The Geometry on Hyper-Kähler Manifolds of Type A_∞

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Abstract Hyper-Kähler manifolds of type A_∞ are noncompact complete Ricci-flat Kähler manifolds of complex dimension 2, constructed by Anderson, Kronheimer, LeBrun (Commun. Math. Phys., **125**, 637–642, 1989) and Goto (Geom. Funct. Anal., **4**(4), 424–454, 1994). We review the asymptotic behavior, the holomorphic symplectic structures and period maps on these manifolds.

1 Introduction

Hyper-Kähler manifolds of type A_∞ were first constructed by Anderson, Kronheimer and LeBrun in [1], as the first example of complete Ricci-flat Kähler manifolds with infinite topological type. Here, infinite topological type means that their homology groups are infinitely generated. The construction in [1] is due to Gibbons-Hawking ansatz, and Goto [5] has constructed these manifolds in another way, using hyper-Kähler quotient construction. Some of the topological and geometric properties of hyper-Kähler manifolds of type A_∞ were studied well in the above papers. In this article, we focus on the volume growth of the hyper-Kähler metrics, the holomorphic symplectic structures, and the period maps.

The construction of hyper-Kähler manifolds of type A_∞ is similar to that of ALE spaces of type A_k, where k is a nonnegative integer. Moreover, their topological properties and complex geometric properties are also similar to type A_k. For example, both of the ALE spaces of type A_k and the hyper-Kähler manifolds of type A_∞ have the parameter naturally given by the construction. We review that they correspond to the cohomology classes of three Kähler forms along [8].

On the other hand, one of the essentially different properties between them appears in their asymptotic behaviors. In fact, the volume growth of ALE spaces is Euclidean, but that of hyper-Kähler manifolds of type A_∞ are less than Euclidean volume growth, which is a main result of [7].

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Moreover, we will review the independence of the volume growth of hyper-Kähler metrics and the complex structures. More precisely, we review the result in [9] to the effect that the volume growth of the hyper-Kähler metric of type A_∞ can be deformed preserving the complex structure.

2 Hyper-Kähler Manifolds of Type A_∞

2.1 Hyper-Kähler Quotient Construction

In this section, we review shortly the construction of hyper-Kähler manifolds of type A_∞ along [5]. For more details, see [1,5] or review in Section 2 of [7].

First of all, hyper-Kähler manifolds are defined as follows.

Definition 1. Let (X, g) be a Riemannian manifold of dimension $4n$ with three integrable complex structures I_1, I_2, I_3, and g be a hermitian metric with respect to each I_i. Then (X, g, I_1, I_2, I_3) is a hyper-Kähler manifold if (I_1, I_2, I_3) satisfying the relations $I_1^2 = I_2^2 = I_3^2 = I_1I_2I_3 = -1$ and each $\omega_i := g(I_i\cdot, \cdot)$ being closed.

Denote by $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k = \mathbb{C} \oplus \mathbb{C}j$ the quaternion and denote by $\mathrm{Im}\mathbb{H} = \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ its Imaginary part. Then an $\mathrm{Im}\mathbb{H}$-valued 2-form $\omega := i\omega_1 + j\omega_2 + k\omega_3 \in \Omega^2(X) \otimes \mathrm{Im}\mathbb{H}$ characterizes the hyper-Kähler structure (g, I_1, I_2, I_3). Accordingly, we call ω the hyper-Kähler structure on X instead of (g, I_1, I_2, I_3).

Now we construct hyper-Kähler quotient method introduced in [9]. Put

$$(\mathrm{Im}\mathbb{H})_0^{\mathbb{N}} := \{\lambda = (\lambda_n)_{n\in\mathbb{N}} \in (\mathrm{Im}\mathbb{H})^{\mathbb{N}};\ \sum_{n\in\mathbb{N}} \frac{1}{1+|\lambda_n|} < +\infty\},$$

where $\mathbb{N}$ is the set of positive integers. Here, we denote by $S^{\mathbb{N}}$ the set of all maps from $\mathbb{N}$ to a set S.

Let

$$M_{\mathbb{N}} := \{v \in \mathbb{H}^{\mathbb{N}};\ \|v\|_{\mathbb{N}}^2 < +\infty\},$$

where

$$\langle u, v\rangle_{\mathbb{N}} := \sum_{n\in\mathbb{N}} u_n \bar{v}_n, \quad \|v\|_{\mathbb{N}}^2 := \langle v, v\rangle_{\mathbb{N}}$$

for $u, v \in \mathbb{H}^{\mathbb{N}}$. Here, the quaternionic conjugate of v_n is denoted by $\bar{v}_n$.

For each $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{N}}$, $\Lambda \in \mathbb{H}^{\mathbb{N}}$ can be taken so that $\Lambda_n i \overline{\Lambda}_n = \lambda_n$. Put

$$M_\Lambda := \Lambda + M_{\mathbb{N}} = \{\Lambda + v;\ v \in M_{\mathbb{N}}\},$$
$$G_\lambda := \{g \in (S^1)^{\mathbb{N}};\ \sum_{n\in\mathbb{N}} (1+|\lambda_n|)|1-g_n|^2 < +\infty,\ \prod_{n\in\mathbb{N}} g_n = 1\}.$$

Here, $\prod_{n\in\mathbb{N}} g_n$ always converges by the condition

$$\sum_{n\in\mathbb{N}} \frac{1}{1+|\lambda_n|} < +\infty.$$

Then G_λ is an infinite dimensional Lie group, and G_λ acts on M_Λ by $xg := (x_n g_n)_{n\in\mathbb{N}}$ for $x \in M_\Lambda$, $g \in G_\lambda$.

Now G_λ acts on

$$N_\Lambda = \{x \in M_\Lambda;\ x_n i \overline{x}_n - \lambda_n = x_m i \overline{x}_m - \lambda_m \text{ for all } n, m \in \mathbb{N}\}$$

and we obtain the quotient space N_Λ/G_λ which is called the hyper-Kähler quotient. Here, N_Λ corresponds to the level set of the hyper-Kähler moment map.

Definition 2. $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{N}}$ is generic if $\lambda_n - \lambda_m \neq 0$ for all distinct $n, m \in \mathbb{N}$.

Theorem 1 ([5]). *If $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{N}}$ is generic, then N_Λ/G_λ is a smooth manifold of real dimension 4, and the hyper-Kähler structure on M_Λ induces a hyper-Kähler structure ω_λ on N_Λ/G_λ.*

Although the hyper-Kähler quotient N_Λ/G_λ seems to depend on the choice of $\Lambda \in \mathbb{H}^{\mathbb{N}}$, the induced hyper-Kähler structure on N_Λ/G_λ depends only on λ by the argument of Section 2 of [7]. Accordingly we may put

$$\begin{aligned} X(\lambda) &:= N_\Lambda/G_\lambda \\ &= \{x \in M_\Lambda; x_n i \bar{x}_n - \lambda_n \text{ is independent of } n \in \mathbb{N}\}/G_\lambda, \end{aligned}$$

and call it a hyper-Kähler manifold of type A_∞

If $\mathbb{N}$ is replaced by a finite set in the above construction, $(X(\lambda), \omega_\lambda)$ becomes an ALE hyper-Kähler manifold of type A_k [4].

2.2 S^1-actions and Moment Maps

An S^1-action on $X(\lambda)$ preserving the hyper-Kähler structure is defined as follows. (See also [5].) Let $[x] \in N_\Lambda/G_\lambda$ be the equivalence class represented by $x \in N_\Lambda$. Take $m \in \mathbb{N}$ arbitrarily and let

$$[x]g := [x_m g, (x_n)_{n\in\mathbb{N}\setminus\{m\}}]$$

for $x = (x_m, (x_n)_{n\in\mathbb{N}\setminus\{m\}}) \in N_\Lambda$ and $g \in S^1$. This definition does not depend on the choice of $m \in \mathbb{N}$. Then we obtain the hyper-Kähler moment map

$$\mu_\lambda([x]) := x_n i \bar{x}_n - \lambda_n \in \mathrm{Im}\mathbb{H}.$$

The right hand side is independent of the choice of $n \in \mathbb{N}$ since x is an element of N_Λ.

We have a principal S^1-bundle $\mu_\lambda|_{X(\lambda)^*} : X(\lambda)^* \to Y(\lambda)$, where

$$X(\lambda)^* := \{[x] \in X(\lambda);\ x_n \neq 0 \text{ for all } n \in \mathbb{N}\},$$
$$Y(\lambda) := \mathrm{Im}\mathbb{H} \backslash \{-\lambda_n;\ n \in \mathbb{N}\}.$$

By the Gibbons-Hawking construction [1], we can check easily that $X(\lambda)$ and $X(\lambda')$ are isomorphic as hyper-Kähler manifolds if λ and λ' satisfy one of the following conditions; (i) $\lambda'_n - \lambda_n \in \mathrm{Im}\mathbb{H}$ is independent of n, (ii) $\lambda'_n = \lambda_{a(n)}$ for some bijective maps $a : \mathbb{N} \to \mathbb{N}$, (iii) $\lambda = -\lambda'$.

3 The Volume Growth

Here we focus on the Riemannian geometric aspects of $X(\lambda)$, especially their volume growth.

For a Riemannian manifold (X, g), denote by $V_g(p, r)$ the volume of the geodesic ball of radius $r > 0$ centered at $p \in X$. By the volume comparison theorem [2, 6], we can deduce that

$$\lim_{r\to\infty} \frac{V_g(p_0, r)}{V_g(p_1, r)} = 1$$

for any Ricci flat manifold (X, g) and any $p_0, p_1 \in X$. Thus the volume growth of g is the invariant for Ricci flat manifolds.

Theorem 2 ([7]). *For each $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{N}}$ and $p_0 \in X(\lambda)$, the function $V_{g_\lambda}(p_0, r)$ satisfies*

$$0 < \liminf_{r\to+\infty} \frac{V_{g_\lambda}(p_0, r)}{r^2 \tau_\lambda^{-1}(r^2)} \leq \limsup_{r\to+\infty} \frac{V_{g_\lambda}(p_0, r)}{r^2 \tau_\lambda^{-1}(r^2)} < +\infty,$$

where the function $\tau_\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is defined by

$$\tau_\lambda(R) := \sum_{n\in\mathbb{N}} \frac{R^2}{R + |\lambda_n|}$$

for $R \geq 0$. Moreover, we have

$$\lim_{r\to+\infty} \frac{V_{g_\lambda}(p_0, r)}{r^4} = 0, \qquad \lim_{r\to+\infty} \frac{V_{g_\lambda}(p_0, r)}{r^3} = +\infty.$$

Next we see some examples computed in [7].

Example 1. Fix $\gamma > 1$ and put $\lambda_n^\gamma := i \cdot n^\gamma \in \mathrm{Im}\mathbb{H}$. Then there exist positive constants $A, B > 0$ such that

$$Ar^{4-\frac{2}{\gamma+1}} \leq V_{g_{\lambda^\gamma}}((p_0, r) \leq Br^{4-\frac{2}{\gamma+1}}.$$

Example 2. Put $\lambda_n := i \cdot e^n \in \mathrm{Im}\mathbb{H}$. Then there exist positive constants $A, B > 0$ such that

$$A\frac{r^4}{\log r} \leq V_{g_\lambda}(p_0, r) \leq B\frac{r^4}{\log r}$$

for any $\alpha < 4$.

4 Period Maps

4.1 Holomorphic Curves

In this subsection, we see that there are several compact minimal submanifolds in $X(\lambda)$ following [8].

Definition 3. (i) Let X be a complex manifold of dimension $2n$ and $\omega_{\mathbb{C}}$ be a holomorphic 2-form on X. Then $(X, \omega_{\mathbb{C}})$ is called a holomorphic symplectic manifold if $d\omega_{\mathbb{C}} = 0$ and $\omega_{\mathbb{C}}^n$ is nowhere vanishing. (ii) An n dimensional complex submanifold L of a holomorphic symplectic manifold $(X, \omega_{\mathbb{C}})$ is holomorphic Lagrangian submanifold if $\omega_{\mathbb{C}}|_L = 0$.

Let (X, ω) be a hyper-Kähler manifold of real dimension $4n$. For each $y \in \mathrm{Im}\mathbb{H}$ with $|y| = 1$, $\mathrm{Im}\mathbb{H}$ is decomposed into y-component and its orthogonal complement. Then we denote by $\omega_y \in \Omega^2(X)$ the y-component of $\omega \in \Omega^2(X) \otimes \mathrm{Im}\mathbb{H}$. Let I_y be the complex structure corresponding to the Kähler form ω_y.

Let $\eta = (\eta_1\ \eta_2\ \eta_3) \in SO(3)$, where $\langle \eta_1, \eta_2, \eta_3 \rangle$ is an orthonormal basis of $\mathbb{R}^3$. Then η gives the orthogonal decomposition $\mathrm{Im}\mathbb{H} = \mathbb{R}^3 = \mathbb{R}\eta_1 \oplus \mathbb{R}\eta_2 \oplus \mathbb{R}\eta_3$, and the hyper-Kähler structure $\omega \in \Omega^2(X) \otimes \mathrm{Im}\mathbb{H}$ can be written as $\omega = \eta_1\omega_{\eta_1} + \eta_2\omega_{\eta_2} + \eta_3\omega_{\eta_3}$ for every $\eta \in SO(3)$. Now we regard (X, I_{η_1}) as a complex manifold. Then a holomorphic symplectic structure on X is given by $\omega_{\eta\mathbb{C}} := \omega_{\eta_2} + i\omega_{\eta_3}$.

Proposition 1. *Let (X, ω) be a hyper-Kähler manifold and take $\eta \in SO(3)$. Then each holomorphic Lagrangian submanifold $L \subset X$ with respect to $\omega_{\eta\mathbb{C}}$ gives the minimum volume in their homology class.*

Proof. The pair of a Kähler form ω_{η_3} and a holomorphic volume form $(\omega_{\eta_1} + i\omega_{\eta_2})^n$ gives the Calabi-Yau structure on (X, I_{η_3}). Here, n is the half of the complex dimension of X. Now, assume that $L \subset X$ is a holomorphic Lagrangian submanifold with respect to $\omega_{\eta\mathbb{C}}$. Then $\omega_{\eta_2}|_L = \omega_{\eta_3}|_L = 0$, hence L is lagrangian with respect to ω_{η_3}. Since $\mathrm{Im}(\omega_{\eta_1} + i\omega_{\eta_2})^n$ is the multiplication of ω_{η_2} and some differential forms, we also have $\mathrm{Im}(\omega_{\eta_1} + i\omega_{\eta_2})^n|_L = 0$, which means L is a special Lagrangian submanifold. The volume minimizing property of special Lagrangian submanifolds [11] gives the assertion. □

Take a generic $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{N}}$ and consider the hyper-Kähler manifold $(X(\lambda), \omega_\lambda)$ as constructed in Sect. 2. Put

$$[a, b] := \{ta + (1-t)b \in \mathrm{Im}\mathbb{H};\ 0 \le t \le 1\},$$
$$(a, b] := \{ta + (1-t)b \in \mathrm{Im}\mathbb{H};\ 0 \le t < 1\},$$
$$[a, b) := \{ta + (1-t)b \in \mathrm{Im}\mathbb{H};\ 0 < t \le 1\},$$
$$(a, b) := \{ta + (1-t)b \in \mathrm{Im}\mathbb{H};\ 0 < t < 1\}$$

for $a, b \in \mathrm{Im}\mathbb{H}$.

Proposition 2. *Let $n, m \in \mathbb{N}$ satisfy $n \neq m$ and $(-\lambda_n, -\lambda_m) \subset Y(\lambda)$. The inverse image $\mu_\lambda^{-1}([-\lambda_n, -\lambda_m]) \cong \mathbb{C}P^1$ is a complex submanifold of $X(\lambda)$ with respect to I_y and gives the minimum volume in its homology class, where $y := \frac{\lambda_n - \lambda_m}{|\lambda_n - \lambda_m|}$.*

Proof. Let $\eta \in SO(3)$ satisfies $\eta i = y$. If we write $\mu_\lambda = (\mu_{\lambda,1}, \mu_{\lambda,2}, \mu_{\lambda,3})$ with respect to the decomposition $\mathrm{Im}\mathbb{H} = \mathbb{R}\eta_1 \oplus \mathbb{R}\eta_2 \oplus \mathbb{R}\eta_3$, then $\mu_{\lambda,2}$ and $\mu_{\lambda,3}$ are constant on $\mu_\lambda^{-1}([-\lambda_n, -\lambda_m])$. Hence we have $d\mu_{\lambda,\alpha}|_{\mu_\lambda^{-1}([-\lambda_n, -\lambda_m])} = 0$ for $\alpha = 2, 3$, which gives $\omega_{\lambda,\eta\mathbb{C}}|_{\mu_\lambda^{-1}([-\lambda_n, -\lambda_m])} = 0$. □

4.2 Topology

In this subsection we review the construction of the deformation retracts of $X(\lambda)$ following [3, 5]. See also [8]. In the case of toric hyper-Kähler varieties, the deformation retracts are constructed in [3].

For $(-\lambda_n, -\lambda_m) \subset Y(\lambda)$, the orientation of $\mu_\lambda^{-1}([-\lambda_n, -\lambda_m])$ is determined as follows. By taking a smooth section $(-\lambda_n, -\lambda_m) \to \mu_\lambda^{-1}((-\lambda_n, -\lambda_m))$ of μ_λ, a coordinate (s, t) on $\mu_\lambda^{-1}((-\lambda_n, -\lambda_m))$ is naturally given where $t \in \mathbb{R}/2\pi\mathbb{Z}$ is the parameter of S^1-action and a function $s : \mu_\lambda^{-1}((-\lambda_n, -\lambda_m)) \to \mathbb{R}$ is given by

$$s(p) := \frac{\lambda_n + \mu_\lambda(p)}{\lambda_n - \lambda_m}$$

for $p \in \mu_\lambda^{-1}((-\lambda_n, -\lambda_m))$. Then the orientation of $\mu_\lambda^{-1}([-\lambda_n, -\lambda_m])$ is given by $ds \wedge dt$. Therefore, $\mu_\lambda^{-1}([-\lambda_n, -\lambda_m])$ and $\mu_\lambda^{-1}([-\lambda_m, -\lambda_n])$ are same as manifolds but have opposite orientations.

For $n, m, l \in \mathbb{N}$, $\mu_\lambda^{-1}([-\lambda_n, -\lambda_m]) \cup \mu_\lambda^{-1}([-\lambda_m, -\lambda_l])$ and $\mu_\lambda^{-1}([-\lambda_n, -\lambda_l])$ determines the same homology class since the boundary of $\mu_\lambda^{-1}(\triangle_{n,m,l})$ is given by $\mu_\lambda^{-1}([-\lambda_n, -\lambda_m]) \cup \mu_\lambda^{-1}([-\lambda_m, -\lambda_l]) \cup \mu_\lambda^{-1}([-\lambda_l, -\lambda_n])$, where

$$\triangle_{n,m,l} := \{-\alpha\lambda_n - \beta\lambda_m - \gamma\lambda_l \in \mathrm{Im}\mathbb{H};\ \alpha + \beta + \gamma = 1,\ \alpha, \beta, \gamma \geq 0\}.$$

We denote by $C_{n,m}$ the homology class determined by $\mu_\lambda^{-1}([-\lambda_n, -\lambda_m])$. Then the above observation implies

$$C_{n,m} + C_{m,l} + C_{l,n} = C_{n,m} + C_{m,n} = 0$$

for $n, m, l \in \mathbb{N}$.

If $n, m, l, h \in \mathbb{N}$ satisfies $n \neq h$, $n \neq m$ and $l \neq h$ then the intersection number $C_{n,m} \cdot C_{l,h}$ is given by

$$C_{n,m} \cdot C_{l,h} = \begin{cases} 1\ (m = l) \\ 0\ (m \neq l) \end{cases}$$

and $C_{n,m} \cdot C_{n,m} = -2$.

Since the subset of $(\mathrm{Im}\mathbb{H})_0^{\mathbb{N}}$ consisting of generic elements is connected in $(\mathrm{Im}\mathbb{H})_0^{\mathbb{N}}$, the topological structure of $X(\lambda)$ does not depend on λ. Consequently, it suffices to study $X(\hat{\lambda})$ for investigating the topology of $X(\lambda)$, where $\hat{\lambda}$ is the special one defined by $\hat{\lambda}_n := (n^2, 0, 0) \in \mathrm{Im}\mathbb{H}$.

Proposition 3. *There exists a deformation retract of* $\mu_{\hat{\lambda}}^{-1}(\bigcup_{n\in\mathbb{N}}[-\hat{\lambda}_n, -\hat{\lambda}_{n+1}]) \subset X(\hat{\lambda})$.

Proof. There is a deformation retract

$$F : \mathrm{Im}\mathbb{H} \times [0, 1] \to \mathrm{Im}\mathbb{H}$$

which satisfy $F(\cdot, 0) = id_{\mathrm{Im}\mathbb{H}}$, $F(\mathrm{Im}\mathbb{H}, 1) = \bigcup_{n\in\mathbb{N}}[-\hat{\lambda}_n, -\hat{\lambda}_{n+1}]$ and $F(\zeta, 1) = \zeta$ for $\zeta \in \bigcup_{n\in\mathbb{N}}[-\hat{\lambda}_n, -\hat{\lambda}_{n+1}]$. Then we have the horizontal lift $\tilde{F} : X(\hat{\lambda}) \times [0, 1] \to X(\hat{\lambda})$ of F by using the S^1-connection on $X(\hat{\lambda})^*$ naturally induced from the hyper-Kähler metric on $X(\hat{\lambda})^*$. The map $\tilde{F}$ is a deformation retract as we expect. □

Corollary 1. *The second homology group* $H_2(X(\lambda), \mathbb{Z})$ *is generated by* $\{C_{n,m};\ n, m \in \mathbb{N}\}$.

Thus we obtain the followings.

Theorem 3. *Let* $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{N}}$ *be generic. Then* $H_2(X(\lambda), \mathbb{Z})$ *is a free* $\mathbb{Z}$*-module generated by* $\{C_{n,m};\ n, m \in \mathbb{N}\}$ *with relations*

$$C_{n,m} + C_{m,l} + C_{l,n} = 0, C_{n,m} + C_{m,n} = 0$$

for all $n, m, l \in \mathbb{N}$*. Moreover the intersection form on* $H_2(X(\lambda), \mathbb{Z})$ *is given by*

$$C_{n,m} \cdot C_{l,h} = \begin{cases} 1 \ (m = l) \\ 0 \ (m \neq l) \end{cases}$$

and $C_{n,m} \cdot C_{n,m} = -2$ *for* $n, m, l, h \in \mathbb{N}$ *taken to be* $n \neq h$*,* $n \neq m$ *and* $l \neq h$*.*

4.3 Period Maps

Let $[\omega_\lambda] \in H^2(X(\lambda), \mathbb{R}) \otimes \mathrm{Im}\mathbb{H}$ be the cohomology class of ω_λ. In this subsection we compute $[\omega_\lambda]$, that is, compute the value of $\langle [\omega_\lambda], C_{n,m} \rangle := \int_{C_{n,m}} \omega_\lambda \in \mathrm{Im}\mathbb{H}$ for all $n, m \in \mathbb{N}$ along [8]. In the case of finite topological type of toric hyper-Kähler varieties, the period maps are computed in [12].

Theorem 4. *Let* $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{N}}$ *be generic. Then*

$$\langle [\omega_\lambda], C_{n,m} \rangle = \lambda_n - \lambda_m$$

for all $n, m \in \mathbb{N}$*.*

Proof. Take a smooth path $\gamma : [0, 1] \to \mathrm{Im}\mathbb{H}$ such that $\gamma(0) = -\lambda_n$, $\gamma(1) = -\lambda_m$ and $\gamma(s) \in Y(\lambda)$ for $s \in (0, 1)$. Since the homology class represented by $\mu_\lambda^{-1}(\gamma([0, 1]))$ is $C_{n,m}$, we have

$$\langle [\omega_\lambda], C_{n,m} \rangle = \int_{\mu_\lambda^{-1}(\gamma([0,1]))} \omega_\lambda.$$

Take the local coordinate $(t, \mu_{\lambda,1}, \mu_{\lambda,2}, \mu_{\lambda,3})$ of an open subset of $X(\lambda)^*$, where $\mu_\lambda = (\mu_{\lambda,1}, \mu_{\lambda,2}, \mu_{\lambda,3})$ and t is the coordinate of S^1-action. Then the local coordinate (s, t) on $\mu_\lambda^{-1}(\gamma([0, 1]))$ is given by $(t, \mu_{\lambda,1} \circ \gamma(s), \mu_{\lambda,2} \circ \gamma(s), \mu_{\lambda,3} \circ \gamma(s))$. By using this, we can see that

$$\omega_{\lambda,\alpha} = \gamma_\alpha'(s) \frac{1}{2\pi} ds \wedge dt$$

for $\alpha = 1, 2, 3$, where $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s)) \in \mathrm{Im}\mathbb{H} = \mathbb{R}^3$. Hence we have

$$\begin{aligned} \int_{\mu_\lambda^{-1}(\gamma([0,1]))} \omega_{\lambda,\alpha} &= \int_{\mu_\lambda^{-1}(\gamma([0,1]))} \gamma_\alpha'(s) \frac{1}{2\pi} ds \wedge dt \\ &= \int_0^{2\pi} \frac{1}{2\pi} dt \int_0^1 \gamma_\alpha'(s) ds \\ &= \gamma_\alpha(1) - \gamma_\alpha(0) = \lambda_{n,\alpha} - \lambda_{m,\alpha}. \end{aligned}$$

□

5 Holomorphic Symplectic Structures

In this section we regard a hyper-Kähler manifold (X, g, I_1, I_2, I_3) as a complex manifold by I_1. Then the holomorphic 2-form $\omega_{\mathbb{C}} = \omega_2 + \sqrt{-1}\omega_3$ is called the holomorphic symplectic structure, and the cohomology class $[\omega_2 + \sqrt{-1}\omega_3]$ is called the holomorphic symplectic class.

Let λ^γ as in Example 1 of Sect. 3. Then, we can see that the holomorphic symplectic class $[\omega_{\lambda^\gamma,\mathbb{C}}]$ is independent of γ by Theorem 4.

Theorem 5 ([9]). *The holomorphic symplectic structures $\omega_{\lambda^\gamma,\mathbb{C}}$ are independent of γ. In particular, $X(\lambda^\gamma)$ and $X(\lambda^{\hat{\gamma}})$ are biholomorphic for all $\gamma, \hat{\gamma} > 1$.*

Since the function $4 - \frac{2}{\gamma+1}$ gives one-to-one correspondence between open intervals $(1, \infty)$ and $(3, 4)$, we have the following conclusion by combining Theorems 2 with 5.

Theorem 6. *Let $\alpha \in (3, 4)$. Then there is a complex manifold X and the family of Ricci-flat Kähler metrics $\{g_\alpha\}_{3<\alpha<4}$ whose volume growth satisfies*

$$Ar^\alpha \le V_{g_\alpha}(p_0, r) \le Br^\alpha$$

for some positive constants A, B.

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