A New Technique for the Study of Complete Maximal Hypersurfaces in Certain Open Generalized Robertson–Walker Spacetimes

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Abstract An $(n + 1)$-dimensional Generalized Robertson–Walker (GRW) spacetime such that the universal Riemannian covering of the fiber is parabolic (thus so is the fiber) is said to be spatially parabolic. This class of spacetimes allows to model open relativistic universes which extend to the spatially closed GRW spacetimes from the viewpoint of the geometric-analysis of the fiber and which are not incompatible with certain cosmological principle. We explain here a new technique for the study of non-compact complete spacelike hypersurfaces in such spacetimes. Thus, a complete spacelike hypersurface in a spatially parabolic GRW spacetime inherits the parabolicity, whenever some boundedness assumptions on the restriction of the warping function to the spacelike hypersurface and on the hyperbolic angle between the unit normal vector field and a certain timelike vector field are assumed. Conversely, the existence of a simply connected parabolic spacelike hypersurface, under the previous assumptions, in a GRW spacetime also leads to its spatial parabolicity. Then, all the complete maximal hypersurfaces in a spatially parabolic GRW spacetime are determined in several cases, extending known uniqueness results. Finally, all the entire solutions of the maximal hypersurface equation on a parabolic Riemannian manifold are found in several cases, solving new Calabi–Bernstein problems.

1 Introduction

In the study of complete spacelike surfaces M in certain three-dimensional GRW spacetimes $\overline{M}$, whose mean curvature function H satisfies: $H = 0$, $H =$ constant or $H^2 \leq \frac{f'(t)^2}{f(t)^2}$, one arrives to the parabolicity of the surface as an intermediate technical step. Normally, it follows from a property of the Gauss curvature of the surface (obtained via the Gauss equation) and an intrinsic result to get the parabolicity on a two-dimensional (non-compact) complete Riemannian manifold (see for instance [15]). In fact, parabolicity for two-dimensional Riemannian manifolds is

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very close to the classical parabolicity for Riemann surfaces. Moreover, it is strongly related to the behavior of the Gauss curvature of the surface. For instance, by a classical result by Ahlfors and Blanc–Fiala–Huber [11], a complete Riemannian surface (M, g) with non-negative Gauss curvature K must be parabolic. The same conclusion is attained if (M, g) is complete and we have either $K \geq -1/(r^2 \log r)$, for r, the distance to a fixed point sufficiently large [8] or if the negative part of K is integrable on M, [12], i.e., $\int_M K_- < \infty$, where $K_-(p) := \max\{-K(p), 0\}$, for any $p \in M$.

Parabolicity has no clear relation with curvature for bigger dimension and, therefore, other techniques are normally used in that case. However, the parabolicity of a complete spacelike hypersurface in a certain spacetime, may be obtained in another way independent of the dimension and of any curvature assumption [16]. Thus, our main aim here is to explain this new approach and to show, based on [16, 17], how it can be applied to prove several uniqueness results on complete maximal hypersurfaces.

2 Parabolicity of Riemannian Manifolds

An n-dimensional non-compact complete Riemannian manifold (M, g) is said to be parabolic if it admits no non-constant non-negative superharmonic function, i.e., if $u \in C^\infty(M)$ satisfies $\Delta_g u \leq 0$ and $u \geq 0$, then $u =$ constant.

To be parabolic is clearly a property invariant under (global) isometries. Even more, a Riemannian manifold (M, g) is said to be quasi-isometric to another one (M', g') if there exists a diffeomorphism $\phi : M \to M'$ and a constant $c \geq 1$ such that

$$c^{-1}|v|_g \leq |d\phi(v)|_{g'} \leq c\,|v|_g\,,$$

for all $v \in T_p M$, $p \in M$ (see for instance [12]). Obviously, isometric Riemannian manifolds are also quasi-isometric and to be quasi-isometric is an equivalence relation. Moreover, we have [10, 18],

Theorem 1. *Let (M, g) and (M', g') be quasi-isometric Riemannian manifolds. Then, (M, g) is parabolic if and only if (M', g') is parabolic.*

Remark 1. (a) The universal Riemannian covering map $\mathbb{R}^3 \to \mathbb{S}^1 \times \mathbb{R}^2$ is a local isometry. Note that $\mathbb{S}^1 \times \mathbb{R}^2$ is parabolic and $\mathbb{R}^3$ is not. Therefore, in the notion of quasi-isometry, the diffeomorphism cannot be relaxed to be a local diffeomorphism, (however, note that if a Riemannian covering $\tilde{M}$ of a Riemannian manifold M is parabolic, then M is also parabolic). (b) Theorem 1 also holds if the exterior of some compact subset in M is quasi-isometric to the exterior of a compact subset in M' [9, Cor. 5.3]. (c) There exists a notion much weaker than quasi-isometry: the so-called rough isometry (roughly isometric manifolds are not homeomorphic, in general). Under this hypothesis, it is necessary to impose extra geometric assumptions (in terms of the Ricci curvature and the injectivity radius) to get that parabolicity is preserved by rough isometries [10].

3 Set Up

For a Generalized Robertson–Walker (GRW) spacetime we mean a product manifold $I \times F$, of an open interval I of the real line $\mathbb{R}$ and an $n(\geq 2)$-dimensional (connected) Riemannian manifold (F, g_F), endowed with the Lorentzian metric

$$\overline{g} = -\pi_I^*(dt^2) + f(\pi_I)^2\, \pi_F^*(g_F)\,, \tag{1}$$

where π_I and π_F denote the projections onto I and F, respectively, and f is a positive smooth function on I. We will denote this Lorentzian manifold by $(\overline{M}, \overline{g})$. The $(n+1)$-dimensional spacetime $\overline{M}$ is a warped product, with base $(I, -dt^2)$, fiber (F, g_F) and warping function f.

On $\overline{M}$, there exists a distinguished vector field $\xi = f(\pi_I)\,\partial_t$, where ∂_t denotes ∂/∂_t, which is timelike and satisfies

$$\overline{\nabla}_X \xi = f\,'(\pi_I)\, X, \tag{2}$$

for any $X \in \mathfrak{X}(\overline{M})$, where $\overline{\nabla}$ is the Levi-Civita connection of $\overline{g}$, from the relationship between the Levi-Civita connections of M and those of the base and the fiber [14, Cor. 7.35]. Therefore, ξ is conformal with $\mathscr{L}_\xi\, \overline{g} = 2\, f\,'(\pi_I)\, \overline{g}$ and its metrically equivalent 1-form is closed. If the warping function of $\overline{M}$ is constant, i.e., $\overline{M}$ is a Lorentzian product, the GRW spacetime is called static. Contrary, if there is no open subinterval J of I such that $f_{|J}$ is constant, then $\overline{M}$ is said to be proper. Any GRW spacetime has a global time function (in particular, it is time orientable) and then it is stably causal [3, p. 64].

Given an n-dimensional manifold M, an immersion $x : M \to \overline{M}$ is said to be spacelike if the metric g on M, induced from the Lorentzian metric (1), is Riemannian. In this case, M is called a spacelike hypersurface in $\overline{M}$. Let $N \in \mathfrak{X}^{\perp}(M)$ be the unitary timelike normal vector field in the same time-orientation of the vector field $-\partial_t$, i.e., such that $\overline{g}(N, -\partial_t) < 0$.

From the wrong-way Schwarz inequality (see [14, Prop. 5.30], for instance) we have $\overline{g}(N, \partial_t) \geq 1$, and the equality holds at $p \in M$ if and only if $N = -\partial_t$ at p. In fact, $\overline{g}(N, \partial_t) = \cosh\theta$, where θ is the hyperbolic angle, at any point, between the unit timelike vectors N and $-\partial_t$. We will refer to θ as the hyperbolic angle function on M. If we denote by $\partial_t^T := \partial_t + \overline{g}(N, \partial_t)N$ the tangential component of ∂_t along x, then we have the following formula for the gradient on M of the function $\tau := \pi_I \circ x$,

$$\nabla\tau = -\partial_t^T, \tag{3}$$

and therefore

$$g(\nabla\tau, \nabla\tau) = \sinh^2\theta\,. \tag{4}$$

If a GRW spacetime admits a compact spacelike hypersurface then its fiber is compact [2, Prop. 3.2(i)]. A GRW spacetime whose fiber is compact is called spatially closed. Classically, the family of spatially closed GRW spacetimes has been very useful to get closed cosmological models. Moreover, from a geometric point of view, to deal with compact spacelike hypersurfaces in a spatially closed GRW spacetime is natural, indeed, the a complete spacelike hypersurface in a spatially closed GRW spacetime must be compact if some natural assumptions are satisfied [2, Prop. 3.2(ii)]. From a physical point of view, spatially closed cosmological models have been being criticized, and open cosmological models have been suggested instead [7]. More recently, it has been argued that the existence of a compact spacelike hypersurface in a spacetime makes it unsuitable in a possible quantum theory of gravity [4].

We will consider here an $(n + 1)$-dimensional GRW spacetime such that the universal Riemannian covering of the fiber is parabolic (thus so is the fiber) and call it a spatially parabolic GRW spacetime.[1] This class of spacetimes extends to spatially closed GRW spacetimes from the point of view of geometric analysis of the fiber, and allows to model open relativistic universes.

4 Parabolicity of Spacelike Hypersurfaces

Let $x : M \to \overline{M}$ be a spacelike hypersurface in a GRW spacetime $(\overline{M}, \overline{g})$ and assume the induced metric g on M is complete. Suppose in addition that there exists a positive constant c such that $f(\tau) \leq c$. Then, we have that the projection of M on the fiber F, $\pi := \pi_F \circ x$, is a covering map [2, Lemma 3.1].

Now, from (1) we have for any $v \in T_p M$,

$$\begin{aligned} g(v, v) &= -g(\nabla\tau, v)^2 + f(\tau)^2 g_F(d\pi(v), d\pi(v)) \\ &\leq c^2\, g_F(d\pi(v), d\pi(v)). \end{aligned}$$

Now, the classical Schwarz inequality $g(\nabla\tau, v)^2 \leq g(\nabla\tau, \nabla\tau)\, g(v, v)$, gives,

$$g(v, v) \geq -g(\nabla\tau, \nabla\tau)\, g(v, v) + f(\tau)^2 g_F(d\pi(v), d\pi(v)),$$

which implies

$$g(v, v) \geq \frac{f(\tau)^2}{\cosh^2\theta}\, g_F(d\pi(v), d\pi(v)).$$

Thus, we arrive to

[1]This definition simplifies the one given in [16] where each GRW spacetime considered was explicitly assumed with parabolic universal Riemannian covering of its fiber.

Lemma 1. *Let M be a spacelike hypersurface in a GRW spacetime $\overline{M}$. If*

(i) *The hyperbolic angle is bounded,*
(ii) $\sup f(\tau) < \infty$, *and*
(iii) $\inf f(\tau) > 0$,

then, there exists a constant $c \geq 1$ such that

$$c^{-1}\,|v|_g \;\leq\; |d\pi(v)|_{g_F} \;\leq c\,|v|_g\,,$$

for all $v \in T_pM,\ p \in M$.

Proposition 1. *Let $\overline{M}$ be a GRW spacetime whose warping function f satisfies* $\sup f < \infty$ *and* $\inf f > 0$. *If $\overline{M}$ admits a simply connected parabolic spacelike hypersurface M and the hyperbolic angle of M is bounded, then $\overline{M}$ is spatially parabolic.*

Proof. Let $\tilde{\pi} : M \to \tilde{F}$ be a lift of the mapping $\pi : M \to F$, where $\tilde{F}$ is the universal Riemannian covering of F. The map $\tilde{\pi}$ is a diffeomorphism [2, Lemma 3.1] and Lemma 1 asserts that it is a quasi-isometry.

Theorem 2. *Let M be a complete spacelike hypersurface in a spatially parabolic GRW spacetime $\overline{M}$. If*

(i) *The hyperbolic angle is bounded*
(ii) $\sup f(\tau) < \infty$, *and*
(iii) $\inf f(\tau) > 0$,

then, M is parabolic.

Proof. Let $\tilde{M}$ be the universal Riemannian covering of M with projection $\pi_M : \tilde{M} \to M$. The map $\pi \circ \pi_M : \tilde{M} \to F$ may be lifted to a diffeomorphism $\tilde{\pi} : \tilde{M} \to \tilde{F}$, where $\tilde{F}$ is the universal Riemannian covering of F, which is, in fact, a quasi-isometry, leading to the parabolicity of $\tilde{M}$ and, hence, M is also parabolic.

Remark 2. The boundedness assumption on the hyperbolic angle has a physical interpretation. In fact, along M there exist two families of instantaneous observers $\mathcal{T}_p$, where $\mathcal{T} := -\partial_t,\ p \in M$, and the normal observers N_p. The quantities $\cosh\theta(p)$ and $v(p) := (1/\cosh\theta(p))\,N_p^F$, where N_p^F is the projection of N_p onto F, are respectively the energy and the velocity that $\mathcal{T}_p$ measures for N_p [19, pp. 45, 67], and on M we have $|v| = \tanh\theta$. Therefore the relative speed function $|v|$ is bounded on M and, hence, it does not approach to speed of light in vacuum.

5 The Restriction of the Warping Function on M

Denote by ∇ the Levi-Civita connection of the induced metric g on M. The Gauss and Weingarten formulas of M in $\overline{M}$ are

$$\overline{\nabla}_X Y = \nabla_X Y - g(AX, Y)N \quad \text{and} \quad AX = -\overline{\nabla}_X N\,, \tag{5}$$

for all $X, Y \in \mathfrak{X}(M)$, where A is the shape operator associated to N. The mean curvature function relative to N is defined by $H := -(1/n)\,\mathrm{trace}(A)$. The mean curvature is zero if and only if the spacelike hypersurface is, locally, a critical point of the n-dimensional area functional for compactly supported normal variations. A spacelike hypersurface with $H = 0$ is called a maximal hypersurface.

In any GRW spacetime the level hypersurfaces of the projection $\pi_I : \overline{M} \to I$ constitute a distinguished family of spacelike hypersurfaces, the so-called spacelike slices. We will represent by $t = t_0$ the spacelike slice $\{t_0\} \times F$. For a spacelike hypersurface $x : M \to \overline{M}$, $x(M)$ is contained in a spacelike slice $t = t_0$ if and only if $\tau = t_0$ on M. When $x(M)$ equals to $t = t_0$, for some $t_0 \in I$, we will say that M is a spacelike slice. The shape operator and the mean curvature of the spacelike slice $t = t_0$ are respectively $A = f'(t_0)/f(t_0)\, I$ and $H = -f'(t_0)/f(t_0)$, where I denotes the identity transformation Thus, a spacelike slice $t = t_0$ is maximal if and only if $f'(t_0) = 0$ (and hence, totally geodesic).

Given a spacelike hypersurface M in $\overline{M}$, from (2) and (5) we get

$$\nabla_Y \xi^T + f(\tau)\,\overline{g}(N, \partial_t)\, AY = f'(\tau)\, Y\,, \tag{6}$$

for any $Y \in \mathfrak{X}(M)$, where $\xi^T = \xi + \overline{g}(\xi, N)N$ is the tangential component of ξ along x, $f(\tau) = f \circ \tau$ and $f'(\tau) = f' \circ \tau$. From (3) and (6) we have

$$\Delta\tau = -\frac{f'(\tau)}{f(\tau)}\left\{n + |\nabla\tau|^2\right\} - nH\,\overline{g}(N, \partial_t)\,, \tag{7}$$

where Δ denotes the Laplacian on M. Therefore

$$\Delta f(\tau) = -n\,\frac{f'(\tau)^2}{f(\tau)} + f(\tau)(\log f)''(\tau)|\nabla\tau|^2 - nH\, f'(\tau)\overline{g}(N, \partial_t)\,. \tag{8}$$

If we assume $(\log f)''(\tau) \leq 0$ and $H\, f'(\tau) \leq 0$, then the positive function $f(\tau)$ on M is superharmonic.

Remark 3. Clearly, the assumption $(\log f)''(\tau) \leq 0$ holds on M if the function $-\log f$ is convex. With respect to this assumption: (a) It was proved that in a GRW spacetime whose warping function f satisfies that $-\log f$ is convex, the only compact CMC spacelike hypersurfaces are the spacelike slices [1]. This result was later extended to a wider class of spacetimes in [5]. On the other hand, the assumption $-\log f$ is convex is related to certain natural one on the Ricci tensor $\overline{\mathrm{Ric}}$ of $\overline{M}$, the so called Null Convergence Condition (NCC): $\overline{\mathrm{Ric}}(w, w) \geq 0$ for any null tangent vector w. (Namely, if $\overline{M}$ obeys the NCC then $-\log f$ is convex). (b) If $-\log f$ is convex, f is not locally constant and it has a critical point, then the assumption $\sup f < \infty$ is automatically satisfied. In fact, if there exists $t_0 \in I$ such that $f'(t_0) = 0$, then t_0 is the unique critical point of f and $\sup f = f(t_0)$. (c) Consider the reference frame $\mathcal{T} := -\partial_t$ (which defines the time orientation we have considered in $\overline{M}$). We have $\mathrm{div}(\mathcal{T}) = -n\frac{f'}{f}$. Thus, $f' < 0$ (resp. $f' > 0$)

may be interpreted saying that the observers in $\mathscr{T}$ are on average spreading apart (resp. coming together). If we assume $-\log f$ convex then $\frac{d}{ds}\big(\mathrm{div}(\mathscr{T}) \circ \gamma\big)(s) \geq 0$, for any observer γ in $\mathscr{T}$. If in addition we assume there is a proper time s_0 of γ such that $\mathrm{div}(\mathscr{T})_{\gamma(s_0)} > 0$, then $\mathrm{div}(\mathscr{T})_{\gamma(s)} > 0$ for any $s > s_0$. Therefore, the assumption $-\log f$ is convex, favors that $\overline{M}$ models an expanding universe.

6 Uniqueness Results in the Parametric Case

Theorem 3. *Let $\overline{M}$ be a proper spatially parabolic GRW spacetime such that $-\log f$ is convex. The only complete spacelike hypersurface M in $\overline{M}$ whose mean curvature function satisfies $H\ f'(\tau) \leq 0$ (in particular, with $H = 0$), such that*

(i) *The hyperbolic angle is bounded*
(ii) $\sup f(\tau) < \infty$, *and*
(iii) $\inf f(\tau) > 0$,

is the spacelike slice $t = t_0$ with $f'(t_0) = 0$.

If the warping function is allowed to be constant on an open subinterval, we have

Theorem 4. *Let $\overline{M}$ be a spatially parabolic GRW spacetime such that $-\log f$ is convex. The only complete maximal hypersurfaces M in $\overline{M}$ such that*

(i) The hyperbolic angle is bounded, and
(ii) which are bounded between two spacelike slices,

are the spacelike slices $t = t_0$ with $f'(t_0) = 0$.

Proof. From the assumption $x(M) \subset [t_0, t_1] \times F$, the function $f(\tau)$ is upper bounded and satisfies $\inf f(\tau) > 0$. As in the previous result, we arrive to $f(\tau)$ constant. Therefore, from (8), we get $f'(\tau) = 0$ and, hence, the function τ is harmonic making use of (7). Since $\tau(M) \subset [t_0, t_1]$, the function τ must be constant.

Remark 4. In order to illustrate the range of application of the two previous results, note that F may be taken as $\mathbb{S}^{n-1} \times \mathbb{R}$, $n \geq 2$, with $g_F = g + ds^2$, being g an arbitrary metric on $\mathbb{S}^{n-1}$. Assume g has non-negative Ricci curvature. Thus, g_F has the same property. When the fiber (F, g_F) has non-negative Ricci curvature, the convexity of $-\log f$ leads that the Ricci tensor of the GRW spacetime satisfies the NCC (and hence, $\overline{M}$, in the case $n = 4$, could be a candidate to represent a solution to the Einstein equation).

The previous result may be specialized to the static case ($f = 1$), i.e., when the GRW spacetime is fact a Lorentzian product. However, we will see that under the assumption that the Ricci tensor of the fiber is positive semi-definite the boundedness assumption of $x(M)$ can be dropped. In order to do that, recall the Bochner–Lichnerowicz formula (see [6, p. 83], for instance)

$$\frac{1}{2}\Delta|\nabla u|^2 = |\mathrm{Hess}(u)|^2 + \mathrm{Ric}(\nabla u, \nabla u) + g(\nabla u, \nabla \Delta u)$$

which holds true for any Riemannian manifold (M, g) and any $u \in C^\infty(M)$. The idea is to apply it to the function $u = \tau$ on a maximal hypersurface M in a static GRW spacetime $\overline{M}$. Using (6), we have $|\mathrm{Hess}(\tau)|^2 = \cosh^2\theta\,\mathrm{trace}(A^2)$. Moreover, from (7), τ is now harmonic. On the other hand, taking into account (4) and $\mathrm{Ric}(\nabla\tau, \nabla\tau) = \cosh^2\theta\,\mathrm{Ric}^F(N^F, N^F) + g(A\nabla\tau, A\nabla\tau)$, which follows from the Gauss equation of M in $\overline{M}$ and [14, Props. 7.42, 7.43], we get

Lemma 2. *For any maximal hypersurface M in a static GRW spacetime $\overline{M}$ whose fiber has non-negative Ricci curvature, we have*

$$\Delta \sinh^2\theta \geq 2\cosh^2\theta\ \mathrm{trace}(A^2)\,,$$

and, hence, $\sinh^2\theta$ *is subharmonic. Moreover, if it is constant, then M is totally geodesic.*

Theorem 5. *Let M be a complete maximal hypersurface in a spatially parabolic static GRW spacetime $\overline{M}$. If the Ricci curvature of the fiber is non-negative and the hyperbolic angle of M is bounded, then M must be totally geodesic.*

Remark 5. It should be recalled that a complete maximal hypersurface in a locally symmetric Lorentzian manifold $\overline{M}$ whose Ricci tensor satisfies $\overline{\mathrm{Ric}}(w, w) \geq 0$ for any timelike tangent vector w (the Timelike Convergence Condition (TCC)) must be totally geodesic [13]. Note that the spacetime in previous result satisfies the TCC but is not locally symmetric, in general.

7 Calabi–Bernstein Type Problems

Let (M, g_M) be a Riemannian manifold and let $f : I \to \mathbb{R}$ be a positive smooth function. For each $u \in C^\infty(M)$ such that $u(M) \subset I$ we can consider its graph $\Sigma_u = \{(u(p), p) : p \in M\}$ in the GRW spacetime $\overline{M}$ with base $(I, -dt^2)$, fiber (M, g_M) and warping function f. The graph inherits a metric from (1), given by

$$g_u = -du^2 + f(u)^2 g_M\,, \tag{9}$$

on M, which is Riemannian (i.e., positive definite) if and only if u satisfies $|\, Du \,| < f(u)$, everywhere on M, where Du denotes the gradient of u in (M, g_M) and $|\, Du \,|^2 = g_M(Du, Du)$. Note that $\tau(u(p), p) = u(p)$ for any $p \in M$, and so, τ and u may be naturally identified on Σ_u. When Σ_u is spacelike, the unitary normal vector field on Σ_u satisfying $\overline{g}(N, \partial_t) > 0$ is

$$N = -\frac{1}{f(u)\sqrt{f(u)^2 - |\, Du \,|^2}}\left(f(u)^2\partial_t + (0, Du)\right), \tag{10}$$

and the corresponding mean curvature function

$$H(u) = -\operatorname{div}\left(\frac{Du}{nf(u)\sqrt{f(u)^2 - \mid Du\mid^2}}\right) - \frac{f'(u)}{n\sqrt{f(u)^2 - \mid Du\mid^2}}\left(n + \frac{\mid Du\mid^2}{f(u)^2}\right).$$

The differential equation $H(u) = 0$ with the constraint $\mid Du \mid < f(u)$ is called the maximal hypersurface equation in $\overline{M}$, and its solutions give the maximal graphs in $\overline{M}$. This equation is elliptic since the constraint holds. We will apply the previous uniqueness results in the parametric case to determine all the entire solutions of the maximal hypersurface equation

$$\operatorname{div}\left(\frac{Du}{f(u)\sqrt{f(u)^2 - \mid Du\mid^2}}\right) = -\frac{f'(u)}{\sqrt{f(u)^2 - \mid Du\mid^2}}\left(n + \frac{\mid Du\mid^2}{f(u)^2}\right), \tag{E.1}$$

$$\mid Du \mid < \lambda f(u), \quad 0 < \lambda < 1. \tag{E.2}$$

in several cases.

Remark 6. (a) The constraint (E.2) means that the differential equation (E) is in fact uniformly elliptic. (b) Note that (E.2) may be written as $\cosh\theta < 1/\sqrt{1-\lambda^2}$, where θ is the hyperbolic angle of Σ_u. Conversely, if $\cosh\theta < \mu$, with $\mu > 1$, then $\mid Du \mid < \lambda f(u)$, where $\lambda = \sqrt{1-(1/\mu^2)}$. Therefore, (E.2) means that Σ_u has bounded hyperbolic angle. (c) If in addition to (E.2) we have $\inf f(u) > 0$ then $L_u(\gamma) \geq \sqrt{1-\lambda^2}\,\inf f(u)\, L(\gamma)$, where $L(\gamma)$ and $L_u(\gamma)$ are the lengths of a smooth curve γ on M with respect to the metrics g_M and g_u, respectively. Therefore, if a divergent curve in M has infinite g_M-length then it has also infinite g_u-length. Hence, if (M, g_M) is complete, then (M, g_u) is so.

As an application of Theorems 3 and 4, we have

Theorem 6. *Let $f : I \to \mathbb{R}$ be a non-locally constant positive smooth function (resp. a positive smooth function). Assume f satisfies $(\log f)'' \leq 0$, $\sup f < \infty$ and $\inf f > 0$ (resp. f satisfies $(\log f)'' \leq 0$). The only entire solutions (resp. The only bounded entire solutions) of the equation* (E) *on a parabolic Riemannian manifold M are the constant functions $u = c$, with $f'(c) = 0$.*

Finally, as a consequence of Theorem 5, we obtain

Theorem 7. *The only entire solutions of the equation*

$$\operatorname{div}\left(\frac{Du}{\sqrt{1 - \mid Du\mid^2}}\right) = 0 \tag{E$'$.1}$$

$$\mid Du \mid < \lambda, \quad 0 < \lambda < 1, \tag{E$'$.2}$$

on $\mathbb{S}^{2m} \times \mathbb{R}$, endowed with a product Riemannian metric $g + ds^2$, where g is a Riemannian metric on $\mathbb{S}^{2m}$ with non-negative Ricci curvature, are the functions $u(x,s) = as + b$, with $a, b \in \mathbb{R}$, $|a| < \lambda$.

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