On the Hamiltonian Minimality of Normal Bundles

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Abstract A Hamiltonian minimal (shortly, H-minimal) Lagrangian submanifold in a Kähler manifold is a critical point of the volume functional under all compactly supported Hamiltonian deformations. We show that any normal bundle of a principal orbit of the adjoint representation of a compact simple Lie group G in the Lie algebra $\mathfrak{g}$ of G is an H-minimal Lagrangian submanifold in the tangent bundle $T\mathfrak{g}$ which is naturally regarded as $\mathbb{C}^m$. Moreover, we specify these orbits with this property in the class of full irreducible isoparametric submanifolds in the Euclidean space.

1 Introduction

A Lagrangian submanifold L is an m-dimensional submanifold in a $2m$-dimensional symplectic manifold (M, ω) on which the pull-back of the symplectic form ω vanishes. When M is a Kähler manifold, extrinsic properties of Lagrangian submanifolds have been studied by many authors. Since the Lagrangian property is preserved by Hamiltonian flows, namely, flows generated by Hamiltonian vector fields on M, it is natural to consider the variational problem under the Hamiltonian constraint. A Lagrangian submanifold which attains an extremal of the volume functional under Hamiltonian deformations is called *Hamiltonian minimal* (shortly, H-minimal). This was first investigated by Oh [26], where he gave some basic examples. Many more examples have been constructed in Kähler manifolds by various methods (see Sect. 2).

In this note, we review some basic results of H-minimal Lagrangian submanifolds in a general Kähler manifold (Sect. 2). Furthermore, we focus on constructions of H-minimal Lagrangian submanifolds in the complex Euclidean space $\mathbb{C}^m$ (Sect. 3.1). In particular, we give a new family of non-compact, complete H-minimal Lagrangian submanifolds in the complex Euclidean space $\mathbb{C}^m$ (Sect. 3.2 through 3.4).

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Let N^n be a submanifold in $\mathbb{R}^{n+k}$. Our examples are given by the normal bundle νN of N in $T\mathbb{R}^{n+k} \simeq \mathbb{C}^{n+k}$. It is known that the normal bundle νN is a Lagrangian submanifold in $T\mathbb{R}^{n+k}$. Harvey–Lawson [10] first showed that νN is a minimal Lagrangian submanifold if and only if N is an austere submanifold, namely, the set of principal curvatures of N with respect to any unit normal vector is invariant under the multiplication by -1. In their context, the condition that a Lagrangian submanifold is minimal is equivalent to that it is a special Lagrangian submanifold of some phase. Hence, one can construct examples of special Lagrangian submanifold in $\mathbb{C}^{n+k}$ from austere submanifolds. On the other hand, in [18], the author proves that any normal bundle over the principal orbit of the adjoint action of a compact semi-simple Lie group G is a non-minimal, H-minimal Lagrangian submanifold. Such an orbit is called the complex flag manifold or regular Kähler C-space. Moreover, we show that this property characterizes regular C-spaces among the class of full and irreducible isoparametric submanifolds in the Euclidean space (Sect. 3.3, Theorem 1). In Sect. 3.4, we review a proof of this result which is given in [18].

2 Hamiltonian Minimal Lagrangian Submanifolds

Let $\iota : L \to M$ be a Lagrangian immersion into a Kähler manifold (M, ω, J), where ω is the Kähler form and J is the complex structure on M. An infinitesimal deformation $\iota_t : L \times (-\varepsilon, \varepsilon) \to M$ of ι is called a *Hamiltonian deformation* if $\alpha_{\tilde{V}_t} \in \Omega^1(L)$ is an exact form for $t \in (-\varepsilon, \varepsilon)$, namely, $\alpha_{V_t} = df_t$ for some functions $f_t \in C_0^\infty(L)$, where $\tilde{V}_t := d\iota_t/dt$ is the variational vector field of ι_t. Define the *mean curvature form* of ι by $\alpha_H := \iota^*(\omega(H, \cdot))$, where H is the mean curvature vector of ι. A Lagrangian immersion ι is called *minimal* if $\alpha_H = 0$, or equivalently $H = 0$. When M is Kähler–Einstein, the mean curvature form α_H is a closed 1-from by the result of Dazord [9], and hence, it defines a real cohomology class $[\alpha_H] \in H^1(L, \mathbb{R})$. It is known that any Hamiltonian isotopy preserves $[\alpha_H]$, namely, under any global Hamiltonian isotopy $\{\iota_t\}_{0 \le t \le 1}$ of $\iota = \iota_0$, the 1-forms α_{H_t} on L represent the same cohomology class, where α_{H_t} is the mean curvature form of ι_t (see [27]). In particular, for a Lagrangian immersion ι into a Kähler–Einstein manifold M, if there exist a minimal Lagrangian immersion in its Hamiltonian isotopy class, then $[\alpha_H] = 0$. Therefore, there exist an obstruction for the existence of *minimal* Lagrangian submanifold in the Hamiltonian isotopy class (see also [30]).

A Lagrangian immersion ι is called *Hamiltonian minimal* (shortly, *H-minimal*) if it is a critical point of the volume functional under all compactly supported Hamiltonian deformations. It is known that ι is H-minimal if and only if the mean curvature form $\alpha_H \in \Omega^1(L)$ satisfies the equation $\delta\alpha_H = 0$, where δ is the codifferential acting on $\Omega^1(L)$ (see [26]). When M is Kähler–Einstein, the maximum principle implies that if $\iota : L \to M$ is a non-minimal, H-minimal immersion of a compact manifold L into M, then $H^1(L, \mathbb{R}) \neq 0$ ([26]).

Example 1. (1) Any *minimal* Lagrangian immersion is H-minimal. Thus, the notion of H-minimality is an extension of minimal submanifold.

(2) Any Lagrangian immersion with the *parallel mean curvature vector* (i.e., $\nabla^{\perp} H = 0$) is H-minimal.

(3) A curve with constant geodesic curvature in a Riemann surface.

(4) Any compact extrinsically homogeneous Lagrangian submanifold in a Kähler manifold.

An H-minimal Lagrangian immersion $\iota : L^n \to M^{2n}$ is *Hamiltonian stable (or H-stable)* if the second variation of the volume functional of the immersion is non-negative for all Hamiltonian deformations $\{\iota_t\}_t$. In [26], Oh derived the second variation under a Hamiltonian deformation for a compact Lagrangian submanifold in a Kähler manifold as follows:

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathrm{Vol}(\iota_t(L)) = \int_L \left\{ |\Delta f|^2 - \overline{\mathrm{Ric}}(\nabla f) - 2g(B(\nabla f, \nabla f), H) + g(J\nabla f, H)^2 \right\} dv_L,$$

where $\alpha_{V_0} = df$, $\overline{\mathrm{Ric}}$ is the Ricci curvature of M, and B is the second fundamental form of L. When M is Kähler–Einstein, and L is a compact minimal Lagrangian submanifold, it turns out that the H-stability is equivalent to $\lambda_1 \geq c$, where λ_1 is the first eigenvalue of Δ acting on $C^{\infty}(L)$ and c is the Einstein constant of M. In particular, any compact minimal Lagrangian submanifold in a Kähler–Einstein manifold with non-positive Ricci curvature is H-stable.

Example 2. The following examples are H-stable.

(1) Einstein real forms (i.e., the fixed point sets of an anti-holomorphic involution on M) in a Hermitian symmetric space of compact type [25].

(2) The standard tori $T^m = S^1(r_1) \times \cdots S^1(r_m)$ in $\mathbb{C}^m$ [26].

(3) Lagrangian submanifolds with parallel second fundamental form in $\mathbb{C}^m$ or $\mathbb{C}P^m$ [1, 2].

For more examples of H-stable Lagrangian submanifold, we refer to [20, 21] and a survey article by Ohnita [28].

A diffeomorphism ϕ on M is called a *Hamiltonian diffeomorphism* of M if ϕ satisfies the following conditions:

(i) ϕ is *symplectic*, namely, $\phi^* \omega = \omega$.

(ii) ϕ is represented by the flow $\{\phi_t\}_{t\in[0,1]}$ of a time dependent Hamiltonian vector field $\{X_{F_t}\}$ on M, namely, $d/dt(\phi_t(x)) = X_{F_t}(\phi_t(x))$ with $\phi_0 = Id_M$ and $\phi_1 = \phi$, where $\omega(X_{F_t}, \cdot) = dF_t$ for $F_t \in C_0^{\infty}(M)$.

We denote the set of all Hamiltonian diffeomorphisms by $\mathrm{Ham}(M, \omega)$. A Lagrangian submanifold L in M is called *Hamiltonian volume minimizing* (or shortly, *H.V.M.* Lagrangian submanifold) if L is a volume minimizer of any Hamiltonian diffeomorphism, namely, L satisfies the inequality $\mathrm{Vol}(\phi(L)) \geq \mathrm{Vol}(L)$ for any $\phi \in \mathrm{Ham}(M, \omega)$. By definition, it follows that an H.V.M. Lagrangian submanifold is necessarily H-minimal and H-stable. We know only a few examples of H.V.M. Lagrangian submanifolds.

Example 3. (1) The totally geodesic $\mathbb{R}P^n$ in $\mathbb{C}P^n$ (Kleiner–Oh, cf. [25]).
(2) The product of two equators $S^1 \times S^1$ in $S^2 \times S^2$ (Iriyeh–Ono–Sakai, [16]).
(3) The totally geodesic S^n in $Q_n(\mathbb{C})$ (cf. Iriyeh–Sakai–Tasaki, [17]).

Note that all known examples of H.V.M. Lagrangians belong to Einstein real forms in a Hermitian symmetric space. Based on these examples, Oh posed the following conjecture:

Conjecture 1. Let L be a real form, i.e., the fixed point sets of an anti-holomorphic involution of a Kähler–Einstein manifold M. If L is Einstein, then L is H.V.M.

More generally, we consider the following problem:

Problem 1. Construct and classify H-minimal Lagrangian submanifolds, H-stable Lagrangian submanifolds and H.V.M. Lagrangian submanifolds in a specific Kähler manifold.

3 Hamiltonian Minimality of Normal Bundles in $T\mathbb{R}^{n+k}$

3.1 H-Minimal Lagrangian Submanifolds in $\mathbb{C}^m$

Let L be an oriented Lagrangian submanifold in the complex Euclidean space $\mathbb{C}^m$. The *Lagrangian angle* of L is an S^1-valued function $\theta : L \to S^1 = \mathbb{R}/2\pi\mathbb{Z}$ on L defined by

$$e^{\sqrt{-1}\theta(p)} = dz_1 \wedge \ldots \wedge dz_m(e_1, \ldots, e_m)(p),$$

where $z_i = x_i + \sqrt{-1}y_i$ and $\{e_1, \ldots, e_m\}$ is an oriented orthonormal basis of L. Then one can show that the mean curvature form α_H of L satisfies the relation

$$\alpha_H = -d\theta. \tag{1}$$

Recall that a Lagrangian submanifold L in $\mathbb{C}^m$ is *special Lagrangian* with phase $e^{\sqrt{-1}\theta}$ if L is calibrated by the calibration $\mathrm{Re}(e^{-\sqrt{-1}\theta}\Omega)$, where $\Omega = dz_1 \wedge \ldots \wedge dz_m$. A special Lagrangian submanifold is automatically volume minimizing in its homology class.

Proposition 1. *For an oriented, connected Lagrangian submanifold L in $\mathbb{C}^m$, we have (i) ι is special Lagrangian if and only if θ is constant, and (ii) ι is H-minimal if and only if θ is harmonic (as a S^1-valued function), namely, $\Delta\theta = 0$.*

The minimality of a Lagrangian submanifold L in $\mathbb{C}^m$ is equivalent to that L is a special Lagrangian submanifold of some phase (see Proposition 2.17 in [10]).

We also note that there exist no compact minimal submanifolds in the (complex) Euclidean space.

On the other hand, Oh [26] pointed out that the standard tori $T^m = S^1(r_1) \times \cdots \times S^1(r_m)$ are H-minimal. Generalizing Oh's results [26], Dong [8] showed that the pre-image of an H-minimal Lagrangian submanifold in the complex projective space $\mathbb{C}P^{m-1}$ via the Hopf fibration $\pi : S^{2m-1} \to \mathbb{C}P^{m-1}$ is H-minimal Lagrangian in $\mathbb{C}^m$. We note that there are some known H-minimal Lagrangian submanifolds in $\mathbb{C}P^{m-1}$. For instance, any compact, extrinsically homogeneous Lagrangian submanifolds in $\mathbb{C}P^{m-1}$ are H-minimal, and Bedulli and Gori [4] gives the complete classification of Lagrangian orbits which are obtained by a simple Lie group of isometries acting on $\mathbb{C}P^{m-1}$. On the other hand, Anciaux and Castro [3] gave examples of H-minimal Lagrangian immersions of manifolds with various topology by taking a product of a Lagrangian surface and Legendrian immersions in odd-dimensional unit spheres. Note that these examples are compact and contained in a sphere. In [18], we give a new family of non-compact, complete H-minimal Lagrangian submanifolds in $\mathbb{C}^m$, which is described in the following subsections. For more examples in $\mathbb{C}^m$ (and $\mathbb{C}P^{m-1}$), we refer to [1–3, 11, 12] and references therein.

3.2 Normal Bundles in $T\mathbb{R}^{n+k}$

Let $\mathbb{R}^{n+k}$ be the Euclidean space with the standard flat metric $\langle , \rangle$. Denote the tangent bundle of $\mathbb{R}^{n+k}$ by $T\mathbb{R}^{n+k}$. Since $T\mathbb{R}^{n+k}$ is trivial, it is identified with the direct sum $\mathbb{R}^{n+k} \oplus \mathbb{R}^{n+k}$ on which we can define the flat metric $g(,)$ induced from $\langle , \rangle$. Moreover, we define the complex structure J by $J(X, Y) = (-Y, X)$ for $(X, Y) \in T_p\mathbb{R}^{n+k} \oplus T_u\mathbb{R}^{n+k}$ where $(p, u) \in \mathbb{R}^{n+k} \oplus \mathbb{R}^{n+k}$. By this identification, we regard $T\mathbb{R}^{n+k}$ as the complex Euclidean space $\mathbb{C}^{n+k}$ with the standard Kähler form $\omega := g(J\cdot, \cdot)$. Let $\iota : N^n \to \mathbb{R}^{n+k}$ be an isometric embedding of an n-dimensional smooth manifold into $\mathbb{R}^{n+k}$. In the following, we always identify N with its image under ι, and call it a submanifold in $\mathbb{R}^{n+k}$. Define the *normal bundle* of N by $\nu N := \{(p, u) \in T\mathbb{R}^{n+k};\ p \in N, u \perp T_pN\}$. This is an $(n + k)$-dimensional submanifold in $T\mathbb{R}^{n+k}$. Moreover, one can check that νN is Lagrangian in $T\mathbb{R}^{n+k}$ with respect to the standard symplectic form.

We denote the Levi–Civita connections on $\mathbb{R}^{n+k}$ and $T\mathbb{R}^{n+k}$ by ∇ and $\tilde{\nabla}$, respectively. For a normal vector $u \in \nu_pN$ at $p \in N$, the *shape operator* $A^u \in \mathrm{End}(T_pN)$ is defined by $A^u(X) := -(\nabla_X u)^\top$ for $X \in T_pN$, where $\top$ denotes the tangent component of the vector. Since A^u is represented by a symmetric matrix, the eigenvalues of A^u are real, and we denote it by $\kappa_i(p, u)$ for $i = 1, \ldots, n$. If u is an unit normal vector, these eigenvalues are called the *principal curvatures* of N with respect to the normal direction u.

Lemma 1 ([18]). *Let N^n be an oriented submanifold in $\mathbb{R}^{n+k}$. Then the Lagrangian angle of the normal bundle νN in $T\mathbb{R}^{n+k} \simeq \mathbb{C}^{n+k}$ is given by*

$$\theta(p,u) = -\sum_{i=1}^{n} \mathrm{Arctan}\kappa_i(p,u) + \frac{k\pi}{2} \pmod{2\pi}, \tag{2}$$

where $\mathrm{Arctan}\kappa_i(p,u)$ *denotes the principal value of* $\arctan\kappa_i(p,u)$.

By the relation (1) and (2), the mean curvature form of the normal bundle can be written by

$$\alpha_H = d\Big(\sum_{i=1}^{n} \arctan\kappa_i\Big). \tag{3}$$

For convenience, we put $\tilde{\theta} := \sum_{i=1}^{n} \arctan\kappa_i$.

Remark 1. We remark that a similar formula of (3) has been obtained by Palmer [33] in a different situation.

The following necessary and sufficient conditions for the minimality of normal bundles in $\mathbb{C}^{n+k}$ was first given by Harvey–Lawson [10]:

Proposition 2 (Theorem 3.11 in [10]). *Let* N^n *be a connected submanifold in* $\mathbb{R}^{n+k}$. *Then the normal bundle* νN *is a minimal Lagrangian submanifold in* $T\mathbb{R}^{n+k} \simeq \mathbb{C}^{n+k}$ *if and only if* N *is austere, namely, the set of principal curvatures* $\{\kappa_i(p,u)\}_{i=1}^{n}$ *is invariant under the multiplication by* -1.

By using this result, one can produce examples of special Lagrangian submanifolds in $\mathbb{C}^{n+k}$ from austere submanifolds in $\mathbb{R}^{n+k}$.

By the explicit formulation of the Lagrangian angle of νN given in Lemma 2.1, we improve Harvey–Lawson's result a bit as follows:

Proposition 3 ([18]). *Let* N^n *be a submanifold in* $\mathbb{R}^{n+k}$. *If the mean curvature vector of the normal bundle* νN *is parallel in* $T\mathbb{R}^{n+k} \simeq \mathbb{C}^{n+k}$, *then* νN *is minimal.*

By Proposition 2 and 3, we obtain the following.

Corollary 1. *Let* N^n *be a submanifold in* $\mathbb{R}^{n+k}$. *Then the following three are equivalent: (i)* N *is austere, (ii) the normal bundle* νN *is minimal in* $T\mathbb{R}^{n+k} \simeq \mathbb{C}^{n+k}$, *(iii)* νN *has parallel mean curvature vector.*

In the following, we investigate the H-minimality of a Lagrangian submanifold in the complex Euclidean space $\mathbb{C}^{n+k}$ obtained as the normal bundle of a submanifold N^n in $\mathbb{R}^{n+k}$. By Lemma 1, the H-minimality of the normal bundle νN in $\mathbb{C}^{n+k}$ is equivalent to

$$\Delta\tilde{\theta} = 0, \ \text{where } \tilde{\theta} := \sum_{i=1}^{n} \arctan\kappa_i. \tag{4}$$

We recall that there are no non-minimal, H-minimal Lagrangian normal bundles in $\mathbb{C}^{n+k}$ with parallel mean curvature vector by Corollary 1.

Besides, one can show that the normal bundle of the Riemannian product $N_1 \times N_2 \to \mathbb{R}^{n_1+k_1} \times \mathbb{R}^{n_2+k_2}$ of two embeddings $N_i \to \mathbb{R}^{n_i+k_i}$ $(i = 1, 2)$ is H-minimal if and only if each of νN_i is H-minimal. Thus, in the following, our concern is always irreducible ones.

3.3 Normal Bundles over Isoparametric Submanifolds

In [18], we classify isoparametric submanifolds in $\mathbb{R}^{n+k}$ with H-minimal normal bundles. Before describing the main result, we briefly review the isoparametric submanifolds in $\mathbb{R}^{n+k}$ (For more details, refer to [5, 39] and references therein).

Let N^n be a submanifold in $\mathbb{R}^{n+k}$ of an arbitrary codimension k. There are several ways to define the notion of isoparametric submanifolds (see [39]). In this article, we consider the following two conditions.

(i) For any parallel normal vector field $u(t)$ along a piece-wise smooth curve $c(t)$ on N, the shape operator $A^{u(t)}$ has constant eigenvalues.
(ii) The normal bundle of N is flat, namely, $R^{\perp} = 0$, where $R^{\perp}$ denotes the curvature tensor with respect to the normal connection of N.

If N satisfies the condition (i), we say N *has constant principal curvatures*. If N satisfies both conditions, we call N an *isoparametric submanifold*. It is known that any non-compact complete isoparametric submanifold is a product of compact isoparametric submanifolds and the Euclidean space (see [37]). Since the Euclidean factor is obviously austere, we may assume that an isoparametric submanifold N is compact for our purpose.

In the following, we consider an isoparametric submanifold N^n in $\mathbb{R}^{n+k}$.

The isoparametric hypersurfaces in $\mathbb{R}^{n+1}$ are classified by Somigliana [35] for $n = 3$, and Segre [34] for the general dimension. We denote the number of distinct principal curvatures by g. Then g is at most two, and an isoparametric hypersurface in $\mathbb{R}^{n+1}$ is one of the following:

$g = 1$: An affine hyperplane $\mathbb{R}^n$ or a hypersphere $S^n(r)$, where $r > 0$.
$g = 2$: A spherical cylinder $\mathbb{R}^k \times S^{n-k}(r)$, i.e., a tube around an affine plane $\mathbb{R}^k$, where $r > 0$.

The codimension two isoparametric submanifolds in $\mathbb{R}^{n+2}$ are known as isoparametric hypersurfaces in the unit sphere $S^{n+1}(1)$. One of large subclasses of these hypersurfaces are extrinsically homogeneous hypersurfaces in $S^{n+1}(1)$ and these are classified by Hsiang–Lawson [10]. This result asserts that all homogeneous hypersurfaces in $S^{n+1}(1)$ are obtained by principal orbits of s-representations of symmetric spaces of rank 2, where the s-representation is the isotropy representation of a symmetric space U/K (see Sect. 3.4). Other classes includes infinitely many non-homogeneous examples due to Ozeki–Takeuchi and Ferus–Karcher–Münzner. These are the so called isoparametric hypersurfaces of *OT-FKM type* (for more details, refer to monographs [6, 39] and references therein). The classification of isoparametric hypersurfaces in $S^{n+1}(1)$ has not been completed yet. Let N be an isoparametric hypersurface in the unit sphere $S^{n+1}(1)$, and ν the unit normal vector field on N. We denote the distinct principal curvatures of N with respect to ν by $\kappa_i = \cot\theta_i$ with $0 < \theta_1 < \cdots < \theta_g < \pi$, and these multiplicities by m_i for $i = 1, \ldots, g$, respectively. Then, Münzner showed the following ([23]):

$$\theta_i = \theta_1 + \frac{i-1}{g}\pi, \text{ for } i = 1, \ldots, g, \tag{5}$$

$$m_i = m_{i+2}, \text{ modulo } g \text{ indexing.} \tag{6}$$

In particular, $0 < \theta_1 < \pi/g$, and the multiplicities are same if g is odd. Münzner also proved that the number of distinct principal curvatures g is equal to $1, 2, 3, 4$ or 6 [24].

On the other hand, Thorbergsson [38] proved that any full, irreducible, isoparametric submanifold in $\mathbb{R}^{n+k}$ with $k \geq 3$ is extrinsically homogeneous (see also Olmos [29]). Moreover, combining it with the results of Dadok [7] and Palais-Terng [32], they are principal orbits of an s-representation, namely, an isotropy orbit of semi-simple symmetric space U/K.

Let us describe the main results in [18]. For the H-minimality of normal bundles of isoparametric submanifolds, we prove the following:

Theorem 1 ([18]). *Let N be a full, irreducible isoparametric submanifold in the Euclidean space $\mathbb{R}^{n+k}$. Then the normal bundle νN is H-minimal in $T\mathbb{R}^{n+k} \simeq \mathbb{C}^{n+k}$ if and only if N is a principal orbit of the adjoint action of a compact simple Lie group G.*

In particular, we obtain:

Corollary 2 ([18]). *Let G be a compact, connected, semi-simple Lie group, $\mathfrak{g}$ the Lie algebra of G, and $N^n = \mathrm{Ad}(G)w$ a principal orbit of the adjoint action of G on $\mathfrak{g} \simeq \mathbb{R}^{n+k}$ through $w \in \mathfrak{g}$. Then the normal bundle νN of N is an H-minimal Lagrangian submanifold in the tangent bundle $T\mathfrak{g} \simeq \mathbb{C}^{n+k}$.*

The principal orbit N is diffeomorphic to G/T, where T is a maximal torus of G, and N is called a *complex flag manifold* or *regular Kähler C-space*. Since $N = \mathrm{Ad}(G)w$ is compact, N is never austere in $\mathbb{R}^{n+k}$, and hence, νN is not minimal. Moreover, it does not have parallel mean curvature vector (see Proposition 3). We also note that the normal bundle of $N = \mathrm{Ad}(G)w$ is always trivial, namely, νN is homeomorphic to $N \times \mathbb{R}^k$.

3.4 Outline of a Proof of Theorem 1

The strategy of the proof of Theorem 1 in [18] is as follows. When N is an isoparametric submanifold, the differential equation (4) on νN is expressed in terms of the eigenvalues of the shape operators of N. If the codimension of isoparametric submanifold is equal to 1, by using the classification results, we specify submanifolds with (4). The full and irreducible (or compact) one is the hypersphere. Then we have the following:

Proposition 4. *The normal bundle of the n-dimensional hypersphere $N^n = S^n(r)$ with radius $r > 0$ in $\mathbb{R}^{n+1}$ is H-minimal if and only if $n = 2$.*

When the codimension is 2, they are isoparametric hypersurfaces in the sphere, and the known examples consist of principal orbits of s-representations and non-homogeneous ones. By applying the relations (5) and (6) to (4), we obtain the following crucial lemma:

Lemma 2. *Let N^n be an isoparametric hypersurface in the unit sphere $S^{n+1}(1) \subset \mathbb{R}^{n+2}$. Suppose that the normal bundle νN of N as a submanifold in $\mathbb{R}^{n+2}$ is H-minimal in $\mathbb{C}^{n+2} \simeq T\mathbb{R}^{n+2}$. Then the multiplicities of the distinct principal curvatures in $\{\kappa_i\}_{i=1}^n$ are all equal to 2.*

In particular, it turns out that N is a homogeneous hypersurface. In fact, Cartan proved this for $g \leq 3$, and Ozeki–Takeuchi for the case $(g, m) = (4, 2)$ [31] . The remaining case $(g, m) = (6, 2)$ was settled by R. Miyaoka [22], where m is the same multiplicity. Therefore, together with the results of Hsiang–Lawson [14] and the fact that isoparametric submanifolds of codimension grater than three are homogeneous (Thorbergsson [38]), it is sufficient to consider the normal bundle of principal orbits of s-representations.

The eigenvalues of the shape operators of these orbits are given by the restricted root systems of associated symmetric spaces. Let (U, K) be a Riemannian symmetric pair of compact type, where U is a compact, connected real semi-simple Lie group and K a closed subgroup of U such that there exist an involutive automorphism σ of U so that $\mathrm{Fix}(\sigma, U)^0 \subset K \subset \mathrm{Fix}(\sigma, U)$, where $\mathrm{Fix}(\sigma, U) := \{g \in U;\ \sigma(g) = g\}$ and $\mathrm{Fix}(\sigma, U)^0$ is the identity component of $\mathrm{Fix}(\sigma, U)$. Denote the Lie algebra of U and K by $\mathfrak{u}$ and $\mathfrak{k}$, respectively. Let $(\mathfrak{u}, \sigma)$ be the orthogonal symmetric Lie algebra which corresponds to (U, K), namely, σ is an involution on $\mathfrak{u}$ such that the $+1$-eigenspace coincides with $\mathfrak{k}$ and $\mathfrak{k}$ is a compactly embedded Lie algebra in $\mathfrak{u}$.

We take an inner product $\langle , \rangle$ of $\mathfrak{u}$ which is invariant under σ and $\mathrm{Ad}(U)$ on $\mathfrak{u}$. Then we have the orthogonal decomposition $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$. Since the subspace $\mathfrak{p}$ is invariant under $\mathrm{Ad}(K)|_{\mathfrak{p}}$, K acts on $\mathfrak{p}$ as an orthogonal transformation. We call this action of K the *s-representation* of the symmetric space U/K.

Choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. For an 1-form λ on $\mathfrak{a}$, set

$$\mathfrak{k}_\lambda := \{X \in \mathfrak{k};\ (\mathrm{ad}H)^2 X = -\lambda(H)^2 X \text{ for all } H \in \mathfrak{a}\},$$

$$\mathfrak{p}_\lambda := \{X \in \mathfrak{p};\ (\mathrm{ad}H)^2 X = -\lambda(H)^2 X \text{ for all } H \in \mathfrak{a}\}.$$

Then $\mathfrak{p}_{-\lambda} = \mathfrak{p}_\lambda$, $\mathfrak{k}_{-\lambda} = \mathfrak{k}_\lambda$, $\mathfrak{p}_0 = \mathfrak{a}$, and $\mathfrak{k}_0$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. A non-zero 1-form λ is called a *(restricted) root* of $(\mathfrak{u}, \sigma)$ with respect to $\mathfrak{a}$ if $\mathfrak{p}_\lambda \neq \{0\}$. We denote the set of all roots of $(\mathfrak{u}, \sigma)$ by R, and call R the *restricted root system* on $\mathfrak{a}$. We take a basis of the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$ and define the lexicographic ordering on $\mathfrak{a}^*$ with respect to the basis. We call a root $\lambda \in R$ a *positive root* if $\lambda > 0$, and put $R_+ := \{\lambda \in R;\ \lambda > 0\}$. Then we have decompositions

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\lambda \in R_+} \mathfrak{k}_\lambda, \quad \mathfrak{p} = \mathfrak{a} + \sum_{\lambda \in R_+} \mathfrak{p}_\lambda. \tag{7}$$

These are orthogonal direct sums with respect to $\langle , \rangle$. We put $m_\lambda := \dim_{\mathbb{R}} \mathfrak{p}_\lambda$, and call it the *multiplicity* of $\lambda \in R_+$.

Let us consider orbits of the s-representation. Since any s-representation is polar (see [5]) and the section is given by $\mathfrak{a}$, it is sufficient to consider the orbits through a point $w \in \mathfrak{a}$. The point w is called a *regular* element if $\lambda(w) \neq 0$ for any $\lambda \in R$ (otherwise, it is called *singular*). We note that regular orbits are orbits of maximal dimension [36]. Since the isotropy action does not have any exceptional orbit, an orbit is regular if and only if it is principal.

When w is a regular element, we have the following [36] (See also [15]):

(i) The tangent space $T_w N_w$ and the normal space $\nu_w N_w$ of N_w at w in $\mathfrak{p}$ are given by

$$T_w N_w = \sum_{\lambda \in R_+} \mathfrak{p}_\lambda, \quad \nu_w N_w = \mathfrak{a}.$$

In particular, $\mathrm{codim} N_w = \dim \mathfrak{a}$.

(ii) The shape operator A^u of N_w in $\mathfrak{p}$ in the direction $u \in \nu_w N_w$ satisfies

$$A^u(X_\lambda) = -\frac{\lambda(u)}{\lambda(w)} X_\lambda \text{ for } X_\lambda \in \mathfrak{p}_\lambda \text{ and } \lambda \in R_+.$$

By using these, we characterize the H-minimality of normal bundles over the principal orbits of s-representations as follows: For the root system R, we set

$$r := \{\lambda \in R;\ \lambda/2 \notin R\}, \text{ and } r_+ := r \cap R_+.$$

Then r is a reduced root system, namely, if two roots $\lambda, \mu \in r$ are proportional, then $\mu = \pm\lambda$. We also set $l_\lambda := m_\lambda + m_{2\lambda}$, where $m_{2\lambda} = 0$ unless $2\lambda \in r$. By using an argument of the reduced root system, we prove the following.

Proposition 5. *Let $N^n = N_w$ be a regular orbit of an s-representation through an element $w \in \mathfrak{p} \simeq \mathbb{R}^{n+k}$. Then the normal bundle νN is H-minimal in $T\mathfrak{p} \simeq \mathbb{C}^{n+k}$ if and only if $l_\lambda = 2$ for all $\lambda \in r_+$ (In fact, this is equivalent to $m_\lambda = 2$ for all $\lambda \in R_+$).*

On the other hand, we have the following characterization of symmetric spaces of Type II due to Loos [19]:

Proposition 6 (cf. [19]). *Let* $(\mathfrak{u}, \sigma)$ *be an effective, irreducible orthogonal symmetric Lie algebra of compact type and* $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ *the* ± 1*-eigenspace decomposition with respect to* σ*. Then the following statements are equivalent:*

(i) For the restricted root system R *of* $(\mathfrak{u}, \sigma)$*,* $m_\lambda = 2$ *for all* $\lambda \in R_+$*.*
(ii) The dual $\mathfrak{u}^* := \mathfrak{k} + \sqrt{-1}\mathfrak{p}$ *of* $\mathfrak{u}$ *has a complex structure (i.e., there exist a complex structure* J *on* $\mathfrak{u}$ *such that* $J[X, Y] = [X, JY]$ *for any* $X, Y \in \mathfrak{u}$*).*
(iii) $(\mathfrak{u}, \sigma)$ *is isomorphic to an irreducible orthogonal symmetric Lie algebra of Type II (in the sense of [13]).*

The compact Lie group G is regarded as a symmetric space of the Riemannian symmetric pair $(G \times G, \Delta G)$, where $\Delta G = \{(g, g) \in G \times G; g \in G\} \simeq G$, and the isotropy representation is equivalent to the adjoint representation of G. Since the associated globally symmetric space with $(\mathfrak{u}, \sigma)$ of Type II is a compact, connected simple Lie group G, the assertion of Theorem 1 follows from Proposition 5 and 6.

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