NS}}(\xi_{p_3,j_3},\xi_{p_2,j_2},\xi_{p_1,j_1}|1)$$

where C^{NS} is a *chiral* three point super-Liouville structure constants.

The three point blocks $\rho^{\text{A}}_{\text{NS}}$ can be explicitly calculated by applying free-field representation of involved fields, the so-called reflection map on the A_{NS} algebra and computing Selberg averages of some symmetric polynomials.

On the double Liouville side the matrix element

$$\left\langle \nu^{\text{L}}_{p_3,j_3} \right| \otimes \left\langle \nu^{\Gamma}_{p_3,j_3} \right| V^{\text{L}}_{p_2,j_2} \otimes V^{\Gamma}_{p_2,j_2} \left| \nu^{\text{L}}_{p_3,j_3} \right\rangle \otimes \left| \nu^{\Gamma}_{p_3,j_3} \right\rangle$$

is given by a product of "ordinary" and "imaginary" Liouville *chiral* structure constants

$$\mathsf{C}^{\text{L}}_{b^{\text{L}}}(\alpha^{\text{L}}_3 + \frac{j_3 b^{\text{L}}}{2}, \alpha^{\text{L}}_2 + \frac{j_2 b^{\text{L}}}{2}, \alpha^{\text{L}}_1 + \frac{j_1 b^{\text{L}}}{2})\mathsf{C}^{\Gamma}_{b^{\Gamma}}(\alpha^{\Gamma}_3 + \frac{j_3}{2b^{\Gamma}}, \alpha^{\Gamma}_2 + \frac{j_2}{2b^{\Gamma}}, \alpha^{\Gamma}_1 + \frac{j_1}{2b^{\Gamma}}).$$

The equality

$$\frac{\mathsf{C}^{\text{L}}_{b^{\text{L}}}(\alpha^{\text{L}}_3 + \frac{j_3 b^{\text{L}}}{2}, \alpha^{\text{L}}_2 + \frac{j_2 b^{\text{L}}}{2}, \alpha^{\text{L}}_1 + \frac{j_1 b^{\text{L}}}{2})\mathsf{C}^{\Gamma}_{b^{\Gamma}}(\alpha^{\Gamma}_3 + \frac{j_3}{2b^{\Gamma}}, \alpha^{\Gamma}_2 + \frac{j_2}{2b^{\Gamma}}, \alpha^{\Gamma}_1 + \frac{j_1}{2b^{\Gamma}})}{\mathsf{C}^{\text{NS}}_b(\alpha_3,\alpha_2,\alpha_1)}$$

$$= \left(\prod_{k=1}^{3} \frac{(-1)^{\frac{j_k^2}{2}}}{\sqrt{l(2\alpha_k, 2j_k)\, l(2\alpha_k - Q, 2j_k)}} \right) \rho^{\text{A}}_{\text{NS}}(\xi_{p_3,j_3},\xi_{p_2,j_2},\xi_{p_1,j_1}|1)$$

where

$$l(x,n) = \prod_{0 \leqslant r+s < n} \left(x + rb + sb^{-1} \right), \qquad r,s \in \mathbb{N}, \quad r+s \in 2\mathbb{N},$$

then follows thanks to some identities satisfied by the Barnes special functions.

After a final check that for $L^{\text{L}}_n, L^{\Gamma}_n$ generators one can use the same Virasoro Ward identities on both sides of the correspondence, we conclude that the map

$$L^{\text{L}}_{-M} L^{\Gamma}_{-N} \,|\, p, j \,\rangle_n \quad \longrightarrow \quad L^{\text{L}}_{-M} |\, \nu^{\text{L}}_{p,j} \rangle \otimes L^{\Gamma}_{-N} |\, \nu^{\Gamma}_{p,j} \rangle,$$

an isomorphism of $\mathsf{Vir} \oplus \mathsf{Vir}$ representations, together with its counterpart for the corresponding chiral operators

$$\mathcal{L}^{\text{L}}_{-M} \mathcal{L}^{\Gamma}_{-N} V_{p,j} \quad \longrightarrow \quad \mathcal{L}^{\text{L}}_{-M} V^{\text{L}}_{p,j} \otimes \mathcal{L}^{\Gamma}_{-N} V^{\Gamma}_{p,j},$$

provides an equivalence of the SLiouville $\times$ fermion and double Liouville field theories.

Let us finally note some unusual features of the correspondence: it is chiral down to the level of structure constants and on the double-Liouville side there appear operators with arbitrary integer, two-dimensional "spins".

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Multi-Point Virtual Structure Constants and Mirror Computation of CP^2-Model

Masao Jinzenji

Abstract This article is a brief summary of the results presented in the paper (Jinzenji, M., Shimizu, M.: Multi-point virtual structure constants and mirror computation of CP^2-model. Communications in Number Theory and Physics, **7**(3), 411–468 (2013)) with the same title, which is a joint work with Dr. M. Shimizu.

1 Introduction

In [8], we gave a geometrical construction of the mirror map used in the mirror computation of the genus 0 Gromov-Witten invariants of projective hypersurfaces.

Let M_N^k be a degree k hypersurface in CP^{N-1}. The two-pointed Gromov-Witten invariant $\langle \mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_{0,d}$ is a rational number geometrically defined by the following formula:

$$\langle \mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_{0,d} = \int_{\overline{M}_{0,2}(CP^{N-1},d)} ev_1^*(h^a)\wedge ev_2^*(h^b)\wedge c_{top}(R^0\pi_* ev_3^*\mathcal{O}_{CP^{N-1}}(k)). \tag{1}$$

h is the hyperplane class in $H^{1,1}(CP^{N-1},\mathbf{C})$. $\overline{M}_{0,n}(CP^{N-1},d)$ is the moduli space of stable maps of degree d from genus 0 semi-stable curves with n marked points to CP^{N-1}. $ev_i : \overline{M}_{0,n}(CP^{N-1},d) \to CP^{N-1}$ is the evaluation map at the i-th marked point. $\pi : \overline{M}_{0,3}(CP^{N-1},d) \to \overline{M}_{0,2}(CP^{N-1},d)$ is the forgetful map that forgets the third marked point. Roughly speaking, this invariant counts the number of rational curves in M_N^k that intersect Poincaré dual cycles $PD(h^a)$ and $PD(h^b)$. The first and the second factors in the integrand of (1) represent the condition that the image of the stable map intersect $PD(h^a)$ and $PD(h^b)$, and the third factor guarantees that the image curve lies inside the hypersurface. If the topological selection rule:

$$a+b = N-3+(N-k)d, \tag{2}$$

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is satisfied, this invariant becomes non-trivial (otherwise, it is trivially 0). If $N > k$, it is believed to be an integer and to count indeed the number of rational curves that satisfies the above conditions.

Mirror computation of the genus 0 Gromov-Witten invariant $\langle \mathcal{O}_{h^a}\mathcal{O}_{h^b}\rangle_{0,d}$ is a way of computing it by using the following differential equation as a starting point:

$$\Big((\partial_x)^{N-1} - k\cdot e^x\cdot(k\partial_x + k-1)(k\partial_x+k-2)\cdots(k\partial_x+1)\Big)w(x) = 0. \quad (3)$$

Let us briefly review the process of mirror computation in the case when $N = k$, i.e., when the hypersurface is a Calabi-Yau manifold. In this case, we consider the following decomposition of the differential operator in the differential equation (3):

$$\Big(\big(\partial_x\big)^{k-1} - ke^x\prod_{j=1}^{k-1}\big(k\partial_x + j\big)\Big)w(x) = \quad (4)$$

$$= \frac{1}{\tilde{L}^{k,k}_{k-1}(e^x)}\partial_x(\frac{1}{\tilde{L}^{k,k}_{k-2}(e^x)}\partial_x(\frac{1}{\tilde{L}^{k,k}_{k-3}(e^x)}\partial_x(\cdots(\frac{1}{\tilde{L}^{k,k}_{1}(e^x)}\partial_x(\frac{w(x)}{\tilde{L}^{k,k}_{0}(e^x)})).\cdots)))$$

where $\tilde{L}^{k,k}_j(e^x) = 1 + \sum_{d=1}^{\infty}\tilde{L}^{k,k,d}_j e^{dx};\ (j = 0,1,\cdots,k-1)$ is a power series in e^x. We call the expansion coefficient $\tilde{L}^{k,k,d}_j$ "virtual structure constants". These power series are efficiently determined by the solution of the differential equation. Let $w_j(x)\ \ (j = 0,1,2,\cdots,k-1)$ be a set of functions given by,

$$w(x,y) := \sum_{d=0}^{\infty}\frac{\prod_{j=1}^{kd}(j+ky)}{\prod_{j=1}^{d}(j+y)^k}e^{(d+y)x}, \quad (5)$$

$$w_j(x) := \frac{1}{j!}\frac{\partial^j w}{\partial y^j}(x,0).$$

$w_j(x)$ $(j = 0,1,\cdots,k-2)$ are solutions of (3) with $N = k$. Then $\tilde{L}^{k,k}_j(e^x)$ is determined inductively by the following relations[1]:

$$\tilde{L}^{k,k}_0(e^x) = w_0(x),\quad \tilde{L}^{k,k}_1(e^x) = \frac{d}{dx}\frac{w_1(x)}{w_0(x)} = \frac{d}{dx}\frac{w_1(x)}{\tilde{L}^{k,k}_0(e^x)}, \quad (6)$$

$$\tilde{L}^{k,k}_j(e^x) = \frac{d}{dx}(\frac{1}{\tilde{L}^{k,k}_{j-1}(e^x)}\frac{d}{dx}(\frac{1}{\tilde{L}^{k,k}_{j-2}(e^x)}\frac{d}{dx}(\frac{1}{\tilde{L}^{k,k}_{j-3}(e^x)}\cdots\times$$

$$\times\frac{d}{dx}(\frac{1}{\tilde{L}^{k,k}_{1}(e^x)}\frac{d}{dx}\frac{w_j(x)}{\tilde{L}^{k,k}_{0}(e^x)})\cdots))).$$

[1] In (6), we need to use formally $w_{k-1}(x)$ to determine $\tilde{L}^{k,k}_{k-1}(e^x)$ though it is not a solution of (3).

We define the mirror map $t = t(x)$ which plays a crucial role in the mirror computation from $\tilde{L}_1^{k,k}(e^x)$:

$$t = t(x) = \int \tilde{L}_1^{k,k}(e^x)dx = x + \sum_{d=1}^{\infty} \frac{\tilde{L}_1^{k,k,d}}{d} e^{dx}. \tag{7}$$

By inverting the mirror map in the form $x = x(t)$, we can obtain the generating function of $\langle \mathcal{O}_{h^{j-1}} \mathcal{O}_{h^{k-2-j}} \rangle_{0,d}$ from $\int \tilde{L}_j^{k,k}(e^x)dx \quad (j = 1, 2, \cdots, k-2)$ by the following equality:

$$kt + \sum_{d=1}^{\infty} \langle \mathcal{O}_{h^{j-1}} \mathcal{O}_{h^{k-2-j}} \rangle_{0,d} e^{dt} = kx(t) + \sum_{d=1}^{\infty} \frac{k\tilde{L}_j^{k,k,d}}{d} e^{dx(t)}. \tag{8}$$

This equality follows from the mirror computation of three point genus 0 Gromov-Witten invariants of M_k^k [7]:

$$k + \sum_{d=1}^{\infty} \langle \mathcal{O}_h \mathcal{O}_{h^{j-1}} \mathcal{O}_{h^{k-2-j}} \rangle_{0,d} e^{dt} = \frac{d}{dt}\left(kt + \sum_{d=1}^{\infty} \langle \mathcal{O}_{h^{j-1}} \mathcal{O}_{h^{k-2-j}} \rangle_{0,d} e^{dt} \right)$$
$$= k \cdot \frac{\tilde{L}_j^{k,k}(e^{x(t)})}{\tilde{L}_1^{k,k}(e^{x(t)})}. \tag{9}$$

Our motivation of the work [8] comes from the formula (8). In (8), $\langle \mathcal{O}_{h^{j-1}} \mathcal{O}_{h^{k-2-j}} \rangle_{0,d}$ and $\frac{k\tilde{L}_j^{k,k,d}}{d}$ differ only by coordinate change. Therefore, there must be a possibility to construct $\frac{k\tilde{L}_j^{k,k,d}}{d}$ as an intersection number of some moduli space of holomorphic maps from genus 0 curve to CP^{N-1}, with different compactification. In [8], we consider a moduli space of quasi-maps with two marked points from genus 0 curve to CP^{N-1}, compactified by $\mathbf{C}^{\times}$ geometric invariant theory. We denote this space by $\widetilde{Mp}_{0,2}(N,d)$. Detailed construction of this moduli space is given in [8]. Boundary components of $\widetilde{Mp}_{0,2}(N,d)$ consist only of quasi-maps from genus 0 semi-stable curves whose components are arranged in a line shape. Therefore, combinatorial structure of boundaries is much simpler than the moduli space of stable maps: $\overline{M}_{0,2}(CP^{N-1}, d)$. In [8], we defined the following intersection number of $\widetilde{Mp}_{0,2}(N,d)$, whose geometrical meaning is analogous to $\langle \mathcal{O}_{h^a} \mathcal{O}_{h^b} \rangle_{0,d}$ of M_N^k.

$$w(\mathcal{O}_{h^a} \mathcal{O}_{h^b})_{0,d} = \int_{\widetilde{Mp}_{0,2}(N,d)} ev_1^*(h^a) \wedge ev_2^*(h^b) \wedge c_{top}(\mathcal{E}_k). \tag{10}$$

The three factors in the integrand has the same geometrical meaning as the ones in (1). Since combinatorial structure of $\widetilde{Mp}_{0,2}(N,d)$ is much simpler than

$\overline{M}_{0,2}(CP^{N-1}, d)$, we can derive an explicit closed formula of $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,d}$ with the aid of localization technique. With this formula, we proved the following theorem:

Theorem 1 ([8]).

$$w(\mathcal{O}_{h^{j-1+(N-k)d}}\mathcal{O}_{h^{N-2-j}})_{0,d} = \frac{k\tilde{L}_j^{N,k,d}}{d}. \tag{11}$$

Here, $\tilde{L}_j^{N,k,d}$ is a "virtual structure constant" constructed from the differential equation (3), and it is translated into genus 0 Gromov-Witten invariants of M_N^k via "generalized mirror transformation". The expansion coefficients of the mirror map used in the generalized mirror transformation are given by $\frac{\tilde{L}_{1+(k-N)d}^{N,k,d}}{d}$ in the case of general k and N. Since $\tilde{L}_j^{N,k,d} = 0 \ \ (j < 0)$, the mirror map becomes trivial when $N-k > 1$. In this case, we obtain,

$$w(\mathcal{O}_{h^{j-1+(N-k)d}}\mathcal{O}_{h^{N-2-j}})_{0,d} = \frac{k\tilde{L}_j^{N,k,d}}{d} = \langle\mathcal{O}_{h^{j-1+(N-k)d}}\mathcal{O}_{h^{N-2-j}}\rangle_{0,d}, \quad (N-k>1). \tag{12}$$

Let us look back at the $k = N$ case. If we combine (7) with (11), we obtain

$$t = t(x) = x + \sum_{d=0}^{\infty}\frac{w(\mathcal{O}_1\mathcal{O}_{h^{k-3}})_{0,d}}{k}e^{dt}. \tag{13}$$

This formula says that the mirror map can be interpreted as a generating function of intersection numbers of $\widetilde{Mp}_{0,2}(N,d)$. [2] We can also rewrite (8) in the following form:

$$kt + \sum_{d=1}^{\infty}\langle\mathcal{O}_{h^{j-1}}\mathcal{O}_{h^{k-2-j}}\rangle_{0,d}e^{dt} = kx(t) + \sum_{d=1}^{\infty}w(\mathcal{O}_{h^{j-1}}\mathcal{O}_{h^{k-2-j}})_{0,d}e^{dx(t)}. \tag{14}$$

(13) and (14) say that we can reconstruct the mirror computation of genus 0 Gromov-Witten invariants by using $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_{0,d}$ as a starting point. Our aim of this article is to generalize the above results to construct a kind of "mirror computation" of genus 0 Gromov-Witten invariants of CP^2.

[2] Recently, some works that can be regarded as generalization of this result appeared [3, 5].

2 Multi-Point Virtual Structure Constants

In the CP^2 case, interesting genus 0 Gromov-Witten invariants are: $\langle(\mathcal{O}_{h^2})^{3d-1}\rangle_{0,d}$ $(d \geq 1)$. Therefore, in order to generalize the formalism in Sect. 1 to the CP^2 case, we have to construct w-intersection numbers with more than two operator insertions. In [10], we constructed $\widetilde{Mp}_{0,2|n}(N,d)$, the moduli space of quasi-maps from genus 0 curve to CP^{N-1} with $2+n$ marked points, compactified by $\mathbf{C}^{\times}$ geometric invariant theory. In construction of $\widetilde{Mp}_{0,2|n}(N,d)$, we use semi-stable genus 0 curves whose component CP^1's are arranged in a line shape and are connected at 0 and ∞. The 0 of the left end CP^1 and the ∞ of the right end CP^1 are two special marked points and correspond to 2 in the notation $2|n$. These marked points are special and distinguished from th other n marked points. The remaining n marked points are distributed to CP^1 components randomly, but should not lie on 0 and ∞ of each component CP^1. In contrast to the construction of $\widetilde{Mp}_{0,2}(N,d)$, we allow existence of some component CP^1's mapped to a point in CP^{N-1}. To describe this kind of situation, we use $\overline{M}_{0,2|m}$, the moduli space of complex structure of genus 0 curve with $2+m$ marked points, compactified by $\mathbf{C}^{\times}$ geometric invariant theory [1,13]. Detailed construction of $\widetilde{Mp}_{0,2|n}(N,d)$ is given in [10]. With this moduli space, we introduced the following intersection number that can be regarded as an analogue of the genus 0 Gromov-Witten invariant of CP^{N-1}: $\langle\mathcal{O}_{h^a}\mathcal{O}_{h^b}\prod_{j=1}^{n}\mathcal{O}_{h^{m_j}}\rangle_{0,d}$.

Definition 1.

$$w(\mathcal{O}_{h^a}\mathcal{O}_{h^b}|\prod_{j=1}^{n}\mathcal{O}_{h^{m_j}})_{0,d} := \int_{\widetilde{Mp}_{0,2|n}(N,d)} ev_0^*(h^a)\cdot ev_{\infty}^*(h^b)\cdot\prod_{j=1}^{n} ev_j^*(h^{m_j}), \qquad (15)$$

where $\cdot$ is the product of the cohomology ring $H^*(\widetilde{Mp}_{0,2|n}(N,d))$.

In (15), ev_0 is the evaluation map at 0 of the left end CP^1, and ev_{∞} is the evaluation map at ∞ of the right end CP^1. ev_j is also the evaluation map at the j-th marked point of the remaining n marked points. In the same way as the $\widetilde{Mp}_{0,2}(N,d)$ case, we can derive an explicit closed formula of this intersection number.

Theorem 2.

$$\begin{aligned} & w(\mathcal{O}_{h^a}\mathcal{O}_{h^b}|\prod_{i=1}^{n}(\mathcal{O}_{h^{m_i}}))_{0,d} \\ &= \frac{1}{(2\pi\sqrt{-1})^{d+1}}\oint_{E_{(0)}^0}\frac{dz_0}{(z_0)^N}\oint_{E_{(0)}^1}\frac{dz_1}{(z_1)^N}\cdots\oint_{E_{(0)}^d}\frac{dz_d}{(z_d)^N} \\ &\times(z_0)^a\cdot\left(\prod_{j=1}^{d-1}\frac{1}{(2z_j-z_{j-1}-z_{j+1})}\right)\cdot(z_d)^b\cdot\prod_{i=1}^{n}\left(\sum_{j=1}^{d}w_{m_i}^1(z_{j-1},z_j)\right), \quad (d>0). \end{aligned} \qquad (16)$$

where $w^1_m(z,w) = \frac{z^m-w^m}{z-w}$ *and* $\frac{1}{2\pi\sqrt{-1}}\oint_{E^j_{(0)}} dz_j$ *means the operation of taking residues at* $z_j = 0$ *and* $z_j = \frac{z_{j-1}+z_{j+1}}{2}$ *for* $j = 1, 2, \cdots, l-1$ *(resp.* $z_j = 0$ *for* $j = 0, d$*).*

The proof of this theorem is given by localization technique, but in contrast to the case of $\widetilde{Mp}_{0,2}(N,d)$, some results on intersection numbers of gravitational descendants on $\overline{M}_{0,2|n}$ are needed. For this purpose, we used the results given in [1, 13]. In (16), effect of the first two operator insertions $\mathcal{O}_{h^a}\mathcal{O}_{h^b}$ is reflected in the terms $(z_0)^a$ and $(z_d)^b$ in the r.h.s., but effect of remaining operator insertions $\prod_{i=1}^n(\mathcal{O}_{h^{m_i}})$ is represented by $\prod_{i=1}^n\left(\sum_{j=1}^d w^1_{m_i}(z_{j-1},z_j)\right)$. This fact tells us that these two kinds of operator insertions have different characteristics. Therefore we insert "|" in the notation :$w(\mathcal{O}_{h^a}\mathcal{O}_{h^b}|\prod_{i=1}^n(\mathcal{O}_{h^{m_i}}))_{0,d}$. From this formula, we can easily see that the puncture axiom:

$$w(\mathcal{O}_{h^a}\mathcal{O}_{h^b}|\mathcal{O}_1\prod_{j=1}^n\mathcal{O}_{h^{m_j}})_{0,d} = 0, \tag{17}$$

and the divisor axiom:

$$w(\mathcal{O}_{h^a}\mathcal{O}_{h^b}|\mathcal{O}_h\prod_{j=1}^n\mathcal{O}_{h^{m_j}})_{0,d} = d\cdot w(\mathcal{O}_{h^a}\mathcal{O}_{h^b}|\prod_{j=1}^n\mathcal{O}_{h^{m_j}})_{0,d}, \tag{18}$$

hold for the latter type of operator insertions.

3 Mirror Computation

In this section, we focus on the intersection number $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b}|\prod_{i=1}^n(\mathcal{O}_{h^{m_i}}))_{0,d}$ for CP^2 and use it as the starting point of mirror computation of genus 0 Gromov-Witten invariants of CP^2 in the spirit of the formulas (13) and (14). Since $H^{*,*}(CP^2,\mathbf{C})$ is spanned by $1 = h^0$, h and h^2, we introduce the following generating function of $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b}|\prod_{j=0}^2(\mathcal{O}_{h^j})^{m_j})_{0,d}$:

Definition 1.

$$\begin{aligned}
&w(\mathcal{O}_{h^a}\mathcal{O}_{h^b}|(x^0,x^1,x^2))_0\\
&:= x^c\cdot\int_{CP^2}h^{a+b+c} + \sum_{d>0,\{m_j\}} w(\mathcal{O}_{h^a}\mathcal{O}_{h^b}|\prod_{j=0}^2(\mathcal{O}_{h^j})^{m_j})_{0,d}\cdot\prod_{j=0}^2\frac{(x^j)^{m_j}}{m_j!},
\end{aligned} \tag{19}$$

where x^j $(j = 0, 1, 2)$ is the variable associated with insertion of $\mathcal{O}_{h^j}$.

Since the puncture axiom (17) and the divisor axiom (18) hold for operator insertions on the right side of "|", the generating function is simplified into the following form:

$$w(\mathcal{O}_{h^a}\mathcal{O}_{h^b}|(x^0,x^1,x^2))_0 = x^c\cdot\int_{CP^2} h^{a+b+c} \; + \tag{20}$$
$$+\sum_{d>0,m} w(\mathcal{O}_{h^a}\mathcal{O}_{h^b}|(\mathcal{O}_{h^2})^m)_{0,d}\cdot e^{dx^1}\cdot\frac{(x^2)^m}{m!}.$$

We also introduce the corresponding generating function of genus 0 Gromov-Witten invariants of CP^2.

Definition 2. Let $\langle\prod_{j=0}^2(\mathcal{O}_{h^j})^{m_j}\rangle_{0,d}$ be the rational Gromov-Witten invariant of degree d of CP^2.

$$\langle\mathcal{O}_{h^a}\mathcal{O}_{h^b}(t^0,t^1,t^2)\rangle_0 \;:=$$
$$:=\; t^c\cdot\int_{CP^2}h^{a+b+c}+\sum_{d>0,\{m_j\}}\langle\mathcal{O}_{h^a}\mathcal{O}_{h^b}\prod_{j=0}^{2}(\mathcal{O}_{h^j})^{m_j}\rangle_{0,d}\cdot\prod_{j=0}^{2}\frac{(t^j)^{m_j}}{m_j!} \;=$$
$$=\; t^c\cdot\int_{CP^2}h^{a+b+c}+\sum_{d>0,m}\langle\mathcal{O}_{h^a}\mathcal{O}_{h^b}(\mathcal{O}_{h^2})^m\rangle_{0,d}\cdot e^{dt^1}\cdot\frac{(t^2)^m}{m!}, \tag{21}$$

where t^j $(j=0,1,2)$ is the variable associated with insertion of $\mathcal{O}_{h^j}$.

With this set-up, we state the following conjecture, which is a generalization of the results given by (13) and (14) to the CP^2 case.

Conjecture 1. If we define the mirror map,

$$t^j(x^0,x^1,x^2) := w(\mathcal{O}_{h^{2-j}}\mathcal{O}_1|(x^0,x^1,x^2))_0, \tag{22}$$

we have the following equality:

$$\langle\mathcal{O}_{h^a}\mathcal{O}_{h^b}(t^0(x^0,x^1,x^2),t^1(x^0,x^1,x^2),t^2(x^0,x^1,x^2))\rangle_0 \;=$$
$$=\; w(\mathcal{O}_{h^a}\mathcal{O}_{h^b}|(x^0,x^1,x^2))_0. \tag{23}$$

Conversely, if we invert the mirror map,

$$x^j = x^j(t^0,t^1,t^2), \tag{24}$$

we obtain the mirror formula to compute the genus 0 Gromov-Witten invariants of CP^2 from the multi-point virtual structure constants:

$$\langle \mathcal{O}_{h^a}\mathcal{O}_{h^b}(t^0,t^1,t^2)\rangle_0 = w(\mathcal{O}_{h^a}\mathcal{O}_{h^b}|(x^0(t^0,t^1,t^2),x^1(t^0,t^1,t^2),x^2(t^0,t^1,t^2)))_0. \tag{25}$$

Since we have the formula (16) to compute $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b}|\prod_{j=0}^{2}(\mathcal{O}_{h^j})^{m_j})_{0,d}$, we can write down numerically the mirror maps.

$$\begin{aligned} t^2 &= x^2+\frac{1}{4}q(x^2)^4+\frac{33}{70}q^2(x^2)^7+\frac{16589}{12600}q^3(x^2)^{10}+\frac{143698921}{32432400}q^4(x^2)^{13}+\cdots,\\ t^1 &= x^1+\frac{1}{2}(x^2)^3q+\frac{7}{10}(x^2)^6q^2+\frac{2593}{1512}q^3(x^2)^9+\frac{2668063}{498960}q^4(x^2)^{12}+\cdots,\\ t^0 &= x^0+\frac{1}{2}(x^2)^2q+\frac{8}{15}(x^2)^5q^2+\frac{983}{840}q^3(x^2)^8+\frac{4283071}{1247400}q^4(x^2)^{11}+\cdots,\\ &\qquad (q:=e^{x^1}). \end{aligned} \tag{26}$$

Of course, we can also compute one of the generating function,

$$\begin{aligned} &w(\mathcal{O}_h\mathcal{O}_h|(x^0,x^1,x^2))_0 =\\ &= x^0+(x^2)^2q+\frac{16}{15}(x^2)^5q^2+\frac{961}{420}q^3(x_2)^8+\frac{4105537}{623700}q^4(x^2)^{11}+\cdots. \end{aligned} \tag{27}$$

If we invert the mirror maps and substitute them to (27),

$$\begin{aligned} &w(\mathcal{O}_h\mathcal{O}_h|(x^0(t^0,t^1,t^2),x^1(t^0,t^1,t^2),x^2(t^0,t^1,t^2)))_0 =\\ &= t^0+\frac{1}{2}(t^2)^2Q+\frac{1}{30}(t^2)^5Q^2+\frac{3}{1120}(t^2)^8Q^3+\frac{31}{124740}(t^2)^{11}Q^4+\cdots\\ &= t^0+\frac{1}{2!}(t^2)^2Q+\frac{2^2}{5!}(t^2)^5Q^2+\frac{3^2\cdot 12}{8!}(t^2)^8Q^3+\frac{4^2\cdot 620}{11!}(t^2)^{11}Q^4+\cdots\\ &\qquad (Q:=e^{t^1}), \end{aligned} \tag{28}$$

the result coincides with $\langle\mathcal{O}_h\mathcal{O}_h(t^0,t^1,t^2)\rangle_0$ computed from the associativity equation [12]. If we compute,

$$\begin{aligned} &w(\mathcal{O}_{h^2}\mathcal{O}_{h^2}|(x^0,x^1,x^2))_0 =\\ &= q+\frac{2}{3}(x^2)^3q^2+\frac{17}{15}q^3(x^2)^6+\frac{6455}{2268}q^4(x^2)^9+\cdots, \end{aligned} \tag{29}$$

we obtain $\langle \mathcal{O}_{h^2}\mathcal{O}_{h^2}(t^0,t^1,t^2)\rangle_0$.

$$\begin{aligned}
&w(\mathcal{O}_{h^2}\mathcal{O}_{h^2}|(x^0(t^0,t^1,t^2),x^1(t^0,t^1,t^2),x^2(t^0,t^1,t^2)))_0 =\\
&= Q+\frac{1}{6}(t^2)^3Q^2+\frac{1}{60}Q^3(t^2)^6+\frac{31}{18144}Q^4(t^2)^9+\frac{1559}{8553600}Q^5(t^2)^{12}+\cdots\\
&= Q+\frac{1}{3!}(t^2)^3Q^2+\frac{12}{6!}(t^2)^6Q^3+\frac{620}{9!}(t^2)^9Q^4+\cdots. \qquad (30)
\end{aligned}$$

In this way, we can confirm numerically the validity of Conjecture 1. We can prove the conjecture up to the $d = 3$ case by using the technique of manipulation of residue integrals, that was presented in Sect. 5 of [9], but for general proof, we have to overcome difficulties of taking non-equivariant limit of the localization formula of Gromov-Witten invariants obtained from [11]. In [4], a different type of mirror computation of genus 0 Gromov-Witten invariants of CP^2 is presented. It is a modern refinement of the results in [2,6] and it starts from extended I-function. We compared our mirror map (26) with the mirror map obtained from the extended I-function. Surprisingly, these two mirror maps turn out to be different. Therefore, we have to answer the question whether these two types of mirror computations are essentially different or not.

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N-Conformal Galilean Group as a Maximal Symmetry Group of Higher-Derivative Free Theory

Krzysztof Andrzejewski and Joanna Gonera

Abstract It is shown that for N odd the N-conformal Galilean algebra is the algebra of maximal Noether symmetry group, both on the classical and quantum level, of free higher derivative dynamics.

1 Introduction

Contrary to the relativistic case, the structure of non-relativistic space-time is more complicated. Instead of being pseudo-Riemannian manifold it is equipped with a foliation of codimension one together with a torsionless affine connection obeying certain compatibility conditions. This implies that the notion of conformal invariance is more subtle. As a result there exists a variety of transformation groups G_N numbered by integer N which can be viewed as the counterparts of relativistic conformal symmetry [1, 2]. They have a common structure of direct product of $SU(2)$ and $SL(2,\mathcal{R})$ groups acting on Abelian normal subgroup and differ only by the choice of the latter. For N odd the non-relativistic conformal group admits central extension. The centrally extended $N = 1$ group is the well-known Schrodinger group, which is the maximal symmetry group for free motion both on the classical and quantum level [3–5]. The natural question is what are the simplest (i.e. such that our group acts transitively) dynamics invariant under the action of G_N with N-odd, $N > 1$. On the Hamiltonian level the answer has been given in [6] and, in general case of non-trivial internal degrees of freedom, in [7]. It appeared that the relevant dynamics is the free motion described by $(N+1)$-th order equation. Below, following our common paper [8] we complete the picture by showing that the G_N group is maximal symmetry group of the above higher-derivative dynamics.

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2 The Niederer's Result

The Schrodinger group can be defined as the most general group of transformations (of wave function) of the form

$$\psi(t,\mathbf{q}) = f(t,\mathbf{q})\psi(g^{-1}(t,\mathbf{q})) \tag{1}$$

where

$$(t,\mathbf{q}) \longrightarrow g(t,\mathbf{q}) \tag{2}$$

is the group action on configuration space, which leaves invariant free Schrodinger equation,

$$i\partial_t\psi = H\psi, \qquad \psi = \psi(t,\mathbf{q}). \tag{3}$$

The Lie algebra of the group of these transformations consists of the operators X

$$-iX = \mathbf{a}\frac{\partial}{\partial\mathbf{q}} + a\frac{\partial}{\partial t} + c, \tag{4}$$

The invariance condition can be now written in the form

$$[i\partial_t - H, X] = i\lambda(i\partial_t - H), \tag{5}$$

for a certain function $\lambda = \lambda(t,\mathbf{q})$ (see [5]).
The transformation (1) provides the quantum counterpart of point transformations. On the classical level the general infinitesimal point transformation reads

$$\mathbf{q}'(t') = \mathbf{q}(t) + \delta\mathbf{q}(\mathbf{q}(t),t), \quad t' = t + \delta t(t) \tag{6}$$

and yields the following conserved charge

$$G = \delta\mathbf{q}\frac{\partial L}{\partial\dot{\mathbf{q}}} - \delta t H - \delta f = (\delta\mathbf{q})\mathbf{p} - \delta t H - \delta f, \tag{7}$$

where $\delta\dot{f}$ is the (infinitesimal) change of the Lagrangian.
G obeys the well known condition

$$\{G,H\} + \frac{\partial G}{\partial t} = 0. \tag{8}$$

On the quantum level one has (assuming there are no ordering problems)

$$\psi' = \hat{V}\psi, \quad \hat{V} \simeq 1 + i\epsilon\hat{G}, \tag{9}$$

with $\hat{G}$ obeying quantum counterpart of classical condition

$$[\hat{G}, \hat{H}] + i\frac{\partial \hat{G}}{\partial t} = 0. \tag{10}$$

In order to write out the generator $\tilde{\mathcal{G}}$ acting "on shell" (i.e. on the set of solution to Schrodinger equation) one has to replace $\hat{H}$ by $i\frac{\partial}{\partial t}$ in $\hat{G}$. Therefore, the resulting relation reads

$$\hat{G} = \tilde{\mathcal{G}} + \delta t(i\frac{\partial}{\partial t} - \hat{H}). \tag{11}$$

Equations (10) and (11) give

$$[\tilde{\mathcal{G}}, i\frac{\partial}{\partial t} - \hat{H}] = i\delta\dot{t}\left(i\frac{\partial}{\partial t} - \hat{H}\right), \tag{12}$$

so we arrive at the Niederer's condition with $X = \tilde{\mathcal{G}}$ and $\lambda = -\delta\dot{t}$. This implies that λ depends on time only; however, we admitted more general form of λ (as Niederer did) to provide the additional consistency check.

3 Symmetries of Higher Derivative Free Theory

3.1 The Classical Case

We consider the free higher derivative theory defined by the Lagrangian

$$L = \frac{m}{2}\left(\frac{d^n\mathbf{q}}{dt^n}\right)^2, \tag{13}$$

where m is a "mass" parameter.

Let us first look at the symmetry on the classical level. We are looking for all point Noether symmetries of the Lagrangian (13), i.e. for the transformations

$$t' = t'(t), \quad \mathbf{q}'(t') = \mathbf{q}'(\mathbf{q}, t), \tag{14}$$

obeying

$$L(\mathbf{q}', \frac{d\mathbf{q}'}{dt'}, \ldots, \frac{d^n\mathbf{q}'}{dt'^n})\frac{dt'}{dt} = \\ L(\mathbf{q}, \frac{d\mathbf{q}}{dt}, \ldots, \frac{d^n\mathbf{q}}{dt^n}) + \frac{df}{dt}(\mathbf{q}, \frac{d\mathbf{q}}{dt}, \ldots, \frac{d^{(n-1)}\mathbf{q}}{dt^{(n-1)}}). \tag{15}$$

We write (14) in infinitesimal form

$$t' = t + \epsilon\psi(t), \quad \mathbf{q}'(t') = \mathbf{q} + \epsilon \mathbf{ffi}(\mathbf{q}, t) \tag{16}$$

and insert in Eq. (15).

Note that $f = f(\mathbf{q}, \dot{\mathbf{q}}, \ldots, \mathbf{q}^{(n-1)})$. This gives the following forms of the functions ψ and **ffi** entering Eq. (16)

$$\begin{aligned} \psi &= \tau + \lambda t + ct^2, \\ \phi_\alpha &= \left(\frac{2n-1}{2}\right)\dot{\psi} q_\alpha + \omega_{\alpha\beta} q_\beta + \sum_{k=0}^{2n-1} v_{\alpha k} t^k. \end{aligned} \tag{17}$$

Writing out Eq. (16) in differential form and identifying the coefficients related to arbitrary parameters, one finds the generators of the most general point Noether symmetry.

$$\begin{aligned} H = i\frac{\partial}{\partial t}, \quad D &= -it\frac{\partial}{\partial t} - i\left(\frac{2n-1}{2}\right)\mathbf{q}\frac{\partial}{\partial \mathbf{q}}, \\ K &= it^2\frac{\partial}{\partial t} + i\,(2n-1)\,t\mathbf{q}\frac{\partial}{\partial \mathbf{q}}, \\ \mathbf{J} = -i\mathbf{q} \times \frac{\partial}{\partial \mathbf{q}}, \quad \mathbf{C}_k &= i(-1)^k t^k \frac{\partial}{\partial \mathbf{q}}. \end{aligned} \tag{18}$$

It is straightforward to check that they obey the following algebra

$$\begin{aligned} [D, H] &= iH, \\ [D, K] &= -iK, \\ [K, H] &= 2iD, \\ [J_\alpha, J_\beta] &= i\epsilon_{\alpha\beta\gamma} J_\gamma, \\ [J_\alpha, C_{\beta k}] &= i\epsilon_{\alpha\beta\gamma} C_{\gamma k}, \\ [H, C_{\alpha k}] &= -ikC_{\alpha k-1}, \\ [D, C_{\alpha k}] &= i\left(\tfrac{2n-1}{2} - k\right) C_{\alpha k}, \\ [K, C_{\alpha k}] &= i(2n-1-k)C_{\alpha k+1}. \end{aligned} \tag{19}$$

which is N-conformal Galilean algebra with $N = 2n - 1$ [1,2,9–11].

It is known that for N odd this algebra admits central extension. However, the central charge appears only on Hamiltonian level.

3.2 The Quantum Case

Lagrangian dynamics defined by the Lagrangian

$$L = \frac{m}{2}\left(\frac{d^n\mathbf{q}}{dt^n}\right)^2, \tag{20}$$

can be put in the Hamiltonian form using the Ostrogradski formalism [12]. To this end we enlarge the configuration space. The new coordinates are

$$\mathbf{q}_1 = \mathbf{q},\ \mathbf{q}_2, \dots, \mathbf{q}_n \tag{21}$$

The Ostrogradski Hamiltonian is

$$H = \sum_{j=1}^{n-1} \mathbf{p}_j \mathbf{q}_{j+1} + \frac{1}{2m}\mathbf{p}_n^2. \tag{22}$$

To pass to the quantum description we write out the Schrodinger equation

$$i\partial_t \psi = H\psi, \quad \psi = \psi(t, \mathbf{q}_1, \dots, \mathbf{q}_n). \tag{23}$$

Again we look for transformations of the form

$$\begin{aligned} \psi(t, \mathbf{q}_1, \dots, \mathbf{q}_n) &\rightarrow (T_g\psi)(t, \mathbf{q}_1, \dots, \mathbf{q}_n) = \\ &= f_g(t, \mathbf{q}_1, \dots, \mathbf{q}_n)\psi(g^{-1}(t, \mathbf{q}_1, \dots, \mathbf{q}_n)), \end{aligned} \tag{24}$$

leaving the Schrodinger equation invariant. The relevant symmetry generators read now

$$-iX = \sum_{j=1}^{n} \mathbf{a}_j \frac{\partial}{\partial \mathbf{q}_j} + a\frac{\partial}{\partial t} + c, \tag{25}$$

and the symmetry condition remains unchanged

$$[i\partial_t - H, X] = i\lambda(i\partial_t - H), \tag{26}$$

except that $\lambda = \lambda(t, \mathbf{q}_1, \dots, \mathbf{q}_n)$. In general, one can argue as previously that λ depends on time only. We keep the $\mathbf{q_i}$ dependence in order to check this explicitly.

The form of the symmetry generators and the symmetry condition implies the following set of equations on a coefficient of the operator X

$$
\begin{aligned}
\lambda &= Qa, \\
\lambda \mathbf{q}_{j+1} &= Q\mathbf{a}_j - \mathbf{a}_{j+1}, \quad j = 1, \dots, n-1, \\
0 &= iQc + \tfrac{1}{2m}\tfrac{\partial^2 c}{\partial \mathbf{q}_n^2}, \\
0 &= iQ\mathbf{a}_n + \tfrac{1}{2m}\tfrac{\partial^2 \mathbf{a}_n}{\partial \mathbf{q}_n^2} + \tfrac{1}{m}\tfrac{\partial c}{\partial \mathbf{q}_n}, \\
0 &= \tfrac{\partial a}{\partial \mathbf{q}_n}, \\
0 &= \tfrac{\partial a_{j\beta}}{\partial q_{n\alpha}}, \quad j = 1, \dots, n-1, \\
\delta^\alpha_\beta &= \tfrac{\partial a_{n\beta}}{\partial q_{n\alpha}} + \tfrac{\partial a_{n\alpha}}{\partial q_{n\beta}},
\end{aligned}
\tag{27}
$$

where α, β etc. are vector indeces while $Q = \frac{\partial}{\partial t} + \sum_{k=1}^{n-1} \mathbf{q}_{k+1} \frac{\partial}{\partial \mathbf{q}_k}$. The detailed analysis of Eq. (27) [13] leads to the following conclusion:

The most general form of the operator X is a linear combination of generators

$$
\begin{aligned}
H &= -i\frac{\partial}{\partial t} \, ; \\
D &= -it\frac{\partial}{\partial t} - i\sum_{j=1}^{n}(\frac{n}{2} + \frac{1}{2} - j)\mathbf{q}_j \frac{\partial}{\partial \mathbf{q}_j} - i\frac{3}{4}n^2 \, ; \\
K &= -it^2\frac{\partial}{\partial t} - i\frac{3}{2}n^2 t - i\sum_{i=1}^{n}\Bigg((j-1)(2n+1-j)\mathbf{q}_{j-1}\frac{\partial}{\partial \mathbf{q}_j} - \\
&\quad -it(2n-2j+1)\mathbf{q}_j\frac{\partial}{\partial \mathbf{q}_j}\Bigg) - m\frac{n^2}{2}(\mathbf{q}_n)^2 \, ; \\
\mathbf{C}_l &= -il!(\sum_{k=0}^{l}\frac{t^{l-k}}{(l-k)!}\frac{\partial}{\partial \mathbf{q}_{k+1}}), \quad l = 0, \dots, n-1 \, ; \\
\mathbf{C}_j &= -ij!\sum_{l=0}^{n-1}\frac{t^{j-l}}{(j-l)!}\frac{\partial}{\partial \mathbf{q}_{l+1}} - mj!\sum_{k=n}^{j}\mathbf{q}_{2n-k}(-1)^{n-k}\frac{t^{j-k}}{(j-k)!} \, , \\
&\quad j = n, \dots, 2n+1 \, ; \\
\mathbf{J} &= -i\sum_{j=1}^{n}\mathbf{q}_j \times \frac{\partial}{\partial \mathbf{q}_j} \, .
\end{aligned}
\tag{28}
$$

Again it is not difficult to check by the explicit calculation that they close to the centrally extended N-conformal Galilean algebra ($N = 2n - 1$) with m being the central charge.

Conclusions

- For N-odd the N-conformal Galilean algebra/group has transparent interpretation. It is the **maximal group** of point transformations which are the symmetry transformations, both on classical and quantum level, of higher derivative free dynamics.

  This result extends the one obtained in [5,6] where has been shown that for N odd N-conformal Galilean algebra is the symmetry algebra of higher derivative free theory. For N even such simple interpretation is lacking.
- The equation of motion resulting from the Lagrangian

$$L = \frac{m}{2}\left(\frac{d^n\mathbf{q}}{dt^n}\right)^2, \tag{29}$$

is of the form

$$\frac{d^{N+1}\mathbf{q}}{dt^{N+1}} = 0, \quad n = \frac{N+1}{2}, \tag{30}$$

The maximal set of point transformations which leaves the above equation invariant takes the same form for all N. The main difference is that for N-even equation (30) is of odd order and has no simple Lagrangian form.

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Part VI
Vertex Algebras and Superalgebras

Virasoro Structures in the Twisted Vertex Algebra of the Particle Correspondence of Type C

Iana I. Anguelova

Abstract In this paper we study the existence of Virasoro structures in the twisted vertex algebra describing the particle correspondence of type C. We show that this twisted vertex algebra has at least two distinct Virasoro structures: one with central charge 1, and a second with central charge -1.

1 Introduction

This paper is part of a series studying various particle correspondences from the point of view of vertex algebras. There are several types of particle correspondences, such as the boson-fermion and boson-boson correspondences, known in the literature. The best known is the charged free fermion-boson correspondence, also known as the boson-fermion correspondence of type A (the name "type A" is due to the fact that this correspondence is canonically related to the basic representations of the Kac-Moody algebras of type A, see [8, 9, 14]). The correspondence of type A is an isomorphism of super vertex algebras, but most boson-fermion correspondences cannot be described by the concept of a super vertex algebra due to the more general singularities in their operator product expansions. In order to describe the more general cases, including the correspondences of types B, C and D-A, in [1] and [2] we defined the concept of a twisted vertex algebra which generalizes super vertex algebra. In [2] we showed that the correspondences of types B, C and D-A are isomorphisms of twisted vertex algebras. As expected in chiral conformal field theory, many examples of super vertex algebras were shown to have a Virasoro structure. Super vertex algebras with a Virasoro field are called vertex operator algebras (see e.g. [11, 12, 18], subject also to additional axioms), or conformal vertex algebras [10, 15]; and are extensively studied. In particular, the boson-fermion correspondence of type A has a one-parameter family of Virasoro fields, $L^{A,\lambda}(z)$, parameterized by $\lambda \in \mathbb{C}$, with central charge $-12\lambda^2 + 12\lambda - 2$ (see e.g. [14, 15]). As was done for super vertex algebras, in a series of papers we study the existence

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of Virasoro structures (see Definition 2.7) in important examples of twisted vertex algebras, such as the correspondences of types B, C and D-A. We started with the correspondence of type D-A, and as we show in [3] the twisted vertex algebra describing the correspondence of type D-A has **two distinct** types of Virasoro structures. These structures are distinct in two ways: first, they have different central charges (correspondingly $\frac{1}{2}$ and 1). But also, the Virasoro fields with central charge $\frac{1}{2}$ are 1-point local, however the Virasoro field with central charge 1 is N-point local (see Definition 2.1 for N-point locality), although it could be reduced to the usual 1-point locality by a change of variables z^N to z. In this second paper we continue with the study of the existence of Virasoro structures for the correspondence of type C. The correspondence of type C was introduced in [7], and further studied in [20]. In [2] we interpret it as an isomorphism of twisted vertex algebras and in [4] we study some properties of its space of fields. In this paper we show that the twisted vertex algebra describing the correspondence of type C is conformal, i.e., it has Virasoro structures. In particular, we show that it has (at least) two distinct Virasoro structures, one with central charge 1, and a second with central charge -1. Both these Virasoro structures are 2-point local, but could be reduced to a 1-point locality by a change of variables z^2 to z.

2 Notation and Background

We work over the field of complex numbers $\mathbb{C}$. Let N be a positive integer, and let ϵ be a primitive N-th root of unity. Recall that in two-dimensional chiral field theory a **field** $a(z)$ on a vector space V is a series of the form

$$a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}, \quad a_{(n)} \in \mathrm{End}(V),$$
$$\text{such that } \ a_{(n)}v = 0 \ \text{ for any } v \in V, \ n \gg 0.$$

The coefficients $a_{(n)} \in \mathrm{End}(V)$ are called modes. (See e.g. [11, 12, 15, 18]).

Definition 2.1 ([4]) (N-Point Local Fields). We say that a field $a(z)$ on a vector space V is **even** and N-point self-local at $1, \epsilon, \epsilon^2, \ldots, \epsilon^{N-1}$, if there exist $n_0, n_1, \ldots, n_{N-1} \in \mathbb{Z}_{\geq 0}$ such that

$$(z-w)^{n_0}(z-\epsilon w)^{n_1} \cdots (z-\epsilon^{N-1}w)^{n_{N-1}}[a(z), a(w)] = 0.$$

In this case we set the **parity** $p(a(z))$ of $a(z)$ to be 0.

We set $\{a, b\} = ab + ba$. We say that a field $a(z)$ on V is N-point self-local at $1, \epsilon, \epsilon^2, \ldots, \epsilon^{N-1}$ and **odd** if there exist $n_0, n_1, \ldots, n_{N-1} \in \mathbb{Z}_{\geq 0}$ such that

$$(z-w)^{n_0}(z-\epsilon w)^{n_1} \cdots (z-\epsilon^{N-1}w)^{n_{N-1}}\{a(z), a(w)\} = 0.$$

In this case we set the **parity** $p(a(z))$ to be 1. For brevity we will just write $p(a)$ instead of $p(a(z))$. If a field $a(z)$ is even or odd, we say that $a(z)$ is homogeneous. If $a(z), b(z)$ are homogeneous fields on V, we say that $a(z)$ and $b(z)$ are *N-point mutually local* at $1, \epsilon, \epsilon^2, \ldots, \epsilon^{N-1}$ if there exist $n_0, n_1, \ldots, n_{N-1} \in \mathbb{Z}_{\geq 0}$ such that

$$(z-w)^{n_0}(z-\epsilon w)^{n_1}\cdots(z-\epsilon^{N-1}w)^{n_{N-1}}\left(a(z)b(w)-(-1)^{p(a)p(b)}b(w)a(z)\right)=0.$$

For a rational function $f(z,w)$ with poles only at $z=0, z=\epsilon^i w,\ 0\leq i\leq N-1$, we denote by $i_{z,w}f(z,w)$ the expansion of $f(z,w)$ in the region $|z|\gg|w|$ (the region in the complex z plane outside of all the points $z=\epsilon^i w,\ 0\leq i\leq N-1$), and correspondingly for $i_{w,z}f(z,w)$. Let

$$a(z)_- := \sum_{n\geq 0} a_n z^{-n-1}, \quad a(z)_+ := \sum_{n<0} a_n z^{-n-1}. \tag{1}$$

Definition 2.2 (Normal Ordered Product) ([4, 11, 15, 18]). Let $a(z), b(z)$ be homogeneous fields on a vector space V. Define

$$:a(z)b(w):=a(z)_+b(w)+(-1)^{p(a)p(b)}b(w)a_-(z). \tag{2}$$

We extend by linearity, and we call this the normal ordered product of $a(z)$ and $b(w)$.

Remark 1. Let $a(z), b(z)$ be fields on a vector space V. Then $:a(z)b(\epsilon^i z):$ and $:a(\epsilon^i z)b(z):$ are well defined fields on V for any $i=0,1,\ldots,N-1$.

The mathematical background of the well-known and often used in physics notion of Operator Product Expansion (OPE) of product of two fields for the case of usual locality ($N=1$) has been established for example in [15,18]. The following lemma extended the mathematical background to the case of N-point locality:

Lemma 2.3 ([4]) (Operator Product Expansion (OPE)). *Suppose $a(z)$, $b(w)$ are N-point mutually local. Then exists fields $c_{jk}(w)$, $j=0,1,\ldots,N-1$; $k=0,\ldots,n_j-1$, such that we have*

$$a(z)b(w)=i_{z,w}\sum_{j=0}^{N-1}\sum_{k=0}^{n_j-1}\frac{c_{jk}(w)}{(z-\epsilon^j w)^{k+1}}+:a(z)b(w):. \tag{3}$$

We call the fields $c_{jk}(w)$, $j=0,\ldots,N-1$; $k=0,\ldots,n_j-1$, OPE coefficients. We will write the above OPE as

$$a(z)b(w)\sim\sum_{j=1}^{N}\sum_{k=0}^{n_j-1}\frac{c_{jk}(w)}{(z-\epsilon_j w)^{k+1}}. \tag{4}$$

The $\sim$ signifies that we have only written the singular part, and also we have omitted writing explicitly the expansion $i_{z,w}$, which we do acknowledge tacitly.

The OPE expansion in the multi-local case allowed us to extend the Wick's Theorem (see e.g., [5, 13]) to the case of multi-locality (see [4]). We further have the following expansion formula extended to the multi-local case, which we will use extensively in what follows:

Lemma 2.4 ([4]) (Taylor Expansion for Normal Ordered Products). *Let $a(z), b(z)$ be N-point mutually local fields on a vector space V. Then*

$$i_{z,z_0} : a(\epsilon^i z + z_0)b(z) : = \sum_{k\geq 0} \Big(: (\partial^{(k)}_{\epsilon^i z} a(\epsilon^i z))b(z) : \Big) z_0^k;$$
$$\textit{for any} \quad i = 0, 1, \ldots, N-1.$$

Definition 2.5 (The Field Descendants Space $\mathfrak{FD}\{\mathbf{a^0(z)}, \mathbf{a^1(z)}, \ldots, \mathbf{a^p(z)}\}$). Let $a^0(z), a^1(z), \ldots, a^p(z)$ be given homogeneous fields on a vector space W, which are self-local and pairwise N-point local with points of locality $1, \epsilon, \ldots, \epsilon^{N-1}$. Denote by $\mathfrak{FD}\{a^0(z), a^1(z), \ldots, a_p(z)\}$ the subspace of all fields on W obtained from the fields $a^0(z), a^1(z), \ldots, a^p(z)$ as follows:

1. $Id_W, a^0(z), a^1(z), \ldots, a^p(z) \in \mathfrak{FD}\{a^0(z), a^1(z), \ldots, a^p(z)\}$;
2. If $d(z) \in \mathfrak{FD}\{a^0(z), a^1(z), \ldots, a^p(z)\}$, then $\partial_z(d(z)) \in \mathfrak{FD}\{a^0(z), \ldots, a^p(z)\}$;
3. If $d(z) \in \mathfrak{FD}\{a^0(z), a^1(z), \ldots, a^p(z)\}$, then $d(\epsilon^i z)$ are also elements of $\mathfrak{FD}\{a^0(z), a^1(z), \ldots, a^p(z)\}$ for $i = 0, \ldots, N-1$;
4. If $d_1(z), d_2(z)$ are both in $\mathfrak{FD}\{a^0(z), a^1(z), \ldots, a^p(z)\}$, then $: d_1(z)d_2(z) :$ is also an element of $\mathfrak{FD}\{a^0(z), a^1(z), \ldots, a^p(z)\}$, as well as all OPE coefficients in the OPE expansion of $d_1(z)d_2(w)$.
5. All finite linear combinations of fields in $\mathfrak{FD}\{a^0(z), a^1(z), \ldots, a^p(z)\}$ are still in $\mathfrak{FD}\{a^0(z), a^1(z), \ldots, a^p(z)\}$.

We will not remind here the definition of a twisted vertex algebra as it is rather technical, see instead [1] and [2]. A twisted vertex algebra is a generalization of a super vertex algebra, in the sense that any super vertex algebra is an $N = 1$-twisted vertex algebra, and vice versa. A major difference is that in twisted vertex algebras the space of fields V is allowed to be strictly larger than the space of states W on which the fields act (i.e., the field-state correspondence is not necessarily a bijection as for super vertex algebras, but only a projective surjection). We have the following construction theorem for twisted vertex algebras:

Proposition 2.6 ([4]). *Let $a^0(z), a^1(z), \ldots, a^p(z)$ be given homogeneous fields on a vector space W, which are N-point local with points of locality ϵ^i, $i = 0, \ldots, N-1$, where ϵ is a primitive root Nth of unity. Then any two fields in $\mathfrak{FD}\{a^0(z), a^1(z), \ldots, a_p(z)\}$ are self and mutually N-point local. Further, if the fields $a^0(z), a^1(z), \ldots, a^p(z)$ satisfy the conditions for generating fields for a twisted vertex algebra (see [4]), then there exists a twisted vertex algebra structure with a space of states W and a space of fields $V = \mathfrak{FD}\{a^0(z), a^1(z), \ldots, a_p(z)\}$.*

Recall the well-known Virasoro algebra Vir, the central extension of the complex polynomial vector fields on the circle. The Virasoro algebra Vir is the Lie algebra with generators L_n, $n \in \mathbb{Z}$, and central element C, with commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{(m^3-m)}{12}C; \quad [C, L_m] = 0,\ m, n \in \mathbb{Z}. \tag{5}$$

Equivalently, the Virasoro field $L(z) := \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ has OPE with itself given by:

$$L(z)L(w) \sim \frac{C/2}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{(z-w)}. \tag{6}$$

Definition 2.7. We say that a twisted vertex algebra with a space of fields V has a Virasoro structure if there is field in V such that its modes are the generators of the Virasoro algebra Vir.

We want to mention that the Virasoro field is of conformal weight 2 (for a precise definition of conformal weight see e.g. [11, 15, 18]).

Remark 2. The boson-fermion correspondence of type A, which is generated from two odd 1-point local fields $\psi^+(z)$ and $\psi^-(z)$, has a one-parameter family of Virasoro fields with central charge $-12\lambda^2 + 12\lambda - 2$ (see e.g. [15], Chap. 5):

$$L^{A,\lambda}(z) = \frac{1}{2} : \alpha(z)^2 : + (\frac{1}{2} - \lambda)\partial_z \alpha(z) \tag{7}$$

$$= (1-\lambda) : (\partial_z \psi^+(z))\psi^-(z) : + \lambda : (\partial_z \psi^-(z))\psi^+(z) :, \tag{8}$$

where $\alpha(z) =: \psi^+(z)\psi^-(z) :$ is the Heisenberg field for the correspondence of type A, $\lambda \in \mathbb{C}$. $\alpha(z)$ is of conformal weight 1 (roughly speaking the normal order products and the derivatives behave as expected with respect to the conformal weight). We would like to underline that the two components of the Virasoro field come from the only two possibilities for conformal-weight-2-fields: first, $: \alpha(z)^2 :$, and second, a linear combination of the normal ordered products $: (\partial_z \psi^+(z))\psi^-(z) :$ and $: (\partial_z \psi^-(z))\psi^+(z) :$, which in this particular case is their difference and equals $\partial_z \alpha(z)$. Moreover, for the correspondence of type A we also have

$$: \alpha(z)\alpha(z) :=: (\partial_z \psi^+(z))\psi^-(z) : + : (\partial_z \psi^-(z))\psi^+(z) :, \tag{9}$$

hence the Virasoro field $L^{A,\lambda}(z)$ is purely a linear combination of the normal ordered products $: (\partial_z \psi^+(z))\psi^-(z) :$ and $: (\partial_z \psi^-(z))\psi^+(z) :$. The equivalent of (9) does **not** hold for the correspondence of type C, as we will show.

3 Virasoro Structure for the Correspondence of Type C

For the correspondences of type C we only need $N = 2$, i.e., the points of locality are at $z = w$ and $z = -w$. Since this correspondence is an isomorphism of twisted vertex algebras (see [2, 4]), it is enough to only consider one of the sides of the correspondence to determine the existence of a Virasoro structure.

The correspondence of type C is strictly speaking not a boson-fermion correspondence: the first side is generated by an **even** twisted **boson** field $\phi^C(z)$, which is then bosonized further to get the second, also **bosonic**, side of the correspondence of type C [7, 20]. The field $\phi^C(z)$, $\phi^C(z) := \sum_{n\in\mathbb{Z}+1/2} \phi_n^C z^{n-1/2}$ (the half-integers are commonly used when indexing in this case); has OPE:

$$\phi^C(z)\phi^C(w) \sim \frac{1}{z+w}, \quad \text{in modes:} \quad [\phi_m^C, \phi_n^C] = (-1)^{n-\frac{1}{2}}\delta_{m,-n}1. \tag{10}$$

The modes of the field $\phi^C(z)$ form a Lie algebra which we denote by L_C. Let F_C be the highest weight module of L_C with the vacuum vector $|0\rangle$, such that $\phi_n^C|0\rangle = 0$ for $n < 0$. By Proposition 2.6 there is a twisted vertex algebra structure with a space of fields $\mathfrak{FD}\{\phi^C(z)\}$, acting on the space of states F_C. We now study the existence of Virasoro structures in this twisted vertex algebra. First, a prominent element of the space of fields $\mathfrak{FD}\{\phi^C(z)\}$ is the normal ordered product $:\phi^C(z)\phi^C(-z):$, and we have the following bosonization lemma:

Lemma 3.1. *Let $h^C(z) = \frac{1}{2} :\phi^C(z)\phi^C(-z): \in \mathfrak{FD}\{\phi^C(z)\}$. We have $h^C(z) = h^C(-z)$, thus we index $h^C(z)$ as $h^C(z) = \sum_{n\in\mathbb{Z}+1/2} h_n^C z^{-2n-1}$ (note the half-integers). The field $h^C(z)$ has OPE with itself given by:*

$$h^C(z)h^C(w) \sim -\frac{z^2+w^2}{2(z^2-w^2)^2} \sim -\frac{1}{4}\frac{1}{(z-w)^2} - \frac{1}{4}\frac{1}{(z+w)^2}, \tag{11}$$

and its modes, h_n^C, $n \in \mathbb{Z}+1/2$, generate a ***twisted*** *Heisenberg algebra $\mathcal{H}_{\mathbb{Z}+1/2}$ with relations $[h_m^C, h_n^C] = -m\delta_{m+n,0}1$, $m, n \in \mathbb{Z}+1/2$.*

The above result appears in [7] (proof by brute force using the modes directly), we prove it here to illustrate the use of the combination of Wick's Theorem and the Taylor expansion Lemma 2.4 in the multi-local case.

Proof. The fact that $h^C(z) = h^C(-z)$ follows immediately from the fact that the field $\phi^C(z)$ is even, as $:\phi^C(-w)\phi^C(w): = :\phi^C(w)\phi^C(-w):$. Next, Wick's theorem applies here (see e.g. [4, 5, 13]) and we have

$$\begin{aligned} :\phi^C(z)\phi^C(-z)::\phi^C(w)\phi^C(-w): &\sim \frac{1}{z+w}\cdot\frac{-1}{z+w} + \frac{-1}{z-w}\cdot\frac{1}{z-w} \\ &+ \frac{-1}{z-w}:\phi^C(z)\phi^C(-w): + \frac{-1}{z+w}:\phi^C(z)\phi^C(w): \\ &+ \frac{1}{z+w}:\phi^C(-z)\phi^C(-w): + \frac{1}{z-w}:\phi^C(-z)\phi^C(w):. \end{aligned}$$

Now we apply Taylor expansion formula from Lemma 2.4:

$$
\begin{aligned}
:\phi^C(z)\phi^C(-z)::\phi^C(w)\phi^C(-w):\sim -\frac{1}{(z+w)^2}-\frac{1}{(z-w)^2} \\
+\frac{-1}{z-w}:\phi^C(w)\phi^C(-w):+\frac{-1}{z+w}:\phi^C(-w)\phi^C(w): \\
+\frac{1}{z+w}:\phi^C(w)\phi^C(-w):+\frac{1}{z-w}:\phi^C(-w)\phi^C(w):.
\end{aligned}
$$

The other summands from the Taylor expansion will produce nonsingular terms and thus do not contribute to the OPE. Using that $\phi^C(z)$ is even finishes the proof. □

Remark 3. We know from the OPE expansion, Lemma 2.3, and by continuing the Taylor expansion in the calculation above, that we can express the normal ordered product $:h^C(z)h^C(z):$ as follows:

$$
\begin{aligned}
h^C(z)h^C(w) = &-i_{z,w}\left(\frac{1}{(z+w)^2}+\frac{1}{(z-w)^2}\right) \\
&+i_{z,w}\left(\frac{1}{z+w}-\frac{1}{z-w}\right)(:\phi^C(w)\phi^C(-w):-:\phi^C(-w)\phi^C(w):) \\
&-2\left(:(\partial_w\phi^C(w))\phi^C(-w):+:(\partial_{-w}\phi^C(-w))\phi^C(w):\right) \\
&+2:\phi^C(w)\phi^C(-w)\phi^C(w)\phi^C(-w):+O(z-w,z+w).
\end{aligned}
$$

That gives us, in contrast to (9), not only second order terms, but a fourth order **non-vanishing** term as well:

$$
:h^C(w)h^C(w): = -2\left(:(\partial_w\phi^C(w))\phi^C(-w):+:(\partial_{-w}\phi^C(-w))\phi^C(w):\right) \tag{12}
$$

$$
+2:\phi^C(w)\phi^C(-w)\phi^C(w)\phi^C(-w):. \tag{13}
$$

The analogous fourth order term vanishes in the case of the correspondence of type A (see [15], Chap. 3.6), hence (9) holds. Further, in this case of type C we have

$$
\partial_z h^C(z) =:(\partial_z\phi^C(z))\phi^C(-z):-:(\partial_{-z}\phi^C(-z))\phi^C(z):. \tag{14}
$$

For the correspondence of type C the field $h^C(z)$ is the equivalent of the field $\alpha(z)$ (for notation see e.g. [15]) from the correspondence of type A. The field $\frac{1}{2}:\alpha(z)^2:$ is the field whose modes produce the well-known oscillator representation of Vir (see e.g. [11, 14]). Similarly we expect the field $\frac{1}{2}:h^C(z)^2:$ to be related to the **twisted** oscillator representation of Vir (see e.g. [11]). But as the Remark above shows, here we cannot treat the term $\frac{1}{2}:h^C(z)^2:$ as part of the linear combination of the second order terms $:(\partial_w\phi^C(w))\phi^C(-w):$ and $:(\partial_{-w}\phi^C(-w))\phi^C(w):$, as opposed to the case of the boson-fermion correspondence of type A (see Remark 2). Thus we have two separate cases in the following:

Proposition 3.2. ***I.*** *Let* $L^{C,1}(z) = -\frac{1}{2z^2} :h^C(z)h^C(z): + \frac{1}{16z^4}$. *We have* $L^{C,1}(z) = L^{C,1}(-z)$, *and we index this field as* $L^{C,1}(z) = \sum_{n\in\mathbb{Z}} L_n^{C,1} z^{-2n-4}$. *The modes* $L_n^{C,1}$ *of the field* $L^{C,1}(z)$ *satisfy the Virasoro commutation relations with central charge* 1. *Equivalently, we can write* $L^{C,1}(z) = L_1^C(z^2)$, *where the field* $L_1^C(z^2)$ *is a Virasoro field with central charge 1, and has the OPE* (6) *with variables z, w changed correspondingly to* z^2, w^2.

II. *Let* $L^{C,-1}(z) = -\frac{1}{8z^2}\left(:(\partial_z\phi^C(z))\phi^C(-z): + :(\partial_{-z}\phi^C(-z))\phi^C(z):\right) - \frac{1}{32z^4}$. *We have* $L^{C,-1}(z) = L^{C,-1}(-z)$, *and we index this field as* $L^{C,-1}(z) = \sum_{n\in\mathbb{Z}} L_n^{C,-1} z^{-2n-4}$. *The modes* $L_n^{C,-1}$ *of the field* $L^{C,-1}(z)$ *satisfy the Virasoro commutation relations with central charge* -1. *Equivalently, we can write* $L^{C,-1}(z) = L_{-1}^C(z^2)$, *where the field* $L_{-1}^C(z^2)$ *is a Virasoro field with central charge* -1, *and has the OPE* (6) *with variables z, w changed correspondingly to* z^2, w^2.

Proof. **I.** Wick's Theorem applies here, and we have

$$
\begin{aligned}
L^{C,1}(z)L^{C,1}(w) &\sim \frac{1}{4z^2w^2} :h^C(z)h^C(z)::h^C(w)h^C(w): \\
&\sim \frac{(z^2+w^2)^2}{8z^2w^2(z^2-w^2)^4} - \frac{z^2+w^2}{2z^2w^2(z^2-w^2)^2} :h^C(z)h^C(w): \\
&\sim \frac{1}{8z^2w^2(z^2-w^2)^2} + \frac{1/2}{(z^2-w^2)^4} \\
&\quad - \frac{1}{4z^2w^2(z-w)^2} :h^C(z)h^C(w): \\
&\quad - \frac{1}{4z^2w^2(z+w)^2} :h^C(z)h^C(w): \\
&\sim \frac{1}{8z^2w^2(z^2-w^2)^2} + \frac{1/2}{(z^2-w^2)^4} \\
&\quad - \left(\frac{1}{4w^4(z-w)^2} - \frac{1}{2w^5(z-w)}\right) :h^C(z)h^C(w): \\
&\quad - \left(\frac{1}{4w^4(z+w)^2} + \frac{1}{2w^5(z+w)}\right) :h^C(z)h^C(w): .
\end{aligned}
$$

We now apply Taylor's expansion formula (Lemma 2.4), noting that $h^C(w) = h^C(-w)$ and thus $\partial_{-w}h^C(-w) = -\partial_w h^C(w)$:

$$
\begin{aligned}
L^{C,1}(z)L^{C,1}(w) &\sim \frac{1}{8z^2w^2(z^2-w^2)^2} + \frac{1/2}{(z^2-w^2)^4} \\
&\quad - \left(\frac{1}{4w^4} \left(\frac{1}{(z-w)^2} + \frac{1}{(z+w)^2} \right) \right. \\
&\quad \left. + \frac{1}{2w^5} \left(\frac{1}{z+w} - \frac{1}{z-w} \right) \right) : h^C(w)h^C(w) : \\
&\quad - \frac{1}{4w^4} \left(\frac{1}{z-w} - \frac{1}{z+w} \right) : \partial_w h^C(w)h^C(w) : \\
&\sim \frac{1/2}{(z^2-w^2)^4} + \frac{1}{8z^2w^2(z^2-w^2)^2} \\
&\quad - \frac{1}{(z^2-w^2)^2} \cdot \frac{1}{w^2} : h^C(w)h^C(w) : \\
&\quad + \frac{1}{z^2-w^2} \cdot \frac{1}{2w^4} : h^C(w)h^C(w) : \\
&\quad - \frac{1}{z^2-w^2} \cdot \frac{1}{2w^3} : \partial_w h^C(w)h^C(w) : \\
&\sim \frac{1/2}{(z^2-w^2)^4} + \frac{1}{8w^4(z^2-w^2)^2} - \frac{1}{8w^6(z^2-w^2)} \\
&\quad + \frac{1}{(z^2-w^2)^2} \left(2L_1^C(w^2) - \frac{1}{8w^4} \right) \\
&\quad + \frac{1}{z^2-w^2} \left(\partial_{w^2} L_1^C(w^2) + \frac{1}{8w^6} \right) \\
&\sim \frac{1/2}{(z^2-w^2)^4} + \frac{1}{(z^2-w^2)^2} 2L_1^C(w^2) + \frac{1}{z^2-w^2} \partial_{w^2} L_1^C(w^2).
\end{aligned}
$$

This proves part **I** of the Proposition. To shorten the calculations in part **II** denote

$$
A(z) :=: (\partial_z \phi^C(z))\phi^C(-z) :, \qquad B(z) =: (\partial_{-z}\phi^C(-z))\phi^C(z) : . \tag{15}
$$

A combination of Wick's Theorem and Taylor expansion Lemma 2.4 gives us

$$
\begin{aligned}
A(z)A(w) &\sim \frac{2}{(z+w)^3} : \phi^C(w)\phi^C(-w) : - \frac{4(z^2+w^2)}{(z^2-w^2)^2} A(w) \\
&\quad - \frac{2w}{z^2-w^2} \partial_w A(w) - \frac{2}{(z+w)^4} + \frac{1}{(z-w)^4}; \\
B(z)B(w) &\sim -\frac{2}{(z+w)^3} : \phi^C(-w)\phi^C(w) : - \frac{4(z^2+w^2)}{(z^2-w^2)^2} B(w)
\end{aligned}
$$

$$- \frac{2w}{z^2 - w^2} \partial_w B(w) - \frac{2}{(z+w)^4} + \frac{1}{(z-w)^4};$$

$$A(z)B(w) \sim \frac{2}{(z-w)^3} : \phi^C(-w)\phi^C(w) : - \frac{4(z^2+w^2)}{(z^2-w^2)^2} B(w)$$
$$- \frac{2w}{z^2 - w^2} \partial_w B(w) - \frac{2}{(z-w)^4} + \frac{1}{(z+w)^4};$$

$$B(z)A(w) \sim - \frac{2}{(z-w)^3} : \phi^C(w)\phi^C(-w) : - \frac{4(z^2+w^2)}{(z^2-w^2)^2} A(w)$$
$$- \frac{2w}{z^2 - w^2} \partial_w A(w) - \frac{2}{(z-w)^4} + \frac{1}{(z+w)^4}.$$

We have

$$L^{C,-1}(z)L^{C,-1}(w) \sim \frac{1}{64z^2w^2} \left(A(z)+B(z)\right)\left(A(w)+B(w)\right)$$
$$\sim \frac{1}{64z^2w^2} \left(\frac{-2}{(z-w)^4} + \frac{-2}{(z+w)^4} \right) + \frac{1}{(z-w)^3} \cdot 0 + \frac{1}{(z+w)^3} \cdot 0$$
$$- \frac{8(z^2+w^2)}{64z^2w^2(z^2-w^2)^2} \left(A(w)+B(w)\right)$$
$$- \frac{4w}{64z^2w^2(z^2-w^2)} \left(\partial_w A(w) + \partial_w B(w)\right)$$
$$\sim \frac{-1/2}{(z^2-w^2)^4} - \frac{1}{16z^2w^2(z^2-w^2)^2}$$
$$- \frac{1}{4w^2(z^2-w^2)^2} \left(A(w)+B(w)\right)$$
$$+ \frac{1}{8w^4(z^2-w^2)} \left(A(w)+B(w)\right)$$
$$- \frac{1}{16w^3(z^2-w^2)} \left(\partial_w A(w) + \partial_w B(w)\right)$$

$$\sim \frac{-1/2}{(z^2-w^2)^4} - \frac{1}{16w^4(z^2-w^2)^2} + \frac{1}{16w^6(z^2-w^2)}$$
$$+ \quad \frac{1}{(z^2-w^2)^2} \left(2L^C_{-1}(w^2) + \frac{1}{16w^4} \right)$$

$$+\frac{1}{w^2(z^2-w^2)}\left(-L^C_{-1}(w^2)-\frac{1}{32w^4})\right)$$
$$+\frac{1}{16w^3(z^2-w^2)}\left(-\frac{1}{2w^3}+16wL^C_{-1}(w^2)+16w^3\partial_{w^2}L^C_{-1}(w^2)\right)$$
$$\sim\frac{-1/2}{(z^2-w^2)^4}+\frac{2L^C_{-1}(w^2)}{(z^2-w^2)^2}+\frac{\partial_{w^2}L^C_{-1}(w^2)}{z^2-w^2}.$$

□

As we mentioned, the first of these Virasoro structures is not entirely surprising—the field $\frac{1}{2}:h^C(z)^2:$ is related to the twisted oscillator representation of Vir. But the second Virasoro structure was completely unexpected, especially the fact that it is 2-point local (the analogous field in the correspondence of type D-A is 1-point-local). The linear combination of the fields $A(z)$ and $B(z)$ producing the Virasoro field is not arbitrary, in fact it is the only linear combination possible. Due to the multi-locality, we have four conformal-weight-two fields which can potentially contribute to a Virasoro structure: $A(z)=:(\partial_z\phi^C(z))\phi^C(-z):$, $B(z)=:(\partial_{-z}\phi^C(-z))\phi^C(z):$, $:(\partial_z\phi^C(z))\phi^C(z):$ and $:(\partial_{-z}\phi^C(-z))\phi^C(-z):$. A very long calculation which we omit here shows that of all the complex linear combinations only the one above will produce a Virasoro field. Furthermore, there are no one-point-local Virasoro fields in the correspondence of type C. This is due perhaps to the absence of a super vertex algebra structure on each of the two "sheets" that the twisted vertex algebra structure "glues" together, as opposed to the case of type D-A (although here each "sheet" is a twisted module for an appropriate super vertex algebra). The two-point local Virasoro fields are due to the overall twisted vertex algebra structure responsible for the bosonization of type C. We expect that there is a genuine (non-splitting) representation of a version of a two-point Virasoro algebra (see e.g. [6, 16, 17, 19]) arising from a linear combination of these four weight-two fields.

To summarize: this paper is part of a series studying the Virasoro structures in various particle correspondences. We show in [3] that the twisted vertex algebra describing the correspondence of type D-A has two distinct types of Virasoro structures: the Virasoro fields with central charge $\frac{1}{2}$ are one-point local, however the Virasoro field with central charge 1 is N-point local. In this paper we show that in the twisted vertex algebra describing the correspondence of type C there are two Virasoro structures, both are two-point local; there are no one-point local Virasoro fields in this twisted vertex algebra. In the next paper we will continue with the Virasoro structures in the twisted vertex algebra of the boson-fermion correspondence of type B. Although similar to the type C, the OPEs in the correspondence of type B do not allow for a direct application of Wick's theorem [4], thus a more complicated modification has to be used.

In conclusion, we would like to thank the organizers of the International Workshop "Lie Theory and its Applications in Physics" for a most enjoyable and productive workshop, and we look forward to the next one!

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On the Correspondence Between Mirror-Twisted Sectors for N = 2 Supersymmetric Vertex Operator Superalgebras of the Form $V \otimes V$ and N = 1 Ramond Sectors of V

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Abstract Using recent results of the author along with Vander Werf, we present the classification and construction of mirror-twisted modules for N $=$ 2 supersymmetric vertex operator superalgebras of the form $V \otimes V$ for the signed transposition mirror map automorphism. In particular, we show that the category of such mirror-twisted sectors for $V \otimes V$ is isomorphic to the category of N $=$ 1 Ramond sectors for V.

1 Introduction

In [5, 6], the author studied twisted modules for N $=$ 2 supersymmetric vertex operator superalgebras (N $=$ 2 VOSAs) for finite order VOSA automorphisms arising from automorphisms of the N $=$ 2 Neveu-Schwarz algebra of N $=$ 2 infinitesimal superconformal transformations. Among such automorphisms is the mirror map. In [5], mirror maps were given for N $=$ 2 VOSAs of the form $V \otimes V$ where V is an N $=$ 1 supersymmetric VOSA of the form $V_L \otimes V_{fer}$, where V_L is a rank d lattice VOSA or the d free boson vertex operator algebra and V_{fer} is the d free fermion VOSA. In particular, we showed that one of the mirror maps for such an N $=$ 2 VOSA, $V \otimes V$, is given by the signed transposition map

$$\tilde{\kappa} = (1\ 2)\ :\ V \otimes V \longrightarrow V \otimes V, \qquad u \otimes v \ \mapsto\ (-1)^{|u||v|} v \otimes u \tag{1}$$

where $|v| = j \bmod 2$ for $v \in V^{(j)}$, with the $\mathbb{Z}_2$-grading of V given by $V = V^{(0)} \oplus V^{(1)}$.

In [7] and [10], the author along with Vander Werf constructed and classified the cyclic permutation-twisted $V^{\otimes k}$-modules, where V is any VOSA and k is

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a positive integer. For k even, this classification is in terms of parity-twisted V-modules where the parity automorphism of a VOSA is the map

$$\sigma : V \longrightarrow V, \qquad v \mapsto (-1)^{|v|}v. \tag{2}$$

In this note, we apply the results of [10] to the setting of the mirror map (1) acting on an N = 2 supersymmetric VOSA of the form $V \otimes V$, to show that the category of $\tilde{\kappa}$-twisted $(V \otimes V)$-modules is isomorphic to the category of σ-twisted V-modules, which are the N = 1 Ramond sectors for the N = 1 supersymmetric VOSA, V. This classification also provides an explicit construction of these modules.

In particular, our result shows that if a representation M_σ of the N = 1 Ramond algebra is also a parity-twisted modules for a VOSA V, where $V \otimes V$ is N = 2 supersymmetric, then M_σ is also naturally a representation of the mirror-twisted N = 2 Neveu-Schwarz algebra. These results can be used to calculate the graded dimensions for one module in terms of the graded dimensions for the other as shown in Corollary 2 below. Note that for our results, we do not need to make any assumptions about, for instance, the values of the central charge, the complete reducibility of the representations, or the rationality of the VOSAs.

Certain representations of the N = 1 Ramond algebra and related VOSA constructions have previously been studied in, e.g., [1, 16, 17, 19–22, 24, 25, 27, 30, 31]. Certain representations of the mirror-twisted N = 2 Neveu-Schwarz algebra have previously been studied in, e.g., [12, 13, 15, 18, 23, 26, 28, 29]. In particular, the relationship between characters of certain modules for the N = 1 Ramond algebra and certain modules for the mirror-twisted N = 2 Neveu-Schwarz algebra had previously been observed. Our explicit isomorphism between mirror-twisted sectors for $V \otimes V$ and N = 1 Ramond sectors for V, gives a constructive and overarching explanation of this phenomenon through the theory of VOSAs.

2 The Notions of VOSA and Twisted Module

Following the notation of [7, 10], we recall the notion of VOSA and the notions of weak, weak admissible and ordinary g-twisted V-module for a VOSA, V, and an automorphism g of V of finite order.

Let x, x_0, x_1, x_2, denote commuting independent formal variables. Let $\delta(x) = \sum_{n\in\mathbb{Z}} x^n$. Expressions such as $(x_1 - x_2)^n$ for $n \in \mathbb{C}$ are to be understood as formal power series expansions in nonnegative integral powers of the second variable.

Definition 1. A *vertex operator superalgebra* is a $\frac{1}{2}\mathbb{Z}$-graded (by weight) vector space $V = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}} V_n$, satisfying $\dim V_n < \infty$ and $V_n = 0$ for n sufficiently negative, that is also $\mathbb{Z}_2$-graded by *sign*, $V = V^{(0)} \oplus V^{(1)}$, with $V^{(j)} = \bigoplus_{n\in\mathbb{Z}+\frac{j}{2}} V_n$, and equipped with a linear map

$$V \longrightarrow (\mathrm{End}\, V)[[x, x^{-1}]], \qquad v \mapsto Y(v, x) = \sum_{n\in\mathbb{Z}} v_n x^{-n-1}, \tag{3}$$

and with two distinguished vectors $\mathbf{1} \in V_0$, (the *vacuum vector*) and $\omega \in V_2$ (the *conformal element*) satisfying the following conditions for $u, v \in V$: $u_n v = 0$ for n sufficiently large; $Y(\mathbf{1}, x)v = v$; $Y(v, x)\mathbf{1} \in V[[x]]$, and $\lim_{x\to 0} Y(v, x)\mathbf{1} = v$;

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u, x_1)Y(v, x_2) - \\ - (-1)^{|u||v|}x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y(v, x_2)Y(u, x_1) = \\ = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u, x_0)v, x_2) \tag{4}$$

(the *Jacobi identity*), where $|v| = j$ if $v \in V^{(j)}$ for $j \in \mathbb{Z}_2$; writing $Y(\omega, x) = \sum_{n\in\mathbb{Z}} L(n)x^{-n-2}$, i.e., $L(n) = \omega_{n+1}$, for $n \in \mathbb{Z}$, then the $L(n)$ give a representation of the Virasoro algebra with central charge $c \in \mathbb{C}$ (the *central charge* of V); for $n \in \frac{1}{2}\mathbb{Z}$ and $v \in V_n$, then $L(0)v = nv = (\mathrm{wt}\, v)v$; and the $L(-1)$-*derivative property* holds: $\frac{d}{dx}Y(v, x) = Y(L(-1)v, x)$.

An *automorphism* of a VOSA, V, is a linear map g from V to itself, preserving $\mathbf{1}$ and ω such that the actions of g and $Y(v, x)$ on V are compatible in the sense that $gY(v, x)g^{-1} = Y(gv, x)$, for $v \in V$. Then $gV_n \subset V_n$ for $n \in \frac{1}{2}\mathbb{Z}$.

Let $\mathbb{Z}_+$ denote the positive integers. If g has finite order, V is a direct sum of the eigenspaces V^j of g, i.e., $V = \bigoplus_{j\in\mathbb{Z}/k\mathbb{Z}} V^j$, where $k \in \mathbb{Z}_+$ is a period of g (i.e., $g^k = 1$) and $V^j = \{v \in V \mid gv = \eta^j v\}$, for η a fixed primitive k-th root of unity.

Definition 2. Let $(V, Y, \mathbf{1}, \omega)$ be a VOSA and g an automorphism of V of period $k \in \mathbb{Z}_+$. A *weak g-twisted V-module* is a vector space M equipped with a linear map

$$V \longrightarrow (\mathrm{End}\, M)[[x^{1/k}, x^{-1/k}]], \qquad v \mapsto Y^g(v, x) = \sum_{n\in\frac{1}{k}\mathbb{Z}} v_n^g x^{-n-1}, \tag{5}$$

with $v_n^g \in (\mathrm{End}\, M)^{(|v|)}$, and satisfying the following conditions for $u, v \in V$ and $w \in M$: $v_n^g w = 0$ for n sufficiently large; $Y^g(\mathbf{1}, x)w = w$;

$$x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y^g(u, x_1)Y^g(v, x_2) - \\ - (-1)^{|u||v|}x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right) Y^g(v, x_2)Y^g(u, x_1) = \\ = x_2^{-1}\frac{1}{k}\sum_{j\in\mathbb{Z}/k\mathbb{Z}} \delta\left(\eta^j \frac{(x_1 - x_0)^{1/k}}{x_2^{1/k}}\right) Y^g(Y(g^j u, x_0)v, x_2) \tag{6}$$

(the *twisted Jacobi identity*) where η is a fixed primitive k-th root of unity.

As a consequence of the definition, we have that $Y^g(v,x) = \sum_{n\in\mathbb{Z}+\frac{j}{k}} v_n^g x^{-n-1}$ for $j \in \mathbb{Z}/k\mathbb{Z}$ and $v \in V^j$, and for $v \in V$, we have

$$Y_g(gv,x) = \lim_{x^{1/k}\to\eta^{-1}x^{1/k}} Y_g(v,x).$$

It also follows that writing $Y^g(\omega,x) = \sum_{n\in\mathbb{Z}} L^g(n)x^{-n-2}$, i.e., setting $L^g(n) = \omega_{n+1}^g$, for $n \in \mathbb{Z}$, then the $L^g(n)$ satisfy the relations for the Virasoro algebra with central charge c the central charge of V.

If we take $g = 1$, then we obtain the notion of weak V-module. The term "weak" means we are making no assumptions about a grading on M.

A *weak admissible* g-twisted V-module is a weak g-twisted V-module M which carries a $\frac{1}{2k}\mathbb{N}$-grading $M = \bigoplus_{n\in\frac{1}{2k}\mathbb{N}} M(n)$, such that $v_m^g M(n) \subseteq M(n + \mathrm{wt}\, v - m - 1)$ for homogeneous $v \in V$, $n \in \frac{1}{2k}\mathbb{N}$, and $m \in \frac{1}{k}\mathbb{Z}$. If $g = 1$, then a weak admissible g-twisted V-module is called a weak admissible V-module.

An (ordinary) g-twisted V-module is a weak g-twisted V-module M graded by $\mathbb{C}$ induced by the spectrum of $L(0)$. That is, we have $M = \bigoplus_{\lambda\in\mathbb{C}} M_\lambda$, where $M_\lambda = \{w \in M \,|\, L(0)^g w = \lambda w\}$, for $L(0)^g = \omega_1^g$. Moreover we require that $\dim M_\lambda$ is finite and $M_{n/2k+\lambda} = 0$ for fixed λ and for all sufficiently small integers n. If $g = 1$, then a g-twisted V-module is a V-module.

3 The Construction and Classification of $(1\ 2\ \cdots\ k)$-Twisted $V^{\otimes k}$-Modules

Let $V = (V, Y, \mathbf{1}, \omega)$ be a VOSA, and let k be a fixed positive integer. Then $V^{\otimes k}$ is also a VOSA, and the permutation group S_k acts naturally on $V^{\otimes k}$ as signed automorphisms. In particular, taking the action of S_k on $V^{\otimes k}$ to be a right action, we have the action of $(1\ 2\ \cdots\ k)$ given by

$$\begin{aligned}(1\ 2\ \cdots\ k) : V\otimes V\otimes\cdots\otimes V &\longrightarrow V\otimes V\otimes\cdots\otimes V \\ v_1\otimes v_2\otimes\cdots\otimes v_k &\mapsto (-1)^{|v_1|(|v_2|+\cdots+|v_k|)} v_2\otimes v_3\otimes\cdots\otimes v_k\otimes v_1.\end{aligned} \tag{7}$$

Let $g = (1\ 2\ \cdots\ k)$. Below, we will recall the classification and construction of g-twisted $V^{\otimes k}$-modules from [7] and [10]. This construction is based on a certain operator $\Delta_k(x)$ first defined in [8], (see also [11]) which we now recall.

Consider the polynomial $\frac{1}{k}(1+x)^k - \frac{1}{k}$ in $x\mathbb{Q}[x]$. Following [8], for $k \in \mathbb{Z}_+$, we define $a_j \in \mathbb{Q}$ for $j \in \mathbb{Z}_+$, by

$$\exp\left(-\sum_{j\in\mathbb{Z}_+} a_j x^{j+1}\frac{\partial}{\partial x}\right)\cdot x = \frac{1}{k}(1+x)^k - \frac{1}{k}. \tag{8}$$

For example, $a_1 = (1-k)/2$ and $a_2 = (k^2-1)/12$. Let $V = (V, Y, \mathbf{1}, \omega)$ be a VOSA. In $(\mathrm{End}\, V)[[x^{1/2k}, x^{-1/2k}]]$, define

$$\Delta_k(x) = \exp\left(\sum_{j\in\mathbb{Z}_+} a_j x^{-\frac{j}{k}} L(j)\right)(k^{\frac{1}{2}})^{-2L(0)} \left(x^{\frac{1}{2k}(k-1)}\right)^{-2L(0)}. \tag{9}$$

For $v \in V$, and k any positive integer, denote by $v^j \in V^{\otimes k}$, for $j = 1, \ldots, k$, the vector whose j-th tensor factor is v and whose other tensor factors are $\mathbf{1}$. Then for $g = (1\ 2\ \cdots\ k)$, we have $gv^j = v^{j-1}$ for $j = 1, \ldots, k$ where 0 is understood to be k.

Let (M, Y_M) be a V-module, and (M_σ, Y_σ) a σ-twisted V-module, where σ is the parity map on V. We define the g-twisted vertex operators for $V^{\otimes k}$ on M, for k odd, and on M_σ, for k even, as follows: Set

$$Y_g(v^1, x) = \begin{cases} Y_M(\Delta_k(x)v, x^{1/k}) & \text{for } k \text{ odd} \\ Y_\sigma(\Delta_k(x)v, x^{1/k}) & \text{for } k \text{ even} \end{cases} \tag{10}$$

and for $j = 0, \ldots, k-1$, define

$$Y_g(v^{j+1}, x) = \lim_{x^{1/k} \to \eta^j x^{1/k}} Y_g(v^1, x). \tag{11}$$

Let V be an arbitrary VOSA and h an automorphism of V of finite order. Denote the categories of weak, weak admissible and ordinary h-twisted V-modules by $\mathcal{C}_w^h(V)$, $\mathcal{C}_a^h(V)$ and $\mathcal{C}^h(V)$, respectively. If $h = 1$, we habitually remove the index h.

Now again consider the VOSA, $V^{\otimes k}$, and the k-cycle $g = (1\ 2\ \cdots\ k)$. For k odd, define

$$T_g^k : \mathcal{C}_w(V) \longrightarrow \mathcal{C}_w^g(V^{\otimes k}), \qquad (M, Y_M) \mapsto (T_g^k(M), Y_g) = (M, Y_g). \tag{12}$$

For k even, define

$$T_g^k : \mathcal{C}_w^\sigma(V) \longrightarrow \mathcal{C}_w^g(V^{\otimes k}), \qquad (M_\sigma, Y_\sigma) \mapsto (T_g^k(M_\sigma), Y_g) = (M_\sigma, Y_g). \tag{13}$$

The following theorem is proved in [7] for k odd, and in [10] for k even.

Theorem 1 ([7, 10]).

(1) For k odd, the functor T_g^k is an isomorphism from the category $\mathcal{C}_w(V)$ of weak V-modules to the category $\mathcal{C}_w^g(V^{\otimes k})$ of weak $g = (1\ 2\ \cdots\ k)$-twisted $V^{\otimes k}$-modules.

(2) For k even, the functor T_g^k is an isomorphism from the category $\mathcal{C}_w^\sigma(V)$ of weak parity-twisted V-modules to the category $\mathcal{C}_w^g(V^{\otimes k})$ of weak $g = (1\ 2\ \cdots\ k)$-twisted $V^{\otimes k}$-modules.

(3) For any $k \in \mathbb{Z}_+$, the functor T_g^k restricted to the respective subcategories of weak admissible, ordinary or irreducible modules in $\mathcal{C}_w(V)$ or $\mathcal{C}_w^\sigma(V)$, respectively, is an isomorphism between these subcategories and the corresponding subcategory of weak admissible, ordinary or irreducible g-twisted $V^{\otimes k}$-modules.

4 N = 2 Supersymmetric VOSAs, Ramond Sectors, and Mirror-Twisted Sectors

In this section, we recall the notions of N = 1 or N = 2 supersymmetric VOSA, following the notation and terminology of, for instance, [2, 3] and [4]. First we will need the notion of several superextensions of the Virasoro algebra.

The $N = 1$ *Neveu-Schwarz algebra* or $N = 1$ *superconformal algebra* is the Lie superalgebra with basis consisting of the central element d, even elements L_n for $n \in \mathbb{Z}$, and odd elements G_r for $r \in \mathbb{Z} + \frac{1}{2}$, and supercommutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}\, d, \tag{14}$$

$$[L_m, G_r] = \left(\frac{m}{2} - r\right) G_{m+r}, \tag{15}$$

$$[G_r, G_s] = 2L_{r+s} + \frac{1}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}\, d, \tag{16}$$

for $m, n \in \mathbb{Z}$, and $r, s \in \mathbb{Z} + \frac{1}{2}$. The $N = 1$ *Ramond algebra* is the Lie superalgebra with basis the central element d, even elements L_n for $n \in \mathbb{Z}$, and odd elements G_r for $r \in \mathbb{Z}$, and supercommutation relations given by (14)–(16), where now $r, s \in \mathbb{Z}$.

The N = 2 *Neveu-Schwarz Lie superalgebra* or N = 2 *superconformal algebra* is the Lie superalgebra with basis consisting of the central element d, even elements L_n and J_n for $n \in \mathbb{Z}$, and odd elements $G_r^{(j)}$ for $j = 1, 2$ and $r \in \mathbb{Z} + \frac{1}{2}$, and such that the supercommutation relations are given as follows: L_n, d and $G_r^{(j)}$ satisfy the supercommutation relations for the N = 1 Neveu-Schwarz algebra (14)–(16) for both $G_r = G_r^{(1)}$ and for $G_r = G_r^{(2)}$; the remaining relations are given by

$$[L_m, J_n] = -nJ_{m+n}, \qquad [J_m, J_n] = \frac{1}{3}m\delta_{m+n,0}d \tag{17}$$

$$\left[J_m, G_r^{(1)}\right] = -iG_{m+r}^{(2)}, \quad \left[J_m, G_r^{(2)}\right] = iG_{m+r}^{(1)}, \tag{18}$$

$$\left[G_r^{(1)}, G_s^{(2)}\right] = i(s-r)J_{r+s}. \tag{19}$$

The $N = 2$ *Ramond algebra* is the Lie superalgebra with basis consisting of the central element d, even elements L_n and J_n for $n \in \mathbb{Z}$, and odd elements $G_r^{(j)}$ for $r \in \mathbb{Z}$ and $j = 1, 2$, and supercommutation relations given by those of the N = 2 Neveu-Schwarz algebra but with $r, s \in \mathbb{Z}$, instead of $r, s \in \mathbb{Z} + \frac{1}{2}$.

Note that there is an automorphism of the N = 2 Neveu-Schwarz algebra given by

$$\kappa: \ G_r^{(1)} \mapsto G_r^{(1)}, \quad G_r^{(2)} \mapsto -G_r^{(2)}, \quad J_n \mapsto -J_n, \quad L_n \mapsto L_n, \quad d \mapsto d, \tag{20}$$

called the *mirror map* automorphism of the N = 2 Neveu-Schwarz algebra.

Let $(V, Y, \mathbf{1}, \omega)$ be a VOSA, and suppose there exists $\tau \in V_{3/2}$ such that writing $Y(\tau, z) = \sum_{n\in\mathbb{Z}} \tau_n x^{-n-1} = \sum_{n\in\mathbb{Z}} G(n + 1/2)x^{-n-2}$, the $G(n + 1/2) = \tau_{n+1} \in (\mathrm{End}\, V)^{(1)}$ generate a representation of the N = 1 Neveu-Schwarz Lie superalgebra such that the $L(n)$ are the modes of ω. Then we call $(V, Y, \mathbf{1}, \tau)$ an $N = 1$ *Neveu-Schwarz VOSA*, or an $N = 1$ *supersymmetric VOSA*, or just an N = 1 VOSA for short.

Suppose a VOSA, V, has two vectors $\tau^{(1)}$ and $\tau^{(2)}$ such that $(V, Y, \mathbf{1}, \tau^{(j)})$ is an N = 1 VOSA for both $j = 1$ and $j = 2$, and the $\tau_{n+1}^{(j)} = G^{(j)}(n + 1/2)$ generate a representation of the N = 2 Neveu-Schwarz Lie superalgebra. Then we call such a VOSA an N = 2 *Neveu-Schwarz VOSA* or an N = 2 *supersymmetric VOSA*, or for short, an N = 2 VOSA.

For the case of the parity map, σ, a σ-twisted V-module, for V an N = 1 or N = 2 VOSA, is naturally a representation of the N = 1 or N = 2 Ramond algebra, respectively. (See for instance [5,6], as well as references therein).

Suppose V is an N = 2 VOSA such that V has an automorphism g_κ which is a lift of the mirror map κ for the N = 2 Neveu-Schwarz algebra. That is letting g_κ act by conjugation on End V, then g_κ restricts to the mirror map κ on the elements $L(n)$, $J(n)$, and $G^{(j)}(r)$, for $n \in \mathbb{Z}$, $j = 1, 2$, and $r \in \mathbb{Z} + \frac{1}{2}$, which give the N = 2 Neveu-Schwarz algebra representation on the N = 2 VOSA, V. Following [5,6], we call such an automorphism g_κ of an N = 2 VOSA, V, a *mirror map*. Then a g_κ-twisted V-module is naturally a representation of the "mirror-twisted N = 2 Neveu-Schwarz algebra". The *mirror-twisted* N = 2 *Neveu-Schwarz algebra* is the Lie superalgebra with basis consisting of even elements L_n, and J_r and central element d, odd elements $G_r^{(1)}$ and $G_n^{(2)}$, for $n \in \mathbb{Z}$ and $r \in \mathbb{Z} + \frac{1}{2}$, and supercommutation relations given as follows: The L_n and $G_r^{(1)}$ satisfy the supercommutation relations for the N = 1 Neveu-Schwarz algebra with central charge d; the L_n and $G_n^{(2)}$ satisfy the supercommutation relations for the N = 1 Ramond algebra with central charge d; and the remaining supercommutation relations are

$$[L_n, J_r] = -rJ_{n+r}, \quad [J_r, J_s] = \frac{1}{3} r\delta_{r+s,0}d, \quad \left[G_r^{(1)}, G_n^{(2)}\right] = -i(r-n)J_{r+n} \tag{21}$$

$$\left[J_r, G^{(1)}_s\right] = -iG^{(2)}_{r+s}, \qquad \left[J_r, G^{(2)}_n\right] = iG^{(1)}_{r+n}, \tag{22}$$

for $n \in \mathbb{Z}$ and $r \in \mathbb{Z} + \frac{1}{2}$. Note that this mirror-twisted N $=$ 2 Neveu-Schwarz algebra is not isomorphic to the ordinary N $=$ 2 Neveu-Schwarz algebra [32].

5 Mirror-Twisted Modules for the Class of N = 2 VOSAs of the Form $V \otimes V$

There are large classes of N $=$ 2 VOSAs of the form $V \otimes V$ such that V is an N $=$ 1 VOSA, and $\tilde{\kappa} = (1\ 2)$, the signed transposition map given by Eq. (1), is a mirror map for $V \otimes V$. Examples of such N $=$ 2 VOSAs, were studied in [5]. These include the following examples: Let V_L be a rank d positive definite integral lattice VOSA or the d free boson vertex operator algebra, and let V^d_{fer} be the d free fermion VOSA. As noted in [5], the VOSA $V = V_L \otimes V^d_{fer}$, is naturally an N $=$ 1 VOSA, and $V \otimes V$ is naturally an N $=$ 2 VOSA. This uses the construction of a VOSA from a positive definite integral lattice, following for instance [9, 14, 33]. Such N $=$ 2 VOSAs have more than one mirror map as was shown in [5], where the author constructed mirror-twisted modules for these VOSAs for the mirror map which is not $\tilde{\kappa}$.

For such N $=$ 2 VOSAs of the form $V \otimes V$, and for the signed transposition mirror-map $\tilde{\kappa}$, we have the following immediate corollary to Theorem 1.

Corollary 1. *The category of weak mirror-twisted $(V \otimes V)$-modules for the signed transposition mirror map automorphism of an* N $=$ 2 *VOSA of the form $V \otimes V$ is isomorphic to the category of weak* N $=$ 1 *Ramond-twisted V-modules (i.e., parity-twisted V-modules). In addition, the subcategories of weak admissible, ordinary, or irreducible modules are isomorphic.*

In particular, it follows that if M_σ is a representation of the N $=$ 1 Ramond algebra such that M_σ is a weak parity-twisted module for an N $=$ 1 VOSA, V, and such that $V \otimes V$ is an N $=$ 2 VOSA, then M_σ is also naturally a representation of the mirror-twisted N $=$ 2 superconformal algebra and is a weak $\tilde{\kappa}$-twisted module for $V \otimes V$.

Furthermore, from the construction of such modules given by the functor T^k_g for $k = 2$ as in (10), (11), (13), (see also [10]), we have as a consequence of Corollary 6.5 in [10], the following:

Corollary 2. *$M_{\tilde{\kappa}}$ is an ordinary $\tilde{\kappa}$-twisted $(V \otimes V)$-module with graded dimension*

$$\dim_q M_{\tilde{\kappa}} = tr_{M_{\tilde{\kappa}}} q^{-2c/24 + L^{\tilde{\kappa}}(0)} = q^{-c/12} \sum_{\lambda \in \mathbb{C}} \dim(M_\lambda) q^\lambda$$

if and only if $(T_{\tilde{\kappa}}^2)^{-1}(M_{\tilde{\kappa}}) = M_{\tilde{\kappa}}$ *is an ordinary* σ*-twisted* V*-module with graded dimension*

$$\dim_q (T_{\tilde{\kappa}}^2)^{-1}(M_{\tilde{\kappa}}) = tr_{M_{\tilde{\kappa}}} q^{-c/24+L^{\sigma}(0)} = \dim_{q^2} M_{\tilde{\kappa}},$$

where c *is the central charge of* V.

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Operadic Bridge Between Renormalization Theory and Vertex Algebras

Nikolay M. Nikolov

Abstract A construction is presented that provides a correspondence between renormalization groups in models of perturbative massless Quantum Field Theory and models of vertex algebras.

The aim of this talk is to show how two different areas in Quantum Field Theory (QFT) are governed by one and the same algebraic structure. This opens perspectives of transferring constructions in both directions via this common structure. The two connected fields are the theory of *Operator Product Expansion* (OPE) *algebras* (called also *vertex algebras*) and the renormalization theory in perturbative QFT and more concretely, the *renormalization group* and its action. The bridge between these two structures is an *operad*, which we call the *expansion operad* $\mathcal{E}$, and whose algebras are the vertex (or OPE) algebras, while the group associated to this operand is the renormalization group. Thus, our plan in this lecture is to consider the following topics:

A. What is a vertex algebra?
B. What is an operad?
C. What is the renormalization group and its action (i.e., a representation by formal diffeomorphisms on the physical parameters)?

A. Starting with the firs topic, a vertex algebra is the structure that is closed by the OPE. The OPE in turn was introduced for the analysis of the short distance behavior in QFT [10]. According to the general principles of locality and causality in QFT one expects that the product of two local quantum fields posses an asymptotic expansion at short distances $\mathrm{x}-\mathrm{y}\to 0$ of the form

$$\phi(\mathrm{x})\,\psi(\mathrm{y}) \underset{\mathrm{x}\to\mathrm{y}}{\sim} \sum_A \theta_A(\mathrm{y})\,C_A(\mathrm{x}-\mathrm{y}),$$

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for a suitable system of two-point numerical functions (distributions) $C_A(\mathrm{x}-\mathrm{y})$ that describes the local behavior of the product, and the coefficients $\theta_A(\mathrm{y})$ are again local fields (the sign $\underset{\mathrm{x}\to\mathrm{y}}{\sim}$ stands for the asymptotic expansion at short distances). For instance, in perturbative massless QFT one can choose

$$C_A(\mathrm{x}-\mathrm{y}) \,=\, \big((\mathrm{x}-\mathrm{y})^2\big)^{\nu}\,\big((\log(\mathrm{x}-\mathrm{y}))\big)^{\ell}\,h_{m,\sigma}(\mathrm{x}-\mathrm{y})\,, \quad A \,=\, (\nu,\ell,m,\sigma)\,,$$

where $\nu\in\mathbb{R}$, $\ell\in\{0,1,\dots\}$ and $\{h_{m,\sigma}(\mathrm{x})\}_{\sigma}$ is a basis of harmonic homogeneous polynomials (spherical functions) of degree $m=0,1,\dots$. Thus, for every index A we obtain a binary operation

$$\theta_A \;=:\; \phi \underset{A}{*} \psi \qquad \Longrightarrow \qquad \big\{\underset{A}{*}\big\}_A$$

in the vector space of all local quantum fields (this space is called "Borchers class"). A vertex algebra is determined as the algebraic structure defined by this infinite system of binary products $\{\underset{A}{*}\}_A$. The main condition on the latter system of operations comes from the operator product associativity:

$$\phi_1(\mathrm{x}_1)\,\big(\phi_2(\mathrm{x}_2)\,\phi_3(\mathrm{x}_3)\big) \;=\; \big(\phi_1(\mathrm{x}_1)\,\phi_2(\mathrm{x}_2)\big)\,\phi_3(\mathrm{x}_3)\,.$$

However, it is rather nontrivial to reformulate this associativity in a purely algebraic way for the system of binary products $\{\underset{A}{*}\}_A$. This is completely understood only in the following cases:

- In space-time dimension $D = 1$ (chiral) Conformal Field Theory ("on a light ray") the OPE takes the form

$$\phi(z)\,\psi(w) \;=\; \sum_{n\in\mathbb{Z}} \big(\phi_{(n)}\psi\big)(w)(z-x)^{-n-1}$$

  and its associativity and further properties was first axiomatized by R. Borcherds [1].
- A generalization to higher D was introduced in [2] but in the context of QFT vertex algebras have been considered in [6]. It has been shown it the latter paper that these algebras are in one–to–one correspondence with models of Wightman axioms possessing the so called Global Conformal Invariance [8].

B. We proceed by considering vertex algebras as algebras over an operad. So first, what is an operad? Besides one of the first references on this topic [5] we shall mention one recent book [4], from which we follow the definitions and conventions.

One can think of an operad as a generalized type of algebras. An algebra of a certain type is determined by introducing a set of multilinear operations subject to certain identities that use compositions of these operations, eventually combined with permutations of the input arguments. Instead of this one can consider the spaces of all possible multilinear operations obtained under compositions and the action of permutations (and all this quotient by the relations). This will be the operad corresponding to the considered type of algebras.

In more details, an operad includes

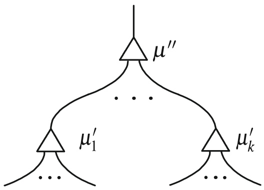
- a sequence of vector spaces $\{\mathcal{M}(n)\}_{n=1}^{\infty}$ ($\mathcal{M}(2)$ being the space of binary operations, …).
- The structure is endowed by various structure maps called operadic compositions,

$$\begin{array}{rcl} \mathcal{M}(k) \otimes \mathcal{M}(j_1) \otimes \cdots \otimes \mathcal{M}(j_k) & \longrightarrow & \mathcal{M}(n) \\ \mu'' \otimes \mu'_1 \otimes \cdots \otimes \mu'_k & \longmapsto & \mu'' \circ (\mu'_1, \cdots, \mu'_k)\,, \end{array}$$

where $n = j_1 + \cdots + j_k$, and permutation actions

$$\mathcal{M}(n) \times \mathcal{S}_n \ni \mu \times \sigma \mapsto \mu^{\sigma} \in \mathcal{M}(n)\,, \quad (\mu^{\sigma_1})^{\sigma_2} = \mu^{\sigma_1 \sigma_2}\,.$$

The operadic composition $\mu'' \circ (\mu'_1, \cdots, \mu'_k)$ is pictorially drawn as:



One of the main examples of an operad is the *endomorphism operad* $\mathcal{E}nd_V$ for a vector space V:

$$\mathcal{E}nd_V(n) := Hom\big(V^{\otimes n}, V\big)\,,$$

where $\mu'' \circ (\mu'_1, \cdots, \mu'_k)$ is the actual composition of multilinear maps and

$$\mu^{\sigma}(v_1, \cdots, v_n) := \mu(v_{\sigma_1}, \cdots, v_{\sigma_n})\,.$$

Morphisms of operads are defined as follows:

$$\{\mathcal{M}(n)\}_{n=1}^{\infty} \to \{\mathcal{N}(n)\}_{n=1}^{\infty} \equiv \big\{\mathcal{M}(n) \to \mathcal{N}(n)\big\}_{n=1}^{\infty}$$

plus compatibility with all structure maps. In particular, morphisms from an operad to the endomorphisms operads have a meaning of "representations" but are called *algebras over the corresponding operad*:

$$\text{Representation} \equiv \text{Algebra over an operad,}$$

$$\text{i.e., } \{\mathcal{M}(n)\}_n \to \{\mathcal{E}nd_V(n)\}_n \text{ – morphism of operads,}$$

$$\text{i.e., } \mathcal{M}(n) \to Hom\big(V^{\otimes n}, V\big)$$

(the abstract operations in $\mathcal{M}(n)$ become actual n–linear maps on V that is the underlined space of the algebra).

Example. The Lie operad $\mathcal{L}ie$ corresponds the class of Lie algebras and is defined as:

$$\mathcal{L}ie(1) = Span_{\mathbb{C}}\{1\} \xrightarrow{\pi_1} Hom(V,V)\,,$$
$$\mathcal{L}ie(2) = Span_{\mathbb{C}}\{\lambda\} \xrightarrow{\pi_2} Hom(V^{\otimes 2},V)\,,$$
$$\pi_2(\lambda)(a,b) = [a,b]\,,$$
$$\mathcal{L}ie(3) = Span_{\mathbb{C}}\{\, \lambda \circ (1,\lambda)\,,\ \lambda \circ (\lambda,1)\,\}$$
$$\downarrow \qquad \downarrow$$
$$\big(\lambda \circ (\lambda,1)\big)^{(1,3,2)} \to [[a,c],b] = [a,[b,c]] - [[a,b],c]\,,$$
$$\big(\lambda \circ (\lambda,1)\big)^{(1,3,2)} = \lambda \circ (\lambda,1) - \lambda \circ (1,\lambda)\,,$$

where μ^σ for an element μ in the nth operadic space and a permutation $\sigma \in \mathcal{S}_n$ stands for the (right) actions of the permutation groups on the operad (that is one of the basic structures in the operad).

The main construction in this work is based on a particular example of an operad, which we call the **expansion operad** $\mathcal{E} = \big\{\mathcal{E}(n)\big\}_n$. It is defined for a sequence of *graded* function spaces

$$\mathcal{O}_n \subseteq C^\infty\big((\mathbb{R}^D)^{\times n} \setminus \text{all diagonals}\big)$$

for $n = 2,3,\ldots$ admitting expansions

$$G(\mathrm{x}_1,\ldots,\mathrm{x}_n) = \sum_\ell G'_\ell(\mathrm{x}_j,\ldots,\mathrm{x}_{j+k})\, G''_\ell(\mathrm{x}_1,\ldots,\mathrm{x}_{j-1},\mathrm{x}_{j+k},\ldots,\mathrm{x}_n)$$

for $|\mathrm{x}_a - \mathrm{x}_{j+k}| \ll |\mathrm{x}_b - \mathrm{x}_{j+k}|$ when $a \in \{j,\ldots,j+k\} \not\ni b$. We set

$$\mathcal{E}(n) = \mathcal{O}_n'\,,$$

which is the *graded dual*. In the applications to vertex algebras and renormalization theory of massless fields:

$\mathcal{O}_n$ = The algebra of rational n–point functions $\dfrac{P(\mathrm{x}_1-\mathrm{x}_2,\ldots,\mathrm{x}_{n-1}-\mathrm{x}_n)}{\prod\limits_{1\leqslant j<k\leqslant n}\big((\mathrm{x}_j-\mathrm{x}_k)^2\big)^{\nu_{j,k}}}$ on $\mathbb{R}^D \ni \mathrm{x}_1,\ldots,x_n$ with light–cone singularities, graded by the degree of homogeneity.

The key relation between the operad $\mathcal{E}$ and the vertex algebras is that every vertex algebra induces a system of linear maps

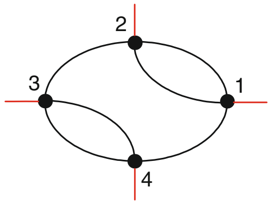
$$\begin{array}{ccc} \mathcal{E}(n) & \longrightarrow & \mathcal{E}nd_V(n) \\ ||| & & ||| \\ \mathcal{O}_n' & \longrightarrow & Hom_{\mathbb{C}}\big(V^{\otimes n},V\big) \cong V'^{\otimes n}\otimes V, \end{array}$$

where the down arrow is the dual of the correlation functions maps:

$$\begin{aligned} V^{\otimes n} \otimes V' &\longmapsto \mathcal{O}_n \\ a_1 \otimes \cdots \otimes a_n \otimes \lambda &\longmapsto \lambda\big(a_1(\mathrm{x}_1 - \mathrm{x}_n) \cdots a_{n-1}(\mathrm{x}_{n-1} - \mathrm{x}_n)\, a_n\big) \\ &\equiv \big\langle \lambda \big| a_1(\mathrm{x}_1 - \mathrm{x}_n) \cdots a_{n-1}(\mathrm{x}_{n-1} - \mathrm{x}_n)\, a_n \big\rangle \end{aligned}$$

(here we assume that the graded pieces of V are finite dimensional). Thus, the operadic structure on $\mathcal{E}$ is such that the above system maps $\mathcal{E}(n) \longrightarrow \mathcal{E}nd_V(n)$ gives an operadic morphism. On the other hand, one can show that this operadic structure can be described entirely in terms of the expansions' operations in $\mathcal{O}_n$.

C. Passing to the renormalization let us mention first that the same rational functions belonging to $\mathcal{O}_n$ appear as "Feynman amplitudes" (= integrands in the Feynman integrals) in massless field theories. Here is an example of such a Feynman amplitude in the ϕ^4–theory:



$$\longleftrightarrow \quad \frac{1}{\big((\mathrm{x}_1 - \mathrm{x}_2)^2\big)^2} \frac{1}{(\mathrm{x}_2 - \mathrm{x}_3)^2} \times \frac{1}{\big((\mathrm{x}_3 - \mathrm{x}_4)^2\big)^2} \frac{1}{(\mathrm{x}_1 - \mathrm{x}_4)^2} \in \mathcal{O}_4$$

It is important for the present construction that we consider the ultraviolet renormalization on *configuration space*. In terms of Feynman amplitudes the renormalization is given by a system of linear maps

$$\mathcal{O}_n \to \mathcal{D}'\big((\mathbb{R}^D)^{\times(n-1)}\big)$$

subject to (recursive) conditions (cf. [7, 9] and references therein). In particular, the renormalization ambiguity at order n is described by a linear map: $\mathcal{O}_n \to \mathcal{D}'[0_n]$, where $\mathcal{D}'[0_n]$ stands for the space of distributions on $(\mathbb{R}^D)^{\times(n-1)}$ supported at the origin. We obtain a sequence of vector spaces

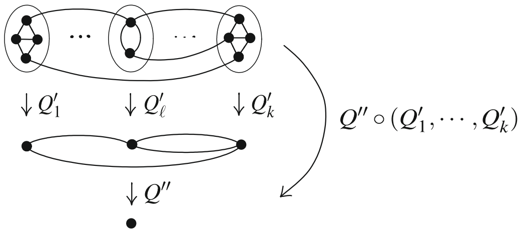
$$\mathcal{R}(n) := \big\{ Q : \mathcal{O}_n \to \mathcal{D}'[0_n] \,\big|\, \text{commuting with multiplication by polynomials} \big\}$$

where the condition comes from the requirements on the renormalization maps (as explained in [7] and [9]).

The bridge between the theory of the vertex algebras and renormalization is based on an existence of a natural isomorphism [7]

$$\mathcal{E}(n) \cong \mathcal{R}(n)\,.$$

Furthermore, the operadic compositions in $\mathcal{E}(n)$ have an interpretation on $\mathcal{R}(n)$ that corresponds to basic operations used in the renormalization group composition. The later has a very natural pictorial illustration



and its combinatorial version was described in [3].

The role of the operad $\mathcal{R}$ in renormalization theory is that it describes the Stückelberg–Bogoliubov renormalization group. The latter group is formed by all possible changes in the renormalization:

$$\begin{array}{c} \big\{\mathcal{O}_n \to \mathcal{D}'\big((\mathbb{R}^D)^{\times n}\big)\big\} \\ \downarrow \{Q'_n\} \\ \big\{\mathcal{O}_n \to \mathcal{D}'\big((\mathbb{R}^D)^{\times n}\big)\big\} \\ \downarrow \{Q''_n\} \\ \big\{\mathcal{O}_n \to \mathcal{D}'\big((\mathbb{R}^D)^{\times n}\big)\big\} \end{array} \Bigg) \; \{Q'''_n \in \mathcal{R}(n)\}\,,$$

where $\{Q'_n\}$ and $\{Q''_n\}$ are arbitrary sequences of changes of the renormalization $Q'_n, Q''_n \in \mathcal{R}(n)$.

In the paper [3] a functor was constructed

$$\Big\{\text{Operads}\Big\} \longrightarrow \Big\{\text{Groups}\Big\}\,,$$

which produces:

- the Renormalization group when applied to $\mathcal{E}$;
- the group of formal diffeomorphisms when applied on $\mathcal{E}nd_V$;
- the renormalization group action via an operadic morphism $\mathcal{E} \to \mathcal{E}nd_V$.

Our conclusion is summarized in the following scheme:

The $\mathcal{E}$-algebras are vertex algebras

↑

Expansion operad $\mathcal{E}$ $\cong$ Renormalization operad $\mathcal{R}$

↓

The group associated to $\mathcal{R}$ is the renormalization group

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Superfields and Vertex Algebras in Four Dimensions

Dimitar Nedanovski

Abstract This contribution is short presentation of the work (Nedanovski, D, Superconformal vertex algebras in four dimensions. arXiv:1401.0884v1 [hep-th]) in which the vertex algebra techniques in four dimensions are used for developing a superfield formalism for quantum fields with extended superconformal symmetry.

1 Introduction

The vertex algebra, first introduced by Borcherds [3], reflects the concept of operator product expansion in the case of two-dimensional conformal filed theories.

In [8] (for an earlier work see also [4]) the notion of vertex algebra was generalized for higher spacetime dimensions in one–to–one correspondence with models of Wightman fields obeying the so called Global Conformal Invariance (GCI) as introduced in [9]. GCI is an invariance of the Wightman functions under finite transformations of the two-fold spin covering of the geometric conformal group.

We use the ideas of [8] to extend the vertex algebra techniques to superconformal field theories in four spacetime dimensions.

Basic Notations. As mentioned, we work in four dimensions, but some of the constructions we use are valid in arbitrary dimension and for them the definitions are given for general dimension D. Vectors in the D-dimensional Minkowski space will be denoted by $\mathrm{x} = \left(x^{\mu}\right)_{\mu=0}^{D-1}$, x_j $(j = 1, 2, \ldots)$. We shall use also vectors in the complexified Euclidean space denoted by $\mathrm{z} = \left(z^{\mu}\right)_{\mu=0}^{D-1}$, z_j $(j = 1, 2, \ldots)$, etc. . The corresponding metrics (and scalar products) are, $\mathrm{x}^2 \equiv \mathrm{x} \cdot \mathrm{x} \equiv -(x^0)^2 + (x^1)^2 + \cdots + (x^{D-1})^2$ and $\mathrm{z}^2 \equiv \mathrm{z} \cdot \mathrm{z} \equiv (z^0)^2 + \cdots + (z^{D-1})^2$. Einstein summation convention is assumed.

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2 Vertex Algebras

GCI allows us to extend the QFT models with such an invariance on a compactification of the real spacetime. The latter is the (conformally) compactified Minkowski space $\overline{M}$. There are special complex coordinates that are obtained by a complex conformal transformation, which globally cover $\overline{M}$ [8]. Vertex operators correspond to local quantum fields in these new coordinates. This formalism is known in the literature as *compact picture*, because of the compactness of the real spacetime in this representation.

Let us stress two important technical features related to the formalism of vertex algebras in GCI QFT.

The first is that in the complex coordinates that parameterize $\overline{M}$ it is natural to change the signature of the metric to a complexified Euclidean metric. In this way, the vertex operator depends on a formal complex Euclidean variable $\mathrm{z} = (z^0, \dots, z^{D-1}) \in \mathbb{C}^D$. Furthermore, the natural generators of the conformal symmetry in these coordinates form a real basis of the Euclidean conformal Lie algebra. However, this ***does not mean*** that we are considering Euclidean fields in the sense of Euclidean field theory! The point is that the relevant real structure in the compact picture, which comes from the initial theory on the Minkowski space is not an ordinary complex conjugation related to the new coordinates or the symmetry generators.

Second, the vertex operators are not exactly quantum fields in the usual sense of Wightman axioms as they are not distributions. They are considered as formal power series in the spatial coordinates (the above complex coordinates). This is just for convenience and it can be considered as a topological lighten of the formalism: instead of with actual distributions we work with formal distributions (as these formal series are called in [5]). However, the axioms of vertex algebras are strong enough to allow us to prove that the vertex operators are not only formal distributions but determine also actual distributions.

The theory of vertex algebras is based on the formalism of formal Laurent–Taylor series with "light–cone poles". This formalism can be found in [8, Sect. 1] or in [1, Sect I]. However, in the present work these techniques will not play a central role and so, we shall not review them.

We follow the definition of a vertex algebra as given in [8, Definition 2.1], partially stated below. For a short review of the definitions and especially for a comparison with the one-dimensional, chiral case, we refer the reader to Sects. 1 and 2 of [2].

Thus, a *vertex algebra* is a $(\mathbb{Z}/2\mathbb{Z})$–graded vector space V endowed with an even (i.e., parity preserving) (bi)linear map[1]

[1]For a fixed first argument it defines a map $Y(a,z) : V \to V[[\mathrm{z}]][(\mathrm{z}^2)^{-1}]$ called *vertex operator*. Sometimes $Y(a,\mathrm{z})\, b$ is also denoted by $a(\mathrm{z})b$. The parity p_a of $a \in V$ is called parity of the vertex operator $Y(a,\mathrm{z})$ and it coincides with the parity of the above map $V \to V[[\mathrm{z}]][(z^2)^{-1}]$ with respect to the induced $(\mathbb{Z}/2\mathbb{Z})$-gradings.

$$V \otimes V \ni a \otimes b \mapsto Y(a, \mathrm{z})\, b \in V[[\mathrm{z}]][(\mathrm{z}^2)^{-1}],$$

($V[[\mathrm{z}]][(\mathrm{z}^2)^{-1}]$ stands for the space of formal Laurent–Taylor series with poles at $\mathrm{z}^2 = 0$, $\mathrm{z} \in \mathbb{C}^D$), a set of mutually commuting even endomorphisms T_μ, $\mu = 0, \ldots, D-1$, called *(infinitesimal) translation endomorphisms*, and an even vector $|0\rangle \in V$ called *vacuum*. These data are subject to certain axioms: (a) *locality*, (b) *translation invariance* and (c) *vacuum axiom*.

Let us remark that instead of defining individually the vertex operators $Y(a, \mathrm{z})$ like two sided infinite formal series, as it was done in [8, Definition 2.1], one can use, without any loss of generality, the approach of [2, Sect. 1]. In this way, $Y(a, \mathrm{z})$ can be defined not individually but only when applied on $b \in V$ and then the result lies in the space of Laurent–Taylor formal series $V[[\mathrm{z}]][(\mathrm{z}^2)^{-1}]$.

A *field* acting on the $(\mathbb{Z}/2\mathbb{Z})$–graded vector space V is a linear map

$$\begin{array}{rcl} \phi : V & \longrightarrow & V[[\mathrm{z}]][(\mathrm{z}^2)^{-1}] \\ \Uplus & & \Uplus \\ a & \longmapsto & \phi(\mathrm{z})a. \end{array} \tag{1}$$

Let the map (1) has defined parity (called *parity of the filed*) p_ϕ. It is said that the field ϕ is *mutually local* with the vertex operators if the supercommutator

$$[\phi(\mathrm{z}_1), Y(b, \mathrm{z}_2)]\, a := \phi(\mathrm{z}_1)\, Y(b, \mathrm{z}_2)\, a - (-1)^{p_\phi p_b}\, Y(b, \mathrm{z}_2)\, \phi(\mathrm{z}_1)\, a, \quad a \in V, \tag{2}$$

is *local* in the sense that it vanishes when multiplied with a sufficiently large power of $(\mathrm{z}_1 - z_2)^2$ [1, Sect. IV.A]:

$$\big((\mathrm{z}_1 - \mathrm{z}_2)^2)\big)^{N_{\phi,b}}\, [\phi(\mathrm{z}_1), Y(b, \mathrm{z}_2)]\, a = 0. \tag{3}$$

The field $\phi(\mathrm{z})$ is additionally called *translation–invariant* if

$$[T_\mu, \phi(\mathrm{z})]\, a := T_\mu\, (\phi(\mathrm{z})\, a) - \phi(\mathrm{z})\, (T_\mu\, a) = \partial_{z^\mu}\, (\phi(\mathrm{z})\, a)$$

for all $\mu = 0, \ldots, D-1$ and $a \in V$.

We consider fields which are within the class of the translation invariant fields mutually local with all the vertex operators (i.e. local with respect to a translation invariant local complete system of fields).

Every field $a \mapsto \phi(\mathrm{z})\, a$ from this class is of a form $a \mapsto Y(b, \mathrm{z})\, a$, for some $b \in V$, i.e. it can be represented by a vertex operator [1, Corollary 4.3]. In fact, $b = \phi(\mathrm{z})|0\rangle \Big|_{\mathrm{z}\,=\,0}$.

Translation–invariance of the vertex operators gives that [8, Proposition 3.2 (b)]:

$$Y(a, \mathrm{z})\, |0\rangle = e^{\mathrm{z} \cdot \mathrm{T}}\, a, \tag{4}$$

where $\mathrm{z} \cdot \mathrm{T} := z^\mu T_\mu$.

3 Superconformal Vertex Algebras

Some Preliminary Notations The Grassmann variables attached to the complexified four-dimensional Euclidean space are denoted by $\boldsymbol{\theta} = (\theta_A^{\alpha})$ and $\bar{\boldsymbol{\theta}} = (\bar{\theta}_{\dot{\alpha}}^A)$, where $\alpha = 1, 2$, $\dot{\alpha} = \dot{1}, \dot{2}$ are chiral spinorial indices and $A = 1, \ldots, N$ is an $su(N)$-index. Grassmann variables with undotted spinorial indices are related to $(\frac{1}{2}, 0)$ and those with doted ones to $(0, \frac{1}{2})$ representations of the orthogonal Lie algebra.

If $V = V_0 \oplus V_1$ is a $(\mathbb{Z}/2\mathbb{Z})$-graded vector space, then $V[\boldsymbol{\theta}, \bar{\boldsymbol{\theta}}]$ is the space of polynomials in the anti-commuting variables $(\boldsymbol{\theta}, \bar{\boldsymbol{\theta}})$, which naturally is a super vector space. Note that if, in addition, V is a Lie superalgebra, then $V[\boldsymbol{\theta}, \bar{\boldsymbol{\theta}}]$ is again a Lie superalgebra.

Conformal Lie Algebra generators:

- $T_0, \ldots, T_{D-1}$—generators of translations in the compact picture.
- $\Omega_{\mu,\nu}$ $(0 \leqslant \mu < \nu \leqslant D-1)$ – generators of rotations in the compact picture.
- H—generator of dilatations in the compact picture. It is called *conformal Hamiltonian*. The eigenvalues of the H are called scaling dimensions of the corresponding eigenstates (or fields).
- $C_0, \ldots, C_{D-1}$—generators of special conformal transformations in the compact picture.

Generators of the N-extended superconformal Lie algebra:

This Lie superalgebra is extension of the four dimensional conformal Lie algebra (which is contained in the even sector) with the following additional generators:

- Odd generators Q_α^A and $\bar{Q}_A^{\dot{\alpha}}$ called *supertranslations*. ($\alpha = 1, 2, \dot{\alpha} = \dot{1}, \dot{2}, A = 1, \ldots, N$, as already explained.)
- Odd generators S_A^α and $\bar{S}_{\dot{\alpha}}^A$ called *super special conformal translations*. The indices are as above.
- An even $U(1)$-generator $\mathcal{R}$ called *R-charge*.
- Even generators $\mathcal{A}_B^A$ spanning the Lie algebra *su*(N) (i.e., $sl(N, \mathbb{C})$, since we consider the complexified *su*(N)). They are called *R-symmetry generators*.

For short description of the well-known N-extended superconformal Lie algebra, coordinated with notations we use in our work, see [7, Appendix A.].

We adopt the following conventions: $\boldsymbol{\theta} \cdot \mathrm{Q} := \theta_A^\alpha\, Q_\alpha^A$ and $\bar{\boldsymbol{\theta}} \cdot \bar{\mathrm{Q}} := \bar{\theta}_{\dot{\alpha}}^A\, \bar{Q}_A^{\dot{\alpha}}$.

Superconformal Vertex Operators A *superfield* $\phi(\mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}})$ acting on the $(\mathbb{Z}/2\mathbb{Z})$-graded vector space V is a linear map

$$\begin{array}{rcl} \phi : V & \longrightarrow & V[[\mathrm{z}]][(\mathrm{z}^2)^{-1}][\boldsymbol{\theta}, \bar{\boldsymbol{\theta}}] \\ \Cup & & \Cup \\ a & \longmapsto & \phi(\mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}})\, a . \end{array}$$

Actually, the superfields are polynomials in $\boldsymbol{\theta}$ and $\bar{\boldsymbol{\theta}}$, such that their coefficients are fields acting on V. We consider superfields whose coefficient fields are translation invariant and mutually local with all the vertex operators, and thus representable by vertex operators [1, Corollary 4.3]. Therefore such classes of superfields can be obtained in the following manner.

Let $Y(a, \mathrm{z}), \mathrm{z} \in \mathbb{C}^4$ be a vertex operator from a vertex algebra whose underlying vector space V is endowed with an action of the N-extended superconformal algebra via linear endomorphisms, such that this action annihilates the vacuum vector.[2] We define *superconformal vertex operators*

$$Y(a, \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}}) := e^{\boldsymbol{\theta} \cdot \mathrm{Q} + \bar{\boldsymbol{\theta}} \cdot \bar{\mathrm{Q}}} \, Y(a, \mathrm{z}) \, e^{-\boldsymbol{\theta} \cdot \mathrm{Q} - \bar{\boldsymbol{\theta}} \cdot \bar{\mathrm{Q}}} \in End\big(V[[\mathrm{z}]][(\mathrm{z}^2)^{-1}][\boldsymbol{\theta}, \bar{\boldsymbol{\theta}}]\big). \tag{5}$$

Note that $Y(a, \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}})$ is a polynomial in $\boldsymbol{\theta}, \bar{\boldsymbol{\theta}}$ with coefficients that are vertex operators. Therefore, using the state-field correspondence (4), $Y(a, \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}})$ can be reconstructed from its action on the vacuum,

$$Y(a, \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}}) \, |0\rangle = e^{\boldsymbol{\theta} \cdot \mathrm{Q} + \bar{\boldsymbol{\theta}} \cdot \bar{\mathrm{Q}}} \, Y(a, \mathrm{z}) \, |0\rangle = e^{\mathrm{z} \cdot \mathrm{T} + \boldsymbol{\theta} \cdot \mathrm{Q} + \bar{\boldsymbol{\theta}} \cdot \bar{\mathrm{Q}}} \, a. \tag{6}$$

This allows us to deduce the covariance properties of the so defined superconformal vertex operators.

Let X be a generator[3] of the superconformal Lie algebra. Commutators $\big[X, Y(a, \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}})\big]$ are computed from their action on the vacuum and using the general formula [6] [4]

$$X \, Y(a, \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}}) \, |0\rangle = Y(e^{-\mathrm{ad}(\mathrm{z} \cdot \mathrm{T} + \boldsymbol{\theta} \cdot \mathrm{Q} + \bar{\boldsymbol{\theta}} \cdot \bar{\mathrm{Q}})}\big(X\big) \, a, \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}}) \, |0\rangle. \tag{7}$$

We have $\big[X \,,\, Y(a, \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}})\big] \, |0\rangle = X \, Y(a, \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}}) \, |0\rangle$. Recalling that $Y(a, \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}})$ is a polynomial in $\boldsymbol{\theta}, \bar{\boldsymbol{\theta}}$ with coefficients being translation invariant fields mutually local with all the vertex operators, we apply the vertex algebra analog of the Reeh-Schlider property, i.e. [8, Theorem 3.1], which gives

$$\big[X \,,\, Y(a, \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}})\big] = Y(e^{-\mathrm{ad}(\mathrm{z} \cdot \mathrm{T} + \boldsymbol{\theta} \cdot \mathrm{Q} + \bar{\boldsymbol{\theta}} \cdot \bar{\mathrm{Q}})}\big(X\big) \, a, \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}}). \tag{8}$$

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[2] In other words V is a module for the N-extended superconformal algebra with an action that annihilates the vacuum.

[3] In all the text we use the same notation for the generators of the N-extended superconformal algebra and their representations as elements of $End(V)$. The meaning of the notation is clear from the context in which it is used.

[4] One can deduce relation (7) by using the definition of superconformal vertex operator, state-field correspondence (4) and axioms of vertex algebra.

Due to the nilpotency of ad(z · T + $\boldsymbol{\theta}$ · Q + $\bar{\boldsymbol{\theta}}$ · $\bar{\mathrm{Q}}$), $e^{-\mathrm{ad}(\mathrm{z}\cdot \mathrm{T} + \boldsymbol{\theta}\cdot \mathrm{Q} + \bar{\boldsymbol{\theta}}\cdot \bar{\mathrm{Q}})}(X)$ is a polynomial in z, $\boldsymbol{\theta}, \bar{\boldsymbol{\theta}}$ with coefficients in the superconformal Lie algebra and linearly depending of X. Further, it can be shown [7, Sect. 3] that the general form of the commutators is

$$\big[X\,,\, Y(a, \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}})\big] \,=\, \mathcal{Z}(X; \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}}) Y(a, \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}}) \,+\, Y\big(\mathcal{M}(X; \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}})\, a, \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}}\big), \tag{9}$$

where $\mathcal{Z}(X; \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}})$ is first order differential operator (i.e., a vector field) in z, $\boldsymbol{\theta}, \bar{\boldsymbol{\theta}}$ with polynomial coefficients in z, $\boldsymbol{\theta}, \bar{\boldsymbol{\theta}}$ and $\mathcal{M}(X; \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}})$ has coefficients belonging to Span$\{C_\mu, S^\alpha_A, \bar{S}^A_{\dot\alpha}, \Omega_{\mu,\nu}, H, \mathcal{A}^A_B, \mathcal{R}\}$.

Using (9), the super Jacobi identity

$$\big[[X,\, X'],\, Y(a, \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}})\big] =$$
$$-(-1)^{p_X\, p_{X'}}\big[X',\, [X,\, Y(a, \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}})]\big] + \big[X,\, [X',\, Y(a, \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}})]\big]$$

can be written as

$$\begin{aligned} &-\mathcal{Z}([X, X']; \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}}) \,+\, \mathcal{M}([X, X']; \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}}) \,=\\ &\big[-\mathcal{Z}(X; \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}}) \,+\, \mathcal{M}(X; \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}}),\, -\mathcal{Z}(X'; \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}}) \,+\, \mathcal{M}(X'; \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}})\big] \end{aligned} \tag{10}$$

(commutators are understood as $(\mathbb{Z}/2\mathbb{Z})$–graded commutators). Note that in Eq. (10) commutators like $\big[\mathcal{Z}(X; \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}}), \mathcal{M}(X'; \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}})\big]$ are understood as a commutator of first and zeroth order differential operators in $(\mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}})$.

We calculate $\mathcal{M}(X; \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}})$ and $\mathcal{Z}(X; \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}})$ for each generator X (see [7, Eq. 3.25] for the results) and thereby obtain an action of the N-extended superconformal Lie algebra on the superconformal vertex operators.

Thus, we arrive to the following notion of a *superconformal vertex algebra*. It is a $(\mathbb{Z}/2\mathbb{Z})$–graded vector space V endowed with an even (bi)linear map

$$V \otimes V \,\ni\, a \otimes b \,\mapsto\, Y(a, \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}})\, b \,\in\, V[[\mathrm{z}]][(\mathrm{z}^2)^{-1}][\boldsymbol{\theta}, \bar{\boldsymbol{\theta}}],$$

an even vector $|0\rangle \in V$ called a *vacuum*, and a linear action on V of the superconformal Lie algebra annihilating the vacuum. For the coefficient fields in the expansion of $Y(a, \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}})$ in $\boldsymbol{\theta}$ and $\bar{\boldsymbol{\theta}}$ we require to fulfil all the axioms of vertex algebra. We also require to have an action of the N-extended superconformal Lie algebra on the vector space V for which the superconformal vertex operators are *equivariant* in the sense $\big[X\,,\, Y(a, \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}})\big] \,=\, \mathcal{Z}(X; \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}}) Y(a, \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}}) + Y\big(\mathcal{M}(X; \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}})\, a, \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}}\big)$, with $\mathcal{M}(X; \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}})$ and $\mathcal{Z}(X; \mathrm{z}, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}})$ given by [7, Eq. 3.25], for every generator X.

Conclusion

We developed an algebraic formalism for quantum superfields with extended superconformal symmetry analogous to vertex algebras.

This can have various applications. First, in direction of cohomological analysis of anomalies in the perturbative models of such theories. Second, it gives a framework for constructing on shell models (i.e., models in a Hilbert space).

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Part VII
Representation Theory

Special Reduced Multiplets and Minimal Representations for SO(p,q)

Vladimir Dobrev

Abstract Using our previous results on the systematic construction of invariant differential operators for non-compact semisimple Lie groups we classify the special reduced multiplets and minimal representations in the case of SO(p,q).

1 Introduction

In a recent paper [1] we started the systematic explicit construction of invariant differential operators. We gave an explicit description of the building blocks, namely, the *parabolic subgroups and subalgebras* from which the necessary representations are induced. Thus we have set the stage for study of different non-compact groups.

Since the study and description of detailed classification should be done group by group we had to decide which groups to study first. We decided to start with a subclass of the hermitian-symmetric algebras which share some special properties of the conformal algebra $so(n,2)$. That is why, in view of applications to physics, we called these algebras '*conformal Lie algebras*' (CLA), (or groups) [2]. Later we gave a natural way to go beyond this subclass using essentially the same results. For this we introduce the new notion of *parabolic relation* between two non-compact semisimple Lie algebras $\mathcal{G}$ and $\mathcal{G}'$ that have the same complexification and possess maximal parabolic subalgebras with the same complexification [3].

Thus, for example, using results for the conformal algebra $so(n,2)$ (for fixed n) we can obtain results for all pseudo-orthogonal algebras $so(p,q)$ such that $p+q = n+2$. In this way, in [3] (among other things) we gave the main and the reduced multiplets of indecomposable elementary representations for $so(p,q)$ including the necessary data for all relevant invariant differential operators. We specially stressed that the classification of all invariant differential operators includes as special cases all possible *conservation laws* and *conserved currents*, unitary or not. In the present paper we give explicitly the conservation laws in the case of $so(p,q)$.

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This paper is a short sequel of [3]. Due to the lack of space we refer to [3] for motivations and extensive list of literature on the subject.

2 Preliminaries

Let $\mathcal{G} = so(p,q)$, $p \geq q$, $p+q > 4$. We choose a maximal parabolic $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$ such that:

$$\mathcal{M} = so(p-1, q-1), \quad \dim \mathcal{A} = 1, \quad \dim \mathcal{N} = p+q-2. \tag{1}$$

With this choice we get for the conformal algebra $so(n,2)$ the Bruhat decomposition $\mathcal{G} = \mathcal{P} \oplus \tilde{\mathcal{N}}$ with direct physical meaning ($\tilde{\mathcal{N}} \cong \mathcal{N}$) [3].

We label the signature of the representations of $\mathcal{G}$ as follows:

$$\begin{aligned} \chi &= \{ n_1, \ldots, n_h ; c \}, \\ & n_j \in \mathbb{Z}/2, \quad c = d - \frac{p+q-2}{2}, \quad h \equiv [\frac{p+q-2}{2}], \\ & 0 \leq |n_1| < n_2 < \cdots < n_h, \quad p+q \ \text{ even}, \\ & 0 < n_1 < n_2 < \cdots < n_h, \quad p+q \ \text{ odd}, \end{aligned} \tag{2}$$

where the parameter c (related to the conformal weight d) labels the characters of $\mathcal{A}$, and the first h entries are labels of the finite-dimensional (nonunitary for $q \neq 1$) irreps μ of $\mathcal{M}$.

Following [4] we call the above induced representations $\chi = \mathrm{Ind}_{\mathcal{P}}^{\mathcal{G}}(\mu \otimes \nu \otimes 1)$ *elementary representations* (ERs) of $G = SO(p,q)$. Their spaces of functions are:

$$\mathcal{C}_\chi = \{ \mathcal{F} \in C^\infty(G, V_\mu) \mid \mathcal{F}(gman) = e^{-\nu(H)} \cdot D^\mu(m^{-1}) \, \mathcal{F}(g) \}$$

where $a = \exp(H)$, $H \in \mathcal{A}$, $m \in M = SO(p-1, q-1)$, $n \in N = \exp \mathcal{N}$. The representation action is the *left regular action*:

$$(\mathcal{T}^\chi(g)\mathcal{F})(g') = \mathcal{F}(g^{-1}g'), \quad g, g' \in G. \tag{3}$$

Remark. Note that the group M has more general irreps representing the centre of M. However, these are discrete parameters which are not essential for the classification of the reducible ERs, cf. [5,6].◇

- An important ingredient in our considerations are the *highest/lowest weight representations* of $\mathcal{G}^{\mathbb{C}}$. These can be realized as (factor-modules of) Verma modules V^Λ over $\mathcal{G}^{\mathbb{C}}$, where $\Lambda \in (\mathcal{H}^{\mathbb{C}})^*$, $\mathcal{H}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathcal{G}^{\mathbb{C}}$, weight $\Lambda = \Lambda(\chi)$ is determined uniquely from χ [6].

  Actually, since our ERs are induced from finite-dimensional representations of $\mathcal{M}$ the Verma modules are always reducible. Thus, it is more convenient to use *generalized Verma modules* $\tilde{V}^{\Lambda}$ such that the role of the highest/lowest weight vector v_0 is taken by the (finite-dimensional) space $V_\mu \, v_0$. For the generalized Verma modules (GVMs) the reducibility is controlled only by the value of the conformal weight d, or the parameter c. Relatedly, for the intertwining differential operators only the reducibility w.r.t. non-compact roots is essential. Thus, from now on we shall consider the ERs factored by the maximal invariant subspace generated by reducibilities w.r.t. compact roots. We shall call these factored ERS: *compactly restricted ERs*.

- One main ingredient of our approach is as follows. We group the (reducible) ERs with the same Casimirs in sets called *multiplets* [5]. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the *vertices* of which correspond to the reducible ERs and the *lines (arrows)* between the vertices correspond to intertwining operators. The multiplets contain explicitly all the data necessary to construct the intertwining differential operators. Actually, the data for each intertwining differential operator consists of the pair (β, m), where β is a (non-compact) positive root of $\mathcal{G}^{\mathbb{C}}$, $m \in \mathbb{N}$, such that the *BGG Verma module reducibility condition* [7] (for highest weight modules) is fulfilled:

$$(\Lambda + \rho, \beta^\vee) \;=\; m\,, \quad \beta^\vee \equiv 2\beta/(\beta,\beta) \tag{4}$$

where ρ is half the sum of the positive roots of $\mathcal{G}^{\mathbb{C}}$. When the above holds then the Verma module with shifted weight $V^{\Lambda - m\beta}$ (or $\tilde{V}^{\Lambda - m\beta}$ for GVM and β non-compact) is embedded in the Verma module V^{Λ} (or $\tilde{V}^{\Lambda}$). This embedding is realized by a singular vector v_s expressed by a polynomial $\mathcal{P}_{m,\beta}(\mathcal{G}^-)$ in the universal enveloping algebra $(U(\mathcal{G}_-))\, v_0$, $\mathcal{G}^-$ is the subalgebra of $\mathcal{G}^{\mathbb{C}}$ generated by the negative root generators [8]. More explicitly, [6] $v^s_{m,\beta} \;=\; \mathcal{P}_{m,\beta}\, v_0$ (or $v^s_{m,\beta} = \mathcal{P}_{m,\beta}\, V_\mu\, v_0$ for GVMs).[1]

Then there exists [6] an *intertwining differential operator* of order $m = m_\beta$:

$$\mathcal{D}^m_\beta : \mathcal{C}_{\chi(\Lambda)} \;\longrightarrow\; \mathcal{C}_{\chi(\Lambda - m\beta)} \tag{5}$$

given explicitly by:

$$\mathcal{D}^m_\beta \;=\; \mathcal{P}^m_\beta(\hat{\mathcal{G}^-}) \tag{6}$$

where $\hat{\mathcal{G}^-}$ denotes the $right$ action on the functions $\mathcal{F}$, cf. (3).

Thus, in each such situation we have an *invariant differential equation* of order $m = m_\beta$:

$$\mathcal{D}^m_\beta\, f \;=\; f'\,, \qquad f \in \mathcal{C}_{\chi(\Lambda)}\,, \quad f' \in \mathcal{C}_{\chi(\Lambda - m\beta)}\,. \tag{7}$$

[1]For explicit expressions for singular vectors we refer to [9].

In many such situations the invariant operator $\mathcal{D}^m_\beta$ has a non-trivial invariant kernel. These kernels are very important since in them are realized the (irreducible) subrepresentations of $\mathcal{G}$ as solutions of the equations:

$$\mathcal{D}^m_\beta \, f \;=\; 0, \qquad f \in \mathcal{C}_{\chi(\Lambda)}, \tag{8}$$

Furthermore, in some physical applications in the case of first order differential operators, i.e., for $m = m_\beta = 1$, Eq. (8) are called *conservation laws*, and the elements $f \in \ker \mathcal{D}_{m,\beta}$ are called *conserved currents*.

3 Classification of Reducible ERs for $so(p,q)$

The reducible ERs are grouped in various multiplets. We start with the so-called main multiplets (which contain the maximal number of ERs with this parabolic). We present them with the following explicit parametrization of the ERs in the multiplets (following [10], see also [11]):

$$\begin{aligned}
\chi_1^\pm &= \{\epsilon\, n_1,\, \ldots,\, n_h\, ;\, \pm n_{h+1}\}, \quad n_h < n_{h+1}, \\
\chi_2^\pm &= \{\epsilon\, n_1,\, \ldots,\, n_{h-1},\, n_{h+1}\, ;\, \pm n_h\} \\
\chi_3^\pm &= \{\epsilon\, n_1, \ldots, n_{h-2}, n_h, n_{h+1}\, ;\, \pm n_{h-1}\} \\
&\ldots \\
\chi_{h-1}^\pm &= \{\epsilon\, n_1,\, n_2,\, n_4,\, \ldots,\, n_h,\, n_{h+1}\, ;\, \pm n_3\} \\
\chi_h^\pm &= \{\epsilon\, n_1,\, n_3,\, \ldots,\, n_h,\, n_{h+1}\, ;\, \pm n_2\} \\
\chi_{h+1}^\pm &= \{\epsilon\, n_2,\, n_3,\, \ldots,\, n_h,\, n_{h+1}\, ;\, \pm n_1\} \\
\epsilon &= \begin{cases} \pm, & p+q \ \ even \\ 1, & p+q \ \ odd \end{cases}
\end{aligned} \tag{9}$$

($\epsilon = \pm$ is correlated with $\chi^\pm$). Clearly, the multiplets correspond 1-to-1 to the finite-dimensional irreps of $so(p+q,\mathbb{C})$ with signature $\{n_1,\ldots,n_h,n_{h+1}\}$ and we are able to use previous results due to the parabolic relation between the $so(p,q)$ algebras for $p+q$ -fixed. Note that the two representations in each pair $\chi^\pm$ are called *shadow fields*.

Further, we denote by $\mathcal{C}_i^\pm$ the representation space with signature $\chi_i^\pm$.

The ERs in the multiplet are related by *intertwining integral and differential operators*.

The *integral operators* were introduced by Knapp and Stein [12]. Here these operators intertwine the pairs $\mathcal{C}_i^\pm$ (cf. (9)):

$$G_i^\pm : \mathcal{C}_i^\mp \longrightarrow \mathcal{C}_i^\pm, \quad i \;=\; 1,\ldots,h+1. \tag{10}$$

The *intertwining differential operators* correspond to non-compact positive roots of the root system of $so(p+q,\mathbb{C})$, cf. [6]. In the current context, compact roots of $so(p+q,\mathbb{C})$ are those that are roots also of the subalgebra $\mathcal{M}^{\mathbb{C}}$, the rest of the roots are non-compact. We denote the differential operators by d_i, d'_i. The spaces from (9) they intertwine are:

$$\begin{aligned}
&d_i : \mathcal{C}^-_i \longrightarrow \mathcal{C}^-_{i+1}, \quad i = 1,\ldots,h\,;\\
&d'_i : \mathcal{C}^+_{i+1} \longrightarrow \mathcal{C}^+_i, \quad i = 1,\ldots,h-1\,;\\
&d_h : \mathcal{C}^+_{h+1} \longrightarrow \mathcal{C}^+_h, \quad (p+q) - \text{even}\,;\\
&d'_h : \mathcal{C}^-_h \longrightarrow \mathcal{C}^+_{h+1}, \quad (p+q) - \text{even}\,;\\
&d'_h : \mathcal{C}^-_{h+1} \longrightarrow \mathcal{C}^+_h, \quad (p+q) - \text{even}\,;\\
&d'_h : \mathcal{C}^+_{h+1} \longrightarrow \mathcal{C}^+_h, \quad (p+q) - \text{odd}\,;\\
&d_{h+1} : \mathcal{C}^-_{h+1} \longrightarrow \mathcal{C}^+_{h+1}, \quad (p+q) - \text{odd}\,.
\end{aligned} \tag{11}$$

The degrees of these intertwining differential operators are given just by the differences of the c entries [10]:

$$\begin{aligned}
&\deg d_i = \deg d'_i = n_{h+2-i} - n_{h+1-i} = m_{h+2-i}, \qquad i = 1,\ldots,h,\\
&\deg d'_h = n_2 + n_1 = m_1, \quad (p+q) - \text{even},\\
&\deg d_{h+1} = 2n_1 = m'_{h+1} = m_1, \quad (p+q) - \text{odd}.
\end{aligned} \tag{12}$$

where d'_h is omitted from the first line for $(p+q)$ even.

4 Multiplets and Representations for $p+q$ Odd

4.1 Reduced Multiplets for $p+q$ Odd

In this section we consider the case $p+q$ odd, thus $h = \frac{1}{2}(p+q-3)$. First we rewrite the main multiplets from (9) in the following parametrization:

$$\begin{aligned}
\chi^{\pm}_1 &= [m_1,\ldots,m_h\,;\ \pm\tfrac{1}{2}(m_1 + 2m_{2,h+1})\,], \qquad (13)\\
\chi^{\pm}_2 &= [m_1,\ldots,m_{h-1},\, m_{h,h+1}\,;\ \pm\tfrac{1}{2}(m_1 + 2m_{2,h})\,]\\
\chi^{\pm}_3 &= [m_1,\ldots,m_{h-2}, m_{h-1,h}, m_{h+1}\,;\ \pm\tfrac{1}{2}(m_1 + 2m_{2,h-1})\,]\\
&\ldots\\
\chi^{\pm}_i &= [m_1,\ldots,m_{h-i+1}, m_{h-i+2,h-i+3}, m_{h+4-i},\ldots,m_h,\, m_{h+1}\,;\\
&\qquad \pm\tfrac{1}{2}(m_1 + 2m_{2,h+2-i})\,]
\end{aligned}$$

$$\begin{aligned}
&\dots \\
\chi^\pm_{h-1} &= [m_1,\, m_2,\, m_{34},\, m_5,\, \dots,\, m_h,\, m_{h+1}\,;\ \pm\tfrac{1}{2}(m_1 + 2m_{2,3})\,] \\
\chi^\pm_{h} &= [m_1,\, m_{23},\, m_4,\, \dots,\, m_h,\, m_{h+1}\,;\ \pm\tfrac{1}{2}(m_1 + 2m_2)\,] \\
\chi^\pm_{h+1} &= [m_1 + 2m_2,\, m_3,\, \dots,\, m_h,\, m_{h+1}\,;\ \pm\tfrac{1}{2}m_1\,]
\end{aligned}$$

where the last entry (as before) is the value of c, while $m_i \in \mathbb{N}$ are the Dynkin labels (as in (12)):

$$m_1 = 2n_1 = 2\ell_1 + 1, \qquad m_j = n_j - n_{j-1} = \ell_j - \ell_{j-1} + 1, \quad j = 2, \dots, h+1. \tag{14}$$

and we use the shorthand notation:

$$m_{r,s} \equiv m_r + \dots + m_s, \quad r < s, \qquad m_{r,r} \equiv m_r, \qquad m_{r,s} \equiv 0, \quad r > s, \tag{15}$$

and we have also introduced the labels ℓ_k (in order to facilitate comparison with the literature):

$$\ell_k = n_k - k + \tfrac{1}{2}, \qquad 0 \le \ell_1 \le \dots \le \ell_{h+1}. \tag{16}$$

We know that the ERs in a pair are related by the KS operators $G_i^\pm$ (10), however for $p+q$ odd the operator G^+_{h+1} degenerates to a differential operator of degree m_1 corresponding to the only short non-compact root ε_1 . The main multiplets are given the Fig. 1. Note that following [3] we do not give the KS integral operators. Their presence is assumed by the symmetry w.r.t the bullet in the centre of the figure.

In this case there are $h+1$ reduced multiplets which may be obtained by formally setting one Dynkin label to zero. For $m_j = 0$ we denote the signatures by ${}^j\chi^\pm_k$.

We shall see that in every multiplet there is only one pair (which we mark with ♠) whose representations are of direct physical relevance (including finite-dimensional irreps of the $\mathcal{M}$ subalgebra). Yet we list the others since they are related by invariant differential operators which we record in each case.

In detail, the signatures are given similarly to (13):

- $m_{h+1} = 0$ equiv $n_{h+1} = n_h$

$$\begin{aligned}
{}^{h+1}\chi^\pm_1 = {}^{h+1}\chi^\pm_2 &= [m_1,\, \dots,\, m_h\,;\ \pm\tfrac{1}{2}(m_1 + 2m_{2,h})\,], \quad ♠ \\
{}^{h+1}\chi^\pm_3 &= [m_1, \dots, m_{h-2}, m_{h-1,h}, 0\,;\ \pm\tfrac{1}{2}(m_1 + 2m_{2,h-1})\,] \\
&\dots \\
{}^{h+1}\chi^\pm_{h-1} &= [m_1,\, m_2,\, m_{34},\, m_5,\, \dots,\, m_h,\, 0\,;\ \pm\tfrac{1}{2}(m_1 + 2m_{2,3})\,] \\
{}^{h+1}\chi^\pm_{h} &= [m_1,\, m_{23},\, m_4,\, \dots,\, m_h,\, 0\,;\ \pm\tfrac{1}{2}(m_1 + 2m_2)\,] \\
{}^{h+1}\chi^\pm_{h+1} &= [m_1 + 2m_2,\, m_3,\, \dots,\, m_h,\, 0\,;\ \pm\tfrac{1}{2}m_1\,]
\end{aligned} \tag{17}$$

Here there are two differential operators involving physically relevant representations, cf. Fig. 2:

$$\begin{aligned} \mathcal{D}^{m_h}_{\varepsilon_1-\varepsilon_3} &: \mathcal{C}^-_1 = \mathcal{C}^-_2 \longrightarrow \mathcal{C}^-_3 \\ \mathcal{D}^{m_h}_{\varepsilon_1+\varepsilon_3} &: \mathcal{C}^+_3 \longrightarrow \mathcal{C}^+_1 = \mathcal{C}^+_2 \end{aligned} \tag{18}$$

- $m_h = 0$ equiv $n_h = n_{h-1}$

$$\begin{aligned} {}^h\chi^\pm_1 &= [m_1, \ldots, m_{h-1}, 0\,;\, \pm\tfrac{1}{2}(m_1 + 2m_{2,h-1} + 2m_{h+1})\,], \\ {}^h\chi^\pm_2 = {}^h\chi^\pm_3 &= [m_1, \ldots, m_{h-1}, m_{h+1}\,;\, \pm\tfrac{1}{2}(m_1 + 2m_{2,h-1})\,], \quad \spadesuit \\ &\ldots \\ {}^h\chi^\pm_{h-1} &= [m_1, m_2, m_{34}, m_5, \ldots, m_{h-1}, 0, m_{h+1}\,;\, \pm\tfrac{1}{2}(m_1 + 2m_{2,3})\,] \\ {}^h\chi^\pm_h &= [m_1, m_{23}, m_4, \ldots, m_{h-1}, 0, m_{h+1}\,;\, \pm\tfrac{1}{2}(m_1 + 2m_2)\,] \\ {}^h\chi^\pm_{h+1} &= [m_1 + 2m_2, m_3, \ldots, m_{h-1}, 0, m_{h+1}\,;\, \pm\tfrac{1}{2}m_1\,] \end{aligned} \tag{19}$$

Here there are four differential operators involving physically relevant representations, cf. Fig. 3:

$$\begin{aligned} \mathcal{D}^{m_{h+1}}_{\varepsilon_1-\varepsilon_2} &: \mathcal{C}^-_1 \longrightarrow \mathcal{C}^-_2 = \mathcal{C}^-_3 \\ \mathcal{D}^{m_{h-1}}_{\varepsilon_1-\varepsilon_4} &: \mathcal{C}^-_2 = \mathcal{C}^-_3 \longrightarrow \mathcal{C}^-_4 \\ \mathcal{D}^{m_{h-1}}_{\varepsilon_1+\varepsilon_4} &: \mathcal{C}^+_4 \longrightarrow \mathcal{C}^+_2 = \mathcal{C}^+_3 \\ \mathcal{D}^{m_{h+1}}_{\varepsilon_1+\varepsilon_2} &: \mathcal{C}^+_2 = \mathcal{C}^+_3 \longrightarrow \mathcal{C}^+_1 \end{aligned} \tag{20}$$

The above case is typical for $m_k = 0$ for $k > 2$. Then for $k = 2, 1$ we have:

- $m_2 = 0$ equiv $n_2 = n_1$

$$\begin{aligned} {}^2\chi^\pm_1 &= [m_1, 0, m_3, \ldots, m_h\,;\, \pm\tfrac{1}{2}(m_1 + 2m_{3,h+1})\,], \\ {}^2\chi^\pm_2 &= [m_1, 0, m_3, \ldots, m_{h-1}, m_{h,h+1}\,;\, \pm\tfrac{1}{2}(m_1 + 2m_{3,h})\,] \\ {}^2\chi^\pm_3 &= [m_1, 0, m_3, \ldots, m_{h-2}, m_{h-1,h}, m_{h+1}\,;\, \pm\tfrac{1}{2}(m_1 + 2m_{3,h-1})\,] \\ &\ldots \\ {}^2\chi^\pm_{h-1} &= [m_1, 0, m_{34}, m_5, \ldots, m_h, m_{h+1}\,;\, \pm\tfrac{1}{2}(m_1 + 2m_3)\,] \\ {}^2\chi^\pm_h = {}^2\chi^\pm_{h+1} &= [m_1, m_3, m_4, \ldots, m_h, m_{h+1}\,;\, \pm\tfrac{1}{2}m_1\,], \quad \spadesuit \end{aligned} \tag{21}$$

Here there are three differential operators involving physically relevant representations, cf. Fig. 4

$$\begin{aligned}
\mathcal{D}^{m_3}_{\varepsilon_1-\varepsilon_h} &: \mathcal{C}^-_{h-1} \longrightarrow \mathcal{C}^-_h = \mathcal{C}^-_{h+1} \\
\mathcal{D}^{m_1}_{\varepsilon_1} &: \mathcal{C}^-_h = \mathcal{C}^-_{h+1} \longrightarrow \mathcal{C}^+_h = \mathcal{C}^+_{h+1} \\
\mathcal{D}^{m_3}_{\varepsilon_1+\varepsilon_h} &: \mathcal{C}^+_h = \mathcal{C}^+_{h+1} \longrightarrow \mathcal{C}^+_{h-1}
\end{aligned} \tag{22}$$

• $m_1 = 0$ equiv $n_1 = 0$

$$\begin{aligned}
{}^1\chi^\pm_1 &= [0,\, m_2,\, \ldots,\, m_h\,;\, \pm m_{2,h+1}\,], \\
{}^1\chi^\pm_2 &= [0,\, m_2,\, \ldots,\, m_{h-1},\, m_{h,h+1}\,;\, \pm m_{2,h}\,] \\
{}^1\chi^\pm_3 &= [0,\, m_2, \ldots, m_{h-2}, m_{h-1,h}, m_{h+1}\,;\, \pm m_{2,h-1}\,] \\
&\ldots \\
{}^1\chi^\pm_{h-1} &= [0,\, m_2,\, m_{34},\, m_5,\, \ldots,\, m_h,\, m_{h+1}\,;\, \pm m_{2,3}\,] \\
{}^1\chi^\pm_h &= [0,\, m_{23},\, m_4,\, \ldots,\, m_h,\, m_{h+1}\,;\, \pm m_2\,] \\
{}^1\chi_{h+1} &= [2m_2,\, m_3,\, \ldots,\, m_h,\, m_{h+1}\,;\, 0\,], \quad \spadesuit
\end{aligned} \tag{23}$$

Here there are two differential operators involving physically relevant representations, cf. Fig. 5:

$$\begin{aligned}
\mathcal{D}^{m_2}_{\varepsilon_1-\varepsilon_{h+1}} &: \mathcal{C}^-_{h-1} \longrightarrow \mathcal{C}^+_{h+1} = \mathcal{C}^-_{h+1} \\
\mathcal{D}^{m_2}_{\varepsilon_1+\varepsilon_{h+1}} &: \mathcal{C}^+_{h+1} = \mathcal{C}^-_{h+1} \longrightarrow \mathcal{C}^+_{h-1}.
\end{aligned} \tag{24}$$

For future reference we summarize the pairs of direct physical relevance reparametrizing for more natural presentation and introducing uniform notation ${}_r\chi^\pm_k$:

$$\begin{aligned}
{}_r\chi^\pm_1 &= {}^{h+1}\chi^\pm_1 = [m_1,\, \ldots,\, m_h\,;\, \pm\tfrac{1}{2}(m_1 + 2m_{2,h})], \\
&\qquad d^+ \geq 2h,\;\; d^- \leq 1, \\
{}_r\chi^\pm_2 &= {}^{h}\chi^\pm_2 = [m_1,\, \ldots,\, m_h\,;\, \pm\tfrac{1}{2}(m_1 + 2m_{2,h-1})], \\
&\qquad d^+ \geq 2h-1,\;\; d^- \leq 2, \\
&\ldots \\
{}_r\chi^\pm_j &= {}^{h-j+2}\chi^\pm_j = [m_1, \ldots, m_h\,;\, \pm\tfrac{1}{2}(m_1 + 2m_{2,h+1-j})], \\
&\qquad d^+ \geq 2h-j+1,\;\; d^- \leq j, \quad 1 \leq j \leq h-1 \\
&\ldots \\
{}_r\chi^\pm_{h-1} &= {}^{3}\chi^\pm_{h-1} = [m_1,\, \ldots,\, m_h\,;\, \pm\tfrac{1}{2}(m_1 + 2m_2)], \\
&\qquad d^+ \geq h+2,\;\; d^- \leq h-1, \\
{}_r\chi^\pm_h &= {}^{2}\chi^\pm_h = [m_1,\, \ldots,\, m_h\,;\, \pm\tfrac{1}{2}m_1], \quad d^+ \geq h+1,\;\; d^- \leq h, \\
{}_r\chi_{h+1} &= {}^{1}\chi_{h+1} = [2m_1,\, m_2,\, \ldots,\, m_h\,;\, 0], \quad d = h + \tfrac{1}{2}
\end{aligned} \tag{25}$$

where we have introduced notation $d^{\pm}$ corresponding to the "$\pm$" occurrences:

$$d^{\pm} \;=\; h + \tfrac{1}{2} \pm |c|. \qquad (26)$$

4.2 Special Reduced Multiplets for $p + q$ Odd

In addition to the standardly reduced multiplets discussed in the previous subsection, there are special reduced multiplets which may be formally obtained by formally setting one or two Dynkin labels to a positive half integer. Again from each main multiplet only one pair is of physical relevance but unlike the standardly reduced multiplets discussed in the previous subsection these pairs are not related by differential operators to the rest of the reduced multiplet (though having the same Casimirs). Thus, we present only the physically relevant pairs.

- $m_{h+1} \mapsto \frac{1}{2}\mu, \quad \mu \in 2\mathbb{N} - 1$

$$_s\chi_1^{\pm} = [m_1,\, \dots,\, m_h\, ;\; \pm\tfrac{1}{2}(m_1 + 2m_{2,h} + \mu)\,] \qquad (27)$$

- $m_h \mapsto \frac{1}{2}\mu,\;\; m_{h+1} \mapsto \frac{1}{2}\mu', \quad \mu, \mu' \in 2\mathbb{N} - 1$

$$_s\chi_2^{\pm} = [m_1,\, \dots,\, m_{h-1},\, \tfrac{1}{2}(\mu + \mu')\, ;\; \pm\tfrac{1}{2}(m_1 + 2m_{2,h-1} + \mu)\,] \qquad (28)$$

- $m_{h-1} \mapsto \frac{1}{2}\mu,\;\; m_h \mapsto \frac{1}{2}\mu', \quad \mu, \mu' \in 2\mathbb{N} - 1$

$$_s\chi_3^{\pm} = [m_1, \dots, m_{h-2}, \tfrac{1}{2}(\mu + \mu'), m_{h+1}\, ;\; \pm\tfrac{1}{2}(m_1 + 2m_{2,h-2} + \mu)\,] \qquad (29)$$

- $m_{h-j+2} \mapsto \frac{1}{2}\mu,\;\; m_{h-j+3} \mapsto \frac{1}{2}\mu', \quad \mu, \mu' \in 2\mathbb{N} - 1, \quad 2 \le j \le h$

$$\begin{aligned} _s\chi_j^{\pm} = [&m_1, \dots, m_{h-j+1}, \tfrac{1}{2}(\mu + \mu'), m_{h+4-j}, \dots, m_h,\, m_{h+1}\, ; \\ &\pm\tfrac{1}{2}(m_1 + 2m_{2,h+1-j} + \mu)\,] \end{aligned} \qquad (30)$$

- $m_3 \mapsto \frac{1}{2}\mu,\;\; m_4 \mapsto \frac{1}{2}\mu', \quad \mu, \mu' \in 2\mathbb{N} - 1$

$$_s\chi_{h-1}^{\pm} = [m_1,\, m_2,\, \tfrac{1}{2}(\mu + \mu'),\, m_5,\, \dots,\, m_h,\, m_{h+1}\, ;\; \pm\tfrac{1}{2}(m_1 + 2m_2 + \mu)\,] \qquad (31)$$

- $m_2 \mapsto \frac{1}{2}\mu,\;\; m_3 \mapsto \frac{1}{2}\mu', \quad \mu, \mu' \in 2\mathbb{N} - 1$

$$_s\chi_h^{\pm} = [m_1,\, \tfrac{1}{2}(\mu + \mu'),\, m_4,\, \dots,\, m_h,\, m_{h+1}\, ;\; \pm\tfrac{1}{2}(m_1 + \mu)\,] \qquad (32)$$

- $m_2 \mapsto \frac{1}{2}\mu, \quad \mu \in 2\mathbb{N} - 1$

$$_s\chi_{h+1}^{\pm} = [m_1 + \mu,\, m_3,\, \dots,\, m_h,\, m_{h+1}\, ;\; \pm\tfrac{1}{2}m_1\,] \qquad (33)$$

In each pair there are the standard KS integral operators $G^{\pm}_k$ between ${}_r\chi^{\mp}_k$. Furthermore, the ERs in a pair are reducible w.r.t. the compact roots and in addition the ERs ${}_r\chi^{-}_k$ are reducible w.r.t. the only short noncompact root ε_1 . Actually, the corresponding differential operators are degenerations of the corresponding KS operators G^{+}_k (10). (In the main multiplets the same was happening but only for $k = h + 1$.) Thus, we have:

$$\mathcal{D}^{2|c_k|}_{\varepsilon_1} :_r \mathcal{C}^{-}_k \longrightarrow {}_r\mathcal{C}^{+}_k , \qquad G^{+}_k \sim \mathcal{D}^{2|c_k|}_{\varepsilon_1} \tag{34}$$

where c_k is the value of c of the ER ${}_r\chi^{-}_k$.

Finally, we give a doubly reduced case originating from (33) setting $m_1 = 0$:

$${}_{rs}\chi^{\pm}_{h+1} = [\mu, m_2, \ldots, m_h \,;\, 0\,], \quad m_k \in \mathbb{N}, \quad \mu \in 2\mathbb{N} - 1. \tag{35}$$

This is a singlet and the ER is reducible only w.r.t. the compact roots, there are no non-trivial differential operators, thus, the corresponding generalized Verma module and the compactly restricted ER are irreducible.

4.3 Special Cases for $p + q$ Odd

The ERS χ^{-}_1 are the only ones in the multiplet that contain as irreducible subrepresentations the finite-dimensional irreducible representations of $\mathcal{G}$. More precisely, the ER χ^{-}_1 contains the finite-dimensional irreducible representation of $\mathcal{G}$ with signature $(m_1, \ldots, m_{h+1})$. (Certainly, the latter is non-unitary except the case of the trivial one-dimensional obtained for $m_i = 1, \forall\, i$.)

Another important case is the ER with signature χ^{+}_1 . It contains a unitary discrete series representation of $so(p, q)$ realized on an invariant subspace $\mathcal{D}$ of the ER χ^{+}_1 . That subspace is annihilated by the KS operator G^{-}_1 , and is the image of the KS operator G^{+}_1 .

Furthermore when $p > q = 2$ the invariant subspace $\mathcal{D}$ is the direct sum of two subspaces $\mathcal{D} = \mathcal{D}^{+} \oplus \mathcal{D}^{-}$, in which are realized a *holomorphic discrete series representation* and its conjugate *anti-holomorphic discrete series representation*, resp. Note that the corresponding lowest weight GVM is infinitesimally equivalent only to the holomorphic discrete series, while the conjugate highest weight GVM is infinitesimally equivalent to the anti-holomorphic discrete series.

Thus, the signatures of the (holomorphic) discrete series are:

$$\chi^{+}_1 = [m_1, \ldots, m_h \,;\, d = h + \tfrac{1}{2}(m_1 + 1) + m_{2,h} + \nu\,], \quad \nu \in \mathbb{N} \tag{36}$$

More (non-holomorphic) discrete series representations are contained in χ^{+}_k for $1 < k \leq h$.

The next important case are the *limits of (holomorphic) discrete series* which are contained in the reduced case (17):

$$ {}_r\chi_1^+ \;=\; [m_1, \ldots, m_h \,;\, d \;=\; h + \tfrac{1}{2}(m_1+1) + m_{2,h}\,] \tag{37}$$

(with conformal weight obtained from (36) as "limit" for $\nu = 0$).

Finally, we mention the so called *first reduction points* (FRP). For $q = 2$ these are the boundary values of d from below of the positive energy UIRs. Most of the FRPs are contained in χ_{h+1}^+, cf. (13), which we give with suitable reparametrization:

$$ \chi_{h+1}^+ \;=\; [m_1, m_2, \ldots, m_h \,;\, d \;=\; h + \tfrac{1}{2}m_1 - \tfrac{1}{2}\,], \qquad m_1 \geq 3. \tag{38}$$

The FRP cases for $m_1 = 1, 2$ (with the same values of d by specializing m_1) are found in (21), (23), resp:

$$ {}_r\chi_h^- \;=\; [1, m_2, \ldots, m_h \,;\, d \;=\; h\,], \tag{39}$$

$$ {}_r\chi_{h+1} \;=\; [2, m_2, \ldots, m_h \,;\, d \;=\; h + \tfrac{1}{2}\,]. \tag{40}$$

Finally, we give some discrete unitary points below the FRP which are found in the special reduced ERs (32) (used for $m_1 = \mu = 1$, $\mu' = 2m_2 - 1$), and then (33) used first for $m_1 = 2$, $\mu = 2k-1$, and then for $m_1 = 1$, $\mu = 2k-1$:

$$ {}_s\chi_h^- \;=\; [1, m_2, \ldots, m_h \,;\, d \;=\; h - \tfrac{1}{2}\,], \tag{41}$$

$$ {}_s\chi_{h+1}^- \;=\; [2k+1, m_2, \ldots, m_h \,;\, d \;=\; h - \tfrac{1}{2}\,], \quad k \in \mathbb{N} \tag{42}$$

$$ {}_s\chi_{h+1}^- \;=\; [2k, m_2, \ldots, m_h \,;\, d \;=\; h\,], \quad k \in \mathbb{N} \tag{43}$$

4.4 Minimal Irreps for $p + q$ Odd

First we give the minimal irreps occurring in standardly reduced multiplets displaying together only the physically relevant representations:

$$ \begin{aligned} {}_r\chi_1^- &= [\,1, \ldots, 1 \,;\, d \;=\; 1\,], \\ &\qquad {}_rL_1^- \;=\; \{\,\varphi \in {}_r\mathcal{C}_1^- : \mathcal{D}^1_{\varepsilon_1-\varepsilon_3}\,\varphi \;=\; 0, \quad G_1^+\,\varphi \;=\; 0\,\}, \\ &\ldots \\ {}_r\chi_j^- &= [\,1, \ldots, 1 \,;\, d \;=\; j\,], \\ &\qquad {}_rL_j^- \;=\; \{\,\varphi \in {}_r\mathcal{C}_j^- : \mathcal{D}^1_{\varepsilon_1-\varepsilon_{j+2}}\,\varphi \;=\; 0, \quad G_j^+\,\varphi \;=\; 0\,\}, \\ &\qquad\qquad 1 \leq j \leq h-1, \\ &\ldots \end{aligned} \tag{44}$$

$$
\begin{aligned}
{}_r\chi_h^- &= [\,1,\,\ldots,\,1\,;\; d_{\rm FRP} \;=\; h\,], \\
&\qquad {}_rL_h^- \;=\; \{\,\varphi \in {}_r\mathcal{C}_h^- : \mathcal{D}^1_{\varepsilon_1}\,\varphi \;=\; 0\,\}, \qquad G_h^+ \;=\; \mathcal{D}^1_{\varepsilon_1}, \\
{}_r\chi_{h+1} &= [\,2,\,1,\,\ldots,\,1\,;\; d_{\rm FRP} \;=\; h + \tfrac{1}{2}\,], \\
&\qquad {}_rL_{h+1} \;=\; \{\,\varphi \in {}_r\mathcal{C}_{h+1}^- : \mathcal{D}^1_{\varepsilon_1+\varepsilon_{h+1}}\,\varphi \;=\; 0\,\}
\end{aligned}
$$

(In the last case there is no KS operator since $c = 0$.)

We see that for $h \geq 2$ there are discrete unitary points *below* the FRPs. For fixed $h \geq 2$ these are in ${}_r\chi_j^-$ with conformal weight $d = j$ (and trivial $\mathcal{M}$ inducing irreps) for $j = 1, \ldots, h-1$. Furthermore, as evident from (61) for $h \geq 3$ there are discrete unitary points below those displayed. For fixed $h \geq 3$ these are in ${}_r\chi_j^-$ with conformal weight $1 \leq d < j$ (and non-trivial $\mathcal{M}$ inducing irreps) for $j = 2, \ldots, h-2$. It seems that all this picture is consistent with [13]. More details will be given elsewhere.

Next we give the case of special reduced multiplets displaying together the physically relevant representations:

$$
\begin{aligned}
{}_s\chi_1^- &= [\,1,\,\ldots,\,1\,;\; d \;=\; \tfrac{1}{2}\,], \\
&\qquad {}_sL_1^- \;=\; \{\,\varphi \in {}_s\mathcal{C}_1^- : \mathcal{D}^{2h}_{\varepsilon_1}\,\varphi \;=\; 0\,\}, \qquad G_1^+ \;=\; \mathcal{D}^{2h}_{\varepsilon_1}, \\
&\ldots \\
{}_s\chi_j^- &= [\,1,\,\ldots,\,1\,;\; d \;=\; j - \tfrac{1}{2}\,], \qquad 1 \leq j \leq h \\
&\qquad {}_sL_j^- \;=\; \{\,\varphi \in {}_s\mathcal{C}_j^- : \mathcal{D}^{2(h+1-j)}_{\varepsilon_1}\,\varphi \;=\; 0\,\}, \qquad G_j^+ \;=\; \mathcal{D}^{2(h+1-j)}_{\varepsilon_1}, \\
&\ldots \\
{}_s\chi_h^- &= [\,1,\,\ldots,\,1\,;\; d \;=\; h - \tfrac{1}{2}\,], \\
&\qquad {}_sL_h^- \;=\; \{\,\varphi \in {}_s\mathcal{C}_h^- : \mathcal{D}^{2}_{\varepsilon_1}\,\varphi \;=\; 0\,\}, \qquad G_h^+ \;=\; \mathcal{D}^{2}_{\varepsilon_1}, \\
{}_s\chi_{h+1} &= [\,2,\,1,\,\ldots,\,1\,;\; d \;=\; h\,], \\
&\qquad {}_sL_{h+1}^- \;=\; \{\,\varphi \in {}_s\mathcal{C}_{h+1}^- : \mathcal{D}^{1}_{\varepsilon_1}\,\varphi \;=\; 0\,\}, \qquad G_{h+1}^+ \;=\; \mathcal{D}^{1}_{\varepsilon_1},
\end{aligned}
\tag{45}
$$

Here all irreps are below the FRP. The "most" minimal representations are the last two cases of (45). For $h = 1$, i.e., $so(3,2)$ these are the so-called *singletons* discovered by Dirac [14].

4.5 Singular Vectors Needed for the Invariant Differential Operators

The mostly used case is $\varepsilon_1 = \alpha_1 + \cdots + \alpha_\ell$, $\ell = h+1$. The corresponding singular vector of weight $m\varepsilon_1$ is given in (13) [9] (noting that this is an $sl(n)$ formula in quantum group setting, thus, one should take $q = 1$):

$$
\begin{aligned}
v^m_{\varepsilon_1} &= \sum_{k_1=0}^{m} \cdots \sum_{k_{\ell-1}=0}^{m} a_{k_1 \dots k_{\ell-1}} (X_1^-)^{m-k_1} \cdots (X_{\ell-1}^-)^{m-k_{\ell-1}} \times \\
&\quad \times (X_\ell^-)^m \, (X_{\ell-1}^-)^{k_{\ell-1}} \, \cdots \, (X_1^-)^{k_1} \, \otimes v_0, \qquad (46) \\
a_{k_1 \dots k_{\ell-1}} &= (-1)^{k_1+\dots+k_{\ell-1}} \, a^\ell \binom{m}{k_1} \cdots \binom{m}{k_{\ell-1}} \times \\
&\quad \times \frac{[(\lambda+\rho)(H^1)]}{[(\lambda+\rho)(H^1)-k_1]} \cdots \frac{[(\lambda+\rho)(H^{\ell-1})]}{[(\lambda+\rho)(H^{\ell-1})-k_{\ell-1}]},
\end{aligned}
$$

where $X_k^\pm$ are the simple root vectors, H_k are the long Chevalley Cartan elements $H_k = [X_k^+, X_k^-]$, $k < \ell$, $H^s = H_1 + H_2 + \dots + H_s$, ρ is the half-sum of the positive roots.

Other cases are: $\varepsilon_1 - \varepsilon_j = \alpha_1 + \dots + \alpha_{j-1}$. Clearly, one uses again formula (46) replacing $\ell \mapsto j-1$.

The last case is: $\varepsilon_1 + \varepsilon_\ell = \alpha_1 + \dots + \alpha_{\ell-1} + 2\alpha_\ell$, $\ell = h+1$. The singular vector is given in (19) of [9]:

$$
\begin{aligned}
v^m_{\varepsilon_1+\varepsilon_\ell} &= \sum_{k_1=0}^{m} \cdots \sum_{k_{\ell-2}=0}^{m} \sum_{k_{\ell-1}=0}^{2m} b_{k_1 \dots k_{\ell-1}} \, (X_1^-)^{m-k_1} \cdots \, (X_{\ell-2}^-)^{m-k_{\ell-2}} \times \\
&\quad \times (X_\ell^-)^{2m-k_{\ell-1}} (X_{\ell-1}^-)^m \, (X_\ell^-)^{k_{\ell-1}} \, (X_{\ell-2}^-)^{k_{\ell-2}} \cdots (X_1^-)^{k_1} \, \otimes v_0 \,, \qquad (47) \\
b_{k_1 \dots k_{\ell-1}} &= (-1)^{k_1+\dots+k_{\ell-1}} \, b^\ell \binom{m}{k_1} \cdots \binom{m}{k_{\ell-2}} \binom{2m}{k_{\ell-1}} \times \\
&\quad \times \frac{[(\lambda+\rho)(H^1)]}{[(\lambda+\rho)(H^1)-k_1]} \cdots \frac{[(\lambda+\rho)(H^{\ell-2})]}{[(\lambda+\rho)(H^{\ell-2})-k_{\ell-2}]} \frac{[(\lambda+\rho)(H_\ell)]}{[(\lambda+\rho)(H_\ell)-k_{\ell-1}]}
\end{aligned}
$$

5 Multiplets and Representations for $p+q$ Even

5.1 Reduced Multiplets for $p+q$ Even

In this section we consider the case $p+q$ odd, thus $h = \frac{1}{2}(p+q-2)$. First we introduce the Dynkin labels parametrization of the multiplets:

$$
\begin{aligned}
\chi_1^\pm &= [(m_1, \dots, m_h)^\pm \, ; \, \pm(\tfrac{1}{2}m_{12} + m_{3,h+1}) \,], \qquad (48) \\
\chi_2^\pm &= [(m_1, \dots, m_{h-1}, m_{h,h+1})^\pm \, ; \, \pm(\tfrac{1}{2}m_{12} + m_{3,h}) \,] \\
\chi_3^\pm &= [(m_1, \dots, m_{h-2}, m_{h-1,h}, m_{h+1})^\pm \, ; \, \pm(\tfrac{1}{2}m_{12} + m_{3,h-1}) \,]
\end{aligned}
$$

$$\begin{aligned}
&\dots \\
\chi_j^\pm &= [(m_1, \dots, m_{h-j+1}, m_{h-j+2,h-j+3}, m_{h+4-j}, \dots, m_h,\, m_{h+1})^\pm\, ; \\
&\qquad \pm(\tfrac{1}{2}m_{12} + m_{3,h+2-j})\,], \quad 2 \le j \le h-1, \\
&\dots \\
\chi_{h-1}^\pm &= [(m_1,\, m_2,\, m_{34},\, m_5,\, \dots,\, m_h,\, m_{h+1})^\pm\, ;\, \pm(\tfrac{1}{2}m_{12} + m_3)\,] \\
\chi_h^\pm &= [(m_{1'3},\, m_{23},\, m_4,\, \dots,\, m_h,\, m_{h+1})^\pm\, ;\, \pm\tfrac{1}{2}m_{12}\,] \\
\chi_{h+1}^\pm &= [(m_{13},\, m_3,\, \dots,\, m_h,\, m_{h+1})^\pm\, ;\, \pm\tfrac{1}{2}(m_1 - m_2)\,]
\end{aligned}$$

where the conjugation of the $\mathcal{M}$ labels interchanges the first two entries:

$$\begin{aligned}
(m_1,\, \dots,\, m_h)^- &= (m_1,\, \dots,\, m_h), \\
(m_1,\, m_2,\, \dots,\, m_h)^+ &= (m_2,\, m_1,\, \dots,\, m_h),
\end{aligned} \tag{49}$$

the last entry (as before) is the value of c, while $m_i \in \mathbb{N}$ are the Dynkin labels (as in (12)):

$$\begin{aligned}
m_1 &= n_1 + n_2 = \ell_1 + \ell_2 + 1, \\
m_j &= n_j - n_{j-1} = \ell_j - \ell_{j-1} + 1, \quad j = 2, \dots, h+1,
\end{aligned} \tag{50}$$

finally, $m_{1'3} \equiv m_1 + m_3$.

The main multiplets are given in Fig. 6. Note that as in the odd case we do not give the KS integral operators.

Then we give the reduced multiplets:

- $m_{h+1} = 0$ equiv $n_{h+1} = n_h$

$$\begin{aligned}
\chi_1^\pm = \chi_2^\pm &= [(m_1,\, \dots,\, m_h)^\pm\, ;\, \pm(\tfrac{1}{2}m_{12} + m_{3,h})\,], \qquad \spadesuit \\
\chi_3^\pm &= [(m_1, \dots, m_{h-2}, m_{h-1,h}, 0)^\pm\, ;\, \pm(\tfrac{1}{2}m_{12} + m_{3,h-1})\,] \\
&\dots \\
\chi_i^\pm &= [(m_1, \dots, m_{h-i+1}, m_{h-i+2,h-i+3}, m_{h+4-i}, \dots, m_h,\, 0)^\pm\, ; \\
&\qquad \pm(\tfrac{1}{2}m_{12} + m_{3,h+2-i})\,] \\
&\dots \\
\chi_{h-1}^\pm &= [(m_1,\, m_2,\, m_{34},\, m_5,\, \dots,\, m_h,\, 0)^\pm\, ;\, \pm(\tfrac{1}{2}m_{12} + m_3)\,] \\
\chi_h^\pm &= [(m_{1'3},\, m_{23},\, m_4,\, \dots,\, m_h,\, 0)^\pm\, ;\, \pm\tfrac{1}{2}m_{12}\,] \\
\chi_{h+1}^\pm &= [(m_{13},\, m_3,\, \dots,\, m_h,\, 0)^\pm\, ;\, \pm\tfrac{1}{2}(m_1 - m_2)\,]
\end{aligned} \tag{51}$$

Here there are two differential operators involving physically relevant representations, cf. Fig. 7:

$$\begin{aligned} \mathcal{D}^{m_h}_{\varepsilon_1-\varepsilon_3} &: \mathcal{C}^-_1 = \mathcal{C}^-_2 \longrightarrow \mathcal{C}^-_3 \\ \mathcal{D}^{m_h}_{\varepsilon_1+\varepsilon_3} &: \mathcal{C}^+_3 \longrightarrow \mathcal{C}^+_1 = \mathcal{C}^+_2 \end{aligned} \tag{52}$$

- $m_h = 0$ equiv $n_h = n_{h-1}$

$$\begin{aligned} \chi^\pm_1 &= [(m_1, \dots, m_{h-1}, 0)^\pm ; \pm(\tfrac{1}{2}m_{12}+m_{3,h-1}+2m_{h+1})], \\ \chi^\pm_2=\chi^\pm_3 &= [(m_1, \dots, m_{h-1}, m_{h+1})^\pm ; \pm(\tfrac{1}{2}m_{12}+m_{3,h-1})] \qquad \spadesuit \\ &\dots \\ \chi^\pm_i &= [(m_1, \dots, m_{h-i+1}, m_{h-i+2,h-i+3}, m_{h+4-i}, \dots, m_{h-1}, 0, \\ &\quad m_{h+1})^\pm ; \pm (\tfrac{1}{2}m_{12}+m_{3,h+2-i})] \\ &\dots \\ \chi^\pm_{h-1} &= [(m_1, m_2, m_{34}, m_5, \dots, m_{h-1}, 0, m_{h+1})^\pm ; \pm(\tfrac{1}{2}m_{12}+m_3)] \\ \chi^\pm_h &= [(m_{1'3}, m_{23}, m_4, \dots, m_{h-1}, 0, m_{h+1})^\pm ; \pm\tfrac{1}{2}m_{12}] \\ \chi^\pm_{h+1} &= [(m_{13}, m_3, \dots, m_{h-1}, 0, m_{h+1})^\pm ; \pm\tfrac{1}{2}(m_1 - m_2)] \end{aligned} \tag{53}$$

Here there are four differential operators involving physically relevant representations, cf. Fig. 8:

$$\begin{aligned} \mathcal{D}^{m_{h+1}}_{\varepsilon_1-\varepsilon_2} &: \mathcal{C}^-_1 \longrightarrow \mathcal{C}^-_2 = \mathcal{C}^-_3 \\ \mathcal{D}^{m_{h-1}}_{\varepsilon_1-\varepsilon_4} &: \mathcal{C}^-_2 = \mathcal{C}^-_3 \longrightarrow \mathcal{C}^-_4 \\ \mathcal{D}^{m_{h-1}}_{\varepsilon_1+\varepsilon_4} &: \mathcal{C}^+_4 \longrightarrow \mathcal{C}^+_2 = \mathcal{C}^+_3 \\ \mathcal{D}^{m_{h+1}}_{\varepsilon_1+\varepsilon_2} &: \mathcal{C}^+_2 = \mathcal{C}^+_3 \longrightarrow \mathcal{C}^+_1 \end{aligned} \tag{54}$$

The above case is typical for $m_k = 0$ for $k > 3$. Then for $k = 3, 2, 1$ we have:

- $m_3 = 0$ equiv $n_3 = n_2$

$$\begin{aligned} \chi^\pm_1 &= [(m_1, m_2, 0, m_4, \dots, m_h)^\pm ; \pm(\tfrac{1}{2}m_{12} + m_{4,h+1})], \\ \chi^\pm_2 &= [(m_1, m_2, 0, m_4, \dots, m_{h-1}, m_{h,h+1})^\pm ; \pm(\tfrac{1}{2}m_{12} + m_{4,h})] \\ &\dots \\ \chi^\pm_i &= [(m_1, m_2, 0, m_4, \dots, m_{h-i+1}, m_{h-i+2,h-i+3}, m_{h+4-i}, \dots, \\ &\quad m_h, m_{h+1})^\pm ; \pm(\tfrac{1}{2}m_{12} + m_{4,h+2-i})] \\ &\dots \end{aligned} \tag{55}$$

$$\chi^\pm_{h-1} = \chi^\pm_h = [(m_1,\, m_2,\, m_4,\, \ldots,\, m_h,\, m_{h+1})^\pm\,;\ \pm\tfrac{1}{2}m_{12}\,], \qquad \spadesuit$$
$$\chi^\pm_{h+1} = [(m_{12}, 0,\ m_4,\, \ldots,\, m_h,\, m_{h+1})^\pm\,;\ \pm\tfrac{1}{2}(m_1 - m_2)\,]$$

Here there are six differential operators involving physically relevant representations, cf. Fig. 9:

$$\begin{aligned}
\mathcal{D}^{m_4}_{\varepsilon_1-\varepsilon_{h-1}} &: \mathcal{C}^-_{h-2} \longrightarrow \mathcal{C}^-_h = \mathcal{C}^-_{h-1} \\
\mathcal{D}^{m_2}_{\varepsilon_1-\varepsilon_{h+1}} &: \mathcal{C}^-_h = \mathcal{C}^-_{h-1} \longrightarrow \mathcal{C}^-_{h+1} \\
\mathcal{D}^{m_1}_{\varepsilon_1+\varepsilon_{h+1}} &: \mathcal{C}^-_h = \mathcal{C}^-_{h-1} \longrightarrow \mathcal{C}^+_{h+1} \\
\mathcal{D}^{m_1}_{\varepsilon_1+\varepsilon_{h+1}} &: \mathcal{C}^-_{h+1} \longrightarrow \mathcal{C}^+_h = \mathcal{C}^+_{h-1} \\
\mathcal{D}^{m_2}_{\varepsilon_1-\varepsilon_{h+1}} &: \mathcal{C}^+_{h+1} \longrightarrow \mathcal{C}^+_h = \mathcal{C}^+_{h-1} \\
\mathcal{D}^{m_4}_{\varepsilon_1+\varepsilon_{h-1}} &: \mathcal{C}^+_h = \mathcal{C}^+_{h-1} \longrightarrow \mathcal{C}^+_{h-2}
\end{aligned} \tag{56}$$

- $m_2 = 0$ equiv $n_2 = n_1$,

$$\begin{aligned}
{}_2\chi^\pm_1 &= [(m_1,\, 0, m_3,\, \ldots,\, m_h)^\pm\,;\ \pm(\tfrac{1}{2}m_1 + m_{3,h+1})\,], \qquad (57)\\
{}_2\chi^\pm_2 &= [(m_1,\, 0, m_3,\, \ldots,\, m_{h-1},\, m_{h,h+1})^\pm\,;\ \pm(\tfrac{1}{2}m_1 + m_{3,h})\,] \\
{}_2\chi^\pm_3 &= [(m_1, 0, m_3,\, \ldots, m_{h-2}, m_{h-1,h}, m_{h+1})^\pm\,;\ \pm(\tfrac{1}{2}m_1 + m_{3,h-1})\,] \\
&\ldots \\
{}_2\chi^\pm_i &= [(m_1, 0, m_3,\, \ldots, m_{h-i+1}, m_{h-i+2,h-i+3}, m_{h+4-i}, \ldots, \\
&\qquad m_h,\, m_{h+1})^\pm\,;\ \pm(\tfrac{1}{2}m_1 + m_{3,h+2-i})\,] \\
&\ldots \\
{}_2\chi^\pm_{h-1} &= [(m_1,\, 0,\, m_{34},\, m_5,\, \ldots,\, m_h,\, m_{h+1})^\pm\,;\ \pm(\tfrac{1}{2}m_1 + m_3)\,] \\
{}_2\chi^\pm_h = {}_2\chi^\pm_{h+1} &= [(m_1{+}m_3,\, m_3,\, \ldots,\, m_h,\, m_{h+1})^\pm\,;\ \pm\tfrac{1}{2}m_1\,], \qquad \spadesuit
\end{aligned}$$

Here there are three differential operators involving physically relevant representations, cf. Fig. 10:

$$\begin{aligned}
\mathcal{D}^{m_3}_{\varepsilon_1-\varepsilon_h} &: \mathcal{C}^-_{h-1} \longrightarrow \mathcal{C}^-_h = \mathcal{C}^-_{h+1} \\
\mathcal{D}^{m_1}_{\varepsilon_1+\varepsilon_{h+1}} &: \mathcal{C}^-_h = \mathcal{C}^-_{h+1} \longrightarrow \mathcal{C}^+_h = \mathcal{C}^+_{h+1} \\
\mathcal{D}^{m_3}_{\varepsilon_1+\varepsilon_h} &: \mathcal{C}^+_h = \mathcal{C}^+_{h+1} \longrightarrow \mathcal{C}^+_{h-1}
\end{aligned} \tag{58}$$

- $m_1 = 0$ equiv $n_2 = -n_1$,

$$\begin{aligned}
{}_1\chi_1^{\pm} &= [(0,\, m_2,\, \ldots,\, m_h)^{\pm}\ ;\ \pm(\tfrac{1}{2}m_2 + m_{3,h+1})\,], \qquad (59)\\
{}_1\chi_2^{\pm} &= [(0,\, m_2,\, \ldots,\, m_{h-1},\, m_{h,h+1})^{\pm}\ ;\ \pm(\tfrac{1}{2}m_2 + m_{3,h})\,]\\
{}_1\chi_3^{\pm} &= [(0,\, m_2, \ldots, m_{h-2}, m_{h-1,h}, m_{h+1})^{\pm}\ ;\ \pm(\tfrac{1}{2}m_2 + m_{3,h-1})\,]\\
&\ldots\\
{}_1\chi_i^{\pm} &= [(0,\, m_2, \ldots, m_{h-i+1}, m_{h-i+2,h-i+3}, m_{h+4-i}, \ldots,\\
&\qquad m_h,\, m_{h+1})^{\pm}\ ;\ \pm(\tfrac{1}{2}m_2 + m_{3,h+2-i})\,]\\
&\ldots\\
{}_1\chi_{h-1}^{\pm} &= [(0,\, m_2,\, m_{34},\, m_5,\, \ldots,\, m_h,\, m_{h+1})^{\pm}\ ;\ \pm(\tfrac{1}{2}m_2 + m_3)\,]\\
{}_1\chi_h^{\pm} = {}_1\chi_{h+1}^{\mp} &= [(m_3,\, m_2 + m_3,\, m_4,\, \ldots,\, m_h,\, m_{h+1})^{\pm}\ ;\ \pm\tfrac{1}{2}m_2\,], \qquad \spadesuit
\end{aligned}$$

Here there are three differential operators involving physically relevant representations, cf. Fig. 11:

$$\begin{aligned}
\mathcal{D}^{m_3}_{\varepsilon_1-\varepsilon_h} &: \mathcal{C}^-_{h-1} \longrightarrow \mathcal{C}^-_h = \mathcal{C}^+_{h+1}\\
\mathcal{D}^{m_2}_{\varepsilon_1-\varepsilon_{h+1}} &: \mathcal{C}^-_h = \mathcal{C}^+_{h+1} \longrightarrow \mathcal{C}^+_h = \mathcal{C}^-_{h+1}\\
\mathcal{D}^{m_3}_{\varepsilon_1+\varepsilon_h} &: \mathcal{C}^+_h = \mathcal{C}^-_{h+1} \longrightarrow \mathcal{C}^+_{h-1} \qquad (60)
\end{aligned}$$

Note that the last two cases: (57) and (59) are conjugate to each other through the $\mathcal{M}$ labels (${}_1\chi_i^{\pm}$ has the same expressions for c as ${}_2\chi_i^{\pm}$, but the $\mathcal{M}$ labels are conjugate).

For future reference we summarize the physically relevant pairs reparametrizing for more natural presentation and introducing uniform notation ${}_r\chi_k^{\pm}$:

$$\begin{aligned}
{}_r\chi_1^{\pm} &= [(m_1,\, \ldots,\, m_h)^{\pm}\ ;\ \pm(\tfrac{1}{2}m_{12} + m_{3,h})\,], \quad d^+ \geq 2h-1,\ d^- \leq 1,\\
{}_r\chi_2^{\pm} &= [(m_1,\, \ldots,\, m_h)^{\pm}\ ;\ \pm(\tfrac{1}{2}m_{12} + m_{3,h-1})\,], \quad d^+ \geq 2h-2,\ d^- \leq 2,\\
&\ldots\\
{}_r\chi_j^{\pm} &= [(m_1,\, \ldots,\, m_h)^{\pm}\ ;\ \pm(\tfrac{1}{2}m_{12} + m_{3,h+1-j})\,],\\
&\qquad d^+ \geq 2h-j,\ d^- \leq j, \quad 1 \leq j \leq h-2, \qquad (61)\\
&\ldots
\end{aligned}$$

$$\begin{aligned}
{}_r\chi^\pm_{h-1} &= [(m_1,\, \dots,\, m_h)^\pm\,;\; \pm\tfrac{1}{2}m_{12}\,], \quad d^+ \geq h+1,\;\; d^- \leq h-1,\\
{}_r\chi^\pm_{h} &= [(m+m_2,\, m_2,\, m_3,\, \dots,\, m_h)^\pm\,;\; \pm\tfrac{1}{2}m\,],\\
&\qquad\qquad\qquad\qquad d^+ \geq h+\tfrac{1}{2},\;\; d^- \leq h-\tfrac{1}{2},\\
{}_r\chi^\pm_{h+1} &= [(m_2,\, m+m_2,\, m_3,\, \dots,\, m_h)^\pm\,;\; \pm\tfrac{1}{2}m\,],\\
&\qquad\qquad\qquad\qquad d^+ \geq h+\tfrac{1}{2},\;\; d^- \leq h-\tfrac{1}{2}.
\end{aligned}$$

Note a last reduction obtained by setting $m = 0$ when the last two pairs in (61) coincide and become further a singlet (being $\mathcal{M}$ self-conjugate):

$$ {}_r\chi^s = [m_2,\, m_2,\, m_3,\, \dots,\, m_h\,;\, 0\,], \quad d = h. \tag{62}$$

5.2 Special Cases for $p + q$ Even

The ERS χ_1^- are the only ones in the multiplet that contain as irreducible subrepresentations the finite-dimensional irreducible representations of $\mathcal{G}$. More precisely, the ER χ_1^- contains the finite-dimensional irreducible representation of $\mathcal{G}$ with signature $(m_1, \dots, m_{h+1})$. (Certainly, the latter is non-unitary except the case of the trivial one-dimensional obtained for $m_i = 1, \forall\, i$.)

Another important case is the ER with signature χ_1^+. For $pq \in 2\mathbb{N}$ it contains a unitary discrete series representation of $so(p,q)$ realized on an invariant subspace $\mathcal{D}$ of the ER χ_1^+. That subspace is annihilated by the KS operator G_1^-, and is the image of the KS operator G_1^+.

Furthermore when $p > q = 2$ the invariant subspace $\mathcal{D}$ is the direct sum of two subspaces $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^-$, in which are realized a *holomorphic discrete series representation* and its conjugate *anti-holomorphic discrete series representation*, resp. Note that the corresponding lowest weight GVM is infinitesimally equivalent only to the holomorphic discrete series, while the conjugate highest weight GVM is infinitesimally equivalent to the anti-holomorphic discrete series.

Thus, the signatures of the (holomorphic) discrete series are:

$$\chi_1^+ = [m_1,\, \dots,\, m_h\,;\, d \;=\; h + \tfrac{1}{2}m_{12} + m_{3,h} \;+\; \nu\,], \quad \nu \in \mathbb{N} \tag{63}$$

More (non-holomorphic) discrete series representations are contained in χ_k^+ for $1 < k \leq h+1$.

The next important case of positive energy UIRs are the *limits of (holomorphic) discrete series* which are contained in the reduced case (61):

$$ {}_r\chi_1^+ \;=\; [m_1,\, \dots,\, m_h\,;\, d \;=\; h + \tfrac{1}{2}m_{12} + m_{3,h}\,] \tag{64}$$

(with conformal weight obtained from (63) as "limit" for $\nu = 0$).

Further we discuss the so called *first reduction points* (FRP). These are the boundary values of d from below of the positive energy UIRs. Most of the FRPs are contained in χ_h^+, cf. (48), which we give with suitable reparametrization:

$$\chi_h^+ \ = \ [m_1, m_2, \dots, m_h \, ; \, d \ = \ h + \tfrac{1}{2}m_{12} - 1 \,], \qquad m_1, m_2 \geq 2. \tag{65}$$

Some FRP cases when only one of m_1, m_2 is equal to 1 are found in $\chi_{h+1}^\pm$:

$$\begin{aligned} \chi_{h+1}^- \ &= \ [m_1, \, 1, \, m_3, \, \dots, m_h \, ; \, d \ = \ h + \tfrac{1}{2}(m_1 - 3) \,], \quad m_1 \geq 3, \\ \chi_{h+1}^+ \ &= \ [1, \, m_2, \, \dots, m_h \, ; \, d \ = \ h + \tfrac{1}{2}(m_2 - 3) \,], \quad m_2 \geq 3. \end{aligned} \tag{66}$$

Finally the last three FRP cases $(m_1, m_2) \ = \ (1,1), (2,1), (1,2)$ are found in ${}_r\chi_{k=,h-1h,h+1}^-$:

$$\begin{aligned} {}_r\chi_{h-1}^- \ &= \ [1, \, 1, \, m_3, \, \dots, m_h \, ; \, d \ = \ h - 1 \,], \\ {}_r\chi_h^- \ &= \ [2, \, 1, \, m_3, \, \dots, m_h \, ; \, d \ = \ h - \tfrac{1}{2} \,], \\ {}_r\chi_{h+1}^- \ &= \ [1, \, 2, \, m_3, \, \dots, m_h \, ; \, d \ = \ h - \tfrac{1}{2} \,]. \end{aligned} \tag{67}$$

5.3 Minimal Irreps for $p + q$ Even

The minimal irreps in this case happen to be related to the ERs in the reduced multiplets. We define the minimal irreps L_Λ as positive energy UIRs which involve the lowest dimensional representation of $\mathcal{M}$. Besides the signature we display the equations that are obeyed by the functions of the irrep. Typically, the irrep is the intersection of the kernel of the corresponding KS operator G^+ and of one or two intertwining differential operators that were already displayed in the subsection on reduced multiplets.

Below we denote by ${}_rL_i^\pm$ the irreducible subrepresentation of the ER ${}_r\mathcal{C}_i^\pm$. The list is:

$$\begin{aligned} &{}_r\chi_1^- = [(1, \dots, 1) \, ; \, d \ = \ 1 \,], \\ &\qquad {}_rL_1^- \ = \ \{ \, \varphi \in {}_r\mathcal{C}_1^- : \mathcal{D}^1_{\varepsilon_1 - \varepsilon_3} \varphi \ = \ 0, \quad G_1^+ \varphi \ = \ 0 \, \}, \\ &{}_r\chi_2^- = [(1, \dots, 1) \, ; \, d \ = \ 2 \,], \\ &\qquad {}_rL_2^- \ = \ \{ \, \varphi \in {}_r\mathcal{C}_2^- : \mathcal{D}^1_{\varepsilon_1 - \varepsilon_4} \varphi \ = \ 0, \quad G_2^+ \varphi \ = \ 0 \, \}, \\ &\dots \\ &{}_r\chi_j^- = [(1, \dots, 1) \, ; \, d \ = \ j \,], \quad 1 \leq j \leq h - 2, \\ &\qquad {}_rL_j^- \ = \ \{ \, \varphi \in {}_r\mathcal{C}_j^- : \mathcal{D}^1_{\varepsilon_1 - \varepsilon_{j+2}} \varphi \ = \ 0, \quad G_j^+ \varphi \ = \ 0 \, \}, \end{aligned} \tag{68}$$

$$\begin{aligned}
&\dots \\
{}_r\chi^-_{h-1} &= [(1,\,\dots,\,1)\,;\; d_{\rm FRP} = h-1\,], \\
&\quad {}_rL^-_{h-1} \;=\; \{\,\varphi\in\, {}_r\mathcal{C}^-_{h-1} : \mathcal{D}^1_{\varepsilon_1-\varepsilon_{h+1}}\varphi \;=\; 0, \quad \mathcal{D}^1_{\varepsilon_1+\varepsilon_{h+1}}\varphi \;=\; 0, \\
&\qquad\qquad G^+_{h-1}\varphi \;=\; 0\,\}, \\
{}_r\chi^-_h &= [(2,\,1,\,\dots,\,1)\,;\; d^-_{\rm FRP} = h-\tfrac12\,], \\
&\quad {}_rL^-_h \;=\; \{\,\varphi\in\, {}_r\mathcal{C}^-_h : \mathcal{D}^2_{\varepsilon_1+\varepsilon_{h+1}}\varphi \;=\; 0\,\}, \quad G^+_h \;\sim\; \mathcal{D}^1_{\varepsilon_1+\varepsilon_{h+1}}, \\
{}_r\chi^-_{h+1} &= [(1,\,2,\,1,\,\dots,\,1)\,;\; d_{\rm FRP} = h-\tfrac12\,], \\
&\quad {}_rL^-_{h+1} \;=\; \{\,\varphi\in\, {}_r\mathcal{C}^-_{h+1} : \mathcal{D}^2_{\varepsilon_1-\varepsilon_{h+1}}\varphi \;=\; 0\,\}, \\
&\qquad G^+_{h+1} \;\sim\; \mathcal{D}^1_{\varepsilon_1-\varepsilon_{h+1}},
\end{aligned}$$

where we have indicated (in the last two cases) the degeneration of KS integral operators to differential operators.

We see in (68) that for $h \geq 3$ there are discrete unitary points *below* the FRPs.[2] For fixed $h \geq 3$ these are in ${}_r\chi^-_j$ with conformal weight $d = j$ (and trivial $\mathcal{M}$ inducing irreps) for $j = 1, \dots, h-2$. Furthermore, as evident from (61) for $h \geq 4$ there are discrete unitary points below those displayed. For fixed $h \geq 4$ these are in ${}_r\chi^-_j$ with conformal weight $1 \leq d < j$ (and non-trivial $\mathcal{M}$ inducing irreps) for $j = 2, \dots, h-2$. It seems that all this picture is consistent with [13]. More details will be given elsewhere.

Singular Vectors Needed for the Invariant Differential Operators

The necessary cases are:

$$\begin{aligned}
\varepsilon_1 - \varepsilon_j \;&=\; \alpha_{h+3-j} + \cdots + \alpha_{h+1}, \quad 2 \leq j \leq h+1, \\
\varepsilon_1 + \varepsilon_{h+1} \;&=\; \alpha_1 + \alpha_3 + \cdots + \alpha_{h+1}.
\end{aligned} \tag{69}$$

These are roots of $sl(n)$ subalgebras ($n < h+1$). Thus, we can use f-la (46) after suitable change of enumeration.

[2] Thus, the most famous case $so(4, 2)$ is excluded.

Figures

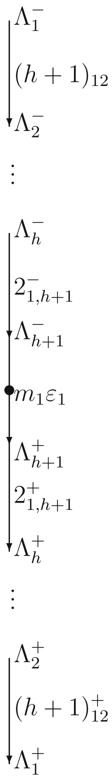


Fig. 1 Main multiplets for $so(p,q)$, $p+q=2h+3$, odd, $p,q \geq 1$ i^-_{1k} corresponds to weight $m_i(\varepsilon_1-\varepsilon_k)$, i^+_{1k} corresponds to weight $m_i(\varepsilon_1+\varepsilon_k)$

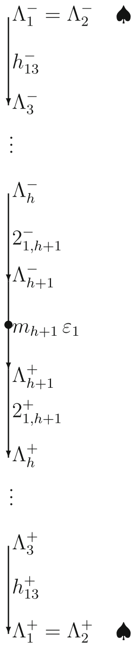


Fig. 2 Reduced multiplets R°_{h+1} for $so(p,q)$, $p+q=2h+3$, odd, $p,q \geq 1$ i_{1k}^- corresponds to weight $m_i(\varepsilon_1 - \varepsilon_k)$, i_{1k}^+ corresponds to weight $m_i(\varepsilon_1 + \varepsilon_k)$

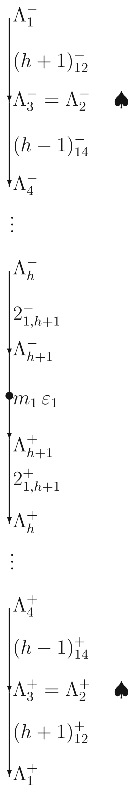


Fig. 3 Reduced multiplets $R^{\circ}_{h+1}(m_h = 0)$ for $so(p,q)$, $p+q=2h+3$, odd, $p,q \geq 1$ i_{1k}^- corresponds to weight $m_i(\varepsilon_1 - \varepsilon_k)$, i_{1k}^+ corresponds to weight $m_i(\varepsilon_1 + \varepsilon_k)$

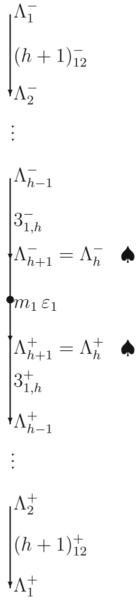


Fig. 4 Reduced multiplets $R_{h+1}^{\circ}(m_2 = 0)$ for $so(p,q)$, $p+q = 2h+3$, odd, $p, q \geq 1$ i_{1k}^- corresponds to weight $m_i(\varepsilon_1 - \varepsilon_k)$, i_{1k}^+ corresponds to weight $m_i(\varepsilon_1 + \varepsilon_k)$

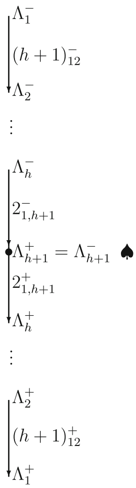


Fig. 5 Reduced multiplets $R_{h+1}^{\circ}(m_2 = 0)$ for $so(p,q)$, $p+q = 2h+3$, odd, $p, q \geq 1$ i_{1k}^- corresponds to weight $m_i(\varepsilon_1 - \varepsilon_k)$, i_{1k}^+ corresponds to weight $m_i(\varepsilon_1 + \varepsilon_k)$

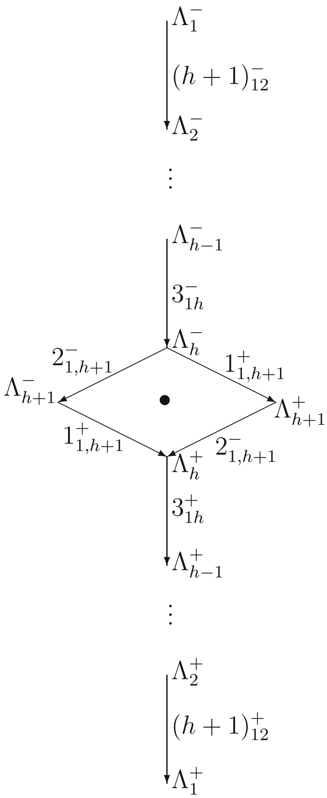


Fig. 6 Main multiplets for $so(p,q)$, $p+q=2h+2$, odd, $p,q \geq 1$ i_{1k}^- corresponds to weight $m_i(\varepsilon_1 - \varepsilon_k)$, i_{1k}^+ corresponds to weight $m_i(\varepsilon_1 + \varepsilon_k)$

Fig. 7 Reduced multiplets $R^{\circ}_{h+1}(m_{h+1} = 0)\, p + q = 2h + 2$, even, $p, q \geq 1$ i^-_{1k} corresponds to weight $m_i(\varepsilon_1 - \varepsilon_k)$, i^+_{1k} corresponds to weight $m_i(\varepsilon_1 + \varepsilon_k)$

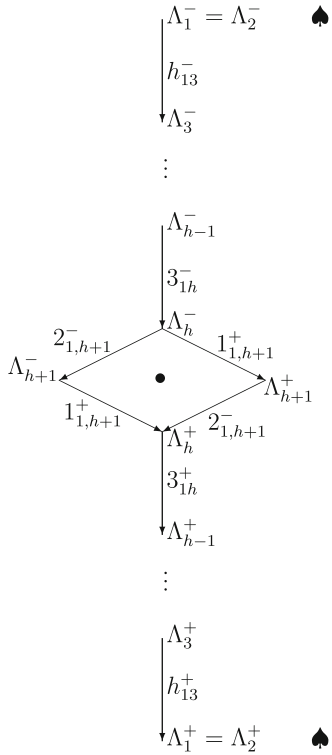


Fig. 8 Reduced multiplets $R_h^e(m_h = 0)\, p + q = 2h + 2$, even, $p, q \geq 1$ i_{1k}^- corresponds to weight $m_i(\varepsilon_1 - \varepsilon_k)$, i_{1k}^+ corresponds to weight $m_i(\varepsilon_1 + \varepsilon_k)$

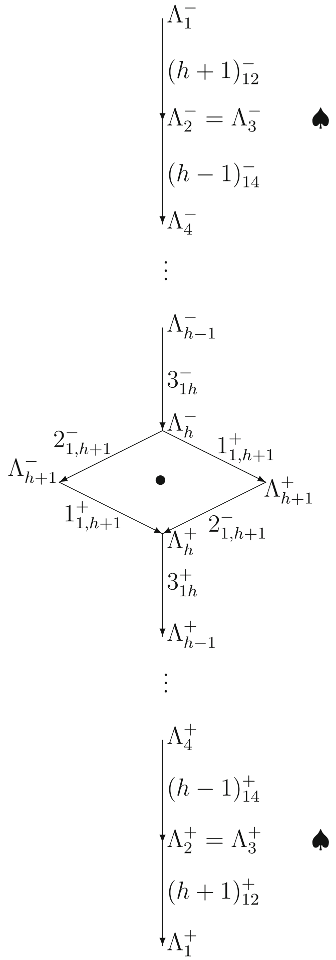


Fig. 9 Reduced multiplets $R_2^e(m_3 = 0)$ for $so(p,q)$, $p + q = 2h + 2$, even, $p, q \geq 1$ i_{1k}^- corresponds to weight $m_i(\varepsilon_1 - \varepsilon_k)$, i_{1k}^+ corresponds to weight $m_i(\varepsilon_1 + \varepsilon_k)$

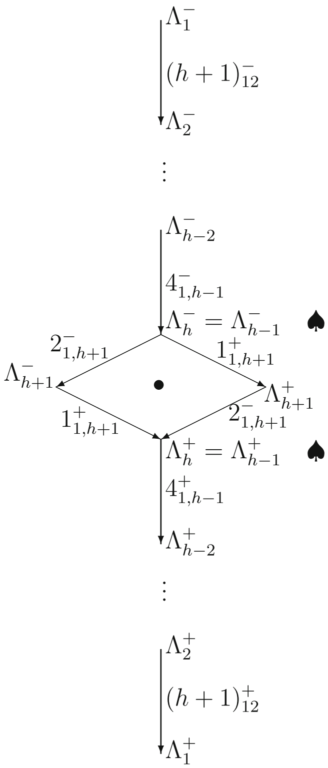
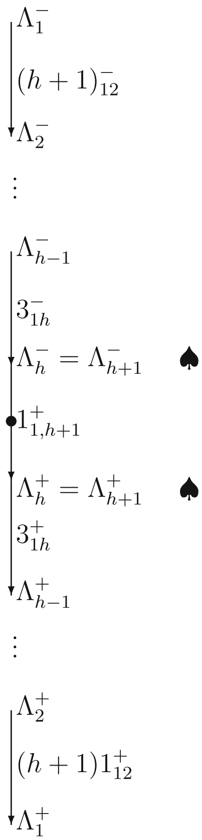


Fig. 10 Reduced multiplets $R_2^e(m_2 = 0)$ for $so(p,q)$, $p + q = 2h + 2$, $p, q \geq 1$ i_{1k}^- corresponds to weight $m_i(\varepsilon_1 - \varepsilon_k)$, i_{1k}^+ corresponds to weight $m_i(\varepsilon_1 + \varepsilon_k)$



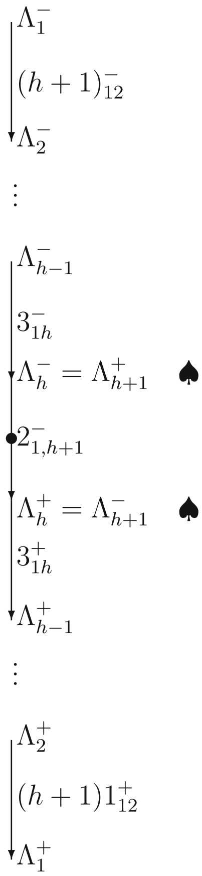


Fig. 11 Reduced multiplets $R_1^e(m_1 = 0)$ for $so(p,q)$, $p+q = 2h+2$, $p,q \geq 1$ i_{1k}^- corresponds to weight $m_i(\varepsilon_j - \varepsilon_k)$, i_{1k}^+ corresponds to weight $m_i(\varepsilon_1 + \varepsilon_k)$

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On the Structure of Green's Ansatz

Igor Salom

Abstract It is well known that the symmetric group has an important role (via Young tableaux formalism) both in labelling of the representations of the unitary group and in construction of the corresponding basis vectors (in the tensor product of the defining representations). We show that orthogonal group has a very similar role in the context of positive energy representations of $osp(1|2n,\mathbb{R})$. In the language of parabose algebra, we essentially solve, in the parabosonic case, the long standing problem of reducibility of Green's Ansatz representations.

1 Introduction

The $osp(1|2n,\mathbb{R})$ superalgebra attracts nowadays significant attention, primarily as a natural generalization of the conformal supersymmetry in higher dimensions [1–9]. In the context of space-time supersymmetry, knowing and understanding unitary irreducible representations (UIR's) of this superalgebra is of extreme importance, as these should be in a direct relation with the particle content of the corresponding physical models.

And the most important from the physical viewpoint are certainly, so called, positive energy UIR's, which are the subject of this paper. More precisely, the goal of the paper is to clarify how these representations can be obtained by essentially tensoring the simplest nontrivial positive energy UIR (the one that corresponds to oscillator representation). This parallels the case of the UIR's of the unitary group $U(n)$ constructed within the tensor product of the defining (i.e. "one box") representations. In both cases the tensor product representation is reducible, and while this reduction in the $U(n)$ case is governed by the action of the commuting group of permutations, in the osp case,[1] as we will show, the role of permutations is played by an orthogonal group. We will clarify the details of this reduction.

[1] We will often write shortly $osp(1|2n)$ or osp for the $osp(1|2n,\mathbb{R})$.

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The $osp(1|2n)$ superalgebra is also known by its direct relation to parabose algebra [10, 11]. In the terminology of parastatistics, the tensor product of oscillator UIR's is known as the Green's Ansatz [12]. The problem of the decomposition of parabose Green's Ansatz space to parabose (i.e. $osp(1|2n)$) UIR's is an old one [12], that we here solve by exploiting additional orthogonal symmetry of a "covariant" version of the Green's Ansatz.

2 Covariant Green's Ansatz

Structural relations of $osp(1|2n)$ superalgebra can be compactly written in the form of trilinear relations of odd algebra operators a_α and $a^\dagger_\alpha$:

$$[\{a_\alpha, a^\dagger_\beta\}, a_\gamma] = -2\delta_{\beta\gamma} a_\alpha, \qquad [\{a^\dagger_\alpha, a_\beta\}, a^\dagger_\gamma] = 2\delta_{\beta\gamma} a^\dagger_\alpha, \tag{1}$$

$$[\{a_\alpha, a_\beta\}, a_\gamma], \qquad [\{a^\dagger_\alpha, a^\dagger_\beta\}, a^\dagger_\gamma] = 0, \tag{2}$$

where operators $\{a_\alpha, a^\dagger_\beta\}$, $\{a_\alpha, a_\beta\}$ and $\{a^\dagger_\alpha, a^\dagger_\beta\}$ span the even part of the superalgebra and Greek indices take values $1, 2, \ldots n$ (relations obtained from these by use of Jacobi identity are also implied). This compact notation emphasises the direct connection [11] of $osp(1|2n)$ superalgebra with the parabose algebra of n pairs of creation/annihilation operators [10].

If we (in the spirit of original definition of parabose algebra [10]) additionally require that the dagger symbol † above denotes hermitian conjugation in the algebra representation Hilbert space (of positive definite metrics), then we have effectively constrained ourselves to the, so called, positive energy UIR's of $osp(1|2n)$.[2] Namely, in such a space, "conformal energy" operator $E \equiv \frac{1}{2}\sum_\alpha \{a_\alpha, a^\dagger_\alpha\}$ must be a positive operator. Operators a_α reduce the eigenvalue of E, so the Hilbert space must contain a subspace that these operators annihilate. This subspace is called vacuum subspace: $V_0 = \{|v\rangle, a_\alpha|v\rangle = 0\}$. If the positive energy representation is irreducible, all vectors from V_0 have the common, minimal eigenvalue ϵ_0 of E: $E|v\rangle = \epsilon_0|v\rangle, |v\rangle \in V_0$. Representations with one dimensional subspace V_0 are called "unique vacuum" representations.

In this paper we will constrain our analysis to UIR's with integer and half-integer values of ϵ_0 (in principle, ϵ_0 has also continuous part of the spectrum—above the, so called, first reduction point of the Verma module). It turns out that all representations from this class can be obtained by representing the odd superalgebra operators a and $a^\dagger$ as the following sum:

$$a_\alpha = \sum_{a=1}^{p} b^a_\alpha\, e^a, \qquad a^\dagger_\alpha = \sum_{a=1}^{p} b^{a\dagger}_\alpha\, e^a. \tag{3}$$

[2]Omitting a short proof, we note that in such a Hilbert space all superalgebra relations actually follow from one single relation—the first or the second of (1).

In this expression integer p is known as the order of the parastatistics, e^a are elements of a real Clifford algebra:

$$\{e^a, e^b\} = 2\delta^{ab} \tag{4}$$

and operators b_α^a together with adjoint $b_\alpha^{a\dagger}$ satisfy ordinary bosonic algebra relations. There are total of $n \cdot p$ mutually commuting pairs of bosonic annihilation-creation operators $(b_\alpha^a, b_\alpha^{a\dagger})$:

$$[b_\alpha^a, b_\beta^{b\dagger}] = \delta_{\beta\alpha}\delta^{ab}; \quad [b_\alpha^a, b_\beta^b] = 0. \tag{5}$$

Indices $a, b, \ldots$ from the beginning of the Latin alphabet will, throughout the paper, take values $1, 2, \ldots p$. Relation (3) is a slight variation, more precisely, realization, of a more common form of the Green's Ansatz [10, 13].

The representation space of operators (3) can be seen as tensor product of p multiples of Hilbert spaces $\mathcal{H}_a$ of ordinary linear harmonic oscillator in n-dimensions multiplied by the representation space of the Clifford algebra:

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_p \otimes \mathcal{H}_{CL}. \tag{6}$$

A single factor Hilbert space $\mathcal{H}_a$ is the space of unitary representation of n dimensional bose algebra of operators $(b_\alpha^a, b_\alpha^{a\dagger}), \alpha = 1, 2, \ldots n$: $\mathcal{H}_a \cong \mathcal{U}(b^{a\dagger})|0\rangle_a$, where $|0\rangle_a$ is the usual Fock vacuum of factor space $\mathcal{H}_a$. The representation space $\mathcal{H}_{CL}$ of real Clifford algebra (4) is of dimension $2^{[p/2]}$, i.e. isomorphic with $\mathbb{C}^{2^{[p/2]}}$ (matrix representation). Positive definite scalar product is introduced in usual way in each of the factor spaces, endowing entire space $\mathcal{H}$ also with positive definite scalar product. The space is spanned by the vectors:

$$\mathcal{H} = l.s.\{\mathcal{P}(b^\dagger)|0\rangle \otimes \omega\}, \tag{7}$$

where $\mathcal{P}(b^\dagger)$ are monomials in mutually commutative operators $b_\alpha^{a\dagger}$, $|0\rangle \equiv |0\rangle_1 \otimes |0\rangle_2 \otimes \cdots \otimes |0\rangle_p$ and $w \in \mathcal{H}_{CL}$.

In the case $p = 1$ (the Clifford part becomes trivial) we obtain the simplest positive energy UIR of $osp(1|2n)$—the n dimensional harmonic oscillator representation. The order p Green's Ansatz representation of $osp(1|2n)$ is, effectively, representation in the p-fold tensor product of oscillator representations [12], with the Clifford factor space taking care of the anticommutativity properties of odd superalgebra operators. It is easily verified that even superalgebra elements act trivially in the Clifford factor space and that their action is simply sum of actions in each of the factor spaces.

The space (6) is highly reducible under action of osp superalgebra. It necessarily decomposes into direct sum of positive energy representations (both unique vacuum and non unique vacuum representations) and thus, from the aspect of osp transformation properties, space $\mathcal{H}$ is spanned by:

$$\mathcal{H} = l.s.\{|(\Lambda, l), \eta_\Lambda\rangle\}, \tag{8}$$

where Λ labels $osp(1|2n)$ positive energy UIR, l uniquely labels a concrete vector within the UIR Λ, and $\eta_\Lambda = 1, 2, \dots N_\Lambda$ labels possible multiplicity of UIR Λ in the representation space $\mathcal{H}$. If some UIR Λ does not appear in decomposition of $\mathcal{H}$, then the corresponding N_Λ is zero. Label Λ in (8) runs through all (integer and halfinteger positive energy) UIR's of $osp(1|2n)$ such that $N_\Lambda > 0$ and l runs through all vectors from UIR Λ.

3 Gauge Symmetry of the Ansatz

Green's Ansatz in the form (3) possesses certain intrinsic symmetries. First, we note that hermitian operators

$$G^{ab} \equiv \sum_{\alpha=1}^{n} i(b_\alpha^{a\dagger} b_\alpha^b - b_\alpha^{b\dagger} b_\alpha^a) + \frac{i}{4}[e^a, e^b] \tag{9}$$

commute with entire osp superalgebra, which immediately follows after checking that $[G^{ab}, a_\alpha] = 0$. Operators G^{ab} themselves satisfy commutation relations of $so(p)$ algebra. The second term in (9) acts in the Clifford factor space, generating a faithful representation of $Spin(p)$ (i.e. spinorial representation of double cover of $SO(p)$ group). Action of the first terms from (9) generate $SO(p)$ group action in the space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_p$. In the entire space $\mathcal{H}$ operators G generate $Spin(p)$ group and all vectors belong to spinorial unitary representations of this symmetry group. The two terms in (9) thus resemble orbital and spin parts of rotation generators and we will often use that terminology. In particular $\mathcal{H} \equiv \mathcal{H}^{\mathbf{o}} \otimes \mathcal{H}^{\mathbf{s}}$, where $\mathcal{H}^{\mathbf{o}} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_p$ and $\mathcal{H}^{\mathbf{s}} = \mathcal{H}_{CL}$. Furthermore, due to existence of operators $I^a \equiv -i exp(i\pi \sum_\alpha b_\alpha^{a\dagger} b_\alpha^a)\overline{e}e^a$ where $\overline{e} \equiv i^{[p/2]}e^1 e^2 \cdots e^p$, for even values of p, the symmetry can be extended to $Pin(p)$ group (the double cover of orthogonal group $O(p)$). We will refer to the symmetry group of the Green's ansatz as the gauge group.

Vectors in space $\mathcal{H}$ carry quantum numbers also according to their transformation properties under the gauge group. As the gauge group commutes with $osp(1|2n)$, these numbers certainly remove at least a part of degeneracy of osp representations in $\mathcal{H}$, in the sense that relation (8) can be rewritten as:

$$\mathcal{H} = l.s.\{|(\Lambda, l), (M, m), \eta_{(\Lambda,M)}\rangle\}, \tag{10}$$

where $(\Lambda,\ l)$ uniquely label vector l within $osp(1|2n)$ positive energy UIR Λ, $(M,\ m)$ uniquely label vector m within finite dimensional UIR M of the gauge group, and $\eta_{(\Lambda,M)} = 1, 2, \dots N_{(\Lambda,M)}$ labels possible remaining multiplicity of tensor product of these two representations $\mathcal{D}_\Lambda^{osp} \otimes \mathcal{D}_M^{gauge}$ in the space $\mathcal{H}$. Again, if some combination (Λ, M) does not appear in decomposition of $\mathcal{H}$, then the corresponding $N_{(\Lambda,M)}$ is zero.

Important property of the gauge symmetry is that it actually removes all degeneracy in decomposition of $\mathcal{H}$ to $osp(1|2n)$ UIR's, i.e. that the multiplicity of $osp(1|2n)$ UIR's is fully taken into account by labeling transformation properties of the vector w.r.t. the gauge symmetry group. Furthermore, there is one-to-one correspondence between UIR's of $osp(1|2n)$ and of the gauge group that appear in the decomposition, meaning that transformation properties under the gauge group action automatically fix the $osp(1|2n)$ representation. We formulate this more precisely in the following theorem.

Theorem 1. *The following statements hold for the basis (10) of the Hilbert space $\mathcal{H}$:*

1. *All multiplicities $N_{(\Lambda,M)}$ are either 1 or 0.*
2. *Let the $\mathcal{N}$ be the set of all pairs (Λ, M) for which $N_{(\Lambda,M)} = 1$, i.e. $\mathcal{N} = \{(\Lambda, M)|N_{(\Lambda,M)} = 1\}$ and let the $\mathcal{L}$ and $\mathcal{M}$ be sets of all Λ and M, respectively, that appear in any of the pairs from $\mathcal{N}$. Then pairs from $\mathcal{N}$ naturally define bijection from $\mathcal{L}$ to $\mathcal{M}$, $\mathcal{N}: \mathcal{L} \to M$.*

The theorem is proved by explicit construction of the bijection $\mathcal{N}$. First we must go through some preliminary definitions and lemmas.

Corollary 1. *If $osp(1|2n)$ representation Λ appears in the decomposition of the space $\mathcal{H}$, then its multiplicity in the decomposition is given by the dimension of the gauge group representation $\mathcal{N}(\Lambda)$.*

4 Root Systems

At this point we must introduce root systems, both for $osp(1|2n)$ superalgebra and for the $so(p)$ algebra of the gauge group.

We choose basis of a Cartan subalgebra $\mathfrak{h}_{osp}$ of (complexified) $osp(1|2n)$ as:

$$\mathfrak{h}_{osp} = l.s.\left\{\frac{1}{2}\{a^\dagger_\alpha, a_\alpha\}, \alpha = 1, 2, \ldots n\right\}. \tag{11}$$

Positive roots, expressed using elementary functionals, are:

$$\begin{aligned}\Delta^+_{osp} = \{&+\delta_\alpha, 1 \le \alpha \le n; +\delta_\alpha + \delta_\beta, 1 \le \alpha < \beta \le n;\\ &+\delta_\alpha - \delta_\beta, 1 \le \alpha < \beta \le n; +2\delta_\alpha, 1 \le \alpha \le n\}\end{aligned} \tag{12}$$

and the corresponding positive root vectors, spanning subalgebra $\mathfrak{g}^+_{osp}$, are (in the same order):

$$\begin{aligned}\Big\{&a^\dagger_\alpha, 1 \le \alpha \le n; \{a^\dagger_\alpha, a^\dagger_\beta\}, 1 \le \alpha < \beta \le n;\\ &\{a^\dagger_\alpha, a_\beta\}, 1 \le \alpha < \beta \le n; \{a^\dagger_\alpha, a^\dagger_\alpha\}, 1 \le \alpha \le n\Big\}.\end{aligned} \tag{13}$$

Simple root vectors are:

$$\Big\{\{a_1^\dagger, a_2\}, \{a_2^\dagger, a_3\}, \dots, \{a_{n-1}^\dagger, a_n\}, a_n^\dagger\Big\}. \tag{14}$$

With this choice of positive roots, positive energy UIR's of $osp(1|2n)$ become lowest weight representations. Thus, we will label positive energy UIR's of $osp(1|2n)$ either by their lowest weight

$$\underline{\lambda} = (\underline{\lambda}_1, \underline{\lambda}_2, \dots, \underline{\lambda}_n), \tag{15}$$

or by its signature

$$\Lambda = [d; \Lambda_1, \Lambda_2, \dots, \Lambda_{n-1}] \tag{16}$$

related to the lowest weight $\underline{\lambda}$ by $d = \underline{\lambda}_1$, $\Lambda_\alpha = \underline{\lambda}_{\alpha+1} - \underline{\lambda}_\alpha$. Λ_α are nonnegative integers [14] and spectrum of d is positive and dependant of Λ_α values.

As a basis of Cartan subalgebra $\mathfrak{h}_{so}$ of $so(p)$ we take:

$$\mathfrak{h}_{so} = l.s.\Big\{G^{(k)} \equiv G^{2k-1,2k}, k = 1, 2, \dots q\Big\}, \tag{17}$$

where $q = [p/2]$ is the dimension of Cartan subalgebra (indices $k, l, \dots$ from the middle of alphabet will take values $1, 2, \dots, q$). Positive roots in case of even p are:

$$\Delta_{so}^+ = \{+\delta_k + \delta_l, 1 \le k < l \le q; +\delta_k - \delta_l, 1 \le k < l \le q\}, \tag{18}$$

while in the odd case we additionally have $\{+\delta_k, 1 \le k \le q\}$.

In accordance with the choice of Cartan subalgebra $\mathfrak{h}_{so}$ it is more convenient to use the following linear combinations:

$$B_{\alpha\pm}^{(k)\dagger} \equiv \frac{1}{\sqrt{2}}(b_\alpha^{2k-1\dagger} \pm i b_\alpha^{2k\dagger}), \qquad B_{\alpha\pm}^{(k)} = \frac{1}{\sqrt{2}}(b_\alpha^{2k-1} \mp i b_\alpha^{2k}), \tag{19}$$

instead of $b^\dagger$ and b, as $[G^{(k)}, B_{\alpha\pm}^{(l)\dagger}] = \pm\delta^{kl} B_{\alpha\pm}^{(l)\dagger}$ and $[G^{(k)}, B_{\alpha\pm}^{(l)}] = \mp\delta^{kl} B_{\alpha\pm}^{(l)}$. Similarly, we introduce $e_\pm^{(k)} \equiv \frac{1}{\sqrt{2}}(e^{2k-1} \pm i e^{2k})$ that satisfy:

$$[G^{(k)}, e_\pm^{(l)}] = \pm\delta^{kl} e_\pm^{(l)}. \tag{20}$$

Odd superalgebra operators take form:

$$a_\alpha^\dagger = \Big(\sum_{k=1}^{q} B_{\alpha+}^{(k)\dagger} e_-^{(k)} + B_{\alpha-}^{(k)\dagger} e_+^{(k)}\Big) + \epsilon\, b_\alpha^{p\dagger} e^p, \tag{21}$$

$$a_\alpha = \Big(\sum_{k=1}^{q} B_{\alpha+}^{(k)} e_+^{(k)} + B_{\alpha-}^{(k)} e_-^{(k)}\Big) + \epsilon\, b_\alpha^p e^p, \tag{22}$$

where $\epsilon = p \mod 2$.

The space $\mathcal{H}$ decomposes to spinorial UIR's of $so(p)$ with the highest weight $\overline{\mu} = (\overline{\mu}^1, \overline{\mu}^2, \ldots, \overline{\mu}^q)$ satisfying $\overline{\mu}^1 \geq \overline{\mu}^2 \geq \cdots \geq \overline{\mu}^{q-1} \geq |\overline{\mu}^q| \geq \frac{1}{2}$ with all $\overline{\mu}^q$ taking half-integer values ($\overline{\mu}^q$ can take negative values when p is even). However, since the gauge symmetry group in the case of even p is enlarged to $Pin(p)$ group, any highest weight of UIR of the gauge group satisfies: $\overline{\mu}^1 \geq \overline{\mu}^2 \geq \cdots \geq \overline{\mu}^q \geq 0$. As the gauge group representation in $\mathcal{H}$ is spinorial, all $\overline{\mu}^k$ take half-integer values greater or equal to $\frac{1}{2}$. To label UIR's of the gauge group we will also use signature

$$M = [M^1, M^2, \ldots, M^q] \tag{23}$$

with $M^k = \overline{\mu}^k - \overline{\mu}^{k+1}, k < q$ and $M^q = \overline{\mu}^q - \frac{1}{2}$. All M^k are nonnegative integers.

The "spin" factor space $\mathcal{H}^{\mathbf{s}}$ is irreducible w.r.t. action of the gauge group. Gauge group representation in the space $\mathcal{H}^{\mathbf{s}}$ has the highest weight $\overline{\mu}_{\mathbf{s}} = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$. Weight spaces of this representation are one dimensional, meaning that basis vectors can be fully specified by weights $\mu_{\mathbf{s}}$:

$$\mathcal{H}^{\mathbf{s}} = l.s.\{\omega_{\mu_{\mathbf{s}}} \equiv \omega(\mu_{\mathbf{s}}^1, \mu_{\mathbf{s}}^2, \ldots, \mu_{\mathbf{s}}^q) | \mu_{\mathbf{s}}^k = \pm\frac{1}{2}\}. \tag{24}$$

An action of operators $e_+^{(k)}, e_-^{(k)}$ and e^p in this basis is given by:

$$e_\pm^{(k)} \omega(\mu_{\mathbf{s}}^1, \mu_{\mathbf{s}}^2, \ldots, \mu_{\mathbf{s}}^q) = \sqrt{2}\left(\prod_{l=1}^{k-1} 2\mu_{\mathbf{s}}^l\right) \omega(\mu_{\mathbf{s}}^1, \ldots, \mu_{\mathbf{s}}^{k-1}, \mu_{\mathbf{s}}^k \pm 1, \mu_{\mathbf{s}}^{k+1}, \ldots, \mu_{\mathbf{s}}^q) \tag{25}$$

and, when p is odd, also:

$$e^p \omega(\mu_{\mathbf{s}}^1, \mu_{\mathbf{s}}^2, \ldots, \mu_{\mathbf{s}}^q) = \left(\prod_{l=1}^{q} 2\mu_{\mathbf{s}}^l\right) \omega(\mu_{\mathbf{s}}^1, \mu_{\mathbf{s}}^2, \ldots, \mu_{\mathbf{s}}^q). \tag{26}$$

In these definitions it is implied that $\omega(\mu_{\mathbf{s}}^1, \mu_{\mathbf{s}}^2, \ldots, \mu_{\mathbf{s}}^q) \equiv 0$ if any $|\mu_{\mathbf{s}}^k| > \frac{1}{2}$.

Gauge group representation in "orbital" factor space $\mathcal{H}^{\mathbf{o}}$ decomposes to highest weight $\overline{\mu}_{\mathbf{o}}$ UIR's such that all $\overline{\mu}_{\mathbf{o}}^k$ are nonnegative integers. Besides, it is not difficult to verify that, if $n < q$, then

$$\overline{\mu}_{\mathbf{o}}^{n+1} = \overline{\mu}_{\mathbf{o}}^{n+2} = \cdots = \overline{\mu}_{\mathbf{o}}^q = 0 \tag{27}$$

(since maximally n operators (19) can be antisymmetrized).

5 Decomposition of the Green's Ansatz Space

Now we can formulate the following lemma that is the remaining step necessary for the proof of Theorem 1.

Lemma 1. *The vector* $|(\underline{\lambda}, \underline{\lambda}), (\overline{\mu}, \overline{\mu}), \eta_{(\underline{\lambda},\overline{\mu})}\rangle \in \mathcal{H}$ *that is the lowest weight vector of* $osp(1|2n)$ *positive energy UIR* $\underline{\lambda}$ *and the highest weight vector of the gauge group UIR* $\overline{\mu}$ *exists if and only if signatures* Λ *and* M *(16, 23) satisfy:*

$$M_k = \Lambda_{n-k}, \tag{28}$$

where $\Lambda_0 \equiv d - p/2$ *and it is implied that* $M_k = 0, k > q$ *and* $\Lambda_\alpha = 0, \alpha < 0$. *In that case this vector has the following explicit form (up to multiplicative constant) in the basis (7):*

$$|(\underline{\lambda}, \underline{\lambda}), (\overline{\mu}, \overline{\mu}), \eta_{(\underline{\lambda},\overline{\mu})}\rangle = \left(B_{n+}^{(1)\dagger}\right)^{\Lambda_{n-1}} \left(B_{n+}^{(1)\dagger} B_{n-1+}^{(2)\dagger} - B_{n+}^{(2)\dagger} B_{n-1+}^{(1)\dagger}\right)^{\Lambda_{n-2}} \cdots$$

$$\cdot \left(\sum_{k_1,k_2,\dots k_n=1}^{\min(n,q)} \varepsilon_{k_1 k_2 \dots k_n} B_{n+}^{(k_1)\dagger} B_{n-1+}^{(k_2)\dagger} \cdots B_{1+}^{(k_n)\dagger} \right)^{\Lambda_0} |0\rangle \otimes \omega(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}). \tag{29}$$

We will omit a rather lengthy proof of the lemma.

Note that the Lemma 1 also determines whether an osp representation Λ appears or not in the decomposition of Green's Ansatz of order p: UIR Λ appears in the decomposition if and only if the condition (28) can be satisfied by allowed integer values of M_k. However, if q is not sufficiently high, the first $n - q$ of the Λ components $\Lambda_0, \Lambda_1, \dots \Lambda_{n-q-1}$ are bound to be zero.

Corollary 2. *All (half)integer positive energy UIR's of* $osp(1|2n)$ *can be constructed in space* $\mathcal{H}$ *with* $p \leq 2n + 1$.

Proof. Due to relation (28), values $\Lambda_0, \Lambda_1, \dots \Lambda_{n-1}$ can be arbitrary integers when $q \geq n$: choice $p = 2n$ contains integer values of d UIR's while $p = 2n + 1$ contains half-integer values. That spaces $\mathcal{H}$ for some $p < 2n$ also contain all UIR's with $d < n$, can be verified by checking the list of all positive energy UIR's of $osp(1|2n)$ will be given elsewhere. □

In other words, the above corollary states that no additional (half)integer energy UIR's of $osp(1|2n)$ appear when considering $p > 2n + 1$, i.e. it is sufficient to consider only $p \leq 2n + 1$.

The proof of the **Theorem 1** now follows from the Lemma 1.

Proof. Lemma 1 gives the explicit form of the vector that is the lowest weight vector of $osp(1|2n)$ positive energy UIR $\underline{\lambda}$ and the highest weight vector of the gauge group UIR $\overline{\mu}$, when such vector exists. It follows that there can be at most one such vector. Therefore, the multiplicity $N_{(\underline{\lambda},\overline{\mu})}$ can be either 1 or 0. The relation between $\underline{\lambda}$ and $\overline{\mu}$ is given by (28) and it defines bijection $\mathcal{N}$. □

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Parafermionic Algebras, Their Modules and Cohomologies

Todor Popov

Abstract We explore the Fock spaces of the parafermionic algebra introduced by H.S. Green. Each parafermionic Fock space allows for a free minimal resolution by graded modules of the graded two-step nilpotent subalgebra of the parafermionic creation operators. Such a free resolution is constructed with the help of a classical Kostant's theorem computing Lie algebra cohomologies of the nilpotent subalgebra with values in the parafermionic Fock space. The Euler-Poincaré characteristic of the parafermionic Fock space free resolution yields some interesting identities between Schur polynomials. Finally we briefly comment on parabosonic and general parastatistics Fock spaces.

1 Introduction

The parafermionic and parabosonic algebras were introduced by H.S. Green as inhomogeneous cubic algebras having as quotients the fermionic and bosonic algebras with canonical (anti)commutation relations. In an attempt to find a new paradigm for quantization of classical fields H.S. Green introduced the parabosonic and parafermionic algebras [5] encompassing the bosonic and fermionic algebras based on the canonical quantization scheme. Here we are dealing with the Fock spaces of the parafermionic algebra $\mathfrak{g}$ of creation and annihilation operators. These Fock spaces are particular parafermionic algebra modules built at the top of a unique vacuum state by the creation operators. The creation operators close a free graded two-step nilpotent algebra $\mathfrak{n}$, $\mathfrak{n} \subset \mathfrak{g}$. The Fock space of a parafermionic algebra $\mathfrak{g}$ is then defined as a quotient module of the free $\mathfrak{n}$-module, where the quotient ideal stems from the generalization of the Pauli exclusion principle. In this note we calculate the cohomologies $H^\bullet(\mathfrak{n}, \mathcal{V}(p))$ of the nilpotent subalgebra $\mathfrak{n}$ with coefficients in the parafermionic Fock space $\mathcal{V}(p)$ (taken as a $\mathfrak{n}$-module). The cohomology ring $H^\bullet(\mathfrak{n}, \mathcal{V}(p))$ is obtained due to by now classical Kostant's theorem [8]. With the data of $H^\bullet(\mathfrak{n}, \mathcal{V}(p))$ one is able to construct a minimal resolution by free $\mathfrak{n}$-module of the Fock space $\mathcal{V}(p)$. Its existence is guaranteed by

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the Henri Cartan's results on graded algebras. It turns out that the Schur polynomials identities which have been recently put forward [9, 13] by Neli Stoilova and Joris Van der Jeugt stem from the Euler-Poincaré characteristic of the minimal free resolutions of the parafermionic and parabosonic Fock space.

2 Parafermionic and Parabosonic Algebras

The parafermionic algebra $\mathfrak{g}$ with finite number n degrees of freedom is a Lie algebra with a Lie bracket $[\bullet, \bullet]$ generated by the creation $a_i^\dagger$ and annihilation a^j operators $(i, j = 1, \ldots, n)$ having the following exchange relations

$$\begin{aligned} [[a_i^\dagger, a^j], a_k^\dagger] &= 2\delta_k^j a_i^\dagger \ , \quad [[a_i^\dagger, a^j], a^k] = -2\delta_i^k a^j \ , \\ [[a_i^\dagger, a_j^\dagger], a_k^\dagger] &= \quad 0 \ , \qquad [[a^i, a^j], a^k] = \quad 0 \ . \end{aligned} \tag{1}$$

The parafermionic algebra $\mathfrak{g}$ with finite number degrees of freedom n is isomorphic to the semi-simple Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_\alpha \ , \tag{2}$$

for a root system $\Delta = \Delta_+ \cup \Delta_-$ of type B_n with positive roots Δ_+ given by

$$\Delta_+ = \{e_i\}_{1 \le i \le n} \cup \{e_i + e_j, e_i - e_j\}_{1 \le i < j \le n} \ , \quad \text{and} \quad \Delta_- = -\Delta_+ \ .$$

Here $\{e_i\}_{i=1}^n$ stands for the orthogonal basis in the root space, $(e_i|e_j) = \delta_{ij}$. One concludes that the parafermionic algebra $\mathfrak{g}$ with n degrees of freedom is isomorphic to the orthogonal algebra $\mathfrak{g} \cong \mathfrak{so}_{2n+1}$ endowed with the anti-involution †. The physical generators correspond to the Cartan-Weyl basis $a_i^\dagger := E^{e_i}$ and $a^j := E^{-e_j}$.

Similarly one defines the parabosonic algebra $\tilde{\mathfrak{g}}$ with exchange relations (1) as the Lie super-algebra endowed with a Lie super-bracket $[\bullet, \bullet]$ whose generators $a_i^\dagger$ and a^j are taken to be odd generators. The parabosonic algebra $\tilde{\mathfrak{g}}$ with m degrees of freedom is shown [3] to be isomorphic to the Lie super algebra of type $B_{0,m}$ in the Kac table, i.e., $\mathfrak{osp}_{1|2m}$. More generally, one defines the parastatistics algebra as the Lie super-algebra with n even parafermionic and m odd parabosonic degrees of freedom. The parastatistics algebra is shown to be isomorphic to the super-algebra of type $B_{n,m}$, i.e., $\mathfrak{osp}_{2n+1|2m}$ [12]. Throughout this note we will concentrate on the parafermionic algebra and its representations.

3 Parafermionic Fock Space

The parafermionic relations (1) imply that the generators $E_i^j = \frac{1}{2}[a_i^\dagger, a^j]$ are the matrix units satisfying

$$[E_i^j, E_k^l] = \delta_k^j E_i^l - \delta_i^l E_k^j \ .$$

These generators close the real form $\mathfrak{u}$ of a linear algebra $\mathfrak{gl}_n$ with $(E_i^j)^\dagger = E_j^i$.

One has decomposition of the parafermionic Lie algebra into reductive algebra $\mathfrak{u}$ and nilpotent Lie algebras, $\mathfrak{n}$ and $\mathfrak{n}^*$

$$\mathfrak{g} = \mathfrak{n}^* \rtimes \mathfrak{u} \ltimes \mathfrak{n}$$

where $\mathfrak{u}$ is the real form of the linear algebra $\mathfrak{gl}_n$. The free two-step nilpotent Lie subalgebra $\mathfrak{n} \subset \mathfrak{g}$ is generated in degree 1 by the *creation* operators $a_i^\dagger$, $V := \bigoplus_i \mathbb{C}a_i^\dagger$

$$\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 = V \oplus \wedge^2 V \ .$$

Analogously the annihilation operators a_i generate the subalgebra $\mathfrak{n}^* = V^* \oplus \wedge^2 V^*$.

The vector space $V = \mathfrak{n}_1$ is the fundamental representation for the left action of the algebra $\mathfrak{gl}_n$, $E_i^j \cdot a_k^\dagger = \delta_k^j a_i^\dagger$. Similarly $V^* = \mathfrak{n}_1^*$ is the fundamental representation for the right $\mathfrak{gl}_n$-action, $a^k \cdot E_i^j = \delta_i^k a^j$. The linear algebra $\mathfrak{gl}_n$ acts on the algebras $\mathfrak{n}$ and $\mathfrak{n}^*$ by automorphisms.

Definition 1. The parafermionic Fock space is the unitary representation $\mathcal{V}(p)$ of the parafermionic algebra $\mathfrak{g} \cong \mathfrak{so}_{2n+1}$ built on a unique vacuum vector $|0\rangle$ such that

$$a_i\,|0\rangle = 0\ , \qquad [a_i, a_j^\dagger]\,|0\rangle = p\delta_{ij}\,|0\rangle\ . \tag{3}$$

The non-negative integer p is called the order of the parastatistics.

Let us single out a particular parabolic subalgebra $\mathfrak{p} = \mathfrak{gl} \ltimes \mathfrak{n}$. In the Fock representation the vacuum module $\mathbb{C}|0\rangle$ is the trivial module for the subalgebra $\mathfrak{p}^* = \mathfrak{n}^* \rtimes \mathfrak{gl}$. The representation induced by $\mathfrak{p}^*$ acting on the vacuum module is isomorphic the universal enveloping algebra of the creation algebra $\mathfrak{n}$

$$\mathrm{Ind}_{\mathfrak{p}^*}^{\mathfrak{g}} \mathbb{C}|0\rangle = U\mathfrak{g} \otimes_{\mathfrak{p}^*} \mathbb{C}|0\rangle \cong U\mathfrak{n}\ .$$

Hence the Fock representation $\mathcal{V}(p)$ which we now describe is a particular quotient of the algebra $U\mathfrak{n}$ created by the free action of the creation algebra $\mathfrak{n}$.

The $\mathcal{V}(p)$ of parastatistics order p is a finite-dimensional $\mathfrak{g}$-module with a unique Lowest Weight vector $|0\rangle$ of weight $-\frac{p}{2}\sum_{i=1}^{n} e_i$ and a unique Highest Weight (HW) vector

$$|\Lambda\rangle = (a_1^\dagger)^p \dots (a_n^\dagger)^p |0\rangle \tag{4}$$

thus the $\mathfrak{so}_{2n+1}$-module $\mathcal{V}(p)$ is a highest weight module of weight Λ

$$V^\Lambda = \mathcal{V}(p) \qquad\qquad \Lambda = \frac{p}{2}\sum_{i=1}^{n} e_i \ .$$

The parafermionic algebra of order $p = 1$ coincides with the canonical fermionic Fock space, i.e., the HW representation $\mathcal{V}(1) = V^\theta$ with $\theta = \frac{1}{2}\sum_{i=1}^{n} e_i$. The physical meaning of the order p for the parafermionic algebra is the number of particles that can occupy one and the same state, that is, we deal with a Pauli exclusion principle of order p. The symmetric submodule $S^{p+1}\mathfrak{n}_1 \subset \mathfrak{n}_1^{\otimes p+1}$ is spanned by the "exclusion condition" $(a_i^\dagger)^{p+1} = 0$ and it generates an ideal $(S^{p+1}\mathfrak{n}_1)$. The parafermionic Fock space $\mathcal{V}(p)$ is a Lowest Weight module isomorphic to the factor module of $U\mathfrak{n}$ by the "exclusion" ideal $(S^{p+1}\mathfrak{n}_1)$

$$\mathcal{V}(p) \cong U\mathfrak{n}/(S^{p+1}\mathfrak{n}_1) \ .$$

On the other hand the parafermionic Fock space $\mathcal{V}(p) = V^\Lambda$ is a HW $\mathfrak{g}$-module with HW vector $|\Lambda\rangle$ (4)

$$V^\Lambda \cong U\mathfrak{n}^*/(S^{p+1}\mathfrak{n}_1^*)) = \mathcal{V}(p) \ .$$

Theorem 1 (A.J. Bracken, H.S. Green[1]). *The HW $\mathfrak{so}_{2n+1}$-module $V^\Lambda \cong \mathcal{V}(p)$ of HW vector $|\Lambda\rangle = |p\theta\rangle$ splits into a sum of irreducible $\mathfrak{gl}_n$-modules V^λ*

$$V^\Lambda \downarrow^{\mathfrak{so}_{2n+1}}_{\mathfrak{gl}_n} = \bigoplus_{\lambda:\lambda\subseteq(p^n)} V^{\lambda-(p/2)^n} \ , \qquad\qquad \Lambda = \frac{p}{2}\sum_{i=1}^{n} e_i \tag{5}$$

where the sum runs over all partitions which match inside the Young diagram (p^n).

Proof. The Weyl character formula applied to a Schur module V^λ yields the Schur polynomial

$$s_\lambda(x_1,\dots,x_n) = \sum_{w\in W_1} \varepsilon(w)e^{w(\rho_1+\lambda)} / \sum_{w\in W_1} \varepsilon(w)e^{w(\rho_1)} \qquad W_1 := S_n \ ,$$

where the variables are $x_i := \exp(-e_i)$ and the vector $\rho_1 = \frac{1}{2}\sum_{i=1}^{n}(n-2i+1)e_i$. Alternatively the Schur polynomial is written as a quotient of determinants

$$s_\lambda(x_1,\dots,n_n) = \frac{\det||x_j^{\rho_{1i}+\lambda_i}||}{\det||x_j^{\rho_{1i}}||}\,. \tag{6}$$

The Weyl character formula applied to the $\mathfrak{so}_{2n+1}$-module V^Λ reads

$$\chi^\Lambda = D_{\rho+p\theta}/D_\rho = e^{p\theta}\sum_{\lambda:\,l(\lambda')\le p} s_\lambda(x_1,\dots,x_n)\,,\qquad e^{p\theta} = (x_1\dots x_n)^{-\frac{p}{2}} \tag{7}$$

where $W = S_n \ltimes \mathbb{Z}_2^n$ is the Weyl group of the root system of Dynkin type B_n and $D_\rho = \sum_{w\in W}\varepsilon(w)e^{w\rho}$ with $\rho = \frac{1}{2}\sum_{i=1}^n(2n-2i+1)e_i$. The quotient of determinants $D_{\rho+p\theta}/D_\rho$ can be further expanded as a sum over the Schur polynomials with no more than p columns (see p. 84 in the book of Macdonald [11]). Here λ' stands for the partition conjugated to λ and $l(\mu)$ is the length of the partition μ. The Schur polynomials $s_\lambda(x)$ are characters of the $\mathfrak{gl}_n$-modules thus the expansion of the $\mathfrak{so}_{2n+1}$-character χ^Λ implies the branching formula (5). We are done. □

4 Kostant's Theorem and the Cohomology $H^\bullet(\mathfrak{n},\mathcal{V}(p))$

The Kostant theorem is a powerful tool helping to calculate cohomologies. Let's have a semi-simple algebra $\mathfrak{g}$ and its Borel subalgebra $\mathfrak{b} = \mathfrak{h}\oplus\bigoplus_{\alpha\in\Delta_+}\mathfrak{g}_\alpha$. Any parabolic subalgebra $\mathfrak{p}$, $\mathfrak{g}\supset\mathfrak{p}\supseteq\mathfrak{b}$ has a Levi decomposition $\mathfrak{p} = \mathfrak{g}_1\ltimes\mathfrak{n}$ where $\mathfrak{g}_1$ is a reductive algebra and $\mathfrak{n}$ is the nilradical (largest nilpotent ideal) of $\mathfrak{p}$. Consider the $\mathfrak{g}$-module V^Λ of weight Λ and the cohomology $H^\bullet(\mathfrak{n},V^\Lambda)$ with coefficients in the restriction $\mathfrak{n}$-module $V^\Lambda\downarrow^{\mathfrak{g}}_{\mathfrak{n}}$. The Kostant's theorem gives the decomposition of $H^\bullet(\mathfrak{n},V^\Lambda)$ as a sum of irreducibles $\mathfrak{g}_1$-modules V^μ.

Theorem 2 (Kostant). *Let W be the Weyl group of the algebra $\mathfrak{g}$ and the subset $\Phi_\sigma\subseteq\Delta_+$ be*

$$\Phi_\sigma := \sigma\Delta_-\cap\Delta_+\subseteq\Delta_+\,.$$

Let ρ be the Weyl vector $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$. The roots of the nilpotent radical $\mathfrak{n}$ are denoted as $\Delta(\mathfrak{n})$ and the subset $W^1 = \{\sigma\in W|\Phi_\sigma\subset\Delta(\mathfrak{n})\}$ is a cross section of the coset $W_1\backslash W$. The cohomology $H^\bullet(\mathfrak{n},V^\Lambda)$ has a decomposition into irreducible $\mathfrak{g}_1$-modules V^μ

$$H^\bullet(\mathfrak{n},V^\Lambda) = \bigoplus_{\sigma\in W^1} V^{\sigma(\rho+\Lambda)-\rho}$$

where the cohomological degree of $H^j(\mathfrak{n})$ is the number of the elements $j := \#\Phi_\sigma$.

J. Grassberger, A. King and P. Tirao [4] applied Kostant's theorem to cohomology $H^\bullet(\mathfrak{n}, \mathbb{C})$ with trivial coefficients. Here we extend their method for cohomologies with coefficients in the parafermionic Fock space $\mathcal{V}(p)$, $H^\bullet(\mathfrak{n}, \mathcal{V}(p))$.

Theorem 3. *Let* $\mathfrak{n}$ *be the free two-step nilpotent Lie algebra* $\mathfrak{n} = V \oplus \wedge^2 V$ *and* V^Λ *be the parafermionic Fock space,* $V^\Lambda = \mathcal{V}(p)$ *. The cohomology* $H^\bullet(\mathfrak{n}, V^\Lambda)$ *with values in the* $\mathfrak{n}$*-module* $V^\Lambda \downarrow^{\mathfrak{g}}_{\mathfrak{n}}$ *has a decomposition into irreducible* $\mathfrak{gl}(V)$*-modules*

$$H^k(\mathfrak{n}, \mathcal{V}(p)) \cong \bigoplus_{\mu:\mu=\mu'} V^{*\mu^{(p)}-(\frac{p}{2})^n}, \qquad k = \frac{1}{2}(|\mu| + r(\mu)) \,, \tag{8}$$

where the sum is over self-conjugated Young diagrams $\mu = (\alpha|\alpha)$ *and the notation* $\mu^{(p)}$ *stays for the* p*-augmented diagram* $\mu^{(p)} = (\alpha + p|\alpha)$.

We recall the Frobenius notation for a Young diagram η

$$\eta := (\alpha_1, \ldots, \alpha_r|\beta_1, \ldots, \beta_r) \qquad r = r(\eta)$$

where the *rank* $r(\eta)$ is the number of boxes on the diagonal of η, the arm-length α_i is the number of boxes on the right of the ith diagonal box, and the leg-length β_i is the number of boxes below the ith diagonal box. The overall number of boxes in η is $|\eta| = r + \sum_{i=1}^r \alpha_i + \sum_{i=1}^r \beta_i$. The conjugated diagram η' is the diagram in which the arms and legs are exchanged

$$\eta' := (\beta_1, \ldots, \beta_r|\alpha_1, \ldots, \alpha_r) \,.$$

Proof. The parafermionic algebra $\mathfrak{g} \cong \mathfrak{so}_{2n+1}$ has Cartan decomposition (2). Consider its parabolic subalgebra $\mathfrak{p} = \bigoplus_{i>j} \mathfrak{g}_{e_i - e_j} \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha \subset \mathfrak{g}$. From the parafermionic relations (1) is readily seen that the Levi decomposition of the parabolic subalgebra $\mathfrak{p} = \mathfrak{g}_1 \ltimes \mathfrak{n}$ has reductive component

$$\mathfrak{g}_1 = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathfrak{g}_{e_i - e_j} \;\cong \mathfrak{gl}_n \tag{9}$$

acting by automorphisms on the free two-step nilpotent algebra $\mathfrak{n}$ (the space $\mathfrak{n}_1 = V$ being the fundamental representation of $\mathfrak{g}_1 = \mathfrak{gl}_n$)

$$\mathfrak{n} = \bigoplus_i \mathfrak{g}_{e_i} \oplus \bigoplus_{i<j} \mathfrak{g}_{e_i + e_j} \cong V \oplus \wedge^2 V \,. \tag{10}$$

The Weyl group W_1 of $\mathfrak{g}_1 = \mathfrak{gl}_n$ is the symmetric group S_n operating on $\{e_1, \ldots, e_n\}$ by permutations. The Weyl group of $\mathfrak{g} = \mathfrak{so}_{2n+1}$ is $W = S_n \ltimes \mathbb{Z}_2^n$. The $\mathbb{Z}_2^n$ is generated by operators τ_i, $i = 1, \ldots, n$ such that $\tau_i^2 = 1$ acting by

$$\tau_i(e_j) = \begin{cases} -e_j & i = j \\ e_j & i \neq j \end{cases} .$$

The elements $\tau_I \in \mathbb{Z}_2^n$ are indexed by subsets $I \subseteq \{1, \ldots, n\}$, $\tau_I \in \prod_{i \in I} \tau_i$.

Let us describe the subset W^1 which has order $|W^1| = 2^n$. Both W^1 and $\mathbb{Z}_2^n$ are cross sections of $W_1 \backslash W$ thus for each $\tau_I \in \mathbb{Z}_2^n$ exists a unique permutation $\omega_I \in S_n$ such that $\omega_I \tau_I \in W^1$.

Let $\mathfrak{b}^0$ be the nilpotent part of the Borel algebra $\mathfrak{b}^0 = \mathfrak{b}/\mathfrak{h}$ and the complement be $\mathfrak{m}_1 = \mathfrak{g}_1 \cap \mathfrak{b}^0 = \mathfrak{b}^0/\mathfrak{n}$. The subset $W^1 = \{\sigma \in W | \Phi_\sigma \subseteq \Delta(\mathfrak{n})\}$ keeps stable also the complement of $\Delta(\mathfrak{n})$

$$\sigma\Delta(\mathfrak{n}) \subseteq \Delta_+ \qquad \Leftrightarrow \qquad \sigma^{-1}\Delta(\mathfrak{b}^0/\mathfrak{n}) \subseteq \Delta_+ .$$

The root system of $\mathfrak{m}_1$ is $\Delta(\mathfrak{m}_1) = \{e_i - e_j, i < j\}$ therefore $\omega_I \tau_I \in W^1$ implies $\tau_I^{-1}\omega_I^{-1}\Delta(\mathfrak{m}_1) \subseteq \Delta_+$ or $\tau_I \omega_I^{-1}(e_i - e_j) > 0$ for $i < j$. These inequalities are satisfied for $\omega_I \in S_n$ defined by

$$\omega_I(a) > \omega_I(b) \quad \text{when} \quad \begin{cases} a < b & a \in I \;\; b \in I \\ a > b & a \notin I \;\; b \notin I \\ & a \in I \;\; b \notin I \end{cases} .$$

The permutation places all elements of $I = \{i_1, \ldots i_r\}$ after all the elements of its complement $\bar{I}$ preserving the order of $\bar{I}$ and reversing the order of I, that is,

$$\omega_I(1, \ldots, i_1, \ldots, i_r, \ldots, n) = (1, \ldots, \hat{i}_1, \ldots, \hat{i}_r, \ldots, n, i_r, \ldots, i_2, i_1) . \tag{11}$$

The permutation ω_I can be represented as a product of cyclic permutations $\omega_I = \zeta_{i_r} \ldots \zeta_{i_2}\zeta_{i_1}$ where ζ_{i_k} is the cycle (of length $n - i_k + 1$) from positions $i_k - k + 1$ to $n - k + 1$. Therefore the action of ω_I is represented by the sequence of steps

$$\begin{aligned}
\zeta_{i_1}(1, \ldots, i_1, \ldots, i_k, \ldots n) &= (1, \ldots, \hat{i}_1, i_1 + 1, \ldots, n, i_1), \\
\zeta_{i_2}(1, \ldots, \underbrace{i_2}_{\text{place } i_2 - 1}, \ldots, n, i_1) &= (1, \ldots, \hat{i}_2, \ldots, n, i_2, i_1), \\
&\ldots \\
\zeta_{i_k}(1, \ldots, \underbrace{i_k}_{\text{place } i_k - k + 1}, \ldots, n, i_{k-1}, \ldots, i_1) &= (1, \ldots, \hat{i}_k, \ldots, n, i_k, \ldots, i_1) .
\end{aligned}$$

Note that after the jth step, the last j places are not touched by the next cyclings.

The Weyl vector ρ associated to $\mathfrak{g} = \mathfrak{so}_{2n+1}$ reads $\rho = \frac{1}{2}\sum_{i=1}^{n}(2n - 2i + 1)e_i$. Note that the components of ρ are strictly decreasing with step $1 = \rho_{i+1} - \rho_i$. The cohomology ring $H^\bullet(\mathfrak{n}, V^\Lambda)$ decomposes into $\mathfrak{gl}(V)$-modules with HW weights $\sigma(\rho + \Lambda) - \rho$ for $\sigma \in W^1$. We are interested in the case $\Lambda = \frac{p}{2}\sum e_i$, $V^\Lambda = \mathcal{V}(p)$.

Consider first the case $p = 0$, i.e., the cohomology with trivial coefficients $H^\bullet(\mathfrak{n}, \mathbb{C})$ following [4]. The highest weights $\lambda_I = \sigma(\rho) - \rho$ for $\sigma \in W^1$ are non-positive due to $\sigma(\rho)_i \leq \rho_i$. The cycling structure of ω_I implies

$$\lambda_I = \sum \lambda_j e_j , \qquad \lambda_j = -(n - i_{n-j+1} + 1)\chi_{(n-r+1 \leq j \leq n)} - \sum_{k=1}^{r} \chi_{(i_k - k + 1 \leq j \leq n-k)} .$$

One has an isomorphism between a HW $\mathfrak{gl}_n$-module V^{λ_I} with negative weight $\lambda_I \leq 0$ and the dual representation $V^{*\mu_I}$ with reflected weight $\mu_I \geq 0$

$$V^{\lambda_I} \cong V^{*\mu_I} \qquad\qquad \mu_I := \sum_{i=1}^{n} \mu_i e_i = -\sum_{i=1}^{n} \lambda_{n-i+1} e_i \geq 0 .$$

The components of μ_I are decreasing positive integers $\mu_1 \geq \ldots \geq \mu_n \geq 0$

$$\mu_j = (n - i_j + 1)\chi_{(1 \leq j \leq r)} + \sum_{k=1}^{r} \chi_{(k+1 \leq j \leq n - i_k + k)} , \tag{12}$$

and these components code a self-conjugated Young diagram $\mu'_I = \mu_I$

$$\mu_I = (\alpha_I | \alpha_I) \qquad \alpha_I = (\alpha_1, \ldots, \alpha_r), \quad \text{for} \quad \alpha_j = n - i_j .$$

Roughly speaking the jth cyclic permutation ζ_{i_k} in ω_I creates a self-conjugated hook subdiagram of μ_I with $\alpha_j = n - i_j$.

By virtue of the Kostant's theorem [8] the cohomology $H^\bullet(\mathfrak{n}, \mathbb{C})$ of the nilpotent Lie algebra $\mathfrak{n}$ has decomposition into Schur modules with HW vector $|\mu_I\rangle$

$$H^\bullet(\mathfrak{n}, \mathbb{C}) = \bigoplus_{\mu_I : \mu'_I = \mu_I} V^{*\mu_I} , \qquad |\mu_I\rangle = E^{-\Phi_\sigma} , \quad \sigma \in W^1$$

labelled by self-conjugated Young diagrams. All self-conjugated Young diagrams $\{\mu_I : \mu'_I = \mu_I\}$ are in bijection with elements of W^1 (with cardinality $|W^1| = 2^n$), all these diagrams are included into the maximal square diagram, $\mu_I \subseteq (n^n)$.

Consider now the cohomology ring $H^\bullet(\mathfrak{n}, V^\Lambda)$ where $\Lambda = \frac{p}{2} \sum e_i$. It decomposes into $\mathfrak{gl}_n$-modules with HW weights $\lambda_I^{(p)} = \sigma(\rho + \Lambda) - \rho$ where $\sigma = \omega_I \tau_I \in W^1$. Given a set $I = \{i_1, \ldots, i_r\}$ the shift Λ modifies the dominant weight $\nu_I = \sum \nu_i e_i$ to

$$\nu_j^{(p)} = -\lambda_{n-j+1}^{(p)} , \qquad \nu_j^{(p)} = -\frac{p}{2} + (n - i_j + 1 + p)\chi_{(1 \leq j \leq r)} + \sum_{k=1}^{r} \chi_{(k+1 \leq j \leq n - i_k + k)} .$$

The weights $\nu_I^{(p)} = \mu_I^{(p)} - \frac{p}{2}\sum e_i$ fix the HW vectors in the $\mathfrak{gl}_n$-modules $V^{*\nu_I^{(p)}}$

$$V^{*\nu_I^{(p)}} = V^{*\mu_I^{(p)}} \otimes |\Lambda\rangle \quad \text{where} \qquad \mu_I^{(p)} = (\alpha_I + p|\alpha_I) \qquad \alpha_j = n - i_j$$

from where the decomposition of $H^\bullet(\mathfrak{n}, \mathcal{V}(p))$ (8) follows, the sum over $\sigma \in W^1$ in Kostant's theorem being replaced by the sum over self-conjugated Young diagrams $\mu = \mu'$. The arm p-augmented diagram $\mu_I^{(p)}$ stems from the self-conjugated diagram $\mu_I = (\alpha_I|\alpha_I)$ cf. Eq. (12) by augmenting the arm-lengths, $\mu_I^{(p)} = (\alpha_I + p|\alpha_I)$.

The cohomological degree k of the elements in $V^{*\mu_I^{(p)}} \otimes |0\rangle \subset H^k(\mathfrak{n}, \mathcal{V}(p))$ do not depend on p but only on $\sigma = \omega_I \tau_I \in W^1$ (or equivalently on μ_I). In view of $\Phi_\sigma = \Delta_- \cap \sigma^{-1}\Delta_+$ a root $\xi \in \Phi_\sigma \subseteq \Delta(\mathfrak{n})$ whenever $\sigma^{-1}\xi < 0$. But the set $\Delta(\mathfrak{n})$ is stable under permutations and $\tau_I^{-1} = \tau_I$ thus

$$\begin{aligned}\#\Phi_\sigma &= \#\{\xi \in \Delta(\mathfrak{n}), \tau_I\xi < 0\} \\ &= \#\{\mathfrak{g}_{e_i}, i \in I\} + \#\{\mathfrak{g}_{e_i+e_j} : i < j, i \in I\} \\ &= \sum_{i\in I}(1 + n - i) = r + \sum_{k=1}^{r}(n - i_k) = r + s = \deg \mu_I \,.\end{aligned}$$

Thus the cohomological degree $k = \deg \mu_I = \#\Phi_\sigma$ is the total degree $k = (r + s)$ of the bi-complex $\wedge^s(\wedge^2 V^*) \otimes \wedge^s V^*$. The number of boxes above the diagonal in μ_I is $s = \frac{1}{2}(|\mu_I| - r)$ so finally one gets $k = \deg \mu_I = \frac{1}{2}(r(\mu_I) + |\mu_I|)$. We are done. □

5 Resolution of $\mathcal{V}(p)$

A general result of Henri Cartan [2] states that every positively graded $\mathcal{A}$-module M of a graded algebra $\mathcal{A} = \oplus_{n\geq 0}\mathcal{A}_n$ allows for a minimal projective resolution by projective $\mathcal{A}$-modules. Moreover the notions of a projective and a free module coincide in the graded category. Thus for every positively graded $\mathcal{A}$-module M there exists a minimal resolution by free $\mathcal{A}$-modules.

The universal enveloping algebra $U\mathfrak{n}$ is a graded associative algebra and the parafermionic Fock space $\mathcal{V}(p) = V^\Lambda$ is a positively graded $U\mathfrak{n}$-module. There exists [2] a minimal free resolution $P_\bullet = \bigoplus_{k=0}^N P_k$ of the right $U\mathfrak{n}$-module $\mathcal{V}(p)^*$

$$0 \to P_N \to \ldots \to P_1 \to P_0 \to \mathcal{V}(p)^* \to 0 \tag{13}$$

by free right $U\mathfrak{n}$-modules $P_k = E_k \otimes U\mathfrak{n}$. We apply the functor $- \otimes_{U\mathfrak{n}} \mathbb{C}$ on the complex $P_\bullet$, where $\mathbb{C}$ is the trivial $U\mathfrak{n}$-module. The minimality of the resolution

$P_\bullet$ implies [2] that the differentials of the complex $P_\bullet \otimes_{U\mathfrak{n}} \mathbb{C}$ vanish. Hence the multiplicity spaces E_k coincide with the homologies

$$E_k \cong \mathrm{Tor}_k^{U\mathfrak{n}}(\mathcal{V}(p)^*, \mathbb{C}) = H_k(\mathfrak{n}, \mathcal{V}(p)^*) \qquad \Rightarrow \qquad E_k^* \cong H^k(\mathfrak{n}, \mathcal{V}(p)) \ ,$$

where we used the isomorphism $H_k(\mathfrak{n}, M)^* = H^k(\mathfrak{n}, M^*)$. Theorem 3 gives us the spaces $E_k \cong H^k(\mathfrak{n}, \mathcal{V}(p))^*$ so we have constructed the minimal free resolution (13).

Theorem 4. *The Euler-Poincaré characteristic of the free minimal resolution of the (dual of the) parafermionic Fock space* $\mathcal{V}(p)$ *(13) yields the identity*

$$\frac{\sum_{\mu:\mu=\mu'}(-1)^{\frac{1}{2}(|\mu|+r(\mu))} s_{\mu^{(p)}}(x)}{\prod_i (1-x_i)\prod_{i<j}(1-x_ix_j)} = \sum_{\lambda: l(\lambda')\leq p} s_\lambda(x) \ . \tag{14}$$

Proof. In general, the mapping of modules of an algebra into its Grothendieck ring of characters is an example of Poincaré-Euler characteristic. The free resolution (13) is naturally a (reducible) $\mathfrak{gl}(V)$-module and the Schur functions (6) span the ring of $\mathfrak{gl}(V)$-characters. All the homology of a resolution is concentrated in degree 0, hence on the RHS of (14) stays the character of the self-conjugated[1] module $\mathcal{V}(p)$(7)

$$ch\mathcal{V}(p) = ch\mathcal{V}(p)^* = e^{-p\theta}\sum_{\lambda\subseteq(p^n)} s_\lambda(x) \qquad x_i := exp(e_i) \ .$$

From the Poincaré-Birkhoff-Witt theorem follows that the character of P_k reads

$$ch\ P_k = ch(E_k \otimes U\mathfrak{n}) = \frac{e^{-\Lambda} s_{\mu^{(p)}}(x)}{\prod_i(1-x_i)\prod_{i<j}(1-x_ix_j)} \ .$$

Thus the alternating sum on the LHS comes from the characters of the $\mathfrak{gl}(V)$-modules $E_k \otimes U\mathfrak{n}$ taken with alternating signs corresponding to the homological degree. The factor $e^{p\theta} = e^{\Lambda}$ accounting for the weight of the HW vector $|\Lambda\rangle$ cancels which proves the parafermionic sign-alternating identity (14). □

Remark. The free minimal resolution of the trivial module $\mathbb{C}$ constructed by Józefiak and Weyman [6] with the help of the homologies $H_k(\mathfrak{n}, \mathbb{C})$ corresponds to the resolution $P_\bullet$ (13) of $\mathbb{C} \cong \mathcal{V}(p = 0)$.

The parafermionic sign-alternating identity (14) was proposed by Stoilova and Van der Jeugt in their study of parafermionic Fock space [13]. The parabosonic

[1] The self-conjugacy $\mathcal{V}(p) \cong \mathcal{V}(p)^*$ allows to switch between $x_i := exp(\pm e_i)$ without a conflict.

Fock space has been explored in [9] where the "super-symmetric partner" of the identities (14) has been proposed (for a combinatorial proof see [7])

$$\frac{\sum_{\mu:\mu=\mu'}(-1)^{\frac{1}{2}(|\mu|+r(\mu))}s_{[\mu^{(p)}]'}(x)}{\prod_i(1-x_i)\prod_{i<j}(1-x_ix_j)} = \sum_{\lambda:l(\lambda)\leq p} s_\lambda(x)\,. \tag{15}$$

The parity functor Π switches parafermionic *even* generators to parabosonic *odd* generators, thus $\mathfrak{g} = \mathfrak{so}_{2n+1} \xrightarrow{\Pi} \tilde{\mathfrak{g}} = \mathfrak{osp}_{1|2n}$. The effect of Π is the passage $\lambda \xrightarrow{\Pi} \lambda'$. The identity (15) is rooted into a minimal free resolution of the parabosonic Fock space $\tilde{\mathcal{V}}(p) = \Pi\mathcal{V}(p)$ by free $U\tilde{\mathfrak{n}}$-modules of the nilpotent Lie super-algebra $\tilde{\mathfrak{n}} \subset \tilde{\mathfrak{g}}$.

More generally, one can consider the parastatistics Fock space $\mathcal{V}_{n|m}(p)$ of the parastatistics Lie super-algebra $\mathfrak{g}_{n|m} := \mathfrak{osp}_{2n+1|2m}$ with n parafermionic and m parabosonic modes. We conjecture that there exists a complex of free $U\mathfrak{n}_{n|m}$-modules of the maximal nilpotent Lie superalgebra $\mathfrak{n}_{n|m} \subset \mathfrak{osp}_{2n+1|2m}$ whose cohomology is $\mathcal{V}_{n|m}(p)$. Then the Euler-Poincaré characteristics of such a complex will yield one more identity (which was obtained by different method in [10])

$$\frac{\prod_{i<j\,,\hat{\imath}\neq\hat{\jmath}}(1+x_ix_j)\sum_{\mu:\mu=\mu'}(-1)^{\frac{1}{2}(|\mu|+r(\mu))}hs_{\mu^{(p)}}(x)}{\prod_i(1-x_i)\prod_{i<j\,,\hat{\imath}=\hat{\jmath}}(1-x_ix_j)} = \sum_{\lambda:\,\lambda_1\leq p} hs_\lambda(x)\,.$$

Here the $(n|m)$-hook Schur polynomial $hs_\lambda(x)$ is the character of the irreducible $\mathfrak{gl}_{n|m}$-module V^λ, $hs_\lambda(x) = ch\,V^\lambda$. The non-trivial $\mathfrak{gl}_{n|m}$-modules V^λ are labelled by diagrams λ such that $\lambda_{n+1} \leq m$.

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On Non-local Representations of the Ageing Algebra in $d \geq 1$ Dimensions

Stoimen Stoimenov and Malte Henkel

Abstract Non-local representations of the ageing algebra for generic dynamical exponents z and for any space dimension $d \geq 1$ are constructed. The mechanism for the closure of the Lie algebra is explained. The Lie algebra generators contain higher-order differential operators or the Riesz fractional derivative. Covariant two-time response functions are derived. An application to phase-separation in the conserved spherical model is described.

1 Introduction: Ageing Systems and Ageing Algebra

Ageing behaviour has been first studied in structural glasses quenched from a molten state to below "glass-transition temperature" by Struik [32]. Nowadays, ageing has been seen in non-equilibrium relaxations in other glassy and non-glassy system far from equilibrium (see e.g. [6, 16] for surveys). Schematically, one may characterise ageing systems by (1) a slow relaxation dynamics, (2) absence of time-translation-invariance and (3) dynamical scaling.

In this work,[1] we consider the dynamical symmetries of ageing systems undergoing "*simple ageing*", with a dynamics characterised by a single length scale, $L(t) \sim t^{1/z}$ at large times, which defines the *dynamical exponent* z. One may ask if the naturally present dynamical scaling in the long-time limit $t \to \infty$ can be extended to a larger set of local scale transformation, called *"local scale-invariance"* (LSI). The current state of LSI-theory, with its explicit predictions for two-time responses and correlators, has been recently reviewed in detail in [16].

[1]This paper contains the main results from [18], presented by the first author at LT-10.

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Here, we describe an algebraic technique in order to extend known representations of LSI algebras with dynamical exponents $z = 2$ (or $z = 1$) to more general values.

The analysis of the ageing of several simple magnetic systems, without disorder nor frustrations, without any macroscopic conservation law of the dynamics, and undergoing ageing when quenched to a temperature $T < T_c$ below the critical temperature $T_c > 0$ is characterised by the dynamical exponent $z = 2$ [5]. Then, the detailed scaling form of the two-time correlators and responses can be obtained by an extension of simple dynamical scaling with $z = 2$ towards a larger Lie group [13]. Its Lie algebra is known as "*ageing algebra*" $\mathfrak{age}(d) = \left\langle X_{0,1}, Y^{(i)}_{\pm\frac{1}{2}}, M_0, R_{ij}\right\rangle_{1\le i<j\le d}$ and can be defined by the following non-vanishing commutators [15]

$$
\begin{aligned}
&[X_n, Y^{(i)}_m] = \left(\frac{n}{2} - m\right) Y^{(i)}_{n+m}, \quad [X_n, X_{n'}] = (n - n')X_{n+n'},\\
&[Y^{(i)}_{\frac{1}{2}}, Y^{(j)}_{-\frac{1}{2}}] = \delta_{ij} M_0, \quad [R_{ij}, R_{k\ell}] = \delta_{i\ell}R_{jk} + \delta_{jk}R_{i\ell} - \delta_{ik}R_{j\ell} - \delta_{j\ell}R_{ik},\\
&[R_{ij}, Y^{(k)}_m] = \delta_{jk}Y^{(i)}_m - \delta_{ik}Y^{(j)}_m
\end{aligned} \tag{1.1}
$$

with $n, n' = 0, 1$, $m = \pm\frac{1}{2}$ and $1 \le i \le j \le d$. When acting on time-space coordinates $(t, \mathbf{r})$, a representation of (1.1) in terms of affine differential operators is:

$$
\begin{aligned}
X_0 &= -t\partial_t - \frac{1}{2}(\mathbf{r}\cdot\partial_{\mathbf{r}}) - \frac{x}{2}, \quad X_1 = -t^2\partial_t - t(\mathbf{r}\cdot\partial_{\mathbf{r}}) - \frac{\mathcal{M}}{2}\mathbf{r}^2 - (x+\xi)t\\
Y^{(i)}_{-\frac{1}{2}} &= -\partial_{r_i}, \quad Y^{(i)}_{\frac{1}{2}} = -t\partial_{r_i} - \mathcal{M}r_i, \quad M_0 = -\mathcal{M}\\
R_{ij} &= r_i\partial_{r_j} - r_j\partial_{r_i} = -R_{ji}.
\end{aligned} \tag{1.2}
$$

The above representation has a dynamical exponent $z = 2$ and acts *locally* on the time-space coordinates. Furthermore, it generates a set of dynamical symmetries of the Schrödinger (or diffusion) equation:

$$
\hat{S}\phi(t,\mathbf{r}) = \left(2\mathcal{M}\partial_t + \frac{2\mathcal{M}}{t}(x + \xi - d/2) - \nabla^2_{\mathbf{r}}\right)\phi(t,\mathbf{r}) = 0, \tag{1.3}
$$

in the sense that each of the generators of $\mathfrak{age}(d)$ maps a solution of (1.3) onto another solution. The triplet $(\mathcal{M}, x, \xi)$ characterises the solution $\phi = \phi_{(\mathcal{M},x,\xi)}$ of this equation.[2] Furthermore, x and ξ are two *independent* scaling dimensions.

For systems undergoing simple ageing with $z = 2$, LSI as described by the representation (1.2) of $\mathfrak{age}(d)$ indeed gives an appropriate description, including several exactly solved examples where $\xi \neq 0$ is required [15, 16]. The best-known example is the $1D$ Glauber-Ising model quenched to $T = 0$. A main prediction is

[2] $\mathcal{M} \in \mathbb{R}$ is interpreted as an inverse diffusion constant, or as a non-relativistic mass if $\mathcal{M} \in \mathrm{i}\mathbb{R}$.

the form of the two-time (linear) response $R = R(t,s) = \left.\frac{\delta\langle\phi(t)\rangle}{\delta h(s)}\right|_{h=0}$ of the order parameter ϕ with respect to its conjugate magnetic field.

In statistical physics, a common formulation uses a stochastic Langevin equation

$$\partial_t \phi(t,\mathbf{r}) = -D\frac{\delta \mathcal{H}[\phi]}{\delta \phi(t,\mathbf{r})} + \eta(t,\mathbf{r}) \tag{1.4}$$

with a Ginzburg-Landau functional $\mathcal{H}$ and a centred gaussian noise η with a δ-correlated second moment. The standard Janssen-de Dominicis formalism [20, 34] relates this to the equation of motion derived from a dynamical functional $\mathcal{J}[\tilde{\phi},\phi] = \mathcal{J}_0[\tilde{\phi},\phi] + \mathcal{J}_\eta[\tilde{\phi}]$, written in terms of order parameter $\phi = \phi_{\mathcal{M},x,\xi}$ and its conjugate *response operator* $\tilde{\phi} = \tilde{\phi}_{-\mathcal{M},\tilde{x},\tilde{\xi}}$ such that the "deterministic part" $\mathcal{J}_0$ is invariant under the action of the Galilei sub-algebra $\mathfrak{gal}(d) = \left\langle Y^{(i)}_{\pm\frac{1}{2}}, M_0, R_{ij}\right\rangle_{1\leq i<j\leq d}$. This implies the Bargmann super-selection rules [1].

Theorem 1 ([16, 29]). *All n-point functions of "noisy theory" described by $\mathcal{J}$ can be reduced to averages $\langle\cdot\rangle_0$ calculable from the deterministic part $\mathcal{J}_0$ alone.*

In particular the response function $R(t,s) = \langle\phi(t)\tilde{\phi}(s)\rangle = \langle\phi(t)\tilde{\phi}(s)\rangle_0$ (see e.g. [20, 34] for introductions and detailed references), *is independent of the noise* η and can be derived from covariance under $\mathfrak{age}(d)$. These calculations have been carried out for a long list of models undergoing simple ageing with $z = 2$ [2, 8, 16, 30].

Can one extend this procedure, at least for linear stochastic Langevin equations of motion, to arbitrary values of the dynamical exponent z? If we were to restrict to locally realised algebras, the recent classification of the non-relativistic limits of the conformal algebra [7, 9] would only admit the cases (1) $z = 1$: the conformal algebra $conf(d)$ or the conformal Galilean algebra ($cga(d)$) [11, 12, 28], eventually with the exotic central extension for $d = 2$ [23] (2) $z = 2$: the Schrödinger algebra and (3) $z = \infty$; all along with their sub-algebras. Further examples can only be found when looking at non-local representation, of known abstract algebras, that is generators more general than first-order linear (affine) differential operators. Some partial information is already available to serve as a guide:

1. the Galilei-invariance of the non-relativistic equation of motion $\hat{S}\phi = 0$ should be kept (this guarantees the validity of the Bargmann superselection rule, hence the applicability of the theorem above):

$$[Y^{(i)}_{\frac{1}{2}}, Y^{(j)}_{-\frac{1}{2}}] = \delta_{ij} M_0, \quad [\hat{S}, Y^{(j)}_{\pm\frac{1}{2}}] = \lambda^{(j)}_{\pm}\hat{S}, \tag{1.5}$$

   Computation of two-point functions requires some kind of conformal invariance.
2. In the context of LSI, different realisations of generalised symmetry algebras have been constructed by using certain fractional derivatives [13, 14, 16]. The closure of these sets of generators can only be achieved by taking a quotient with

respect to a certain set of "physical" states. Although this has been successfully applied to certain physical models [3, 8] the closing procedure is not completely determined and it is not clear how to obtain the group (finite) transformations.

A distinct and potentially more promising method has been explored in [17]. Therein, new non-local representations of $\mathfrak{age}(1)$ for an integer-valued dynamical exponent $z = n \in N$ were constructed. This reads

$$
\begin{aligned}
X_0 &= -\frac{n}{2}t\partial_t - \frac{1}{2}r\partial_r - \frac{x}{2}, \quad Y_{-\frac{1}{2}} = -\partial_r, \quad M_0 = -\mu \\
Y_{\frac{1}{2}} &= -t\partial_r^{n-1} - \mu r, \quad 2 \le z = n \in N \\
X_1 &= \left(-\frac{n}{2}t^2\partial_t - tr\partial_r - (x+\xi)t\right)\partial_r^{n-2} - \frac{1}{2}\mu r^2
\end{aligned} \tag{1.6}
$$

The commutation relations (1.1) are satisfied except the following

$$
[X_1, Y_{\frac{1}{2}}] = \frac{n-2}{2}t^2\partial_r^{n-3}\hat{S}, \tag{1.7}
$$

Consequently, the algebra is "on shell" algebra that is closed only on quotients with respect to the solution space of the equation

$$
\hat{S}\phi(t,r) = \left(z\mu\partial_t - \partial_r^z + \frac{2\mu}{t}\left(x + \xi - \frac{z-1}{2}\right)\right)\phi(t,r) = 0. \tag{1.8}
$$

The generators (1.6) act as dynamical symmetries [17] of the Eq. (1.8), for $z \in \mathbb{N}$. In the limit $z \to 2$, the usual representation of the ageing algebra is recovered.

In Sect. 2 we shall generalise the above construction to any spatial dimension $d \ge 1$. This transition is not trivial because of non-locality of the generators (1.6). Covariant two-point functions are computed from these non-local representations in Sect. 3. In Sect. 4, we shall apply these results to some simple physical models, namely the kinetic spherical model with a conserved order-parameter and quenched to $T = T_c$ and the Mullins-Herring (or Wolf-Villain) equations of interface growth with mass conservation. The time-space responses are calculated from the non-local representations of $age(d)$, to be compared with the known exact results [3, 22, 24, 31]. We conclude in section "Conclusions".

2 Non-local Representations of $\mathfrak{age}(d)$ in Dimensions $d \geq 1$

It turns out that *only for* $z = 2n$ *even, it is possible to extend the non-local representation of ageing algebra (1.6) to* $d \ge 1$ *dimensions, while this do not work for* $z = 2n+1$ *odd.* A common treatment of both cases requires the use of the *Riesz fractional derivative* [16, 25]. It is defined as a linear operator $\nabla_{\mathbf{r}}^{\alpha}$ acting as follows

$$\nabla_{\mathbf{r}}^{\alpha} f(\mathbf{r}) = \mathrm{i}^{\alpha} \int_{R}^{d} \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} |\mathbf{k}|^{\alpha}\, e^{\mathrm{i}\mathbf{r}\cdot\mathbf{k}}\, \hat{f}(\mathbf{k}), \tag{2.9}$$

where the right-hand side as to be understood in a distribution sense and $\hat{f}(\mathbf{k})$ denotes the Fourier transform. Some elementary properties are: [16]

$$\nabla_{\mathbf{r}}^{\alpha}\nabla_{\mathbf{r}}^{\beta} = \nabla_{\mathbf{r}}^{\alpha+\beta}, \quad \nabla_{\mathbf{r}}^{2} = \sum_{i=1}^{d} \partial_i^2 = \Delta_{\mathbf{r}}, \quad [\nabla_{\mathbf{r}}^{\alpha}, r_i] = \alpha\partial_i \nabla_{\mathbf{r}}^{\alpha-2}$$
$$[\nabla_{\mathbf{r}}^{\alpha}, \mathbf{r}^2] = 2\alpha(\mathbf{r}\cdot\partial_{\mathbf{r}})\nabla_{\mathbf{r}}^{\alpha-2} + \alpha(d+\alpha-2)\nabla_{\mathbf{r}}^{\alpha-2}, \quad \nabla_{\mu\mathbf{r}}^{\alpha} f(\mu\mathbf{r}) = |\mu|^{-\alpha}\nabla_{\mathbf{r}}^{\alpha} f(\mu\mathbf{r}).$$

The Riesz fractional derivative can be viewed as a "square root" of the Laplacian.

Now consider the generators:

$$\begin{aligned} X_0 &:= -\frac{z}{2}t\partial_t - \frac{1}{2}(\mathbf{r}\cdot\partial_{\mathbf{r}}) - \frac{x}{2}, \\ X_1 &:= \left(-\frac{z}{2}t^2\partial_t - t(\mathbf{r}\cdot\partial_{\mathbf{r}}) - (x+\xi)t\right)\nabla_{\mathbf{r}}^{z-2} - \frac{\mu}{2}\mathbf{r}^2 \\ Y_{-1/2}^{(i)} &:= -\partial_i, \quad Y_{+1/2}^{(i)} := -t\partial_i\nabla_{\mathbf{r}}^{z-2} - \mu r_i, \quad M_0 := -\mu \\ R_{ij} &:= r_i\partial_j - r_j\partial_i = -R_{ji}. \end{aligned} \tag{2.10}$$

The commutators (1.1) of $\mathfrak{age}(d)$ are seen to hold true, except for

$$[X_1, Y_{\frac{1}{2}}^{(i)}] = \frac{1}{2}(z-2)t^2\partial_i\nabla_{\mathbf{r}\hat{S}}^{z-4}. \tag{2.11}$$

Hence, the above generators close into a Lie algebra $\mathfrak{age}(d)$ only in the quotient space over solutions of "Schrödinger equation"

$$\hat{S}\phi(t,\mathbf{r}) = \left(z\mu\partial_t - \nabla_{\mathbf{r}}^{z} + 2\mu t^{-1}\left(x+\xi-\frac{1}{2}(d+z-2)\right)\right)\phi(t,\mathbf{r}) = 0. \tag{2.12}$$

This representation of $\mathfrak{age}(d)$ generates dynamical symmetries of the Eq. (2.12) since $[\hat{S}, Y_{-\frac{1}{2}}^{(i)}] = [\hat{S}, Y_{\frac{1}{2}}^{(i)}] = [\hat{S}, M_0] = [\hat{S}, R_{ij}] = 0$ and

$$[\hat{S}, X_0] = -\frac{1}{2}z\hat{S}, \quad [\hat{S}, X_1] = -zt\,\nabla_{\mathbf{r}}^{z-2}\hat{S}.$$

Some comments are in order:

1. the non-locality only enters into the Galilei $Y_{+\frac{1}{2}}^{i}$ and special transformations X_1. For $z = 2n$ even, these non-local generators, as well as invariant equation (2.12) are expressed in powers of the Laplacian

$$Y^{(i)}_{+1/2} := -t\partial_i \Delta_{\mathbf{r}}^{n-1} - \mu r_i$$

$$X_1 := \left(-nt^2\partial_t - t(\mathbf{r}\cdot\partial_{\mathbf{r}}) - (x+\xi)t\right)\Delta_{\mathbf{r}}^{n-1} - \frac{\mu}{2}\mathbf{r}^2, \tag{2.13}$$

$$\hat{S}\phi(t,\mathbf{r}) = \left(2n\mu\partial_t - \Delta^n + 2\mu t^{-1}\left(x+\xi-\frac{1}{2}(d+2n-2)\right)\right)\phi(t,\mathbf{r}) = 0.$$

2. for a dynamical exponent $z \neq 2n$, use of the Riesz fractional derivatives (2.10) is necessary and there is no simple relation to the representations of $\mathfrak{age}(1)$.

Summarising, the representation of $\mathfrak{age}(d)$ proposed here explicitly uses generators acting non-locally on space. In Fourier space, the generators become local, but non-analytic. The special case of an even-valued dynamical exponent appears to have rather special and possibly non-generic properties.

3 Covariant Two-Point Function

Covariance under (2.10) gives the two-point function (with $\phi_i = \phi_{i,(\mu_1,x_1,\xi_1)}(t_i,\mathbf{r}_i)$)

$$F(t_1,t_2,\mathbf{r}_1,\mathbf{r}_2) = \langle\phi_1(t_1,\mathbf{r}_1)\phi_2(t_2,\mathbf{r}_2)\rangle \tag{3.14}$$

The result is (with $\tau = t_1 - t_2$, $y = t_1/t_2$):

$$F = \delta(\mu_1+\mu_2)t_2^{-\frac{x_1+x_2}{z}}(y-1)^{-\frac{2}{z}[\frac{x_1+x_2}{2}+\xi_1+\xi_2-z+2]}y^{-\frac{1}{z}[x_2-x_1+2\xi_2-z+2]}f\left(|\mathbf{r}|^z\tau^{-1}\right). \tag{3.15}$$

where f still has to be found from Galilei-covariance.

Even dynamical exponent $z = 2n$ If $p := |\mathbf{r}|^z/\tau$, Galilei-covariance gives

$$(\tau\partial_{r_j}\Delta_{\mathbf{r}}^{n-1} + \mu r_i)f(p) = r_j\left((2n)^n p^{\frac{n-1}{n}}\partial_p\Delta_p^{n-1} + \mu\right)f(p) = 0. \tag{3.16}$$

and $j = 1,\ldots d$. In particular if

$$\text{n} = 2,$$

a Frobénius series representation leads to

$$f(p) = f_0\, {}_0F_2\left(\frac{1}{2},\frac{1}{2}+\frac{d}{4};-\frac{\mu p}{64}\right) + f_1\, p^{1/2}\, {}_0F_2\left(\frac{3}{2},\frac{d}{4}+1;-\frac{\mu p}{64}\right)$$
$$+ f_2\, p^{1/2-d/4}\, {}_0F_2\left(1-d/4, 3/2-d/4; -\mu p/64\right). \tag{3.17}$$

Generic dynamical exponent matters become simple in Fourier space

$$(\mu\partial_{k_j} + \mathrm{i}^{z-2}\tau k_j|\mathbf{k}|^{z-2})\hat{f}(\tau,\mathbf{k}) = 0 \Rightarrow \hat{f}(\tau,\mathbf{k}) = f_0(\tau)\exp\left[-\frac{\mathrm{i}^{z-2}}{z}\frac{\tau}{\mu}|\mathbf{k}|^z\right] \tag{3.18}$$

This is rewritten in the direct space as follows

$$f(\tau,\mathbf{r}) = \frac{f_0(\tau)}{(2\pi)^d}\int_{\mathbb{R}^d}\mathrm{d}\mathbf{k}\,\exp\left[\mathrm{i}\mathbf{k}\cdot\mathbf{r} - \frac{\mathrm{i}^{z-2}}{z}\frac{\tau}{\mu}|\mathbf{k}|^z\right] = \frac{f_0(\tau)}{(2\pi)^d}I_\beta(\mathbf{r})$$
$$\beta := \alpha\tau = \frac{\mathrm{i}^{z-2}}{z\mu}\tau \in \mathbb{C}, \quad I_\beta(\mathbf{r}) := \int_{\mathbb{R}^d}\mathrm{d}\mathbf{k}\,\exp\left[\mathrm{i}\mathbf{k}\cdot\mathbf{r} - \beta|\mathbf{k}|^z\right] \tag{3.19}$$

Finally we have (with an infinite radius of convergence for $z > 1$)

$$f(\tau,\mathbf{r}) = f_{00}\frac{\Gamma(d/2)}{\Gamma(d/z)}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma\left(\frac{2n+d}{z}\right)}{n!\Gamma\left(n+\frac{d}{2}\right)}\left(\frac{\mathbf{r}^2}{4(\alpha\tau)^{2/z}}\right)^n. \tag{3.20}$$

4 Conserved Spherical Model. Field-Theoretical Description

The spherical model [4] is defined through spin variable $S(t,\mathbf{x}) \in \mathbb{R}$, attached to each site $\mathbf{x}$ of the hyper-cubic lattice $\Lambda \subset \mathbb{Z}^d$ and which satisfy the mean spherical constraint $\left\langle\sum_{\mathbf{x}\in\Lambda} S(t,\mathbf{x})^2\right\rangle = \mathcal{N}$, where $\mathcal{N}$ is the number of sites. The Hamiltonian is $\mathcal{H} = -\sum_{(\mathbf{x},\mathbf{y})} S_\mathbf{x}S_\mathbf{y}$, where the sum is over pairs of nearest neighbours. At equilibrium, a second-order phase transition is observed for $d > 2$ at some $T_c > 0$. The critical exponents have non-mean-field values for $d < 4$ [21]. The dynamics is given by a Langevin equation with a conserved order parameter (model B) [19]

$$\partial_t S(t,\mathbf{x}) = -\nabla_\mathbf{x}^2[\nabla_\mathbf{x}^2 S(t,\mathbf{x}) + \mathfrak{z}(t)S(t,\mathbf{x}) + h(t,\mathbf{x})] + \eta(t,\mathbf{x})$$
$$\langle\eta(t,\mathbf{x})\eta(t',\mathbf{x}')\rangle = -2T_c\nabla_\mathbf{x}^2\delta(t-t')\delta(\mathbf{x}-\mathbf{x}'). \tag{4.21}$$

This is a simple but physically reasonable model (since $\mathfrak{z}(t) \sim 1/t$ for $t \to \infty$) for the kinetics of phase-separation (for example in alloys). A simple variant is the Mullins-Herring/Wolf-Villain model, where one fixes the Lagrange multiplier $\mathfrak{z}(t) = 0$, and which describes the growth of interfaces on a substrate with a conservation of particles along the interface [27, 35]. The correlators and response are studied in detail [3, 10, 22, 24, 31]. Recall the full time-space response in the conserved spherical model for $d > 4$, or equivalently in the Mullins-Herring model for any d

$$R(t,s;\mathbf{r}) = \frac{\sqrt{\pi}}{2^{3d/2}\pi^{d/2}\Gamma(d/4)}(t-s)^{-(d+2)/4}\left[{}_0F_2\left(\frac{1}{2},\frac{d}{4};\frac{\mathbf{r}^4}{256(t-s)}\right)\right.$$
$$\left. -\frac{8}{d}\frac{\Gamma(\frac{d}{4}+1)}{\Gamma(\frac{d}{4}+\frac{1}{2})}\left(\frac{\mathbf{r}^2}{16\sqrt{t-s}}\right){}_0F_2\left(\frac{3}{2},\frac{d}{4}+\frac{1}{2};\frac{\mathbf{r}^4}{256(t-s)}\right)\right], \tag{4.22}$$

which we want to compare with the $\mathfrak{age}(d)$-covariant two-point function, obtained above from the non-local representation (3.17) with $z = 4$.

In order to do this, adapt, to the present non-local case, the standard methods of Janssen-de Dominicis theory in non-equilibrium field theory [3], to find a relation between a dynamical symmetry of a deterministic equation with the properties of a solution of a stochastic Langevin equation. The Langevin equation

$$\partial_t\phi = -\frac{1}{4\mu}\nabla_{\mathbf{r}}^2\left(-\nabla_{\mathbf{r}}^2\phi + v(t)\phi + h(t,\mathbf{r})\right) + \eta \tag{4.23}$$
$$\langle\eta(t,\mathbf{r})\eta(t',\mathbf{r}')\rangle = -\frac{T_c}{2\mu}\nabla_{\mathbf{r}}^2\delta(t-t')\delta(\mathbf{r}-\mathbf{r}')$$

can be viewed as equation of motion of the Janssen-de Dominicis action, decomposed into deterministic and stochastic parts $\mathcal{J}(\phi,\tilde{\phi}) = \mathcal{J}_0(\phi,\tilde{\phi}) + \mathcal{J}_\eta(\tilde{\phi})$

$$\mathcal{J}_0(\phi,\tilde{\phi}) = \int \mathrm{d}u\mathrm{d}\mathbf{R}\left[\tilde{\phi}\left(\partial_u - \frac{1}{4\mu}\nabla_{\mathbf{R}}^2(\nabla_{\mathbf{R}}^2 - v(u))\right)\phi + h\nabla_{\mathbf{R}}^2\tilde{\phi}\right] \tag{4.24}$$
$$\mathcal{J}_\eta(\tilde{\phi}) = \frac{T}{4\mu}\int \mathrm{d}u\mathrm{d}\mathbf{R}\,\tilde{\phi}(u,\mathbf{R})(\nabla^2\tilde{\phi}(u,\mathbf{R})) + \mathcal{J}_{init}. \tag{4.25}$$

The averages of an observable $\mathcal{A}$ is given by the functional integral:

$$\langle\mathcal{A}\rangle = \int \mathcal{D}[\phi]\mathcal{D}[\tilde{\phi}]\,\mathcal{A}[\phi]\exp(-\mathcal{J}(\phi,\tilde{\phi})) =: \langle\mathcal{A}\exp(-\mathcal{J}_\eta)\rangle_0. \tag{4.26}$$

In particular for the linear response function we obtain[3]

$$R(t,s;\mathbf{x}-\mathbf{y}) := \left.\frac{\langle\phi(t,\mathbf{x})\rangle}{\delta h(s,\mathbf{y})}\right|_{h=0} = \langle\phi(t,\mathbf{x})\nabla_{\mathbf{y}}^2\tilde{\phi}(s,\mathbf{y})\exp(-\mathcal{J}_\eta)\rangle_0$$
$$= \nabla_{\mathbf{y}}^2\langle\phi(t,\mathbf{x})\tilde{\phi}(s,\mathbf{y})\exp(-\mathcal{J}_\eta)\rangle_0 = \nabla_{\mathbf{r}}^2F^{(2)}(t,s;\mathbf{x}-\mathbf{y}),$$

[3]In order to compute response function, we must introduce small perturbation h (conjugate magnetic field) in the right-hand side of the Eq. (4.23), which respects the conservation law. This generates respectively an additional term in the Janssen-de Dominicis action, which we have written explicitly (4.24).

where $F^{(2)}(t, s; \mathbf{r})$ is the two-point function, found in Sect. 3 with identification $\phi = \phi_{\mu,x,\xi}$ as order parameter and $\tilde{\phi} = \phi_{-\mu,\tilde{x},\tilde{\xi}}$ as response field. In the last line we have used the Bargmann super-selection rule [1], which holds in terms of the "mass" parameter μ, that is $\langle \phi_1(t_1, \mathbf{r}_1) \ldots \phi_n(t_n, \mathbf{r}_n) \rangle_0 = 0$ unless $\mu_1 + \ldots \mu_n = 0$. It is enough to consider the case $v = 0$ which gives rise to conserved spherical model for $d > 4$ and Mullins-Herring model for any d.

We see that the deterministic part of Eq. (4.23) coïncides with "Schrödinger equation" for $z = 4$, if in addition the time-translation invariance is taken into account (i.e. $\mathfrak{z}(t) = 0$), that is the parameters of non-local representation of the ageing algebra must satisfy $x + \xi = \tilde{x} + \tilde{\xi} = (d + 2)/2$. Then

$$
\begin{aligned}
R(t, s; \mathbf{r}) &= (t - s)^{-d/4} \nabla_{\mathbf{r}}^2 f\left(\frac{\mathbf{r}^4}{t - s}\right) = (t - s)^{-(d+2)/4} \Delta_p f(p) \\
&= 4(t - s)^{-(d+2)/4} ((d + 2) p^{\frac{1}{2}} \partial_p + 4 p^{\frac{3}{2}} \partial_p^2) f(p) \\
&= (t - s)^{-(d+2)/4} \times \\
&\quad \times \left[f_1' \; {}_0F_2\left(\frac{1}{2}, \frac{d}{4}; -\frac{\mu p}{64}\right) + f_0' p^{1/2} \; {}_0F_2\left(\frac{3}{2}, \frac{d}{4} + \frac{1}{2}; -\frac{\mu p}{64}\right) \right. \\
&\quad \left. + f_2' p^{1-d/4} \; {}_0F_2\left(\frac{3}{2} - \frac{d}{4}, 2 - \frac{d}{4}; -\frac{\mu p}{64}\right) \right]. \qquad (4.27)
\end{aligned}
$$

Since the response function must be regular at $\mathbf{r} = 0$ and vanish for $|\mathbf{r}| \to \infty$, the third term is eliminated, viz. $f_2' = 0$. The constants f_0' and f_1' can be related by the known long-term behaviour of the hyper-geometric function [17, 36]. Hence one reproduces the exact result (4.22), but now from the covariance under non-local representation of ageing algebra with dynamical exponent $z = 4$.

Conclusions

When trying to construct a closed Lie algebra for generalised scale-transformations with an arbitrary dynamical exponent $z \in \mathbb{R}$, we have been led to consider non-local representations of the ageing algebra $\mathfrak{age}(d)$, for general $d \geq 1$ [17, 18].

It was necessary to slightly extend the usual definition of the notion of *dynamical symmetry*. Conventionally, the infinitesimal generator X of a dynamical symmetry of the equation of motion $\hat{S}\phi = 0$ must satisfy $[\hat{S}, X] = \lambda_X \hat{S}$ as an operator, where λ_X should be a scalar or a function. Here, λ_X may be an *operator* itself. The Lie algebra closes on the quotient space with respect to $\hat{S}\phi = 0$.

(continued)

Several details depend on the value of z :

1. For an odd dynamical exponent $z \geq 2$, the generalisation from the one-dimensional case requires the explicit introduction of some kind of fractional derivative. For our purposes, the Riesz fractional derivative turned out to have the required algebraic properties. In addition, the result derived for the covariant two-point function is compatible with the directly treatable case when z is even, but we are not aware of confirmed physical applications in this case.
2. For z even, the algebra (2.13) contains $d + 1$ non-local generators of generalised Galilei-transformation and special transformations, constructed with linear differential operators of order $z - 1$. By analogy with the $1D$ case [17], we suspect that these might be interpreted as generating transformation of distribution functions of the positions, rather than *bona fide* coordinate transformations. The example studied here (conserved spherical model for $d > 4$ or equivalently in the Mullins-Herring equation for any d) might be the first step towards an understanding how to use such non-local transformations in applications to the non-equilibrium physics of strongly interacting particles.

Extensions to more general representations may be of interest [26].

Recall that in the context of interface growth with conserved dynamics, exactly the kind of non-local generalised Galilei-transformation we have studied here has already been introduced in analysing the stochastic equation (related to molecular beam epitaxy (MBE)), with constants ν, λ and a white noise η

$$\partial_t \phi = -\nabla^2 \left[\nu \nabla^2 \phi + \frac{\lambda}{2} (\nabla \phi)^2 \right] + \eta \tag{4.28}$$

It can be shown that Galilei-invariance leads to a non-trivial hyper-scaling relation, expected to be exact [33]. In particular, they obtain $z = 4$ in $d = 2$ space dimensions. We hope to return to a symmetry analysis of these non-linear equations in the future. In any case, the available evidence that generalised Galilei-invariance could survive the loop expansion is very encouraging.

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Part VIII
Various Mathematical Results

The Quantum Closet

Alon E. Faraggi

Abstract The equivalence postulate approach to quantum mechanics entails a derivation of quantum mechanics from a fundamental geometrical principle. Underlying the formalism there exists a basic cocycle condition, which is invariant under D-dimensional finite Möbius transformations. The invariance of the cocycle condition under finite Möbius transformations implies that space is compact. Additionally, it implies energy quantisation and the undefinability of quantum trajectories. I argue that the decompactification limit coincides with the classical limit. Evidence for the compactness of the universe may exist in the Cosmic Microwave Background Radiation.

1 Introduction

The synthesis of quantum mechanics and general relativity continues to pose an important challenge in the basic understanding of physics. While quantum mechanics accounts with astonishing success for physical observations at the smallest distance scales, general relativity accomplishes a similar feat at the largest. Yet these two mathematical modellings of the observed data are mutually incompatible. This is seen most clearly in relation to the vacuum. The first predicts a value that is off by orders of magnitude from the observed value, which is determined by using the second. To date there is no solution to this problem. In view of this calamity it seems prudent to explore the foundations of each of these theories, and the fundamental principles that underly them. General relativity follows from a basic geometrical principle, the equivalence principle, whereas the basic tenant of quantum mechanics is the probability interpretation of the wave function.

The question arises whether quantum mechanics can follow from a basic geometrical principle, akin to the geometrical principle that underlies relativity. Starting in [1] we embarked on a rigorous derivation of quantum mechanics from a geometrical principle. The equivalence postulate of quantum mechanics hypothesises that any

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two physical states can be connected by a coordinate transformation. This includes states which arise under different potentials. In particular, any state may be transformed so as to correspond to that of a free particle at rest. This bears close resemblance to Einstein's equivalence principle that underlies general relativity with an important caveat. While in the case of Einstein's equivalence principle it is the gravitational field which is "locally balanced" by a coordinate transformation, in the equivalence postulate approach to quantum mechanics it is an arbitrary external potential which is "globally balanced" by a coordinate transformation. The equivalence postulate of quantum mechanics is naturally formulated in the framework of Hamilton–Jacobi theory.

The implementation of the equivalence postulate in the context of the Hamilton–Jacobi theory yields a Quantum Hamilton–Jacobi equation. The Classical Hamilton–Jacobi Equation is obtained by requiring the existence of a canonical transformation from one set of phase space variables to a second set of phase space variable such that the Hamiltonian is mapped to a trivial Hamiltonian. Consequently, the new phase-space variable are constants of the motion, i.e.

$$H(q,p) \longrightarrow K(Q,P) \equiv 0 \qquad \Longrightarrow \qquad \dot{Q} = \frac{\partial K}{\partial P} \equiv 0\,,\ \dot{P} = -\frac{\partial K}{\partial Q} \equiv 0. \tag{1}$$

The solution to this problem is given by the Classical Hamilton–Jacobi equation (CHJE). Since the transformations are canonical the phase space variables are taken as independent variables and their functional dependence is only extracted from the solution of the CHJE via the functional relation

$$p = \frac{\partial S(q)}{\partial q}, \tag{2}$$

where $S(q)$ is Hamilton's principal function. The fundamental uncertainty relations of quantum mechanics imply that the phase-space variables are not independent. The equivalence postulate of quantum mechanics therefore requires the existence of trivialising coordinate transformations for any physical system, but the phase-space variables are not independent in the application of the trivialising transformations. They are related by a generating function, via (2), which transforms as a scalar function under the transformations. That is,

$$(\, q\ ,\ S_0(q)\ ,\ p\ =\ \frac{\partial S_0}{\partial q}) \ \longrightarrow\ (\, q^v\ ,\ S_0^v(q^v)\ ,\ p^v\ =\ \frac{\partial S_0^v}{\partial q^v}\,), \tag{3}$$

where $S_0(q)$ is the generating function in the stationary case. It is instrumental to study the stationary case in order to see the symmetry structure that underlies quantum mechanics. The consistency of the equivalence hypothesis implies that the Hamilton–Jacobi equation retains its form under coordinate transformations. However, this cannot be implemented in classical mechanics. The CSHJE for a particle moving under the influence of a velocity independent potential $V(q)$ is given by

$$\frac{1}{2m}\sum_{i=1}^{N}\left(\frac{\partial S}{\partial q_i}\right)^2 + \mathcal{W}(q) = 0, \tag{4}$$

where $\mathcal{W}(q) \equiv V(q) - E$. Under a change of coordinates $q \to q^v$ we have (by (3))

$$\frac{\partial S^v(q^v)}{\partial q_j^v} = \frac{\partial S(q)}{\partial q_j^v} = \sum_i \frac{\partial S(q)}{\partial q_i}\frac{\partial q_i}{\partial q_j^v}, \tag{5}$$

which we can write as $\boldsymbol{p}^v = \mathbf{J}^v \boldsymbol{p}$, where $J_{ij}^v = \frac{\partial q_i}{\partial q_j^v}$ is the Jacobian matrix connecting the coordinate systems q and q^v, and where, $p_i = \frac{\partial S}{\partial q_i}$. Then

$$\sum_j \left(\frac{\partial S^v}{\partial q_j^v}\right)^2 = |\boldsymbol{p}^v|^2 = \left(\frac{|\boldsymbol{p}^v|^2}{|\boldsymbol{p}|^2}\right)|\boldsymbol{p}|^2 = (p^v|p)|\boldsymbol{p}|^2, \tag{6}$$

where we have defined

$$(p^v|p) \equiv \frac{|\boldsymbol{p}^v|^2}{|\boldsymbol{p}|^2} = \frac{\boldsymbol{p}^{v\top}\boldsymbol{p}^v}{\boldsymbol{p}^\top \boldsymbol{p}} = \frac{\boldsymbol{p}^\top \mathbf{J}^{v\top}\mathbf{J}^v \boldsymbol{p}}{\boldsymbol{p}^\top \boldsymbol{p}}. \tag{7}$$

It is seen that the first term in Eq. (4) transforms as a quadratic differential under the v-map Eq. (3). Since $S_0^v(q^v)$ must satisfy the CSHJE, covariance of the HJ equation under the v-transformations implies that the second term in Eq. (4) transforms as a quadratic differential. That is

$$\mathcal{W}^v(q^v) = (p^v|p)\mathcal{W}(q). \tag{8}$$

In particular, for the $\mathcal{W}^0(q^0) \equiv 0$ state we have,

$$\mathcal{W}^0(q^0) \longrightarrow \mathcal{W}^v(q^v) = (p^v|p^0)\mathcal{W}^0(q^0) = 0. \tag{9}$$

This means that $\mathcal{W}^0$ is a *fixed point* under v-maps, i.e. it cannot be connected to other states. Hence, we conclude that the equivalence postulate cannot be implemented consistently in classical mechanics.

2 The Cocycle Condition

Consistent implementation of the equivalence postulate necessitates the modification of classical mechanics, which entails adding a yet to be determined function, $\mathcal{Q}(q)$, to the CSHJE. This augmentation produces the Quantum Stationary Hamilton–Jacobi Equation (QSHJE)

$$\frac{1}{2m}\left(\frac{\partial S(q)}{\partial q}\right)^2 + \mathcal{W}(q) + \mathcal{Q}(q) = 0\,, \tag{10}$$

where $\mathcal{W}(q) = V(q) - E$. It is noted that the combination $\mathcal{W}(q) + \mathcal{Q}(q)$ transforms as a quadratic differential under coordinate transformations, whereas each of the functions $\mathcal{W}(q)$ and $\mathcal{Q}(q)$ transforms as a quadratic differential up to an additive term, i.e. under $q^a \to q^v(q)$ we have,

$$\begin{aligned}\mathcal{W}^a(q^a) \to \mathcal{W}^v(q^v) &= (p^v|p^a)\,\mathcal{W}^a(q^a) + (q^a;q^v)\\ \mathcal{Q}^a(q^a) \to \mathcal{Q}^v(q^v) &= (p^v|p^a)\,\mathcal{Q}^a(q^a) - (q^a;q^v).\end{aligned}$$

and

$$(\mathcal{W}(q^a) + \mathcal{Q}(q^a)) \to (\mathcal{W}^v(q^v) + \mathcal{Q}^v(q^v)) = (p^v|p^a)\,(\mathcal{W}^a(q^a) + \mathcal{Q}^a(q^a)) \tag{11}$$

All physical states with a non-trivial $\mathcal{W}(q)$ then arise from the inhomogeneous part in the transformation of the trivial state $\mathcal{W}^0(q^0) \equiv 0$, i.e. $\mathcal{W}(q) = (q^0;q)$. Considering the transformation $q^a \to q^b \to q^c$ versus $q^a \to q^c$ gives rise to the cocycle condition on the inhomogeneous term

$$(q^a;q^c) = \left(p^c|p^b\right)\left[(q^a;q^b) - (q^c;q^b)\right]. \tag{12}$$

The *cocycle condition* Eq. (12) embodies the essence of quantum mechanics in the equivalence postulate approach. Furthermore, it reveals the basic symmetry properties that underly quantum mechanics. It is proven [1, 2] that the cocycle condition is invariant under D-dimensional Möbius transformations, which include translations, dilatations, rotations and, most crucially, inversions, or reflections, in the unit sphere. The Möbius transformations are, hence, defined on the compactified space $\hat{\mathbb{R}}^D = \mathbb{R}^D \cup \{\infty\}$. Whereas translations, dilatation and rotations map ∞ to itself, inversions exchange $0 \leftrightarrow \infty$. We argue that energy quantisation and the existence of a fundamental length scale in the formalism, together with the invariance of the cocycle condition Eq. (12) under the Möbius group $M(\hat{\mathbb{R}}^D)$ of transformations, implies that space is compact. The more general situation may be considered in the decompactification limit.

The cocycle condition fixes the functional form of the quantum potential $\mathcal{Q}(q)$. In one dimension the cocycle condition (12) fixes the inhomogeneous term

$$(q^a;q^b) = -\beta^2\{q^a,q^b\}/4m,$$

where $\{f, q\} = f'''/f' - 3(f''/f')^2/2$ the Schwarzian derivative and β is a constant with the dimension of an action. In one dimension the Quantum Hamilton–Jacobi equation is given in terms of a basic Schwarzian identity,

$$\left(\frac{\partial S(q)}{\partial q}\right)^2 = \frac{\beta^2}{2}\left(\left\{e^{\frac{2i\beta S}{,}} q\right\} - \{S, q\}\right) \tag{13}$$

Making the identification

$$\mathcal{W}(q) = V(q) - E = -\frac{\beta^2}{4m}\left\{e^{\frac{(i2S_0)}{\beta}}, q\right\}, \tag{14}$$

and

$$\mathcal{Q}(q) = \frac{\beta^2}{4m}\{S_0, q\}, \tag{15}$$

we have that S_0 is the solution of the Quantum Stationary Hamilton–Jacobi equation (QSHJE),

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 + V(q) - E + \frac{\hbar^2}{4m}\{S_0, q\} = 0. \tag{16}$$

The Schwarzian identity, Eq. (13), is generalised in higher dimensions by the basic quadratic identity

$$\alpha^2(\nabla S_0)^2 = \frac{\Delta(Re^{\alpha S_0})}{Re^{\alpha S_0}} - \frac{\Delta R}{R} - \frac{\alpha}{R^2}\nabla\cdot(R^2\nabla S_0), \tag{17}$$

which holds for any constant α and any functions R and S_0. Then, if R satisfies the continuity equation $\nabla\cdot(R^2\nabla S_0) = 0$, and setting $\alpha = i/\hbar$, we have

$$\frac{1}{2m}(\nabla S_0)^2 = -\frac{\hbar^2}{2m}\frac{\Delta(Re^{\frac{i}{\hbar}S_0})}{Re^{\frac{i}{\hbar}S_0}} + \frac{\hbar^2}{2m}\frac{\Delta R}{R}. \tag{18}$$

In analogy with the one dimensional case we make identifications,

$$\mathcal{W}(q) = V(q) - E = \frac{\hbar^2}{2m}\frac{\Delta(Re^{\frac{i}{\hbar}S_0})}{Re^{\frac{i}{\hbar}S_0}}, \tag{19}$$

$$\mathcal{Q}(q) = -\frac{\hbar^2}{2m}\frac{\Delta R}{R}. \tag{20}$$

Equation (19) implies the D-dimensional Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\Delta + V(q)\right]\Psi = E\Psi. \tag{21}$$

and the general solution

$$\Psi = R(q)\left(Ae^{\frac{i}{\hbar}S_0} + Be^{-\frac{i}{\hbar}S_0}\right). \tag{22}$$

We note that consistency of the equivalence postulate formalism necessitates that the two solutions of the second order Schrödinger equation are retained. This is reminiscent of relativistic quantum mechanics in which both the positive and negative energy solutions are retained. We can replace the gradient in Eq. (17) by a four vector derivative in Minkowski space. This produces the generalisation of the formalism to the relativistic case and the Schrödinger equation, Eq. (21), is replaced by the Klein–Gordon equation. The time-dependent Schrödinger equation arises in the limit $c \to \infty$. Similarly, the cocycle condition Eq. (12) generalises to Minkowski space by replacing the Euclidean metric with the Minkowski metric. It is important to emphasize that the equivalence postulate approach to quantum mechanics does not represent a modification or interpretation of quantum mechanics but its derivation from a basic geometrical principle. As such it reveals the geometrical structures underlying quantum mechanics and in that respect provides an intrinsic framework to explore the quantum space-time. It is further noted that the cocycle condition, Eq. (12), is completely universal. Hence, its generalisation to curved space provides a background independent approach to quantum gravity. In this respect the equivalence postulate approach reveals the interplay between quantum variables, encoded $R(q)$ and $S(q)$, versus the classical background parameters. For example, in [3] we showed that the QHJE does not admit a consistent time parameterisation of quantum trajectories. In this respect, therefore, time cannot be defined as a quantum observable, but is merely a classical background parameter. Generalising this observation to relativistic space-time entails that space-time cannot be consistently defined as a quantum observable. Instead, the quantum data is encoded in the cocycle condition and the corresponding quadric identity in the relevant domain, i.e. in curved space-time. In this respect, we note that the inhomogeneous term can be written in the general form [2],

$$(q^a; q^b) = (p^b|p^a)\mathcal{Q}^a(q^a) - \mathcal{Q}^b(q^b) = -\frac{\hbar^2}{2m}\left[(p^b|p^a)\frac{\Delta^a R^a}{R^a} - \frac{\Delta^b R^b}{R^b}\right], \tag{23}$$

which shows how the information on the inhomogeneous term is encoded in the functions $R(q)$ and $S(q)$.

3 The Quantum Closet

The invariance of the cocycle condition under Möbius transformations implies that space is compact. Let us gather the evidence for this claim. In the one dimensional case we see from Eq. (19) that the QSHJE is equivalent to the equation $\{w, q\} = -4m(V(q) - E)/\hbar^2$ where w is the ratio of the two solutions of the Schrödinger equation. It follows from the Möbius invariance of the cocycle condition that $w \neq const$, $w \in C^2(\hat{\mathbb{R}})$ with w'' differentiable on $\mathbb{R}$, where $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, and

$$w(-\infty) = \begin{cases} +w(+\infty) \text{ if } w(-\infty) \neq \pm\infty \,, \\ -w(+\infty) \text{ if } w(-\infty) = \pm\infty \,. \end{cases} \tag{24}$$

Furthermore, denoting by q_- (q_+) the lowest (highest) q for which $V(q) - E$ changes sign, we prove the general theorem [1],
If

$$V(q) - E \geq \begin{cases} P_-^2 > 0 \,, \, q < q_- \,, \\ P_+^2 > 0 \,, \, q > q_+ \,, \end{cases} \tag{25}$$

then $w = \psi^D/\psi$ *is a local self-homeomorphism of* $\hat{\mathbb{R}}$ *iff the Schrödinger equation has an* $L^2(\mathbb{R})$ *solution.*

Since the QSHJE is defined if and only if w is a local self-homeomorphism of $\hat{\mathbb{R}}$, this theorem implies that energy quantisation *directly* follows from the geometrical gluing conditions of w at $q = \pm\infty$, as implied by the equivalence postulate, which in turn imply that the one dimensional space is compact. In turn the compactness of space implies that the energy of the free quantum particle is quantised and that time parameterisation of trajectories is ill defined either via Bohm–de Broglie mechanical definition, or via Floyd's definition by using Jacobi's theorem [4]. The Möbius invariance of the cocycle condition in D dimensions then implies that the D dimensional space is compact.

Generalisation of the cocycle condition to curved space suggests a background independent approach to quantum gravity. The connection with gravity and with an internal structure of elementary particles is implied due to the existence of an intrinsic fundamental length scale in the formalism, and the association of the quantum potential, $\mathcal{Q}(q)$, with a curvature term [1,5,6]. To see the origin of that we can again examine the stationary one dimensional case with $\mathcal{W}^0(q^0) \equiv 0$. In this case the Schrödinger equation takes the form

$$\frac{\partial^2}{\partial q^2}\psi = 0,$$

with the two linearly independent solutions being $\psi^D = q^0$ and $\psi = const$. Consistency of the equivalence postulate dictates that both solutions of the Schrödinger equation must be retained. The solution of the corresponding QHJE is given by [1]

$$e^{\frac{2i}{\hbar}S_0^0} = e^{i\alpha}\frac{q^0 + i\bar{\ell}_0}{q^0 - i\ell_0},$$

where ℓ_0 is a constant with the dimension of length [1], and the conjugate momentum $p_0 = \partial_{q^0} S_0^0$ takes the form

$$p_0 = \pm \frac{\hbar(\ell_0 + \bar{\ell}_0)}{2|q^0 - i\ell_0|^2}. \tag{26}$$

It is seen that p_0 vanishes only for $q^0 \to \pm\infty$. The requirement that in the classical limit $\lim_{\hbar\to 0} p_0 = 0$ implies that we can set [1]

$$\mathrm{Re}\,\ell_0 = \lambda_p = \sqrt{\frac{\hbar G}{c^3}}, \tag{27}$$

i.e. we identify $\mathrm{Re}\,\ell_0$ with the Planck length. The interpretation of the quantum potential as a curvature term [1, 6] implies that elementary particles possess an internal structure, i.e. points do not have curvatures. This suggests possible connection with theories of extended objects.

If the universe is compact it would imply the existence of an intrinsic energy scale reminiscent of the Casimir effect. Taking the present size of the observable universe would imply a very small energy scale, which is essentially unobservable [6]. However, given the indication of a larger energy scale in the Cosmic Microwave Background (CMB) Radiation suggests the possibility of observing the imprints of compactness of the universe in the CMB in the current [7] or future CMB observatories. Indeed, the possibility of signatures of a non-trivial topology in the CMB has been of recent interest [8]. Additional experimental evidence for the equivalence postulate approach to quantum mechanics may arise from modifications of the relativistic energy-momentum relation [9], which affects the propagation of light from gamma ray bursts [10].

4 The Decompactification Limit

The Möbius invariance of cocycle condition may only be implemented if space is compact. We may contemplate that the decompactification limit represents the case when the spectrum of the free quantum particle becomes continuous. In that case time parameterisation of quantum trajectories is consistent with the definition of time by using Jacobi's theorem [1,3,4]. However, I argue that the decompactification limit in fact concides with the classical limit. To see that this may be the case we examine again the case of the free particle in one dimension. The quantum potential associated with the state $W^0 \equiv 0$ is given by

$$\mathcal{Q}^0 = \frac{\hbar^2}{4m}\{S_0^0, q^0\} = -\frac{\hbar^2(\mathrm{Re}\,\ell_0)^2}{2m}\frac{1}{|q^0 - i\ell_0|^4}. \tag{28}$$

We note that the limit $q^0 \to \infty$ concides with the limit $\mathcal{Q}^0 \to 0$, i.e. with the classical limit.

Conclusions

Heisenberg's uncertainty principle mandates that the phase-space variables cannot be treated as independent variables. The classical Hamilton–Jacobi trivialising transformations are in direct conflict with this fact. Reconciling the Hamilton–Jacobi theory to quantum mechanics leads to the quantum Hamilton–Jacobi equation (QHJE). In turn, the QHJE implies a basic cocycle condition that underlies quantum mechanics. The cocycle condition holds in any background and provides a framework for the background independent formulation of quantum gravity. The cocycle condition is invariant under D-dimensional finite Möbius transformations with respect to the Euclidean or Minkowski metrics. Its invariance under D-dimensional Möbius transformations implies that space is compact, which may have an imprint in the cosmic microwave background radiation.

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Shape-Invariant Orbits and Their Laplace-Runge-Lenz Vectors for a Class of "Double Potentials"

Jamil Daboul

Abstract We derive exact $E = 0$ classical solutions for the following class of Hamiltonians with "*double potentials*"

$$H_D := \frac{\mathbf{p}^2}{2m} + V_D(r),$$

where

$$V_D := -\frac{\gamma}{r^{2+2\mu}} + \frac{\lambda}{r^{2+4\mu}}, \quad \forall\ 0 \neq \mu \in \mathbb{R}.$$

For $\mu = -1/2$ and $\mu = -1$ the H_D yields the Kepler and oscillator systems for $E \neq 0$, respectively. The classical orbits of H_D are *shape invariant* for a wide range of γ and λ, in the sense that each maximum of their orbits $r(\varphi)$ is followed by a minimum after an angular shift of $\Delta\varphi = \pi/2\mu$. We map the LRL vector $\mathbf{M} := (M_1, M_2)$ of the Kepler problem to a complex expression $M_\mu \in \mathbb{C}$, which is conserved for every μ. We use M_μ to derive a general expression for the orbit $r(\varphi, \mu; \gamma, \lambda)$ for all $\mu \neq 0$. We also contrast the limit of the above orbits as $\lambda \to 0$ with those considered by Daboul and Nieto for the power-law potentials $V_P := -\gamma/r^{2+2\mu}$.

1 Introduction

Levi-Civita [1] in 1920 mapped the 2-dim harmonic-oscillator system with positive energies E_{osc} onto a 2-dim Kepler system with negative energies. This map can be formulated by using 2×2 matrices and real variables. All attempts to generalize this map to three dimensions did not succeed.

In the present paper we use complex variables to define a canonical transformation which maps the Kepler Hamiltonian in two dimensions with arbitrary energy E_{kep} to Hamiltonians with the following class of potentials

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$$V_D(r) := -\frac{\gamma}{r^{2+2\mu}} + \frac{\lambda}{r^{2+4\mu}}\ , \quad \forall\ 0 \neq \mu \in \mathbb{R}, \tag{1}$$

but for zero energy, $E_D = 0$. The only exception is for $\mu = -1$, where the first potential, $-\gamma/r^{2+2\mu}$, becomes a constant equal to $-\gamma$. By interpreting γ as the energy of the oscillator E_{osc}, the Levi-Civita map [1] follows as a special case. The potentials in (1) will be called *"double potentials"*, since they are sums of two power potentials.

In principle, we can now study the following Hamiltonian with the double potential classically and quantum mechanically for any dimension, and not just for two dimensions, and for any energy E_D:

$$H_D := \frac{p^2}{2m} - \frac{\gamma}{r^{2+2\mu}} + \frac{\lambda}{r^{2+4\mu}}\ , \quad \forall\ 0 \neq \mu \in \mathbb{R}. \tag{2}$$

Since the potential in (2) is spherical, $V_D = V_D(r)$, the classical orbits take place in a 2-dim plane, which we choose as the (x_1, x_2) plane. Therefore, the orbits $r(\varphi, \mu)$ of (2) in two and higher dimensions will look the same as those of two dimensions. We shall determine these orbits by using the conserved image M_μ of the *Laplace-Runge-Lenz* (LRL) vector $M_{-1/2}$ of the Kepler orbit.

2 Complex Canonical Transformations and Mapping of Hamiltonians

In this section I describe a canonical transformation which enables us to transform a Hamiltonian system defined in two dimensions $H(w, \pi) = \pi^2/2m + V_w(w)$ for arbitrary energy E to a system with an additional potential, but for zero energy $E = 0$.

It is useful to use complex canonical variables,

$$z := x_1 + ix_2 \quad \text{and} \quad p := p_1 + ip_2, \tag{3}$$

for dealing with 2-dim Hamiltonian problems.

The usual commutation relations

$$\{x_i, x_j\} = \{p_i, p_j\} = 0, \quad \text{and} \quad \{x_i, p_j\} = \delta_{i,j} \tag{4}$$

become

$$\{z, \bar{p}\} = \{x_1 + ix_2, p_1 - ip_2\} = 2, \quad \text{so that } \{z, p\} = 0.$$

2.1 *Complex Canonical Transformation*

Let

$$w = f(z)\,, \quad w, z \in \mathbb{C}\,, \tag{5}$$

be a complex map. To complete this map to a canonical transformation $(w, \pi) \rightarrow (z, p)$, we use Poisson brackets

$$\{A, B\}_{z,p} = \{A, B\}_{w,\pi}\,.$$

Calculating $\{z, p\}_{w,\pi}$ yields

$$\frac{\partial \bar{p}}{\partial \bar{\pi}} = \frac{dw}{dz}\,, \quad \text{so that } \pi = p\frac{d\bar{z}}{d\bar{w}}\,. \tag{6}$$

2.2 *Transformation of* General *Hamiltonians*

We now transform the Hamiltonians by using (6), as follows

$$\begin{aligned} 0 = H_w - E_w &= \frac{|\pi|^2}{2m} + V_w(w) - E_w \\ &= \left|\frac{dz}{dw}\right|^2 \left(\frac{|p|^2}{2m} + \left|\frac{dw}{dz}\right|^2 (V_w(w) - E_w)\right) =: \left|\frac{dz}{dw}\right|^2 \; H_z. \end{aligned} \tag{7}$$

The H_z in (7) is a new Hamiltonian

$$H_z := \frac{|p|^2}{2m} + V_z(z), \quad \text{where} \quad V_z(z) = \left|\frac{dw}{dz}\right|^2 (V_w(w) - E_w) \tag{8}$$

Note that $V_w(w)$ in (7) need not be a central potential. Moreover, $\left|\frac{dw}{dz}\right|^2$ for a general map $w = f(z)$ need not be a function of $|z|$ only.

However, if the complex transformation $f(z)$ is a *power* of z, as in (9) below, then $\left|\frac{dw}{dz}\right|^2$ becomes a function of $|z|$ only. In this case, the map in (7) transforms a Hamiltonian with a central potential $V_w(|w|)$ to a Hamiltonian H_z, also with central potential $V_z(|z|)$.

2.3 Derivation of the Double Potential V_D from the Kepler Potential

In this subsection I shall derive the Hamiltonian H_D in (2), by applying (7) to map the Kepler potential $V_w(|w|) = -\alpha/|w|$ and $E_w = E_{kep}$: Starting with

$$w = f(z) := \frac{c}{z^{2\mu}}\,, \tag{9}$$

where c is a dimensional constant, which is helpful to check formulas. Its dimension is $[c] = [\text{length}]^{2\mu+1}$. Differentiating, we obtain

$$\frac{dw}{dz} = -2\mu\,\frac{c}{z^{2\mu+1}} = -2\mu\frac{w}{z}\,. \tag{10}$$

Substituting (10) in (8), we obtain the following double potential

$$\begin{aligned} V_D(r) = V_z(|z|) &= \left|\frac{dw}{dz}\right|^2 (V_{kep}(|w|) - E_w) = \frac{4\mu^2|w|^2}{|z|^2}(-\frac{\alpha}{|w|} - E_{kep}) \\ &= -\frac{\gamma}{r^{2+2\mu}} + \frac{\lambda}{r^{2+4\mu}}\,, \end{aligned} \tag{11}$$

where

$$\gamma := 4\mu^2 c\,\alpha\,, \quad \text{and} \quad \lambda := -4\mu^2 c^2\, E_{kep}\,. \tag{12}$$

2.4 The Classical Orbits of the Hamiltonian H_D

In the last section we derived a new Hamiltonian H_z with an new potential V_z. We can solve this system for any energy E_D. In general, this can be done mainly by numerical calculations. However, the solutions for $E_D = 0$ can be obtained directly as images of the solutions of the original system H_w.

3 "Generalized" Laplace-Runge-Lenz "Vector" for the Double Potentials V_D and all μ

It is well known that the N-dimensional Kepler problem and the spherical harmonic oscillator have dynamical symmetry of $\mathfrak{so}(N,1)$ and $\mathfrak{su}(N)$ respectively. The generators of these symmetries can be expressed in terms of vectors and tensors in N-dimensions.

However for the double potential V_D I do not know what these symmetries are for general μ and whether it is possible to write the generators in matrix or tensor form.

What I shall do in the following is to write the LRL vector of the Kepler problem as a complex variable and then write it in terms of the canonical (z, p)-variables.

3.1 A Complex Expression M_μ for the LRL of V_D

The LRL vector of the 3-dim Kepler problem is given by

$$\mathbf{M}_{kep} = \mathbf{p} \times \mathbf{L} - m\alpha\hat{\mathbf{r}}\ . \tag{13}$$

$\mathbf{M}_{kep}$ vector lies in the plane of motion, which we define as the (w_1, w_2) plane. This enables us to write it as a complex variable $M_{kep} = (M_1 + iM_2)$ in a complex w-plane, as follows

$$M_{kep}(w, \pi) = -i\pi L_{kep} - m\alpha\frac{w}{|w|}. \tag{14}$$

We now write the M_{kep} in (14) as a function of the new canonical variables (z, p): By substituting $dw/dz = -2\mu c/z^{2\mu+1}$ from (10) in Eq. (6) for π, we obtain

$$\pi = p\frac{d\bar{z}}{d\bar{w}} = -\frac{p}{2\mu c}\bar{z}^{2\mu+1} = -\frac{p}{2\mu}\ \frac{\bar{z}}{\bar{w}}\ , \tag{15}$$

so that

$$L_{kep} := w_1\pi_2 - w_2\pi_1 = \text{Im}\,(\bar{w}\pi) = -\frac{1}{2\mu}\text{Im}\,(\bar{z}p) = -\frac{1}{2\mu}L_\mu\ . \tag{16}$$

Substituting (16) and (15) into (14) and noting the equality $\alpha = \gamma/(4\mu^2 c)$ in (12), we obtain a *complex* expression $M_\mu \in \mathbb{C}$ for the LRL of the H_D-system:

$$M_\mu := f(M_{kep}) = -i\,\frac{L_\mu}{4\mu^2 c}\,p\bar{z}^{(1+2\mu)} - m\alpha\left(\frac{z}{|z|}\right)^{-2\mu} \tag{17a}$$

$$= -m\alpha\left[\frac{ip\bar{z}\,L_\mu}{m\gamma}\,r^{2\mu} + 1\right]e^{-i\,2\mu\varphi} \tag{17b}$$

$$= -m\alpha\left[\frac{(i\,\mathbf{r}\cdot\mathbf{p} - L_\mu)\,L_\mu}{m\gamma}\,r^{2\mu} + 1\right]e^{-i\,2\mu\varphi}\ , \tag{17c}$$

where we first used (12) and then $ip\,\bar{z} = i\mathbf{r}\cdot\mathbf{p} - L_\mu$.

3.2 Derivation of the $E = 0$ Orbits Using LRL Vector

From now on we shall often use $M \equiv M_\mu$ for simplicity. Instead of obtaining the orbits of H_D by solving differential equations, it is more interesting to use the LRL M in (17c): Let φ_M denotes the *phase* of M, i.e.

$$M \equiv M_\mu =: |M_\mu| \exp[i\varphi_M] \,. \tag{18}$$

By multiplying M in (17c) by $\exp[2i\mu\varphi]$, we obtain

$$|M|\ \exp[i(2\mu\varphi + \varphi_M)] = -m\alpha \left[\frac{(i\mathbf{r}\cdot\mathbf{p} - L_\mu)\,L_\mu}{m\gamma}\, r^{2\mu} + 1 \right] . \tag{19}$$

By taking the *real part* of (19), we immediately obtain a general expression for the $E_D = 0$ orbits of the double-potential system

$$r^{2\mu}(\varphi) = \frac{m\gamma}{L_\mu^2}[1+\epsilon\cos(2\mu\varphi+\varphi_M)] = \frac{m\gamma}{L_\mu^2}[1+\epsilon\cos 2\mu(\varphi-\varphi_0)]\,,\ \ \varphi_0 := -\frac{\varphi_M}{2\mu}\,,$$

so that

$$r(\varphi,\mu) = \left(\frac{m\gamma}{L_\mu^2}[1+\epsilon\cos 2\mu(\varphi-\varphi_0)] \right)^{1/2\mu}, \quad \text{with} \quad \epsilon := \frac{|M|}{m\alpha}\,. \tag{20}$$

3.3 Calculating $\epsilon = |M|/m\alpha$ in Terms of L^2, γ and λ

We now calculate $\epsilon = |M|/m\alpha$ in (20). Using $|a+b|^2 = |a|^2 + |b|^2 + 2Re(a\bar{b})$, we obtain from (17b)

$$\begin{aligned} \epsilon^2 = \frac{|M|^2}{m^2\alpha^2} &= 1 + \frac{1}{(m\gamma)^2}\left[L^2|z|^{2+4\mu}|p|^2 + 2m\gamma L\ Re\left(i\bar{z}^{2\mu+1}\pi\frac{z^\mu}{\bar{z}^\mu} \right) \right] \\ &= 1 + \frac{2L^2}{m\gamma^2} r^{2+4\mu}\left(\frac{|p|^2}{2m} - \frac{\gamma}{r^{2+2\mu}} \right) = 1 + \frac{2L^2}{m\gamma^2} r^{2+4\mu}\left(E_D - \frac{\lambda}{r^{2+4\mu}} \right) \\ &= 1 + \frac{2L^2}{m\gamma^2}\left(E_D\, r^{2+4\mu} - \lambda \right) \ \ \Rightarrow\ \ 1 - \frac{2\lambda L^2}{m\gamma^2} \quad (\text{for}\ \ E_D = 0)\,. \end{aligned} \tag{21}$$

Note that $|M|, \varphi_M$ and thus $\epsilon = |M|/m\alpha$ are constants of motion *only* for $E_D = 0$.

By noting that for the Kepler problem ($\mu = -1/2$) we obtain $\lambda = -E_{kep}$ whereas for the oscillator ($\mu = -1$) we obtain $\gamma = E_{osc}$ and λr^2 is the potential of the spherical oscillator. Hence, for the above two cases ϵ becomes

$$\epsilon = \sqrt{1 - \frac{2\lambda L^2}{m\gamma^2}} = \begin{cases} \sqrt{1 + \frac{2E_{kep}L^2}{m\gamma^2}} & \text{for} \quad Kepler \\ \sqrt{1 - \frac{2\lambda L^2}{mE_{osc}^2}} & \text{for} \quad oscillator \end{cases} \tag{22}$$

4 Limit of the Orbits for $E_D = 0$ as $\lambda \to 0$

From (21) we see that for $E_D = 0$ and $\lambda \to 0$, we obtain $\epsilon \to 1$. Hence, by substituting $1 + \epsilon \cos 2\varphi \to 1 + \cos 2\varphi = 2\cos^2\varphi$ into (20), the orbit $r(\varphi, \mu)$ becomes

$$r_h(\varphi, \mu) = (\frac{m\gamma}{L_\mu^2}[1 + \cos 2\mu(\varphi - \varphi_0)])^{1/2\mu} = \left(\frac{\sqrt{2m\gamma}}{L_\mu}|\cos\mu(\varphi - \varphi_0)|\right)^{1/\mu}. \tag{23}$$

The *limit orbits* $r_h(\varphi, \mu) := \lim_{\lambda\to 0} r(\varphi, \mu; \lambda)$ in (23) will be called *hard orbits*, since for $\mu > 0$ they are produced by elastic collision with the infinite repulsive potential $\lambda/r^{2+4\mu}$ at the origin. These orbits are illustrated in Fig. 1 for $\mu = 3$. Note that $r_h(-\mu, \varphi) = 1/r_h(\mu, \varphi)$, so that if an orbit is bound for a given μ, it is infinite for $-\mu$.

In contrast, the orbits $r_s(\varphi)$ illustrated in Fig. 2 will be called *soft orbits*, because they were obtained for a single power-law potential $V_P(r) = -\gamma/r^{2+2\mu}$, without imposing any repulsive potential [2, 3]. Daboul and Nieto argued that for $\mu > 0$ the particle is attracted to the origin $r = 0$, and since $L = rp = $ const. it passes by the origin in a straight line with infinite speed. And since $V_P(r)$ is a central

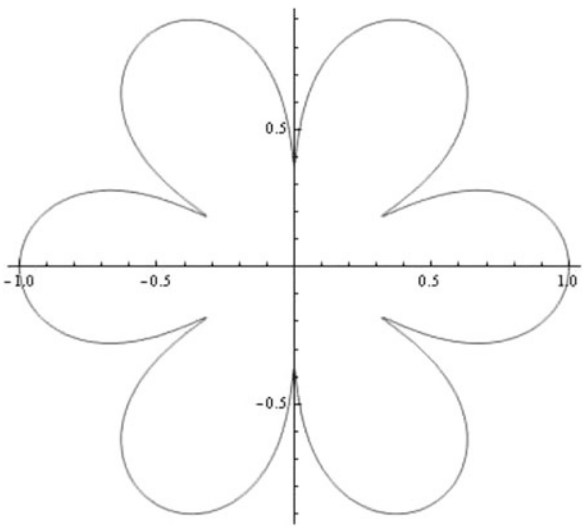
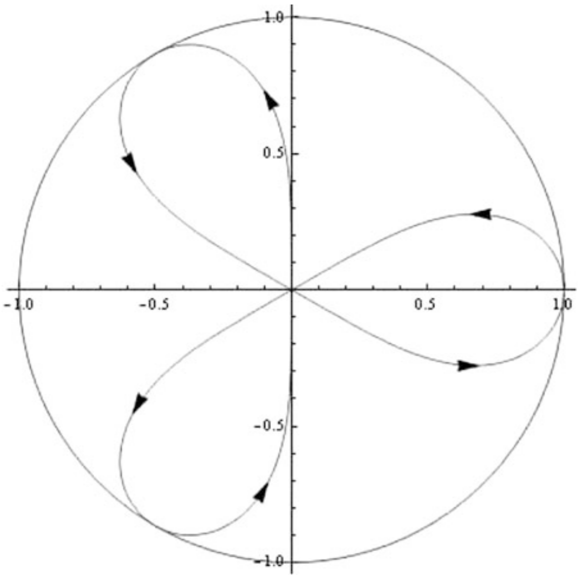


Fig. 1 Orbit $r(\varphi, 3)$ of $V_D = -\gamma/r^8 + \lambda/r^{14}$

Fig. 2 Soft orbit $r_s(\varphi, 3)$ of $V_P = -\gamma/r^8$



potential, the angular momentum **L** does not change its direction, so that direction of motion (clockwise or counter-clockwise) should remain the same. This argument leads to the soft orbits, defined such that the momentum $\mathbf{p}_s(\varphi)$ keeps its direction when it passes by the origin. Thus, if the orbit $r_s(\varphi)$ enters into the origin at an angle $\varphi_{in} = \varphi_n$, then it comes out on the opposite side at $\varphi_{out} = \varphi_n + \pi$. The orbits $r_s(\varphi, \mu)$ are illustrated in Fig. 2. For more details, see [2, 3].

Clearly, for $E_D \neq 0$ the LRL vector M_μ in (17) changes its direction, and with it the hard $\mathbf{r}_h(\varphi)$ and soft orbits $\mathbf{r}_s(\varphi)$ precess accordingly, as we verified numerically.

5 The $E_D = 0$ Quantum Solutions for Double Potential V_D

In the present section I briefly report on preliminary results on the quantum solutions of the Hamiltonian H_D in three dimensions:

$$\psi_{lm}(\mathbf{r}, \mu) = \text{const.}\ R_l(r, \mu) Y_{lm}(\theta, \varphi)\,, \tag{24}$$

where the radial function $R_l(r, \mu; \gamma, \lambda)$ reproduces the well-known solutions Kepler and the spherical oscillator for $\mu = -1/2$ and $\mu = -1$, respectively.

Moreover, the limit of the quantum solutions in (24) for $\lambda \to 0$ yield those for $V_P(r) = -\gamma/r^{2+2\mu}$ in [3, 5]. This indicates that the later solutions correspond to the hard orbits $r_h(\varphi, \mu)$ (23), and not to the soft orbits $r_s(\varphi, \mu)$, which are illustrated in Fig. 2. It is not even clear whether quantum solutions exist which correspond to

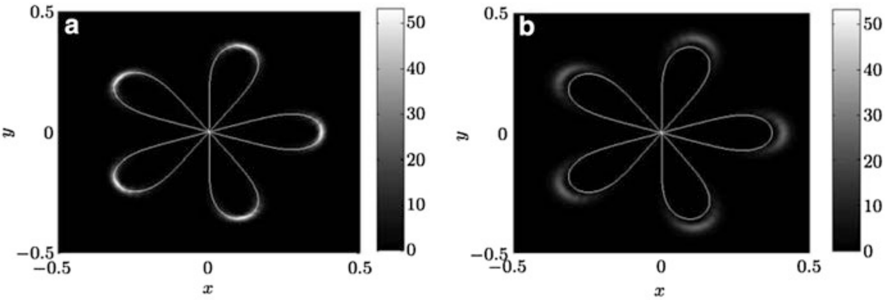


Fig. 3 Image of the probability density $|\Psi_{\mu,N}(r,\varphi)|^2$ of the coherent state in Eq. (22) in [4] and the closed classical orbit *(solid white curve*; more clear in color) for $V_P(r) = -\gamma/r^{12}$, i.e. for $\mu = 5$

the soft orbits $r_s(\varphi)$. Perhaps the quantum-classical correspondence can be resolved by using coherent states, as was done by Xin and Liang [4] (see Fig. 3).

6 Summary

I derived in Sect. 3.2 the orbits $r(\varphi,\mu)$ for all μ by using the conserved LRL vector.

In my oral presentation I presented a calculation on the rotation of the LBL vector for $E \neq 0$ and showed numerically that the orbits precess accordingly.

Even though the double potential $V_D(r)$ in (1) was derived by a *two* dimensional complex map (9), the quantum solutions of the Hamiltonian $H_D := p^2/2m + V_D(z)$ in (2) can be obtained *analytically* for $E_D = 0$, for every d-dimensions.

Thus, a general complex map in two dimensions, as in (5), acts as a *bridge* which connects the quantum solutions of Hamiltonians H_w and H_z for any dimension.

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Quantization on Co-adjoint Group Orbits and Second Class Constraints

Michail Stoilov

Abstract We make a comparison between two schemes for quantization of dynamical systems with non-trivial phase space—the geometric quantization based on co-adjoint group orbits and second class constraints method. It is shown that the Hilbert space of a system with second class constraints always has, contrary to the geometric quantization, infinite dimension.

1 Introduction

During the years the co-adjoint orbit method [1] proved to be a powerful and unified method for quantization of systems with complex symplectic structure. The same can be said for the approach based on second class constraints [2]. It is shown [3] that for the orthogonal and unitary groups the symplectic form defined on the co-adjoint groups orbits can be constructed using a system of first and second class constraints. However some singular orbit's points have to be excluded in order the procedure to be correct. This suggests that the two methods are not equivalent. Here we shall prove this conjecture using the dimension of the Hilbert space of the quantized system as a probe. It turns out that the Hilbert space is always infinite dimensional for systems with second class constraints while it can be with finite dimension in the co-adjoint orbit approach.

2 Co-adjoint Group Orbits

The power of the co-adjoint group orbits is due to a theorem for the universality of the co-adjoint orbits: Any co-adjoint Lie group orbit is a homogeneous symplectic space and vise versa provided some global criteria are satisfied. In addition the method offers a different approach to the physics. Usually we start with the model

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Lagrangian, then we determine its symmetries and finally we quantize it. The co-adjoint orbits method allows to start directly from the symmetry and to obtain automatically the corresponding Lagrangian and quantization. However, the method has not to be overestimated: All interesting examples of the quantization on co-adjoint orbits have been solved without using this method.

Here we use only some basic results of the co-adjoint orbits method. Suppose we have a Lie group $\mathfrak{G}$ and its algebra $\mathfrak{g}$. Having these two objects we can define the adjoint group action $Ad(g)$ on the algebra ($Ad(g)a = gag^{-1}$ in the case of matrix groups). Let us introduce the space $\mathfrak{g}^*$ of linear functionals over $\mathfrak{g}$. Now we can define the co-adjoint group representation $Ad^*(g)$ with the following relation

$$\langle Ad^*(g)\alpha, a\rangle = \langle \alpha, Ad(g^{-1})a\rangle. \tag{1}$$

It turns out that the group orbits in $\mathfrak{g}^*$ are symplectic manifolds: On each group orbit Ω_α there is a $\mathfrak{G}$-invariant symplectic form σ such that

$$\sigma(\alpha)(p_{\alpha*}(a), p_{\alpha*}(b)) = \langle \alpha, [a,b]\rangle. \tag{2}$$

Here p_α is the projection from the group to the orbit through α and p_* is the corresponding algebra projection (in fact, it is the ad^* action). A basic theorem states that

$$p_\alpha^*(\sigma) = -d\langle \alpha, \Theta\rangle, \quad \Theta = g^{-1}dg, \tag{3}$$

where p^* is the pull back of p and Θ is the Maurer–Cartan one form.

Using the symplectic form on the group orbits and an appropriate vacuum and coherent states based on it, we can construct the Lagrangian of the model with the symmetry in question and to write down the corresponding path integral, i.e. to quantize the model. Some nice examples of the outlined procedure, including relativistic particle, string, Chern–Simon and many more can be found in [4].

In what follows we need the notion of Poisson manifold as well. The Poisson manifold is a pair $\{M, c\}$ of manifold M and bi-vector c, $c = c^{ij}\partial_i \wedge \partial_j$ such that the Poisson bracket $\{f, g\} = c(f, g)$ defines a Lie algebra structure on $C^\infty(M)$. The following three theorems give the global structure of any Poisson manifold as a foliation into disjoint union. They are in the core of our consideration.

Theorem 1. *$\{M, c\} = \sum_i \{M_i, c_i\}$ where $\{M_i, c_i^{-1}\}$ are symplectic and $\{f, g\}(m_i) = \{f|_{M_i}, g|_{M_i}\}_i(m_i)$ for $m_i \in M_i$.*

Theorem 2. *$\{\mathfrak{g}^*, c_{ij}^k X_k \partial^i \wedge \partial^j\}$ where c_{ij}^k are the $\mathfrak{g}$ structure constants in the basis $\{X_i\}$ is a Poisson manifold. (note that linear functions on $\mathfrak{g}^*$ form an algebra isomorphic to $\mathfrak{g}$.)*

Theorem 3. *The symplectic leaves of $\{\mathfrak{g}^*, c_{ij}^k X_k \partial^i \wedge \partial^j\}$ are coadjoint orbits.*

3 Dynamical Systems with Second Class Constraints

The constraints in a dynamical model are some identities which are due to the definition of the momenta in it. Basically the constraints are first and second class and we are interested here by the latter ones because they change the Poisson structure. Suppose we have a dynamical model with $2n$-dimensional phase space and suppose we have identified the constraints $\chi_a = 0, a = 1, \ldots, k$ in it. They are second class if

$$\det\left(\{\chi_a, \chi_b\}|_{\chi=0}\right) \neq 0. \tag{4}$$

As a consequence the second class constraints are always even number $k = 2m$. The case $m = n$ corresponds to a trivial system with no dynamical degrees of freedom at all. In the models with second class constraints the Poisson bracket is replaced by the so called Dirac bracket

$$\{f, g\}_D = \{f, g\} - \{f, \chi_a\}\Delta^{-1}_{ab}\{\chi_b, g\}. \tag{5}$$

Here $\Delta_{ab} = \{\chi_a, \chi_b\}$. Note that Δ is by definition invertible.

The Dirac bracket can be very simple in some cases and very complicated in others. The example $\chi_1 = p_1, \quad \chi_2 = x_1$ describes phase space reduction. The example $\chi_1 = x_i x^i - r^2;\ \chi_2 = p_i x^i$ describes a model with compact configuration space. A natural question arise at this point: Can we represent any dynamical system as a system with second class constraints?

4 Quantization

The quantization of a classical dynamical system, considered as a mathematical problem, is a map from the real functions on the system phase space to self-adjoin operators in some Hilbert space. The Hilbert space can be with finite or infinite dimension depending on the model we quantize. For example, if we have a flat phase space with globally separated coordinates and momenta then the resulting Hilbert space is infinite dimensional. On the other hand, if we have a compact phase space, then the Hilbert space is finite dimensional [5].

During the quantization we cannot map consistently all functions on the phase space to self-adjoint operators in a Hilbert space. But we can a map a set of functions $\{f\}$, which is as large as possible to a set $\{\hat{f}\}$ of self-adjoint operators. The functions $\{f\}$ are called primary quantities and the correspondence $f \rightarrow \hat{f}$ has the following properties:

1. $\{f\}$ forms a closed algebra under Poisson brackets
2. The constants are primary quantities and $1 \rightarrow \mathfrak{I}$ where $\mathfrak{I}$ is the identity operator in the Hilbert space.

3. The operator image of the Poisson bracket is the commutator of the corresponding operators:

$$\widehat{\{f, g\}} = i\hbar[\hat{f}, \hat{g}], \tag{6}$$

As a result $\{\hat{f}\}$ is a Lie algebra representation of $\{f\}$.

Here we give a simple demonstration how one can deduce whether the dimension of the Hilbert space is finite or infinite depending on the quantized system.

Consider a model with flat phase space R^{2n}. Let $x_i, i = 1, \ldots, n$ are the coordinates and $p_i, i = 1, \ldots, n$ are corresponding momenta. Let $\omega = \sum_i dx_i \wedge dp_i$ is the canonical symplectic form on R^{2n}. Using this form we define the Poisson bracket between (C^∞) functions on the phase space

$$\{f(x, p), g(x, p)\} = \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x_i}. \tag{7}$$

In particular from Eq. (7) we have

$$\{x_i, p_j\} = \delta_{ij}. \tag{8}$$

In the classical case Eqs. (7) and (8) are equivalent. It is not the same when we quantize the system.

In the flat phase space example which we are considering now the primary quantities can be either linear functions of momenta and arbitrary functions of coordinates [1], or quadratic polynomials of coordinates and momenta [6]. In both cases we have

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}\mathfrak{I} \tag{9}$$

plus other commutation relations depending on what is our choice for $\{f\}$. So, according to Eq. (9), we have a representation of the Heisenberg algebra in the Hilbert space. This fact allows us to demonstrate that the Hilbert space is with infinite dimension. Indeed, if it is with finite dimension D then, taking the trace of both sides of Eq. (9) we will obtain a contradiction $0 = i\hbar\delta_{ij}D$.

Consider now a dynamical system which is symmetric with respect to the action of some Lie group. The symmetry acts by definition as canonical transformations and its algebra has a representation in the functions on the phase space which is closed under time evolution, i.e.

$$\{g_a, g_b\} = c_{abc}g_c \tag{10}$$

$$\{g_a, H\} = h_{ab}g_b, \tag{11}$$

where g_a are the generators of the Lie algebra and H is the Hamiltonian of the system. Usually the Hamiltonian is a combination of the symmetry generators and we can skip the second equation in this case. Even if the Hamiltonian is an independent generator Eqs. (10, 11) guarantee that all generators together form a closed Lie algebra. But as we have seen already we can realize any matrix Lie algebra with linear functions over a suitable Poisson manifold. In this manifold the linear functions which correspond to the symmetry generators are the primary quantities. Therefore the quantization of Eqs. (10, 11) look as follows

$$[\hat{g}_a, \hat{g}_b] = i\hbar c_{abc}\hat{g}_c \tag{12}$$

$$\left[\hat{g}_a, \hat{H}\right] = i\hbar h_{ab}\hat{g}_b. \tag{13}$$

If the symmetry algebra can be represented with trace-less matrices, e.g. it is a simple Lie algebra, then the same arguments which show that the representations of the Heisenberg algebra are infinite dimensional lead us to the conclusion that a system which primary quantities satisfy Eqs. (12, 13) can have a finite dimensional Hilbert space. In this case the primary quantities are mapped into constant matrices. This is exactly the reason why we can use Pauli matrices as the electron spin operators.

5 Results

Here we consider the quantization of systems with second class constraints. Any system of this type exhibits properties which allow us to think that it can interpolate between models with flat and compact phase spaces: it is defined on a flat phase space but this space is larger than the real one; the real phase space can be very complicated and with highly non trivial analog of the Poisson bracket on it thus resembling a system with compact phase space.

The quantization of systems with second class constraints follows the same rules as standard quantization, but now everywhere the Poisson brackets are replaced by Dirac ones. We make a very mild assumption that the linear functions on the initial phase space are primary quantities and so there Dirac brackets have to be mapped into commutators. In order to determine the dimension of the Hilbert space of a quantized system with second class constraints we consider the quantity $\{x_i, p^i\}_D$. Using the identity $\{\chi_b, p^i\}\{x_i, \chi_a\} = \partial\chi_b/\partial x_i \partial\chi_a/\partial p^i$ and the skew-symmetry of the matrix Δ we get

$$\{x_i, p^i\}_D = n - \{\chi_b, p^i\}\{x_i, \chi_a\}\Delta^{-1}_{ab} = n - m \tag{14}$$

Corollary 1. *There is always a Heisenberg subalgebra in the algebra of the primary quantities of any system with second class constraints. In this way we have proved the following Lemma:*

Lemma 1. *The Hilbert space of any Bose-Einstein quantized dynamical system with second class constraints is infinite dimensional.*

Conclusions

Our considerations show that we can use second class constraints only to construct local Darboux coordinates in the case of compact symplectic manifolds. As in the $U(n)$ and $O(n)$ examples considered in [3] there is no global construction and always some point in the group orbit has to be removed in order to view the orbit as a phase space of a system with second class constraints.

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Some Kind of Stabilities and Instabilities of Energies of Maps Between Kähler Manifolds

Tetsuya Taniguchi and Seiichi Udagawa

Abstract We treat the variational problem of the energy of the map between two Riemannian manifolds. It is known that any holomorphic or anti-holomorphic map $f: M \to N$ between compact Kähler manifolds is stable for the variation f_t of f with fixed Kähler metrics compatible with the holomorphic structures. Is this also stable for the variation g_t of the metric g of M with fixed volume of M and fixed isometric map f? In this paper, we show that the answer is no if the dimension of M is no less than 3. This paper is a expositary note of Taniguchi and Udagawa (Characterizations of Ricci flat metrics and Lagrangian submanifolds in terms of the variational problem. To appear in Glasgow Math. J).

1 Harmonic Maps

The geometric variational problems in Riemannian geometry have the long history. For example, the minimum path between two points on the earth is obtained by minimizing the length of arbitrary paths between them. It is a subset of a great circle going through the two points. It is called a geodesic. This is a solution for the variational problem of the length function. Solutions for the variational problem of the area function is called minimal surfaces. The concept of these two variational problems are extended to that of the variational problem of the energy of the map between two Riemannian manifolds. Solutions for it are called harmonic maps. In Appendix, we introduce a new variational problem on the space of the tensor product of symmetric (0, 2)-tensors and positive-definite symmetric (0, 2)-tensors.

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In particular, we review the first and second variational formulaes for the new variational problem. In Sect. 2, by applying the variational formulaes, we shall get some stabilities and instabilities of maps between Kähler manifolds.

Next, we review the definition of harmonic maps.
Let $f : (M, g) \longrightarrow (N, h)$ be a smooth map from an m-dimensional Riemannian manifold into an n-dimensional Riemannian manifold. Choose a local coordinates $(x^1, x^2, \cdots, x^m)$ for M and $(y^1, y^2, \cdots, y^n)$ for N. Then, both Riemannian metrics are expressed as $g = \sum_{i,j=1}^{m} g_{ij}\, dx^i dx^j$ and $h = \sum_{\alpha,\beta=1}^{n} h_{\alpha\beta} dy^\alpha dy^\beta$. The energy of the map f is given by

$$E(f) = \int_M \frac{1}{2} \sum_{i,j=1}^{m} \sum_{\alpha,\beta=1}^{n} g^{ij} h_{\alpha\beta}(f) f_i^\alpha f_j^\beta \, d\mu_g, \tag{1}$$

where g^{ij} is the component of the inverse matrix of the matrix (g_{ij}) and f_i^α is the component of the differential df of f, i.e., $df = \sum_{i=1}^{m} \sum_{\alpha=1}^{n} f_i^\alpha dx^i \otimes \frac{\partial}{\partial y^\alpha}$ and $d\mu_g$ is the volume element of (M, g) given by $d\mu_g = \sqrt{\det(g_{ij})}\, dx^1 dx^2 \cdots dx^m$. Let R and $\tilde{R}$ be the curvature tensors of (M, g) and (N, h), respectively. In terms of the local coordinates, they are expressed as

$$R = \sum_{i,j,k,l=1}^{m} R_{ijk}^{l} dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^l},$$

$$\tilde{R} = \sum_{\alpha,\beta,\gamma,\delta=1}^{n} \tilde{R}_{\alpha\beta\gamma}^{\delta} dy^\alpha \otimes dy^\beta \otimes dy^\gamma \otimes \frac{\partial}{\partial y^\delta}$$

We write $R(X, Y)Z = \sum_{i,j,k,l=1}^{m} R_{ijk}^{l} \xi^j \eta^k \tau^i \frac{\partial}{\partial x^l}$ for $X = \sum_{j=1}^{m} \xi^j \frac{\partial}{\partial x^j}$, $Y = \sum_{k=1}^{m} \eta^k \frac{\partial}{\partial x^k}$, $Z = \sum_{i=1}^{m} \tau^i \frac{\partial}{\partial x^i}$. Similarly, $\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z}$ is defined. It is well-known that when one fixes both metrics g and h and vary the map f, the 1st variational formulae of the variation $\{f_t\}$ with $f_0 = f$ is given by $\frac{d}{dt} E(f_t)\big|_{t=0} = -\int_M h\,(\tau(f), V)\; d\mu_g$, where V is the variational vector field given by

$$V = \sum_{\alpha=1}^{n} \left.\frac{\partial f_t^\alpha}{\partial t}\right|_{t=0} \left(\frac{\partial}{\partial y^\alpha} \circ f \right) \in C^\infty(f^{-1}TN),$$

and $\tau(f)$ is the tension field along f given by

$$\begin{cases} \tau(f) = \sum_{\alpha=1}^{n} \sum_{i,j=1}^{m} g^{ij} f_{ij}^{\alpha} \left(\frac{\partial}{\partial y^{\alpha}} \circ f \right) \in C^{\infty}(f^{-1}TN), \\ f_{ij}^{\alpha} \quad = \frac{\partial}{\partial x^i} f_j^{\alpha} + \sum_{k=1}^{m} \Gamma_{ij}^{k} f_k^{\alpha} - \sum_{\beta,\gamma=1}^{n} f_i^{\beta} f_j^{\gamma} \tilde{\Gamma}_{\beta\gamma}^{\alpha}(f), \end{cases} \tag{2}$$

where Γ_{ij}^k and $\tilde{\Gamma}_{\beta\gamma}^{\alpha}$ are the Christoffel symbols for (M, g) and (N, h), respectively. We remark that $\Delta f^{\alpha} := \sum_{i,j=1}^{m} g^{ij} f_{ij}^{\alpha}$ is called the non-linear Laplacian of f. The Euler-Lagrange equation of the variation coincides with the equation $\tau(f) = 0$ which the critical point of the variation satisfies. This equation is called *harmonic map equation* and f is called *harmonic map*. When f is a harmonic map, the second variational formulae is given by

$$\frac{d^2}{dt^2} E(f_t)\bigg|_{t=0}$$
$$= \int_M \sum_{i,j=1}^{m} g^{ij} \left(h(\nabla_{\frac{\partial}{\partial x^i}} V, \nabla_{\frac{\partial}{\partial x^j}} V) - h(\tilde{R}(df(\frac{\partial}{\partial x^i}), V)V, df(\frac{\partial}{\partial x^j})) \right) d\mu_g.$$

In particular, if (N, h) is of non-positive sectional curvature then any harmonic map into (N, h) is (weakly) stable. Note that any holomorphic or anti-holomorphic map between compact Kähler manifolds is energy-minimizing map with respect any Kähler metrics compatible with the holomorphic structures in its homotopy class, whence it is a stable harmonic map. With reference to these facts, we may consider the problem "When is a stable harmonic map holomorphic or anti-holomorphic ? For the related results on this problem, see [1–4].

2 Instabilities of Holomorphic or Anti-Holomorphic Maps Between Kähler Manifolds

In this section, we discuss some stabilities and instabilities. In Sect. 1, a energy is defined for a smooth map $f : (M, g) \to (N, h)$. Note that, the energy is also depend on the metric g. From now on, fixing f, we consider the energy as a functional $E(g)$ on the space $\mathbf{T}_2^{0+}$ of Riemannian metrics on M with fixed volume.

Theorem 1. *Let f be a holomorphic or anti-holomorphic map from a Riemann surface (M, g) to a Kähler manifold (N, h). Here g is compatible with the holomorphic structure of M. Then g is a critical point of E. Moreover g is stable.*

In contrast with it, we have the following theorem:

Theorem 2. *Let f be a isometric holomorphic or anti-holomorphic map from a Kähler manifold (M, g) to a Kähler manifold (N, h) and $m \geq 3$. Then g is a critical point of E. Moreover g is unstable.*

We first prove Theorem 1.

Proof. Set $P = f^*h$ and $\eta = g$. Since f is holomorphic or anti-holomorphic, f is, in particular, conformal. So, there exists a function $\lambda \in C^\infty(M)$ such that $P = \lambda\eta$ and hence $P = \mathbf{p}\eta$, where $\mathbf{p}$ is as in Theorem 3. From Theorem 4, we see that $\delta I(\eta) = 0$. Since $2E(\eta) = V^{(m-2)/m} I(\eta)$, $\delta I(\eta) = 0$ implies that $\delta E(g) = 0$ for the variation on $\mathbf{T}_\mathbf{2}^{\mathbf{0}+}$. Using Theorem 5, it is straightforward to see that g is stable. □

Next, we prove Theorem 2.

Proof. Set $P = f^*h$ and $\eta = g$. Since f is isometric, $P = \eta$. Hence $P = 2\mathbf{p}\eta/m$, where $\mathbf{p}$ is as in Theorem 3. From Theorem 4, we see that $\delta I(\eta) = 0$. We can also see that g is a critical point of E in the same way as Theorem 1. Note that $\mathbf{p} \neq 0$ because f is isometric. By using the conditions $\mathbf{p} \neq 0$ and $m \geq 3$ and applying Theorem 6, we see that I is unstable at $\eta = g$. Since I is invariant under the homothetic transformation, the instability of I implies that of E. □

Appendix

We recall a weak form of results in [5]. Let $\mathbf{F}_\mathbf{2}^\mathbf{0}$ be the set of all smooth symmetric $(0, 2)$-tensors on M. Denote by $\mathbf{F}_\mathbf{2}^{\mathbf{0}+}$ the subset of all smooth positive definite symmetric $(0, 2)$-tensors. Define a function I on $\mathbf{F}_\mathbf{2}^\mathbf{0} \times \mathbf{F}_\mathbf{2}^{\mathbf{0}+}$ by

$$I(P, \eta) = \int_M \sum_{i,j=1}^{m} P_{ij}\eta^{ij}\, d\mu_\eta \quad \text{for } (P, \eta) \in \mathbf{F}_\mathbf{2}^\mathbf{0} \times \mathbf{F}_\mathbf{2}^{\mathbf{0}+}. \tag{3}$$

We normalize I so that it is invariant under the homothetic transformation. Next, fixing $P \in \mathbf{F}_\mathbf{2}^\mathbf{0}$, we define $I(\eta)$ by

$$I(\eta) := \frac{\displaystyle\int_M \sum_{i,j=1}^{m} P_{ij}\eta^{ij}\, d\mu_\eta}{\left(\displaystyle\int_M d\mu_\eta\right)^{\frac{m-2}{m}}} \tag{4}$$

Theorem 3 (1st Variation Formula). $\delta I(\eta) = \frac{1}{V^{c+1}} \langle\langle V\,(\mathbf{p}\eta - P) - cU\eta, \delta\eta \rangle\rangle$,

$$\text{where} \quad V = \int_M d\mu_\eta, \quad \mathbf{p} = \frac{1}{2}\sum_{i,j=1}^{m} P_{ij}\eta^{ji}, \quad U = \int_M \mathbf{p}\, d\mu_\eta, \quad c = \frac{m-2}{m},$$

$$< p, q > = \sum_{i,j,k,l=1}^{m} p_{ij}\eta^{jk}q_{kl}\eta^{li},$$

$$\langle\langle p, q \rangle\rangle = \int_M < p, q > \; d\mu_\eta, \qquad (p, q \in \mathbf{F}_\mathbf{2}^\mathbf{0}).$$

Theorem 4. *$\delta I(\eta) = 0$ if and only if $P = 2\mathbf{p}\eta/m$. Moreover $\mathbf{p}$ is constant if $\delta I(\eta) = 0$ and $m \neq 2$.*

Theorem 5 (2nd Variation Formula). *Assume that $\delta I(\eta) = 0$. Then,*

$$\delta^2 I(\eta) = \frac{1}{V^{c+1}} \left\{ \frac{m-2}{m^2 V} \langle\langle \mathbf{p} \rangle\rangle \langle\langle \mathrm{trace}_\eta \delta\eta \rangle\rangle^2 - \frac{V}{m} \langle\langle \mathbf{p}(\mathrm{trace}_\eta \delta\eta)^2 \rangle\rangle + \right.$$
$$\left. + \frac{2V}{m} \langle\langle \mathbf{p}\delta\eta, \delta\eta \rangle\rangle \right\}. \tag{5}$$

In particular, if $m = 1, 2$ and $\mathbf{p}$ is a non-negative(resp. non-positive) then $\delta^2 I(\eta) \geq 0$ (resp. $\delta^2 I(\eta) \leq 0$) holds.

Theorem 6. *Assume that $m \geq 3$ and the critical point η of I is stable. Then $\mathbf{p} = 0$.*

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