Totally Geodesic Surfaces of Riemannian Symmetric Spaces

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Abstract A submanidfold S of a Riemannian manifold is called a *totally geodesic submanifold* if every geodesic of S is also a geodesic of M. Totally geodesic submanifolds of Riemannian symmetric spaces have long been studied by many mathematicians. We give a classification of non-flat totally geodesic surfaces of the Riemannian symmetric space of type AI, $AIII$ and BDI.

1 Introduction

Let G be a compact simple Lie group and θ be an involutive automorphism of G. We denote by $\mathfrak{g}$ the Lie algebra of G and denote also by θ the differential of θ. Let $\mathfrak{k}$ be the set of all θ-invariant elements of $\mathfrak{g}$ and K be a Lie subgroup of G of which Lie algebra coincides with $\mathfrak{k}$.

Let $\langle , \rangle$ be an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$ and $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$. We extend the restriction of $\langle , \rangle$ on $\mathfrak{p}$ to the G-invariant Riemannian metric on G/K and denote it also by $\langle , \rangle$.

A subspace $\mathfrak{s}$ of $\mathfrak{p}$ is called a *Lie triple system* if it satisfies $[[\mathfrak{s}, \mathfrak{s}]\mathfrak{s}] \subset \mathfrak{s}$. There exits a one-to-one correspondence between the set of totally geodesic submanifold of M through the origin $o = eK$ and the set of Lie triple systems in $\mathfrak{p}$ [1].

Important constructions and classification results of totally geodesic submanifolds in Riemannian symmetric spaces are summarized in an expository article by S. Klein [2].

In [3] the author classified non-flat totally geodesic surfaces in irreducible Riemannian symmetric spaces where G is $SU(n)$, $Sp(n)$ or $SO(n)$. The main tool used in [3] is the representation theory of $SU(2)$. The aim of this article is to introduce the outline of the contents of [3].

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2 Irreducible Representation of $SU(2)$

In this section, we review real and complex irreducible representations of $SU(2)$.

Let H, X, Y be a basis of the complexification of the Lie algebra $\mathfrak{su}(2)$ of $SU(2)$ satisfying

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H. \tag{1}$$

2.1 Complex Irreducible Representations

If we denote by V_d the set of polynomial functions on $\mathbb{C}^2$ and by ρ_d the contragradient action of $SU(2)$ on V_d, then (V_d, ρ_d) is a complex irreducible representation of $SU(2)$. On the other hand, every finite dimensional complex irreducible representation of $SU(2)$ is equivalent to (V_d, ρ_d) for some positive integer d.

The next proposition plays an important role in our classification.

Proposition 1. *Let (V, ρ) be a $(d + 1)$-dimensional complex irreducible representation of $SU(2)$ and $\langle\, ,\, \rangle$ be an $SU(2)$-invariant Hermitian inner product on V. If we put λ the largest eigenvalue of $\rho(H)$ and $v_0 \in V$ be a corresponding eigen vector, then we have $\lambda = d$ and $\rho(Y)^i(v_0)$ is an eigen vector of $\rho(H)$ corresponding to the eigenvalue $(\lambda - 2i)$.*

Let ε_i $(0 \le i \le d)$ be arbitrary complex numbers with $|\varepsilon_i| = 1$, and put $v_i = \dfrac{\varepsilon_i}{|\rho(Y)^i v_0|}\rho(Y)^i v_0$ $(0 \le i \le d)$. Then $v_0, v_1, \cdots, v_d$ is an orthonormal basis of V_d and the matrix representations of $\rho(H)$, $\rho(X)$, $\rho(Y)$ with respect to $v_0, \cdots, v_d$ are as follows

$$\rho(H) = \begin{bmatrix} d & 0 & \cdots & 0 \\ 0 & d-2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -d \end{bmatrix}, \quad \rho(X) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ c_1 & 0 & \cdots & 0 & 0 \\ 0 & c_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_d & 0 \end{bmatrix},$$

$$\rho(Y) = \begin{bmatrix} 0 & c'_1 & 0 & \cdots & 0 \\ 0 & 0 & c'_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c'_d \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad \text{where} \quad c'_i = \overline{c_i}, \quad |c_i| = \sqrt{i(d-i+1)}.$$

2.2 Real Irreducible Representations

Let (V, ρ) be a complex representation of $SU(2)$ and $v_1, \cdots, v_N$ be a basis of V. We denote by $\overline{V}$ the complex vector space, which is V itself as an additive group and the scalar multiplication is defined by $c * x = \overline{c}\, x$ $(c \in \mathbb{C},\ x \in V)$. Define the action $\overline{\rho}$ of $SU(2)$ on $\overline{V}$ so that

$$\overline{\rho}\left(\sum z_i * v_i\right) = \sum z_i * \rho(v_i).$$

The representation $(\overline{V}, \overline{\rho})$ is called the *conjugate* representation of (V, ρ).

A complex irreducible representation (V, ρ) of G is said to be a *self-conjugate* representation if there exists a conjugate-linear automorphism $\hat{j} : V \to V$ which commute with $\rho(g)$ for any $g \in SU(2)$. A conjugate-linear automorphism commuting with ρ is called a *structure map* of (V, ρ).

Let (V, ρ) be a self-conjugate representation and $\hat{j}$ be a structure map. By Schur's lemma, $\hat{j}^2 = c$ for some constant. It is known that the constant c is a real number and (V, ρ) is said to be of *index* 1 (resp. -1) if $c > 0$ (resp. $c < 0$).

Each complex irreducible representation (V_d, ρ_d) of $SU(2)$ is a self-conjugate representation and its index is equal to $(-1)^d$. If d is an even integer, the subspace of V_d invariant under the structure map $\hat{j}$ is a real irreducible representation of $SU(2)$. If d is an odd integer, V_d (viewed as a real representation by restricting the coefficient field from $\mathbb{C}$ to $\mathbb{R}$) is also a Real irreducible representation and V_d admits a structure of vector space over the field of quaternions.

3 Classification

The standard orthonormal basis of $\mathbb{R}^N$ or $\mathbb{C}^N$ will be denote by $e_1, \cdots, e_N$. We denote by G_{ij} $(i \neq j)$ the skew-symmetric endomorphism satisfying

$$G_{ij}(e_j) = e_i, \quad G_{ij}(e_i) = -e_j, \quad G_{ij}(e_k) = 0 \quad (k \neq i, j),$$

and by S_{ij} the symmetric endomorphism

$$S_{ij}(e_j) = e_i, \quad S_{ij}(e_i) = e_j, \quad S_{ij}(e_k) = 0 \quad (k \neq i, j).$$

3.1 $AI : SU(n)/SO(n)$

We denote by τ the conjugation on $\mathbb{C}^N$ with respect to $\mathbb{R}^N$ and denote by θ the involutive automorphism on $SU(N)$ defined by $\theta(g) = \tau \circ g \circ \tau$ $(g \in SU(n))$.

Theorem 1. *Let M be a non-flat totally geodesic surface of $SU(n)/SO(n)$ and U be the set of all elements in $SU(n)$ leaving M invariant.*

(i) There exists an orthogonal direct sum decomposition of $\mathbb{C}^n$ by τ-invariant and U-invariant subspaces.

(ii) Let X_2, X_3 be a basis of the Lie triple system corresponding to M with

$$[[X_2, X_3], X_2] = 4\,X_3, \quad [[X_2, X_3], X_3] = -4\,X_2.$$

Assume that $\mathbb{C}^n$ is U-invariant. There exists an element $g = [u_1, \cdots, u_n] \in SO(n)$ such that

$$\mathrm{Ad}(g)X_2 = \sqrt{-1}\sum_{i=1}^{n}(n-2\,i)\,E_{i,i} \tag{2}$$

$$\mathrm{Ad}(g)X_3 = -\sqrt{-1}\left[\sum_{i=1}^{n-2}\sqrt{i(n-i)}S_{i,i+1} + \varepsilon\sqrt{n-1}S_{n-1,n}\right] \tag{3}$$

where

$$\varepsilon = \begin{cases} 1 & \text{if } n \equiv 1 \text{ (mod2)}, \\ \pm 1 & \text{if } n \equiv 0 \text{ (mod2)}. \end{cases}$$

Proof. We omit the proof of (i) and assume that the action of U on $\mathbb{C}^n$ is irreducible.

Note that $\mathfrak{k} = \{X : \theta(X) = X\} = \mathrm{Skew}(n;\mathbb{R})$ and $\mathfrak{p} = \{X : \theta(X) = -X\} = \sqrt{-1}\,\mathrm{Sym}(n;\mathbb{R})$.

If we put $a_1 \geq a_2 \geq \cdots \geq a_n$ the set of eigenvalues of $H = -\sqrt{-1}\,X_2 \in \mathrm{Sym}(n;\mathbb{R})$, then by the action of $\mathrm{Ad}(SO(n))$ we may assume that $H = \mathrm{Diag}(a_1, a_2, \cdots, a_n)$.

If we put

$$H = [X_2, X_3], \quad X = \frac{1}{2}(\sqrt{-1}\,X_3 + X_1), \quad Y = \frac{1}{2}(\sqrt{-1}\,X_3 - X_1),$$

we have

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

Since a_i are weights of the complex irreducible representation of U we have

$$a_1 - a_2 = a_2 - a_3 = \cdots = a_{n-1} - a_n = 2.$$

Put $n = d + 1$ and $v_0 = e_1$. Since each eigenspace (the weight space) of H is one-dimensional there exists ε_i $(1 \leq i \leq d)$ such that $e_i = \dfrac{\varepsilon_i}{|H^i v_0|} H^i v_0$. Thus the matrix H, X and Y are of the form given in the Proposition 1. We can choose unit complex numbers ε'_i $(0 \leq i \leq d)$ such that by a change of basis $\{e_i\} \to \{\varepsilon'_i e_i\}$ all the components of X, Y in the Proposition 1 are changed to real numbers. We omit further detail. □

3.2 $AIII : SU(p+q)/S(U(p) \times U(q))$

We denote by I_n the unit matrix of order n and put $I_{p,q} = \begin{bmatrix} I_p & O \\ O & -I_q \end{bmatrix}$.

Theorem 2. *Let M be a non-flat totally geodesic surface of $SU(p+q)/S(U(p) \times U(q))$ and U be the set of all elements of $SU(p+q)$ which leave M invariant.*

(i) There exists an orthogonal direct sum decomposition of $\mathbb{C}^{p+q}$ by $I_{p,q}$-invariant, U-irreducible subspaces.
(ii) If V is an $I_{p,q}$-invariant, U-irreducible subspace of $\mathbb{C}^{p+q}$, then we have

$$\left|\dim\{v \in V : I_{p,q}(v) = v\} - \dim\{v \in V : I_{p,q}(v) = -v\}\right| \leq 1.$$

(iii) Assume that the action of $SU(2)$ on $\mathbb{C}^{p+q}$ is irreducible. Let X_2, X_3 be a basis of the Lie triple system corresponding to M with

$$[[X_2, X_3], X_2] = 4\,X_3, \quad [[X_2, X_3], X_3] = -4\,X_2.$$

There exists an element $g = [u_1, \cdots, u_{p+q}] \in S(U(p) \times U(q))$ such that

$$\mathrm{Ad}(g)X_2 = \sum_{i=1}^{q} \sqrt{(2i-1)(p+q+1-2i)}\, G_{i,p+i} + \sum_{i=1}^{p-1} \sqrt{2i(p+q-2i)}\, G_{p+i,i+1} \tag{4}$$

$$\mathrm{Ad}(g)X_3 = \sqrt{-1}\left[\sum_{i=1}^{q} \sqrt{(2i-1)(p+q+1-2i)}\, S_{p+i,i} + \sum_{i=1}^{p-1} \sqrt{2i(p+q-2i)}\, S_{i+1,p+i}\right] \tag{5}$$

Proof. We omit the proof of (i).

Assume that the action of U on $\mathbb{C}^{p+q}$ is irreducible.

Take a basis X_1, X_2, X_3 of the Lie algebra $\mathfrak{u}$ of U which satisfy

$$I_{p,q} \circ X_1 = X_1 \circ I_{p,q}, \quad I_{p,q} \circ X_i = -X_i \circ I_{p,q} \quad (i = 2, 3),$$
$$[X_1, X_2] = 2X_3, \quad [X_2, X_3] = 2X_1, \quad [X_3, X_1] = 2X_2,$$

and put

$$H = -\sqrt{-1}X_1,\ X = \frac{1}{2}(X_2 - \sqrt{-1}X_3),\ Y = -\frac{1}{2}(X_2 + \sqrt{-1}X_3) = {}^t\overline{X}.$$

Since H is a Hermitian matrix, there exists an element $g \in S(U(p) \times U(q))$ such that

$$\mathrm{Ad}(g)H = \mathrm{diag}(a_1, \cdots, a_p; b_1, \cdots, b_q)$$

where $a_1 > \cdots > a_p$ and $b_1 > \cdots > b_q$ holds. We denote by ξ_i the i-th column vector of g. The set of eigenvalues of H coincides with the set of weights of the $(p+q)$-dimensional complex irreducible representation of $SU(p+q)$), namely we have

$$\{a_1, \cdots, a_p, b_1, \cdots, b_q\} = \{p+q-1,\ p+q-2,\ \cdots,\ 1-p-q\}.$$

We assume that $a_1 > b_1$ holds.

- We have $a_1 = p+q-1$ and $I_{p,q}\xi_1 = \xi_1$, $H \cdot \xi_1 = (p+q-1)\,\xi_1$ hold.
- From $I_{p,q} \circ Y = -Y \circ I_{p,q}$, we have $I_{p,q}(Y \cdot \xi_1) = -Y \cdot \xi_1$ and from $[H, Y] = -2Y$ we have $H(Y \cdot \xi_1) = (p+q-3)\, Y \cdot \xi_1$. Thus we have $b_1 = p+q-3$ and there exists a complex number γ_i with

$$Y \cdot \xi_1 = \gamma_1\, \xi_{p+1}, \quad |\gamma_1| = \sqrt{p+q-1}.$$

- Similarly we have

$$Y \cdot \xi_{p+1} = \gamma_2\, \xi_2, \quad |\gamma_2| = \sqrt{2(p+q-2)}$$

  etc.

Finally we have $p - q = 0, 1$ and the matrix representation of Y with respect to the basis $\xi_1, \cdots, \xi_p, \xi_{p+1}, \cdots, \xi_{p+q}$ is

$$\mathrm{Ad}(g)Y = \sum_{i=1}^{q} \gamma_{2i-1}\, E_{p+i,i} + \sum_{i=1}^{p-1} \gamma_{2i}\, E_{i+1,p+i}.$$

Let ε_i $(1 \leq i \leq p+q)$ be unit complex numbers and put $g = (\varepsilon_1 \xi_1, \cdots, \varepsilon_{p+q} \xi_{p+q})$. We can choose ε_i so that the all of the coefficients γ_{2i} and γ_{2i-1} in the representation of $\mathrm{Ad}(g)Y$ above are positive real numbers. From

$$X_2 = {}^t\overline{Y} - Y, \quad X_3 = \sqrt{-1}\left({}^t\overline{Y} + Y\right)$$

we obtain (4) and (5). □

3.3 BDI : $SO(p+q)/S(O(p) \times O(q))$

Let θ be the involutive automorphism on $G = SO(p+q)$ defined by

$$\theta(g) = I_{p,q} \circ g \circ I_{p,q}$$

and put

$$K = \{g \in SO(p+q) : \theta(g) = g\} = S(O(p) \times O(q)).$$

We can classify totally geodesic surfaces of $SO(p+q)/S(O(p) \times O(q))$ by similar argument to that on $SU(p+q)/S(U(p) \times U(q))$. But, since there are two types of real irreducible representations of $SU(2)$, the classification result is divided into two cases; (iii) and (iv) in the following theorem. Since it is troublesome to give the representation matrix of the action of $\mathfrak{su}(2)$ on the odd-dimensional real irreducible representation ((iii) in the following theorem), we give only the result without proof.

Theorem 3. *Let M be a non-flat totally geodesic surface of $SO(p+q)/S(O(p)\times O(q))$ and U be the set of all elements in $SO(p+q)$ leaving M invariant.*

(i) There exists an orthogonal direct sum decomposition of $\mathbb{R}^{p+q}$ by $I_{p,q}$-invariant and U-irreducible subspaces.

(ii) For each $I_{p,q}$-invariant, U-irreducible subspace V of $\mathbb{R}^{p+q}$, we have

$$\left|\dim\{v \in V : I_{p,q}(v) = v\} - \dim\{v \in V : I_{p,q}(v) = -v\}\right| \leq 1.$$

(iii) Assume that the action of U on $\mathbb{R}^{p+q}$ is irreducible and $p = q+1 \geq 3$.
We denote by p' the integer part of $p/2$ and by q' the integer part of $q/2$. There exists an element $g \in S(O(p) \times O(q))$ such that

$$\mathrm{Ad}(g)X_2 = -\sum_{i=1}^{q'} \sqrt{(2i-1)(p+q+1-2i)}\left(G_{p+2i-1,2i-1} + G_{p+2i,2i}\right)$$

$$+\sum_{i=1}^{p'-1}\sqrt{2i(p+q-2i)}\left(G_{p+2i-1,2i+1}+G_{p+2i,2i+2}\right)$$

$$+\begin{cases}(-\sqrt{2})\sqrt{p\,q}G_{p+q,q} & (\text{if } p=0 \mod 2)\\ \sqrt{2}\sqrt{p\,q}G_{p+q-1,p} & (\text{if } p=1 \mod 2)\end{cases} \tag{6}$$

$$\mathrm{Ad}(g)X_3=\sum_{i=1}^{q'}\sqrt{(2i-1)(p+q+1-2i)}\left(G_{p+2i,2i-1}-G_{p+2i-1,2i}\right)$$

$$+\sum_{i=1}^{p'-1}\sqrt{2i(p+q-2i)}\left(G_{p+2i,2i+1}-G_{p+2i-1,2i+2}\right)$$

$$+\begin{cases}(-\sqrt{2})\sqrt{p\,q}G_{p+q,p} & (\text{if } p=0 \mod 2)\\ \sqrt{2}\sqrt{p\,q}G_{p+q,p} & (\text{if } p=1 \mod 2)\end{cases} \tag{7}$$

(iv) Assume that the action of U on $\mathbb{R}^{p+q}$ is irreducible and $p=q$. Then p is an even integer, say $p=2p'$, and there exists an element $g\in S(O(p)\times O(q))$ such that

$$\mathrm{Ad}(g)X_2=\sum_{i=1}^{p'-1}\sqrt{2i(p-2i)}\left(G_{p+i,i+1}+G_{p+p'+i,p'+i+1}\right)$$

$$-\sum_{i=1}^{p'}\sqrt{(2i-1)(p+1-2i)}\left(G_{p+i,i}+G_{p+p'+i,p'+i}\right) \tag{8}$$

$$\mathrm{Ad}(g)X_3=\sum_{i=1}^{p'-1}\sqrt{2i(p-2i)}\left(G_{p+p'+i,i+1}-G_{p+i,p'+i+1}\right)$$

$$+\sum_{i=1}^{p'}\sqrt{(2i-1)(p+1-2i)}\left(G_{p+p'+i,i}-G_{p+i,p'+i}\right) \tag{9}$$

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