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Real and Complex Submanifolds

Daejeon, Korea, August 2014

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Editors

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Preface

The ICM 2014 Satellite Conference on Real and Complex Submanifolds and the 18th International Workshop on Differential Geometry was held at NIMS, Daejeon in Korea during the period from August 10 (Sun.) to August 12 (Tue.), 2014.

This conference was mainly organized by National Institute for Mathematical Sciences (2014 NIMS Hot Topic Workshop) and Grassmann Research Group supported from National Research Foundation (Proj. No. NRF-2011-220-C00002 and Proj. No. NRF-2012-R1A2A2A-01043023).

Until now our Grassmann Research Group has organized many kinds of international workshops on Differential Geometry and Related Fields. About 18 years ago 1996, the 1st international workshop was held at Kyungpook National University on Dec. 21–22, 1996. Since then, our research group has organized several kinds of conferences from a mini international workshop to a great national joint meeting between KMS and AMS held at Ewha Womans Univ. in Seoul on Dec. 16–20, 2009, (SS28, Differential and Integral Geometry) and another joint meeting between KMS and CMS at Southwest Univ. in Chongqing, China, on May 10–13, 2010. Here the editors want to say their thanks to great contributions and enthusiasm of Prof. Jiazu Zhou who was an organizer of such two joint meetings. All these workshops are supported by NRF, KRF (Korea Research Foundation), NIMS, KNU and KMS. Recently, for the last 6 years our NIMS has given their best efforts to make this conference nicely and has given their constant financial supports and invariant encouragements.

On behalf of the organizing committee (Prof. J. Berndt, Y. Ohnita and B.H. Kim), the editors would like to express their sincere gratitude to all participants and invited speakers from all over the world, in particular, to our 6 plenary speakers Prof. Alfonso Romero (Univ. of Granada, Spain), Prof. Zhizhou Tang (Beijing Normal University, China), Prof. David E. Blair (Michigan State University, USA), Prof. Jürgen Berndt (King's College London, UK), Prof. Katsuei Kenmotsu (Tohoku University, Japan), and Prof. Yong-Geun Oh (POSTECH & IBS-CGP, Korea). For the best efforts to edit the manuscripts submitted to Proceedings of Mathematics and Statistics (Springer), first of all the editors want to say their thanks to the Program Committee as follows:

Prof. Jürgen Berndt (King's College London, UK), Prof. Yoshihiro Ohnita (Osaka City University & OCAMI, Japan), Prof. Juan de Dios Pérez (University of Granada, Spain), Prof. Jiazu Zhou (Southwest University, China), Prof. Reiko Miyaoka (Tohoku University, Japan), Prof. Kastuei Kenmotsu (Tohoku University, Japan), Prof. Carlos Enrique Olmos (Univeridad Nacional de Cordoba, Argentina) and Prof. Masaaki Umehara (Tokyo Institute of Technology, Japan).

The Program Committee have referred all of manuscripts submitted to our Proceedings and have given a great contribution to publish our proceedings in Springer.

Moreover, the editors also want to say their thanks to the Scientific Committee who have recommended nice invited speakers from all over the world. Here I want to mention their names as follows:

Prof. Chuan-Lian Terng (University of California at Irvine, USA), Prof. Gudlaugur Thorbergsson (University of Cologne, Germany), Prof. Carlos Enrique Olmos (Univeridad Nacional de Cordoba, Argentina), Prof. Juan de Dios Pérez (University of Granada, Spain), Prof. Jiazu Zhou (Southwest University, China), Prof. Reiko Miyaoka (Tohoku University, Japan), Prof. Kastuei Kenmotsu (Tohoku University, Japan) and Prof. Masaaki Umehara (Tokyo Institute of Technology, Japan).

Hopefully, our ICM 2014 Satellite Conference on Real & Complex Submanifolds will be a bifurcation point that our GRG, OCAMI and together with many famous differential geometers from all over the world could take a leap in more outstanding level in the field of differential geometry and related fields.

The editors would like to say their thanks to Dr. Chang Hwa Woo and the secretary in chief Miss Ahram Lee for their best efforts to accomplish this book by using AMS LATEX files.

Finally the editors would be willing to say their hearty gratitude to our National Research Foundation and all the staff working at the National Institute for Mathematical Science in Korea.

The editors would be really happy if this kind of Proceedings of Mathematics and Statistics (Springer) will be helpful for graduated students to study their research more creatively and successfully. Thanks a lot.

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Contents

Part I
Plenary Talks

Generalizations of the Catenoid and the Helicoid

David E. Blair

Abstract In this lecture we will discuss various generalizations of the catenoid and the helicoid as well as related differential geometric notions including minimality, quasi-umbilicity and conformal flatness.

1 Historical Remarks

In 1744 Euler showed that a catenoid is a minimal surface and in 1766 Meusnier showed that a right helicoid is a minimal surface. The converse that the catenoid is the only surface of revolution that is minimal is due to Meusnier in 1785 ([17] or see [8]). That the helicoid is the only ruled minimal surface aside from the plane was proved by Catalan in 1842. For more history see Chen [12, esp. pp. 207–208].

A remarkable feature of these surfaces is that they are locally isometric. In fact one can easily construct an isometric family of minimal surfaces depending on a parameter λ such that $\lambda = 0$ is a helicoid and $\lambda = \frac{\pi}{2}$ is a catenoid:

$$x^1(u,v) = \cos\lambda \sin u \sinh v + \sin\lambda \cos u \cosh v,$$
$$x^2(u,v) = -\cos\lambda \cos u \sinh v + \sin\lambda \sin u \cosh v,$$
$$x^3(u,v) = u\cos\lambda + v\sin\lambda.$$

2 Quasi-umbilicity

We will begin with a discussion of quasi-umbilicity. For an n-dimensional hypersurface of Euclidean space quasi-umbilicity means that the Weingarten map has at least $n-1$ eigenvalues equal and we have the following theorem of Cartan [7] dating from 1917, also attributed to Schouten [19] (1921).

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Theorem 1. *A conformally flat hypersurface M^n, $n \geq 4$, in Euclidean space E^{n+1} is quasi-umbilical.*

Common examples, also due to Cartan, are the canal hypersurfaces, i.e. envelopes of one-parameter families of hyperspheres. Thus conformal flatness can, at least in some situations, be viewed as a natural generalization of a surface of revolution.

For submanifolds of general codimension the notion of quasi-umbilicity is more subtle. Consider an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold. A unit normal vector field is a *quasi-umbilical section* of the normal bundle if the corresponding Weingarten map has at least $n-1$ eigenvalues equal. A submanifold is said to be *quasi-umbilical* if there exist p mutually orthogonal quasi-umbilical normal sections (see e.g. Chen [10, pp. 147–148], or [12, p. 308]).

An early result relating quasi-umbilicity and conformal flatness is the following theorem of Chen and Yano of 1972 [13].

Theorem 2. *A quasi-umbilical submanifold of dimension ≥ 4 of a conformally flat manifold is conformally flat.*

In general the converse of this theorem is not true; e.g. we will note below that the Lagrangian catenoid in $\mathbb{C}^n$ of Castro and Urbano [9] is conformally flat but not quasi-umbilical. Other examples of conformally flat submanifolds of dimension $n > 3$ in a Euclidean space which are not quasi-umbilical can be found in the work of Ü. Lumiste and M. Väljas [16]; the codimensions of their examples are integers $\geq n-2$. However if the codimension is not too large, we do have a converse of the Chen-Yano theorem due to Moore and Morvan in 1978 [18].

Theorem 3. *If $p \leq min\{4, n-3\}$, a conformally flat submanifold M^n of Euclidean space E^{n+p} is quasi-umbilical.*

To illustrate the theorem of Moore and Morvan we give a simple, though not complete, example of a conformally flat submanifold of codimension 2 in E^7.

Let $\mathscr{U}(-1)$ be a piece of the pseudo-sphere

$$\mathbf{x}(s,\theta) = (e^s\cos\theta, e^s\sin\theta, \int_0^s \sqrt{1-e^{2t}}\,dt);$$

$\mathscr{U}(-1)$ can be viewed as a piece of the hyperbolic plane with constant curvature -1, realized as a surface in E^3. Let $S^3(1)$ be the unit sphere in E^4.

Now consider $M^5 = \mathscr{U}(-1) \times S^3(1)$ as an incomplete conformally flat submanifold of E^7 which by the Moore-Morvan theorem must be quasi-umbilical. With respect to the usual normal fields, ζ_1, ζ_2, in the factor spaces, the Weingarten maps are of the form

$$A_1 = \begin{pmatrix} -\frac{e^s}{\sqrt{1-e^{2s}}} & & & & \\ & \frac{\sqrt{1-e^{2s}}}{e^s} & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}.$$

Thus ζ_1 and ζ_2 are not quasi-umbilical sections of the normal bundle.

Let

$$\eta_1 = \cos\phi\, \zeta_1 + \sin\phi\, \zeta_2,$$
$$\eta_2 = -\sin\phi\, \zeta_1 + \cos\phi\, \zeta_2$$

where $\phi = \tan^{-1} \dfrac{\sqrt{1-e^{2s}}}{e^s}$. Then the Weingarten maps with respect to η_1, η_2 are

$$B_1 = \begin{pmatrix} -\frac{e^s \cos\phi}{\sqrt{1-e^{2s}}} & & & & \\ & \sin\phi & & & \\ & & \sin\phi & & \\ & & & \sin\phi & \\ & & & & \sin\phi \end{pmatrix},$$

$$B_2 = \begin{pmatrix} \cos\phi & & & & \\ & -\frac{\sqrt{1-e^{2s}}\sin\phi}{e^s} & & & \\ & & \cos\phi & & \\ & & & \cos\phi & \\ & & & & \cos\phi \end{pmatrix}$$

both of which have four equal eigenvalues illustrating the quasi-umbilicity.

In the course of their proof, Moore and Morvan show the existence of an orthonormal basis $e_1, \ldots, e_n$ of the tangent space of M^n with respect to which the second fundamental form is given by a matrix of the form

$$\begin{pmatrix} \zeta_{ab} & & & \\ & f\zeta & & \\ & & \ddots & \\ & & & f\zeta \end{pmatrix}$$

where (ζ_{ab}) is a $p \times p$ matrix of normal vectors and ζ is a unit normal vector.

In particular one can show that a conformally flat submanifold of E^{n+p} with $p \leq min\{4, n-3\}$ admits a foliation by spaces of constant curvature of dimension $\geq n - p$.

Lumiste and Väljas [16] also defined the notion of a *completely quasi-umbilical submanifold*, meaning that every unit normal vector field is a quasi-umbilical section of the normal bundle. They show that a submanifold M^n of a Euclidean space E^{n+p} is completely quasi-umbilical if and only if it is a canal submanifold, i.e. the envelope of a one-parameter family of n-spheres.

3 Generalized Catenoids

In 1975 (see [2]) the author proved the following theorem.

Theorem 4. *A conformally flat, minimal hypersurface M^n, $n \geq 4$, in Euclidean space E^{n+1} is either totally geodesic or a hypersurface of revolution $S^{n-1} \times \gamma(s)$ where the profile curve is a plane curve γ determined by its curvature κ as a function of arc length by $\kappa = (1-n)/u^n$ and $s = \int \frac{u^{n-1}du}{\sqrt{Cu^{2n-2}-1}}$, C being a constant.*

If $n = 3$, replacing conformal flatness by quasi-umbilicity gives the same result with the same proof. For $n = 2$, the profile curve is a catenary and hence these hypersurfaces are called *generalized catenoids*. In 1991 Jagy [15] gave an independent study of this question by assuming that the minimal hypersurface is foliated by spheres from the outset.

Turning to the higher codimension case we have the following 2006 result of the author [4]. First recall that the Schouten tensor L of a Riemannian manifold (M^n, g) is defined by

$$L = -\frac{Q}{n-2} + \frac{\tau}{2(n-1)(n-2)} I,$$

Q being the Ricci operator and τ the scalar curvature. For a survey of ideas surrounding the Schouten tensor and conformal flatness, see the essay by K. Bang and the author [1].

Theorem 5. *Let M^n, be a conformally flat, minimal submanifold of E^{n+p} with $p \leq min\{4, n-3\}$. If the Schouten tensor has at most two eigenvalues, then either M^n is flat and totally geodesic or a generalized catenoid lying in some $(n+1)$-dimensional Euclidean space.*

4 Lagrangian Catenoids

Let $(v^1 + iv^2, \dots, v^{2n-1} + iv^{2n})$ be the coordinates on $\mathbb{C}^n$. An n-dimensional submanifold M^n of $\mathbb{C}^n$ is said to be *Lagrangian* if the restriction of the canonical symplectic form $\Omega = \sum_{i=1}^n dv^{2i-1} \wedge dv^{2i}$ to M^n vanishes.

In 1999 I. Castro and F. Urbano introduced the Lagrangian catenoid. The manifold itself, M_0, was introduced earlier by Harvey and Lawson in 1982 [14] as an example of a minimal Lagrangian submanifold. It is defined as the set of points $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \equiv \mathbb{C}^n$ satisfying

$$|x|y = |y|x, \quad \Im(|x| + i|y|)^n = 1, \quad |y| < |x| \tan(\pi/n).$$

Topologically M_0 is $\mathbb{R}\times S^{n-1}$. To describe it precisely, let S^{n-1} be the unit sphere in $\mathbb{R}^n$ and view a point $p \in S^{n-1}$ as an n-tuple in $\mathbb{R}^n$ giving its coordinates. Define a map

$$\phi_0 : \mathbb{R} \times S^{n-1} \longrightarrow \mathbb{C}^n \equiv \mathbb{R}^n \times \mathbb{R}^n$$

by

$$\phi_0(u, p) = \cosh^{1/n}(nu)e^{i\beta(u)}p$$

where

$$\beta(u) = \frac{\pi}{2n} - \frac{2}{n}\arctan(\tanh\frac{nu}{2}) \in (0, \frac{\pi}{n})$$

and the multiplication $e^{i\beta}p$ multiplies each coordinate of p by $e^{i\beta}$ and lists the real and imaginary parts as a $2n$-tuple in $\mathbb{C}^n \equiv \mathbb{R}^n \times \mathbb{R}^n$.

For example, for $n = 2$, writing p as $(\cos\theta, \sin\theta)$,

$$\phi_0(u, p) = \sqrt{\cosh 2u}\big(\cos\beta\cos\theta, \cos\beta\sin\theta, \sin\beta\cos\theta, \sin\beta\sin\theta\big)$$

Now $\beta = \frac{\pi}{4} - \arctan(\tanh u)$, so with some simplification

$$\phi_0(u, p) = \Big(\frac{e^u}{\sqrt{2}}\cos\theta, \frac{e^u}{\sqrt{2}}\sin\theta, \frac{e^{-u}}{\sqrt{2}}\cos\theta, \frac{e^{-u}}{\sqrt{2}}\sin\theta\Big)$$

Let g_0 be the standard metric of constant curvature +1 on S^{n-1}; then the metric induced on $\mathbb{R} \times S^{n-1}$ by ϕ_0 is

$$ds^2 = \cosh^{2/n}(nu)(du^2 + g_0)$$

which is clearly conformally flat. This Lagrangian submanifold defined by the mapping

$$\phi_0 : \mathbb{R} \times S^{n-1} \longrightarrow \mathbb{C}^n$$

together with its induced metric is known as the *Lagrangian catenoid*. A detailed proof that the Lagrangian catenoid is not quasi-umbilical can be found in [5].

The main result of Castro and Urbano is the following.

Theorem 6. *Let $\phi : M^n \longrightarrow \mathbb{C}^n$ be a minimal (non-flat) Lagrangian immersion. Then M^n is foliated by pieces of round $(n-1)$-spheres in $\mathbb{C}^n$ if and only if ϕ is congruent (up to dilations) to an open subset of the Lagrangian catenoid.*

Again considering conformal flatness and minimality, we have the following result of the author [5].

Theorem 7. *Let $\phi : M^n \longrightarrow \mathbb{C}^n$, $n \geq 4$ be a conformally flat, minimal, Lagrangian submanifold of $\mathbb{C}^n$. If the Schouten tensor has at most two eigenvalues, then either M^n is flat and totally geodesic or is homothetic to (a piece of) the Lagrangian catenoid.*

It would be natural to ask for a complete classification of the conformally flat, minimal, Lagrangian submanifolds in $\mathbb{C}^n$. However this seems to be a difficult, but potentially interesting, question.

5 A Generalized Helicoid

The generalization of the helicoid is quite different as seen from the following theorem of Vanstone and the author [6].

Theorem 8. *Let M^n be a complete, minimal hypersurface of E^{n+1} and suppose that M^n admits a codimension 1 foliation by Euclidean $(n-1)$-spaces. Then either M^n is totally geodesic or $M^n = M^2 \times E^{n-2}$ where M^2 is a classical helicoid in E^3.*

This generalized helicoid is not conformally flat and hence contrasts with the classical case since it is not locally isometric to the generalized catenoid, such an isometric deformation would preserve the conformal flatness.

6 A Lagrangian Helicoid

We will see below that the only minimal, Lagrangian submanifolds in $\mathbb{C}^n$ that are foliated by pieces of $(n-1)$-planes are pieces of n-planes. Thus we will drop the minimality and study Lagrangian submanifolds in $\mathbb{C}^n$ that are foliated by Euclidean $(n-1)$-planes. The four theorems of this section are from the author's paper [3].

First, however, we need one additional notion. A Lagrangian submanifold of a Kähler manifold is said to be *Lagrangian H-umbilical* if its second fundamental form σ satisfies

$$\sigma(X,Y) = \alpha \langle JX, H\rangle\langle JY, H\rangle H + \beta|H|^2\{\langle X,Y\rangle H + \langle JX, H\rangle JY + \langle JY, H\rangle JX\}$$

for some functions α and β, H being the mean curvature vector.

Theorem 9. *Let M^n be a complete Lagrangian submanifold of $\mathbb{C}^n$ which is foliated by $(n-1)$-planes, then M^n is either totally geodesic, Lagrangian H-umbilical and flat or the product of a ruled Lagrangian surface in $\mathbb{C}^2$ and a Lagrangian $(n-2)$-plane in $\mathbb{C}^{n-2}$.*

Flat, Lagrangian H-umbilical submanifolds of $\mathbb{C}^n$ were completely classified by B.-Y. Chen [11]. The description of these submanifolds is quite technical and depends on some special curves. The submanifolds are of form

$$L : I \times \mathbb{R}^{n-1} \longrightarrow \mathbb{C}^n.$$

If t denotes the coordinate on the interval I and $u_2, \ldots, u_n$ the coordinates on $\mathbb{R}^{n-1}$, the induced metric is of the form

$$g = f^2 dt^2 + du_2^2 + \cdots du_n^2$$

where f is a function of all the variables but linear in $u_2, \ldots, u_n$. The mapping L is linear in $u_2, \ldots, u_n$ and hence such Lagrangian submanifolds are foliated by $(n-1)$-planes. Thus our main point here is to consider the case $n = 2$ in detail.

Theorem 10. *Let M^2 be a non-flat, Lagrangian submanifold in $\mathbb{C}^2$ that is foliated by lines. Then there exist local coordinates (t, x) such that the induced metric takes the form $ds^2 = f^2dt^2 + dx^2$ where f^2 is a positive function which is quadratic in x and the Weingarten maps A_1, A_2 corresponding to the normals $\zeta_1 = \frac{1}{f} J \frac{\partial}{\partial t}$, $\zeta_2 = J \frac{\partial}{\partial x}$ are given by $A_1 = \begin{pmatrix} b & a \\ a & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, where $a = A(t)/f^2$, $-4A(t)^2$ is the discriminant of f^2 and*

$$b = \frac{1}{f}\Big[\int \frac{A'(t)f^2 - A(t)(f^2)_t}{f^4}\,dx + B(t)\Big]$$

for some function $B(t)$. Conversely let M^2 be a simply connected domain in the (tx)-plane and

$$f^2 = F(t)x^2 + G(t)x + H(t)$$

a positive function, quadratic in x, on M^2. Then there exists an isometric Lagrangian immersion of M^2 into $\mathbb{C}^2$ that is foliated by line segments whose first and second fundamental forms are given as above.

At the beginning of this section we indicated that the only minimal, Lagrangian submanifolds in $\mathbb{C}^n$ that are foliated by pieces of $(n-1)$-planes are pieces of n-planes, i.e. totally geodesic. We can now see this as a corollary of the above theorems. The first case of Theorem 9 is the totally geodesic case. The second case is Lagrangian H-umbilical and hence, by the definition of Lagrangian H-umbilicity, minimality implies totally geodesic. For the third case in Theorem 9 we turn to Theorem 10 and observe again that the minimality would imply totally geodesic.

We now suppose there is a one-parameter family of Lagrangian surfaces in $\mathbb{C}^2$ connecting a ruled Lagrangian surface to the Lagrangian catenoid. Let (v^1+iv^2, v^3+iv^4) denote the coordinates on $\mathbb{C}^2$ and $\mathbf{v}$ the mapping $\mathbf{v} : M^2 \longrightarrow \mathbb{C}^2$ given by $v^i = v^i(t,x)$.

Theorem 11. *If there exists a one-parameter family of Lagrangian surfaces in $\mathbb{C}^2$ connecting a ruled Lagrangian surface M^2 to the Lagrangian catenoid, then M^2 is given by*

$$v^1 = k(\cos t)x + \beta_1(t), \quad v^2 = l(\cos t)x + \beta_2(t),$$
$$v^3 = k(\sin t)x + \beta_3(t), \quad v^4 = l(\sin t)x + \beta_4(t)$$

where k and l are constants satisfying $k^2 + l^2 = 1$, the quadratic becomes $x^2 + G(t)x + H(t)$ and the β_i's are determined by

$$\beta_1' = -\Big(\frac{kG(t)}{2} + lA(t)\Big)\sin t, \quad \beta_2' = \Big(-\frac{lG(t)}{2} + kA(t)\Big)\sin t,$$
$$\beta_3' = \Big(\frac{kG(t)}{2} + lA(t)\Big)\cos t, \quad \beta_4' = \Big(\frac{lG(t)}{2} - kA(t)\Big)\cos t$$

where $4A(t)^2 = 4H(t) - G(t)^2$.

In particular we have a continuous family of surfaces whose position vectors are

$$P(\lambda)\big(k(\cos t)x + \beta_1(t), l(\cos t)x + \beta_2(t), k(\sin t)x + \beta_3(t), l(\sin t)x + \beta_4(t)\big)$$
$$+Q(\lambda)\Big(\frac{e^x}{\sqrt{2}}\cos t, \frac{e^{-x}}{\sqrt{2}}\cos t, \frac{e^x}{\sqrt{2}}\sin t, \frac{e^{-x}}{\sqrt{2}}\sin t\Big)$$

where for the parameter λ we have $P(0) = 1$, $P(\Lambda) = 0$, $Q(0) = 0$ and $Q(\Lambda) = 1$, P and Q being continuous functions on an interval $[0, \Lambda]$.

We call a surface given as in Theorem 11 a *Lagrangian helicoid*.

The last result that we mention from [3] is that even though the Lagrangian helicoids can be connected to a Lagrangian catenoid through a family of Lagrangian surfaces, the Lagrangian submanifolds of Theorem 9, cannot be locally isometric to a Lagrangian catenoid.

Theorem 12. *Let M^n be a Lagrangian submanifold of $\mathbb{C}^n$ which is foliated by $(n-1)$-planes, then M^n is not locally isometric to a Lagrangian catenoid.*

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Harmonic Functions and Parallel Mean Curvature Surfaces

Katsuei Kenmotsu

Abstract Minimal surfaces in a Euclidean three space are closely related to complex function theory. A constant mean curvature surface is constructed by a harmonic mapping using the generalized Weierstrass representation formula. In this paper, I present a surface that is constructed by a harmonic function. It is immersed in a complex two-dimensional complex space form with parallel mean curvature vector. We prove that the Kaehler angle function of the surface is obtained by a functional transformation of a harmonic function. And, then, the 1st and 2nd fundamental forms of the surface are explicitly expressed by the Kaehler angle function. As a byproduct, we show that any Riemann surface can be locally embedded in the complex projective plane and also in the complex hyperbolic plane as a parallel mean curvature surface.

1 Introduction

Minimal surfaces in a three-dimensional Euclidean space are closely related to complex function theory. A constant mean curvature surface is constructed by a harmonic mapping using the generalized Weierstrass representation formula. In this paper, I will present a surface that is constructed by a harmonic function. This is immersed in a complex two-dimensional complex space form with parallel mean curvature vector. To explain how to construct such surfaces, first in Sect. 2, we give fundamental formulas for the surfaces of parallel mean curvature vector immersed in a complex two-dimensional complex space form. Then, in Sect. 3, we review known results for such surfaces. In Sect. 4, we define a surface of general type by the second fundamental tensor, and prove the main result of this paper: *A surface of general type depends on a real valued harmonic function on the surface and five real constants*. Since the coordinate function of any Riemann surface is harmonic, a direct application of the above is that *any Riemann surface can be locally embedded in the complex projective plane, and in the complex hyperbolic plane as a parallel mean curvature surface*.

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2 Surfaces in a Complex Space Form

Let M be a real two-dimensional Riemannian manifold with the Riemannian metric ds^2, and $\overline{M}[4\rho]$ be a complex two-dimensional complex space form of constant holomorphic sectional curvature 4ρ. Let $x : M \longrightarrow \overline{M}[4\rho]$ be an isometric immersion of parallel mean curvature vector, and let α be the Kaehler angle function of x. α is defined by $\cos\alpha = \langle Jx_*e_1, x_*e_2\rangle$, where e_1, e_2 define the local orthonormal frame on M, J is the complex structure of $\overline{M}[4\rho]$, and $\langle\ ,\ \rangle$ is the Kaehler metric on $\overline{M}[4\rho]$. For the geometric meaning of the Kaehler angle α, we have $\alpha = 0$ (resp. π) if and only if x is holomorphic (resp. anti-holomorphic), and $\alpha = \pi/2$ if and only if x is totally real.

Let H be the mean curvature vector of x. Suppose that $H \neq 0$, and set $|H| = 2b > 0$. According to Ogata [9], there exist unitary coframes $\{w_1, w_2\}$ on $\overline{M}[4\rho]$ and complex-valued functions a, c on M such that $ds^2 = \phi\bar{\phi}$. The structure equations of x are

$$d\alpha = (a+b)\phi + (\bar{a}+b)\bar{\phi}, \tag{1}$$

$$d\phi = (\bar{a}-b)\cot\alpha\cdot\phi\wedge\bar{\phi}, \tag{2}$$

$$K = -4(|a|^2-b^2) + 6\rho\cos^2\alpha, \tag{3}$$

$$da\wedge\phi = -\left(2a(\bar{a}-b)\cot\alpha + \frac{3}{2}\rho\sin\alpha\cos\alpha\right)\phi\wedge\bar{\phi}, \tag{4}$$

$$dc\wedge\bar{\phi} = 2c(a-b)\cot\alpha\cdot\phi\wedge\bar{\phi}, \tag{5}$$

$$|a|^2 - |c|^2 + \frac{\rho}{2}\left(-2+3\sin^2\alpha\right) = 0, \tag{6}$$

where $\phi = \cos\alpha/2\cdot\omega_1 + \sin\alpha/2\cdot\bar{\omega}_2$, and K denotes the Gaussian curvature of M. Formula (3) is the Gauss equation, (4) and (5) are Codazzi equations, and (6) is the Ricci equation of the immersion.

Let us explain the geometric meaning of the functions a and c. For the adapted orthonormal frame $\{e_1, e_2, e_3, e_4\}$ with $e_3 = -H/|H|$, the components of the second fundamental tensor of x are written as h^3_{ij}, h^4_{ij}, $(1 \le i, j \le 2)$. Then, we have

$$\frac{h^3_{11}+h^3_{22}}{2} = -2b,\ \frac{h^4_{11}+h^4_{22}}{2} = 0,$$

$$2a = -\left(\frac{h^3_{11}-h^3_{22}}{2} + h^4_{12}\right) - i\left(\frac{h^4_{11}-h^4_{22}}{2} - h^3_{12}\right), \tag{7}$$

$$2c = -\left(\frac{h^3_{11}-h^3_{22}}{2} - h^4_{12}\right) - i\left(\frac{h^4_{11}-h^4_{22}}{2} + h^3_{12}\right). \tag{8}$$

We remark that the complex functions a and c are local invariants of x, because if we fix the orientations of M and $\overline{M}[4\rho]$, then the adapted frame $\{e_1, e_2, e_3, e_4\}$ is uniquely determined under the condition $e_3 = -H/|H|$.

3 Known Results

First, we review the result for the immersions in which the Kaehler angle α is constant on M. Such an immersion is called a slant immersion, and many results are known. For references, see Sect. 18 of Chap. 3 (by B.Y. Chen) in [3].

Theorem 1 (Chen and Vrancken 1997). *Under the above notation, if α is constant, then $K \in \{-2b^2, 0, 4b^2\}$. Neglecting trivial cases, the only interesting result occurs on $K = -2b^2$, and there is a parallel mean curvature immersion from $RH^2[-2b^2]$ into $CH^2[-12b^2]$ with a constant Kaehler angle.*

Theorem 1 is included in Theorem 1.1 of Hirakawa [5], because if a parallel mean curvature surface has a constant Kaehler angle, then it is of constant Gaussian curvature, which is easily proved by (1) and (3), and Hirakawa has classified such surfaces in [5].

We now assume $d\alpha \neq 0$ on M. The immersions with $d\alpha \neq 0$ were first treated by Ogata [9] in 1995 and Kenmotsu and Zhou [8] in 2000, but these papers are not complete. Later, Hirakawa [5] classified surfaces with $a = \bar{a}$, and proved that those surfaces found in [8, 9] were covered by his classification table.

Theorem 2 (Hirakawa 2006). *Let $x : M \longrightarrow \overline{M}[4\rho]$ be a parallel mean curvature immersion with $a = \bar{a}$. If $\rho \neq 0$ and the Kaehler angle is not constant, then $\rho = -3b^2$ and $\alpha, a,$ and c are functions of a single variable, say u. Moreover, the Kaehler angle $\alpha = \alpha(u)$ satisfies the differential equation*

$$\frac{d\alpha}{du} = \sqrt{b}\sqrt{8 - 9\sin^2\alpha(u)}, \tag{9}$$

and the first fundamental form of x, and the functions a, c can be expressed as

$$\begin{aligned} ds^2 &= \frac{4}{b}\frac{1}{(8 - 9\sin^2\alpha(u))}(du^2 + dv^2), \\ a(u) &= \frac{b}{4}\left(4 - 9\sin^2\alpha(u)\right), \\ c(u) &= \frac{b}{4}\left|8 - 9\sin^2\alpha(u)\right| e^{it}, \ (\text{for some } t \in R). \end{aligned}$$

Conversely, for any $b > 0$ and a given real-valued function $\alpha(u)$ defined on an interval I in R satisfying Eq. (9), there exists a parallel mean curvature immersion from $I \times R$ into $\overline{M}[-12b^2]$ with $|H| = 2b$ such that the Kaehler angle is $\alpha(u)$.

In short, in the case where $a = \bar{a}$, non-trivial results occur only in the negative curvature space, and the surface is isometric to a surface of revolution in R^3.

4 The Case Where $a \neq \bar{a}$

The case $a \neq \bar{a}$ was first studied by Hirakawa [5] under an additional condition $c = 0$. He found a global isometric embedding from $RH^2[-2b^2]$ into $CH^2[-12b^2]$ with $d\alpha \neq 0$. We note that this is not congruent to the surface in Theorem 1.

Definition 1. Let $x : M \longrightarrow \overline{M}[4\rho]$ be a parallel mean curvature immersion. x is of general type if and only if $a \neq \bar{a}$.

Generalizing Hirakawa's work [5] to the case where $c \neq 0$, we obtained [7].

Theorem 3 (Kenmotsu 2013). *Let $x : M \longrightarrow \overline{M}[4\rho]$ be a parallel mean curvature immersion of general type. Then, if $\rho \neq 0$, $H \neq 0$, and α is not constant, x depends on one real-valued harmonic function on the surface and five real constants.*

We now remark on the above assumptions.

1. When $\rho = 0$, the complex Euclidean plane C^2 is identified as R^4, and parallel mean curvature surfaces in R^4 have been independently studied by Chen [1], Hoffman [6], and Yau [10].
2. For the case $H = 0$, we refer readers to Chern-Wolfson [2] and Eschenburg-Guadalupe-Tribuzy [4].
3. When $\alpha =$ constant, we already have Theorem 1.

The following is important in our proof of Theorem 3.

Lemma 1. *If α is not constant and x is of general type, then a is a function of α.*

The proof of Lemma 1 proceeds by considering all prolongations of the structure equations of x. We may write $a = a(\alpha)$ and proceed with the proof of Theorem 3. Now, a, c, α, and ds^2 are explicitly determined as follows. First, we obtain the ordinary differential equation of a for α.

$$\frac{da}{d\alpha} = \frac{t_1(\alpha, a(\alpha))}{\overline{a(\alpha)} + b}, \tag{10}$$

where

$$t_1(\alpha, a) = \frac{\cot \alpha}{-2 + 3\sin^2 \alpha}(-4b + 12b \sin^2 \alpha + 4a + 3a \sin^2 \alpha).$$

The complex-valued function $a = a(\alpha)$ is a solution of the first-order ordinary differential equation (10). Hence, $a(\alpha)$ is determined by two real constants. Next, we determine the function c. Set

$$c = \left(|a|^2 + \rho/2(-2 + 3\sin^2\alpha)\right)^{1/2} e^{i\nu}. \tag{11}$$

Then, ν satisfies

$$d\nu = \frac{1}{2i}\frac{t_2(\alpha, a, \bar{a})\phi - \overline{t_2(\alpha, a, \bar{a})\phi}}{(|a|^2 + \rho/2(-2 + 3\sin^2\alpha))},$$

where

$$t_2(\alpha, a, \bar{a}) = 2a(\bar{a} - b)\cot\alpha + \frac{3}{2}\rho\sin\alpha\cos\alpha.$$

c is fixed by the integration of the real one-form of $d\nu$; hence, it is determined up to a real constant.

Finally, we study the Kaehler angle function α. Set

$$F(\alpha) = \frac{(|a(\alpha) - b|^2 + \frac{3}{2}\rho\sin^2\alpha)}{|a(\alpha) + b|^2}\cot\alpha.$$

Then, α satisfies

$$\alpha_{z\bar{z}} - F(\alpha)\alpha_z\alpha_{\bar{z}} = 0, \tag{12}$$

where we write $\phi = \lambda dz$, and $z = u + iv$ is an isothermal coordinate on M.

Lemma 2. *Any solution* α *of partial differential equation* (12) *is written as* $\alpha = \psi(f(z,\bar{z}))$, *where* $\psi(t)$ *is a solution of*

$$\psi''(t) - F(\psi(t))\psi'(t)^2 = 0 \tag{13}$$

and $f(z,\bar{z})$ *is a harmonic function on* M.

Proof. Define a real-valued function $K(t)$ of one real variable by

$$K(t) = \int e^{-\int F(t)dt}dt,$$

and set $f(z,\bar{z}) = K(\alpha(z,\bar{z}))$. From (12), $f(z,\bar{z})$ is a harmonic function, i.e., f satisfies $\partial^2 f/\partial z\partial\bar{z} = 0$. We set $\psi(t) = K^{-1}(t)$. Then, $\psi(t)$ satisfies the ordinary differential equation (13), proving Lemma 2.

We remark that $\psi(t)$ is fixed up to two real constants. Using these results, the first fundamental form of x can be written as

$$ds^2 = \frac{\psi'(f)^2|f_z|^2}{|a(\psi(f)) + b|^2}|dz|^2.$$

We showed that the first and second fundamental forms of the immersion x of general type are expressed by a harmonic function and five real constants.

Next, we prove that the converse holds. Let $\rho \neq 0$ be real and $b > 0$. Let D be a simply connected domain in $\boldsymbol{C}$ with the complex coordinate z, and $f(z, \bar{z})$ be a harmonic function on D with $f_z \neq 0$. Consider a complex-valued function $a(t)$ of one real variable t as a solution of

$$\frac{da}{dt} + \frac{2\cot t}{\overline{a(t)} + b}\left(ba(t) - |a(t)|^2 - \frac{3\rho}{4}\sin^2 t\right) = 0.$$

Let $\psi = \psi(t)$ be a solution of (13). Define $\alpha = \psi(f(z, \bar{z}))$, $(z \in D)$, and $a = a(\alpha)$. Set $\phi = \alpha_z dz/(a(\alpha) + b)$ and $ds^2 = \phi\bar{\phi}$. To find c, we need the following result.

Lemma 3. *Set*

$$\theta = \frac{1}{2i}\frac{t_2(\alpha, a(\alpha), \overline{a(\alpha)})\phi - \overline{t_2(\alpha, a(\alpha), \overline{a(\alpha)})\phi}}{(|a(\alpha)|^2 + \rho/2(-2 + 3\sin^2\alpha))}.$$

Then, θ is a closed one-form on D.

The proof of Lemma 3 is straightforward. By Lemma 3, there exists $\nu : D \to \boldsymbol{R}$ such that $d\nu = \theta$. We define c by (11). Then, we can prove that these ϕ, α, a, and c satisfy the Gauss, Codazzi, and Ricci equations (1) $\sim$ (6). Hence, by the fundamental theorem of surfaces there exists an isometric immersion of (D, ds^2) into $\overline{M}[4\rho]$ with a parallel mean curvature vector. This is of general type and satisfies $|H| = 2b$, proving Theorem 3.

Since the coordinate function of any Riemann surface is harmonic, we have the following embedding theorem [7].

Theorem 4 (Kenmotsu 2013). *Any Riemann surface can be locally embedded in the complex projective plane, and in the complex hyperbolic plane, as a parallel mean curvature surface.*

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A New Technique for the Study of Complete Maximal Hypersurfaces in Certain Open Generalized Robertson–Walker Spacetimes

Alfonso Romero

Abstract An $(n + 1)$-dimensional Generalized Robertson–Walker (GRW) spacetime such that the universal Riemannian covering of the fiber is parabolic (thus so is the fiber) is said to be spatially parabolic. This class of spacetimes allows to model open relativistic universes which extend to the spatially closed GRW spacetimes from the viewpoint of the geometric-analysis of the fiber and which are not incompatible with certain cosmological principle. We explain here a new technique for the study of non-compact complete spacelike hypersurfaces in such spacetimes. Thus, a complete spacelike hypersurface in a spatially parabolic GRW spacetime inherits the parabolicity, whenever some boundedness assumptions on the restriction of the warping function to the spacelike hypersurface and on the hyperbolic angle between the unit normal vector field and a certain timelike vector field are assumed. Conversely, the existence of a simply connected parabolic spacelike hypersurface, under the previous assumptions, in a GRW spacetime also leads to its spatial parabolicity. Then, all the complete maximal hypersurfaces in a spatially parabolic GRW spacetime are determined in several cases, extending known uniqueness results. Finally, all the entire solutions of the maximal hypersurface equation on a parabolic Riemannian manifold are found in several cases, solving new Calabi–Bernstein problems.

1 Introduction

In the study of complete spacelike surfaces M in certain three-dimensional GRW spacetimes $\overline{M}$, whose mean curvature function H satisfies: $H = 0$, $H =$ constant or $H^2 \leq \frac{f'(t)^2}{f(t)^2}$, one arrives to the parabolicity of the surface as an intermediate technical step. Normally, it follows from a property of the Gauss curvature of the surface (obtained via the Gauss equation) and an intrinsic result to get the parabolicity on a two-dimensional (non-compact) complete Riemannian manifold (see for instance [15]). In fact, parabolicity for two-dimensional Riemannian manifolds is

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very close to the classical parabolicity for Riemann surfaces. Moreover, it is strongly related to the behavior of the Gauss curvature of the surface. For instance, by a classical result by Ahlfors and Blanc–Fiala–Huber [11], a complete Riemannian surface (M, g) with non-negative Gauss curvature K must be parabolic. The same conclusion is attained if (M, g) is complete and we have either $K \geq -1/(r^2 \log r)$, for r, the distance to a fixed point sufficiently large [8] or if the negative part of K is integrable on M, [12], i.e., $\int_M K_- < \infty$, where $K_-(p) := \max\{-K(p), 0\}$, for any $p \in M$.

Parabolicity has no clear relation with curvature for bigger dimension and, therefore, other techniques are normally used in that case. However, the parabolicity of a complete spacelike hypersurface in a certain spacetime, may be obtained in another way independent of the dimension and of any curvature assumption [16]. Thus, our main aim here is to explain this new approach and to show, based on [16, 17], how it can be applied to prove several uniqueness results on complete maximal hypersurfaces.

2 Parabolicity of Riemannian Manifolds

An n-dimensional non-compact complete Riemannian manifold (M, g) is said to be parabolic if it admits no non-constant non-negative superharmonic function, i.e., if $u \in C^\infty(M)$ satisfies $\Delta_g u \leq 0$ and $u \geq 0$, then $u =$ constant.

To be parabolic is clearly a property invariant under (global) isometries. Even more, a Riemannian manifold (M, g) is said to be quasi-isometric to another one (M', g') if there exists a diffeomorphism $\phi : M \to M'$ and a constant $c \geq 1$ such that

$$c^{-1}|v|_g \leq |d\phi(v)|_{g'} \leq c\,|v|_g \,,$$

for all $v \in T_p M$, $p \in M$ (see for instance [12]). Obviously, isometric Riemannian manifolds are also quasi-isometric and to be quasi-isometric is an equivalence relation. Moreover, we have [10, 18],

Theorem 1. *Let (M, g) and (M', g') be quasi-isometric Riemannian manifolds. Then, (M, g) is parabolic if and only if (M', g') is parabolic.*

Remark 1. (a) The universal Riemannian covering map $\mathbb{R}^3 \to \mathbb{S}^1 \times \mathbb{R}^2$ is a local isometry. Note that $\mathbb{S}^1 \times \mathbb{R}^2$ is parabolic and $\mathbb{R}^3$ is not. Therefore, in the notion of quasi-isometry, the diffeomorphism cannot be relaxed to be a local diffeomorphism, (however, note that if a Riemannian covering $\tilde{M}$ of a Riemannian manifold M is parabolic, then M is also parabolic). (b) Theorem 1 also holds if the exterior of some compact subset in M is quasi-isometric to the exterior of a compact subset in M' [9, Cor. 5.3]. (c) There exists a notion much weaker than quasi-isometry: the so-called rough isometry (roughly isometric manifolds are not homeomorphic, in general). Under this hypothesis, it is necessary to impose extra geometric assumptions (in terms of the Ricci curvature and the injectivity radius) to get that parabolicity is preserved by rough isometries [10].

3 Set Up

For a Generalized Robertson–Walker (GRW) spacetime we mean a product manifold $I \times F$, of an open interval I of the real line $\mathbb{R}$ and an $n(\geq 2)$-dimensional (connected) Riemannian manifold (F, g_F), endowed with the Lorentzian metric

$$\overline{g} = -\pi_I^*(dt^2) + f(\pi_I)^2\, \pi_F^*(g_F)\,, \tag{1}$$

where π_I and π_F denote the projections onto I and F, respectively, and f is a positive smooth function on I. We will denote this Lorentzian manifold by $(\overline{M}, \overline{g})$. The $(n+1)$-dimensional spacetime $\overline{M}$ is a warped product, with base $(I, -dt^2)$, fiber (F, g_F) and warping function f.

On $\overline{M}$, there exists a distinguished vector field $\xi = f(\pi_I)\, \partial_t$, where ∂_t denotes ∂/∂_t, which is timelike and satisfies

$$\overline{\nabla}_X \xi = f\,'(\pi_I)\, X, \tag{2}$$

for any $X \in \mathfrak{X}(\overline{M})$, where $\overline{\nabla}$ is the Levi-Civita connection of $\overline{g}$, from the relationship between the Levi-Civita connections of M and those of the base and the fiber [14, Cor. 7.35]. Therefore, ξ is conformal with $\mathscr{L}_\xi\, \overline{g} = 2\, f\,'(\pi_I)\, \overline{g}$ and its metrically equivalent 1-form is closed. If the warping function of $\overline{M}$ is constant, i.e., $\overline{M}$ is a Lorentzian product, the GRW spacetime is called static. Contrary, if there is no open subinterval J of I such that $f_{|J}$ is constant, then $\overline{M}$ is said to be proper. Any GRW spacetime has a global time function (in particular, it is time orientable) and then it is stably causal [3, p. 64].

Given an n-dimensional manifold M, an immersion $x : M \to \overline{M}$ is said to be spacelike if the metric g on M, induced from the Lorentzian metric (1), is Riemannian. In this case, M is called a spacelike hypersurface in $\overline{M}$. Let $N \in \mathfrak{X}^\perp(M)$ be the unitary timelike normal vector field in the same time-orientation of the vector field $-\partial_t$, i.e., such that $\overline{g}(N, -\partial_t) < 0$.

From the wrong-way Schwarz inequality (see [14, Prop. 5.30], for instance) we have $\overline{g}(N, \partial_t) \geq 1$, and the equality holds at $p \in M$ if and only if $N = -\partial_t$ at p. In fact, $\overline{g}(N, \partial_t) = \cosh\theta$, where θ is the hyperbolic angle, at any point, between the unit timelike vectors N and $-\partial_t$. We will refer to θ as the hyperbolic angle function on M. If we denote by $\partial_t^T := \partial_t + \overline{g}(N, \partial_t)N$ the tangential component of ∂_t along x, then we have the following formula for the gradient on M of the function $\tau := \pi_I \circ x$,

$$\nabla\tau = -\partial_t^T, \tag{3}$$

and therefore

$$g(\nabla\tau, \nabla\tau) = \sinh^2\theta\,. \tag{4}$$

If a GRW spacetime admits a compact spacelike hypersurface then its fiber is compact [2, Prop. 3.2(i)]. A GRW spacetime whose fiber is compact is called spatially closed. Classically, the family of spatially closed GRW spacetimes has been very useful to get closed cosmological models. Moreover, from a geometric point of view, to deal with compact spacelike hypersurfaces in a spatially closed GRW spacetime is natural, indeed, the a complete spacelike hypersurface in a spatially closed GRW spacetime must be compact if some natural assumptions are satisfied [2, Prop. 3.2(ii)]. From a physical point of view, spatially closed cosmological models have been being criticized, and open cosmological models have been suggested instead [7]. More recently, it has been argued that the existence of a compact spacelike hypersurface in a spacetime makes it unsuitable in a possible quantum theory of gravity [4].

We will consider here an $(n + 1)$-dimensional GRW spacetime such that the universal Riemannian covering of the fiber is parabolic (thus so is the fiber) and call it a spatially parabolic GRW spacetime.[1] This class of spacetimes extends to spatially closed GRW spacetimes from the point of view of geometric analysis of the fiber, and allows to model open relativistic universes.

4 Parabolicity of Spacelike Hypersurfaces

Let $x : M \to \overline{M}$ be a spacelike hypersurface in a GRW spacetime $(\overline{M}, \overline{g})$ and assume the induced metric g on M is complete. Suppose in addition that there exists a positive constant c such that $f(\tau) \leq c$. Then, we have that the projection of M on the fiber F, $\pi := \pi_F \circ x$, is a covering map [2, Lemma 3.1].

Now, from (1) we have for any $v \in T_pM$,

$$\begin{aligned} g(v, v) &= -g(\nabla\tau, v)^2 + f(\tau)^2 g_F(d\pi(v), d\pi(v)) \\ &\leq c^2\, g_F(d\pi(v), d\pi(v)). \end{aligned}$$

Now, the classical Schwarz inequality $g(\nabla\tau, v)^2 \leq g(\nabla\tau, \nabla\tau)\, g(v, v)$, gives,

$$g(v, v) \geq -g(\nabla\tau, \nabla\tau)\, g(v, v) + f(\tau)^2 g_F(d\pi(v), d\pi(v)),$$

which implies

$$g(v, v) \geq \frac{f(\tau)^2}{\cosh^2\theta}\, g_F(d\pi(v), d\pi(v)).$$

Thus, we arrive to

[1]This definition simplifies the one given in [16] where each GRW spacetime considered was explicitly assumed with parabolic universal Riemannian covering of its fiber.

Lemma 1. *Let M be a spacelike hypersurface in a GRW spacetime $\overline{M}$. If*

(i) *The hyperbolic angle is bounded,*
(ii) $\sup f(\tau) < \infty$, *and*
(iii) $\inf f(\tau) > 0$,

then, there exists a constant $c \geq 1$ such that

$$c^{-1}\,|v|_g \;\leq\; |d\pi(v)|_{g_F} \;\leq\; c\,|v|_g\,,$$

for all $v \in T_pM,\; p \in M$.

Proposition 1. *Let $\overline{M}$ be a GRW spacetime whose warping function f satisfies $\sup f < \infty$ and $\inf f > 0$. If $\overline{M}$ admits a simply connected parabolic spacelike hypersurface M and the hyperbolic angle of M is bounded, then $\overline{M}$ is spatially parabolic.*

Proof. Let $\tilde{\pi} : M \to \tilde{F}$ be a lift of the mapping $\pi : M \to F$, where $\tilde{F}$ is the universal Riemannian covering of F. The map $\tilde{\pi}$ is a diffeomorphism [2, Lemma 3.1] and Lemma 1 asserts that it is a quasi-isometry.

Theorem 2. *Let M be a complete spacelike hypersurface in a spatially parabolic GRW spacetime $\overline{M}$. If*

(i) *The hyperbolic angle is bounded*
(ii) $\sup f(\tau) < \infty$, *and*
(iii) $\inf f(\tau) > 0$,

then, M is parabolic.

Proof. Let $\tilde{M}$ be the universal Riemannian covering of M with projection $\pi_M : \tilde{M} \to M$. The map $\pi \circ \pi_M : \tilde{M} \to F$ may be lifted to a diffeomorphism $\tilde{\pi} : \tilde{M} \to \tilde{F}$, where $\tilde{F}$ is the universal Riemannian covering of F, which is, in fact, a quasi-isometry, leading to the parabolicity of $\tilde{M}$ and, hence, M is also parabolic.

Remark 2. The boundedness assumption on the hyperbolic angle has a physical interpretation. In fact, along M there exist two families of instantaneous observers $\mathcal{T}_p$, where $\mathcal{T} := -\partial_t$, $p \in M$, and the normal observers N_p. The quantities $\cosh\theta(p)$ and $v(p) := (1/\cosh\theta(p))\,N_p^F$, where N_p^F is the projection of N_p onto F, are respectively the energy and the velocity that $\mathcal{T}_p$ measures for N_p [19, pp. 45, 67], and on M we have $|v| = \tanh\theta$. Therefore the relative speed function $|v|$ is bounded on M and, hence, it does not approach to speed of light in vacuum.

5 The Restriction of the Warping Function on M

Denote by ∇ the Levi-Civita connection of the induced metric g on M. The Gauss and Weingarten formulas of M in $\overline{M}$ are

$$\overline{\nabla}_X Y = \nabla_X Y - g(AX, Y)N \quad \text{and} \quad AX = -\overline{\nabla}_X N\,, \tag{5}$$

for all $X, Y \in \mathfrak{X}(M)$, where A is the shape operator associated to N. The mean curvature function relative to N is defined by $H := -(1/n)\,\mathrm{trace}(A)$. The mean curvature is zero if and only if the spacelike hypersurface is, locally, a critical point of the n-dimensional area functional for compactly supported normal variations. A spacelike hypersurface with $H = 0$ is called a maximal hypersurface.

In any GRW spacetime the level hypersurfaces of the projection $\pi_I : \overline{M} \to I$ constitute a distinguished family of spacelike hypersurfaces, the so-called spacelike slices. We will represent by $t = t_0$ the spacelike slice $\{t_0\} \times F$. For a spacelike hypersurface $x : M \to \overline{M}$, $x(M)$ is contained in a spacelike slice $t = t_0$ if and only if $\tau = t_0$ on M. When $x(M)$ equals to $t = t_0$, for some $t_0 \in I$, we will say that M is a spacelike slice. The shape operator and the mean curvature of the spacelike slice $t = t_0$ are respectively $A = f'(t_0)/f(t_0)\, I$ and $H = -f'(t_0)/f(t_0)$, where I denotes the identity transformation Thus, a spacelike slice $t = t_0$ is maximal if and only if $f'(t_0) = 0$ (and hence, totally geodesic).

Given a spacelike hypersurface M in $\overline{M}$, from (2) and (5) we get

$$\nabla_Y \xi^T + f(\tau)\, \overline{g}(N, \partial_t)\, AY = f'(\tau)\, Y\,, \tag{6}$$

for any $Y \in \mathfrak{X}(M)$, where $\xi^T = \xi + \overline{g}(\xi, N)N$ is the tangential component of ξ along x, $f(\tau) = f \circ \tau$ and $f'(\tau) = f' \circ \tau$. From (3) and (6) we have

$$\Delta\tau = -\frac{f'(\tau)}{f(\tau)} \left\{n + |\nabla\tau|^2\right\} - nH\, \overline{g}(N, \partial_t)\,, \tag{7}$$

where Δ denotes the Laplacian on M. Therefore

$$\Delta f(\tau) = -n\, \frac{f'(\tau)^2}{f(\tau)} + f(\tau)(\log f)''(\tau)|\nabla\tau|^2 - nH\, f'(\tau)\overline{g}(N, \partial_t)\,. \tag{8}$$

If we assume $(\log f)''(\tau) \leq 0$ and $H\, f'(\tau) \leq 0$, then the positive function $f(\tau)$ on M is superharmonic.

Remark 3. Clearly, the assumption $(\log f)''(\tau) \leq 0$ holds on M if the function $-\log f$ is convex. With respect to this assumption: (a) It was proved that in a GRW spacetime whose warping function f satisfies that $-\log f$ is convex, the only compact CMC spacelike hypersurfaces are the spacelike slices [1]. This result was later extended to a wider class of spacetimes in [5]. On the other hand, the assumption $-\log f$ is convex is related to certain natural one on the Ricci tensor $\overline{\mathrm{Ric}}$ of $\overline{M}$, the so called Null Convergence Condition (NCC): $\overline{\mathrm{Ric}}(w, w) \geq 0$ for any null tangent vector w. (Namely, if $\overline{M}$ obeys the NCC then $-\log f$ is convex). (b) If $-\log f$ is convex, f is not locally constant and it has a critical point, then the assumption $\sup f < \infty$ is automatically satisfied. In fact, if there exists $t_0 \in I$ such that $f'(t_0) = 0$, then t_0 is the unique critical point of f and $\sup f = f(t_0)$. (c) Consider the reference frame $\mathscr{T} := -\partial_t$ (which defines the time orientation we have considered in $\overline{M}$). We have $\mathrm{div}(\mathscr{T}) = -n\frac{f'}{f}$. Thus, $f' < 0$ (resp. $f' > 0$)

may be interpreted saying that the observers in $\mathcal{T}$ are on average spreading apart (resp. coming together). If we assume $-\log f$ convex then $\frac{d}{ds}\big(\operatorname{div}(\mathcal{T}) \circ \gamma\big)(s) \geq 0$, for any observer γ in $\mathcal{T}$. If in addition we assume there is a proper time s_0 of γ such that $\operatorname{div}(\mathcal{T})_{\gamma(s_0)} > 0$, then $\operatorname{div}(\mathcal{T})_{\gamma(s)} > 0$ for any $s > s_0$. Therefore, the assumption $-\log f$ is convex, favors that $\overline{M}$ models an expanding universe.

6 Uniqueness Results in the Parametric Case

Theorem 3. *Let $\overline{M}$ be a proper spatially parabolic GRW spacetime such that $-\log f$ is convex. The only complete spacelike hypersurface M in $\overline{M}$ whose mean curvature function satisfies $H\, f'(\tau) \leq 0$ (in particular, with $H = 0$), such that*

(i) *The hyperbolic angle is bounded*
(ii) $\sup f(\tau) < \infty$, *and*
(iii) $\inf f(\tau) > 0$,

is the spacelike slice $t = t_0$ with $f'(t_0) = 0$.

If the warping function is allowed to be constant on an open subinterval, we have

Theorem 4. *Let $\overline{M}$ be a spatially parabolic GRW spacetime such that $-\log f$ is convex. The only complete maximal hypersurfaces M in $\overline{M}$ such that*

(i) The hyperbolic angle is bounded, and
(ii) which are bounded between two spacelike slices,

are the spacelike slices $t = t_0$ with $f'(t_0) = 0$.

Proof. From the assumption $x(M) \subset [t_0, t_1] \times F$, the function $f(\tau)$ is upper bounded and satisfies $\inf f(\tau) > 0$. As in the previous result, we arrive to $f(\tau)$ constant. Therefore, from (8), we get $f'(\tau) = 0$ and, hence, the function τ is harmonic making use of (7). Since $\tau(M) \subset [t_0, t_1]$, the function τ must be constant.

Remark 4. In order to illustrate the range of application of the two previous results, note that F may be taken as $\mathbb{S}^{n-1} \times \mathbb{R}$, $n \geq 2$, with $g_F = g + ds^2$, being g an arbitrary metric on $\mathbb{S}^{n-1}$. Assume g has non-negative Ricci curvature. Thus, g_F has the same property. When the fiber (F, g_F) has non-negative Ricci curvature, the convexity of $-\log f$ leads that the Ricci tensor of the GRW spacetime satisfies the NCC (and hence, $\overline{M}$, in the case $n = 4$, could be a candidate to represent a solution to the Einstein equation).

The previous result may be specialized to the static case ($f = 1$), i.e., when the GRW spacetime is fact a Lorentzian product. However, we will see that under the assumption that the Ricci tensor of the fiber is positive semi-definite the boundedness assumption of $x(M)$ can be dropped. In order to do that, recall the Bochner–Lichnerowicz formula (see [6, p. 83], for instance)

$$\frac{1}{2}\Delta|\nabla u|^2 = |\operatorname{Hess}(u)|^2 + \operatorname{Ric}(\nabla u, \nabla u) + g(\nabla u, \nabla \Delta u)$$

which holds true for any Riemannian manifold (M, g) and any $u \in C^\infty(M)$. The idea is to apply it to the function $u = \tau$ on a maximal hypersurface M in a static GRW spacetime $\overline{M}$. Using (6), we have $|\text{Hess}(\tau)|^2 = \cosh^2\theta \,\text{trace}(A^2)$. Moreover, from (7), τ is now harmonic. On the other hand, taking into account (4) and $\text{Ric}(\nabla\tau, \nabla\tau) = \cosh^2\theta \,\text{Ric}^F(N^F, N^F) + g(A\nabla\tau, A\nabla\tau)$, which follows from the Gauss equation of M in $\overline{M}$ and [14, Props. 7.42, 7.43], we get

Lemma 2. *For any maximal hypersurface M in a static GRW spacetime $\overline{M}$ whose fiber has non-negative Ricci curvature, we have*

$$\Delta \sinh^2\theta \;\geq\; 2\cosh^2\theta \;\text{trace}(A^2)\,,$$

and, hence, $\sinh^2\theta$ *is subharmonic. Moreover, if it is constant, then M is totally geodesic.*

Theorem 5. *Let M be a complete maximal hypersurface in a spatially parabolic static GRW spacetime $\overline{M}$. If the Ricci curvature of the fiber is non-negative and the hyperbolic angle of M is bounded, then M must be totally geodesic.*

Remark 5. It should be recalled that a complete maximal hypersurface in a locally symmetric Lorentzian manifold $\overline{M}$ whose Ricci tensor satisfies $\overline{\text{Ric}}(w, w) \geq 0$ for any timelike tangent vector w (the Timelike Convergence Condition (TCC)) must be totally geodesic [13]. Note that the spacetime in previous result satisfies the TCC but is not locally symmetric, in general.

7 Calabi–Bernstein Type Problems

Let $(M, g_{_M})$ be a Riemannian manifold and let $f : I \to \mathbb{R}$ be a positive smooth function. For each $u \in C^\infty(M)$ such that $u(M) \subset I$ we can consider its graph $\Sigma_u = \{(u(p), p) \,:\, p \in M\}$ in the GRW spacetime $\overline{M}$ with base $(I, -dt^2)$, fiber $(M, g_{_M})$ and warping function f. The graph inherits a metric from (1), given by

$$g_u = -du^2 + f(u)^2 g_{_M}\,, \tag{9}$$

on M, which is Riemannian (i.e., positive definite) if and only if u satisfies $\mid Du \mid < f(u)$, everywhere on M, where Du denotes the gradient of u in $(M, g_{_M})$ and $\mid Du \mid^2 = g_{_M}(Du, Du)$. Note that $\tau(u(p), p) = u(p)$ for any $p \in M$, and so, τ and u may be naturally identified on Σ_u. When Σ_u is spacelike, the unitary normal vector field on Σ_u satisfying $\overline{g}(N, \partial_t) > 0$ is

$$N = -\frac{1}{f(u)\sqrt{f(u)^2 - \mid Du \mid^2}}\,\left(f(u)^2\partial_t + (0, Du)\right), \tag{10}$$

and the corresponding mean curvature function

$$H(u) = -\operatorname{div}\left(\frac{Du}{nf(u)\sqrt{f(u)^2 - |\, Du\,|^2}}\right) - \frac{f'(u)}{n\sqrt{f(u)^2 - |\, Du\,|^2}}\left(n + \frac{|\, Du\,|^2}{f(u)^2}\right).$$

The differential equation $H(u) = 0$ with the constraint $|\, Du\,| < f(u)$ is called the maximal hypersurface equation in $\overline{M}$, and its solutions give the maximal graphs in $\overline{M}$. This equation is elliptic since the constraint holds. We will apply the previous uniqueness results in the parametric case to determine all the entire solutions of the maximal hypersurface equation

$$\operatorname{div}\left(\frac{Du}{f(u)\sqrt{f(u)^2 - |\, Du\,|^2}}\right) = -\frac{f'(u)}{\sqrt{f(u)^2 - |\, Du\,|^2}}\left(n + \frac{|\, Du\,|^2}{f(u)^2}\right), \tag{E.1}$$

$$|\, Du\,| < \lambda f(u), \quad 0 < \lambda < 1. \tag{E.2}$$

in several cases.

Remark 6. (a) The constraint (E.2) means that the differential equation (E) is in fact uniformly elliptic. (b) Note that (E.2) may be written as $\cosh\theta < 1/\sqrt{1-\lambda^2}$, where θ is the hyperbolic angle of Σ_u. Conversely, if $\cosh\theta < \mu$, with $\mu > 1$, then $|\, Du\,| < \lambda f(u)$, where $\lambda = \sqrt{1-(1/\mu^2)}$. Therefore, (E.2) means that Σ_u has bounded hyperbolic angle. (c) If in addition to (E.2) we have $\inf f(u) > 0$ then $L_u(\gamma) \geq \sqrt{1-\lambda^2}\, \inf f(u)\, L(\gamma)$, where $L(\gamma)$ and $L_u(\gamma)$ are the lengths of a smooth curve γ on M with respect to the metrics g_M and g_u, respectively. Therefore, if a divergent curve in M has infinite g_M-length then it has also infinite g_u-length. Hence, if (M, g_M) is complete, then (M, g_u) is so.

As an application of Theorems 3 and 4, we have

Theorem 6. *Let $f : I \to \mathbb{R}$ be a non-locally constant positive smooth function (resp. a positive smooth function). Assume f satisfies* $(\log f)'' \leq 0$, $\sup f < \infty$ *and* $\inf f > 0$ (*resp. f satisfies* $(\log f)'' \leq 0$). *The only entire solutions* (*resp. The only bounded entire solutions*) *of the equation* (E) *on a parabolic Riemannian manifold M are the constant functions $u = c$, with $f'(c) = 0$.*

Finally, as a consequence of Theorem 5, we obtain

Theorem 7. *The only entire solutions of the equation*

$$\operatorname{div}\left(\frac{Du}{\sqrt{1 - |\, Du\,|^2}}\right) = 0 \tag{E$'$.1}$$

$$|\, Du\,| < \lambda, \quad 0 < \lambda < 1, \tag{E$'$.2}$$

on $\mathbb{S}^{2m} \times \mathbb{R}$*, endowed with a product Riemannian metric* $g + ds^2$*, where g is a Riemannian metric on* $\mathbb{S}^{2m}$ *with non-negative Ricci curvature, are the functions* $u(x, s) = as + b$*, with* $a, b \in \mathbb{R}$*,* $|a| < \lambda$*.*

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Totally Geodesic Submanifolds of Riemannian Symmetric Spaces

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Abstract The index of a Riemannian manifold is defined as the minimal codimension of a totally geodesic submanifold. In this note we discuss two recent results by the author and Olmos (Berndt and Olmos, On the index of symmetric spaces, preprint, arXiv:1401.3585) and some related topics. The first result states that the index of an irreducible Riemannian symmetric space is bounded from below by the rank of the symmetric space. The second result is the classification of all irreducible Riemannian symmetric spaces of noncompact type whose index is less or equal than three.

1 Introduction

Let M be a connected Riemannian manifold and denote by $\mathscr{S}$ the set of all connected totally geodesic submanifolds Σ of M with $\dim(\Sigma) < \dim(M)$. The index $i(M)$ of M is defined by

$$i(M) = \min\{\dim(M) - \dim(\Sigma) \mid \Sigma \in \mathscr{S}\} = \min\{\mathrm{codim}(\Sigma) \mid \Sigma \in \mathscr{S}\}.$$

This notion was introduced by Onishchik in [11] who also classified the irreducible simply connected Riemannian symmetric spaces M with $i(M) \leq 2$. The author and Olmos developed in [2] a new approach to the index of symmetric spaces. The first main result in [2] is:

Theorem 1. *Let M be an irreducible Riemannian symmetric space. Then*

$$\mathrm{rk}(M) \leq i(M).$$

Thus the index is bounded from below by the rank of the symmetric space. The second main result in [2] is the classification of all irreducible Riemannian symmetric spaces M of noncompact type with $i(M) \leq 3$.

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Theorem 2. *Let M be an irreducible Riemannian symmetric space of noncompact type. Then the following statements hold:*

1. $i(M) = 1$ *if and only if M is isometric to*
   a. *the real hyperbolic space* $\mathbb{R}H^k = SO^o_{1,k}/SO_k$, $k \geq 2$.
2. $i(M) = 2$ *if and only if M is isometric to one of the following spaces:*
   a. *the complex hyperbolic space* $\mathbb{C}H^k = SU_{1,k}/S(U_1U_k)$, $k \geq 2$;
   b. *the Grassmannian* $G^*_2(\mathbb{R}^{k+2}) = SO^o_{2,k}/SO_2SO_k$, $k \geq 3$;
   c. *the symmetric space* $SL_3(\mathbb{R})/SO_3$.
3. $i(M) = 3$ *if and only if M is isometric to one of the following spaces:*
   a. *the Grassmannian* $G^*_3(\mathbb{R}^{k+3}) = SO^o_{3,k}/SO_3SO_k$, $k \geq 3$;
   b. *the symmetric space* G^2_2/SO_4;
   c. *the symmetric space* $SL_3(\mathbb{C})/SU_3$.

The proofs for both theorems can be found in [2]. In this note we present some basic theory and relevant results for this context.

2 Totally Geodesic Submanifolds

A submanifold Σ of a Riemannian manifold M is said to be *totally geodesic* if every geodesic in Σ is also a geodesic in M. The existence and classification of totally geodesic submanifolds are fundamental problems in submanifold geometry. The existence problem is closely related to curvature, as the following result by Élie Cartan [3] shows: Let M be a Riemannian manifold, $p \in M$ and V be a linear subspace of T_pM. Then there exists a totally geodesic submanifold Σ of M with $p \in \Sigma$ and $T_p\Sigma = V$ if and only if there exists a real number $\epsilon \in \mathbb{R}_+$ such that for every geodesic γ in M with $\gamma(0) = p$ and $\dot{\gamma}(0) \in \{v \in V \mid \|v\| < \epsilon\}$ the Riemannian curvature tensor of M at $\gamma(1)$ preserves the parallel translate of V along γ from p to $\gamma(1)$. A detailed proof can be found in [1].

A connected totally geodesic submanifold Σ of a connected Riemannian manifold M is said to be *maximal* if there is no connected totally geodesic submanifold Σ' of M with $\Sigma \subsetneq \Sigma' \subsetneq M$.

A connected submanifold Σ of a connected Riemannian manifold M is said to be *reflective* if the geodesic symmetry of M in Σ is a well-defined global isometry of M. Since every connected component of the fixed point set of an isometry is a totally geodesic submanifold, it follows that every reflective submanifold is totally geodesic. The reflective submanifolds in irreducible simply connected Riemannian symmetric spaces of compact type were classified by Leung in [9, 10].

3 Riemannian Symmetric Spaces

We refer to [4] for details about symmetric spaces and their classification. Let M be a connected Riemannian manifold. We denote by $I(M)$ the isometry group of M and by $I^o(M)$ the connected component of $I(M)$ containing the identity transformation of M. Let ∇ be the Levi Civita connection of M and R be the Riemannian curvature tensor of M. If $\nabla R = 0$, then M is said to be a *Riemannian locally symmetric space*. The terminology is motivated by the following geometric characterization: The equality $\nabla R = 0$ holds if and only if for every point $p \in M$ there exists an open neighborhood U of p in M and an isometric involution $\sigma_p : U \to U$ such that p is an isolated fixed point of σ_p. If for every point $p \in M$ there exists an involution $\sigma_p \in I(M)$ such that p is an isolated fixed point of σ_p, then M is said to be a *Riemannian symmetric space*. The Riemannian universal covering space of a Riemannian locally symmetric space is a Riemannian symmetric space. Thus every Riemannian locally symmetric space is locally isometric to a Riemannian symmetric space. Therefore the existence and classification of totally geodesic submanifolds in Riemannian locally symmetric spaces can be reduced, via covering maps, to that of simply connected Riemannian symmetric spaces.

Let M be a simply connected Riemannian symmetric space. Then its de Rham decomposition is of the form $M = M_0 \times M_1 \times \ldots \times M_d$, where M_0 is isometric to the Euclidean space $\mathbb{R}^k$ for some $k \geq 0$ and $M_1, \ldots, M_d$ are irreducible simply connected Riemannian symmetric spaces. We allow $d = 0$ here, which means that M is isometric to $\mathbb{R}^k$, and M is said to be *of Euclidean type*. The sectional curvature of an irreducible Riemannian symmetric space of dimension ≥ 2 is either nonnegative or nonpositive. If it is nonnegative then M is said to be *of compact type*, and if it is nonpositive then M is said to be *of noncompact type*.

The Riemannian symmetric spaces were classified by Élie Cartan who established a beautiful correspondence with semisimple Lie algebras. Let M be a Riemannian symmetric space without Euclidean factor, that is, $k = 0$. Then the Lie algebra $\mathfrak{g}$ of $G = I^o(M)$ is a semisimple Lie algebra. Let $p \in M$ and $K = G_p$ be the isotropy group of G at p. Then G acts transitively on M and M is isometric to the homogeneous space G/K equipped with a suitable G-invariant Riemannian metric. Then M is irreducible if and only if the isotropy representation $\chi : K \to T_pM,\ k \mapsto d_pk$ is irreducible. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form B of $\mathfrak{g}$. The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is known as a *Cartan decomposition* of $\mathfrak{g}$. There is a natural isomorphism between $\mathfrak{p}$ and T_pM. The Cartan decomposition is reductive and therefore $d_pk = Ad(k) : \mathfrak{p} \to \mathfrak{p}$, that is, the isotropy representation coincides with the adjoint representation of K on $\mathfrak{p}$.

Let $M = G/K$ be a Riemannian symmetric space of noncompact type. Consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and the canonical embedding of $\mathfrak{g}$ into its complexification $\mathfrak{g}^{\mathbb{C}}$. Then $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p}$ is a subalgebra of $\mathfrak{g}^{\mathbb{C}}$ and the corresponding homogeneous space $M^* = G^*/K$ with the induced G^*-invariant Riemannian metric is a Riemannian symmetric space of compact type. This process is known as *duality* between Riemannian symmetric spaces of compact type and of noncompact type.

Table 1 Irreducible Riemannian symmetric spaces M of noncompact type

(RS)	M	$\dim(M)$	$\mathrm{rk}(M)$	Comments
(A_r)	$SO^o_{1,k}/SO_k$	k	1	$k \geq 2$
	$SL_{r+1}(\mathbb{R})/SO_{r+1}$	$\frac{1}{2}r(r+3)$	r	$r \geq 2$
	$SL_{r+1}(\mathbb{C})/SU_{r+1}$	$r(r+2)$	r	$r \geq 2$
	SU^*_{2r+2}/Sp_{r+1}	$r(2r+3)$	r	$r \geq 2$
	E_6^{-26}/F_4	26	2	
(B_r)	$SO^o_{r,r+k}/SO_r SO_{r+k}$	$r(r+k)$	r	$r \geq 2, k \geq 1$
	$SO_{2r+1}(\mathbb{C})/SO_{2r+1}$	$r(2r+1)$	r	$r \geq 2$
(C_r)	$Sp_r(\mathbb{R})/U_r$	$r(r+1)$	r	$r \geq 3$
	$Sp_r(\mathbb{C})/Sp_r$	$r(2r+1)$	r	$r \geq 3$
	$SU_{r,r}/S(U_r U_r)$	$2r^2$	r	$r \geq 3$
	$Sp_{r,r}/Sp_r Sp_r$	$4r^2$	r	$r \geq 2$
	SO^*_{4r}/U_{2r}	$2r(2r-1)$	r	$r \geq 3$
	$E_7^{-25}/E_6 U_1$	54	3	
(D_r)	$SO^o_{r,r}/SO_r SO_r$	r^2	r	$r \geq 4$
	$SO_{2r}(\mathbb{C})/SO_{2r}$	$r(2r-1)$	r	$r \geq 4$
(BC_r)	$SU_{r,r+k}/S(U_r U_{r+k})$	$2r(r+k)$	r	$r \geq 1, k \geq 1$
	$Sp_{r,r+k}/Sp_r Sp_{r+k}$	$4r(r+k)$	r	$r \geq 1, k \geq 1$
	SO^*_{4r+2}/U_{2r+1}	$2r(2r+1)$	r	$r \geq 2$
	$F_4^{-20}/Spin_9$	16	1	
	$E_6^{-14}/Spin_{10} U_1$	32	2	
(E_6)	E_6^6/Sp_4	42	6	
	$E_6(\mathbb{C})/E_6$	78	6	
(E_7)	E_7^7/SU_8	70	7	
	$E_7(\mathbb{C})/E_7$	133	7	
(E_8)	E_8^8/SO_{16}	128	8	
	$E_8(\mathbb{C})/E_8$	248	8	
(F_4)	$F_4^4/Sp_3 Sp_1$	28	4	
	$F_4(\mathbb{C})/F_4$	52	4	
	$E_6^2/SU_6 Sp_1$	40	4	
	$E_7^{-5}/SO_{12} Sp_1$	64	4	
	$E_8^{-24}/E_7 Sp_1$	112	4	
(G_2)	G_2^2/SO_4	8	2	
	$G_2(\mathbb{C})/G_2$	14	2	

The rank $\mathrm{rk}(M)$ of a Riemannian symmetric space M is defined as the maximal possible dimension of a flat Riemannian manifold which can be embedded in M as a totally geodesic submanifold. The rank of M is equal to the maximal dimension of an abelian subspace of $\mathfrak{p}$. In Table 1 we list the irreducible Riemannian symmetric spaces of noncompact type together with the type of root system (RS), dimension and rank.

Table 2 Isometric Riemannian symmetric spaces of noncompact type in low dimensions

dim	rk	M_1	M_2	M_3	M_4
2	1	$SO^o_{1,2}/SO_2$	$SU_{1,1}/S(U_1U_1)$	$SL_2(\mathbb{R})/SO_2$	$Sp_1(\mathbb{R})/U_1$
3	1	$SO^o_{1,3}/SO_3$	$SO_3(\mathbb{C})/SO_3$	$SL_2(\mathbb{C})/SU_2$	$Sp_1(\mathbb{C})/Sp_1$
4	1	$SO^o_{1,4}/SO_4$	$Sp_{1,1}/Sp_1Sp_1$		
5	1	$SO^o_{1,5}/SO_5$	SU^*_4/Sp_2		
6	1	$SU_{1,3}/S(U_1U_3)$	SO^*_6/U_3		
6	2	$SO^o_{2,3}/SO_2SO_3$	$Sp_2(\mathbb{R})/U_2$		
8	2	$SO^o_{2,4}/SO_2SO_4$	$SU_{2,2}/S(U_2U_2)$		
9	3	$SO^o_{3,3}/SO_3SO_3$	$SL_4(\mathbb{R})/SO_4$		
10	2	$SO_5(\mathbb{C})/SO_5$	$Sp_2(\mathbb{C})/Sp_2$		
12	2	$SO^o_{2,6}/SO_2SO_6$	SO^*_8/U_4		
15	3	$SO_6(\mathbb{C})/SO_6$	$SL_4(\mathbb{C})/SU_4$		

In Table 2 we list irreducible Riemannian symmetric spaces M_i of noncompact type which are isometric to each other.

4 Lie Triple Systems

Let $M = G/K$ be a semisimple Riemannian symmetric space and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition. A linear subspace $\mathfrak{m}$ of $\mathfrak{p}$ is said to be a *Lie triple system* if $[[\mathfrak{m}, \mathfrak{m}], \mathfrak{m}] \subset \mathfrak{m}$. If Σ is a connected totally geodesic submanifold of M with $p \in M$, then $\mathfrak{m} = T_p\Sigma \subset \mathfrak{p}$ is a Lie triple system. Conversely, if $\mathfrak{m}$ is a Lie triple system in $\mathfrak{p}$, then $\mathfrak{h} = [\mathfrak{m}, \mathfrak{m}] \oplus \mathfrak{m} \subset \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{g}$ is a subalgebra of $\mathfrak{g}$ and the orbit $\Sigma = H \cdot p$ of the corresponding connected subgroup H of G through p is a connected complete totally geodesic submanifold of M. This establishes a one-to-one correspondence between connected complete totally geodesic submanifolds of M and Lie triple systems in $\mathfrak{p}$.

The geometric problem of classifying totally geodesic submanifolds in a Riemannian symmetric space $M = G/K$ is therefore equivalent to the algebraic problem of classifying Lie triple systems in $\mathfrak{p}$. However, this algebraic reformulation of the classification problem is still very difficult to handle in practice and, up to now, could be solved only in special circumstances, for example when $\mathrm{rk}(M) \leq 2$.

Let $M = G/K$ be a Riemannian symmetric space of noncompact type and Σ be a totally geodesic submanifold of M with $p \in M$. Then $\mathfrak{m} = T_p\Sigma$ is a Lie triple system in $\mathfrak{p}$. Now consider the dual Riemannian symmetric space $M^* = G^*/K$ of compact type. The corresponding Cartan decomposition is $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p}$. The subspace $i\mathfrak{m}$ is a Lie triple system in $i\mathfrak{p}$ and induces a totally geodesic submanifold Σ^* in M^*. In a similar way every totally geodesic submanifold in a Riemannian symmetric space of compact type induces a totally geodesic submanifold in a Riemannian symmetric space of noncompact type. Thus duality between Riemannian symmetric spaces gives a one-to-one correspondence between connected complete totally geodesic submanifolds of dual Riemannian symmetric spaces. It is clear that duality also preserves maximality of totally geodesic submanifolds. For this reason we restrict from now on to Riemannian symmetric spaces of noncompact type.

A Lie triple system $\mathfrak{m}$ is said to be *reflective* if the orthogonal complement $\mathfrak{m}^\perp$ of $\mathfrak{m}$ in $\mathfrak{p}$ is also a Lie triple system. Let $\mathfrak{m}$ be a Lie triple system in $\mathfrak{p}$ and let Σ be the corresponding connected complete totally geodesic submanifold of M. Then $\mathfrak{m}$ is a reflective Lie triple system in $\mathfrak{p}$ if and only if Σ is a reflective submanifold of M.

5 The Index of Riemannian Symmetric Spaces

The Riemannian symmetric spaces of noncompact type whose rank is equal to one are the hyperbolic spaces over the normed real division algebras: $\mathbb{R}H^k = SO^o_{1,k}/SO_k$, $\mathbb{C}H^k = SU_{1,k}/S(U_1U_k)$, $\mathbb{H}H^k = Sp_{1,k}/Sp_1Sp_k$ and $\mathbb{O}H^2 = F_4^{-20}/Spin_9$, where $k \geq 2$. The totally geodesic submanifolds in these symmetric spaces were classified by Wolf [12]. From his classification we get Table 3.

We observe from Table 3 that maximal totally geodesic submanifolds of maximal dimension are not unique (up to conjugacy) in general.

Using the Lie triple system approach, Klein classified in a series of papers [5–8] the totally geodesic submanifolds of irreducible Riemannian symmetric spaces M of noncompact type with $\mathrm{rk}(M) = 2$. From Klein's classifications we can easily deduce the classification of maximal totally geodesic submanifolds of maximal dimension in irreducible Riemannian symmetric spaces of rank 2. We summarize this classification in Table 4.

Onishchik calculated in [11] many of the indices in Tables 3 and 4.

We observe from Table 4 that there are maximal totally geodesic submanifolds Σ of maximal dimension in M with $\mathrm{rk}(\Sigma) < \mathrm{rk}(M)$, e.g.

Table 3 Maximal totally geodesic submanifolds Σ of maximal dimension in Riemannian symmetric spaces M of noncompact type with $\mathrm{rk}(M) = 1$

M	$\dim(M)$	Σ	$\dim(\Sigma)$	$i(M)$	Comments
$\mathbb{R}H^k$	k	$\mathbb{R}H^{k-1}$	$k-1$	1	$k \geq 2$
$\mathbb{C}H^2$	4	$\mathbb{C}H^1, \mathbb{R}H^2$	2	2	
$\mathbb{C}H^k$	$2k$	$\mathbb{C}H^{k-1}$	$2k-2$	2	$k \geq 3$
$\mathbb{H}H^2$	8	$\mathbb{H}H^1, \mathbb{C}H^2$	4	4	
$\mathbb{H}H^k$	$4k$	$\mathbb{H}H^{k-1}$	$4k-4$	4	$k \geq 3$
$\mathbb{O}H^2$	16	$\mathbb{O}H^1, \mathbb{H}H^2$	8	8	

Table 4 Maximal totally geodesic submanifolds Σ of maximal dimension in irreducible Riemannian symmetric spaces M of noncompact type with $\mathrm{rk}(M) = 2$

M	$\dim(M)$	Σ	$\dim(\Sigma)$	$i(M)$	Comments
$SO^o_{2,k}/SO_2SO_k$	2k	$SO^o_{2,k-1}/SO_2SO_{k-1}$	$2k-2$	2	$k \geq 3$
$SU_{2,k}/S(U_2U_k)$	4k	$SU_{2,k-1}/S(U_2U_{k-1})$	$4k-4$	4	$k \geq 3$
$Sp_{2,k}/Sp_2Sp_k$	8k	$Sp_{2,k-1}/Sp_2Sp_{k-1}$	$8k-8$	8	$k \geq 3$
$SL_3(\mathbb{R})/SO_3$	5	$\mathbb{R} \times SL_2(\mathbb{R})/SO_2$	3	2	
G^2_2/SO_4	8	$SL_3(\mathbb{R})/SO_3$	5	3	
$SL_3(\mathbb{C})/SU_3$	8	$SL_3(\mathbb{R})/SO_3$	5	3	
$SO_5(\mathbb{C})/SO_5$	10	$SO^o_{2,3}/SO_2SO_3$, $SO_4(\mathbb{C})/SO_4$	6	4	
SU^*_6/Sp_3	14	$Sp_{1,2}/Sp_1Sp_2$, $SL_3(\mathbb{C})/SU_3$	8	6	
$G_2(\mathbb{C})/G_2$	14	G^2_2/SO_4, $SL_3(\mathbb{C})/SU_3$	08	6	
$Sp_{2,2}/Sp_2Sp_2$	16	$Sp_2(\mathbb{C})/Sp_2$	10	6	
SO^*_{10}/U_5	20	$SO^o_{2,6}/SO_2SO_6$, $SU_{2,3}/S(U_2U_3)$	12	8	
E^{-26}_6/F_4	26	$F^{-20}_4/Spin_9$	16	10	
$E^{-14}_6/Spin_{10}U_1$	32	SO^*_{10}/U_5	20	12	

$$\Sigma = \mathbb{O}H^2 = F^{-20}_4/Spin_9 \subset E^{-26}_6/F_4 = M.$$

As this example shows, the index $i(M)$ cannot always be realized by a totally geodesic submanifold Σ with $\mathrm{rk}(\Sigma) = \mathrm{rk}(M)$.

Let M be a connected Riemannian manifold and denote by $\hat{\mathscr{S}}$ the set of all connected reflective submanifolds Σ of M with $\dim(\Sigma) < \dim(M)$. We define the positive integer $\hat{i}(M)$ of M by

$$\hat{i}(M) = \min\{\dim(M) - \dim(\Sigma) \mid \Sigma \in \hat{\mathscr{S}}\} = \min\{\mathrm{codim}(\Sigma) \mid \Sigma \in \hat{\mathscr{S}}\}.$$

Using the classification by Leung [9, 10] of reflective submanifolds in irreducible Riemannian symmetric spaces of compact type, and duality, we can calculate $\hat{i}(M)$ for each irreducible Riemannian symmetric space M of noncompact type. In Table 5 we list $\hat{i}(M)$ for each irreducible Riemannian symmetric space of noncompact type and the reflective submanifolds Σ for which the codimension is equal to $\hat{i}(M)$.

By comparing Table 3 and Table 5 we see that $i(M) = \hat{i}(M)$ if $\mathrm{rk}(M) = 1$. However, from Table 4 and Table 5 we see that this equality does not always hold if $\mathrm{rk}(M) = 2$. For example, we have

$$i(G^2_2/SO_4) = 3 < 4 = \hat{i}(G^2_2/SO_4).$$

This shows that for higher rank the index $i(M)$ of a symmetric space M can be strictly less than $\hat{i}(M)$. In other words, the index $i(M)$ cannot always be realized by a reflective submanifold. Equivalently, this means that there are maximal totally geodesic submanifolds of maximal dimension which are not reflective.

Table 5 Maximal reflective submanifolds Σ of maximal dimension in irreducible Riemannian symmetric spaces M of noncompact type

M	Σ	$\hat{i}(M)$	Comments
$SO^o_{r,r+k}/SO_rSO_{r+k}$	$SO^o_{r,r+k-1}/SO_rSO_{r+k-1}$	r	$r,k \geq 1$
$SO^o_{r,r}/SO_rSO_r$	$SO^o_{r-1,r}/SO_{r-1}SO_r$	r	$r \geq 4$
$SL_{r+1}(\mathbb{R})/SO_{r+1}$	$\mathbb{R} \times SL_r(\mathbb{R})/SO_r$	r	$r \geq 2$
	$SO^o_{2,3}/SO_2SO_3$		$r = 3$
$Sp_r(\mathbb{R})/U_r$	$Sp_1(\mathbb{R})/U_1 \times Sp_{r-1}(\mathbb{R})/U_{r-1}$	$2r-2$	$r \geq 3$
$SO_{2r}(\mathbb{C})/SO_{2r}$	$SO_{2r-1}(\mathbb{C})/SO_{2r-1}$	$2r-1$	$r \geq 4$
$SL_{r+1}(\mathbb{C})/SU_{r+1}$	$\mathbb{R} \times SL_r(\mathbb{C})/SU_r$	$2r$	$r \geq 4$
$SU_{r,r+k}/S(U_rU_{r+k})$	$SU_{r,r+k-1}/S(U_rU_{r+k-1})$	$2r$	$r,k \geq 1$
	$SO^o_{1,2}/SO_2$		$r = k = 1$
$SU_{r,r}/S(U_rU_r)$	$SU_{r-1,r}/S(U_{r-1}U_r)$	$2r$	$r \geq 4$
$SO_{2r+1}(\mathbb{C})/SO_{2r+1}$	$SO_{2r}(\mathbb{C})/SO_{2r}$	$2r$	$r \geq 2$
	$SO^o_{2,3}/SO_2SO_3$		$r = 2$
$Sp_r(\mathbb{C})/Sp_r$	$Sp_{r-1}(\mathbb{C})/Sp_{r-1}$	$4r-4$	$r \geq 3$
SO^*_{4r}/U_{2r}	SO^*_{4r-2}/U_{2r-1}	$4r-2$	$r \geq 3$
SU^*_{2r+2}/Sp_{r+1}	$\mathbb{R} \times SU^*_{2r}/Sp_r$	$4r$	$r \geq 3$
	$SL_4(\mathbb{C})/SU_4$		$r = 3$
$Sp_{r,r+k}/Sp_rSp_{r+k}$	$Sp_{r,r+k-1}/Sp_rSp_{r+k-1}$	$4r$	$r,k \geq 1$
	$SU_{1,2}/S(U_1U_2)$		$r = k = 1$
$Sp_{r,r}/Sp_rSp_r$	$Sp_{r-1,r}/Sp_{r-1}Sp_r$	$4r$	$r \geq 3$
SO^*_{4r+2}/U_{2r+1}	SO^*_{4r}/U_{2r}	$4r$	$r \geq 2$
	$SU_{2,3}/S(U_2U_3)$		$r = 2$
$SL_3(\mathbb{C})/SU_3$	$SL_3(\mathbb{R})/SO_3$	3	
G^2_2/SO_4	$SO^o_{1,2}/SO_2 \times SO^o_{1,2}/SO_2$	4	
$SL_4(\mathbb{C})/SU_4$	$Sp_2(\mathbb{C})/Sp_2$	5	
SU^*_6/Sp_3	$Sp_{1,2}/Sp_1Sp_2$, $SL_3(\mathbb{C})/SU_3$	6	
$SU_{3,3}/S(U_3U_3)$	$SU_{2,3}/S(U_2U_3)$, $Sp_3(\mathbb{R})/U_3$	6	
$Sp_{2,2}/Sp_2Sp_2$	$Sp_2(\mathbb{C})/Sp_2$	6	
$G_2(\mathbb{C})/G_2$	G^2_2/SO_4	6	
$F^{-20}_4/Spin_9$	$SO^o_{1,8}/SO_8$, $Sp_{1,2}/Sp_1Sp_2$	8	
F^4_4/Sp_3Sp_1	$SO^o_{4,5}/SO_4SO_5$	8	
E^{-26}_6/F_4	$F^{-20}_4/Spin_9$	10	
$E^{-14}_6/Spin_{10}U_1$	SO^*_{10}/U_5	12	
E^2_6/SU_6Sp_1	F^4_4/Sp_3Sp_1	12	
E^6_6/Sp_4	F^4_4/Sp_3Sp_1	14	
$F_4(\mathbb{C})/F_4$	$SO_9(\mathbb{C})/SO_9$	16	
E^{-25}_7/E_6U_1	$E^{-14}_6/Spin_{10}U_1$	22	
$E^{-5}_7/SO_{12}Sp_1$	E^2_6/SU_6Sp_1	24	
$E_6(\mathbb{C})/E_6$	$F_4(\mathbb{C})/F_4$	26	
E^7_7/SU_8	$\mathbb{R} \times E^6_6/Sp_4$	27	
E^{-24}_8/E_7Sp_1	$E^{-5}_7/SO_{12}Sp_1$	48	
$E_7(\mathbb{C})/E_7$	$\mathbb{R} \times E_6(\mathbb{C})/E_6$	54	
E^8_8/SO_{16}	$E^7_7/SU_8 \times Sp_1(\mathbb{R})/U_1$	56	
$E_8(\mathbb{C})/E_8$	$E_7(\mathbb{C})/E_7 \times Sp_1(\mathbb{C})/Sp_1$	112	

We always have the inequalities $\mathrm{rk}(M) \leq i(M) \leq \hat{i}(M)$. Using Table 5 we can identify two series of symmetric spaces for which $\mathrm{rk}(M) = i(M)$, namely

$$i(SO^o_{r,r+k}/SO_r SO_{r+k}) = r = i(SL_{r+1}(\mathbb{R})/SO_{r+1}) \ , \ r \geq 1, k \geq 0.$$

We finish the paper with two questions:

1. What are the irreducible Riemannian symmetric spaces M of noncompact type for which $i(M) = \mathrm{rk}(M)$? As mentioned above, the Riemannian symmetric spaces $SO^o_{r,r+k}/SO_r SO_{r+k}$ ($r \geq 1$, $k \geq 0$) and $SL_{r+1}(\mathbb{R})/SO_{r+1}$ ($r \geq 1$) satisfy this equality.[1]
2. What are the irreducible Riemannian symmetric spaces M of noncompact type for which $i(M) < \hat{i}(M)$? As mentioned above, the Riemannian symmetric space G^2_2/SO_4 satisfies this inequality.

Of course, the general open problem is to determine $i(M)$ for each irreducible Riemannian symmetric spaces M of noncompact type.

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[1]NOTE ADDED IN PROOF: Question 1 has now been answered by the author and Olmos in "Maximal totally geodesic submanifolds and index of symmetric spaces", preprint arXiv:1405.0598. In the same paper the authors calculated the index of some further symmetric spaces and classified all irreducible Riemannian symmetric spaces M of noncompact type with $i(M) \leq 6$.

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Canonical Connection on Contact Manifolds

Yong-Geun Oh and Rui Wang

Abstract We introduce a family of canonical affine connections on the contact manifold (Q, ξ), which is associated to each contact triad (Q, λ, J) where λ is a contact form and $J : \xi \to \xi$ is an endomorphism with $J^2 = -id$ compatible to $d\lambda$. We call a particular one in this family the *contact triad connection* of (Q, λ, J) and prove its existence and uniqueness. The connection is canonical in that the pull-back connection $\phi^*\nabla$ of a triad connection ∇ becomes the triad connection of the pull-back triad $(Q, \phi^*\lambda, \phi^*J)$ for any diffeomorphism $\phi : Q \to Q$. It also preserves both the triad metric $g := d\lambda(\cdot, J\cdot) + \lambda \otimes \lambda$ and J regarded as an endomorphism on $TQ = \mathbb{R}\{X_\lambda\} \oplus \xi$, and is characterized by its torsion properties and the requirement that the contact form λ be holomorphic in the CR-sense. In particular, the connection restricts to a Hermitian connection ∇^π on the Hermitian vector bundle (ξ, J, g_ξ) with $g_\xi = d\lambda(\cdot, J\cdot)|_\xi$, which we call the *contact Hermitian connection* of (ξ, J, g_ξ). These connections greatly simplify tensorial calculations in the sequels (Oh and Wang, The Analysis of Contact Cauchy-Riemann maps I: a priori Ck estimates and asymptotic convergence, preprint. arXiv:1212.5186, 2012; Oh and Wang, Analysis of contact instantons II: exponential convergence for the Morse-Bott case, preprint. arXiv:1311.6196, 2013) performed in the authors' analytic study of the map w, called contact instantons, which satisfy the nonlinear elliptic system of equations

$$\overline{\partial}^\pi w = 0, \ d(w^*\lambda \circ j) = 0$$

of the contact triad (Q, λ, J).

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1 Introduction

Let (Q, ξ) be a $2n+1$ dimensional contact manifold and a contact form λ be given, which means that the contact distribution ξ is given as $\ker\lambda$ and $\lambda \wedge (d\lambda)^n$ nowhere vanishes. On Q, the Reeb vector field X_λ associated to the contact form λ is the unique vector field satisfying $X_\lambda \rfloor \lambda = 1$ and $X_\lambda \rfloor d\lambda = 0$. Therefore the tangent bundle TQ has the splitting $TQ = \mathbb{R}\{X_\lambda\} \oplus \xi$. We denote by $\pi : TQ \to \xi$ the corresponding projection.

Now let J be a complex structure on ξ, i.e., $J : \xi \to \xi$ with $J^2 = -id|_\xi$. We extend J to TQ by defining $J(X_\lambda) = 0$. We will use such $J : TQ \to TQ$ throughout the paper. Then we have $J^2 = -\Pi$ where $\Pi : TQ \to TQ$ is the unique idempotent with $\operatorname{Im}\Pi = \xi$ and $\ker\Pi = \mathbb{R}\cdot X_\lambda$. We note that we have the unique decomposition $h = \lambda(h)X_\lambda + \pi h$ for any $h \in TQ$ in terms of the decomposition $TQ = \mathbb{R}\cdot X_\lambda \oplus \xi$.

Definition 1 (Contact Triad Metric). Let (Q, λ, J) be a contact triad. We call the metric defined by $g(h,k) := \lambda(h)\lambda(k) + d\lambda(\pi h, J\pi k)$ for any $h,\ k \in TQ$ the *contact triad metric* associated to the triad (Q, λ, J).

The main purpose of the present paper is to introduce the notion of the *contact triad connection* of the triad (Q, λ, J) which is the contact analog to the Ehresman-Libermann's notion of *canonical connection* on the almost Kähler manifold (M, ω, J). (See [2, 3, 7, 9, 10] for general exposition on the canonical connection.)

Theorem 1 (Contact Triad Connection). *Let (Q, λ, J) be any contact triad of contact manifold (Q, ξ), and g the contact triad connection.Then there exists a unique affine connection ∇ that has the following properties:*

1. *∇ is a Riemannian connection of the triad metric.*
2. *The torsion tensor of ∇ satisfies $T(X_\lambda, Y) = 0$ for all $Y \in TQ$.*
3. *$\nabla_{X_\lambda} X_\lambda = 0$ and $\nabla_Y X_\lambda \in \xi$, for $Y \in \xi$.*
4. *$\nabla^\pi := \pi\nabla|_\xi$ defines a Hermitian connection of the vector bundle $\xi \to Q$ with Hermitian structure $(d\lambda, J)$.*
5. *The ξ projection, denoted by $T^\pi := \pi T$, of the torsion T has vanishing $(1,1)$-component in its complexification, i.e., satisfies the following properties: for all Y tangent to ξ, $T^\pi(JY, Y) = 0$.*
6. *For $Y \in \xi$, $\nabla_{JY} X_\lambda + J\nabla_Y X_\lambda = 0$.*

We call ∇ the contact triad connection.

Recall that the leaf space of Reeb foliations of the contact triad (Q, λ, J) canonically carries a (non-Hausdorff) almost Kähler structure which we denote by $(\hat{Q}, \widehat{d\lambda}, \hat{J})$. We would like to note that Axioms (4) and (5) are nothing but properties of the canonical connection on the tangent bundle of the (non-Hausdorff) almost Kähler manifold $(\hat{Q}, \widehat{d\lambda}, \hat{J}_\xi)$ lifted to ξ. (In fact, as in the almost Kähler case, vanishing of $(1,1)$-component also implies vanishing of $(2,0)$-component and hence the torsion automatically becomes $(0, 2)$-type.) On the other hand, Axioms (1)–(3) indicate this connection behaves like the Levi-Civita connection when the Reeb direction X_λ get involved. Axiom (6) is an extra requirement to connect the information in ξ part and X_λ part, which is used to dramatically simplify our calculation in [14, 15].

In fact, the contact triad connection is one of the $\mathbb{R}$-family of affine connections satisfying Axioms (1)–(5) with (6) replaced by

$$\nabla_{JY} X_\lambda + J \nabla_Y X_\lambda = c\, Y, \quad c \in \mathbb{R}.$$

Contact triad connection corresponds to $c = 0$ and the connection $\nabla^{LC} + B_1$ (see Sect. 6 for the expression of B_1) corresponds to $c = -1$.

The contact triad connection (and also the whole $\mathbb{R}$-family) we construct here has naturality as stated below.

Corollary 1 (Naturality). *Let ∇ be the contact triad connection of the triad (Q, λ, J). Then for any diffeomorphism $\phi : Q \to Q$, the pull-back connection $\phi^* \nabla$ is the contact triad connection associated to the triad $(Q, \phi^* \lambda, \phi^* J)$.*

While our introduction of Axiom (6) is motivated by our attempt to simplify the tensor calculations [14], it has a nice geometric interpretation in terms of CR-geometry. (We refer to Definition 4 for the definition for CR-holomorphic k-forms.)

Proposition 1. *In the presence of other defining properties of contact triad connection, Axiom (6) is equivalent to the statement that λ is holomorphic in the CR-sense.*

Some motivations of the study of the canonical connection are in order. Hofer-Wysocki-Zehnder [5, 6] derived exponential decay estimates of proper pseudoholomorphic curves with respect to the cylindrical almost complex structure associated to the endomorphism $J : \xi \to \xi$ in symplectization by bruit force coordinate calculations using some special coordinates around the given Reeb orbit which is rather complicated. Our attempt to improve the presentation of these decay estimates, using the tensorial language, was the starting point of the research performed in the present paper.

We do this in [14, 15] by considering a map $w : \dot{\Sigma} \to Q$ satisfying the equation

$$\overline{\partial}^\pi w = 0, \quad d(w^* \lambda \circ j) = 0 \tag{1}$$

without involving the function a on the contact manifold Q or the symplectization. We call such a map a contact instanton. We refer [4] for the origin of this equation in contact geometry, as well as [14, 15] for the detailed analytic study of priori $W^{k,2}$-estimates and asymptotic convergence on punctured Riemann surfaces.

In the course of our studying the geometric analysis of such maps, we need to simplify the tensorial calculations by choosing a special connection as in the (almost) complex geometry. It turns out that for the purpose of taking the derivatives of the map w several times, the contact triad connection on Q is much more convenient and easier to keep track of various terms arising from switching the order of derivatives than the commonly used Levi-Civita connection. The advantage of the contact triad connection will become even more apparent in [13] where the Fredholm theory and the corresponding index computations in relation to Eq. (1) are developed.

There have been several literatures that studied special connections on contact manifolds, such as [11, 17, 18]. We make some rough comparisons between these connections and the contact triad connection introduced in this paper.

Although all the connections mentioned above are characterized by the torsion properties, one big difference between ours and the ones in [11, 17] is that we don't require $\nabla J = 0$, but only $\nabla^\pi J = 0$. Notice that $\nabla J = 0$ is equivalent to both $\nabla^\pi J = 0$ and $\nabla X_\lambda \in \mathbb{R} \cdot X_\lambda$. Together with the metric property, $\nabla J = 0$ also implies $\nabla X_\lambda = 0$, which is the requirement of the contact metric connection studied in [11, Definition 3.1] as well as the so-called adapted connection considered in [17, Sect. 4]. Our contact triad connection doesn't satisfy this requirement in general, and so is not in these families.

The connections considered in [11, 17] become the canonical connection when lifted to the *symplectization* as an almost Kähler manifold, while our connection and the generalized Tanaka-Webster connection considered by Tanno [18] are canonical for the (non-Hausdorff) almost Kähler manifold $(\hat{Q}, \widehat{d\lambda}, \hat{J}_\xi)$ lifted to ξ. (We remark that some other people named their connections the generalized Tanaka-Webster connection with different meanings.)

Difference in our connection and Tanno's shows up in the torsion property of $T(X_\lambda, \cdot)$ among others. It would be interesting to provide the classification of the canonical connections in a bigger family that includes both the contact triad connection and Tanno's generalized Tanaka-Webster connection. Since the torsion of the triad connection is already reduced to the simplest one, we expect that it satisfies better property on its curvature and get better results on the gauge invariant studied in [18].

This paper is a simplified version of [16], to which we refer readers for the complete proofs of various results given in this paper.

2 Review of the Canonical Connection of Almost Kähler Manifold

We recall this construction of the canonical connection for almost Kähler manifolds (M, ω, J). A nice and exhaustive discussion on the general almost Hermitian connection is given by Gauduchon in [3] to which we refer readers for more details. (See also [7], [12, Sect. 7.1].)

Assume (M, J, g) an almost Hermitian manifold, which means J is an almost complex structure J and g the metric satisfying $g(J\cdot, J\cdot) = g(\cdot, \cdot)$. An affine connection ∇ is called J-linear if $\nabla J = 0$. There always exists a J-linear connection for a given almost complex manifold. We denote by T the torsion tensor of ∇.

Definition 2. Let (M, J, g) be an almost Hermitian manifold. A J-linear connection is called a (the) canonical connection (or a (the) Chern connection) if for any for any vector field Y on M there is $T(JY, Y) = 0$.

Recall that any J-linear connection extended to the complexification $T_{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C}$ complex linearly preserves the splitting into $T^{(1,0)}M$ and $T^{(0,1)}M$. Similarly we can extend the torsion tensor T complex linearly which we denote by $T_{\mathbb{C}}$. Following the notation of [7], we denote $\Theta = \Pi' T_{\mathbb{C}}$ the $T^{(1,0)}M$-valued two-form, where Π' is the projection to $T^{(1,0)}M$. We have the decomposition $\Theta = \Theta^{(2,0)} + \Theta^{(1,1)} + \Theta^{(0,2)}$. We can define the canonical connection in terms of the induced connection on the complex vector bundle $T^{(1,0)}M \to M$. The following lemma is easy to check by definition.

Lemma 1. *An affine connection ∇ on M is a (the) canonical connection if and only if the induced connection ∇ on the complex vector bundle $T^{(1,0)}M$ has its complex torsion form $\Theta = \Pi' T_{\mathbb{C}}$ satisfy $\Theta^{(1,1)} = 0$.*

We particularly quote two theorems from Gauduchon [3], Kobayashi [7].

Theorem 2. *On any almost Hermitian manifold (M, J, g), there exists a unique Hermitian connection ∇ on TM leading to the canonical connection on $T^{(1,0)}M$. We call this connection the canonical Hermitian connection of (M, J, g).*

We recall that (M, J, g) is almost-Kähler if the fundamental two-form $\Phi = g(J\cdot, \cdot)$ is closed [8].

Theorem 3. *Let (M, J, g) be almost Kähler and ∇ be the canonical connection of $T^{(1,0)}M$. Then $\Theta^{(2,0)} = 0$ in addition, and hence Θ is of type $(0, 2)$.*

Remark 1. It is easy to check by definition (or see [3, 7] for details) that Θ is of type $(0, 2)$ is equivalent to say that for all vector fields Y, Z on W, $T(JY, Z) = T(Y, JZ)$ and $JT(JY, Z) = T(Y, Z)$.

Now we describe one way of constructing the canonical connection on an almost complex manifold described in [8, Theorem 3.4] which will be useful for our purpose of constructing the contact analog thereof later. This connection has its torsion which satisfies $N = 4T$, where N is the Nijenhuis tensor of the almost complex structure J defined as $N(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]$. In particular, the complexification $\Theta = \Pi' T_{\mathbb{C}}$ is of $(0, 2)$-type.

We now describe the construction of this canonical connection. Let ∇^{LC} be the Levi-Civita connection. Consider the standard averaged connection ∇^{av} of multiplication $J : TM \to TM$,

$$\nabla^{av}_X Y := \frac{\nabla^{LC}_X Y + J^{-1} \nabla^{LC}_X (JY)}{2} = \nabla^{LC}_X Y - \frac{1}{2} J (\nabla^{LC}_X J) Y.$$

We then have the following Proposition stating that this connection becomes the canonical connection. Its proof can be found in [8, Theorem 3.4] or from Sect. 2 [3] with a little more strengthened argument by using (3) for the metric property.

Proposition 2. *Assume that* (M, g, J) *is almost Kähler, i.e, the two-form* $\omega = g(J\cdot, \cdot)$ *is closed. Then the average connection* ∇^{av} *defines the canonical connection of* (M, g, J)*, i.e., the connection is* J*-linear, preserves the metric and its complexified torsion is of* $(0, 2)$*-type.*

In fact, a more general construction of the canonical connection for almost Hermitian manifold is given in [8]. We describe it and in later sections, we will give a contact analog of this construction.

Consider the tensor field Q defined by

$$4Q(X, Y) = (\nabla^{LC}_{JY} J)X + J((\nabla^{LC}_{Y} J)X) + 2J((\nabla^{LC}_{X} J)Y) \tag{2}$$

for vector fields X, Y on M. It turns out that when (M, g, J) is almost Kähler, i.e., the two form $g(J\cdot, \cdot)$ is closed, the sum of the first two terms vanish. In general, $\nabla := \nabla^{LC} - Q$ is the canonical connection of the almost Hermitian manifold. In fact, we have the following lemma which explains the construction above for almost Kähler case.

Lemma 2 ((2.2.10) [3]). *Assume* (M, g, J) *is almost Kähler. Then*

$$\nabla^{LC}_{JY} J + J(\nabla^{LC}_{Y} J) = 0 \tag{3}$$

and so $Q(X, Y) = \frac{1}{2} J((\nabla^{LC}_{X} J)Y)$*.*

3 Definition of the Contact Triad Connection and Its Consequences

In this section, we associate a particular type of affine connection on Q to the given contact triad (Q, λ, J) which we call *the contact triad connection* of the triple.

We recall $TQ = \mathbb{R}\{X_\lambda\} \oplus \xi$, and denote by $\pi : TQ \to \xi$ the projection. Under this splitting, we may regard a section Y of $\xi \to Q$ as a vector field $Y \oplus 0$. We will just denote the latter by Y with slight abuse of notation. Define ∇^π the connection of the bundle $\xi \to Q$ by $\nabla^\pi Y = \pi \nabla Y$.

Definition 3 (Contact Triad Connection). We call an affine connection ∇ on Q the *contact triad connection* of the contact triad (Q, λ, J), if it satisfies the following properties:

1. ∇^π is a Hermitian connection of the Hermitian bundle ξ over the contact manifold Q with Hermitian structure $(d\lambda, J)$.
2. The ξ projection, denoted by $T^\pi := \pi T$, of the torsion T satisfies the following properties: for all Y tangent to ξ, $T^\pi(JY, Y) = 0$.
3. $T(X_\lambda, Y) = 0$ for all $Y \in TQ$.
4. $\nabla_{X_\lambda} X_\lambda = 0$ and $\nabla_Y X_\lambda \in \xi$, for $Y \in \xi$.

5. For $Y \in \xi$, $\nabla_{JY} X_\lambda + J\nabla_Y X_\lambda = 0$.
6. For any $Y,\ Z \in \xi$, $\langle \nabla_Y X_\lambda, Z\rangle + \langle X_\lambda, \nabla_Y Z\rangle = 0$.

It follows from the definition that the contact triad connection is a Riemannian connection of the triad metric. (The statements of this definition are equivalent to those given in the introduction. We state properties of contact triad connection here as above which are organized in the way how they are used in the proofs of uniqueness and existence.)

By the second part of Axiom (4), the covariant derivative ∇X_λ restricted to ξ can be decomposed into $\nabla X_\lambda = \partial^\nabla X_\lambda + \overline{\partial}^\nabla X_\lambda$, where $\partial^\nabla X_\lambda$ (respectively, $\overline{\partial}^\nabla X_\lambda$) is J-linear (respectively, J-anti-linear part). Axiom (6) then is nothing but requiring that $\partial^\nabla X_\lambda = 0$, i.e., X_λ is anti J-holomorphic in the CR-sense. (It appears that this explains the reason why Axiom (5) gives rise to dramatic simplification in our tensor calculations performed in [14].)

One can also consider similar decompositions of one-form λ. For this, we need some digression. Define $J\alpha$ for a k-form α by the formula $J\alpha(Y_1, \cdots, Y_k) = \alpha(JY_1, \cdots, JY_k)$.

Definition 4. Let (Q, λ, J) be a contact triad. We call a k-form is CR-holomorphic if α satisfies

$$\nabla_{X_\lambda} \alpha = 0, \tag{4}$$

$$\nabla_Y \alpha + J\nabla_{JY} \alpha = 0 \quad \text{for } Y \in \xi. \tag{5}$$

Proposition 3. *Axiom (5) is equivalent to the statement that λ is holomorphic in the CR-sense in the presence of other defining properties of contact triad connection.*

Proof. We first prove $\nabla_{X_\lambda} \lambda = 0$ by evaluating it against vector fields on Q. For X_λ, the first half of Axiom (4) gives rise to $\nabla_{X_\lambda} \lambda(X_\lambda) = -\lambda(\nabla_{X_\lambda} X_\lambda) = 0$. For the vector field $Y \in \xi$, we compute

$$\begin{aligned}\nabla_{X_\lambda} \lambda(Y) &= -\lambda(\nabla_{X_\lambda} Y)\\ &= -\lambda(\nabla_Y X_\lambda + [X_\lambda, Y] + T(X_\lambda, Y))\\ &= -\lambda(\nabla_Y X_\lambda) - \lambda([X_\lambda, Y]) - \lambda(T(X_\lambda, Y)).\end{aligned}$$

Here the third term vanishes by Axiom (3), the first term by the second part of Axiom (4) and the second term vanishes since

$$\lambda([X_\lambda, Y]) = \lambda(\mathscr{L}_{X_\lambda} Y) = X_\lambda[\lambda(Y)] - \mathscr{L}_{X_\lambda} \lambda(Y) = 0 - 0 = 0.$$

Here the first vanishes since $Y \in \xi$ and the second because $\mathscr{L}_{X_\lambda} \lambda = 0$ by the definition of the Reeb vector field. This proves (4).

We next compute $J\nabla_Y \lambda$ for $Y \in \xi$. For a vector field $Z \in \xi$,

$$(J\nabla_Y\lambda)(Z) = (\nabla_Y\lambda)(JZ) = \nabla_Y(\lambda(JZ)) - \lambda(\nabla_Y(JZ)) = -\lambda(\nabla_Y(JZ))$$

since $\lambda(JZ) = 0$ for the last equality. Then by the definitions of the Reeb vector field and the triad metric and the skew-symmetry of J, we derive

$$-\lambda(\nabla_Y(JZ)) = -\langle\nabla_Y(JZ), X_\lambda\rangle = \langle JZ, \nabla_Y X_\lambda\rangle = -\langle Z, J\nabla_Y X_\lambda\rangle.$$

Finally, applying (6), we obtain

$$-\langle Z, J\nabla_Y X_\lambda\rangle = \langle Z, \nabla_{JY} X_\lambda\rangle = -\langle\nabla_{JY} Z, X_\lambda\rangle = -\lambda(\nabla_{JY} Z) = (\nabla_{JY}\lambda)(Z).$$

Combining the above, we have derived $J(\nabla_Y\lambda)(Z) = \nabla_{JY}\lambda(Z)$ for all $Z \in \xi$. On the other hand, for X_λ, we evaluate

$$J(\nabla_Y\lambda)(X_\lambda) = \nabla_Y\lambda(JX_\lambda) = \nabla_Y\lambda(0) = 0.$$

We also compute $\nabla_{JY}\lambda(X_\lambda) = \mathscr{L}_{JY}(\lambda(X_\lambda)) - \lambda(\nabla_{JY}X_\lambda))$. The first term vanishes since $\lambda(X_\lambda) \equiv 1$ and the second vanishes since $\nabla_{JY}X_\lambda \in \xi$ by the second part of Axiom (4). Therefore we have derived (5).

Combining (4) and (5), we have proved that Axiom (5) implies λ is holomorphic in the CR-sense. The converse can be proved by reading the above proof backwards.

From now on, when we refer Axioms, we mean the properties in Definition 3. One very interesting consequence of this uniqueness is the following naturality result of the contact-triad connection.

Theorem 4 (Naturality). *Let ∇ be the contact triad connection of the triad (Q, λ, J). For any diffeomorphism $\phi : Q \to Q$, the pull-back connection $\phi^*\nabla$ is the contact triad connection associated to the triad $(Q, \phi^*\lambda, \phi^*J)$.*

Proof. A straightforward computation shows that the pull-back connection $\phi^*\nabla$ satisfies all Axioms (1)–(6) for the triad $(Q, \phi^*\lambda, \phi^*J)$. Therefore by the uniqueness, $\phi^*\nabla$ is the canonical connection.

Remark 2. An easy examination of the proof of Theorem 4 shows that the naturality property stated in Theorem 4 also holds for the one-parameter family of connections for all $c \in \mathbb{R}$ (see Sect. 4) among which the canonical connection corresponds to $c = 0$.

4 Proof of the Uniqueness of the Contact Triad Connection

In this section, we give the uniqueness proof by analyzing the first structure equation and showing how every axiom determines the connection one forms. In the next two sections, we explicitly construct a connection by carefully examining properties of the Levi-Civita connection and modifying the constructions in [7, 8] for the canonical connection, and then show it satisfies all the requirements and thus the unique contact triad connection.

We are going to prove the existence and uniqueness for a more general family of connections. First, we generalize the Axiom (5) to the following Axiom: For $Y \in \xi$,

$$\nabla_{JY} X_\lambda + J\nabla_Y X_\lambda \in \mathbb{R} \cdot Y, \tag{6}$$

and we denote by Axiom (5; c): For a given $c \in \mathbb{R}$,

$$\nabla_{JY} X_\lambda + J\nabla_Y X_\lambda = c\, Y, \quad Y \in \xi. \tag{7}$$

In particular, Axiom (5) corresponds to Axiom (5; 0).

Theorem 5. *For any $c \in \mathbb{R}$, there exists a unique connection satisfies Axiom (1)–(4), (6) and (5; c).*

Proof (Uniqueness). Choose a moving frame of $TQ = \mathbb{R}\{X_\lambda\} \oplus \xi$ given by $\{X_\lambda, E_1, \cdots, E_n, JE_1, \cdots, JE_n\}$ and denote its dual co-frame by $\{\lambda, \alpha^1, \cdots, \alpha^n, \beta^1, \cdots, \beta^n\}$. (We use the Einstein summation convention to denote the sum of upper indices and lower indices in this paper.) Assume the connection matrix is (Ω^i_j), $i, j = 0, 1, \ldots, 2n$, and we write the first structure equations as follows

$$\begin{aligned}
d\lambda &= -\Omega^0_0 \wedge \lambda - \Omega^0_k \wedge \alpha^k - \Omega^0_{n+k} \wedge \beta^k + T^0\\
d\alpha^j &= -\Omega^j_0 \wedge \lambda - \Omega^j_k \wedge \alpha^k - \Omega^j_{n+k} \wedge \beta^k + T^j\\
d\beta^j &= -\Omega^{n+j}_0 \wedge \lambda - \Omega^{n+j}_k \wedge \alpha^k - \Omega^{n+j}_{n+k} \wedge \beta^k + T^{n+j}
\end{aligned}$$

Throughout the section, if not stated otherwise, we let i, j and k take values from 1 to n. Denote

$$\Omega^u_v = \Gamma^u_{0,v}\lambda + \Gamma^u_{k,v}\alpha^k + \Gamma^u_{n+k,v}\beta^k$$

where $u, v = 0, 1, \cdots, 2n$. We will analyze each axiom in Definition 3 and show how they set down the matrix of connection one forms.

We first state that Axioms (1) and (2) uniquely determine $(\Omega^i_j|_\xi)_{i,j=1,\cdots,2n}$. This is exactly the same as Kobayashi's proof for the uniqueness of Hermitian connection given in [7]. To be more specific, we can restrict the first structure equation to ξ and get the following equations for α and β since ξ is the kernel of λ.

$$\begin{aligned}
d\alpha^j &= -\Omega^j_k|_\xi \wedge \alpha^k - \Omega^j_{n+k}|_\xi \wedge \beta^k + T^j|_\xi\\
d\beta^j &= -\Omega^{n+j}_k|_\xi \wedge \alpha^k - \Omega^{n+j}_{n+k}|_\xi \wedge \beta^k + T^{n+j}|_\xi
\end{aligned}$$

We can see $(\Omega^i_j|_\xi)_{i,j=1,\cdots,2n}$ is skew-Hermitian from Axiom (1). We also notice that from the Remark 1 that Axiom (2) is equivalent to say that $\Theta^{(1,1)} = 0$, where $\Theta = \Pi' T_{\mathbb{C}}$. Then one can strictly follow Kobayashi's proof of Theorem 2 in [7] and get $(\Omega^i_j|_\xi)_{i,j=1,\cdots,2n}$ are uniquely determined. For this part, we refer readers to the proofs of [7, Theorems 1.1 and 2.1].

In the rest of the proof, we will clarify how the Axioms (3), (4), (5;c), (6) uniquely determine $\Omega^0_\cdot$, $\Omega^\cdot_0$ and $(\Omega^i_j(X_\lambda))_{i,j=1,\cdots,2n}$. Compute the first equality in Axiom (4) and we get

$$\nabla_{X_\lambda} X_\lambda = \Gamma^0_{0,0} X_\lambda + \Gamma^k_{0,0} E_k + \Gamma^{n+k}_{0,0} J E_k = 0.$$

Hence

$$\Gamma^0_{0,0} = 0, \quad \Gamma^k_{0,0} = 0, \quad \Gamma^{n+k}_{0,0} = 0 \tag{8}$$

The second claim in Axiom (4) is equal to say

$$\nabla_{E_k} X_\lambda \in \xi, \quad \nabla_{JE_k} X_\lambda \in \xi. \tag{9}$$

Similar calculation shows that

$$\Gamma^0_{k,0} = 0, \quad \Gamma^0_{n+k,0} = 0. \tag{10}$$

Now the first vanishing in (8) together with (10) uniquely settle down

$$\Omega^0_0 = \Gamma^0_{0,0}\lambda + \Gamma^0_{k,0}\alpha^k + \Gamma^0_{n+k,0}\beta^k = 0.$$

The vanishing of second and third equality in (8) will be used to determine Ω_0 in the later part. From Axiom (3), we can get

$$\Gamma^k_{j,0} - \Gamma^k_{0,j} = \langle [E_j, X_\lambda], E_k\rangle = -\langle \mathscr{L}_{X_\lambda} E_j, E_k\rangle \tag{11}$$

$$\Gamma^k_{n+j,0} - \Gamma^k_{0,n+j} = \langle [JE_j, X_\lambda], E_k\rangle = -\langle \mathscr{L}_{X_\lambda}(JE_j), E_k\rangle \tag{12}$$

and

$$\Gamma^{n+k}_{j,0} - \Gamma^{n+k}_{0,j} = \langle [E_j, X_\lambda], JE_k\rangle = -\langle \mathscr{L}_{X_\lambda} E_j, JE_k\rangle \tag{13}$$

$$\Gamma^{n+k}_{n+j,0} - \Gamma^{n+k}_{0,n+j} = \langle [E_j, X_\lambda], JE_k\rangle = -\langle \mathscr{L}_{X_\lambda}(JE_j), JE_k\rangle. \tag{14}$$

From Axiom (5; c), we have

$$\Gamma^k_{j,0} + \Gamma^{n+k}_{n+j,0} = 0 \tag{15}$$

$$\Gamma^{n+k}_{j,0} - \Gamma^k_{n+j,0} = -c\delta_{j,k}. \tag{16}$$

Now we show how to determine Ω^j_0 for $j = 1, \ldots, 2n$. For this purpose, we calculate $\Gamma^k_{j,0}$. First, by using (15), we write $\Gamma^k_{j,0} = \frac{1}{2}\Gamma^k_{j,0} - \frac{1}{2}\Gamma^{n+k}_{n+j,0}$.

Furthermore, using (11) and (14), we have

$$\begin{aligned}
\Gamma^k_{j,0} &= \frac{1}{2}\Gamma^k_{j,0} - \frac{1}{2}\Gamma^{n+k}_{n+j,0} \\
&= \frac{1}{2}(\Gamma^k_{0,j} - \langle \mathcal{L}_{X_\lambda} E_j, E_k\rangle) - \frac{1}{2}(\Gamma^{n+k}_{0,n+j} - \langle \mathcal{L}_{X_\lambda}(JE_j), JE_k\rangle) \\
&= \frac{1}{2}(\Gamma^k_{0,j} - \Gamma^{n+k}_{0,n+j}) - \frac{1}{2}(\langle \mathcal{L}_{X_\lambda} E_j, E_k\rangle - \langle \mathcal{L}_{X_\lambda}(JE_j), JE_k\rangle) \\
&= \frac{1}{2}(\Gamma^k_{0,j} - \Gamma^{n+k}_{0,n+j}) - \frac{1}{2}\langle \mathcal{L}_{X_\lambda} E_j + J\mathcal{L}_{X_\lambda}(JE_j), E_k\rangle \\
&= \frac{1}{2}(\Gamma^k_{0,j} - \Gamma^{n+k}_{0,n+j}) - \frac{1}{2}\langle J(\mathcal{L}_{X_\lambda} J)E_j, E_k\rangle \\
&= \frac{1}{2}(\Gamma^k_{0,j} - \Gamma^{n+k}_{0,n+j}) + \frac{1}{2}\langle (\mathcal{L}_{X_\lambda} J)JE_j, E_k\rangle
\end{aligned}$$

Notice the first term vanishes by Axiom (2). In particular, that is from $\nabla_{X_\lambda} J = 0$. Hence we get

$$\Gamma^k_{j,0} = \frac{1}{2}\langle (\mathcal{L}_{X_\lambda} J)JE_j, E_k\rangle. \tag{17}$$

Following the same idea, we use (16) and will get

$$\Gamma^{n+k}_{j,0} = -\frac{1}{2}c\delta_{jk} + \frac{1}{2}\langle (\mathcal{L}_{X_\lambda} J)JE_j, JE_k\rangle.$$

Then substituting this into (15) and (16), we have

$$\Gamma^k_{n+j,0} = \frac{1}{2}c\delta_{jk} + \frac{1}{2}\langle (\mathcal{L}_{X_\lambda} J)JE_j, JE_k\rangle = \frac{1}{2}c\delta_{jk} - \frac{1}{2}\langle (\mathcal{L}_{X_\lambda} J)E_j, E_k\rangle.$$

and

$$\Gamma^{n+k}_{n+j,0} = -\frac{1}{2}\langle (\mathcal{L}_{X_\lambda} J)JE_j, E_k\rangle = \frac{1}{2}\langle (\mathcal{L}_{X_\lambda} J)E_j, JE_k\rangle.$$

Together with (8), Ω_0 is uniquely determined by this way.

Furthermore (11)–(14), uniquely determine $\Omega^i_j(X_\lambda)$ for $i, j = 1, \ldots, 2n$.

Notice that for any $Y \in \xi$, we derive $\nabla_{X_\lambda} Y \in \xi$ from Axiom (3). This is because the axiom implies $\nabla_{X_\lambda} Y = \nabla_Y X_\lambda + \mathcal{L}_{X_\lambda} Y$ and the latter is contained in ξ: the second part of Axiom (4) implies $\nabla_Y X_\lambda \in \xi$ and the Lie derivative along the Reeb vector field preserves the contact structure ξ. It then follows that $\Gamma^0_{0,l} = 0$ for $l = 1, \ldots, 2n$. At the same time, Axiom (6) implies $\Gamma^0_{j,k} = -\Gamma^k_{j,0}$. for j, $k = 1, \ldots, 2n$. Hence together with (10), Ω^0 is uniquely determined. This finishes the proof.

We end this section by giving a summary of the procedure we take in the proof of uniqueness which actually indicates a way how to construct this connection in later sections.

First, we use the Hermitian connection property, i.e., Axiom (1) and torsion property Axiom (2), i.e., $T^\pi|_\xi$ has vanishing $(1,1)$ part, to uniquely fix the connection on ξ projection of ∇ when taking values on ξ.

Then we use the metric property $\langle X_\lambda, \nabla_Y Z\rangle + \langle \nabla_Y X_\lambda, Z\rangle = 0$, for any $Y, Z \in \xi$, to determine the X_λ component of ∇ when taking values in ξ.

To do this, we need the information of $\nabla_Y X_\lambda$. As mentioned before the second part of Axiom (4) enables us to decompose $\nabla X_\lambda = \partial^\nabla X_\lambda + \overline{\partial}^\nabla X_\lambda$. The requirement $\nabla_{X_\lambda} J = 0$ in Axiom (1) implies $\nabla_{X_\lambda}(JY) - J\nabla_{X_\lambda} Y = 0$. Axiom (3), the torsion property $T(X_\lambda, Y) = 0$, then interprets this one into

$$\nabla_{JY} X_\lambda - J\nabla_Y X_\lambda = -(\mathscr{L}_{X_\lambda} J)Y$$

which is also equivalent to saying

$$J\overline{\partial}^\nabla_Y X_\lambda = \frac{1}{2}(\mathscr{L}_{X_\lambda} J)Y \quad \text{or} \quad \overline{\partial}^\nabla_Y X_\lambda = \frac{1}{2}(\mathscr{L}_{X_\lambda} J)JY. \tag{18}$$

It turns out that we can vary Axiom (5) by replacing it to (5;c)

$$\nabla_{JY} X_\lambda + J\nabla_Y X_\lambda = cY, \quad \text{or equivalently } \partial^\nabla_Y X_Y = \frac{c}{2}Y \tag{19}$$

for any given real number c. This way we shall have one-parameter family of affine connections parameterized by $\mathbb{R}$ each of which satisfies Axioms (1)–(4) and (6) with (5) replaced by (5;c).

When c is fixed, i.e., under Axiom (5; c), we can uniquely determine $\nabla_Y X_\lambda$ to be

$$\nabla_Y X_\lambda = -\frac{1}{2}cJY + \frac{1}{2}(\mathscr{L}_{X_\lambda} J)JY.$$

Therefore, ∇_Y, $Y \in \xi$ is uniquely determined in this process by getting the formula of $\nabla_Y X_\lambda$ when combined with the torsion property. Then the remaining property $\nabla_{X_\lambda} X_\lambda = 0$ now completely determines the connection.

5 Properties of the Levi-Civita Connection on Contact Manifolds

From the discussion in previous sections, the only thing left to do for the existence of the contact triad connection is to globally define a connection such that it can patch the ξ part of $\nabla|_\xi$ and the X_λ part of it. In particular, we seek for a connection that satisfies the following properties:

1. it satisfies all the algebraic properties of the canonical connection of almost Kähler manifold [7] when restricted to ξ.
2. it satisfies metric property and has vanishing torsion in X_λ direction.

The presence of such a construction is a manifestation of delicate interplay between the geometric structures ξ, λ, and J in the geometry of contact triads (Q, λ, J). In this regard, the closeness of $d\lambda$ and the definition of Reeb vector field X_λ play important roles. In particular $d\lambda$ plays the role similar to that of the fundamental two-form Φ in the case of almost Kähler manifold [8] (in a non-strict sense) in that it is closed.

This interplay is reflected already in several basic properties of the Levi-Civita connection of the contact triad metric exposed in this section. We list these properties but skip most proofs of them in this section since most results are well-known in Blair's book [1]. We also refer readers to [16] for the complete proof with the same convention.

Recall that we have extend J to TQ by defining $J(X_\lambda) = 0$. Denote by $\Pi : TQ \to TQ$ the idempotent associated to the projection $\pi : TQ \to \xi$, i.e., the endomorphism satisfying $\Pi^2 = \Pi$, $\operatorname{Im} \Pi = \xi$, and $\ker \Pi = \mathbb{R}\{X_\lambda\}$.

We have now $J^2 = -\Pi$. Moreover, for any connection ∇ on Q,

$$(\nabla J)J = -(\nabla \Pi) - J(\nabla J). \tag{20}$$

Notice for $Y \in \xi$, we have

$$\Pi(\nabla \Pi)Y = 0, \quad (\nabla \Pi)X_\lambda = -\Pi \nabla X_\lambda. \tag{21}$$

Denote the triad metric g as $\langle \cdot, \cdot \rangle$. By definition, we have

$$\langle X, Y \rangle = d\lambda(X, JY) + \lambda(X)\lambda(Y)$$
$$d\lambda(X, Y) = d\lambda(JX, JY)$$

which gives rise to the following identities

Lemma 3. *For all X, Y in TQ, $\langle JX, JY \rangle = d\lambda(X, JY)$, $\langle X, JY \rangle = -d\lambda(X, Y)$, and $\langle JX, Y \rangle = -\langle X, JY \rangle$.*

However, we remark $\langle JX, JY \rangle \neq \langle X, Y \rangle$ in general now, and hence there is no obvious analog of the fundamental 2 form Φ defined as in [8] for the contact case. This is the main reason that is responsible for the differences arising in the various relevant formulae between the contact case and the almost Hermitian case.

The following preparation lemma says that the linear operator $\mathscr{L}_{X_\lambda} J$ is symmetric with respect to the metric $g = \langle \cdot, \cdot \rangle$.

Lemma 4 (Lemma 6.2 [1]). *For $Y, Z \in \xi$, $\langle (\mathscr{L}_{X_\lambda} J)Y, Z \rangle = \langle Y, (\mathscr{L}_{X_\lambda} J)Z \rangle$.*

The following simple but interesting lemma shows that the Reeb foliation is a geodesic foliation for the Levi-Civita connection (and so for the contact triad connection) of the contact triad metric.

Lemma 5 ([1]). *For any vector field Z on Q,*

$$\nabla^{LC}_Z X_\lambda \in \xi, \tag{22}$$

and

$$\nabla^{LC}_{X_\lambda} X_\lambda = 0. \tag{23}$$

Next we state the following lemma which is the contact analog to the Prop 4.2 in [8] for the almost Hermitian case. The proof of this lemma can be also extracted from [1, Corollary 6.1] and so we skip it but refer [16] for details.

Lemma 6. *Consider the Nijenhuis tensor N defined by*

$$N(X,Y) = [JX,JY] - [X,Y] - J[X,JY] - J[JX,Y]$$

as in the almost complex case. For all X, Y and Z in TQ,

$$\begin{aligned} 2\langle(\nabla^{LC}_X J)Y, Z\rangle &= \langle N(Y,Z), JX\rangle \\ &\quad -\langle JX, JY\rangle\lambda(Z) + \langle JX, JZ\rangle\lambda(Y) \end{aligned}$$

In particular, we obtain the following corollary.

Corollary 2. *For $Y, Z \in \xi$,*

$$\begin{aligned} 2\langle(\nabla^{LC}_Y J)X_\lambda, Z\rangle &= -\langle(\mathscr{L}_{X_\lambda} J)Z, Y\rangle + \langle Y, Z\rangle \\ 2\langle(\nabla^{LC}_Y J)Z, X_\lambda\rangle &= \langle(\mathscr{L}_{X_\lambda} J)Z, Y\rangle - \langle Y, Z\rangle \\ 2\langle(\nabla^{LC}_X J)Y, Z\rangle &= \langle N(Y,Z), JX\rangle. \end{aligned}$$

Proof. This is a direct corollary from Lemma 6 except that we also use

$$N(X_\lambda, Z) = -J(\mathscr{L}_{X_\lambda} J)Z \tag{24}$$

$$N(Z, X_\lambda) = J(\mathscr{L}_{X_\lambda} J)Z. \tag{25}$$

for the first two conclusions.

Straightforward calculations give the following lemma which is the contact analog of the fact that the Nijenhuis tensor is of $(0,2)$-type.

Lemma 7. *For $Y, Z \in \xi$,*

$$JN(Y, JZ) - \Pi N(Y, Z) = 0$$
$$\Pi N(Y, JZ) + \Pi N(Z, JY) = 0.$$

Together with the last equality in Corollary 2 and Lemma 7, we obtain the following lemma, which is the contact analog to Lemma 2.

Lemma 8.

$$\Pi(\nabla^{LC}_{JY} J)X + J(\nabla^{LC}_{Y} J)X = 0. \tag{26}$$

The following result is an immediate but important corollary of Corollary 2 and the property $\nabla_{X_\lambda} X_\lambda = 0$ of X_λ, which plays an essential role in our construction of the contact triad connection.

Proposition 4 (Corollary 6.1 [1]). $\nabla^{LC}_{X_\lambda} J = 0$.

The following is equivalent to the second part of Lemma 6.2 [1] after taking into consideration of different sign convention of the definition of compatibility of J and $d\lambda$.

Lemma 9 (Lemma 6.2 [1]). *For any $Y \in \xi$, we have $\nabla^{LC}_Y X_\lambda = \frac{1}{2} JY + \frac{1}{2}(\mathscr{L}_{X_\lambda} J)JY$.*

6 Existence of the Contact Triad Connection

In this section, we establish the existence theorem of the contact triad connection in two stages.

Before we give the construction, we first remark the relationship between the connections of two different c's. Denote by $\nabla^{\lambda;c}$ the unique connection associated to the constant c, which we are going to construct. The following proposition shows that $\nabla^{\lambda;c}$ and $\nabla^{\lambda;c'}$ for two different nonzero constants with the same parity are essentially the same in that it arises from the scale change of the contact form. We skip the proof since it is straightforward.

Proposition 5. *Let (Q, λ, J) be a contact triad and consider the triad $(Q, a\lambda, J)$ for a constant $a > 0$. Then $\nabla^{a\lambda;1} = \nabla^{\lambda;a}$.*

In regard to this proposition, one could say that for each given contact structure (Q, ξ), there are essentially two inequivalent ∇^0, ∇^1 (respectively three, ∇^0, ∇^1 and ∇^{-1}, if one fixes the orientation) choice of triad connections for each given projective equivalence class of the contact triad (Q, λ, J). In this regard, the connection ∇^0 is essentially different from others in that this argument of scaling procedure of contact form λ does not apply to the case $a = 0$ since it would lead to the zero form $0 \cdot \lambda$. This proposition also reduces the construction essentially two connections of $\nabla^{\lambda;0}$ and $\nabla^{\lambda;1}$ (or $\nabla^{\lambda;-1}$).

In the rest of this section, we will explicitly construct $\nabla^{\lambda;-1}$ and $\nabla^{\lambda;c}$ in two stages, by construct the potential tensor B from the Levi-Civita connection, i.e., by adding suitable tensors B to get $\nabla^B = \nabla^{LC} + B := \nabla^{LC} + B_1 + B_2$.

In the first stage, motivated by the construction of the canonical connection on almost Kähler manifold and use the properties of the Levi-Civita connection we extracted in the previous section, we construct the connection $\nabla^{tmp;1}$ and show that it satisfies Axioms (1)–(4), (5;−1), (6).

In the second stage, we modify $\nabla^{tmp;1}$ to get $\nabla^{tmp;2}$ by deforming the property (5;−1) thereof to (5;c) leaving other properties of $\nabla^{tmp;1}$ intact. This $\nabla^{tmp;2}$ then satisfies all the axioms in Definition 3.

6.1 Modification 1; $\nabla^{tmp;1}$

Define an affine connection $\nabla^{tmp;1}$ by the formula

$$\nabla^{tmp;1}_{Z_1} Z_2 = \nabla^{LC}_{Z_1} Z_2 - \Pi P(\Pi Z_1, \Pi Z_2)$$

where the bilinear map $P : \Gamma(TQ) \times \Gamma(TQ) \to \Gamma(TQ)$ over $C^\infty(Q)$ is defined by

$$4P(X, Y) = (\nabla^{LC}_{JY} J)X + J((\nabla^{LC}_Y J)X) + 2J((\nabla^{LC}_X J)Y) \tag{27}$$

for vector fields X, Y in Q. (To avoid confusion with our notation Q for the contact manifold and to highlight that P is not the same tensor field as Q but is the contact analog thereof, we use P instead for its notation.) From (26), we have now

$$\Pi P(\Pi Z_1, \Pi Z_2) = \frac{1}{2} J((\nabla^{LC}_{\Pi Z_1} J)\Pi Z_2).$$

According to the remark made in the beginning of the section, we choose B_1 to be

$$B_1(Z_1, Z_2) = -\Pi P(\Pi Z_1, \Pi Z_2) = -\frac{1}{2} J((\nabla^{LC}_{\Pi Z_1} J)\Pi Z_2). \tag{28}$$

First we consider the induced vector bundle connection on the Hermitian bundle $\xi \to Q$, which we denote by $\nabla^{tmp;1,\pi}$: it is defined by

$$\nabla^{tmp;1,\pi}_X Y := \pi \nabla^{tmp;1}_X Y \tag{29}$$

for a vector field Y tangent to ξ, i.e., a section of ξ for arbitrary vector field X on Q. We now prove the J linearity of $\nabla^{tmp;1,\pi}$.

Lemma 10. *Let $\pi : TQ \to \xi$ be the projection. Then $\nabla^{tmp;1,\pi}_X (JY) = J\nabla^{tmp;1,\pi}_X Y$ for $Y \in \xi$ and all $X \in TQ$.*

Proof. For $X \in \xi$,

$$\begin{aligned}\nabla_X^{tmp;1}(JY) &= \nabla_X^{LC}(JY) - \Pi P(X, JY) \\ &= (J\nabla_X^{LC}Y + (\nabla_X^{LC}J)Y) - \frac{1}{2}J((\nabla_X^{LC}J)JY) \\ &= J\nabla_X^{LC}Y + (\nabla_X^{LC}J)Y - \frac{1}{2}\Pi((\nabla_X^{LC}J)Y) + \frac{1}{2}J((\nabla_X^{LC}\Pi)Y) \quad (30) \\ &= J\nabla_X^{LC}Y + (\nabla_X^{LC}J)Y - \frac{1}{2}\Pi((\nabla_X^{LC}J)Y)\end{aligned}$$

where we use (20) to get the last two terms in the third equality and use (21) to see that the last term in (30) vanishes. Hence,

$$\pi\nabla_X^{tmp;1}(JY) = \pi\nabla_X^{tmp;1}(JY) = J\nabla_X^{LC}Y + \frac{1}{2}\pi((\nabla_X^{LC}J)Y).$$

On the other hand, we compute

$$J\pi\nabla_X^{tmp;1}Y = J\left(\nabla_X^{LC}Y - \frac{1}{2}J((\nabla_X^{LC}J)Y)\right) = J\nabla_X^{LC}Y + \frac{1}{2}\pi((\nabla_X^{LC}J)Y).$$

Hence we have now $\pi\nabla_X^{tmp;1}(JY) = J\pi\nabla_X^{tmp;1}Y$ for $X,\ Y \in \xi$.

On the other hand, we notice that $\nabla_{X_\lambda}^{tmp;1}Y = \nabla_{X_\lambda}^{LC}Y$. By using Proposition 4, the equality $\pi\nabla_X^{tmp;1}(JY) = J\pi\nabla_X^{tmp;1}Y$ also holds for $X = X_\lambda$, and we are done with the proof.

Next we study the metric property of $\nabla^{tmp;1}$ by computing $\langle\nabla_X^{tmp;1}Y, Z\rangle + \langle Y, \nabla_X^{tmp;1}Z\rangle$ for arbitrary $X, Y, Z \in TQ$.

Using the metric property of the Levi-Civita connection, we derive

$$\begin{aligned}&\langle\nabla_X^{tmp;1}Y, Z\rangle + \langle Y, \nabla_X^{tmp;1}Z\rangle - X\langle Y, Z\rangle \\ &= \langle\nabla_X^{LC}Y, Z\rangle + \langle Y, \nabla_X^{LC}Z\rangle - X\langle Y, Z\rangle - \langle\Pi P(\Pi X, \Pi Y), Z\rangle - \langle Y, \Pi P(\Pi X, \Pi Z)\rangle \\ &= -\langle\Pi P(\Pi X, \Pi Y), Z\rangle - \langle Y, \Pi P(\Pi X, \Pi Z)\rangle, \quad (31)\end{aligned}$$

The following lemma shows that when $X, Y, Z \in \xi$ this last line vanishes. This is the contact analog to Proposition 2 whose proof is also similar thereto this time based on Lemma 7. Since we work in the contact case for which we cannot directly quote its proof here, we give complete proof for readers' convenience.

Lemma 11. *For $X, Y, Z \in \xi$, $\langle P(X, Y), Z\rangle + \langle Y, P(X, Z)\rangle = 0$. In particular,*

$$\langle\nabla_X^{tmp;1}Y, Z\rangle + \langle Y, \nabla_X^{tmp;1}Z\rangle = X\langle Y, Z\rangle.$$

Proof. We compute for $X, Y, Z \in \xi$,

$$\begin{aligned} & \langle P(X,Y), Z\rangle + \langle Y, P(X,Z)\rangle \\ &= \frac{1}{2}\langle J((\nabla^{LC}_X J)Y), Z\rangle + \frac{1}{2}\langle Y, J((\nabla^{LC}_X J)Z)\rangle \\ &= -\frac{1}{2}\langle (\nabla^{LC}_X J)Y, JZ\rangle - \frac{1}{2}\langle JY, (\nabla^{LC}_X J)Z\rangle \\ &= -\frac{1}{4}\langle N(Y, JZ), JX\rangle - \frac{1}{4}\langle N(Z, JY), JX\rangle \qquad (32) \\ &= -\frac{1}{4}\langle \Pi N(Y, JZ) + \Pi N(Z, JY), JX\rangle = 0, \qquad (33) \end{aligned}$$

where we use the third equality of Corollary 2 for (32) and use the second equality of Lemma 7 for the vanishing of (33).

Now, we are ready to state the following proposition.

Proposition 6. *The vector bundle connection $\nabla^{tmp;1,\pi} := \pi\nabla^{tmp;1}$ is an Hermitian connection of the Hermitian bundle $\xi \to Q$.*

Proof. What is now left to show is that for any $Y,\ Z \in \xi$,

$$\langle \nabla^{tmp;1}_{X_\lambda} Y, Z\rangle + \langle Y, \nabla^{tmp;1}_{X_\lambda} Z\rangle = X_\lambda\langle Y, Z\rangle,$$

which immediately follows from our construction of $\nabla^{tmp;1}$ since

$$\nabla^{tmp;1}_{X_\lambda} Y = \nabla^{LC}_{X_\lambda} Y, \quad \nabla^{tmp;1}_{X_\lambda} Z = \nabla^{LC}_{X_\lambda} Z.$$

With direct calculation, one can check the metric property when the Reeb direction gets involved.

Lemma 12. *For $Y,\ Z \in \xi$, $\langle \nabla^{tmp;1}_Y X_\lambda, Z\rangle + \langle X_\lambda, \nabla^{tmp;1}_Y Z\rangle = 0$.*

Now we study the torsion property of $\nabla^{tmp;1}$. Denote the torsion of $\nabla^{tmp;1}$ by $T^{tmp;1}$. Similar as for the almost Hermitian case, define $\Theta^\pi = \Pi' T^{tmp;1,\pi}_{\mathbb{C}}$. Here we decompose

$$T^{tmp;1}|_\xi = \pi T^{tmp;1}|_\xi + \lambda(T^{tmp;1,\pi}|_\xi)\, X_\lambda$$

and denote $T^{tmp;1,\pi}|_\xi := \pi T^{tmp;1,\pi}|_\xi$, The proof of the following lemma follows essentially the same strategy as that of the proof of [8, Theorem 3.4]. We give the complete proof for readers' convenience.

Lemma 13. *For $Y \in \xi$, $T^{tmp;1}(X_\lambda, Y) = 0$, and*

$$T^{tmp;1,\pi}|_\xi = \frac{1}{4} N^\pi|_\xi, \quad \lambda(T^{tmp;1}|_\xi) = 0.$$

In particular, $\Theta^\pi|_\xi$ *is of* $(0, 2)$ *form.*

Proof. Since $\nabla^{tmp;1} = \nabla^{LC} - \Pi P(\Pi, \Pi)$ and ∇^{LC} is torsion free, we derive for $Y,\ Z \in \xi$,

$$\begin{aligned} T^{tmp;1}(Y, Z) &= T^{LC}(Y, Z) - \Pi P(Y, Z) + \Pi P(Z, Y) \\ &= \frac{1}{2} J \nabla^{LC}_Y JZ - \frac{1}{2} J \nabla^{LC}_Z JY. \end{aligned}$$

from the general torsion formula.

Next we calculate $-\Pi P(\Pi Y, \Pi Z) + \Pi P(\Pi Z, \Pi Y)$ using the formula

$$\begin{aligned} \frac{1}{2} J \nabla^{LC}_Y JZ - \frac{1}{2} J \nabla^{LC}_Z JY &= \frac{1}{4}\pi([JY, JZ] - \pi[Y, Z] - J[JY, Z] - J[Y, JZ]) \\ &= \frac{1}{4}\pi N(Y, Z). \end{aligned}$$

This follows from the general formula

$$-P(Y, Z) + P(Z, Y) = \frac{1}{4}([JY, JZ] - \Pi[Y, Z] - J[JY, Z] - J[Y, JZ]), \qquad (34)$$

whose derivation we refer [16, Appendix].

On the other hand, since the added terms to ∇^{LC} only involves ξ-directions, the X_λ-component of the torsion does not change and so

$$\lambda(T^{tmp;1}|_\xi) = \lambda(T^{LC}|_\xi) = 0.$$

This finishes the proof.

From the definition of $\nabla^{tmp;1}$, we have the following lemma from the properties of the Levi-Civita connection in Proposition 5.

Lemma 14. $\nabla^{tmp;1}_{X_\lambda} X_\lambda = 0$ *and* $\nabla^{tmp;1}_Y X_\lambda \in \xi$ *for any* $Y \in \xi$.

We also get the following property by using Lemma 9 for Levi-Civita connection.

Lemma 15. *For any* $Y \in \xi$, *we have* $\nabla^{tmp;1}_Y X_\lambda = \frac{1}{2} JY + \frac{1}{2}(\mathscr{L}_{X_\lambda} J)JY$.

We end the construction of $\nabla^{tmp;1}$ by summarizing that $\nabla^{tmp;1}$ satisfies Axioms (1)–(4),(6) and (5;−1), i.e., $\nabla^{tmp;1} = \nabla^{\lambda;-1}$.

6.2 Modification 2; $\nabla^{tmp;2}$

Now we introduce another modification $\nabla^{tmp;2}$ starting from $\nabla^{tmp;1}$ to make it satisfy Axiom (5;c) and preserve other axioms for any given constant $c \in \mathbb{R}$. We define $\nabla^{tmp;2} = \nabla^{tmp;1} + B_2$ for the tensor B_2 given as

$$B_2(Z_1, Z_2) = \frac{1}{2}(1+c)\left(-\langle Z_2, X_\lambda\rangle JZ_1 - \langle Z_1, X_\lambda\rangle JZ_2 + \langle JZ_1, Z_2\rangle X_\lambda\right). \tag{35}$$

Proposition 7. *The connection $\nabla^{tmp;2}$ satisfies all the properties of the canonical connection with constant c. In particular $\nabla := \nabla^{tmp;2}$ with $c = 0$ is the contact triad connection.*

Proof. The checking of all Axioms are straightforward, and we only do it for Axiom (5;c) here.

$$\begin{aligned}
&\nabla^{tmp;2}_{JY} X_\lambda + J\nabla^{tmp;2}_{Y} X_\lambda \\
&= \nabla^{tmp;1}_{JY} X_\lambda - \frac{1}{2}(1+c)\, JJY + J\nabla^{tmp;1}_{Y} X_\lambda - J\frac{1}{2}(1+c)\, JY \\
&= -Y + (1+c)Y = cY.
\end{aligned}$$

Before ending this section, we restate the following properties which will be useful for calculations involving contact Cauchy-Riemann maps performed in [14, 15].

Proposition 8. *Let ∇ be the connection satisfying Axiom (1)–(4),(6) and (5; c), then $\nabla_Y X_\lambda = -\frac{1}{2}cJY + \frac{1}{2}(\mathscr{L}_{X_\lambda} J)JY$. In particular, for the contact triad connection, $\nabla_Y X_\lambda = \frac{1}{2}(\mathscr{L}_{X_\lambda} J)JY$.*

Proof. We already gave its proof in the last part of Sect. 3.

Proposition 9. *Decompose the torsion of ∇ into $T = \pi T + \lambda(T)\, X_\lambda$. The triad connection ∇ has its torsion given by $T(X_\lambda, Z) = 0$ for all $Z \in TQ$, and for all $Y, Z \in \xi$,*

$$\begin{aligned}
\pi T(Y, Z) &= \frac{1}{4}\pi N(Y, Z) = \frac{1}{4}\left((\mathscr{L}_{JY} J)Z + (\mathscr{L}_Y J)JZ\right) \\
\lambda(T(Y, Z)) &= d\lambda(Y, Z).
\end{aligned}$$

Proof. We have seen $\pi T^{tmp;2}|_\xi = \pi T^{tmp;1}|_\xi = \frac{1}{4}N^\pi|_\xi$. On the other hand, a simple computation shows $N^\pi(Y, Z) = (\mathscr{L}_{JY} J)Z - J(\mathscr{L}_Y J)Z = (\mathscr{L}_{JY} J)Z + (\mathscr{L}_Y J)JZ$, which proves the first equality.

For the second, a straightforward computation shows

$$\lambda(T^{tmp;2}(Y, Z)) = \lambda(T^{tmp;1}(Y, Z)) + (1+c)\langle JY, Z\rangle = (1+c)\, d\lambda(Y, Z)$$

for general c. Substituting $c = 0$, we obtain the second equality. This finishes the proof.

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Recent Progress in Isoparametric Functions and Isoparametric Hypersurfaces

Chao Qian and Zizhou Tang

Abstract This paper gives a survey of recent progress in isoparametric functions and isoparametric hypersurfaces, mainly in two directions.

(1) Isoparametric functions on Riemannian manifolds, including exotic spheres. The existences and non-existences will be considered.
(2) The Yau conjecture on the first eigenvalues of the embedded minimal hypersurfaces in the unit spheres. The history and progress of the Yau conjecture on minimal isoparametric hypersurfaces will be stated.

1 Introduction

E. Cartan was the pioneer who made a comprehensive study of isoparametric functions (hypersurfaces) on the unit spheres. In the past decades, the study of isoparametric functions (hypersurfaces) has became a highly influential field in differential geometry. For a systematic and complete survey of isoparametric functions (hypersurfaces) and their generalizations, we recommend [7, 43] and [10]. Very recently, Cecil-Chi-Jensen, Immervoll and Chi obtained classification results for isoparametric hypersurfaces with four distinct principal curvatures in the unit spheres, except for one case (c.f. [9, 23] and [11]). As for that with six distinct principal curvatures, Miyaoka showed the homogeneity and hence the classification (c.f. [29]).

The note is organized as follows. In Sect. 2, we first recall some basic notations and fundamental theory of isoparametric functions on Riemannian manifolds. Next we introduce exotic spheres and investigate the existences and non-existences of

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isoparametric functions on exotic spheres. Section 3 will be concerned with the progress of the well known Yau conjecture on the first eigenvalues of embedded minimal hypersurfaces in the unit spheres, especially on the minimal isoparametric case (being isoparametric implies embedding). Moreover, the first eigenvalues of the focal submanifolds are also taken into account. In the end, related topics and applications are described in Sect. 4.

2 Exotic Spheres and Isoparametric Functions

We start with definitions. Let N be a connected complete Riemannian manifold. A non-constant smooth function $f : N \to \mathbb{R}$ is called *transnormal* if there is a smooth function $b : \mathbb{R} \to \mathbb{R}$ such that

$$|\nabla f|^2 = b(f), \tag{1}$$

where ∇f is the gradient of f. If moreover there is a continuous function $a : \mathbb{R} \to \mathbb{R}$ such that

$$\triangle f = a(f), \tag{2}$$

where $\triangle f$ is the Laplacian of f, then f is called *isoparametric* (cf. [44]). Each regular level hypersurface is called an *isoparametric hypersurface*. The two equations of the function f mean that regular level hypersurfaces are parallel and have constant mean curvatures. According to Wang [44], a transnormal function f on a complete Riemannian manifold has no critical value in the interior of $\mathrm{Im} f$. The preimage of the maximum (resp. minimum), if it exists, of an isoparametric (or transnormal) function f is called the *focal set* of f, denoted by M_+ (resp. M_-).

Since the work of Cartan [5, 6] and Münzner [33], the subject of isoparametric hypersurfaces in the unit spheres is rather fascinating to geometers. We refer to [8] for the development of this subject. Up to now, the classification has almost been completed as mentioned in Sect. 1.

In general Riemannian manifolds, the classification problem is far from being touched. Wang [44] firstly took up a systematic study of isoparametric functions on general Riemannian manifolds, and similar to the case in a unit sphere, proved or claimed a series of beautiful results. The structural result for transnormal functions is stated as follows.

Theorem 1 ([44]). *Let N be a connected complete Riemannian manifold and f a transnormal function on N. Then*

- *The focal sets of f are smooth submanifolds (may be disconnected) of N;*
- *Each regular level set of f is a tube over either of the focal sets (the dimensions of the fibers may differ on different connected components).*

The above theorem shows that the existence of a transnormal function on a Riemannian manifold N restricts strongly its topology.

In the first part of [17], Ge and Tang improved the fundamental theory of isoparametric functions on Riemannian manifolds. Given a transnormal function $f : N \to \mathbb{R}$, we denote by $C_1(f)$ the set where f attains its global maximum value or global minimum value, by $C_2(f)$ the union of singular level sets of f, *i.e.*, $C_2(f) = \{p \in N | \nabla f(p) = 0\}$, and for any regular value t of f, by $C_3^t(f)$ the focal set of the level hypersurface $M_t := f^{-1}(t)$, *i.e.*, the set of singular values of the normal exponential map. From [44], it follows that $C_1(f) = C_2(f) = M_- \cup M_+$, and $C_3^{t_1}(f) = C_3^{t_2}(f)$ for any two regular level hypersurfaces which will be thus denoted simply by $C_3(f)$. Moreover, one can see that $C_3(f) \subset C_1(f) = C_2(f)$. Then Ge and Tang proved

Theorem 2 ([17]). *Each component of $M_\pm$ has codimension not less than 2 if and only if $C_3(f) = C_1(f) = C_2(f)$. Moreover in this case, each level set M_t is connected. If in addition N is closed and f is isoparametric, then at least one isoparametric hypersurface is minimal in N.*

Indeed, there exists example of an isoparametric function f satisfying $C_3(f) \subsetneq C_1(f) = C_2(f)$ (c.f. [17]). For this case, the focal sets of the isoparametric function are not really focal sets of the level hypersurface. Hence, in [17], a transnormal (isoparametric) function f is called *proper* if the focal sets have codimension not less than 2. It seems that a properly transnormal (isoparametric) function is exactly what we should concern in geometry. Furthermore, in [17], they observed three elegant ways to construct examples of isoparametric functions, i.e.,

- For a Riemannian manifold (N, ds^2) with an isoparametric function f, take a special conformal deformation $\widetilde{ds^2} = e^{2u(f)}ds^2$. Then f is also isoparametric on $(N, \widetilde{ds^2})$;
- For a cohomogeneity one manifold (N, G) with a G-invariant metric, one can get isoparametric functions on N;
- For a Riemannian submersion $\pi : E \to B$ with minimal fibers, if f is an isoparametric function on B, then so is $F := f \circ \pi$ on E.

Applying these methods, interesting results and abundant examples are acquired, especially, isoparametric functions on Brieskorn varieties and on isoparametric hypersurfaces of spheres are obtained.

As a continuation of [17], they made new contributions in [18]. First, for a properly isoparametric function, they proved that at least one isoparametric hypersurface is minimal if the ambient space N is closed in Theorem 2. By using the Riccati equation, they can further show that such a minimal isoparametric hypersurface is also unique if N has positive Ricci curvature. Next, by expressing the shape operator $S(t)$ of M_t as a power series, they gave a complete proof to Theorem D of [44] (no proof there; compare with L. Ni, 1997, Notes on Transnormal Functions on Riemannian Manifolds, unpublished, http://math.ucsd.edu/~lni/academic/isopara.pdf and [28]).

Theorem 3 ([18]). *The focal sets $M_\pm$ of an isoparametric function f on a complete Riemannian manifold N are minimal submanifolds.*

Meanwhile, Ge and Tang also established the following theorem, which is a generalization of the spherical case to general Riemannian manifolds.

Theorem 4 ([18]). *Suppose that each isoparametric hypersurface M_t has constant principal curvatures with respect to the unit normal vector field in the direction of ∇f. Then each of the focal sets $M_\pm$ has common constant principal curvatures in all normal directions,* i.e.*, the eigenvalues of the shape operator are constant and independent of the choices of the point and unit normal vector of $M_\pm$.*

Owning to the rich and beautiful topological and geometric properties of isoparametric functions on Riemannian manifolds, Ge and Tang initiated the study of isoparametric functions on exotic spheres in [17].

Recall that an n-dimensional smooth manifold Σ^n is called an *exotic n-sphere* if it is homeomorphic but not diffeomorphic to S^n. It is J. Milnor [27] who firstly discovered an exotic 7-sphere which is an S^3-bundle over S^4. Later, Kervaire and Milnor [25] computed the group of homotopy spheres in each dimension greater than four which implies that there exist exotic spheres in infinitely many dimensions and in each dimension there are at most finitely many exotic spheres. In particular, ignoring orientation there exist 14 exotic 7-spheres, 10 of which can be exhibited as S^3-bundles over S^4, the so-called *Milnor spheres*. However, in dimension four, the question of whether an exotic 4-sphere exists remains open, which is the so called *smooth Poincaré conjecture* (c.f. [24]).

Since the discovery of exotic spheres by Milnor, a very intriguing problem is to interpret the geometry of them (c.f. [3,20,21,24]). In [17], using isoparametric (even transnormal) functions to attack the smooth Poincaré conjecture in dimension four, Ge and Tang showed the following theorem.

Theorem 5 ([17]). *Suppose Σ^4 is a homotopy 4-sphere and it admits a transnormal function under some metric. Then Σ^4 is diffeomorphic to S^4.*

Note that a *homotopy n-sphere* is a smooth manifold with the same homotopy type as S^n. Freedman [15] showed that any homotopy 4-sphere is homeomorphic to S^4. As a result of this, the above Theorem 5 says equivalently that there exists no transnormal function on any exotic 4-sphere if it exists. In contrast to the non-existence result in dimension four, Ge and Tang also constructed many examples of isoparametric functions on the Milnor spheres. Furthermore, by projecting an S^3-invariant isoparametric function on the symplectic group $Sp(2)$ with a certain left invariant metric, they constructed explicitly a properly transnormal but not an isoparametric function on the Gromoll-Meyer sphere with two points as the focal sets. Inspired by this example, they posed a question that whether there is an isoparametric function on the Gromoll-Meyer sphere or on any exotic n-sphere ($n > 4$) with two points as the focal sets. More generally, they posed the following:

Problem 1 ([17]). Does there always exist a properly isoparametric function on an exotic sphere Σ^n ($n > 4$) with the focal sets being those occurring on S^n?

To answer the Problem 1, Qian and Tang developed a general way to construct metrics and isoparametric functions on a given manifold in [35], which is based on a simple and useful observation that a transnormal function on a complete Riemannian manifold is necessarily a Morse-Bott function (c.f. [44]). As is well known, a Morse-Bott function is a generalization of a Morse function, and it admits critical submanifolds satisfying a certain non-degenerate condition on normal bundles. In [35], the following fundamental construction is given, whose proof depends heavily on Moser's volume element theorem.

Theorem 6 ([35]). *Let N be a closed connected smooth manifold and f a Morse-Bott function on N with the critical set $C(f) = M_+ \sqcup M_-$, where M_+ and M_- are both closed connected submanifolds of codimensions more than 1. Then there exists a metric on N so that f is an isoparametric function.*

It follows from a theorem of S. Smale that

Corollary 1 ([35]). *Every homotopy n-sphere with $n > 4$ admits a metric and an isoparametric function with 2 points as the focal sets.*

Remark 1. Corollary 1 answers partially the above Problem 1.

Moreover, metrics and isoparametric functions on homotopy spheres and on the Eells-Kuiper projective planes can also be constructed so that at least one component of the critical set is not a single point.

In addition to the above existence theorem on homotopy spheres, on the other side, the following non-existence results was also proved.

Theorem 7 ([35]). *Every odd dimensional exotic sphere admits no totally isoparametric functions with 2 points as the focal set.*

Recall that a *totally isoparametric* function is an isoparametric function so that each regular level hypersurface has constant principal curvatures, as defined in [19]. As it is well known, an isoparametric function on a unit sphere must be totally isoparametric.

Remark 2. According to [22] and [37], there exists at leat one exotic Kervaire sphere Σ^{4m+1} which has a cohomogeneity one action. Consequently, Σ^{4m+1} admits a totally isoparametric function f under an invariant metric (c.f. [17]). However, each component of the focal set of f is not just a point, but a smooth submanifold. Hence, the assumption on the focal set in Theorem 7 is essential.

In light of the above Theorem 7, it is reasonable to ask

Problem 2. Does there exist an even dimensional exotic sphere $\Sigma^{2n}(n > 2)$ admitting a metric and a totally isoparametric function with 2 points as the focal set?

In the last section of [35], both existence and non-existence results of isoparametric functions on some homotopy spheres which also have SC^p-property were investigated. A Riemannian manifold has SC^p-property if every geodesic issuing from the point p is closed and has the same length [2]. For some even dimensional homotopy spheres, the existence theorem in [35] improves a beautiful result of Bérard-Bergery [1].

Recently, Tang and Zhang [40] solved a problem of Bérard-Bergery and Besse. That is, they showed that every Eells-Kuiper quaternionic projective plane carries a Riemannian metric with SC^p-property for a certain point p. Thus, it is interesting to know whether there is a metric on every Eells-Kuiper quaternionic projective plane which not only has the SC^p-property, but also admits a certain isoparametric function (c.f. [35]).

3 Yau Conjecture on the First Eigenvalue and Isoparametric Foliations

The Laplace-Beltrami operator is one of the most important operators acting on C^∞ functions on a Riemannian manifold. Over several decades, research on the spectrum of the Laplace-Beltrami operator has always been a core issue in the study of geometry. For instance, the geometry of closed minimal submanifolds in the unit sphere is closely related to the eigenvalue problem.

Let (M^n, g) be an n-dimensional compact connected Riemannian manifold without boundary and Δ be the Laplace-Beltrami operator acting on a C^∞ function f on M by $\Delta f = -\operatorname{div}(\nabla f)$, the negative of divergence of the gradient ∇f. It is well known that Δ is an elliptic operator and has a discrete spectrum

$$\{0 = \lambda_0(M) < \lambda_1(M) \leq \lambda_2(M) \leq \cdots \leq \lambda_k(M), \cdots, \uparrow \infty\}$$

with each eigenvalue repeated a number of times equal to its multiplicity. As usual, we call $\lambda_1(M)$ the first eigenvalue of M. When M^n is a minimal hypersurface in the unit sphere $S^{n+1}(1)$, it follows from Takahashi Theorem that $\lambda_1(M)$ is not greater than n. Consequently, S.T. Yau posed in 1982 the following conjecture:

Yau conjecture ([45]). *The first eigenvalue of every closed embedded minimal hypersurface M^n in the unit sphere $S^{n+1}(1)$ is just n.*

In 1983, Choi and Wang made the most significant breakthrough to this conjecture [12]. To be precise, they showed that the first eigenvalue of every (embedded) closed minimal hypersurface in $S^{n+1}(1)$ is not smaller than $\frac{n}{2}$. Usually, the calculation of the spectrum of the Laplace-Beltrami operator, even of the first eigenvalue, is rather complicated and difficult. Up to now, the Yau conjecture is far from being solved even in dimension two.

It was proved in [30] that if the Yau conjecture is true for the torus of dimension two, then the Lawson conjecture holds, that is to say, the only minimally embedded torus in $S^3(1)$ is the Clifford torus. In fact, the Lawson conjecture has been a challenging problem for more than 40 years, and recently it was solved by S. Brendle (c.f. [4]).

In this note, we pay attention to a little more restricted problem of the Yau conjecture for closed minimal isoparametric hypersurfaces M^n in $S^{n+1}(1)$.

Recall that an isoparametric hypersurface M^n in the unit sphere $S^{n+1}(1)$ must have constant principal curvatures (c.f. [5, 6, 8]). Let ξ be a unit normal vector field along M^n in $S^{n+1}(1)$, g the number of distinct principal curvatures of M, $\cot\theta_\alpha$ ($\alpha = 1, \ldots, g$; $0 < \theta_1 < \cdots < \theta_g < \pi$) the principal curvatures with respect to ξ and m_α the multiplicity of $\cot\theta_\alpha$. Using a brilliant topological method, Münzner (c.f. [33]) proved the remarkable result that the number g must be $1, 2, 3, 4$ or 6; $m_\alpha = m_{\alpha+2}$ (indices mod g); $\theta_\alpha = \theta_1 + \frac{\alpha-1}{g}\pi$ ($\alpha = 1, \ldots, g$) and when g is odd, $m_1 = m_2$.

In order to attack the Yau conjecture, Muto-Ohnita-Urakawa [32] and Kotani [26] made a breakthrough for some minimal homogeneous (automatically isoparametric) hypersurfaces. More precisely, they verified the Yau conjecture for all minimal homogeneous hypersurfaces with $g = 1, 2, 3, 6$. However, when it came to the case $g = 4$, they were only able to deal with the cases $(m_1, m_2) = (2, 2)$ and $(1, k)$.

Furthermore, Muto [31] proved that the Yau conjecture is also true for some families of minimal inhomogeneous isoparametric hypersurfaces with $g = 4$. This remarkable result contains many inhomogeneous isoparametric hypersurfaces. However, there is no result in [31] for isoparametric hypersurfaces with $\min(m_1, m_2) > 10$.

Based on all results mentioned above and the classification of isoparametric hypersurfaces in $S^{n+1}(1)$ (c.f. [9, 11, 13, 23, 29]), Tang and Yan [39] completely solved the Yau conjecture on the minimal isoparametric case by establishing the following

Theorem 8 ([39]). *Let M^n be a closed minimal isoparametric hypersurface in the unit sphere $S^{n+1}(1)$ with $g = 4$ and $m_1, m_2 \geq 2$. Then*

$$\lambda_1(M^n) = n.$$

Remark 3. Theorem 8 depends only on the values of (m_1, m_2). In particular, it covers the unclassified case $g = 4$, $(m_1, m_2) = (7, 8)$.

Remark 4. A purported conjecture of Chern states that a closed, minimally immersed hypersurface in $S^{n+1}(1)$, whose second fundamental form has constant length, is isoparametric (c.f. [16]). If this conjecture is proven, Theorem 8 would have settled the Yau conjecture for the minimal hypersurface whose second fundamental form has constant length, which gives more confidence in the Yau conjecture.

Indeed, the more fascinating part of [39] was to determine the first eigenvalues of the focal submanifolds in $S^{n+1}(1)$. To state their result clearly, let us recall some notations. Given an isoparametric hypersurface M^n in $S^{n+1}(1)$ and a smooth field ξ of unit normals to M, for each $x \in M$ and $\theta \in \mathbb{R}$, we can define $\phi_\theta : M^n \to S^{n+1}(1)$ by

$$\phi_\theta(x) = \cos\theta\, x + \sin\theta\, \xi(x).$$

Clearly, $\phi_\theta(x)$ is the point at an oriented distance θ to M along the normal geodesic through x. If $\theta \neq \theta_\alpha$ for any $\alpha = 1, \ldots, g$, ϕ_θ is a parallel hypersurface to M at an oriented distance θ, which we will denote by M_θ henceforward. If $\theta = \theta_\alpha$ for some $\alpha = 1, \ldots, g$, it is easy to find that for any vector X in the principal distributions $E_\alpha(x) = \{X \in T_xM \mid A_\xi X = \cot\theta_\alpha X\}$, where A_ξ is the shape operator with respect to ξ, $(\phi_\theta)_* X = 0$. In other words, in case $\cot\theta = \cot\theta_\alpha$ is a principal curvature of M, ϕ_θ is not an immersion, which is actually a *focal submanifold* of codimension $m_\alpha + 1$ in $S^{n+1}(1)$.

Münzner asserted that there are only two distinct focal submanifolds in a parallel family of isoparametric hypersurfaces, regardless of the number of distinct principal curvatures of M; and every isoparametric hypersurface is a tube of constant radius over each focal submanifold. Denote by M_1 the focal submanifold in $S^{n+1}(1)$ at an oriented distance θ_1 along ξ from M with codimension $m_1 + 1$, M_2 the focal submanifold in $S^{n+1}(1)$ at an oriented distance $\frac{\pi}{g} - \theta_1$ along $-\xi$ from M with codimension $m_2 + 1$. In virtue of Cartan's identity, one sees that the focal submanifolds M_1 and M_2 are both minimal in $S^{n+1}(1)$ (c.f. [8]).

Another main result of [39] concerning the first eigenvalues of focal submanifolds in the non-stable range, is now stated as follows.

Theorem 9 ([39]). *Let M_1 be the focal submanifold of an isoparametric hypersurface with $g = 4$ in $S^{n+1}(1)$ with codimension $m_1 + 1$. If $\dim M_1 \geq \frac{2}{3}n + 1$, then*

$$\lambda_1(M_1) = \dim M_1$$

with multiplicity $n + 2$. A similar conclusion holds for M_2 under an analogous condition.

We emphasize that the assumption $\dim M_1 \geq \frac{2}{3}n + 1$ in Theorem 9 is essential. For instance, Solomon [36] constructed an eigenfunction on the specific focal submanifolds M_2 of OT-FKM-type (we will explain it immediately), which has $4m$ as an eigenvalue. In some case, $4m$ is less than the dimension of M_2.

As an example, Theorem 9 implies that each focal submanifold of isoparametric hypersurfaces with $g = 4$, $(m_1, m_2) = (7, 8)$ has its dimension as the first eigenvalue.

We need to recall the construction of the isoparametric hypersurfaces of OT-FKM-type. For a symmetric Clifford system $\{P_0, \cdots, P_m\}$ on $\mathbb{R}^{2l}$, *i.e.*, P_i's are symmetric matrices satisfying $P_iP_j + P_jP_i = 2\delta_{ij}I_{2l}$, Ferus, Karcher and Münzner [14] constructed a polynomial F on $\mathbb{R}^{2l}$:

$$F: \quad \mathbb{R}^{2l} \to \mathbb{R}$$
$$F(x) = |x|^4 - 2\sum_{i=0}^{m} \langle P_i x, x\rangle^2.$$

For $f = F|_{S^{2l-1}}$, define $M_1 = f^{-1}(1)$, $M_2 = f^{-1}(-1)$, which have codimensions $m + 1$ and $l - m$ in $S^{n+1}(1)$, respectively.

For focal submanifold M_1 of OT-FKM-type, the only unsettled multiplicities in [39] are $(m_1, m_2) = (1,1), (4,3), (5,2)$. And for the $(4,3)$ case, there exist only one homogeneous and one inhomogeneous examples.

Finally, Tang and Yan [39] proposed the following problem, which could be regarded as an extension of the Yau conjecture.

Problem 3 ([39]). Let M^d be a closed embedded minimal submanifold in the unit sphere $S^{n+1}(1)$. If the dimension d of M^d satisfies $d \geq \frac{2}{3}n + 1$, then

$$\lambda_1(M^d) = d.$$

Later, Tang, Xie and Yan [42] took chance to solve the unsolved cases in [39] and considered the case with g=6. First, by applying the similiar method as in [39], they got the following theorem for the case with $g = 6$ which contains more information than that in [32] and does not depend on the classification result of Miyaoka [29].

Theorem 10 ([42]). *Let M^{12} be a closed minimal isoparametric hypersurface in $S^{13}(1)$ with $g = 6$ and $(m_1, m_2) = (2,2)$. Then*

$$\lambda_1(M^{12}) = 12$$

with multiplicity 14*. Furthermore, the following inequality holds*

$$\lambda_k(M^{12}) > \frac{3}{7}\,\lambda_k(S^{13}(1)), \qquad k = 1, 2, \cdots .$$

And for focal submanifolds M_1 of OT-FKM-Type, they solved two left cases and proved

Theorem 11 ([42]). *For the focal submanifold M_1 of OT-FKM-type in $S^5(1)$ with $(m_1, m_2) = (1,1)$,*

$$\lambda_1(M_1) = \dim M_1 = 3$$

with multiplicity 6*; for the focal submanifold M_1 of homogeneous OT-FKM-type in $S^{15}(1)$ with $(m_1, m_2) = (4,3)$,*

$$\lambda_1(M_1) = \dim M_1 = 10$$

with multiplicity 16*.*

At last, in the case with $g = 6$, by a deep investigation into the shape operator of the focal submanifolds, they obtained estimates on the first eigenvalue. Particularly, for one of the focal submanifolds with $g = 6$, $m_1 = m_2 = 2$, the first eigenvalue is equal to its dimension. It gives an affirmative answer to Problem 3 in this case.

4 Related Topics and Applications

The connection between geometry of Riemannian manifolds with positive scalar curvatures and surgery theory is quite a deep subject which has attracted widely attention. The most important aspect of this field is the original discovery of Gromov-Lawson and of Schoen-Yau.

Motivated by the Schoen-Yau-Gromov-Lawson surgery theory on metrics of positive scalar curvature, Tang, Xie and Yan [41] constructed a double manifold associated with a minimal isoparametric hypersurface in the unit sphere. The resulting double manifold carries a metric of positive scalar curvature and an isoparametric foliation as well. To investigate the topology of the double manifolds, they used topological K-theory and the representation of the Clifford algebra for the OT-FKM-type, and determined completely the isotropy subgroups of singular orbits for homogeneous case. Here we note that, as it is well known, a homogeneous (isoparametric) hypersurface in the unit sphere can be characterized as a principal orbit of isotropy representation of some symmetric space of rank two.

In the last part of this section, we describe an application of isoparametric foliation to Willmore submanifolds. By definition, a Willmore submanifold (in the unit sphere) is the critical point of the Willmore functional. In particular, every minimal surface in the unit sphere is automatically Willmore; in other words, Willmore surfaces are a generalization of minimal surfaces in the unit sphere. However, examples of Willmore submanifolds in the unit sphere are rare in the literature.

Qian, Tang and Yan [34, 38] proved that each focal submanifold of isoparametric hypersurface (not only OT-FKM-type) in the unit sphere with $g = 4$ is a Willmore submanifold. For $g = 1, 2, 3$, the conclusion above is clearly valid. As for $g = 6$, the conclusion should be also true.

Recall that the focal submanifolds are minimal in unit spheres. It is worth noting that an Einstein manifold minimally immersed in the unit sphere is a Willmore submanifold. A natural problem arises: whether the focal submanifolds are Einstein? To this problem with $g = 4$, [38] and [34] gave a complete answer, depending on the classification results. In other words, they dealt with this problem in two cases–homogeneous type and OT-FKM-type.

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Part II
Invited Talks

Information Geometry of Barycenter Map

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Dedicated to Professor Young Jin Suh on his sixtieth birthday

Abstract Using barycenter of the Busemann function we define a map, called the barycenter map from a space $\mathscr{P}^+$ of probability measures on the ideal boundary ∂X to an Hadamard manifold X. We show that the space $\mathscr{P}^+$ carries a fibre space structure over X from a viewpoint of information geometry. Following the idea of [7, 9] and [8] we present moreover a theorem which states that under certain hypotheses of information geometry a homeomorphism Φ of ∂X induces, via the push-forward for probability measures, an isometry of X whose ∂X-extension coincides with Φ.

1 Introduction

The aim of this article is to present the results relating with a map, called the barycenter map that associates a point in an Hadamard manifold (X, g) to any μ in the space $\mathscr{P}^+$ of probability measures on its ideal boundary ∂X, the boundary of X at infinity. This map is, geometrically, equivariant with respect to the isometry action. One of our main results (see Theorem 3, Sect. 3) asserts that this map, denoted by bar enjoys the projection of a fibre space $\mathscr{P}^+ \to X$, under some adequate assumptions on X. Refer to [15], and also to [9] and [7] for the notion of barycenter. For related topics see [5] and [3].

The barycenter is defined as follows. Let $B_\theta(x)$, $x \in X$, $\theta \in \partial X$ be the Busemann function normalized at a point $o \in X$, whose level hypersurface, a horosphere plays an excellently important role in differential geometry, as in [17]. Using a probability measure $\mu \in \mathscr{P}^+$ we define a μ-average Busemann function

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$\mathbf{B}_\mu : X \to \mathbf{R}$. It is shown in Sect. 2 that the function $\mathbf{B}_\mu$ has a unique critical point in X which is a minimizer.

Following the idea of [7,9] and also of [8], together by the aid of the fibre space structure of $\mathscr{P}^+$ we exhibit another main result as Theorem 4, Sect. 4 in which under certain assumptions of information geometry a homeomorphism Φ of ∂X is assured to induce, via the push-forward for probability measures, an isometry of X whose ∂X-extension coincides with Φ.

We would like to state motivation to our study, before giving preliminaries and describing our results.

MOTIVATION. Let (X, g) be an Hadamard manifold, quasi-isometric with a Damek-Ricci space (X_o, g_o). It is commonly known that any isometry of (X_o, g_o) induces a quasi-isometric map of (X, g) and hence a homeomorphism of ∂X. Investigate under which hypotheses does this homeomorphism induce, via the barycenter map, an isometry of (X, g). For a Damek-Ricci space, refer to [2,6] and [13].

The full detailed arguments together with supplementing results will be appeared elsewhere (see [16]).

2 Barycenter and Barycenter Map

The barycenter of a probability measure is defined as a critical point of the μ-average function with respect to the Busemann function on an Hadamard manifold X. From the existence and uniqueness of the barycenter we define a map, called the barycenter map from the space of probability measures to X.

Let (X, g) be an Hadamard manifold, i.e., a simply connected, complete Riemannian manifold of non-positive curvature. From Cartan-Hadamard theorem X is diffeomorphic to $\mathbf{R}^n$, $n = \dim X$.

We introduce the ideal boundary ∂X of X and the Busemann function which are machinery for defining the barycenter.

Let $\mathscr{R}_X$ be the set of all geodesic rays in X. Two rays σ, $\sigma_1 \in \mathscr{R}_X$ are asymptotically equivalent, denoted by $\sigma \sim_a \sigma_1$, if there exists a constant $C > 0$ such that $d(\sigma(t), \sigma_1(t)) < C$ for any $t \geq 0$. The quotient space $\mathscr{R}_X / \sim_a$ is denoted by ∂X, called the ideal boundary of (X, g). We denote by $[\gamma]$ the equivalence class represented by a geodesic ray γ. For any $x \in X$ and $\theta \in \partial X$ there exists uniquely $\gamma \in \mathscr{R}_X$ such that $\gamma(0) = x$, $[\gamma] = \theta$ and we write symbolically $\gamma(t) \to \theta$ as $t \to \infty$.

For an arbitrary, fixed point $o \in X$, ∂X is identified with $U_o X = \{v \in T_o X \mid |v| = 1\}$; $v \mapsto [\sigma_v]$, where $\sigma_v(t) = \exp_o tv$ is the geodesic ray; $\partial X \cong U_o X$. Via this identification we equip ∂X with a topology from $U_o X$ so that the space $X \cup \partial X$, the compactification of X, carries a natural topology called the cone topology.

To define probability measures on ∂X we denote by $d\theta$ a measure on ∂X induced from the normalized canonical measure on $U_o X \cong S^{n-1}(1)$ so $\int_{\partial X} d\theta = 1$. Let $\mathscr{P}^+ = \mathscr{P}^+(\partial X, d\theta)$ be the space of probability measures on ∂X being absolutely continuous with respect to $d\theta$ and having positive continuous density function. So, $\mu \in \mathscr{P}^+(\partial X, d\theta)$ is written as $\mu(\theta) = f(\theta)d\theta$, $f \in C^0$, $f(\theta) > 0$ for any $\theta \in \partial X$. $\mathscr{P}^+(\partial X, d\theta)$ is equipped with a topology by embedding $\mathscr{P}^+(\partial X, d\theta)$ into $L^2(\partial X, d\theta)$. Note the space $\mathscr{P}^+(\partial X, d\theta)$ is path-connected, since any measure on a path between μ, μ_1; $(1-t)\mu + t\mu_1$, $0 \le t \le 1$, belongs also to $\mathscr{P}^+(\partial X, d\theta)$.

To develop geometrical argument we need to define a C^1-curve in $\mathscr{P}^+(\partial X, d\theta)$ and then tangent vectors to $\mathscr{P}^+(\partial X, d\theta)$. A C^1-curve is a one-parameter family $\{\mu_t = f(\theta, t)d\theta \mid t \in I\}$, where I is an open interval (so we have a map : $I \to \mathscr{P}^+(\partial X, d\theta)$; $t \mapsto \mu_t$), whose density function $f = f(\theta, t)$ is of C^0 with respect to $\theta \in \partial X$ and of C^1 with respect to $t \in I$. A velocity vector $\frac{d\mu}{dt}(t) = \frac{\partial f}{\partial t}(\theta, t)d\theta$ of the curve at $t \in I$ suggests a tangent vector space at μ to the space $\mathscr{P}^+(\partial X, d\theta)$ as

$$T_\mu \mathscr{P}^+(\partial X, d\theta) = \{\nu = h(\theta)d\theta \mid h \in C^0(\partial X), \int_{\partial X} d\nu = 0\}. \tag{1}$$

To define the barycenter map from the space $\mathscr{P}^+(\partial X, d\theta)$ to X we begin with definition of the Busemann function as follows.

Definition 1. A function on X

$$B_\theta(x) = \lim_{t \to \infty} \{d(x, \gamma(t)) - t\}, \quad x \in X$$

is called the *Busemann* function associated to $\theta \in \partial X$, normalized at a reference point o, where $\gamma(t)$ denotes the geodesic starting o and tending to θ.

Example 1. For a real hyperbolic space $X = \mathbf{R}H^n$ of constant curvature -1 modelled on a unit ball $\mathbf{B}^n = \{y \in \mathbf{R}^n \mid |y| < 1\}$ with $\partial X \cong \mathbf{S}^{n-1}(1)$, normalized at the origin o

$$B_\theta(x) = \log \frac{|x-\theta|^2}{1-|x|^2}, \quad x \in X, \ \theta \in \partial X \cong \mathbf{S}^{n-1}(1).$$

Note 1. The *Busemann cocycle formula*

$$B_\theta(\varphi x) = B_{\hat{\varphi}^{-1}\theta}(x) + B_\theta(\varphi o), \qquad \forall\, (x, \theta) \in X \times \partial X$$

holds with respect to an isometry φ of (X, g) (see [12, p. 208]). Here, $\hat{\varphi}$ denotes the ∂X-extension of φ, a homeomorphism of ∂X, defined by $\hat{\varphi}([\gamma]) := [\varphi \circ \gamma]$.

Definition 2. Let $\mu \in \mathscr{P}^+(\partial X, d\theta)$. Then $y \in X$ is called a *barycenter* of μ, if y is a critical point of the function $\mathbf{B}_\mu : X \to \mathbf{R}$, defined by

$$\mathbf{B}_\mu(x) = \int_{\partial X} B_\theta(x) d\mu(\theta), \tag{2}$$

i.e., if the following is satisfied at y

$$(d\mathbf{B}_\mu)_y(U) = \int_{\partial X} (dB_\theta)_y(U)\, d\mu(\theta) = 0, \quad \forall U \in T_yX. \tag{3}$$

Note 2. (i) $\mathbf{B}_\mu(\cdot)$ is normalized at o ; $\mathbf{B}_\mu(o) = 0$, since $B_\theta(o) = 0$. (ii) $\mathbf{B}_\mu(\cdot)$ is Lipschitz continuous and convex. (iii) Restricted to any geodesic γ, $\mathbf{B}_\mu$ is of C^1 from $|\nabla B_\theta(\cdot)| = 1$. (iv) Moreover, the second order differentiability of $\mathbf{B}_\mu$ along a geodesic is assured, under a certain uniform boundedness assumption of ∇dB_θ and in such a case one has $\frac{d^2}{dt^2}\,\mathbf{B}_\mu(\gamma(t)) = \int_{\partial X} \nabla dB_\theta(\gamma'(t), \gamma'(t))\, d\mu(\theta)$.

Theorem 1 ([B-C-G-1]). *Assume that (X, g) satisfies the axiom of visibility and the Busemann function $B_\theta(\cdot)$ is of C^0 with respect to any $\theta \in \partial X$. Then, any $\mu \in \mathscr{P}^+(\partial X, d\theta)$ admits a barycenter.*

For a proof see [7, Appendice A] in which the authors showed that $\mathbf{B}_\mu$ takes a minimal point. Here an Hadamard manifold satisfies the axiom of visibility, if any two distinct points θ, θ' of ∂X can be joined by a geodesic lying in X (see [10]).

Theorem 2. *Assume that for some $\mu_o \in \mathscr{P}^+(\partial X, d\theta)$ the μ_o-average of Hessian*

$$\mathbf{Q}_{\mu_o}(\cdot,\cdot) = \int_{\theta \in \partial X} (\nabla dB_\theta)_x(\cdot,\cdot)\, d\mu_o(\theta) \tag{4}$$

is strictly positive definite on T_xX at any point x. Then, the uniqueness of barycenter is guaranteed for any $\mu \in \mathscr{P}^+(\partial X, d\theta)$.

Theorem 2, a generalization of the uniqueness theorem given in [7, Appendice A] is shown as follows. Suppose that $\mu_o = f_o(\theta)d\theta$ admits at least two points $y, y_1 \in X$ as it barycenters. Then, along a geodesic joining them there exists a point between them at which the value of $\frac{d^2}{dt^2}\mathbf{B}_{\mu_o}(\gamma(t))$ is non-positive. However, $\mathbf{Q}_{\mu_o}(\cdot,\cdot) > 0$ is assumed, so the uniqueness follows.

The positive definiteness of the quadratic form $\mathbf{Q}_\mu$ is assured for any $\mu = f(\theta)d\theta$ and at any point x. In fact, we observe $\mathbf{Q}_\mu(U,U) \geq \frac{c_\mu}{C_{\mu_o}}\mathbf{Q}_{\mu_o}(U,U)$ for any $U \in T_xX$, where $c_\mu = \inf_{\partial X} f(\theta) > 0$ and $C_{\mu_o} = \sup_{\partial X} f_o(\theta) > 0$.

Under those assumptions of Theorems 1 and 2, we are able to define a map

$$bar\ :\ \mathscr{P}^+(\partial X, d\theta) \to X;\ \ \mu \mapsto y,$$

called the *barycenter map*, where a point y is a barycenter of μ.

3 Fibre Space Structure of $\mathscr{P}^+(\partial X, d\theta)$

The barycenter map is regarded as a fibre space projection from the space $\mathscr{P}^+(\partial X, d\theta)$ to X, provided (X, g) carries a Poisson kernel of special type. The Poisson kernel induces a cross section of this fibre space.

Definition 3 ([20, §2, II]). A function $P(x,\theta)$ of $(x,\theta) \in X \times \partial X$ is called *Poisson kernel*, normalized at o, if

(i) $P(x,\theta)$ is of C^∞ with respect to $x \in X$ and of C^0 with respect to $\theta \in \partial X$, and further gives the fundamental solution of the Dirichlet problem at ∂X; $\Delta u = 0$ on X and $u|_{\partial X} = f$ for a given $f \in C^0(\partial X)$ so the solution u is described as

$$u(x) = \int_{\theta \in \partial X} P(x,\theta) f(\theta) d\theta \tag{5}$$

(ii) $P(x,\theta) > 0$ for any (x,θ) and $P(o,\theta) = 1$ for any θ, and

(iii) the measure $P(x,\theta)d\theta \in \mathscr{P}^+(\partial X, d\theta)$ converges to the Dirac measure δ_{θ_0}, when x tends to θ_0 of ∂X in the cone topology; $\lim_{x\to\theta_o} \int_{\partial X} P(x,\theta)h(\theta)d\theta = h(\theta_o)$ for any function h on ∂X.

For later use, we set $\mu_x = P(x,\theta)d\theta$. To get a finer investigation of the map bar we require (X,g) to admit a Poisson kernel of special type, namely a Poisson kernel represented by the Busemann function as

$$P(x,\theta) = \exp\{-Q\ B_\theta(x)\}, \tag{6}$$

($Q = Q(X) > 0$ is the volume entropy of X) called *Busemann-Poisson kernel*, occasionally.

Example 2. The real hyperbolic space $\mathbf{R}H^n$ of curvature -1 carries Busemann-Poisson kernel as the Poisson kernel is represented by

$$P(x,\theta) = \left(\frac{1-|x|^2}{|x-\theta|^2}\right)^{n-1}$$

with $Q = n-1$.

Note 3. Damek-Ricci spaces including the real hyperbolic space and other rank one symmetric spaces of non-compact type carry Busemann-Poisson kernel. See for this [13] and [7].

Proposition 1. *Suppose that an Hadamard manifold (X,g) is equipped with Busemann-Poisson kernel. Then, (X,g) is necessarily asymptotically harmonic (i.e., $\Delta B_\theta \equiv -Q$ for any θ) and satisfies the axiom of visibility. Furthermore the Busemann function B_θ on X turns out to be of C^∞ with respect to $x \in X$ and of C^0 with respect to $\theta \in \partial X$. Therefore, every $\mu \in \mathscr{P}^+(\partial X, d\theta)$ admits a barycenter.*

In fact, the assumption of Proposition 1 implies from (iii) of Definition 3, $\lim_{x\to\theta'\neq\theta} B_\theta(x) = \infty$. So, from [4, Lemma 4.14, Section 4], the axiom of visibility follows. Further one finds from the harmonicity of $P(x,\theta)$ that $\Delta B_\theta(x) = -Q$ for any $(x,\theta) \in X \times \partial X$, which means the asymptotical harmonicity of (X,g) (see [19]).

In what follows, we assume that (X, g) admits a Busemann-Poisson kernel, unless otherwise stated.

Proposition 2. *(i) $bar(\mu_x) = x$ for $\mu_x = P(x,\theta)d\theta$, $x \in X$, that is, any point $x \in X$ is the barycenter of $\mu_x = P(x,\theta)d\theta$.*
(ii) Any $\mu \in \mathscr{P}^+(\partial X, d\theta)$ admits a unique barycenter point.

(i) is derived directly from $\int_{\partial X} d\mu_x = \int P(x,\theta)d\theta = 1$ for any x. (ii) is verified from Theorem 2 by checking that for $\mu_y \in \mathscr{P}^+(\partial X, d\theta)$, $y \in X$, the quadratic form $\mathbf{Q}_{\mu_y}(\cdot,\cdot) = \int_{\partial X} \nabla dB_\theta(\cdot,\cdot)\, d\mu_y(\theta)$ is positive definite at any point $x \in X$. Indeed, we have

$$(\mathbf{Q}_{\mu_y})_x(U,U) \geq \frac{c_y}{C_x}(\mathbf{Q}_{\mu_x})_x(U,U), \quad U \in T_xX, \tag{7}$$

where $c_y = \inf_\theta P(y,\theta) > 0$ and $C_x = \sup_\theta P(x,\theta) > 0$ and moreover

$$\begin{aligned}(\mathbf{Q}_{\mu_x})_x(U,U) &= \int_{\partial X} (\nabla dB_\theta)_x(U,U)P(x,\theta)d\theta \\ &= Q\int_{\partial X} \langle(\nabla B_\theta)_x, U\rangle^2 P(x,\theta)d\theta > 0\end{aligned} \tag{8}$$

for $U \neq 0$. The last equality is shown by taking a geodesic $\gamma(t)$ and differentiating twice the following equality

$$\int_{\partial X} P(\gamma(t),\theta)d\theta = 1.$$

Associated to the Busemann-Poisson kernel we define a map

$$\Theta : X \to \mathscr{P}^+(\partial X, d\theta);\ x \mapsto \mu_x = P(x,\theta)d\theta, \tag{9}$$

called *Poisson kernel map*. From (i), Proposition 2, we see

$$bar \circ \Theta = id_X, \tag{10}$$

so $bar : \mathscr{P}^+(\partial X, d\theta) \to X$ is surjective.

The differential map $d\Theta_x : T_xX \to T_{\mu_x}\mathscr{P}^+(\partial X, d\theta)$ is injective and represented by

$$d\Theta_x(U) = -Q\ (dB_\theta)_x(U)\ P(x,\theta)d\theta = -Q\ (dB_\theta)_x(U)\ \mu_x, \quad U \in T_xX. \tag{11}$$

Theorem 3. *The space $\mathscr{P}^+(\partial X, d\theta)$ enjoys a fibre space structure over an Hadamard manifold (X, g) whose fibre over $x \in X$ is $bar^{-1}(x) = \{\mu \mid bar(\mu)=x\}$. Further the map $\Theta : X \to \mathscr{P}^+(\partial X, d\theta)$ yields a cross section of this fibration.*

This theorem is a consequence of the following arguments.

For each path-connected subset $bar^{-1}(x) \subset \mathscr{P}^+(\partial X, d\theta)$, $x \in X$ the tangent space $T_\mu bar^{-1}(x)$ to $bar^{-1}(x)$ can be characterized as follows.

Lemma 1. *Let $\nu \in T_\mu \mathscr{P}^+(\partial X, d\theta)$. Then, ν belongs to $T_\mu bar^{-1}(x)$ if and only if ν satisfies $\int_{\theta \in \partial X} (dB_\theta)_x(U)\, d\nu(\theta) = 0$ for any $U \in T_x X$.*

If $\mu = \mu_x$, then we have

Proposition 3. *Any tangent vector $\nu \in T_{\mu_x} bar^{-1}(x)$ at $\mu_x = P(x,\theta)d\theta$ fulfills*

$$G_{\mu_x}(\,(d\Theta)_x(U),\, \nu) = 0, \quad \forall U \in T_x X$$

with respect to the Fisher information metric G on the space $\mathscr{P}^+(\partial X, d\theta)$. Therefore, at $\mu_x = P(x,\theta)d\theta \in \mathscr{P}^+(\partial X, d\theta)$ the tangent space $T_{\mu_x}\mathscr{P}^+(\partial X, d\theta)$ enjoys an orthogonal direct sum decomposition as

$$T_{\mu_x}\mathscr{P}^+(\partial X, d\theta) = d\Theta_x(T_x X) \oplus T_{\mu_x} bar^{-1}(x) \tag{12}$$

with respect to the metric G (see for definition of G [1, 14, 18] and [11]).

Remark 1. An orthogonal decomposition similar to (12) also holds at any $\mu \in \mathscr{P}^+(\partial X, d\theta)$ with respect to the metric G. Refer to [16] for the detailed argument.

4 Barycentrically Associated Maps

Now we will introduce a map from X into X, barycentrically associated to a homeomorphism $\Phi : \partial X \to \partial X$, via the barycenter map.

Definition 4. Let $\Phi : \partial X \to \partial X$ be a homeomorphism of ∂X. Then, a map $\varphi : X \to X$ is called *barycentrically associated* to Φ, when φ satisfies the relation $bar \circ \Phi_\sharp = \varphi \circ bar$ in the diagram

$$\begin{array}{ccc} \mathscr{P}^+(\partial X, d\theta) & \xrightarrow{\Phi_\sharp} & \mathscr{P}^+(\partial X, d\theta) \\ \downarrow bar & & \downarrow bar \\ X & \xrightarrow{\varphi} & X \end{array} \tag{13}$$

Here, the map $\Phi_\sharp : \mathscr{P}^+(\partial X, d\theta) \to \mathscr{P}^+(\partial X, d\theta)$ is the push-forward of a homeomorphism, more generally, of a measurable map $\Phi : \partial X \to \partial X$, defined by

$$\int_{\theta \in \partial X} h(\theta)\, d[\Phi_\sharp \mu](\theta) = \int_{\theta \in \partial X} (h \circ \Phi)(\theta)\, d\mu(\theta) \tag{14}$$

for any function $h = h(\theta)$ on ∂X (see [21, p. 4]).

Note 4 (Equivariant action). Let φ be an isometry of (X, g) and $\hat{\varphi}$ the bijective map (homeomorphism) : $\partial X \to \partial X$ induced from φ. Then φ is a map barycentrically associated to $\hat{\varphi}$, due to the equivariance formula

$$bar(\hat{\varphi}_\sharp \mu) = \varphi(bar(\mu)), \quad \forall \mu \in \mathscr{P}^+(\partial X, d\theta). \tag{15}$$

Remark 2. In [8] the authors utilize the barycenter to assert the Mostow rigidity of hyperbolic manifolds. In fact, let $f : \partial X \to \partial X$ be a certain map, where X is a compact smooth quotient of $\mathbf{R}H^n$, $n \geq 3$. Then, there exists a map $F : X \to X$; $F(y) = bar(f_* \sigma_y)$, $y \in X$ associated to the map f and they assert that $F : X \to X$ is an isometry by using Schwarz's inequality lemma. Here σ_y is the Patterson-Sullivan measure.

Note 5. Barycentrically associated maps obey the composition rule [16].

Theorem 4. *Let $\varphi : X \to X$ be a map barycentrically associated to a homeomorphism $\Phi : \partial X \to \partial X$. Assume that φ is of C^1 and satisfies*

$$\Theta \circ \varphi = \Phi_\sharp \circ \Theta, \tag{16}$$

that is, the diagram

$$\begin{array}{ccc} \mathscr{P}^+(\partial X, d\theta) & \xrightarrow{\Phi_\sharp} & \mathscr{P}^+(\partial X, d\theta) \\ \uparrow \Theta & & \uparrow \Theta \\ X & \xrightarrow{\varphi} & X \end{array} \tag{17}$$

is commutative. Then φ is an isometry of (X, g).

In general, the measures $\Phi_\sharp \Theta(x)$ and $\Theta(\varphi(x))$ have necessarily the same barycenter $\varphi(x)$.

For a proof it suffices to show that the differential map $d\varphi$ is a linear isometry. By the way, the diagram (17) asserts

$$\Theta(\varphi x) \;=\; \Phi_\sharp(\Theta(x)), \quad \forall x \in X \tag{18}$$

and hence

$$\mu_{\varphi x} = \Phi_\sharp(\mu_x), \quad \forall x \in X \tag{19}$$

which means as probability measures

$$P(\varphi x, \theta) d\theta = \Phi_\sharp(P(x, \theta) d\theta) = P(x, \Phi^{-1}\theta) \Phi_\sharp d\theta. \tag{20}$$

Here, we notice that for a measure $\mu = f(\theta) d\theta$ its push-forward measure $\Phi_\sharp \mu$ has the form $\Phi_\sharp \mu = f(\Phi^{-1}\theta)\ \Phi_\sharp d\theta$.

Let $\gamma = \gamma(t)$ be a geodesic such that $\gamma(0) = x$ and $\gamma'(0) = U \in T_x X$. Differentiate (18) along γ at $t = 0$ we obtain

$$(d\Theta)_{\varphi x}(d\varphi_x U) = \Phi_\sharp\left((d\Theta)_x(U)\right), \quad \forall U \in T_x X,\ \forall x \in X, \tag{21}$$

and hence

$$(dB_\theta)_{\varphi x}(d\varphi_x U) = (dB_{\Phi^{-1}\theta})_x(U), \quad \forall U \in T_x X,\ \forall x \in X, \tag{22}$$

so that

$$\langle d\varphi_x^*(\nabla B_\theta)_{\varphi x}, U\rangle_x = \langle(\nabla B_{\Phi^{-1}\theta})_x, U\rangle_x,$$

for any U in a gradient field form, where $d\varphi_x^* : T_{\varphi x}X \to T_x X$ is the formal adjoint of $d\varphi_x$. Then, the gradient vector fields ∇B_θ and $\nabla B_{\Phi^{-1}\theta}$ must satisfy

$$d\varphi_x^*(\nabla B_\theta)_{\varphi x} = (\nabla B_{\Phi^{-1}\theta})_x, \quad \forall x \in X,\ \forall \theta \in \partial X. \tag{23}$$

Now take an arbitrary unit vector $V \in T_{\varphi x}X$. Then we have $V = (\nabla B_\theta)_{\varphi x}$ for some θ. Then from the above equation

$$|d\varphi_x^* V| = |d\varphi_x^*(\nabla B_\theta)_{\varphi x}| = |(\nabla B_{\Phi^{-1}\theta})_x| = 1,$$

where we used $|\nabla B_\theta| = 1$ for any $\theta \in \partial X$. This holds for any unit vector so $d\varphi_x^*$ and hence $d\varphi_x : T_x X \to T_{\varphi x}X$ is a linear isometry. Since x is arbitrary, $\varphi : X \to X$ is an isometry of (X, g), i.e., $\varphi^* g = g$.

Remark 3. (23) is an infinitesimal version of the Busemann cocycle formula of note 1.

It is seen that Φ is the extension to ∂X of the above isometry φ with respect to the cone topology; $\Phi = \hat{\varphi}$ so that we obtain a homeomorphism $\varphi \cup \Phi : X \cup \partial X \to X \cup \partial X$.

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Stability of Complete Minimal Lagrangian Submanifold and L^2 Harmonic 1-Forms

Reiko Miyaoka and Satoshi Ueki

Abstract We show that a non-compact complete stable minimal Lagrangian submanifold L in a Kähler manifold with positive Ricci curvature has no non-trivial L^2 harmonic 1-forms, which gives a topological and conformal constraint on L.

1 Introduction

In this paper, all manifolds are complete and oriented. It is well-known that there exist no compact stable minimal hypersurfaces in a Riemannian manifold with positive Ricci curvature [14]. In general, stable minimal submanifolds hardly exist in a positively curved manifold. On the stability of minimal Lagrangian submanifold in a Kähler manifold, we know:

Fact 1 ([5]). *A compact or compact with boundary minimal Lagrangian submanifold in a Kähler manifold with non-positive Ricci curvature is stable.*

Fact 2 ([9]). *A compact (Lagrangian-)stable minimal Lagrangian submanifold L in a Kähler manifold with positive Ricci curvature satisfies $H^1(L, \mathbb{R}) = 0$.*

Fact 2 suggests that there is a constraint on the topology of compact stable minimal Lagrangian submanifolds (Lagragian-stable is weaker than stable). However, both facts do not mention the complete non-compact case. The purpose of this paper is to investigate complete non-compact stable minimal Lagrangian submanifolds in a Kähler manifold with positive Ricci curvature. We obtain:

Theorem 1. *There exist no non-trivial L^2 harmonic 1-forms on a non-compact complete stable minimal Lagrangian submanifold in a Kähler manifold M with positive Ricci curvature.*

By Dodziuk, it is shown:

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Fact 3 ([2]). *When there are no non-trivial L^2 harmonic 1-forms on a complete non-compact Riemannian manifold N, any codimension one cycle of N disconnects N.*

Using this in the surface case, we obtain:

Theorem 2. *Let L be a complete stable minimal Lagrangian surface in a Kähler manifold (M, g) with positive Ricci curvature. Then L is conformally $S^2 \setminus \{s \text{ points}\} \setminus \{l \text{ disks}\}$, where one of the following occurs:*

(1) L is conformally S^2 ($s = l = 0$).
(2) L is conformally $\mathbb{C}$ ($s = 1$, $l = 0$), and there is no $b > 0$ such that $\overline{\mathrm{Ric}} \geq bg$.
(3) $s \geq 3$ and $l = 0$.
(4) $s \geq 1$ and $l = 1$.

Remark. We conjecture that the cases (3) and (4) do not occur.

Example 1. (1) Castro and Urbano [1] show that the diagonal S^2 in $Q_2(\mathbb{C}) = S^2 \times S^2$ is the unique stable minimal Lagrangian surface in $Q_2(\mathbb{C})$.
(2) Consider a complete positive metric on $\mathbb{C}$ (e.g., the induced metric on the paraboloid in $\mathbb{R}^3$) and take $M = \mathbb{C} \times \mathbb{C}$, so that M is a Kähler manifold with positive Ricci curvature. Then the diagonal set $\triangle_M$ is a complex submanifold, and hence volume minimizing by Wirtinger's inequality. Changing the complex and symplectic structure J and ω in the second term by $-J$ and $-\omega$, we see that $\triangle_M$ is a minimal Lagrangian submanifold which has the same volume as before, and hence is stable.
(3) In $M = \mathbb{C}P^1 \times \mathbb{C}$ with natural metric ($\overline{\mathrm{Ric}} \geq 0$), the standard embedding of $S^1 \times \mathbb{R}$ is a totally geodesic Lagrangian surface with two parabolic ends. This is intuitively unstable, since S^1 could be shortened.

Here we remark that a non-compact complete Riemann surface N is classified in two ways; geometrically and function theoretically. In the former sense, if the universal covering of N is $\mathbb{C}$, we say N is parabolic, and if the universal covering is a complex disk $\mathbb{D}$, we say N is hyperbolic.

On the other hand, any dimensional non-compact complete Riemannian manifold N is called parabolic if any non-positive subharmonic function on N is constant, and nonparabolic (or hyperbolic) otherwise.

We call an unbounded component $\mathscr{E}$ of the complement of a sufficiently large compact subset of N an "end". Then $\mathscr{E}$ is called parabolic if there exists a parabolic manifold of whose only end is $\mathscr{E}$. Otherwise, $\mathscr{E}$ is called nonparabolic.

Theorem 1 is inspired by the following fact and is proved in [15]:

Fact 4 ([8, 10]). *Let M be a Riemannian manifold with non-negative sectional curvature and N be a complete non-compact stable minimal hypersurface in M. Then there exist no non-trivial L^2 harmonic 1-forms on N. When $\dim M = 3$, the curvature condition is weakened to non-negative scalar curvature.*

Using this, the first author [8] gives a partial proof to the results of Fischer-Colbrie and Scheon [3] to the effect that a complete orientable stable minimal surface in a Riemannian manifold M with non-negative scalar curvature are topologically S^2, T^2, $\mathbb{C}$ or $\mathbb{C} \setminus \{0\}$. Each case is realized in certain M.

The unique compact stable minimal submanifolds in $\mathbb{C}P^n$ are complex submanifolds [4]. Thus minimal Lagrangian submanifolds in $\mathbb{C}P^n$ are never stable, and the stability in this space should be argued in a weak sense, namely, the Hamiltonian stability (H-stability, for short), see [9]. The standard embeddings of T^n and $\mathbb{R}P^n$ in $\mathbb{C}P^n$ are H-stable [9]. For more results, see [7]. Concerning this, B. Palmer shows:

Fact 5. (1) [11, Section 3] *The Gauss image in $Q_2(\mathbb{C})$ of a minimal surface in S^3 is minimal Lagrangian, and the only H-stable one is S^2 if L is compact.*
(2) [12] *If a non-compact complete minimal Lagrangian surface in a Kähler manifold (M, g) with $\overline{\mathrm{Ric}} \geq bg$, $b > 0$, is H-stable, then the number of nonparabolic ends is less than two.*

Then Palmer conjectures:

Conjecture ([12]). A non-compact complete minimal Lagrangian surface in a Kähler manifold (M, g) with $\overline{\mathrm{Ric}} \geq bg$ for some $b > 0$ is *not* H-stable.

In this paper, we investigate the classical stability, and will discuss the H-stability in a separate paper.

2 Proof of Theorem 1

Let M be a real $2n$-dimensional Kähler manifold, and L be a minimal Lagrangian submanifold of M. We denote the Kähler form, the complex structure and the Kähler metric on M by ω, J and $\langle\, , \rangle$, respectively. We denote the connection, the curvature tensor and the Ricci tensor of M by $\bar{\nabla}$, $\bar{R}$ and $\overline{\mathrm{Ric}}$ respectively. We denote those of L without bar and the normal connection on L by $\nabla^{\perp}$. We adopt $\triangle = d\delta + \delta d$ as the definition of the Laplacian on L.

There is a natural correspondence between $\Lambda^1(L)$ and $\Gamma(T^{\perp}L)$ as follows: For $\alpha \in \Lambda^1(L)$, there exists $\xi \in \Gamma(T^{\perp}L)$ such that $\alpha(X) = \omega(\xi, X)$, $X \in TL$, and for $\xi \in \Gamma(T^{\perp}L)$, there exists $\alpha_{\xi} \in \Lambda^1(L)$ such that $\alpha_{\xi}(X) = \omega(\xi, X)$. Note that $\|\alpha_{\xi}\| = \|\xi\|$ holds.

We consider a deformation $\{\iota_t\}$ of L, namely a smooth family of immersions which satisfies $\iota_0 = \iota$:

$$\iota_t : L \to M,\ t \in (-\varepsilon, \varepsilon).$$

In the following, we assume that the support of the deformation $\{\iota_t\}$ is compact and that the variation vector field

$$V_t := \frac{d}{dt}\iota_t$$

is normal to L. Let $\mathscr{A}_V(t) := \mathrm{Vol}(\iota_t(L))$. Then L is said to be *minimal* if the first variation $\mathscr{A}'_V(0) = (d/dt)|_{t=0}\mathscr{A}_V(t)$ vanishes for any compactly supported normal deformation $\{\iota_t\}$ of L. A minimal submanifold L is said to be *stable minimal* if the second variation $\mathscr{A}''_V(0) = (d^2/dt^2)|_{t=0}\mathscr{A}_V(t)$ is non-negative for any compactly supported normal deformation.

Fact 6 ([9]). *Let M be a Kähler manifold and L be a minimal Lagrangian submanifold in M. Then the second variation formula of L w.r.t. a compactly supported normal variation V is given by*

$$\mathscr{A}''_V(0) = \int_L \{\langle \triangle\alpha_V, \alpha_V\rangle - \overline{\mathrm{Ric}}(V, V)\}.$$

When L is compact, Fact 1 and 2 immediately follows from this second variation formula and the Hodge theory.

When L is non-compact, we need the following fact on L^2 harmonic 1-forms.

Fact 7 ([13]). *Let α be an L^2 form on a Riemannian manifold L. Then α is harmonic, i.e., $\triangle\alpha = 0$, if and only if $d\alpha = 0$ and $\delta\alpha = 0$.*

Moreover, in order to obtain a variation vector field with compact support, we need a cut-off function. For any $r > 0$, we choose a function $f = f_r : L \to [0, 1]$ with the following properties:

(1) f is continuous on L and smooth almost everywhere on L,
(2) $f = 1$ on $B_{r/2}$, $f = 0$ outside B_r,
(3) $\|df\|^2 \leq \dfrac{c}{r^2}$,

where B_r is a geodesic ball with radius r in L centered at a fixed point $p \in L$, and c is a constant independent of r. Such function is easily obtained by using the distance function.

Proof of Theorem 1. Let α be an L^2 harmonic 1-form on L, namely a smooth harmonic 1-form with

$$\int_L \|\alpha\|^2 < \infty.$$

We show that α must be trivial. We use ξ such that $\alpha = \omega(\xi,\ )$, the cut-off function f, and a variation vector field $V = f\xi$ with compact support. Putting $\alpha_V = \omega(V,\ ) = \omega(f\xi,\ ) = f\alpha$, we obtain from Fact 6

$$\begin{aligned}
\mathscr{A}''_V(0) &= \int_L \{\langle \triangle\alpha_V, \alpha_V\rangle - \overline{\mathrm{Ric}}(V, V)\}\\
&= \int_L \{\langle (d\delta + \delta d)(f\alpha), f\alpha\rangle - \overline{\mathrm{Ric}}(V, V)\}\\
&= \int_L \{\|d(f\alpha)\|^2 + (\delta(f\alpha))^2 - \overline{\mathrm{Ric}}(V, V)\}
\end{aligned}$$

since f has a compact support. For the first term, we have

$$\|d(f\alpha)\|^2 = \|df \wedge \alpha\|^2 = \|df\|^2\|\alpha\|^2 - \langle df, \alpha\rangle^2$$

since $d\alpha = 0$ holds by Fact 7. On the other hand, we have

$$\begin{aligned}(\delta(f\alpha)) &= - * d * (f\alpha) \\ &= - * (df \wedge *\alpha) - f * d * \alpha \\ &= -\langle df, \alpha\rangle\end{aligned}$$

since $\delta\alpha = 0$ holds by Fact 7. Combining these, we obtain

$$\begin{aligned}\mathscr{A}_V''(0) &= \int_L \{\|d(f\alpha)\|^2 + (\delta(f\alpha))^2 - \overline{\mathrm{Ric}}(V, V)\} \\ &= \int_L \{\|df\|^2\|\xi\|^2 - f^2\overline{\mathrm{Ric}}(\xi, \xi)\}.\end{aligned}$$

By the properties of the cut-off function f and the stability of L, we have

$$0 \leq \mathscr{A}_V''(0) \leq \frac{c}{r^2}\int_L \|\xi\|^2 - \int_{B_{r/2}} \overline{\mathrm{Ric}}(\xi, \xi).$$

Letting $r \to \infty$, we obtain

$$0 \leq -\int_L \overline{\mathrm{Ric}}(\xi, \xi)$$

since ξ is L^2. Thus $\xi = 0$, namely, $\alpha = 0$ follows. □

3 Surface case

Proof of Theorem 2. When L is a surface, L cannot have positive genus by Theorem 1 and Fact 3. Hence L is conformally $S^2 \setminus \{s \text{ points}\} \setminus \{l \text{ disks}\}$. By Example (1) and (2), $L \cong S^2$ and $\mathbb{C}$ can be stable minimal Lagrangian in certain M.

(2) When $L \cong \mathbb{C}$, we have a harmonic 1-form $\alpha = dx$, where $z = x + iy$ is a complex coordinate of $\mathbb{C}$. Applying the cut-off function f as before to ξ such that $\alpha = \omega(\xi, \ )$, and using that the Dirichlet integral is a conformal invariant, we obtain

$$0 \leq \mathscr{A}_V''(0) \leq \frac{c}{r^2}\int_{B_r \setminus B_{r/2}} \|\xi\|^2 - \int_{B_{r/2}} \overline{\mathrm{Ric}}(\xi, \xi) = \frac{c}{r^2}\frac{3\pi r^2}{4} - \int_{B_{r/2}} \overline{\mathrm{Ric}}(\xi, \xi).$$

If $\overline{\mathrm{Ric}} \geq bg$, $b > 0$, the last term tends to $-b\int_L \|\xi\|^2$, which diverges since there are no L^2 harmonic 1-form on $L \cong \mathbb{C}$. Thus $\overline{\mathrm{Ric}}$ cannot be uniformly positive.

(3) When the universal covering of L is $\mathbb{C}$, L is either $\mathbb{C}$ or $\mathbb{C} \setminus \{0\}$. We show the latter is not stable. In fact, consider the holomorphic function

$$\varphi(z) = \frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

on L. Then

$$\alpha = \frac{x}{x^2 + y^2} dx + \frac{y}{x^2 + y^2} dx$$

is a harmonic 1-form on L. Since the L^2 norm depends only on the conformal structure, we obtain

$$\|\alpha\|_2^2 = \lim_{r \to \infty} \int_{1/r}^{r} \int_0^{2\pi} \|\alpha\|^2 r dr d\theta = 4\pi \lim_{r \to \infty} \log r.$$

The metric on L can be written as $ds^2 = \mu(p)|dz|^2$ where z is the complex parameter. Let f be the cut-off function as before w.r.t. the geodesic ball B_R around a point other than the origin, where we use the flat metric. Hence $R = r - a$ for some constant a. Since the Dirichlet integral depends only on the conformal structure, putting $V = f\xi, \alpha = \omega(\xi, \ )$ as before, we obtain

$$\mathscr{A}_V''(0) \leq \frac{4\pi c}{R^2} \log r - \int_{B_{R/2}} \overline{\mathrm{Ric}}(\xi, \xi).$$

The first term on the right hand side tends to 0 as $R \to \infty$, and the second term tends to $\int_L \overline{\mathrm{Ric}}(\xi, \xi)$. Therefore, $L \cong \mathbb{C} \setminus \{0\}$ cannot be stable.

(4) When the universal covering of L is the disk, either $L \cong S^2 \setminus \{s \text{ points}\}$ where $s \geq 3$, or L has at least one nonparabolic end ($l \geq 1$). Since $\mathbb{D}$ has many L^2 harmonic 1-forms, $L \cong \mathbb{D}$ does not occur. Thus L has more than one ends.

When L has at least two nonparabolic ends ($l \geq 2$), Li-Tam prove that there exists a non-constant bounded harmonic function h with finite Dirichlet integral [6, Theorem 2.1]. Thus $\alpha = dh$ is an L^2 harmonic 1-form on L, and L cannot be stable by Theorem 1. □

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Bonnesen-Style Symmetric Mixed Isoperimetric Inequality

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Abstract For convex domains K_i $(i = 0, 1)$ (compact convex sets with non-empty interiors) in the Euclidean plane R^2. Denote by A_i and P_i areas and circum-perimeters, respectively. The symmetric mixed isoperimetric deficit is $\Delta(K_0, K_1) := P_0^2 P_1^2 - 16\pi^2 A_0 A_1$. In this paper, we give some Bonnesen-style symmetric mixed inequalities, that is, inequalities of the form $\Delta(K_0, K_1) \geq B_{K_0,K_1}$, where B_{K_0,K_1} is a non-negative invariant of geometric significance and vanishes if and only if both K_0 and K_1 are discs. We also obtain some reverse Bonnesen-style symmetric mixed inequalities. Those inequalities are natural generalizations of known geometric inequalities, such as the known classical isoperimetric inequality.

Keywords Convex domain • Containment measure • Symmetric mixed isoperimetric deficit • Symmetric mixed isoperimetric inequality • Bonnesen-style symmetric mixed inequality

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1 Introductions and Preliminaries

Let R^n denote the Euclidean space. A set of points K in R^n is said to be convex if for all $x, y \in K$ and scalar $\lambda (0 \leq \lambda \leq 1)$, $\lambda x + (1-\lambda)y \in K$. The convex hull K^* of K is the intersection of all convex sets that contain K. The **Minkowski sum** of convex sets K and L is defined by

$$K + L = \{x + y : \; x \in K, \; y \in L\},$$

and for $\lambda \geq 0$ the **scalar product (dilation)** of convex set K is defined by

$$\lambda K = \{\lambda x : \; x \in K\}.$$

A **homothety** (translation and a dilation) $h_K(x, \lambda)$ of a convex set K is of the form

$$h_K(x, \lambda) = x + \lambda K; \quad x \in R^n, \quad \lambda > 0. \tag{1}$$

A convex domain is a compact convex set with nonempty interiors.

In integral geometry, one may be interested in the **strong containment problem**: Given convex domains K_0 and K_1, is there a translation x so that $x + \alpha K_1 \subset K_0$ or $x + \alpha K_1 \supset K_0$ for any rotation α. That is, for two domains K_0 and K_1, is there a homothety $h_{K_1}(x, \alpha)$ such that $h_{K_1}(x, \alpha) \subset K_0$ or $h_{K_1}(x, \alpha) \supset K_0$. It should be noted that this containment problem is much stronger than the traditional Hadwiger's one [14, 15, 25]. Therefore the strong containment measure could lead to more general and fundamental geometric inequalities (cf. [14, 15, 25–41]).

The known isoperimetric problem says that the ball encloses the maximum volume among all convex domains of fixed surface area in R^n. Especially, Let Γ be a simple closed curve of length P that encloses a domain K of area A in the Euclidean plane R^2. Then

$$P^2 - 4\pi A \geq 0, \tag{2}$$

where the equality holds if and only if Γ is a circle.

Analytic proofs of (2) root back to centuries ago. For simplified and beautiful proofs that lead to generalizations of the discrete case, higher dimensions, the surface of constant curvature and applications to other branches of mathematics, see [1, 3, 7–14, 16–34, 37–39] for references.

The isoperimetric deficit $\Delta(K) = P^2 - 4\pi A$ measures the deficit of the domain K of area A and perimeter P, and a disc of radius $P/2\pi$. During the 1920s, Bonnesen initiated a series of inequalities of the form

$$\Delta(K) = P^2 - 4\pi A \geq B_K, \tag{3}$$

where the quantity B_K is a non-negative invariant of geometric significance and vanishes only when K is a disc.

Many B_Ks are found during the past. The main interest is still focusing on those unknown invariants of geometric significance. See references [2, 4–7, 19, 20, 23, 25] for more details. The following known Bonnesen's isoperimetric inequality is typical.

Proposition 1. *Let K be a domain of area A and bounded by a simple closed curve of perimeter P in R^2. Let r and R be, respectively, the maximum inscribed radius and minimum circumscribed radius of K. Then*

$$P^2 - 4\pi A \geq \pi^2 (R - r)^2, \tag{4}$$

where the equality holds if and only if K is a disc.

In this paper, we investigate a stronger containment problem: For two domains K_0 and K_1, is there a homothety $h_{K_1}(x, \alpha)$ such that $h_{K_1}(x, \alpha) \subset K_0$ or $h_{K_1}(x, \alpha) \supset K_0$? Then we investigate the mixed isoperimetric deficit $\Delta(K_0, K_1) := P_0^2 P_1^2 - 16\pi^2 A_0 A_1$ of domains K_0 and K_1.

Since the convex hull K^* of a set K in R^2 decreases the circum perimeter and increases the area, we have

$$\Delta(K_0, K_1) = P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \geq P_0^{*2} P_1^{*2} - 16\pi^2 A_0^* A_1^* = \Delta(K_0^*, K_1^*).$$

Therefore we can only consider the convex domains when we estimate the low bound of the symmetric mixed isoperimetric deficit.

By the kinematic formulas of Poincaré and Blaschke in integral geometry, we obtain a sufficient condition for convex domain K_1 to contain, or to be contained in, another convex domain K_0. Via the sufficient condition we are able to obtain the symmetric mixed isoperimetric inequality and some Bonnesen style mixed inequalities. One immediate consequence of our results is the strengthening Bonnesen isoperimetric inequality. These new Bonnesen style symmetric mixed inequalities obtained are fundamental and generalize many known Bonnesen style inequalities. The idea and methods could be applied to the higher dimensional case and the surface of constant curvature.

2 The Containment Measure

Let K_i $(i = 0, 1)$ be simple connected domains of areas A_i with circum perimeters P_i in R^2. Let dg denote the kinematic density of the group G_2 of rigid motions, that is, translations and rotations, in R^2. Let K_1 be convex and tK_1 $(t \in (0, +\infty))$ be a dilation of K_1, then we have the following known kinematic formula of Poincaré (cf. [23, 25])

$$\int_{\{g \in G_2 : \partial K_0 \cap t\partial(gK_1) \neq \emptyset\}} n\{\partial K_0 \cap t\partial(gK_1)\} dg = 4tP_0 P_1, \tag{5}$$

where $n\{\partial K_0 \cap t\partial(gK_1)\}$ denotes the number of points of intersection $\partial K_0 \cap t\partial(gK_1)$.

Let $\chi(K_0 \cap t(gK_1))$ be the Euler-Poincaré characteristics of the intersection $K_0 \cap t(gK_1)$. Then we have the Blaschke's kinematic formula (cf. [23, 25]):

$$\int_{\{g\in G_2: K_0\cap t(gK_1)\neq\emptyset\}} \chi(K_0 \cap t(gK_1))dg = 2\pi(t^2 A_1 + A_0) + tP_0P_1. \tag{6}$$

Since domains K_i $(i = 0, 1)$ are assumed to be connected and simply connected and bounded by simple curves, we have $\chi(K_0 \cap t(gK_1)) = n(g) =$ the number of connected components of the intersection $K_0 \cap t(gK_1)$. The fundamental kinematic formula of Blaschke (6) can be rewritten as

$$\int_{\{g\in G_2: K_0\cap t(gK_1)\neq\emptyset\}} n(g)dg = 2\pi(t^2 A_1 + A_0) + tP_0P_1. \tag{7}$$

Denote by μ the set of all positions of K_1 in which either $t(gK_1) \subset K_0$ or $t(gK_1) \supset K_0$, then the above formula of Blaschke can be rewritten as

$$\int_\mu dg + \int_{\{g\in G_2: \partial K_0\cap t\partial(gK_1)\neq\emptyset\}} n(g)dg = 2\pi(t^2 A_1 + A_0) + tP_0P_1. \tag{8}$$

When $\partial K_0 \cap t\partial(gK_1) \neq \emptyset$, each component of $K_0 \cap t(gK_1)$ is bounded by at least an arc of ∂K_0 and an arc of $t\partial(gK_1)$. Therefore $n(g) \leq n\{\partial K_0 \cap t\partial(gK_1)\}/2$. Then by the formula of Poincaré (5) and the formula of Blaschke (8), we obtain

$$\int_\mu dg \geq 2\pi(t^2 A_1 + A_0) - tP_0P_1. \tag{9}$$

Therefore this inequality immediately lead to a solution of the containment problem (cf. [14, 15, 23, 25, 36, 38, 40–46]):

Containment problem: Let K_i $(i = 0, 1)$ be two domains of areas A_i with simple boundaries of perimeters P_i in R^2. Let K_1 be convex. A sufficient condition for tK_1 to contain, or to be contained in, another domain K_0 for a translation and any rotation, is

$$2\pi A_1 t^2 - P_0 P_1 t + 2\pi A_0 > 0. \tag{10}$$

3 The Symmetric Mixed Isoperimetric Inequality

Let $r_{01} = \max\{t : t(gK_1) \subseteq K_0,\ g \in G_2\}$, the maximum inscribed radius of K_0 with respect to K_1, and $R_{01} = \min\{t : t(gK_1) \supseteq K_0,\ g \in G_2\}$, the minimum circum scribed radius of K_0 with respect to K_1. Note that r_{01}, R_{01} are, respectively, the maximum inscribed radius, the minimum circum radius of K_0 when K_1 is the unit disc. It is obvious that $r_{01} \leq R_{01}$. Therefore for $t \in [r_{01}, R_{01}]$ neither tK_1 contains K_0 nor is contained in K_0. Then the inequality (10) leads to:

Theorem 1. *Let K_i $(i = 0, 1)$ be two convex domains with areas A_i and circum perimeters P_i. Then*

$$2\pi A_1 t^2 - P_0 P_1 t + 2\pi A_0 \leq 0; \quad r_{01} \leq t \leq R_{01}. \tag{11}$$

Theorem 2. *Let K_i $(i = 0, 1)$ be two convex domains in the Euclidean plane R^2 with areas A_k and perimeters P_k. Then*

$$P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \geq 4\pi^2 A_1^2 (R_{01} - r_{01})^2 + [2\pi A_1 (R_{01} + r_{01}) - P_0 P_1]^2, \tag{12}$$

where the equality holds if and only if $r_{01} = R_{01}$, that is, K_0 and K_1 are discs.

Proof. Two special cases of (11) are

$$2\pi A_1 r_{01}^2 - P_0 P_1 r_{01} + 2\pi A_0 \leq 0; \quad 2\pi A_1 R_{01}^2 - P_0 P_1 R_{01} + 2\pi A_0 \leq 0, \tag{13}$$

that is,

$$-8\pi^2 A_0 A_1 \geq 8\pi^2 A_1^2 r_{01}^2 - 4\pi A_1 r_{01} P_0 P_1; \quad -8\pi^2 A_0 A_1 \geq 8\pi^2 A_1^2 R_{01}^2 - 4\pi A_1 R_{01} P_0 P_1.$$

Then

$$\begin{aligned} P_0^2 P_1^2 - 16\pi^2 A_0 A_1 &\geq P_0^2 P_1^2 + 8\pi^2 A_1^2 r_{01}^2 + 8\pi^2 A_1^2 R_{01}^2 - 4\pi A_1 r_{01} P_0 P_1 - 4\pi A_1 R_{01} P_0 P_1 \\ &= 4\pi^2 A_1^2 (R_{01} - r_{01})^2 + (2\pi A_1 r_{01} + 2\pi A_1 R_{01} - P_0 P_1)^2. \end{aligned}$$ □

If K_1 is a disc, then we immediately have the following strengthening Bonnesen isoperimetric inequality:

Corollary 1. *Let K be a domain of area A and bounded by a simple closed curve of length P in R^2. Let r and R be, respectively, the inscribed radius and circumscribed radius of K, then*

$$P^2 - 4\pi A \geq \pi^2 (R - r)^2 + [\pi (R + r) - P]^2, \tag{14}$$

where the equality holds if and only if K is a disc.

Another immediate consequence is the following Kotlyar's inequality (cf. [16, 25]):

Corollary 2 (Kotlyar). *Let K_i $(i = 0, 1)$ be two domains in R^2 with areas A_i and perimeters P_i, then*

$$P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \geq 4\pi^2 A_1^2 (R_{01} - r_{01})^2, \tag{15}$$

where the equality holds if and only if both K_0 and K_1 are discs.

One immediate outcome of is the following **symmetric mixed isoperimetric inequality**:

$$P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \geq 0,$$

where the equality holds if and only if K_0 and K_1 are discs.

We now consider the following **Bonnesen style symmetric mixed inequality**:

$$P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \geq B_{K_0,K_1}, \tag{16}$$

where B_{K_0,K_1} is an invariant of significance (of K_0 and K_1) and hopefully vanishes when both K_0 and K_1 are discs.

The inequality (11) can be rewritten as the following several inequalities:

$$\begin{array}{l} P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \geq (P_0 P_1 - 4\pi A_1 t)^2; \\ P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \geq \left(P_0 P_1 - \frac{4\pi A_0}{t}\right)^2; \qquad r_{01} \leq t \leq R_{01}, \\ P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \geq 4\pi^2 \left(\frac{A_0}{t} - A_1 t\right)^2. \end{array} \tag{17}$$

The following Bonnesen style symmetric mixed inequalities can be proved by similar ways as above.

Theorem 3. *Let $K_i \ (i = 0, 1)$ be two convex domains in the Euclidean plane R^2 with areas A_i and perimeters P_i. Then for $r_{01} \leq t \leq R_{01}$, we have*

$$P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \geq 4\pi^2 A_1^2 (R_{01} - t)^2 + [2\pi A_1 (t + R_{01}) - P_0 P_1]^2;$$

$$P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \geq 4\pi^2 A_1^2 (t - r_{01})^2 + [2\pi A_1 (r_{01} + t) - P_0 P_1]^2;$$

$$\begin{array}{l} P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \geq (P_0 P_1 - 4\pi A_1 r_{01})^2; \\ P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \geq \left(\frac{4\pi A_0}{r_{01}} - P_0 P_1\right)^2; \\ P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \geq 4\pi^2 \left(\frac{A_0}{r_{01}} - A_1 r_{01}\right)^2; \\ P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \geq (P_0 P_1 - 4\pi A_1 t)^2; \\ P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \geq \left(P_0 P_1 - \frac{4\pi A_0}{t}\right)^2; \\ P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \geq 4\pi^2 \left(\frac{A_0}{t} - A_1 t\right)^2; \\ P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \geq (4\pi A_1 R_{01} - P_0 P_1)^2; \\ P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \geq \left(P_0 P_1 - \frac{4\pi A_0}{R_{01}}\right)^2; \\ P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \geq 4\pi^2 \left(A_1 R_{01} - \frac{A_0}{R_{01}}\right)^2. \end{array} \tag{18}$$

Each inequality holds as an equality if and only if both K_0 and K_1 are discs.

Let K_1 be a unit disc. Then immediate consequences of Bonnesen-style symmetric mixed inequalities (18) are following known Bonnesen-style inequalities (cf. [7, 12, 19, 20, 23, 25, 40, 44]):

Corollary 3. *Let K be a plane domain of area A and bounded by a simple closed curve of length P. Let r and R be, respectively, the in-radius and out-radius of K. Then for any disc of radius t, $r \leq t \leq R$, we have the following Bonnesen-type inequalities:*

$$P^2 - 4\pi A \geq (P - 2\pi t)^2;$$

$$P^2 - 4\pi A \geq \pi^2(t-r)^2 + [\pi(t+r) - P]^2;$$

$$P^2 - 4\pi A \geq \pi^2(R-t)^2 + [\pi(R+t) - P]^2;$$

$$\begin{array}{ll} P^2 - 4\pi A \geq \left(P - \frac{2A}{t}\right)^2; & P^2 - 4\pi A \geq \left(\frac{A}{t} - \pi t\right)^2; \\ P^2 - 4\pi A \geq A^2\left(\frac{1}{r} - \frac{1}{R}\right)^2; & P^2 - 4\pi A \geq P^2\left(\frac{R-r}{R+r}\right)^2; \\ P^2 - 4\pi A \geq A^2\left(\frac{1}{r} - \frac{1}{t}\right)^2; & P^2 - 4\pi A \geq P^2\left(\frac{t-r}{t+r}\right)^2; \\ P^2 - 4\pi A \geq A^2\left(\frac{1}{t} - \frac{1}{R}\right)^2; & P^2 - 4\pi A \geq P^2\left(\frac{R-t}{R+t}\right)^2. \end{array} \tag{19}$$

Each equality holds if and only if K is a disc.

4 Reverse Bonnesen Style Symmetric Mixed Inequalities

On the other hand, we may be interested in the so called reverse Bonnesen style symmetric mixed inequalities, that is, inequalities of the form:

$$P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \leq U_{K_0,K_1}, \tag{20}$$

and hopefully the inequality holds as an equality if and only if both K_0 and K_1 are discs.

This question has interested many mathematicians for a while and we were not aware of any such reverse Bonnesen style symmetric mixed inequality until works in [36] and [33]. To obtain the upper limit U_{K_0,K_1} of the symmetric mixed isoperimetric deficit $\Delta_2(K_0, K_1)$ of two convex domains K_0 and K_1, we need the following lemma [44]:

Lemma 1. *Let K_i $(i = 0, 1)$ be convex domains of areas A_i and perimeters P_i. Let R_1 and r_1 be, respectively, the radius of the minimum circumscribed disc and radius of the maximum inscribed disc of K_1. Let $r_{01} = \max\{t : t(gK_1) \subseteq K_0,\ g \in G_2\}$, the maximum inscribed radius of K_0 with respect to K_1, and $R_{01} = \min\{t : t(gK_1) \supseteq K_0,\ g \in G_2\}$, the minimum circumscribed radius of K_0 with respect to K_1. Then we have*

$$r_{01} r_1^2 \leq \frac{4A_0A_1}{P_0P_1} \leq \sqrt{\frac{A_0A_1}{\pi^2}} \leq \frac{P_0P_1}{4\pi^2} \leq R_{01}R_1^2. \tag{21}$$

The first equality sign holds when both K_0 and K_1 are two discs, each other equality sign holds if and only if both K_0 and K_1 are two discs.

Proof. By the symmetric mixed isoperimetric inequality

$$P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \geq 0,$$

we have

$$\frac{4A_0 A_1}{P_0 P_1} \leq \sqrt{\frac{A_0 A_1}{\pi^2}} \leq \frac{P_0 P_1}{4\pi^2},$$

and hence

$$\frac{4A_0 A_1}{P_0 P_1} = \frac{2A_0}{P_0}\frac{2A_1}{P_1} = \frac{1}{r_{01}}\frac{2r_{01}^2 A_0}{r_{01} P_0}\frac{2A_1}{P_1} \geq r_{01} r_1^2,$$

and

$$\frac{P_0 P_1}{4\pi^2} = \frac{P_0}{2\pi}\frac{P_1}{2\pi} = \frac{1}{R_{01}}\frac{R_{01} P_0}{2\pi}\frac{P_1}{2\pi} \leq R_{01} R_1^2.$$

□

Via inequalities

$$r_{01} r_1^2 \leq \frac{4A_0 A_1}{P_0 P_1} \leq \frac{P_0 P_1}{4\pi^2} \leq R_{01} R_1^2,$$

we obtain

$$P_0^2 P_1^2 - 16\pi^2 A_1 A_1 \leq 4\pi^2 P_0 P_1 (r_{01} R_1^2 - r_{01} r_1^2).$$

Then we obtain the following reverse Bonnesen-style symmetric mixed inequality:

Theorem 4. *Let K_i $(i = 0, 1)$ be convex domains of areas A_i and perimeters P_i. Let R_1 and r_1 be, respectively, the radius of the minimum circumscribed disc and radius of the maximum inscribed disc of K_1. Let $r_{01} = \max\{t : t(gK_1) \subseteq K_0,\ g \in G_2\}$, the maximum inscribed radius of K_0 with respect to K_1, and $R_{01} = \min\{t : t(gK_1) \supseteq K_0,\ g \in G_2\}$, the minimum circumscribed radius of K_0 with respect to K_1. Then*

$$P_0^2 P_1^2 - 16\pi^2 A_0 A_1 \leq 4\pi^2 P_0 P_1 (R_{01} R_1^2 - r_{01} r_1^2), \tag{22}$$

with equality if and only if K_0 and K_1 are discs.

On the other hand, by inequalities

$$r_{01}r_1^2 \leq \frac{4A_0A_1}{P_0P_1} \leq \sqrt{\frac{A_0A_1}{\pi^2}} \leq R_{01}R_1^2,$$

we obtain

$$P_o^2P_1^2 - 16\pi^2A_0A_1 \leq \frac{\pi^2P_0^2P_1^2}{A_0A_1}(R_{01}^2R_1^4 - r_{01}^2r_1^4). \tag{23}$$

The following inequalities

$$r_{01}r_1^2 \leq \sqrt{\frac{A_0A_1}{\pi^2}} \leq \frac{P_0P_1}{4\pi^2} \leq R_{01}R_1^2,$$

lead to

$$P_0^2P_1^2 - 16\pi^2A_0A_1 \leq 16\pi^4(R_{01}^2R_1^4 - r_{01}^2r_1^4). \tag{24}$$

Then we obtain the following reverse Bonnesen-style symmetric mixed inequalities:

Theorem 5. *Let K_i $(i = 0, 1)$ be convex domains of areas A_i and perimeters P_i. Let R_1 and r_1 be, respectively, the radius of the minimum circumscribed disc and radius of the maximum inscribed disc of K_1. Let r_{01} and R_{01} be, respectively, the maximum inscribed radius and the minimum circumscribed radius of K_0 with respect to K_1. Then*

$$P_0^2P_1^2 - 16\pi^2A_0A_1 \leq \frac{\pi^2P_0^2P_1^2}{A_0A_1}(R_{01}^2R_1^4 - r_{01}^2r_1^4); \tag{25}$$

$$P_0^2P_1^2 - 16\pi^2A_0A_1 \leq 16\pi^4(R_{01}^2R_1^4 - r_{01}^2r_1^4). \tag{26}$$

Each inequality holds as an equality if and only if K_0 and K_1 are discs.

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Cho Operators on Real Hypersurfaces in Complex Projective Space

Juan de Dios Pérez and Young Jin Suh

Abstract Let M be a real hypersurface in complex projective space. On M we have the Levi-Civita connection and for any nonzero constant k the corresponding generalized Tanaka-Webster connection. For such a k and any vector field X tangent to M we can define from both connections the kth Cho operator $F_X^{(k)}$. We study commutativity properties of these operators with the shape operator and the structure Jacobi operator on M obtaining some characterizations of either Type (A) real hypersurfaces or ruled real hypersurfaces.

1 Introduction

Let $\mathbb{C}P^m$, $m \geq 2$, be a *complex projective space* endowed with the metric g of constant holomorphic sectional curvature 4. Let M be a *connected real hypersurface* of $\mathbb{C}P^m$ without boundary. Let ∇ be the Levi-Civita connection on M and J the complex structure of $\mathbb{C}P^m$. Take a locally defined unit normal vector field N on M and denote by $\xi = -JN$. This is a tangent vector field to M called the structure vector field on M (or the Reeb vector field or the Hopf vector field). On M there exists an almost contact metric structure (ϕ, ξ, η, g) induced by the Kaehlerian structure of $\mathbb{C}P^m$ given in the following way: For any vector field X tangent to M we write $JX = \phi X + \eta(X)N$, where ϕX denotes the tangential component of JX. Then we have

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \qquad (1)$$

for any tangent vectors X, Y to M. From (1) we obtain

$$\phi\xi = 0, \quad \eta(X) = g(X, \xi). \qquad (2)$$

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From the parallelism of J we get

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \tag{3}$$

and

$$\nabla_X \xi = \phi AX \tag{4}$$

for any X, Y tangent to M, where A denotes the shape operator with respect to ξ of the immersion. As the ambient space has holomorphic sectional curvature 4, the equations of Gauss and Codazzi are given, respectively, by

$$\begin{aligned} R(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ -2g(\phi X, Y)\phi Z + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{5}$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \tag{6}$$

for any tangent vectors X, Y, Z to M, where R is the curvature tensor of M. We will call the maximal holomorphic distribution $\mathbb{D}$ on M to the following one: at any $p \in M, \mathbb{D}(p) = \{X \in T_pM | g(X, \xi) = 0\}$.

If $X \in \mathbb{D}$, we will call $\mathbb{D}_X = \{Z \in TM | \eta(Z) = g(X, Z) = g(\phi X, Z) = 0\}$.

We will say that M is Hopf if ξ is principal, that is, $A\xi = \alpha\xi$ for a certain function α on M. On a Hopf real hypersurface we have the following result due to Y. Maeda, [8],

Theorem 1.1. *If ξ is a principal curvature vector with corresponding principal curvature α and $X \in \mathbb{D}$ is principal with principal curvature λ, then ϕX is principal with principal curvature $\frac{\alpha\lambda+2}{2\lambda-\alpha}$.*

The classification of homogeneous real hypersurfaces in $\mathbb{C}P^m$ was obtained by Takagi, see [4, 12–14]. His classification contains six types of real hypersurfaces. Among them we find type (A_1) real hypersurfaces that are geodesic hyperspheres of radius r, $0 < r < \frac{\pi}{2}$ and type (A_2) real hypersurfaces that are tubes of radius r, $0 < r < \frac{\pi}{2}$, over totally geodesic complex projective spaces $\mathbb{C}P^n$, $0 < n < m - 1$. We will call both types of real hypersurfaces type (A) real hypersurfaces. These ones are the hypersurfaces with richest geometry. A characterization of type (A) real hypersurfaces is the following one due to Okumura, [9].

Theorem 1.2. *Let M be a real hypersurface of $\mathbb{C}P^m$, $m \geq 2$. Then the following are equivalent:*

1. *M is locally congruent to either a geodesic hypersphere or a tube of radius r, $0 < r < \frac{\pi}{2}$ over a totally geodesic $\mathbb{C}P^n$, $0 < n < m - 1$.*
2. *$\phi A = A\phi$.*

Ruled real hypersurfaces in $\mathbb{C}P^m$ can be described as follows: Take a regular curve γ in $\mathbb{C}P^m$ with tangent vector field X. At each point of γ there is a unique $\mathbb{C}P^{m-1}$ cutting γ so as to be orthogonal not only to X but also to JX. The union of these hyperplanes is called a ruled real hypersurface. It will be an embedded hypersurface locally, although globally it will in general have self-intersections and singularities. Equivalently, a ruled real hypersurface satisfies that the maximal holomorphic distribution on M, $\mathbb{D}$, is integrable, or $g(A\mathbb{D}, \mathbb{D}) = 0$. For examples of ruled real hypersurfaces see [5] or [7].

The Jacobi operator R_X with respect to a unit vector field X is defined by $R_X = R(., X)X$. R_X is a self-adjoint endomorphism of the tangent space. It is related to Jacobi vector fields which are solutions of the second order differential equation called the Jacobi equation $\nabla_{\dot\gamma}(\nabla_{\dot\gamma}Y) + R(Y, \dot\gamma)\dot\gamma = 0$ along a geodesic γ in M. The Jacobi operator with respect to the structure vector field ξ, R_ξ, is called the structure Jacobi operator on M. By (5) it is given by

$$R_\xi(X) = X - \eta(X)\xi + \eta(A\xi)AX - g(AX, \xi)A\xi \tag{7}$$

for any X tangent to M.

The Tanaka-Webster connection, [15, 17], is the canonical affine connection defined on a non-degenerate, pseudo-Hermitian CR-manifold. As a generalization of this connection, Tanno, [16], defined the generalized Tanaka-Webster connection for contact metric manifolds by

$$\hat\nabla_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi Y. \tag{8}$$

Let k be a nonzero real number. Using the naturally extended affine connection of Tanno's generalized Tanaka-Webster connection, Cho defined the kth g-Tanaka-Webster connection $\hat\nabla^{(k)}$ for a real hypersurface M in $\mathbb{C}P^m$, see [2, 3], by

$$\hat\nabla^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{9}$$

for any X,Y tangent to M. Then $\hat\nabla^{(k)}\eta = 0$, $\hat\nabla^{(k)}\xi = 0$, $\hat\nabla^{(k)}g = 0$, $\hat\nabla^{(k)}\phi = 0$. In particular, if the shape operator of a real hypersurface satisfies $\phi A + A\phi = 2k\phi$, the kth g-Tanaka-Webster connection coincides with the Tanaka-Webster connection.

Now we can consider the tensor field of type (1,2) given by the difference of both connections $F^{(k)}(X, Y) = g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y$, for any X, Y tangent to M, see [6, Proposition 7.10, pp. 234–235]. We will call this tensor the kth Cho tensor on M. Associated to it, for any X tangent to M and any nonnull real number k we can consider the tensor field of type (1,1) $F^{(k)}_X$, given by $F^{(k)}_X Y = F^{(k)}(X, Y)$ for any $Y \in TM$. This operator will be called the kth Cho operator corresponding to X. The torsion of the connection $\hat\nabla^{(k)}$ is given by $\hat T^{(k)}(X, Y) = F^{(k)}_X Y - F^{(k)}_Y X$ for any X, Y tangent to M.

Notice that if $X \in \mathbb{D}$, $F^{(k)}_X$ does not depend on k. In this case we will write simply F_X for $F^{(k)}_X$.

Consider any tensor T of type (1,1) on M. We can study when the covariant and the kth g-Tanaka-Webster derivatives of T coincide, that is, $\nabla T = \hat{\nabla}^{(k)} T$. This is equivalent to the fact that, for any X tangent to M, $TF_X^{(k)} = F_X^{(k)} T$. The meaning of this condition is that every eigenspace of T is preserved by the kth Cho operator $F_X^{(k)}$, for any X tangent to M.

On the other hand, as $TM = Span\{\xi\} \oplus \mathbb{D}$ we can weak the above condition to the cases $X = \xi$ or $X \in \mathbb{D}$.

In [11] we have studied such commutativity conditions in the case $T = A$ and in [10] in the case $T = R_\xi$. In this paper we will present a survey of the results obtained in both papers.

2 Case of Shape Operator

Suppose that $F_X A = AF_X$ for any $X \in \mathbb{D}$. Then we have

$$g(\phi AX, AY)\xi - \eta(AY)\phi AX = g(\phi AX, Y)A\xi - \eta(Y)A\phi AX \tag{10}$$

for any $Y \in TM$. Suppose first that M is non Hopf. Therefore we can write locally $A\xi = \alpha\xi + \beta U$, where U is a unit vector field in $\mathbb{D}$, α and β are functions on M and $\beta \neq 0$.

Taking $Y = \xi$, respectively $Y = U$ or $X \in \mathbb{D}_U$ in (10) we have $A\phi U = 0$, $AU = \beta\xi$, $AX = 0$ for any $X \in \mathbb{D}_U$. This yields M is locally congruent to a ruled real hypersurface.

If now M is Hopf, $A\xi = \alpha\xi$, let $X \in \mathbb{D}$ such that $AX = \lambda X$. From (10) we have two possibilities

1. $\lambda = 0$. From Codazzi equation for such an X, $Y = \xi$ and ϕX, $Y = \xi$ we get

$$1 = g(\nabla_\xi X, A\phi X) = -1 \tag{11}$$

   which gives a contradiction.
2. $\lambda \neq 0$. Then $A\phi X = \alpha\phi X$. In this case Theorem 1.1 and (10) yield M should be totally umbilical. As this is impossible Hopf real hypersurfaces do not satisfy our condition.

Moreover it is easy to see that ruled real hypersurfaces satisfy (10) and we have

Theorem 2.1. *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$. Then $F_X A = AF_X$ for any $X \in \mathbb{D}$ if and only if M is locally congruent to a ruled real hypersurface.*

Let us now suppose that $F_\xi^{(k)} A = AF_\xi^{(k)}$ and M is non Hopf. Then we get

$$\begin{aligned}\alpha\beta g(\phi U, Y)\xi + \beta^2 g(\phi U, Y)U - \beta\eta(Y)A\phi U - kA\phi Y \\ = \beta g(A\phi U, Y)\xi - \beta\eta(AY)\phi U - k\phi AY\end{aligned} \tag{12}$$

for any $Y \in TM$. Taking $Y = \xi$ in (12) we obtain

$$A\phi U = (\alpha + k)\phi U \tag{13}$$

and taking $Y = U$ in (12) we obtain

$$AU = \beta\xi + (\alpha + k - \frac{\beta^2}{k})U. \tag{14}$$

Thus $\mathbb{D}_U$ is holomorphic and A-invariant. If $Y \in \mathbb{D}_U$ satisfies $AY = \lambda Y$, $A\phi Y = \lambda\phi Y$. That is, any eigenspace in $\mathbb{D}_U$ is holomorphic. Moreover by Codazzi equation

$$k(\lambda^2 - \alpha\lambda - 1)(\lambda - \alpha - k) = (-\lambda^2 + (\alpha + k)\lambda + 1)\beta^2. \tag{15}$$

From (15) we can see

$$grad(\alpha) = \frac{3k\beta}{k^2 + \beta^2}\phi U \tag{16}$$

and

$$grad(\beta) = \mu\phi U \tag{17}$$

where $\mu = \frac{3\alpha\beta^2}{k} + k(\alpha + k - \frac{\beta^2}{k}) + 1 - 3\beta^2(\frac{k^2+\beta^2+1}{k^2+\beta^2})$. From the fact that $g(\nabla_X grad(\beta), Y) = g(\nabla_Y grad(\beta), X)$ for any $X, Y \in TM$ we obtain

$$\beta^4 + (3k^2 + k\alpha)\beta^2 + k^3\alpha + 3k^2 + 2k^4 = 0. \tag{18}$$

Taking the derivative of (18) in the direction of ϕU and bearing in mind (17) we obtain that β is a solution of an equation with constant coefficients. Thus β is constant and also α is constant. This and (16) give a contradiction.

Thus we have proved that M must be Hopf. Then our condition applied to $X \in \mathbb{D}$ yields $\phi A = A\phi$ on $\mathbb{D}$ because $k \neq 0$. Thus by Theorem 1.2 M must be locally congruent to a type (A) real hypersurface. The converse is trivial and we have

Theorem 2.2. *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$. Then $F_\xi^{(k)}A = AF_\xi^{(k)}$ for a nonnull constant k if and only if M is locally congruent to a type (A) real hypersurface.*

From these two theorems we have

Theorem 2.3. *There do not exist real hypersurfaces M in $\mathbb{C}P^m$, $m \geq 3$, such that for a nonnull constant k $F_X^{(k)}A = AF_X^{(k)}$ for any $X \in TM$.*

3 Case of Structure Jacobi Operator

Suppose that $F_X R_\xi = R_\xi F_X$ for any $X \in \mathbb{D}$. Then we get

$$\begin{aligned} &g(Y, \phi AX)\xi + \eta(A\xi)g(\phi AX, AY)\xi - \eta(AY)g(\phi AX, A\xi)\xi \\ &\quad + \eta(Y)\phi AX + \eta(Y)\eta(A\xi)A\phi AX - \eta(Y)\eta(A\phi AX)A\xi = 0 \end{aligned} \tag{19}$$

for any $X \in \mathbb{D}$, $Y \in TM$. Let us suppose that M is non Hopf. Thus locally we can write $A\xi = \alpha\xi + \beta U$, as above

By different choices of Y in (19) we obtain

- $g(AU, \phi U) = 0$.
- $\phi AU = (1 - \beta^2)A\phi U$.

From these facts we can write $A\phi U = \delta\phi U + \omega Z_1$, where $Z_1 \in \mathbb{D}_U$ is a unit vector field. Then from (19) taking $Y = \phi Z_1$ we obtain

$$\alpha\beta\omega(\beta^2 - 1) = 0. \tag{20}$$

In the case $\alpha = 0$ it follows that M must be a minimal ruled real hypersurface.

In the case $\beta^2 = 1$ we obtain $\omega = 0$. Thus we study the case $\omega = 0$ with two subcases:

SUBCASE 1. $\beta^2 = 1$ and $AU = \beta\xi$.
SUBCASE 2. $\beta^2 \neq 1$ and $AU = \beta\xi + \sigma U$, where $\sigma = (1 - \beta^2)\delta$.

In both subcases $\mathbb{D}_U$ is A-invariant and holomorphic and either any eigenvalue in $\mathbb{D}_U$ is 0 or there exists a nonnull eigenvalue λ in $\mathbb{D}_U$. In this case $\alpha \neq 0$ and $\lambda = -\frac{1}{\alpha}$. Then the eigenspace T_λ is holomorphic. If for any $Y \in \mathbb{D}_U$ $AY = 0$ we obtain a ruled real hypersurface.

If $A\xi = \alpha\xi + \beta U$, $AU = \beta\xi + \sigma U$, $A\phi U = \delta\phi U$ and there exists $Z \in \mathbb{D}_U$ such that $AZ = -\frac{1}{\alpha}Z$, $A\phi Z = -\frac{1}{\alpha}\phi Z$ we arrive to a contradiction.

Now if M is Hopf and suppose $\alpha = 0$, (19) yields M is totally geodesic which is impossible. If $\alpha \neq 0$ the unique principal curvatures in $\mathbb{D}$ are either 0 or $-\frac{1}{\alpha}$ and the corresponding eigenspaces are holomorphic. This is impossible by Theorem 1.1 and we have

Theorem 3.1. *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$. Then $F_X R_\xi = R_\xi F_X$ for any $X \in \mathbb{D}$ if and only if M is locally congruent to a ruled real hypersurface.*

Suppose now that for some $k \neq 0$ $R_\xi F_\xi^{(k)} = F_\xi^{(k)} R_\xi$. Then for any $Y \in TM$ we get

$$\begin{aligned} &g(Y, \phi A\xi)\xi + g(A\xi, \xi)g(\phi A\xi, AY)\xi + \eta(Y)\phi A\xi + \eta(Y)\eta(A\xi)A\phi A\xi \\ &\quad - \eta(Y)\eta(A\phi A\xi)A\xi - k\phi R_\xi(Y) + kR_\xi(\phi Y) = 0. \end{aligned} \tag{21}$$

Let us suppose that M is non Hopf. From (21) we have

- $\alpha \neq 0$
- $A\phi U = -\frac{1}{\alpha}\phi U$
- $AU = \beta\xi + \frac{\beta^2-1}{\alpha}U$.

Therefore $\mathbb{D}_U$ is A-invariant and its eigenspaces are ϕ-invariant. Let λ be an eigenvalue in $\mathbb{D}_U$. From (21) and the Codazzi equation we obtain

$$(\alpha\lambda + 1)(\lambda^2 - \alpha\lambda - 1) = \beta^2(\lambda^2 - 1). \tag{22}$$

Applying several times Codazzi equation and (22) we arrive to

$$grad(\alpha) = 3\beta\left(\frac{1-\alpha^2}{\alpha}\right)\phi U \tag{23}$$

and

$$grad(\beta) = \left(-3\beta^2 + \frac{\beta^2-1}{\alpha^2}\right)\phi U. \tag{24}$$

From these facts and that $g(\nabla_Y grad(\gamma), Y) = g(\nabla_Y grad(\gamma), X)$ for any X, Y tangent to M and any function γ on M we obtain $\alpha^2 = 1$, $\beta^2 = -3$, which yields that M must be Hopf.

If M is Hopf, from (19) we get $\phi R_\xi = R_\xi\phi$. If $Y \in \mathbb{D}$ satisfies $AY = \lambda Y$, $\alpha\lambda\phi Y = \alpha A\phi Y$. Then either $\alpha = 0$ and M is locally congruent to a tube of radius $\frac{\pi}{4}$ around a complex submanifold of $\mathbb{C}P^m$ (Cecil and Ryan, [1]) or $A\phi = \phi A$ and M is locally congruent to a type (A) real hypersurface. Thus

Theorem 3.2. *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$. Let k be a nonnull constant. Then $F_\xi^{(k)}R_\xi = R_\xi F_\xi^{(k)}$ if and only if M is locally congruent to either a tube of radius $\frac{\pi}{4}$ over a complex submanifold of $\mathbb{C}P^m$ or to a type (A) real hypersurface with radius $r \neq \frac{\pi}{4}$.*

As a direct consequence of these Theorems we have

Theorem 3.3. *There do not exist real hypersurfaces M in $\mathbb{C}P^m$, $m \geq 3$, such that for a nonnull constant k $F_X^{(k)}R_\xi = R_\xi F_X^{(k)}$ for any X tangent to M.*

□

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Geometry of Lagrangian Submanifolds Related to Isoparametric Hypersurfaces

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Abstract In this article we shall provide a survey of our recent works and their environs on differential geometry of Lagrangian submanifolds in specific Kähler manifolds such as complex projective spaces, complex space forms, Hermitian symmetric spaces and so on. We shall emphasis on the relationship between certain minimal Lagrangian submanifold in complex hyperquadrics and isoparametric hypersurfaces in spheres. We shall discuss their properties and related problems of the Gauss images of isoparametric hypersurfaces. This article is mainly based on my joint work with Hui Ma (Tsinghua University, Beijing).

1 Introduction

Theory of *Submanifolds in Riemannian manifolds* is a higher dimensional generalization of curves and surfaces in Euclidean space. It is one of the most fundamental subjects in Differential Geometry. Our main research interests in submanifold theory are

1. Deformations and Moduli Spaces for Submanifolds
2. Geometric Variational Problems for Submanifolds
3. Lie Group Theoretic Methods in Finite and Infinite Dimensions

In this article we shall give attention to Lagrangian submanifolds in Kähler manifolds and discuss Lagrangian submanifolds and the geometric variational problem for the volume under Hamiltonian deformations of Lagrangian submanifolds in Kähler manifolds. Moreover, we shall concentrate on the Lie-theoretic construction and the properties of nice Lagrangian submanifolds in specific Kähler manifolds such as complex Euclidean spaces, complex projective spaces, compact Hermitian symmetric spaces and complex hyperquadrics [24–28].

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Hypersurfaces with constant principal curvatures in the standard sphere have abundant structures and applications, and they are called *isoparametric hypersurfaces*. Isoparametric hypersurface theory is outstanding in submanifold geometry. A nice class of compact minimal Lagrangian submanifolds embedded in complex hyperquadrics is provided by isoparametric hypersurfaces in the unit standard sphere vis the Gauss map construction. The purpose of this article is to explain our recent results on the study of such Lagrangian submanifolds of complex hyperquadrics.

This article is mainly based on my joint work with Hui Ma (Tsinghua University, Beijing).

2 Lagrangian Submanifolds of Symplectic and Kähler Manifolds

Let (M^{2n}, ω) be a $2n$-dimensional symplectic manifold with a symplectic form ω. A smooth immersion $\varphi : L \to M$ of a smooth manifold L into M is called a *Lagrangian immersion* if φ satisfies the two conditions $\varphi^*\omega = 0$ and $\dim L = n$. If φ is a Lagrangian immersion, then the vector bundle map $\alpha : \varphi^{-1}TM/\varphi_* TL \ni v \longmapsto \alpha_v \in T^*L$ becomes a bundle isomorphism, because of the non-degeneracy of φ.

Let $\varphi : L \to M$ be a Lagrangian immersion. By definition a *Lagrangian deformation* $\{\varphi_t\}$ of φ is a one-parameter smooth family of Lagrangian immersions $\varphi_t : L \longrightarrow (M^{2n}, \omega)$ with $\varphi_0 = \varphi$. The variational vector field of $\{\varphi_t\}$ is defined as $V_t := \dfrac{\partial \varphi_t}{\partial t} \in C^\infty(\varphi_t^{-1}TM)$ and let $\alpha_t = \omega(V_t, \varphi_*(\cdot))$ be the 1-form on L corresponding to V_t. It is easy to show that $\{\varphi_t\}$ is a Lagrangian deformation if and only if α_t is closed for each t. More strongly, if α_t is exact for each t, then such a Lagrangian deformation $\{\varphi_t\}$ of φ is called an *Hamiltonian deformation* of φ.

Assume that (M, ω, g, J) is a Kähler manifold. Let $\varphi : L \to M$ be a Lagrangian immersion. Let $B : TL \times TL \to T^\perp L$ denote the second fundamental form of φ. Then the tensor field S of degree 3 on L corresponding to B is defined by $S(X, Y, Z) := \omega(B(X, Y), Z) \quad (\forall\, X, Y, Z \in TL)$. It follows from the Kähler identity that S is a symmetric 3-tensor field on L. The mean curvature vector field H of φ is a normal vector field to L defined by $H := \mathrm{tr}_{\varphi^* g}(B)$. We call the 1-form α_H corresponding to H the *mean curvature form* of φ. Note that the usual *minimal submanifold* is by definition a submanifold vanishing the mean curvature vector field $H = 0$. Then the mean curvature form of φ satisfies the following identity (Dazard): $d\alpha_H = \varphi^* \rho_M$, where ρ_M denoted the Ricci form of the Kähler manifold (M, ω, g, J).

For simplicity we assume that L is compact without boundary. A Lagrangian immersion $\varphi : L \to M$ is called *Hamiltonian minimal* (shortly, *H-minimal*) if for any Hamiltonian deformation $\{\varphi_t\}$ of φ the first variation of the volume vanishes

$$\frac{d}{dt}\mathrm{Vol}(L, \varphi_t^* g)|_{t=0} = 0. \tag{1}$$

Its Euler–Lagrangian equation is the Hamiltonian minimal Lagrangian equation given by $\delta\alpha_H = 0$, where δ denotes the co-differential operator of d relative to $\varphi^* g$. Note that it is weaker than the minimality ($\alpha_H = 0$) for Lagrangian submanifolds in Kähler manifolds.

A Hamiltonian minimal Lagrangian immersion $\varphi : L \to M$ is called *Hamiltonian stable* if for any Hamiltonian deformation $\{\varphi_t\}$ of φ the second variation of the volume is nonnegative

$$\frac{d^2}{dt^2}\mathrm{Vol}(L, \varphi_t^* g)|_{t=0} \geq 0. \tag{2}$$

The Hamiltonian version of the second variational formula is given as follows [22]:

$$\begin{aligned} &\frac{d^2}{dt^2}\mathrm{Vol}(L, \varphi_t^* g)|_{t=0} \\ &\quad = \int_L \left(\langle \triangle_L^1 \alpha, \alpha \rangle - \langle \overline{R}(\alpha), \alpha \rangle - 2\langle \alpha \otimes \alpha \otimes \alpha_H, S \rangle + \langle \alpha_H, \alpha \rangle^2 \right) dv \end{aligned} \tag{3}$$

where $\alpha := \alpha_{V_t} \in B^1(L)$, $\langle \overline{R}(\alpha), \alpha \rangle := \sum_{i,j=1}^n \mathrm{Ric}^M(e_i, e_j)\alpha(e_i)\alpha(e_j)$ for an orthonormal basis $\{e_i\}$ of $T_p L$.

We shall notice about the null space of the second variations on Hamiltonian deformations. Let X be a holomorphic Killing vector field of M. Then $\alpha_X = \omega(X, \cdot)$ is closed. If M is simply connected, more generally $H^1(M, \mathbf{R}) = \{0\}$, then $\alpha_X = \omega(X, \cdot)$ is exact and thus X is a Hamiltonian vector field on M. Hence each holomorphic Killing vector field of M generates a Hamiltonian deformation of φ preserving the metric and thus the volume. We call such a Hamiltonian deformation of φ *trivial*. Obviously a trivial Hamiltonian deformation of φ gives an element of the null space of the second variations.

Assume that φ is an H-minimal Lagrangian immersion. Then φ is called *strictly Hamiltonian stable* if

(1) φ is Hamiltonian stable.
(2) The null space of the second variation on Hamiltonian deformations of φ coincides with the vector subspace induced by trivial Hamiltonian deformations of φ, that is, $n(\varphi) = n_{hk}(\varphi)$. Here $n(\varphi) :=$ dim[the null space] and $n_{hk}(\varphi) := \dim\{\varphi^*\alpha_X \mid X$ a holomorphic Killing vector field of $M\}$.

Note that a strictly Hamiltonian stable Hamiltonian minimal Lagrangian submanifold has local minimum volume under any Hamiltonian deformation. There are nice results on deformation, bifurcation and existence of Hamiltonian minimal Lagrangian submanifolds by Joyce–Y. I. Lee–Schoen [14], Bettiol–Piccione–Siciliano (equivariant case) [7] related to this condition.

Problem 1. What Lagrangian submanifolds are Hamiltonian minimal? What Hamiltonian minimal Lagrangian submanifolds are Hamiltonian stable? Moreover, examine their Hamiltonian rigidity or strict Hamiltonian stability.

Assume that M is an Einstein–Kähler manifold of Einstein constant κ. Let $L \hookrightarrow M$ be a compact minimal Lagrangian submanifold immersed in M. Then it follows from the second fundamental form that L is Hamiltonian stable if and only if $\lambda_1 \geq \kappa$, where λ_1 denotes the first (positive) eigenvalue of the Laplacian of L acting on $C^\infty(L)$ (B. Y. Chen–T. Nagano–P. F. Leung, Y. G. Oh [21]).

Assume that M is compact homogeneous Einstein–Kähler manifold with $\kappa > 0$. Let $L \hookrightarrow M$ be a compact minimal Lagrangian submanifold immersed in M. Then we know that $\lambda_1 \leq \kappa$ (A. Ros, N. Ejiri, F. Urbano, Hajime Ono, Amarzaya–Ohnita cf. [3]). Therefore we can say that λ_1 attains the upper bound κ if and only if L is Hamiltonian stable.

Let $\mathrm{Aut}(M, \omega, J, g)$ be the automorphism group of the Kähler manifold (M, ω, J, g). Assume that $K \subset \mathrm{Aut}(M, \omega, J, g)$ be an analytic Lie subgroup. A Lagrangian submanifold $L = K \cdot x \subset M$ obtained as a Lagrangian orbit of K is called a *homogeneous Lagrangian submanifold* of M. We easily observe that

Proposition 1 (cf. [16]). *Any compact homogeneous Lagrangian submanifolds of a Kähler manifold is always Hamiltonian minimal.*

Here we shall describe all known examples of compact Hamiltonian stable H-minimal or minimal Lagrangian submanifolds embedded in $\mathbb{C}^{n+1}$ and $\mathbb{C}P^n$.

Example 1. (1) Circles $S^1(r)$ in $\mathbb{C}$ $(S \neq 0, \alpha_H \neq 0, \nabla S = 0)$.
(2) $Q_{2,n+1}(\mathbb{R}) = (S^1 \times S^n)/\mathbb{Z}_2 \subset \mathbb{C}^{n+1}$, $U(p)/O(p) \subset \mathbb{C}^{p(p+1)/2}$, $U(p) \subset \mathbb{C}^{p^2}$, $U(2p)/Sp(p) \subset \mathbb{C}^{p(2p-1)}$, $T^1 \cdot E_6/F_4 \subset \mathbb{C}^{27}$ $(S \neq 0, \alpha_H \neq 0, \nabla S = 0)$.
(3) Their Riemannian products $\tilde{L} = L_1 \times \cdots \times L_k$ $(S \neq 0, \alpha_H \neq 0, \nabla S = 0)$.

They all are strictly Hamiltonian stable (Amarzaya–Ohnita [2–4]). Notice that they all have parallel second fundamental form $\nabla S = 0$ and are symmetric R-spaces of $U(r)$ given by orbits of the isotropy representations of Hermitian symmetric spaces.

Nice examples of Lagrangian submanifolds in $\mathbb{C}P^n$ can be obtained from Example 1 via the Hopf fibration. $\pi : \mathbb{C}^{n+1} \supset S^{2n+1} \longrightarrow \mathbb{C}P^n$.

Example 2. (1) Real projective subspaces $\mathbb{R}P^n = \pi(Q_{2,n+1}(\mathbb{R})) \subset \mathbb{C}P^n$ $(S = 0)$.
(2) $SU(p)/SO(p) \cdot \mathbb{Z}_p \subset \mathbb{C}P^{(p-1)(p+2)/2}$, $SU(p)/\mathbb{Z}_p \subset \mathbb{C}P^{p^2-1}$, $SU(2p)/Sp(p) \cdot \mathbb{Z}_{2p} \subset \mathbb{C}P^{(p-1)(2p+1)}$, $E_6/F_4 \cdot \mathbb{Z}_3 \subset \mathbb{C}P^{26}$ [3]. $(S \neq 0, \nabla S = 0, \alpha_H = 0)$,
(3) $L = \pi(\tilde{L})$, where $\tilde{L}$ is one of examples in Example 1 except for $\tilde{L} = Q_{2,n+1}(\mathbb{R})$ $(S \neq 0, \nabla S = 0$, generically $\alpha_H \neq 0)$ [2,4].
(4) $\rho_3(SU(2))[z_0^3 + z_1^3] \subset \mathbb{C}P^3(\nabla S \neq 0, \alpha_H = 0)$ (L. Bedulli–A. Gori [5], independently Ohnita [23]).
(5) $(SU(3) \times SU(3))/T^2 \cdot \mathbb{Z}_4 \subset \mathbb{C}P^5$ $\nabla S \neq 0, \alpha_H = 0$ (Petrecca–Podestá [32]).

They all are also strictly Hamiltonian stable. Notice that Bedulli–Gori [6] classified all compact homogeous Lagrangian submanifolds of $\mathbb{C}P^n$ obtained by (necessarily, minimal) Lagrangian orbits of simple compact Lie subgroups of $SU(n+1)$, by using the prehomogeneous vector space theory due to Mikio Sato and Tatsuo Kimura.

It is known that any compact Hamiltonian stable minimal Lagrangian submanifold L immersed in $\mathbb{C}P^n$ must satisfy $\pi_1(L) \neq \{1\}$, more strongly $H_1(L;\mathbb{Z}) \neq \{1\}$ (cf. [3]). Urbano [34] showed that any Hamiltonian stable minimal Lagrangian torus L immersed in $\mathbb{C}P^2$ must be the Clliford minimal torus.

At present the author does not know whether there is an example of compact minimal Lagrangian submanifold embedded in $\mathbb{C}P^n$ which is *not* Hamiltonian stable. However, the case of compact Hermitian symmetric spaces of rank ≥ 2 are quite different from the complex projective space case.

In general, a *real form* of a Kähler manifold M is by definition a connected component of the fixed point subset of an involutive anti-holomorphic isometry of M, which is a totally geodesic Lagrangian submanifold of M. Masaru Takeuchi classified all real forns of Hermitian symmetric spaces of compact type and he showed that any totally geodesic Lagrangian submanifold of an Hermitian symmetric space M of compact type is a real form of M (see [33]). Let M be a compact irreducible Hermitian symmetric space of rank ≥ 2 and let L be a real form of M. The Hamiltonian stability of L is given as follows (see also [15, p. 755]):

Theorem 1 ([3,33]). *$(L, M) = (Q_{p+1,q+1}(\mathbb{R}) = (S^p \times S^q)/\mathbb{Z}_2,\ Q_{p+q}(\mathbb{C}))$ $(q - p \geq 3)$, $(U(2p)/Sp(p), SO(4p)/U(p))$ $(p \geq 3)$ or $(T \cdot E_6/F_4, E_7/T \cdot E_6)$ if and only if L is NOT Hamiltonian stable.*

3 Isoparametric Hypersurface Geometry and Lagrangian Submanifolds of Complex Hyperquadrics

The complex hyperquadric $Q_n(\mathbb{C})$ is a compact embedded smooth complex hypersurface of the $(n+1)$-dimensional complex projective space $\mathbb{C}P^{n+1}$ defined by the homogeneous quadratic equation $z_0^2 + z_1^2 + \cdots + z_{n+1}^2 = 0$, where $\{z_0, z_1, \cdots, z_{n+1}\}$ denotes the homogeneous coordinate system of $\mathbb{C}P^{n+1}$. The complex hyperquadric $Q_n(\mathbb{C})$ is canonically isometric to the real Grassmann manifold $\widetilde{Gr}_{n+1}(\mathbb{R}^{n+2})$ consisting of all oriented 2-dimensional vector subspaces of $\mathbb{R}^{n+2}$. The $\widetilde{Gr}_{n+1}(\mathbb{R}^{n+2})$ has the natural embedding $\widetilde{Gr}_{n+1}(\mathbb{R}^{n+2}) \subset \bigwedge^2 \mathbb{R}^{n+2}$ by $[W] \mapsto \mathbf{a} \wedge \mathbf{b}$, where $\{\mathbf{a}, \mathbf{b}\}$is an orthonormal basis of $[W] \in \widetilde{Gr}_{n+1}(\mathbb{R}^{n+2})$ compatible with its orientation. The correspondence between $\widetilde{Gr}_{n+1}(\mathbb{R}^{n+2})$ and $Q_n(\mathbb{C})$ is given by

$$\widetilde{Gr}_{n+1}(\mathbb{R}^{n+2}) \ni [W] = \mathbf{a} \wedge \mathbf{b} \longleftrightarrow [\mathbf{a} + \sqrt{-1}\mathbf{b}] \in Q_n(\mathbb{C}), \tag{4}$$

Then one also has a symmetric space expression

$$Q_n(\mathbb{C}) \cong \widetilde{Gr_2}(\mathbb{R}^{n+2}) \cong SO(n+2)/SO(2) \times SO(n), \tag{5}$$

which is a compact Hermitian symmetric space of rank 2.

The Gauss map construction of Lagrangian submanifolds in $Q_n(\mathbb{C})$ is as follows: Let $N^n \hookrightarrow S^{n+1}(1) \subset \mathbb{R}^{n+2}$ be an oriented hypersurface in the unit standard sphere. Let $\mathbf{x}$ denote the position vector of points of N^n and $\mathbf{n}$ denote the unit normal vector field of N^n in $S^{n+1}(1)$. The *Gauss map* of N^n defined by

$$\mathcal{G} : N^n \ni p \longmapsto [x(p) + \sqrt{-1}n(p)] = x(p) \wedge n(p) \in Q_n(\mathbb{C}) = \widetilde{Gr_2}(\mathbb{R}^{n+2})$$

is a Lagrangian immersion of N^n into $Q_n(\mathbb{C})$.

Palmer showed the following formula for the mean curvature form α_{H} of the Gauss map $\mathcal{G}$ in terms of principal curvatures $\kappa_1, \cdots, \kappa_n$ of N^n [31]:

Lemma 1.

$$\alpha_{\mathrm{H}} = d\left(\mathrm{Im}\left(\log \prod_{i=1}^{n}(1 + \sqrt{-1}\kappa_i)\right)\right) = -d\left(\sum_{i=1}^{n} \mathrm{arccot}\kappa_i\right).$$

From this formula we note that if N^2 is a minimal surface of $S^3(1)$, then there are excellent results of Castro and Urbano [8] on the classification of compact orientable or non-orientable Hamiltonian stable minimal Lagrangian surfaces of low genus immersed in $S^2 \times S^2 = Q_2(\mathbb{C})$.

Suppose that N^n is an oriented hypersurface with constant principal curvatures in $S^{n+1}(1)$, so-called *isoparametric hypersurface*. Then by Lemma 1 we have

Proposition 2. *The Gauss map* $\mathcal{G} : N^n \to Q_n(\mathbb{C})$ *is a minimal Lagrangian immersion.*

The fundamental theory of isoparametric hypersurfaces in the standard sphere was established by E. Cartan and Münzner [19]. Let N^n be a hypersurface immersed in the unit standard hypersphere $S^{n+1}(1) \subset \mathbf{R}^{n+2}$ with g distinct constant principal curvatures $k_1 > k_2 > \cdots > k_g$. and corresponding multiplicities $m_1, m_2, \cdots, m_g$. Then $m_\alpha = m_{\alpha+2}$ indices modulo g [19]. We may assume that $m_1 \leq m_2$. There is a homogeneous polynomial function $F : \mathbb{R}^{n+2} \to \mathbb{R}$ of degree g, so-called *Cartan–Münzner polynomial*, which satisfies the system of partial differential equations

$$\Delta F = c\, r^{g-2}, \quad \|\mathrm{grad} F\|^2 = g^2 r^{2g-2},$$

where $c := g^2(m_2 - m_1)/2$ and $r = \|\mathbf{x}\|$ ($\mathbf{x} \in \mathbb{R}^{n+2}$), such that N^n extends to a compact embedded level hypersurface $S^{n+1}(1) \cap F^{-1}(s)$ $(\exists s \in (-1, 1))$. Then each level hypersurface is also a hypersurfaces with constant principal curvatures, which is called an *isoparametric hypersurface*. The family of such level hypersurfaces is called the *isoparametric family*. The Münzner's famous result [19] is that g must be $1, 2, 3, 4$ or 6 and if $g = 6$, then $m_1 = m_2$. Moreover, Abresch [1] showed that if $g = 6$, then $m_1 = m_2 = 1$ or 2.

- Construction of isoparametric hypersurfaces:
  - Principal orbits of the isotropy representations of Riemannian symmetric pairs (U, K) of rank 2 provide all homogeneous isoparametric hypersurfaces (Hsiang–Lawson, R. Takagi–T. Takahashi)

  - Algebraic construction of Cartan–Münzner polynomials by representations of Clifford algebras in case $g = 4$ (Ozeki–Takeuchi [30], Ferus–Karcher–Münzner [12]) provide so many non-homogeneous isoparametric hypersurfaces (OT-FKM type).

- Classification of isoparametric hypersurfaces:
  - $g = 1 : N^n = S^n$, a great or small sphere;
  - $g = 2 : N^n = S^{m_1} \times S^{m_2}$, $(n = m_1 + m_2, 1 \leq m_1 \leq m_2)$, Clifford hypersurfaces;
  - $g = 3$: N^n is homogeous, $N^n = \frac{SO(3)}{\mathbb{Z}_2+\mathbb{Z}_2}, \frac{SU(3)}{T^2}, \frac{Sp(3)}{Sp(1)^3}, \frac{F_4}{Spin(8)}$ (E. Cartan);
  - $g = 6$: N^n is homogeneous.
    - $m_1 = m_2 = 1$: homogeneous (Dorfmeister–Neher, R. Miyaoka).
    - $m_1 = m_2 = 2$: homogeneous (R. Miyaoka [20]).
  - $g = 4$: N^n is either homogeneous or OT-FKM type except for $(m_1, m_2) = (7, 8)$ (Cecil–Chi–Jensen [9], Immervoll [13], Chi [10, 11]).

The Gauss image $\mathcal{G}(N^n)$ of an isoparametric hypersurface has the following properties. It follows from [15, 17, 26] that

Proposition 3. (1) *The Gauss image $\mathcal{G}(N^n)$ is a compact smooth minimal Lagrangian submanifold embedded in $Q_n(\mathbb{C})$.*

(2) *The Gauss map $\mathcal{G}$ gives a covering map $\mathcal{G} : N^n \to \mathcal{G}(N^n)$ over the Gauss image with the deck transformation group $\mathbb{Z}_g$. Note that the $\mathbb{Z}_g$-action does not preserve the induced metric on N^n from $S^{n+1}(1)$ if $g \geq 3$.*

(3) *$\mathcal{G}(N^n)$ is invariant under the deck transformation group $\mathbb{Z}_2$ of the universal covering $Q_n(\mathbb{C}) = \widetilde{Gr}_2(\mathbb{R}^{n+2}) \to Gr_2(\mathbb{R}^{n+2})$.*

(4) *$\dfrac{2n}{g}$ is even (resp. odd) if and only if $\mathcal{G}(N^n)$ is orientable (resp. non-orientable).*

(5) *$\mathcal{G}(N^n)$ is a monotone and cyclic Lagrangian submanifold in $Q_n(\mathbb{C})$ with minimal Maslov number equal to $\dfrac{2n}{g}$.*

4 Hamiltonian Stability of the Gauss Images of Isoparametric Hypersurfaces and Further Problems

In the author's joint work with Hui Ma on the Gauss images of isoparametric hypersurfaces, we have done

(1) Classification of all compact homogeneous Lagrangian submanifolds in complex hyperquadrics [15].

(2) Determination of the Hamiltonian stability, the Hamiltonian rigidity and the strict Hamiltonian stability for the Guass images of all *homogeneous* isoparametric hypersurfaces.

Let $N^n \hookrightarrow S^{n+1}(1)$ be a compact isoparametric hypersurface embedded in $S^{n+1}(1)$. First Palmer showed the Hamiltonian stability result for the Gauss map $\mathcal{G} : N \to Q_n(\mathbb{C})$:

Theorem 2 ([31]). *Its Gauss map $\mathcal{G} : N \to Q_n(\mathbb{C})$ is Hamiltonian stable if and only if $N^n = S^n \subset S^{n+1}$ $(g = 1)$.*

Problem 2. Determine the Hamiltonian stability of the Gauss images $L^n = \mathcal{G}(N^n) \subset Q_n(\mathbb{C})$.

Note that $g = 1$ or 2 if and only if L is a real form $Q_{m_1+1,m_2+1}(\mathbb{R})$ of $Q_n(\mathbb{C})$. In these cases we calculate directly their properties or we may use results of Theorem 1:

If $g = 1$, then L is strictly Hamiltonian stable.
If $g = 2$, then it holds

(a) L is not Hamiltonian stable if and only if $m_2 - m_1 \geq 3$.
(b) L is strictly Hamiltonian stable if and only if $m_2 - m_1 < 2$.
(c) L is Hamiltonian stable but not strictly Hamiltonian stable if and only if $m_2 - m_1 = 2$.

Theorem 3 ([15]). *If $g = 3$, then L is strictly Hamiltonian stable.*

Theorem 4 ([18]). *Assume that $g = 6$, that is, $L = SO(4)/(\mathbb{Z}_2 + \mathbb{Z}_2) \cdot \mathbb{Z}_6$ $(m_1 = m_2 = 1)$ or $L = G_2/T^2 \cdot \mathbb{Z}_6$ $(m_1 = m_2 = 2)$. Then L is strictly Hamiltonian stable.*

Theorem 5 ([17]). *Suppose that $g = 4$ and N^n is homogeneous. Then*

(1) $L = SO(5)/T^2 \cdot \mathbb{Z}_4$ $(m_1 = m_2 = 2)$ is strictly Hamiltonian stable.
(2) $L = U(5)/(SU(2) \times SU(2) \times U(1)) \cdot \mathbb{Z}_4$ $(m_1 = 4, m_2 = 5)$ is strictly Hamiltonian stable.
(3) Assume that $L = (SO(2) \times SO(m))/(\mathbb{Z}_2 \times SO(m-2)) \cdot \mathbb{Z}_4$ $(m_1 = 1, m_2 = m-2, m \geq 3)$. Then

 (a) *$m_2 - m_1 \geq 3$ if and only if L is NOT Hamiltonian stable.*
 (b) *$m_2 - m_1 = 2$ if and only if L is Hamiltonian stable but not strictly Hamiltonian stable.*
 (c) *$m_2 - m_1 = 1$ or 0 if and only if L is strictly Hamiltonian stable.*

(4) Suppose that $L = S(U(2) \times U(m))/S(U(1) \times U(1) \times U(m-2)) \cdot \mathbb{Z}_4$ $(m_1 = 2, m_2 = 2m-3, m \geq 2)$. Then

 (a) *$m_2 - m_1 \geq 3$ if and only if L is NOT Hamiltonian stable.*
 (b) *$m_2 - m_1 = 1$ or -1 if and only if L is strictly Hamiltonian stable.*

(5) Assume that $L = (Sp(2) \times Sp(m))/(Sp(1) \times Sp(1) \times Sp(m-2)) \cdot \mathbb{Z}_4$ $(m_1 = 4, m_2 = 4m-5, m \geq 2)$. Then

 (a) *$m_2 - m_1 \geq 3$ if and only if L is NOT Hamiltonian stable.*
 (b) *$m_2 - m_1 = -1$ if and only if L is strictly Hamiltonian stable.*

Theorem 6 ([18]). *Suppose that* $g = 4$ *and* $L = U(1) \cdot Spin(10)/(S^1 \cdot Spin(6)) \cdot \mathbb{Z}_4$ $(m_1 = 6, m_2 = 9,$ *thus* $m_2 - m_1 = 3)$. *Then* L *is strictly Hamiltonian stable!*

Combining all these results, we conclude

Theorem 7 ([15,17,18]). *Suppose that* (U, K) *is not of type* EIII*, that is,* $(U, K) \neq (E_6, U(1) \cdot Spin(10))$. *Then the Gauss image* $L = \mathcal{G}(N)$ *is NOT Hamiltonian stable if and only if* $m_2 - m_1 \geq 3$. *Moreover if* (U, K) *is of type* EIII*, that is,* $(U, K) = (E_6, U(1) \cdot Spin(10))$*, then* $(m_1, m_2) = (6, 9)$ *but the Gauss image* $L = \mathcal{G}(N)$ *is strictly Hamiltonian stable.*

Problem 3. Study the Hamiltonian stability and related other properties of the Gauss images of non-homogeneous isoparametric hypersurfaces (necessarily, $g = 4$ and of OT-FKM type).

Problem 4. Let N_0 and N_1 be two compact isoparametric hypersurfaces embedded in the unit standard sphere $S^{n+1}(1)$. Let $L_0 = \mathcal{G}_0(N_0) \subset Q_n(\mathbb{C})$ and $L_1 = \mathcal{G}_1(N_1) \subset Q_n(\mathbb{C})$ be their Gauss images. Then it is an interesting problem to study the intersection theory of the Gauss images L_0 and L_1 as Lagrangian submanifolds of $Q_n(\mathbb{C})$ [29]. This is the author's joint work with Hui Ma and Reiko Miyaoka which is in progress now.

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Isometric Deformations of Surfaces with Singularities

Masaaki Umehara

Abstract This is a survey article on isometric deformations of surfaces with singularities. At the end of this paper, the author introduces a new problem on isometric deformations of cross cap singularities.

1 Introduction

Let (M^2, ds^2) be a Riemannian 2-manifold, and $f : M^2 \to \mathbf{R}^3$ an isometric immersion. For each point $p \in M^2$, two principal curvatures λ_1, λ_2 are defined, and their product

$$K := \lambda_1 \lambda_2$$

is called the *Gaussian curvature* of the surface f. An invariant I of surfaces is called *extrinsic* if there exists a neighborhood $U^2 (\subset M^2)$ of p and a smooth map $g : U^2 \to \mathbf{R}^3$ such that $I(g) \neq I(f)$ holds at p and the induced metric of g coincides with that of f. On the other hand, an invariant of surfaces is called *intrinsic* if one can show the following;

(i) setting up a class of local coordinate systems determined by the induced metrics (i.e. the first fundamental forms),
(ii) to give a formula of the invariant in terms of the coefficients of first fundamental form with respect to the above coordinate systems.

By definition, intrinsic invariants are not extrinsic. Intrinsic invariants are usually thought of as ones written in terms of only first fundamental forms. The relationships between our definition and this philosophy will be discussed in the paper [2].

We know that the Gaussian curvature K is a typical intrinsic invariant, since it coincides with the sectional curvature of the first fundamental form ds^2. On the other hand, to prove that a given invariant is extrinsic, it is sufficient to show the

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existence of an isometric deformation of a surface which changes the values of the invariant.

For example, let $\xi(s)$ be a regular curve with arc length parameter on the unit sphere $S^2 := \{\xi \in \mathbf{R}^3\,;\, |\xi| = 1\}$, and set

$$f(u,v) := \gamma(u) + v\xi(u),$$
$$\gamma(u) := \int_0^u \Big(a(u)\xi(u) + b(u)\xi'(u) + c(u)(\xi(u) \times \xi'(u)) \Big) du,$$

where '×' denotes the vector product of $\mathbf{R}^3$, and $a(u), b(u), c(u)$ are arbitrarily given smooth functions of u. Then

$$ds^2 = \big(a^2 + (b+v)^2 + c^2\big)\,du^2 + 2a\,du\,dv + dv^2$$

is the first fundamental form of f. It is a crucial point that ds^2 does not depend on the choice of the initial spherical curve $\xi(s)$. This construction of f was applied to show that several invariants of cuspidal edges and cross caps are extrinsic, in [3] and [7]. Moreover, the following assertions were proved in [3].

(i) If $b(0)^2 + c(0)^2 > 0$, then $(0,0)$ is a regular point of f.
(ii) If two functions a, b vanish identically, $c(0) = 0$ and $c'(0) \neq 0$, then $(0,0)$ corresponds to a cross cap (cf. Fig. 1, right).
(iii) If $a(0) > 0$ and b, c vanish identically, then $(0,0)$ corresponds to a cuspidal edge singularity (cf. Fig. 1, left).

For example, if we set $a(u) = c(u) = 0,\ b(u) = 1$, then by (i),

$$f(u,v) := \gamma(u) + v\xi(u), \qquad \gamma(u) := \int_0^u \xi'(u)du$$

gives a regular surface which has the first fundamental form

$$ds^2 = (1 + v^2)du^2 + dv^2.$$

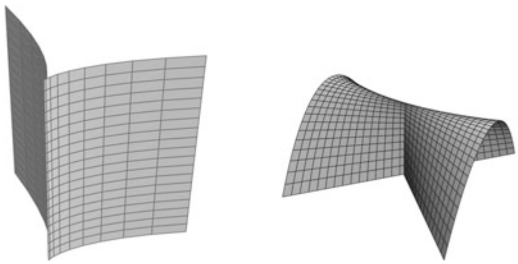

Fig. 1 A cuspidal edge and a cross cap

The unit normal vector field of f is given by $N(u) := \xi'(u) \times \xi(u)$, and the second fundamental form of f is given by

$$h := \kappa_g(u)du^2,$$

where $\kappa_g(u)$ is the geodesic curvature of the spherical curve $\xi(u)$. The principal curvatures of f at $(u, 0)$ are $\lambda_1 = 0$ and $\lambda_2 = \kappa_g(u)$. Since we can deform the curve ξ so that κ_g moves, we get an isometric deformation of f which moves one of its principal curvatures. Thus we can conclude that principal curvatures are extrinsic invariants. In this paper, we will introduce several invariants on cuspidal edges and cross caps, and discuss the existence of isometric deformations of these two singularities.

2 Isometric Deformations of Regular Surfaces

In the introduction, we gave an example of an isometric deformation of ruled surfaces. However, to show that a given invariant is extrinsic, it is better to prepare more general tool for constructing isometric deformations, which is the following classical result:

Theorem 1 (Janet-Cartan). *Let (M^n, ds^2) be a real analytic n-manifold. For each $p \in M^n$, there exist a neighborhood U^n of $p \in M^n$ and a real analytic isometric embedding $f : U^n \to \mathbf{R}^{n(n+1)/2}$.*

This is a local result. It is well-known that any Riemannian manifold can be isometrically embedded in a sufficiently high dimensional Euclidean space. However, we are interested in local properties of surfaces with singularities, and so the Janet-Cartan theorem rather fits our purposes. We now give an outline of the proof of it when $n = 2$ (in this case $n(n + 1)/2 = 3$) as follows[1]: We apply the following classical result:

Fact 1 (Cauchy-Kovalevskaya theorem). *Let*

$$x^i_v(u, v) = \varphi^i(u, v, x^1, \dots, x^k, x^1_u, \dots, x^k_u) \qquad (i = 1, \dots, k) \tag{1}$$

be a PDE having $x^i := x^i(u, v)$ $(i = 1, \dots, k)$ as unknown functions, where $\varphi := (\varphi^1, \dots, \varphi^k)$ is a real analytic map and

$$x^i_u := \frac{\partial x^i}{\partial u}, \qquad x^i_v := \frac{\partial x^i}{\partial v} \qquad (i = 1, \dots, k).$$

[1]The precise proof of this theorem is written in [11].

This equation has a unique real analytic solution $x = (x^1, \dots, x^k)$ *satisfying the following initial conditions*

$$x^i(u,0) = w^i(u) \qquad (i = 1, \dots, k),$$

where w^i $(i = 1, \dots, k)$ *are given real analytic functions.*

(*An outline of the proof of the Janet-Cartan theorem for* $n = 2$.)
We fix a point p on M^2, and let $\gamma(t)$ be a geodesic emanating from $p = \gamma(0)$. Using the exponential mapping Exp_q at $q \in M^2$, we define a real analytic map

$$(u, v) \mapsto \mathrm{Exp}_{\gamma(u)}(v\mathbf{n}(u)),$$

where $\mathbf{n}(t)$ is the unit normal vector field along the geodesic $\gamma(t)$ on M^2. Then (u, v) gives a local coordinate system with the origin p. Regarding $f = (f^1, f^2, f^3)$ as the unknown $\mathbf{R}^3$-valued function, we consider the equation

$$f_u \cdot f_u = E, \quad f_u \cdot f_v = F, \quad f_v \cdot f_v = G, \tag{2}$$

where '$\cdot$' denotes the canonical inner product of $\mathbf{R}^3$ and

$$ds^2 = E du^2 + 2F du dv + G dv^2.$$

The PDE (2) is not a normal form (1) as in Fact 1. Differentiating (2) we get the following three identities

$$f_{vv} \cdot f_v = G_v/2, \tag{3}$$

$$f_{vv} \cdot f_u = F_v - G_u/2, \tag{4}$$

$$f_{vv} \cdot f_{uu} = F_{uv} - E_{vv}/2 - G_{uu}/2 + f_{uv} \cdot f_{uv}. \tag{5}$$

Take a real analytic regular space curve $\Gamma(u)$ with arc length parameter u having non-vanishing curvature. We would like to find a solution with the initial condition $f(u,0) = \Gamma(u)$. Since the two vectors

$$f_u(u,0) = \Gamma'(u), \qquad f_{uu}(u,0) = \Gamma''(u)$$

are linearly independent, (3)–(5) yield the following identity

$$f_{vv} = {}^T(f_v, f_u, f_{uu})^{-1} \begin{pmatrix} G_v/2 \\ F_v - G_u/2 \\ F_{uv} - E_{vv}/2 - G_{uu}/2 + f_{uv} \cdot f_{uv} \end{pmatrix}.$$

Since $\Gamma(u) = f(u,0)$ is a geodesic, f must satisfy the initial condition

$$f_v(u,0) = \Gamma'(u) \times \Gamma''(u).$$

Consequently, we get the following PDE having $f(u,v), r(u,v), q(u,v)$ as unknown $\mathbf{R}^3$-valued functions:

$$f_v = q, \tag{6}$$

$$r_v = q_u, \tag{7}$$

$$q_v = {}^T(q,r,r_u)^{-1}\begin{pmatrix} G_v/2 \\ F_v - G_u/2 \\ F_{uv} - E_{vv}/2 - G_{uu}/2 + q_u \cdot q_u \end{pmatrix} \tag{8}$$

with the initial conditions

$$f(u,0) := \Gamma(u), \quad r(u,0) := \Gamma'(u), \quad q(u,0) := \Gamma'(u) \times \Gamma''(u),$$

where ${}^T(q,r,r_u)^{-1}$ is the transposed inverse matrix of the 3×3-matrix (q,r,r_u). If the solutions f, q, r exist, r must coincide with f_u and (7) corresponds to the identity $f_{uv} = f_{vu}$. Since

$$(q(u,0), r(u,0), r_u(u,0)) = (\Gamma'(u), \Gamma''(u), \Gamma'(u) \times \Gamma''(u))$$

is a regular matrix, we can apply the Cauchy-Kovalevskaya theorem in this situation, and one can easily check that the resulting function f gives the desired isometric embedding. As a consequence, we now get the following assertion:

Corollary 1. *Let p be an arbitrary fixed point on a real analytic Riemannian 2-manifold (M^2, ds^2). Suppose that $\Gamma(t)$ $(|t| \le \epsilon)$ is a real analytic regular space curve in $\mathbf{R}^3$ whose curvature function is positive. Then there exists a real analytic isometric embedding*

$$f : [-\epsilon, \epsilon] \times (-\delta, \delta) \to \mathbf{R}^3,$$

such that $f(u,0) = \Gamma(u)$ holds, where δ is a sufficiently small positive number.

We will apply this technique for the existence of isometric deformations of cuspidal edges in the next section.

3 Isometric Deformations of Cuspidal Edges

A C^∞-map $f : (U^2, p) \to \mathbf{R}^3$ has a *cuspidal edge* singularity at p if there exist local diffeomorphisms φ and Φ on $\mathbf{R}^2$ and $\mathbf{R}^3$ respectively such that

$$\Phi \circ f \circ \varphi(u,v) = f_0(u,v) \qquad (f_0 := (u, v^2, v^3)).$$

The map f_0 is called the *standard cuspidal edge*. Figure 1, left, is the image of the map f_0. Martins–Saji [6] proved that there exists a coordinate system (u, v) such that

$$f(u,v) = \left(u, \frac{a(u)u^2 + v^2}{2}, \frac{b_0(u)u^2 + b_2(u)uv^2}{2} + \frac{b_3(u,v)v^3}{6}\right).$$

We set

$$\kappa_s := a(0) \qquad \text{(called the } \textit{singular curvature}\text{)}, \tag{9}$$

$$\kappa_\nu := b_0(0) \qquad \text{(called the } \textit{limiting normal curvature}\text{)}, \tag{10}$$

$$\kappa_c := b_3(0,0) \qquad \text{(called the } \textit{cuspidal curvature}\text{)}. \tag{11}$$

These three values do not depend on the choice of such a coordinate system (u, v). The singular set of f is given by

$$S_f := \{(u,v) \in U^2 \,;\, v = 0\},$$

that is, the u-axis is the singular set and

$$\hat{\gamma}(t) := f(t, 0)$$

parametrizes the image of S_f. By the definition of cuspidal edges, the curve $\hat{\gamma}(t)$ must be a regular curve. Let $\kappa(t)$ be the curvature function of $\hat{\gamma}(t)$ as a space curve. As shown in [6], it holds that

$$\kappa(t) = \sqrt{\kappa_s(t)^2 + \kappa_\nu(t)^2}, \tag{12}$$

where κ_s is the singular curvature [cf. (9)] and κ_ν is the limiting normal curvature [cf. (10)]. Moreover, the singular curvature κ_s plays an important role for knowing the shape of singularities: An example of a cuspidal edge singularity with $\kappa_s > 0$ (resp. with $\kappa_s < 0$) is shown in Fig. 2, right and left, respectively.

The following fact is known.

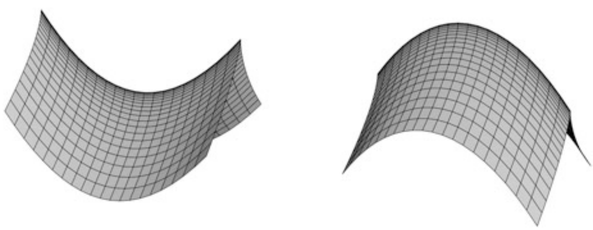

Fig. 2 Cuspidal edges with $\kappa_s < 0$ and $\kappa_s > 0$

Fact 2 ([9]). *The singular curvature κ_s is an intrinsic invariant. Moreover, if the Gaussian curvature is non-negative around the singular set, then $\kappa_s \leq 0$ holds.*

On the other hand, it is well-known that the unit normal vector field N is smoothly defined across the cuspidal edge singularities. Then the limiting normal curvature is defined by

$$\kappa_\nu(t) := \frac{\hat{\gamma}''(t) \cdot N(t)}{\hat{\gamma}'(t) \cdot \hat{\gamma}'(t)},$$

where $\gamma(t)$ is the singular curve in M^2 and

$$\hat{\gamma} := f \circ \gamma(t), \qquad N(t) := N \circ \gamma(t).$$

Also, the following facts are known ([9] and [7]):

- *If $\kappa_\nu > 0$, then K is unbounded and goes to $+\infty$ on one side of the singular curve γ and goes to $-\infty$ on the opposite side of γ.*
- *If the Gaussian curvature K is bounded in a neighborhood of γ, then κ_ν vanishes*[2] *along γ.*
- *A singular point p of f is also a singular point of the Gauss map of f if and only if $\kappa_\nu(p) = 0$.*

Later, we will show that κ_ν is not an intrinsic invariant. In (11), we introduced the cuspidal curvature κ_c at a given cuspidal edge singular point p. Let Π be the plane in $\mathbf{R}^3$ passing through $f(0,0)(= f(p))$ perpendicular to the vector $d\hat{\gamma}(0)/dt$. Then the intersection of the image of the singular set $f(S_f)$ by Π gives a $3/2$-cusp in the plane Π. The value $\kappa_c(p)$ coincides with the cuspidal curvature of this $3/2$-cusp (cf. [7], and for the definition of cuspidal curvature of planar curves, see [10]). The following assertion holds:

Theorem 2 ([7]). *The value $|\kappa_c \kappa_\nu|$ is an intrinsic invariant.*

A cuspidal edge satisfying $\kappa_\nu \neq 0$ is called a *generic* cuspidal edge (cf. Fig. 3).

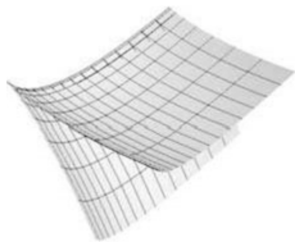

Fig. 3 A generic cuspidal edge

[2]The standard cuspidal edge as in Fig. 1, left, has an identically vanishing limiting normal curvature. In fact, it is a developable surface.

We now fix a real analytic map

$$f : (U^2, p) \to \mathbf{R}^3,$$

having a generic cuspidal edge singularity at p, where U^2 is a neighborhood of p in $\mathbf{R}^2$. Then we can parametrize the singular set S_f by a regular curve $\gamma(t)$ such that $\gamma(0) = p$ and t is the arc length parameter. Since p is a generic cuspidal edge singular point, we may assume that $\kappa_\nu(t)$ is positively valued. We denote by $\kappa(t)$ the curvature function of the regular space curve $\hat{\gamma}(t) := f \circ \gamma(t)$. The following assertion is an analogue of Corollary 1 for surfaces with cuspidal edge singularities:

Theorem 3 ([8]). *Let $\Gamma(t)$ be a real analytic regular space curve whose curvature function $\tilde{\kappa}(t)$ satisfies*[3]

$$|\kappa_s(t)| < \tilde{\kappa}(t). \tag{13}$$

Then there exists a real analytic cuspidal edge $g : (V^2, p) \to \mathbf{R}^3$ $(V^2 \subset U^2)$ such that

1. *the first fundamental form of g coincides with that of f. In particular, the singular set of S_g is a subset of S_f, and*
2. *$g \circ \gamma(t) = \Gamma(t)$ holds.*

Moreover, the number of such maps g is at most two up to motions in $\mathbf{R}^3$.

Remark 1. Kossowski [5] is the first geometer who considered the realizing problem of given first fundamental forms as cuspidal edge singularities. However, in [5], the isometric deformations of cuspidal edge singularities are not discussed, and the above theorem can be considered as a refinement of [5, Theorem 1].

Corollary 2. *The limiting normal curvature κ_ν is an extrinsic invariant.*[4]

Proof. Consider a deformation of γ so that its curvature function changes. Then by Theorem 3, the deformation of γ induces the deformation of cuspidal edge singularities which moves $\kappa_\nu (\neq 0)$ [cf. (12)].

Corollary 3. *There exists a real analytic map g satisfying the following conditions up to a motion of $\mathbf{R}^3$:*

- *g admits only cuspidal edge singularities,*
- *the first fundamental form of g coincides with that of f, and*
- *$g \circ \gamma(t)$ is a planar curve having the same curvature function as $\hat{\gamma}(t)$.*

Moreover, such a map g is uniquely determined up to a motion of $\mathbf{R}^3$.

[3] One cannot replace the condition $|\kappa_s(t)| < \tilde{\kappa}(t)$ by $|\kappa_s(t)| \le \tilde{\kappa}(t)$, see [8].

[4] Since the product $|\kappa_\nu \kappa_c|$ is intrinsic, κ_c is an extrinsic invariant.

4 Isometric Deformations of Cross Caps

A given C^∞-map $f : (U^2, p) \to \mathbf{R}^3$ is called a germ of a *cross cap* singularity if there exist local diffeomorphisms φ and Φ on $\mathbf{R}^2$ and $\mathbf{R}^3$ respectively such that

$$\Phi \circ f \circ \varphi(u, v) = f_0(u, v) \qquad (f_0 := (u, uv, v^2)).$$

The map f_0 is called the standard cross cap (cf. Fig. 1, right).

By a rotation, a translation in $\mathbf{R}^3$ and a suitable orientation preserving coordinate change of the domain $U^2 \subset \mathbf{R}^2$, we have the following Maclaurin expansion of f at a cross cap singularity $(0, 0)$ (cf. [1])

$$f(u, v) = \left(u, uv + \sum_{i=3}^{n} \frac{b_i}{i!} v^i, \sum_{r=2}^{n} \sum_{j=0}^{r} \frac{a_{j\,r-j}}{j!(r-j)!} u^j v^{r-j} \right) + O(u, v)^{n+1}, \tag{14}$$

where a_{02} never vanishes and $O(u, v)^{n+1}$ is a higher order term. By orientation preserving coordinate changes $(u, v) \mapsto (-u, -v)$ and $(x, y, z) \mapsto (-x, y, -z)$, we may assume that

$$a_{02} > 0, \tag{15}$$

where (x, y, z) is the usual Cartesian coordinate system of $\mathbf{R}^3$. After this normalization (15), one can easily verify that all of the coefficients a_{jk} and b_i are uniquely determined. An oriented local coordinate system (u, v) giving such a normal form is called the *canonical coordinate system* of f at the cross cap singularity. This unique expansion of a cross cap implies that the coefficients a_{jk} and b_i can be considered as geometric invariants of the cross cap f.

When $n = 3$, (14) reduces to

$$\begin{aligned} f(u, v) = &\left(u, uv, \frac{a_{20}\, u^2}{2} + a_{11}\, uv + \frac{a_{02}\, v^2}{2} \right) \\ &+ \frac{1}{3!} \left(0, b_3\, v^3, a_{30}\, u^3 + 3a_{21}\, u^2 v + 3a_{12}\, uv^2 + a_{03} v^3 \right) + O(u, v)^4. \end{aligned}$$

We are interested in the coefficients

$$a_{20},\ a_{11},\ a_{02},\ b_3,\ a_{30},\ a_{21},\ a_{12},\ a_{03},$$

which are all considered as invariants of cross caps. In particular, the invariant a_{20} plays a similar role as the singular curvature κ_s for cuspidal edges (compare Figs. 2 and 4).

As shown in the introduction, we can construct an isometric deformation of a cross cap as a ruled surface. In fact, the standard cross cap f_0 is a ruled surface, and

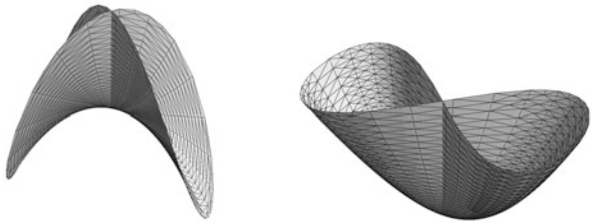

Fig. 4 Cross caps with $a_{20} < 0$ and $a_{20} > 0$, respectively

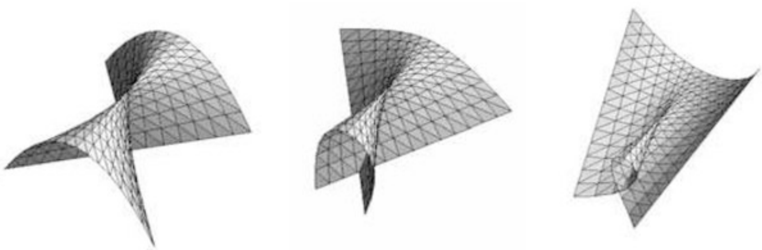

Fig. 5 An isometric deformation of f_0 (figures made by Atsufumi Honda)

Fig. 5 gives the isometric deformation of f_0. Examining this kind of deformation, the following assertion was shown:

Proposition 1 ([3]). *The three invariants* a_{03}, a_{12}, b_3 *are extrinsic.*

On the other hand, the following assertion is proved in [3] to give formulas for a_{20}, a_{11} and a_{02} using only the derivatives of the first fundamental forms:

Theorem 4 ([3]). *The three invariants* a_{02}, $|a_{11}|$, a_{02} *are intrinsic.*

Using these three invariants a_{02}, a_{11}, a_{02} and the canonical coordinate system (u, v), the Gaussian curvature around the origin is given by

$$K = \frac{a_{02}}{r^2 A_\theta^4} \left(a_{20} \cos^2 \theta - a_{02} \sin^2 \theta + O(r) \right),$$

where $u = r \cos \theta$, $v = r \sin \theta$ and

$$A_\theta := \sqrt{\cos^2 \theta + (a_{11} \cos \theta + a_{02} \sin \theta)^2}.$$

We note that the following assertion holds:

Proposition 2. *The set of self-intersections of a given cross cap singularity is contained in a line perpendicular to the tangential direction*[5] *of* f *if and only if*

[5] $df_p(T_pU^2)$ i a 1-dimensional vector space, which is called the tangential direction of f.

$$0 = b_3 = b_4 = b_5 = \cdots$$

holds[6].

We pose the following problem which was discussed in the meeting [4]:

Problem. *Can an arbitrarily fixed real analytic cross cap f admit an isometric deformation to a normal cross cap f_∞?*

If f is written as in (14), the corresponding normal cross cap should be written as

$$f_\infty(u,v) = \left(u, uv, \frac{a_{20}\,u^2}{2} + a_{11}\,uv + \frac{a_{02}\,v^2}{2}\right) + \frac{1}{3!}\left(0, 0, A_{30}\,u^3 + 3A_{21}\,u^2v + 3A_{12}\,uv^2 + A_{03}v^3\right) + O(u,v)^4,$$

where

$$A_{03} := a_{03} + \frac{3a_{11}b_3}{2}, \qquad A_{12} := a_{12} + \frac{(1+a_{11}^2)b_3}{2a_{02}},$$
$$A_{21} := a_{21} - \frac{a_{11}a_{20}b_3}{6a_{02}}, \qquad A_{30} := a_{30} - \frac{(1+a_{11}^2)a_{20}b_3}{2a_{02}^2}.$$

In [4], it is shown that A_{03}, A_{12}, A_{21} and A_{30} are all intrinsic invariants of cross caps, which is a reason for thinking that the normal form of f_∞ might be uniquely determined from f.

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Part III
Contributed Talks

Beyond Generalized Sasakian-Space-Forms!

Alfonso Carriazo

Abstract In this paper we will review some recent advances on the theory of generalized Sasakian-space-forms, as well as some new directions in which this theory is being developed now.

1 Introduction

The study of curvature properties is one of the main problems in Differential Geometry. As S.-S. Chern said in [15], "a fundamental notion is *curvature*, in its different forms". Therefore, the determination of the Riemann curvature tensor constitutes a very important topic.

In this sense, the author, jointly with P. Alegre and D. E. Blair, introduced generalized Sasakian-space-forms in [1], providing a new frame in which many works have been produced since then. In this paper, we will recall some well-known facts about these spaces (some of them can be found in [10]), as well as some new directions in which this theory is being developed now.

For more details and the proofs of the results presented in the following sections, we refer to the corresponding papers included in the references list.

2 Preliminaries

In this section, we recall some definitions and basic formulas which we will use later. For more background on almost contact metric manifolds, we recommend the reference [6].

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An odd-dimensional Riemannian manifold (M, g) is said to be an *almost contact metric manifold* if there exist on M a $(1, 1)$ tensor field ϕ, a vector field ξ (called the *structure vector field*) and a 1-form η such that $\eta(\xi) = 1$, $\phi^2(X) = -X + \eta(X)\xi$ and $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$, for any vector fields X, Y on M. In particular, in an almost contact metric manifold we also have $\phi\xi = 0$ and $\eta \circ \phi = 0$.

Such a manifold is said to be a *contact metric* manifold if $d\eta = \Phi$, where $\Phi(X, Y) = g(X, \phi Y)$ is called the *fundamental 2-form* of M. If, in addition, ξ is a Killing vector field, then M is said to be a *K-contact manifold*. It is well-known that a contact metric manifold is a K-contact manifold if and only if $\nabla_X \xi = -\phi X$, for any vector field X on M. On the other hand, the almost contact metric structure of M is said to be *normal* if $[\phi, \phi](X, Y) = -2d\eta(X, Y)\xi$, for any X, Y, where $[\phi, \phi]$ denotes the Nijenhuis torsion of ϕ, given by $[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$. A normal contact metric manifold is called a *Sasakian manifold*. It can be proved that an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X,$$

for any X, Y.

In [23], J. A. Oubiña introduced the notion of a trans-Sasakian manifold. An almost contact metric manifold M is a *trans-Sasakian manifold* if there exist two functions α and β on M such that

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X),$$

for any X, Y on M. If $\beta = 0$, M is said to be an *α-Sasakian manifold*. Sasakian manifolds appear as examples of α-Sasakian manifolds, with $\alpha = 1$. If $\alpha = 0$, M is said to be a *β-Kenmotsu manifold*. Kenmotsu manifolds are particular examples with $\beta = 1$. If both α and β vanish, then M is a *cosymplectic manifold*. Actually, in [22], J. C. Marrero showed that a trans-Sasakian manifold of dimension greater than or equal to 5 is either α-Sasakian, β-Kenmotsu or cosymplectic.

3 Generalized Sasakian-Space-Forms

A Sasakian manifold (M, ϕ, ξ, η, g) is said to be a *Sasakian-space-form* if all the ϕ-sectional curvatures $K(X \wedge \phi X)$ are equal to a constant c, where $K(X \wedge \phi X)$ denotes the sectional curvature of the section spanned by the unit vector field X, orthogonal to ξ, and ϕX. In such a case, the Riemann curvature tensor of M is given by

$$\begin{aligned} R(X,Y)Z = {} & \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} \\ & + \frac{c-1}{4}\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ & + \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ & + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}. \end{aligned} \tag{1}$$

These spaces can be modeled, depending on $c > -3$, $c = -3$ or $c < -3$.

As a natural generalization of these manifolds, P. Alegre, D. E. Blair and the author introduced in [1] the notion of *generalized Sasakian-space-forms*. They were defined as almost contact metric manifolds with Riemann curvature tensors satisfying an equation similar to (1), in which the constant quantities, $(c+3)/4$ and $(c-1)/4$ are replaced by differentiable functions, i.e., such that

$$\begin{aligned} R(X,Y)Z = {} & f_1\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ & + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}. \end{aligned}$$

We will denote such a space by $M(f_1, f_2, f_3)$ and we will write $R = f_1R_1 + f_2R_2 + f_3R_3$. Let us notice that, despite its name, a generalized Sasakian-space-form is not a Sasakian manifold in general; just an almost contact metric one.

Actually, the following theorem by P. Bueken and L. Vanhecke from [8], which we adapt to our notation, seemed to be an obstruction for the existence of generalized Sasakian-space-forms with non-constant functions:

Theorem 1 ([8]). *Let (M,ϕ,ξ,η,g) be a connected generalized Sasakian-space-form with $f_2 = f_3$ not identically zero. If* $\dim(M) \geq 5$ *and* $g(X,\nabla_X\xi) = 0$ *for any vector field X orthogonal to ξ, then f_1 and f_2 are constant functions and* $f_1 - f_2 \geq 0$. *Moreover, if $f_1 - f_2 = 0$, then (M,ϕ,ξ,η,g) is a cosymplectic-space-form and if $f_1 - f_2 = \alpha^2 > 0$ then (M,ϕ,ξ,η,g) or $(M,-\phi,\xi,\eta,g)$ is an α-Sasakian manifold with constant ϕ-sectional curvature c and a generalized Sasakian-space-form with* $f_1 = (c+3\alpha^2)/4$ *and* $f_2 = f_3 = (c-\alpha^2)/4$.

But we were able to construct in [1] interesting examples in any dimension. In fact, if we consider an almost Hermitian manifold (N,J,G) and we produce the warped product $M = \mathbb{R}\times_f N$, for any real warping function $f > 0$, it is easy to see that (M,ϕ,η,ξ,g_f) is an almost contact metric manifold, where $\phi X = (J\sigma_* X)^*$ for any vector field X in M, $\xi = \partial/\partial t$, $\eta(X) = g_f(X,\xi)$ and g_f is the usual warped metric on M, given by $g_f = \pi^*(g_{\mathbb{R}}) + (f\circ\pi)^2\sigma^*(G)$. It can be proved that M is β-Kenmotsu, with $\beta = f'/f$, if and only if N is Kaehler. In particular, if we consider a complex-space-form $N(c)$ of constant holomorphic curvature c, then M is a generalized Sasakian-space-forms with functions:

$$f_1 = \frac{c - 4f'^2}{4f^2}, \quad f_2 = \frac{c}{4f^2}, \quad f_3 = \frac{c - 4f'^2}{4f^2} + \frac{f''}{f}.$$

Concerning the structure of these spaces, it was proved in [1] that in any Sasakian generalized Sasakian-space-form we have $f_2 = f_3$, and in [2] that any contact metric generalized Sasakian-space-form with dimension greater than or equal to 5 is a Sasakian manifold. Then, it can be deduced from Theorem 1 that the only connected and contact metric generalized Sasakian-space-forms with dimensions greater than or equal to 5 are just Sasakian-space-forms (and so they have constant functions f_1, f_2, f_3). In dimension 3, the situation is different, because the writing of the Riemann curvature tensor is not unique. If we take two different writings $R = f_1 R_1 + f_2 R_2 + f_3 R_3$ and $R = f_1^* R_1 + f_2^* R_2 + f_3^* R_3$, then the functions are related by $f_1^* = f_1 + f$, $f_2^* = f_2 - f/3$, $f_3^* = f_3 + f$, where f is an arbitrary function. We can choose a canonical writing by putting $f_2 = 0$. It was proved in [2] that in any non-Sasakian, three-dimensional and contact metric generalized Sasakian-space-form, $R = -\kappa R_1 - 2\kappa R_3$, where $\kappa < 1$ is a constant.

With respect to trans-Sasakian generalized Sasakian-space-forms, we can emphasize the following results from [1, 2]:

Proposition 1 ([2]). *Let $M(f_1, f_2, f_3)$ be an α-Sasakian generalized Sasakian-space-form, with dimension greater than or equal to 5. Then α depends only on the direction of ξ and the functions f_1, f_3 and α satisfy the equation $f_1 - f_3 = \alpha^2$.*

Theorem 2 ([2]). *Let $M(f_1, f_2, f_3)$ be a connected α-Sasakian generalized Sasakian-space-form, with dimension greater than or equal to 5. Then, f_1 and f_2 are constant functions and, if either $\alpha = 0$ or $\alpha \neq 0$ at every point of M, then f_3 is also a constant function.*

Proposition 2 ([2]). *Let $M(f_1, f_2, f_3)$ be a β-Kenmotsu generalized Sasakian-space-form. Then, β depends only on the direction of ξ and the functions f_1, f_3 and β satisfy the equation $f_1 - f_3 + \xi(\beta) + \beta^2 = 0$.*

Theorem 3 ([1]). *Let $M(f_1, f_2, f_3)$ be a β-Kenmotsu generalized Sasakian-space-form, with dimension greater than or equal to 5. Then, $X(f_i) = 0$ for any X orthogonal to ξ, $i = 1, 2, 3$, and the following equations hold:*

$$\xi(f_1) + 2\beta f_3 = 0, \quad \xi(f_2) + 2\beta f_2 = 0.$$

On the other hand, there are examples of three-dimensional, (α, β)-trans-Sasakian generalized Sasakian-space-forms. In fact, if M is a three-dimensional, (α, β)-trans-Sasakian manifold such that α, β depend only on the direction of ξ, then it is a generalized Sasakian-space-form with functions $f_1 = 3\tau - 2\alpha^2 + 2\xi(\beta) + 2\beta^2$, $f_2 = 0$ and $f_3 = 3\tau - 3\alpha^2 + 3\xi(\beta) + 3\beta^2$, where τ is the scalar curvature of M (see [2]).

The study of generalized Sasakian-space-forms was continued by the author, jointly with P. Alegre, in [3], by analyzing their behavior under generalized D-conformal deformations. With certain conditions, new examples of generalized Sasakian-space-forms can be obtained.

In the next sections, we will examine three different directions to extend this theory. The first idea is to study a similar situation in manifolds with more than one structure vector field: these spaces are known as metric f-manifolds. This will lead to the idea of *generalized S-space-forms* (see Sect. 4). The second direction is motivated by the writing of the curvature tensor of a (κ, μ)-space with constant ϕ-sectional curvature given by T. Koufogiorgos in [19], resulting in the definition of *generalized* (κ, μ)*-space-forms* presented in Sect. 5. Finally, *semi-Riemannian generalized Sasakian-space-forms* are considered as a third possible extension in Sect. 6.

4 Generalized S-Space-Forms

A tensor field f of type (1,1) and rank $2m$ on a manifold M is called an f*-structure* if it satisfies $f^3 + f = 0$, and a $(2m+s)$-dimensional Riemannian manifold (M, g) endowed with an f-structure f is said to be a *metric* f*-manifold* if, moreover, there exist s global vector fields $\xi_1, \ldots, \xi_s$ on M (called *structure vector fields*) such that, if $\eta_1, \ldots, \eta_s$ are the dual 1-forms of $\xi_1, \ldots, \xi_s$, then

$$f\xi_\alpha = 0, \ \eta_\alpha \circ f = 0, \ f^2 = -I + \sum_{\alpha=1}^{s} \eta_\alpha \otimes \xi_\alpha,$$

$$g(X, Y) = g(fX, fY) + \sum_{\alpha=1}^{s} \eta_\alpha(X)\eta_\alpha(Y),$$

for any vector fields X, Y on M and any $\alpha = 1, \ldots, s$. Almost contact metric manifolds are particular cases of metric f-manifolds with $s = 1$. In this frame, there are manifolds playing a similar role to that of Sasakian manifolds; they are called S*-manifolds*. An S-manifold of constant f-sectional curvature is said to be an S*-space-form*, whose curvature tensor is determined.

In [11], the author, jointly with L. M. Fernández and A. M. Fuentes, studied metric f-manifolds with two structure vector fields ξ_1, ξ_2, and it was said that $(M, f, \xi_1, \xi_2, \eta_1, \eta_2, g)$ is a *generalized* S*-space-form* if there exists differentiable functions $F_1, \ldots, F_8$ on M such that the curvature tensor field of M satisfies

$$\begin{aligned}R(X,Y)Z = {} & F_1\{g(Y,Z)X - g(X,Z)Y\}\\ & + F_2\{g(X,fZ)fY - g(Y,fZ)fX + 2g(X,fY)fZ\}\\ & + F_3\{\eta_1(X)\eta_1(Z)Y - \eta_1(Y)\eta_1(Z)X + g(X,Z)\eta_1(Y)\xi_1 - g(Y,Z)\eta_1(X)\xi_1\}\\ & + F_4\{\eta_2(X)\eta_2(Z)Y - \eta_2(Y)\eta_2(Z)X + g(X,Z)\eta_2(Y)\xi_2 - g(Y,Z)\eta_2(X)\xi_2\}\\ & + F_5\{\eta_1(X)\eta_2(Z)Y - \eta_1(Y)\eta_2(Z)X + g(X,Z)\eta_1(Y)\xi_2 - g(Y,Z)\eta_1(X)\xi_2\}\\ & + F_6\{\eta_2(X)\eta_1(Z)Y - \eta_2(Y)\eta_1(Z)X + g(X,Z)\eta_2(Y)\xi_1 - g(Y,Z)\eta_2(X)\xi_1\}\\ & + F_7\{\eta_1(X)\eta_2(Y)\eta_2(Z)\xi_1 - \eta_2(X)\eta_1(Y)\eta_2(Z)\xi_1\}\\ & + F_8\{\eta_2(X)\eta_1(Y)\eta_1(Z)\xi_2 - \eta_1(X)\eta_2(Y)\eta_1(Z)\xi_2\},\end{aligned}$$

for any vector fields X, Y, Z on M. Let us notice how the first terms in the above equation correspond to tensors R_1, R_2, R_3 appearing in the writing of the curvature tensor of a generalized Sasakian-space-form, while the last ones are new; they appear now because of the existence of more than one structure vector field. This definition is justified by the following examples:

Example 1 ([11]). Any S-space-form $M(c)$ with two structure vector fields is a generalized S-space-form with functions:

$$F_1 = \frac{c+6}{4},\ F_2 = F_7 = F_8 = \frac{c-2}{4},\ F_3 = F_4 = \frac{c+2}{4},\ F_5 = F_6 = -1.$$

Example 2 ([11]). If M is a pseudo-umbilical hypersurface of a generalized Sasakian-space-form $\tilde{M}(f_1, f_2, f_3)$, with shape operator given by

$$A = g_1\,(I - \eta_1 \otimes \xi_1) + g_2\eta_2 \otimes \xi_2 - \eta_1 \otimes \xi_2 - \eta_2 \otimes \xi_1,$$

then M is a generalized S-space-form with functions:

$$F_1 = f_1 + g_1^2,\ F_2 = f_2,\ F_3 = f_2 + g_1^2,\ F_4 = -g_1 g_2,$$
$$F_5 = F_6 = g_1,\ F_7 = F_8 = -1 - g_1 g_2.$$

Example 3 ([11]). If M is the warped product of the real line $\mathbb{R}$ times a generalized Sasakian-space-form $\tilde{M}(f_1, f_2, f_3)$, with warping function h, then M is a generalized S-space-form with functions:

$$F_1 = \frac{(f_1\circ\pi_2) - h'^2}{h^2},\ F_2 = \frac{f_2\circ\pi_2}{h^2},\ F_3 = F_7 = F_8 = \frac{f_3\circ\pi_2}{h^2},$$
$$F_4 = \frac{(f_1\circ\pi_2) - h'^2}{h^2} + \frac{h''}{h},\ F_5 = F_6 = 0.$$

In particular, if $\tilde{M} = \mathbb{R}\times_f N(c)$, where $N(c)$ is a complex-space-form, then $M = \mathbb{R}\times_h (\mathbb{R}\times_f N(c))$ is a generalized S-space-form with functions:

$$F_1 = \frac{c - 4(f')^2 - 4f^2(h')^2}{4f^2h^2}, \ F_2 = \frac{c}{4f^2h^2}, \ F_3 = F_7 = F_8 = \frac{c - 4(f')^2 + 4ff''}{4f^2h^2},$$

$$F_4 = \frac{c - 4(f')^2 - 4f^2(h')^2 + 4f^2hh''}{4f^2h^2}, \ F_5 = F_6 = 0.$$

The paper [11] also contains some results about the structure of generalized S-space-forms with two structure vector fields. M. Falcitelli and A. M. Pastore defined in [17] *generalized globally framed f-space-forms*, in the general case of having s structure vector fields. A different extension has been recently given by L. M. Fernández, A. M. Fuentes and A. Prieto-Martín in [18].

5 Generalized (κ, μ)-Space-Forms

Given two functions κ, μ, a contact metric manifold is said to be a generalized (κ, μ)-space if its curvature tensor satisfies the condition

$$R(X, Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\},$$

for any vector fields X, Y, where $2h = L_\xi\phi$ and L is the usual Lie derivative. If κ, μ are constant, the manifold is called a (κ, μ)-space. These spaces were defined by D. E. Blair, T. Koufogiorgos and B. J. Papantoniou in [7] and it was proved in [19] that if a (κ, μ)-space has constant ϕ-sectional curvature c and dimension greater than 3, then its curvature tensor is given by

$$R = \frac{c+3}{4}\,R_1 + \frac{c-1}{4}\,R_2 + \left(\frac{c+3}{4} - \kappa\right)R_3 + R_4 + \frac{1}{2}R_5 + (1-\mu)\ R_6,$$

where R_1, R_2, R_3 are the tensors defined in Sect. 3 and

$$R_4(X, Y)Z = g(Y, Z)hX - g(X, Z)hY + g(hY, Z)X - g(hX, Z)Y,$$

$$R_5(X, Y)Z = g(hY, Z)hX - g(hX, Z)hY + g(\phi hX, Z)\phi hY - g(\phi hY, Z)\phi hX,$$

$$R_6(X, Y)Z = \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX, Z)\eta(Y)\xi - g(hY, Z)\eta(X)\xi,$$

for any vector fields X, Y, Z. Therefore, the author, jointly with V. Martín-Molina and M. M. Tripathi, defined in [14] *generalized* (κ, μ)*-space-forms* as those almost contact metric manifolds whose curvature tensor can be written as $R = f_1R_1 + \cdots + f_6R_6$, with functions $f_1, \ldots, f_6$. Of course, generalized Sasakian-space-forms are particular cases of these new spaces, where $f_4 = f_5 = f_6 = 0$.

It is well-known that $h = 0$ in K-contact manifolds, and hence in Sasakian ones. Therefore, generalized (κ, μ)-space-forms with such a structure are just generalized Sasakian-space-forms. The case of the non-Sasakian, contact metric, generalized

(κ, μ)-space-forms with dimension greater than or equal to 5 is completely studied in [14]: they are just (κ, μ)-space-forms with constant ϕ-sectional curvature $c = 2f_6 - 1 > -3$ such that:

$$f_1 = \frac{f_6+1}{2},\ f_2 = \frac{f_6-1}{2},\ f_3 = \frac{3f_6+1}{2},\ f_4 = 1,\ f_5 = \frac{1}{2},\ \kappa = -f_6 < 1,\ \mu = 1 - f_6.$$

Actually, a nice method to obtain examples for every constant $f_6 > -1$ is also described in [14]. The three-dimensional case is similar to that of generalized Sasakian-space-forms: the writing of the curvature tensor is not unique. Therefore, it has been studied separately. Examples with non-constant functions have also been given in that case.

An important difference of generalized (κ, μ)-space-forms with respect to generalized Sasakian-space-forms is their behavior with respect to deformations. Actually, the author and V. Martín-Molina introduced in [12] a small change in the definition when they proved that, in order to preserve the structure of a contact metric generalized (κ, μ)-space-form under a D_a-homothetic deformation, it was necessary to split the tensor R_5 into two new tensors $R_{5,1}$, $R_{5,2}$ given by:

$$\begin{aligned} R_{5,1}(X,Y)Z &= g(hY,Z)hX - g(hX,Z)hY, \\ R_{5,2}(X,Y)Z &= g(\phi hY,Z)\phi hX - g(\phi hX,Z)\phi hY. \end{aligned}$$

This gave rise to the notion of *generalized* (κ, μ)*-space-forms with divided* R_5, where the writing of the curvature tensor takes the form $R = f_1 R_1 + \cdots + f_{5,1} R_{5,1} + f_{5,2} R_{5,2} + f_6 R_6$. Since $R_5 = R_{5,1} - R_{5,2}$, it is obvious that these spaces include the previous ones.

Going one step further, the author, jointly with K. Arslan, V. Martín-Molina and C. Murathan, introduced in [5] *generalized* (κ, μ, ν)*-space-forms* as those almost contact metric manifolds whose curvature tensor can be written as $R = f_1 R_1 + \cdots + f_8 R_8$, where the new tensors R_7 and R_8 are given by

$$\begin{aligned} R_7(X,Y)Z &= g(Y,Z)\phi hX - g(X,Z)\phi hY + g(\phi hY,Z)X - g(\phi hX,Z)Y, \\ R_8(X,Y)Z &= \eta(X)\eta(Z)\phi hY - \eta(Y)\eta(Z)\phi hX + g(\phi hX,Z)\eta(Y)\xi - g(\phi hY,Z)\eta(X)\xi. \end{aligned}$$

The contact metric case was completely studied.

Of course, this notion is not arbitrary, but was motivated by several previous definitions, results and computations. For example, T. Koufogiorgos, M. Markellos and V. J. Papantoniou introduced in [20] the notion of (κ, μ, ν)*-contact metric manifold*, where now the equation to be satisfied is

$$\begin{aligned} R(X,Y)\xi =& \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\} \\ &+ \nu\{\eta(Y)\phi hX - \eta(X)\phi hY\}, \end{aligned}$$

for some smooth functions κ, μ, ν. Actually, the computation of the curvature tensor of a (κ, μ, ν)-contact metric manifold of dimension 3 motivated the previous writing in terms of tensors $R_1, \dots, R_8$.

In [13], the author, jointly with V. Martín-Molina, studied the (κ, μ, ν)-spaces with almost cosymplectic and almost Kenmotsu structures, giving explicitly the writing of their curvature tensors. That led to the definition of *generalized* (κ, μ, ν)-*space-forms with divided* R_5, of which they provided examples or obstruction results in all possible cases.

6 Semi-Riemannian Generalized Sasakian-Space-Forms

The third extension of the notion of generalized Sasakian-space-form that we are reviewing in this paper rises from a natural question: is it possible to define a similar space with a semi-Riemannian metric?

To give an answer to this question, we first need to recall that an *ε-almost contact metric manifold* [16] or *almost contact pseudo-metric manifold* [9] is an odd-dimensional semi-Riemannian manifold (M^{2n+1}, g), with a structure (ϕ, ξ, η) such that $\eta(\xi) = 1$, $\phi^2 X = -X + \eta(X)\xi$, $g(\phi X, \phi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y)$ and $\eta(X) = \varepsilon g(X, \xi)$, for any vectors fields X, Y on M, where $\varepsilon = \pm 1$. It follows from the above conditions that $g(\xi, \xi) = \varepsilon$, i.e., ε indicates the causal character of ξ: $\varepsilon = 1$ (resp. $\varepsilon = -1$) if ξ is a spacelike (resp. timelike) vector field. Given that X and ϕX have the same causal character, we get M^{2n+1}_{2s} and M^{2n+1}_{2s+1} for $\varepsilon = 1$ and $\varepsilon = -1$, respectively. We will use the name *indefinite almost contact metric manifold*. For index $s = 0$ and $\varepsilon = 1$ we obtain almost contact metric manifolds, and for $s = 0$ and $\varepsilon = -1$, almost contact Lorentzian manifolds.

In [21], J. W. Lee considered *generalized indefinite Sasakian-space-forms* whose curvature tensors can be written as $R = f_1 R_1 + f_2 R_2 + f_3 R_{3,\varepsilon}$, where

$$R_{3,\varepsilon}(X, Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \varepsilon g(X, Z)\eta(Y)\xi - \varepsilon g(Y, Z)\eta(X)\xi.$$

Notice that $R_{3,1} = R_3$. In [4], the author, jointly with P. Alegre, constructed examples of these spaces with non-constant functions in any dimension, by using warped products, and studied their structures.

Another useful notion is that of *ε-almost para-contact metric manifolds*, introduced in [24] as those semi-Riemannian manifolds (M^{2n+1}, g), endowed with an almost para-contact structure (ϕ, ξ, η) such that $\eta(\xi) = -1$, $\phi^2 X = X + \eta(X)\xi$, $g(\phi X, \phi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y)$ and $\eta(X) = -\varepsilon g(X, \xi)$, for any vector fields X, Y on M, where $\varepsilon = \pm 1$. As above, ε indicates the causal character of ξ. In this frame, the similar writing for the curvature tensor of what we can call a *generalized indefinite para-Sasakian-space-form* seems to be $R = f_1 R_1 + f_2 \tilde{R}_2 + f_3 \tilde{R}_{3,\varepsilon}$, where

$$\tilde{R}_2(X,Y)Z = g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X,$$

$$\tilde{R}_{3,\varepsilon}(X,Y)Z = -\varepsilon\eta(X)\eta(Z)Y + \varepsilon\eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi.$$

Examples and some results about these spaces can also be found in [4].

Finally, let us recall that an odd-dimensional manifold with an indefinite metric (M,g) is said to be an *hyperbolic almost contact metric manifold* if there exits on M a $(1,1)$ tensor field ϕ, a vector field ξ and a 1-form η such that $\eta(\xi) = -1$, $\phi^2 X = X + \eta(X)\xi$, $g(\phi X, \phi Y) = -g(X,Y) - \eta(X)\eta(Y)$ and $\eta(X) = g(X,\xi)$, for any vector fields X, Y on M. By using again warped products, P. Alegre and the author gave in [4] examples of *generalized hyperbolic Sasakian-space-forms* with curvature tensor $R = f_1 R_1 + f_2 R_2 + f_3 R_3$, i.e., the same writing than in the original case with which we began this paper.

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Some Geometric Aspects of the Hessian One Equation

Antonio Martínez and Francisco Milán

Abstract The Hessian one equation and its complex resolution provides an important tool in the study of improper affine spheres in $\mathbb{R}^3$ with some kind of singularities. The singular set can be characterized and, in most of the cases, it determines the surface. Here, we show how to obtain improper affine spheres with a prescribed singular set and construct some global examples with the desired singularities. We also classify improper affine spheres admitting a planar singular set.

1 Introduction

Differential geometry of surfaces and partial differential equations (PDEs) are related by a productive tie by means of which both theories out mutually benefited.

Many classic partial differential equations (PDEs) are link to interesting geometric problems, [18, 20, 27]. Sometimes, the geometry allows to establish non trivial properties of the solutions and to determine new solutions in terms of already known solutions.

One of the biggest contributions from geometry to the theory of partial differential equations is the Monge Ampère equation. Among the most outstanding Monge Ampère equation we can quote the Hessian one equation

$$\phi_{xx}\phi_{yy} - \phi_{xy}^2 = \varepsilon, \qquad \varepsilon \in \{-1, 1\}. \tag{1}$$

This is the easiest Monge Ampère equation and it appears, among others, in problems of affine differential geometry, flat surfaces or special Kähler manifolds.

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The Eq. (1) has been studied from a global perspective and the situation changes completely if we take $\varepsilon = 1$ (definite case) or $\varepsilon = -1$ (indefinite case). When $\varepsilon = 1$, Jörgens, [13, 14], proved that revolution surfaces provide the only entire solutions with at most an isolated singularity and solutions in $\mathbb{R}^2$ with a finite set of points removed are classified in [9]. The indefinite case is more complicated and we can not expect a classification result as in the definite case. Actually, $\phi(x, y) = xy + g(x)$ is an entire solution for any function g.

Another important issue in the theory of geometric PDEs is the study of singularities. Concerning with (1), a geometric theory of smooth maps with singularities (improper affine maps) has been developed in [21, 25]. In most of the cases the singular set determines the surface and, generically, the singularities are cuspidal edges and swallowtails, see [1, 6, 12, 23, 24].

In this paper we show how to obtain easily improper affine maps with a prescribed singular set and construct some global examples with the desired singularities. We also classify definite improper affine maps admitting a planar singular set.

The paper is organized as follows. In Sect. 2 we introduce some notations and give a complex resolution for the Eq. (1).

In Sect. 3 we discuss a priori conditions on a curve in $\mathbb{R}^3$ to be a singular curve of an improper affine map with prescribed cuspidal edges and swallowtails. We also study isolated singularities both from a local and a global view.

In Sect. 4 we describe the global behavior of embedded complete definite improper affine maps with a planar singular set and those with only a finite number of isolated singularities.

2 The Conformal Structure

Let $\phi : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a solution to (1) on a planar domain Ω. Then its graph

$$\psi = \{(x, y, \phi(x, y)) : \ (x, y) \in \Omega\}$$

describes an improper affine sphere in the affine 3-space $\mathbb{R}^3$ with constant affine normal $\xi = (0, 0, 1)$, affine metric h,

$$h := \phi_{xx}\, dx^2 + \phi_{yy}\, dy^2 + 2\phi_{xy}\, dxdy, \tag{2}$$

and affine conormal N,

$$N := (-\phi_x, -\phi_y, 1). \tag{3}$$

From (2) and (3) it is easy to check that the following relations hold,

$$h = - < dN, d\psi >, \quad < N, \xi >= 1, \quad < N, d\psi >= 0, \tag{4}$$

$$\sqrt{\det(h)} = \det[\psi_x, \psi_y, \xi] = -\det[N_x, N_y, N], \tag{5}$$

see [19, 26] for more details. Conversely, up to unimodular transformations, any improper affine sphere in $\mathbb{R}^3$ is, locally, the graph over a domain in the x, y-plane of a solution to (1).

When $\varepsilon = 1$ (resp. $\varepsilon = -1$) the affine metric h induces a Riemann (Lorentz) surface structure on Ω known as the *underlying conformal structure of* $\phi(x, y)$.

It follows from (1) that,

$$(d\phi_x)^2 + \varepsilon\, dy^2 = \phi_{xx} h, \qquad (d\phi_y)^2 + \varepsilon\, dx^2 = \phi_{yy} h, \tag{6}$$

and the expression (6) indicates that the two first coordinates of ψ and N provide conformal parameters for h. Actually, consider $\mathbb{C}_\varepsilon$ the complex (split-complex) numbers according to $\varepsilon = 1$ (or $\varepsilon = -1$), that is

$$\mathbb{C}_\varepsilon = \{z = s + j\ t\ :\ s, t \in \mathbb{R},\ j^2 = -\varepsilon,\ j1 = 1j\}, \tag{7}$$

see [4, 11] for more information, then it is not difficult to prove, see [3, 8, 23], that $\Phi : \Omega \longrightarrow \mathbb{C}^3_\varepsilon$,

$$\Phi := N + j\ \xi \times \psi, \tag{8}$$

is a planar holomorphic (split-holomorphic) curve. In fact, $\Phi = (-B, A, 1)$ where

$$A := -\phi_y + j\ x, \qquad B := \phi_x + j\ y, \tag{9}$$

are holomorphic (split-holomorphic) functions on Ω. Moreover, from (1) and (2),

$$|d\Phi|^2 = |dA|^2 + |dB|^2 = (\phi_{xx} + \phi_{yy})h, \tag{10}$$

and $|d\Phi|^2$ and h are in the same conformal class always that $\phi_{xx} + \phi_{yy}$ has a sign.

From (2) and (9), the metric h is given by

$$h := \text{Im}(dA\overline{dB}) = |dG|^2 - |dF|^2 \tag{11}$$

where $2F = -B - \varepsilon j\ A$ and $2G = B - \varepsilon j\ A$, and the immersion ψ may be recovered as

$$\psi := \text{Im}(A, B, \int A d\overline{B}) = -\frac{1}{2}\,\text{Im}\int (\Phi + \overline{\Phi}) \times d\Phi. \tag{12}$$

or

$$\psi := (G + \overline{F}, \frac{|G|^2}{2} - \frac{|F|^2}{2} + 2\,\text{Re}\int G d\overline{F}), \tag{13}$$

where in (13) the two first coordinates of ψ are identified as numbers of $\mathbb{C}_\varepsilon$ in the standard way.

Remark 1. The complex representation (12) is similar to the introduced in [5] and (13) was studied in [7, 8, 21, 25].

3 Allowing Singularities

In this section we discuss improper affine spheres admitting some kind of singularities. First, we study when a prescribed curve of singularities determines the surface and then we deal with the case of isolated singularities both from a local and a global view.

Definition 1. Let Σ be a Riemann (Lorentz) surface and $\psi : \Sigma \longrightarrow \mathbb{R}^3$ be a differentiable map, ψ is called an improper affine map with constant affine normal $\xi = (0,0,1)$, if ψ is given as in (12) for some holomorphic (split-holomorphic) curve $\Phi = (-B, A, 1) : \Sigma \longrightarrow \mathbb{C}^3_\varepsilon$ satisfying that $\mathrm{Im}(dA\overline{dB})$ does not vanish identically on Σ.

Remark 2. Equivalent definitions of improper affine maps (also called improper affine fronts by other authors) have been introduced in [15, 21, 25].

From (3), (9) and (12) one may write $\Phi = N + j\ \xi \times \psi$, where N is the affine conormal of ψ and we have

$$h = \mathrm{Im}(dA\overline{dB}) = -\frac{\varepsilon j}{4}\det[\Phi + \overline{\Phi}, d\Phi, d\overline{\Phi}].$$

The singular set of ψ is the set of points where h degenerates. A singular point z_0 is called non degenerate if, writing $h = \rho|dz|^2$ around z_0, then

$$\rho(z_0) = 0, \qquad d\rho|_{z_0} \neq 0.$$

When z_0 is a non degenerate singular point, $\psi(z_0)$ is either an isolated singularity or the singular set of ψ around z_0 becomes a regular curve $\gamma : I \subset \mathbb{R} \longrightarrow \Sigma$. Generically, the image of these curves are singular curves with cuspidal edges and swallowtails, see [6, 12, 25]. In [17], we have the following criterion for the singular curve $\alpha = \psi \circ \gamma$,

Theorem 1 ([17]). *If η is a vector field along γ, with $\eta(s) \neq 0$ in the kernel of $d\psi_{\gamma(s)}$ for any s in the interval I, then*

1. *$\gamma(0) = z_0$ is a cuspidal edge if and only if $\det[\gamma\,'(0), \eta(0)] \neq 0$, where* det *denotes the usual determinant and prime indicates differentiation with respect to s.*
2. *$\gamma(0) = z_0$ is a swallowtail if and only if $\det[\gamma\,'(0), \eta(0)] = 0$ and*

$$\left.\frac{d}{ds}\right|_{s=0} \det[\gamma\,'(s), \eta(s)] \neq 0.$$

3.1 Prescribing Singular Curves

The *affine Björling problem* of finding an improper affine map containing a curve α with a prescribed affine conormal U along it has been discuss in [1, 23] and its solution is applied to see that a non constant singular curve determines the surface.

Actually, if we assume that $\alpha : I \longrightarrow \mathbb{R}^3$ is an analytic curve, which is in the singular set of a definite (indefinite) improper affine map ψ^ε, then from (4) the affine conormal U along α satisfies

$$< \alpha', U >= 0, \qquad < U, \xi >= 1, \qquad < \alpha'', U >= 0.$$

Hence, if α is non constant but $\det[\alpha', \alpha'', \xi] \equiv 0$ on I, then $\alpha' \times \alpha'' \equiv 0$ and α is an straight line with a constant tangent vector ν. In this case, $< N, \nu >= 0$ and the conormal N of ψ^ε satisfies $\det[N, N_z, N_{\bar{z}}] \equiv 0$ on a neighborhood of α which is a contradiction.

But, if $\det[\alpha', \alpha'', \xi] \neq 0$ on I, then U is uniquely determined by α and it may be written as

$$U = \frac{\alpha' \times \alpha''}{\det[\alpha', \alpha'', \xi]}. \tag{14}$$

Then, ψ^ε is uniquely determined as in (12) by the holomorphic (split-holomorphic) curve

$$\Phi_\varepsilon = \frac{\alpha_z \times \alpha_{zz}}{\det[\alpha_z, \alpha_{zz}, \xi]} + j\ \xi \times \alpha, \tag{15}$$

which is defined in a neighborhood of I in $\mathbb{C}_\varepsilon$ where the holomorphic (split-holomorphic) extension of α is well defined.

Theorem 2. *Let* $\alpha : I \longrightarrow \mathbb{R}^3$ *be an analytic curve satisfying* $\det[\alpha', \alpha'', \xi] \neq 0$ *on* I*. Then the following items hold*

- *there exists a unique definite improper affine map containing* $\alpha(I)$ *in its singular set.*
- *if* $\det[\alpha', \alpha'', \alpha''']^2 \neq \det[\alpha', \alpha'', \xi]^4$ *on* I*, then there exists a unique indefinite improper affine map containing* $\alpha(I)$ *in its singular set.*

Moreover, in both cases $\alpha(s)$ *is a cuspidal edge for all* $s \in I$ *(Figs. 1 and 2).*

Proof. From (12) and (14), we have that along I the improper affine map ψ^ε given by Φ_ε satisfies

$$\begin{aligned}\psi^\varepsilon_z &= \frac{\varepsilon\, j}{4}((\Phi + \overline{\Phi}) \times \Phi_z) = \frac{\varepsilon\, j}{2}\ U \times U' - \frac{1}{2} U \times \left(\xi \times \alpha'\right) \\ &= \frac{1}{2}\alpha' + \frac{\varepsilon\, j}{2}\ U \times U' = \frac{1}{2}\alpha' + \frac{\varepsilon\, j}{2}\ \frac{\det[\alpha', \alpha'', \alpha''']}{\det[\alpha', \alpha'', \xi]^2}\alpha'\end{aligned}$$

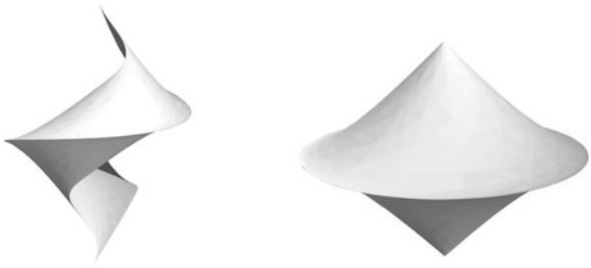

Fig. 1 Indefinite improper affine maps whose singular set contains $\alpha(s) = (\cos(s), \sin(s), as)$ with $a = 0.2$ and $a = 0$

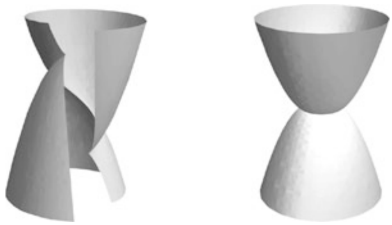

Fig. 2 Definite improper affine maps whose singular set contains $\alpha(s) = (\cos(s), \sin(s), as)$ with $a = 0.5$ and $a = 0$

and ψ^ε contains the curve α with

$$\psi_s^\varepsilon = \alpha', \qquad \psi_t^\varepsilon = -\frac{\det[\alpha', \alpha'', \alpha''']}{\det[\alpha', \alpha'', \xi]^2}\alpha'. \tag{16}$$

Thus, from (4), (5) and (16), we get $\det[\psi_s^\varepsilon, \psi_t^\varepsilon, \xi](s,0) = 0, \quad \forall s \in I$, and

$$\begin{aligned}\frac{d}{dt}\Big|_{(s,0)} \det[\psi_s^\varepsilon, \psi_t^\varepsilon, \xi] &= \det[\psi_{ts}^\varepsilon, \psi_t^\varepsilon, \xi](s,0) - \varepsilon \det[\psi_s^\varepsilon, \psi_{ss}^\varepsilon, \xi](s,0) \\ &= \det[\alpha', \alpha'', \xi]\Big(- \varepsilon - \frac{\det[\alpha', \alpha'', \alpha''']^2}{\det[\alpha', \alpha'', \xi]^4}\Big) \neq 0.\end{aligned}$$

That is, α is a non degenerate singular curve and the kernel of $d\psi^\varepsilon$ at $\gamma(s) = (s,0)$ is spanned by $\eta = (\det[\alpha', \alpha'', \alpha'''], \det[\alpha', \alpha'', \xi]^2)$. We conclude that $\det(\gamma', \eta) = \det[\alpha', \alpha'', \xi]^2 \neq 0$ and $\alpha(s)$ is a cuspidal edge for all $s \in I$ from Theorem 1.

Theorem 3. *Let* $\alpha : I \longrightarrow \mathbb{R}^3$ *be an analytic curve satisfying* $\det[\alpha', \alpha'', \xi] \neq 0$ *on* $I \setminus \{0\}$ *and such that* $0 \in I$ *is a zero of* α', $\alpha' \times \alpha''$, $\det[\alpha', \alpha'', \xi]$ *and* $\det[\alpha', \alpha'', \alpha''']$ *of order* $1, 2, 2$ *and* 3 *respectively. Then the following items hold*

- *there exists a unique definite improper affine map containing* $\alpha(I)$ *in its singular set.*
- *if* $\det[\alpha', \alpha'', \alpha''']^2 \neq \det[\alpha', \alpha'', \xi]^4$ *on* $I \setminus \{0\}$*, then there exists a unique indefinite improper affine map containing* $\alpha(I)$ *in its singular set.*

Moreover, in both cases $\alpha(0)$ *is a swallowtail (Fig. 3).*

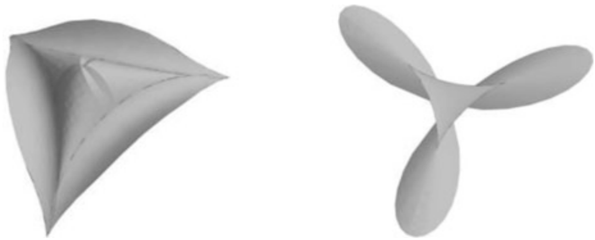

Fig. 3 Improper affine maps with three swallowtails

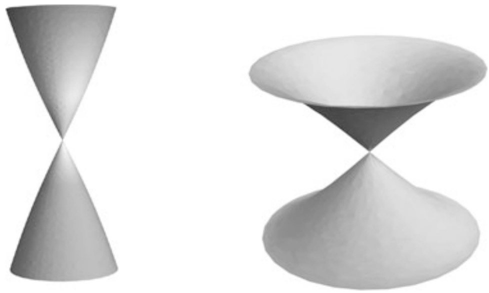

Fig. 4 Improper affine maps with isolated singularities

Proof. Following the same arguments as in the proof of Theorem 2, we have that α is a non degenerate singular curve of ψ^{ε} and the kernel of $d\psi^{\varepsilon}$ at $\gamma(s) = (s, 0)$ is spanned by $\eta = (1, \det[\alpha', \alpha'', \xi]^2 / \det[\alpha', \alpha'', \alpha'''])$. But from the hypothesis, 0 is a zero of order 1 of $\det(\gamma', \eta) = \det[\alpha', \alpha'', \xi]^2 / \det[\alpha', \alpha'', \alpha''']$ and $\alpha(0)$ is a swallowtail from Theorem 1.

3.2 *Isolated Singularities*

It is well known, see [1], that the conformal structure of the affine metric around any isolated singularity of a definite improper affine map is that of an annulus. Moreover, any definite improper affine map must be symmetric with respect to point reflection in $\mathbb{R}^3$ through any isolated embedded singularity.

In the case of indefinite improper affine maps and when the conformal structure of the affine metric around an isolated singularity is that of an annulus we have as application of the affine Björling problem that (see [1, 24]),

Theorem 4 ([1, 24]). *Let $U : \mathbb{R} \longrightarrow \mathbb{R}^2 \times \{1\}$ be a 2π-periodic regular analytic parameterization of a convex curve. Then, there exists a unique (definite) indefinite improper affine map ψ, with a non removable isolated singularity, where the affine conormal is tending to U. Moreover, it is embedded if and only if $U(\mathbb{R})$ is a Jordan curve (see Fig. 4).*

Fig. 5 Entire solutions on the puncture plane obtained by taking $H(z) = z$ in Theorem 5

But in the indefinite case, one may construct improper affine maps with non removable isolated singularities around which the conformal structure of the affine metric is a punctured disk $\mathscr{D}^*$. Actually, from (11) and (12) it is easy to see the following result,

Theorem 5. *Let $A : \mathscr{D} \longrightarrow \mathbb{C}_{-1}$ be a split-holomorphic function satisfying $A_z = H^2$ for some split-holomorphic function $H : \mathscr{D} \longrightarrow \mathbb{C}_{-1}$. If $z_0 \in \mathscr{D}$ is an isolated zero of F, then the indefinite improper affine map $\psi : \mathscr{D} \longrightarrow \mathbb{R}^3$ given, as in (12), by the split-holomorphic curve $\Phi(z) = (j\,z, A(z), 1)$, is well defined on $\mathscr{D}^* = \mathscr{D} - \{z_0\}$ and it has a non removable isolated singularity at z_0 (Fig. 5).*

Remark 3. By using the Theorem 5 we can construct indefinite improper affine map $\psi : \mathbb{C}_{-1} \longrightarrow \mathbb{R}^3$ with a finite number of prescribed isolated singularities at the points $\{z_1, \cdots, z_n\}$. For this is enough to consider a split-holomorphic function $H : \mathbb{C}_{-1} \longrightarrow \mathbb{C}_{-1}$ with zeros at the points $\{z_1, \cdots, z_n\}$.

4 Global Results

The aim of this section is to determine the global behavior of embedded complete definite improper affine maps such that any connected component of its singular set is mapped on a plane in $\mathbb{R}^3$ and those with only a finite number of isolated singularities.

4.1 The Case of Finitely Many Isolated Singularities

In [9] is proved the existence of entire solutions of (1) with any finite number of isolated singularities. The situation is totally different for an embedded complete definite improper affine map, where complete means that the affine metric is complete outside a compact subset.

Actually, from the generalized symmetry principle one has, [1, Theorem 4.2], any definite improper affine map must be symmetric with respect to point reflection in $\mathbb{R}^3$ through any isolated embedded singularity. As immediate consequence we have

Theorem 6. *Any embedded complete definite improper affine map whose singular set is a finite number of isolated singularities must be rotational, see Fig. 4.*

Proof. An easy application of the Maximum Principle let us to see that any embedded complete improper affine map with only one isolated singularity must be rotational. Consequently, it is enough to prove that if a complete improper affine map $\psi : \Sigma \longrightarrow \mathbb{R}^3$ has two different isolated singularities p_1 and p_2, then it has infinitely many isolated singularities.

In fact, having in mind that $\psi(\Sigma)$ is symmetric with respect to the reflections, s_1 and s_2, in $\mathbb{R}^3$ through the points p_1 and p_2, respectively, we get that

$$s_1(p_2), s_2(p_1), s_2 \circ s_1(p_2), s_1 \circ s_2(p_1), s_1 \circ s_2 \circ s_1(p_2), \cdots$$

also are isolated singularities of the map.

4.2 Embedded Complete Definite Improper Affine Map with a Planar Singular Set

We shall prove the following result:

Theorem 7. *Let $\psi : \Sigma \longrightarrow \mathbb{R}^3$ be an embedded complete definite improper affine map with a non-degenerate analytic singular set $\mathscr{S} \subset \Sigma$ such that $\psi(\mathscr{S})$ lies on a plane Π in $\mathbb{R}^3$. Then ψ is a snowman rotational improper affine map (see Fig. 2)*

Proof. Let $\mathscr{K} \subset \Sigma$ be a compact containing $\mathscr{S}$ in its interior. Thanks to a classical result of Huber, [10], $\Sigma \setminus \text{int}(\mathscr{K})$ is conformally a compact Riemann surface with compact boundary and finitely many points removed which are the ends of ψ.

But ψ is an embedding and then each end is asymptotic to one of rotational type (see [8]). Consider Σ^+ a connected component of $\psi(\Sigma) \setminus \psi(\mathscr{S})$, if we add to $\Sigma^+ \cup \partial\Sigma^+$ the planar bounded regions determined by the convex Jordan curves of its boundary, we get a globally convex surface $\tilde{\Sigma}^+$ in $\mathbb{R}^3$.

It is clear than Σ^+ has at least one end, otherwise adding its reflexion respect to the plane Π we get a compact flat improper affine map without boundary, which is impossible, see [21].

Consider $\Phi = (B, A, 1)$ the holomorphic curve associated to Σ^+ and denote by Σ_*^+ the corresponding improper affine map associate to the holomorphic curve $\Phi_*(-jA, -jB, 1)$, then Σ_*^+ has the following properties:

1. The boundary of Σ_*^+ is a singular point $a \in \mathbb{H}^3$.
2. Bearing in mind that any embedded complete end is of rotational type, see [8], any end of Σ_*^+ is also embedded and complete, moreover Σ_*^+ has the same number of ends as Σ^+.

In other words, $\Sigma_*^+ \cup \{a\}$ is a non compact complete definite improper affine map with only one isolated singularity. An easy application of the Maximum Principle says it must be rotational and, consequently, Σ^+ is also rotational and then the Theorem follows easily

Definite improper affine maps with a planar singular set also are symmetric. Actually, we have

Proposition 1. *Any improper affine map containing an analytic singular curve lying on a plane Π in $\mathbb{R}^3$ must be symmetric with respect to the plane Π.*

Proof. Let $\psi : \Sigma \longrightarrow \mathbb{R}^3$ be the improper affine map having

$$\alpha = (\alpha_1, \alpha_2, 0) : I \longrightarrow \mathbb{R}^3$$

as a singular curve with affine conormal $V = (0, 0, 1)$ along α, then from (15), ψ is determined by the holomorphic (split-holomorphic) curve $\Phi_\epsilon(z) = (\alpha_1(z), \alpha_2(z), 1)$, z in a neighborhood Ω_ϵ of I in $\mathbb{C}_\varepsilon$ where the holomorphic (split-holomorphic) extension of α is well determined. Then, from the Riemann-Schwarz symmetry principle we have that $\overline{\alpha_i(z)} = \alpha_i \bar{z}$ and, we conclude $\psi(\Omega_\epsilon)$ is symmetric respect to the plane Π in $\mathbb{R}^3$.

Using this fact, we can generalize the Theorem 7 as follows:

Theorem 8. *Let $\psi : \Sigma \longrightarrow \mathbb{R}^3$ be an embedded complete improper affine map with a non-degenerate analytic singular set $\mathscr{S} \subset \Sigma$ such that any connected component of $\psi(\mathscr{S})$ lies on a plane in $\mathbb{R}^3$. Then ψ is a snowman rotational improper affine map.*

Remark 4. There is a flat metric associated with (1) that connects the equation to another interesting family of surfaces. Actually, if we consider on Ω the Riemannian metric

$$ds^2 = dx^2 + dy^2, \tag{17}$$

one may check, from (2) and (17), that h satisfies the Codazzi-Mainardi equations of classical surface theory with respect to the metric ds^2. In other words, the pair (ds^2, h) of real quadratic forms is a Codazzi pair on Ω (see for instance [2, 16] for more information about Codazzi pairs). Moreover, from (1), (2) and (17), (ds^2, h) has constant extrinsic curvature $K(ds^2, h) = \varepsilon$ and from the existence and uniqueness theorem of surfaces in a space form we have that, locally, (Ω, ds^2) is isometrically immersed in the 3-dimensional space form $\mathbb{M}^3(-\varepsilon)$ of constant sectional curvature $-\varepsilon$. Conversely, any flat surface in $\mathbb{M}^3(-\varepsilon)$ has around any point local coordinates (x, y) such that its second fundamental form may be written as in (2) where ϕ is a solution to (1).

In [22] you can find similar results to the above mentioned theorems for flat surfaces in the hyperbolic space.

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Affine Isoperimetric Inequalities for L_p Geominimal Surface Area

Baocheng Zhu, Jiazu Zhou, and Wenxue Xu

Abstract We present some L_p affine isoperimetric inequalities for L_p geominimal surface area. In particular, we obtain an analogue of Blaschke-Santaló inequality. We give an integral formula of L_p geominimal surface area by the p-Petty body. Furthermore, we introduce the concept of L_p mixed geominimal surface area which is a nature extension of L_p geominimal surface area. We also extend Lutwak's results for L_p mixed geominimal surface area.

Keywords L_p geominimal surface area • L_p affine surface area • Affine isoperimetric inequality • Blaschke-Santaló inequality

Mathematics Subject Classification (2010). 52A20; 52A40.

1 Introduction

The classical isoperimetric inequality states that: among all the compact domains of given surface area S in the Euclidean space $\mathbb{R}^n$, the volume V of a domain K is maximized only at the ball. The isoperimetric inequality is usually written as

$$S(K)^n \geq n^n \omega_n V(K)^{n-1}$$

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with equality if and only if the compact domain is a Euclidean ball. Where ω_n is the volume of the unit ball in $\mathbb{R}^n$. As we know that the surface area of a body may change while the affine surface area of a body is affine invariant under unimodular affine transformations of the body. Therefore, the affine surface area is the essential notion in the affine isoperimetric inequalities.

The classical affine surface area, was first introduced by Blaschke [3], of smooth convex bodies from affine differential geometry, and was extended to arbitrary convex bodies. Its L_p version, called L_p affine surface area, was introduced by Lutwak in his seminal article [14]. The L_p affine surface area has been regarded as a core of the L_p Brunn-Minkowski theory which is a nature extension of the classical Brunn-Minkowski theory. L_p affine surface area has been extended to all $-n \neq p \in \mathbb{R}$ by its nice integral expression, and further extended to all $p \in \mathbb{R}$ and to more general convex bodies (see [17, 23, 24]). There are many researches working on the L_p affine surface area (see e.g., [7, 12, 16, 17, 23–26, 29]). The L_p affine surface area are related closely to the theory of valuation (see [1, 2, 9, 10]), the information theory of convex bodies (see [6, 18, 27, 28]) and the approximation of convex bodies by polytopes (see e.g., [4, 8, 24]). In [14, 29], we can find the affine isoperimetric inequalities for L_p affine surface area: among all the convex bodies with centroid at the origin of given volume in the Euclidean space $\mathbb{R}^n$, the L_p affine surface area is maximized (minimum) only at the ellipsoids.

The classical geominimal surface area was first introduced by Petty [19] in 1974. It is another affine invariant under unimodular affine transformations. The classical geominimal surface area and its L_p extensions, which are introduced by Lutwak, serve as bridges connecting affine differential geometry, relative differential geometry and Minkowski geometry. Affine isoperimetric inequalities related to the geominimal surface area are not only closely connected to many affine isoperimetric inequalities involving affine surface area (see e.g., [7,9,12,14,19,20,22]), but clarify the equality conditions of many of these inequalities. For example, combining Petty's theory of geominimal surface area with the information on the affine surface area, Schneider [21] proved that the affine surface areas of K and L can only be equal if $K = L$. In [32], the authors proved some isoperimetric inequalities for L_p geominimal surface area. The affine isoperimetric inequalities for L_p geominimal surface area can be stated as: among all the convex bodies with centroid at the origin of given volume, the L_p geominimal surface area is maximized (minimum) only at the ellipsoids.

Unlike the L_p affine surface area, L_p geominimal surface area has no nice integral expression (except for some especial bodies [33]). This will lead to a big obstacle on extending the L_p geominimal surface area. Fortunately, motivated by an equivalent formula of the L_p affine surface area, the L_p geominimal surface area was successfully extended to all $-n \neq p \in \mathbb{R}$ by Ye [30]. He obtained the affine isoperimetric inequality and the Santaló style inequality for the generalized L_p geominimal surface area. Recently, the L_p geominimal surface area has been extended to L_p mixed geominimal surface area (see [31, 33]). A big effort of this article will be devoted to the affine isoperimetric inequalities for L_p geominimal surface area.

2 Notation and Definitions

We say that $K \subset \mathbb{R}^n$ is a convex body if K is a compact, convex subset in $\mathbb{R}^n$ with non-empty interior. The set of all convex bodies is written as $\mathscr{K}$, and its subset $\mathscr{K}_0$ denotes the set of convex bodies containing the origin in their interiors. Similarly, we use $\mathscr{K}_c$ for the set of convex bodies with centroid at the origin. We use $B_2^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ and $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ for the unit ball and the unit sphere in Euclidean $\mathbb{R}^n$, respectively. For a subset $K \subset \mathbb{R}^n$, its Hausdorff content is denoted by $V(K)$. In particular, the volume of B_2^n is written as $\omega_n = \pi^{n/2}/\Gamma(1+n/2)$.

For $K \in \mathscr{K}_0$, its support function $h_K(\cdot) = h(K, \cdot) : S^{n-1} \to [0, \infty)$ is defined by $h_K(u) = \max\{\langle x, u\rangle : x \in K\}$. Where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$. Associated with each $K \in \mathscr{K}_0$, one can uniquely define its polar body $K^\circ \in \mathscr{K}_0$ by

$$K^\circ = \{x \in \mathbb{R}^n : \langle x, y\rangle \leq 1, \quad \forall y \in K\}.$$

A set $L \subset \mathbb{R}^n$ is star shape about the origin 0 if for each $x \in L$, the line segment from 0 to x is contained in L. The radial function of L, denoted by $\rho_L(\cdot) = \rho(L, \cdot) : S^{n-1} \to [0, \infty)$, is defined by $\rho_L(u) = \max\{\lambda \geq 0 : \lambda u \in L\}$. If ρ_L is positive and continuous, we call L is a star body about the origin. The set of all star bodies about the origin is denoted by $\mathscr{S}_0$. Two star bodies $L_1, L_2 \in \mathscr{S}_0$ are dilates of one another if $\rho_{L_1}(u)/\rho_{L_2}(u)$ is independent of $u \in S^{n-1}$. The volume of $L \in \mathscr{S}_0$ can be calculated by

$$V(L) = \frac{1}{n} \int_{S^{n-1}} \rho_L^n(u) d\sigma(u),$$

where σ is the spherical measure on S^{n-1}. It is easily verified that $(K^\circ)^\circ = K$ for all $K \in \mathscr{K}_0$. Moreover,

$$h_{K^\circ}(u)\rho_K(u) = 1 \quad \text{and} \quad \rho_{K^\circ}(u) h_K(u) = 1, \quad \text{for all} \quad u \in S^{n-1}.$$

For $K \in \mathscr{K}$, there exists a unique point $s(K)$ in the interior of K, called the Santaló point of K, such that

$$V((-s(K) + K)^\circ) = \min\{V((-x + K)^\circ) : x \in \text{int}K\}.$$

Let $\mathscr{K}_s$ denote the set of convex bodies having their Santaló point at the origin. Thus, we have

$$K \in \mathscr{K}_s \quad \text{if and only if} \quad K^\circ \in \mathscr{K}_c.$$

For real $p \geq 1$, $\lambda, \mu \geq 0$ (not both zero), the Firey linear combination $\lambda \cdot K +_p \mu \cdot L$ of $K, L \in \mathscr{K}_0$ is defined by (see [14])

$$h(\lambda \cdot K +_p \mu \cdot L, \cdot)^p = \lambda h(K, \cdot)^p + \mu h(L, \cdot)^p.$$

For $K, L \in \mathscr{K}_0$ and $p \geq 1$, the p-mixed volume $V_p(K, L)$ of K and L, was defined in [13] as

$$\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \to 0} \frac{V(K +_p \varepsilon \cdot L) - (K)}{\varepsilon}.$$

It was proved in [13] that for each $K \in \mathscr{K}_0$, there is a positive Borel measure $S_p(K, \cdot)$ on S^{n-1} such that, for each $L \in \mathscr{K}_0$ and $p \geq 1$,

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L(u)^p dS_p(K, u).$$

The surface area measure $S_p(K, \cdot)$ of K is a positive Borel measure on S^{n-1}, it is absolutely continuous with respect to the classical surface area $S(K, \cdot)$ of K, and has Radon–Nikodym derivative

$$dS_p(K, \cdot) = h_K(\cdot)^{1-p} dS(K, \cdot).$$

We write $K \in \mathscr{F}_0$ if $K \in \mathscr{K}_0$ has a curvature function, namely, the measure $S(K, \cdot)$ is absolutely continuous with respect to the spherical measure σ. Hence, there is a function $f_K : S^{n-1} \to \mathbb{R}$, the curvature function of K, such that,

$$dS(K, u) = f_K(u) d\sigma(u). \tag{1}$$

For $K \in \mathscr{F}_0$ and $p \geq 1$, the L_p curvature function $f_p(K, u)$ was defined by (see [14])

$$f_p(K, u) = h_K(u)^{1-p} f_K(u). \tag{2}$$

We write $\mathscr{F}_c = \mathscr{F}_0 \cap \mathscr{K}_c$ and $\mathscr{F}_s = \mathscr{F}_0 \cap \mathscr{K}_s$ for convex bodies in $\mathscr{F}_0$ with centroid and Santaló point at the origin respectively. The set of all convex bodies in $\mathscr{F}_0$ with continuous positive curvature function $f_K(\cdot)$ on S^{n-1} is denoted by $\mathscr{F}_0^+$.

3 L_p Geominimal Surface Area

One of the most famous open problems in convex geometric analysis is closely related to volume $V(K^\circ)$ of the polar body of $K \in \mathscr{K}_0$. For example, the Mahler's conjecture asks for the most optimal lower bound of $V(K^\circ)$ among all bodies $K \in \mathscr{K}_0$ of volume one. The Mahler's conjecture is the reverse problem of the celebrated Blaschke–Santaló inequality, which is an important affine isoperimetric inequality in convex geometric analysis. The Blaschke–Santaló inequality can be stated as: if $K \in \mathscr{K}_s$, then

$$V(K)V(K^\circ) \leq \omega_n^2, \tag{3}$$

with equality if and only if K is an ellipsoid.

For each compact star-shaped (about the origin) K in $\mathbb{R}^n$ and for real number $p \geq 1$, the polar L_p centroid body $\Gamma_p^\circ K$ of K was defined by (see [15])

$$\rho_{\Gamma_p^\circ K}^{-p}(u) = \frac{1}{c_{n,p} V(K)} \int_K |u \cdot x|^p \, dx,$$

where the integration is with respect to Lebesgue measure and $c_{n,p} = \omega_{n+p}/\omega_2 \omega_{p-1}$. The normalization above is chosen so that for the unit ball B_2^n, we have $\Gamma_p^\circ B_2^n = B_2^n$. In [15] the authors proved the following L_p centro-affine inequality.

Theorem 1. *If $K \in \mathscr{S}_0$, then for $1 \leq p \leq \infty$,*

$$V(K) V(\Gamma_p^\circ K) \leq \omega_n^2, \tag{4}$$

with equality if and only if K is an ellipsoid centered at the origin .

As explained in [15], if K is an origin-symmetric convex body, then $\Gamma_p^\circ K$ is just the polar body K° of K. Thus, for $p = \infty$, the L_p centro-affine inequality (4) reduces to (3). Recently, Haberl and Schuster [5] showed that there is an interesting asymmetric L_p version of (3).

For $K \in \mathscr{K}_0$ and $p \geq 1$, the L_p geominimal surface area, $G_p(K)$, was defined in [14] (the case $p = 1$ defined in [19]) by

$$\omega_n^{p/n} G_p(K) = \inf_{Q \in \mathscr{K}_0} \{ n V_p(K, Q) V(Q^\circ)^{p/n} \}.$$

Let

$$\mathscr{T} = \{ T \in \mathscr{K} : s(T) = 0, V(T^\circ) = \omega_n \}.$$

Lemma 1 (and Definition). *(see [14]). For each $K \in \mathscr{K}_0$ and $p \geq 1$, there exists a unique body $T_p K \in \mathscr{T}$ with $G_p(K) = n V_p(K, T_p K)$.*

The unique body $T_p K$ is called the p-Petty body of K. When $p = 1$, the subscript will often be suppressed and defined by Petty [19].

Associated with L_p geominimal surface area, Lutwak [14] proved the following L_p affine isoperimetric inequality.

Theorem 2. *If $p \geq 1$ and $K \in \mathscr{K}_0$, then*

$$G_p(K)^n \leq n^n \omega_n^p V(K)^{n-p},$$

with equality if and only if K is an ellipsoid.

In [32], the authors proved the Blaschke-Santaló type inequality for L_p geominimal surface area $G_p(K)$.

Theorem 3. *If $K \in \mathscr{K}_c$ and $1 \le p < n$, then*

$$G_p(K)G_p(K^\circ) \le (n\omega_n)^2,$$

with equality if and only if K is an ellipsoid.

For the case of $1 \le p < \infty$ of Theorem 3, one can refer [33]. Theorem 3 strengthens the following result proved by Lutwak in [14]: If $K \in \mathscr{K}_c$, then for $p \ge 1$

$$\Omega_p(K)\Omega_p(K^\circ) \le (n\omega_n)^2,$$

with equality if and only if K is an ellipsoid. Here, Ω_p is the L_p affine surface area introduced by Lutwak in [14]: For $p \ge 1$ and $K \in \mathscr{F}_0$,

$$\Omega_p(K) = \int_{S^{n-1}} f_p(K,u)^{\frac{n}{n+p}}\, d\sigma(u).$$

4 An Affine Isoperimetric Inequality Between G_p and Ω_p

Call a body $K \in \mathscr{F}_0$ is of p-elliptic type if the function $f_p(K,\cdot)^{\frac{1}{n+p}}$ is the support function of a convex body in $\mathscr{K}_0$; i.e., K is of p-elliptic type if there exists a body $Q \in \mathscr{K}_0$ such that

$$f_p(K,\cdot) = h(Q,\cdot)^{-(n+p)}.$$

Then, Lutwak defined (see [14])

$$\mathscr{V}_p = \{K \in \mathscr{F}_0 : \text{ there exists a } Q \in \mathscr{K}_0 \text{ with } f_p(K,\cdot) = h(Q,\cdot)^{-(n+p)}\}.$$

We say Theorem 3 strengthens Lutwak's result in Sect. 3 is due to the following Petty's theory of L_p geominimal surface area with the information on the general L_p affine surface area proved by Lutwak in [14].

Theorem 4. *If $p \ge 1$ and $K \in \mathscr{F}_0$, then*

$$\Omega_p(K)^{n+p} \le (n\omega_n)^p G_p(K)^n, \tag{5}$$

with equality if and only if K is $\mathscr{V}_p$.

The case $p = 1$ of inequality (5) was proved by Petty [19] for $K \in \mathscr{F}_0$ and extended by Lutwak [14] to $K \in \mathscr{K}_0$ without equality condition. The equality condition of Lutwak's extension for $K \in \mathscr{K}_0$ and for $p = 1$ was proved by Schneider recently in [21].

The equality condition for Theorem 4 was only known under the additional assumption that $K \in \mathscr{F}_0$. Lutwak proved the inequality (5) for $K \in \mathscr{K}_0$ and $p \geq 1$ without the equality condition. Here, we extend Theorem 4 for $K \in \mathscr{F}_0$ to the following result for $K \in \mathscr{K}_0$.

Theorem 5. *If $p \geq 1$ and $K \in \mathscr{K}_0$, then*

$$\Omega_p(K)^{n+p} \leq (n\omega_n)^p G_p(K)^n, \tag{6}$$

with equality if and only if K is $\mathscr{V}_p$.

Proof. Let $K \in \mathscr{K}_0$. By the definition of p-Petty body (Lemma 1), there exists a unique convex body $T_p K \in \mathscr{T}$ with

$$G_p(K) = \int_{S^{n-1}} h^p_{T_p K}(u) dS_p(K,u). \tag{7}$$

With respect to spherical Lebesgue measure, the measure $S_p(K,\cdot)$ has a Lebesgue decomposition into the sum of an absolutely continuous measure $S^a_p(K,\cdot)$ and a singular measure $S^s_p(K,\cdot)$. By (1) and (2), it is known that

$$S^a_p(K,\omega) = \int_\omega dS^a_p(K,u) = \int_\omega f_p(K,u) d\sigma(u), \quad \text{for Borel sets} \quad \omega \subset S^{n-1}. \tag{8}$$

With (7) this gives

$$\begin{aligned} G_p(K) &= \int_{S^{n-1}} h^p_{T_p K}(u) dS^a_p(K,u) + \int_{S^{n-1}} h^p_{T_p K}(u) dS^s_p(K,u) \\ &\geq \int_{S^{n-1}} h^p_{T_p K}(u) f_p(K,u) d\sigma(u). \end{aligned} \tag{9}$$

By Hölder's inequality, we obtain

$$\begin{aligned} \int_{S^{n-1}} h^p_{T_p K}(u) f_p(K,u) d\sigma(u) &\geq \left(\int_{S^{n-1}} h_{T_p K}(u)^{-n} d\sigma(u) \right)^{-\frac{p}{n}} \\ &\quad \times \left(\int_{S^{n-1}} f_p(K,u)^{\frac{n}{n+p}} d\sigma(u) \right)^{\frac{n+p}{n}} \\ &= [nV(T^\circ)]^{-\frac{p}{n}} \Omega_p(K)^{\frac{n+p}{n}} \\ &= [n\omega_n]^{-\frac{p}{n}} \Omega_p(K)^{\frac{n+p}{n}}. \end{aligned}$$

The inequality (6) follows. Suppose that equality holds here. Then equality holds in Hölder's inequality. This implies there is a positive constant c with

$$c h^{-(n+p)}_{T_p K}(u) = f_p(K,u). \tag{10}$$

But since equality holds also in (9), and $h_{T_pK} > 0$ everywhere on S^{n-1}, the singular part $S_p^s(K,\cdot)$ is the zero measure. Now it follows from (8) and (10) that $ch_{T_pK}^{-(n+p)}(u)$ is a density for $S_p(K,\cdot)$. Thus, K has the curvature function $f_p(K,u) = ch_{T_pK}^{-(n+p)}(u)$, and $f_p(K,u)^{-\frac{1}{n+p}}$ is a support function. Therefore, $K \in \mathscr{V}_p$, that is, K is of p-elliptic type, which completes the proof. □

5 L_p Mixed Geominimal Surface Area

In [33], we provided an integral formula of L_p geominimal surface area by the p-Petty body as follows.

Definition 1. For each $K \in \mathscr{F}_0$, there exists a unique convex body $T = T_pK \in \mathscr{T}$ with

$$G_p(K) = \int_{S^{n-1}} h_T^p(u) f_p(K,u) d\sigma(u). \tag{11}$$

Furthermore, we defined the L_p mixed geominimal surface area, $G_p(K_1,\ldots,K_n)$, of $K_1,\ldots,K_n \in \mathscr{F}_0$, for all $p \geq 1$.

Definition 2. For each $K_i \in \mathscr{F}_0$, there exists a unique convex body (Petty of K_i) $T_i = T_pK_i \in \mathscr{T}(i = 1,\ldots,n)$ with

$$G_p(K_1,\ldots,K_n) = \int_{S^{n-1}} [h_{T_1}^p(u) f_p(K_1,u)\cdots h_{T_n}^p(u) f_p(K_n,u)]^{\frac{1}{n}} d\sigma(u). \tag{12}$$

Let $GL(n)$ and $SL(n)$ denote the group of nonsingular linear transformations and special linear transformations, respectively. We proved that the L_p mixed geominimal surface area is affine invariant.

Theorem 6. *If $p \geq 1$ and $K_1,\ldots,K_n \in \mathscr{F}_0$, then for $\phi \in GL(n)$,*

$$G_p(\phi K_1,\ldots,\phi K_n) = |det(\phi)|^{\frac{n-p}{n}} G_p(K_1,\ldots,K_n).$$

In particular, if $\phi \in SL(n)$, then $G_p(K_1,\ldots,K_n)$ is affine invariant, that is,

$$G_p(\phi K_1,\ldots,\phi K_n) = G_p(K_1,\ldots,K_n).$$

Let $V(K_1,\ldots,K_n)$ be the mixed volume of $K_1,\ldots,K_n \in \mathscr{K}$. Then the Minkowski inequality for mixed volume is

$$V(K_1,\ldots,K_n)^n \geq V(K_1)\cdots V(K_n),$$

with equality if and only if K_i $(1 \leq i \leq n)$ are homothetic.

The analogous Minkowski inequality for dual mixed volume $\tilde{V}(K_1, \ldots, K_n)$, introduced by Lutwak in [11], is

$$\tilde{V}(K_1, \ldots, K_n)^n \leq V(K_1) \cdots V(K_n),$$

with equality if and only if K_i $(1 \leq i \leq n)$ are dilates of one another.

We also proved some affine isoperimetric inequalities for L_p mixed geominimal surface areas, such as the following isoperimetric inequality.

Theorem 7. *Let $K_i \in \mathscr{F}_0$, $1 \leq i \leq n$.*

(i). For $p \geq 1$,

$$\left(\frac{G_p(K_1, \ldots, K_n)}{G_p(B_2^n, \ldots, B_2^n)}\right)^n \leq \left(\frac{V(K_1)}{V(B_2^n)} \cdots \frac{V(K_n)}{V(B_2^n)}\right)^{\frac{n-p}{n}},$$

with equality if the K_i are ellipsoids that are dilates of each other.

(ii). For $1 \leq p \leq n$,

$$\frac{G_p(K_1, \ldots, K_n)}{G_p(B_2^n, \ldots, B_2^n)} \leq \left(\frac{V(K_1, \ldots, K_n)}{V(B_2^n, \ldots, B_2^n)}\right)^{\frac{n-p}{n}},$$

with equality if the K_i are ellipsoids that are dilates of each other.

(iii). For $p \geq n$,

$$\frac{G_p(K_1, \ldots, K_n)}{G_p(B_2^n, \ldots, B_2^n)} \leq \left(\frac{\tilde{V}(K_1, \ldots, K_n)}{\tilde{V}(B_2^n, \ldots, B_2^n)}\right)^{\frac{n-p}{n}},$$

with equality if the K_i are ellipsoids that are dilates of each other.

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Real Hypersurfaces in Complex Two-Plane Grassmannians with Commuting Jacobi Operators

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Abstract In this paper, we have considered new commuting conditions, that is, $(R_\xi\phi)S = S(R_\xi\phi)$ (resp. $(\bar{R}_N\phi)S = S(\bar{R}_N\phi)$) between the Jacobi operators R_ξ (resp. $\bar{R}_N$), the structure tensor field ϕ and the Ricci tensor S for real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$. With such a condition we give a complete classification of Hopf hypersurfaces M in $G_2(\mathbb{C}^{m+2})$.

1 Introduction

The geometry of real hypersurfaces in Hermitian symmetry spaces is one of the interesting parts in the field of differential geometry. The complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ is a kind of Hermitian symmetry spaces of compact irreducible type with rank 2. It consists of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. Remarkably, it is equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ (not containing J) satisfying $JJ_\nu = J_\nu J$ $(\nu = 1, 2, 3)$ where $\{J_\nu\}_{\nu=1,2,3}$ is an orthonormal basis of $\mathfrak{J}$. When $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight. When $m = 2$, we note that the isomorphism $\mathrm{Spin}(6) \simeq \mathrm{SU}(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces in $\mathbb{R}^6$. In this paper, we assume $m \geq 3$, (see Berndt and Suh [2] and [3]).

Let us consider a hypersurface M in $G_2(\mathbb{C}^{m+2})$ and denoted by N a local unit normal vector field to M. Hereafter unless otherwise stated, we consider that X and Y are any tangent vector field on M. By using the Kähler structure J of $G_2(\mathbb{C}^{m+2})$, we can define a structure vector field by $\xi = -JN$, which is said to be a *Reeb vector field*. If ξ is invariant under the shape operator A, it is said to be *Hopf*. The one-dimensional foliation of M by the integral manifolds of the Reeb vector field ξ

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is said to be a *Hopf foliation* of M. We say that M is the *Hopf hypersurface* in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of M is totally geodesic. If X is a tangent vector on M, we may put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N$$

where ϕX (resp. $\phi_\nu X$) is the tangential part of JX (resp. $J_\nu X$) and $\eta(X) = g(X, \xi)$ (resp. $\eta_\nu(X) = g(X, \xi_\nu)$) is the coefficient of normal part of JX (resp. $J_\nu X$). In this case, we call ϕ the structure tensor field of M. Using the Gauss and Weingarten formulas in [5, Sects. 1 and 2], the Kähler condition $\bar{\nabla} J = 0$ gives $\nabla_X \xi = \phi AX$ for any tangent vector field X on M, where ∇ (resp. $\bar{\nabla}$) denotes the covariant derivative on M (resp. $G_2(\mathbb{C}^{m+2})$). From this, it can be easily checked that M is Hopf if and only if the Reeb vector field ξ is Hopf.

From the quaternionic Kähler structure $\mathfrak{J}$ of $G_2(\mathbb{C}^{m+2})$, there naturally exist *almost contact three-structure* vector fields $\{\xi_1, \xi_2, \xi_3\}$ defined by $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$. Now let us denote by $\mathscr{Q}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ a three-dimensional distribution in a tangent vector space T_pM at $p \in M$. In addition, $\mathscr{Q}$ stands for the orthogonal complement of $\mathscr{Q}^\perp$ in T_pM. Then it becomes a quaternionic maximal subbundle of T_pM. Thus the tangent space of M consists of the direct sum of $\mathscr{Q}$ and $\mathscr{Q}^\perp$ as follows: $T_pM = \mathscr{Q} \oplus \mathscr{Q}^\perp$.

For these two distributions $[\xi] = \text{Span}\{\xi\}$ and $\mathscr{Q}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$, we may consider two natural invariant geometric properties under the shape operator A of M, that is, $A[\xi] \subset [\xi]$ and $A\mathscr{Q}^\perp \subset \mathscr{Q}^\perp$. By using the result of Alekseevskii [1], Berndt and Suh [2] have classified all real hypersurfaces with the invariant properties in $G_2(\mathbb{C}^{m+2})$ as follows:

Theorem 1. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathscr{Q}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*
(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

In the case of (A), we will say M of Type (A). Similarly in the case of (B), we will say M of Type (B).

Until now, by using Theorem 1, many geometers have investigated some chracterization of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with geometric quantities like shape operator, structure (or normal) Jacobi operator, Ricci tensor, and so on. Commuting Ricci tensor means that the Ricci tensor S and the structure tensor field ϕ commute, that is, $S\phi = \phi S$. From such a point of view, Suh [8] has given characterizations of real hypersurfaces of Type (A) with commuting Ricci tensor as follows:

Theorem 2. *Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with commuting Ricci tensor, $m \geq 3$. Then M is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Lee and Suh [5] gave a characterization of real hypersurfaces of Type (B) in Theorem 1 as follows:

Theorem 3. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb vector field ξ belongs to the distribution $\mathcal{Q}$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, $m = 2n$, where the distribution $\mathcal{Q}$ denotes the orthogonal complement of $\mathcal{Q}^{\perp}$ in T_xM, $x \in M$. In other words, M is locally congruent to of a real hypersurface of Type (B).*

On the other hand, a Jacobi field along geodesics of a given Riemannian manifold $(\bar{M}, \bar{g})$ is an important role in the study of differential geometry. It satisfies a well-known differential equation which inspires Jacobi operators. It is defined by $(\bar{R}_X(Y))(p) = (\bar{R}(Y,X)X)(p)$, where $\bar{R}$ denotes the curvature tensor of $\bar{M}$ and X, Y denote any vector fields on $\bar{M}$. It is known to be a self-adjoint endomorphism on the tangent space $T_p\bar{M}$, $p \in \bar{M}$. Clearly, each tangent vector field X to $\bar{M}$ provides a Jacobi operator with respect to X. Thus the Jacobi operator on a real hypersurface M of $G_2(\mathbb{C}^{m+2})$ with respect to ξ (resp. N) is said to be a *structure Jacobi operator* (resp. *normal Jacobi operator*) and will be denoted by R_ξ (resp. $\bar{R}_N$).

For a commuting problem concerned with structure Jacobi operator R_ξ and structure tensor ϕ of M in $G_2(\mathbb{C}^{m+2})$, that is, $R_\xi\phi = \phi R_\xi$, Suh and Yang [9] gave a characterization of a real hypersurface of Type (A) in $G_2(\mathbb{C}^{m+2})$. Also, related to a commuting problem for the normal Jacobi operator $\bar{R}_N$, Pérez, Jeong and Suh [7] gave a characterization of a real hypersurface of Type (A) in $G_2(\mathbb{C}^{m+2})$. Motivated by these results, we consider in this paper a new commuting condition for three operators; the restricted structure Jacobi operator $R_\xi\phi$ and the Ricci tensor S given by

$$(R_\xi\phi)S = S(R_\xi\phi), \tag{C-1}$$

and give a classification of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying (C-1) as follows:

Theorem 4. *Let M be a Hopf hypersurface in complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ with $R_\xi\phi S = SR_\xi\phi$. If the smooth function $\alpha = g(A\xi, \xi)$ is constant along the direction of ξ, then M is locally congruent with an open part of a tube of some radius $r \in (0, \frac{\pi}{2\sqrt{2}})$ around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Respectively, we may consider a commuting condition between the restricted normal Jacobi operator $\bar{R}_N\phi$ and the Ricci tensor S given by

$$(\bar{R}_N\phi)S = S(\bar{R}_N\phi), \tag{C-2}$$

and give a classification of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying (C-2) as follows:

Theorem 5. *Let M be a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ satisfying (C-2). If the smooth function $\alpha = g(A\xi, \xi)$ is constant along the direction of ξ, then M is locally congruent to an open part of a tube of some radius $r \in (0, \frac{\pi}{2\sqrt{2}})$ around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Actually, according to the geometric meaning of the condition (C-1)(resp. (C-2)), any eigenspaces of the Ricci tensor S on M in $G_2(\mathbb{C}^{m+2})$ are invariant under $R_\xi\phi$ (resp. $\bar{R}_N\phi$). In Sects. 2 and 3, we will give a complete proof of Theorems 4 and 5, respectively. We refer to [1–3] and [5] for Riemannian geometric structures of $G_2(\mathbb{C}^{m+2})$, $m \geq 3$.

2 Proof of Theorem 4

In this section, by using geometric quantities in [8] and [9], we will give a complete proof of our Theorem 4. To prove it, we assume that M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying (C-1), that is,

$$(R_\xi\phi)SX = S(R_\xi\phi)X. \tag{1}$$

From now on, X,Y and Z always stand for any tangent vector fields on M.

Let us introduce the Ricci tensor S and structure Jacobi operator R_ξ, briefly. The curvature tensor $R(X,Y)Z$ of M in $G_2(\mathbb{C}^{m+2})$ can be derived from the curvature tensor $\bar{R}(X,Y)Z$ of $G_2(\mathbb{C}^{m+2})$. Then by contraction and using the geometric structure $JJ_\nu = J_\nu J$ $(\nu = 1,2,3)$, connecting the Kähler structure J and the quaternionic Kähler structure J_ν,$(\nu = 1,2,3)$, we can obtain the Ricci tensor S given by

$$g(SX,Y) = \sum\nolimits_{i=1}^{4m-1} g(R(e_i,X)Y,e_i),$$

where $\{e_1,\cdots,e_{4m-1}\}$ denotes a orthonormal basis of the tangent space T_xM of M, $x \in M$, in $G_2(\mathbb{C}^{m+2})$ (see [8]).

From the definition of the Ricci tensor S and by fundamental formulas in [8, Sect. 2], we have

$$\begin{aligned} SX &= \sum_{i=1}^{4m-1} R(X,e_i)e_i \\ &= (4m+7)X - 3\eta(X)\xi + hAX - A^2X \qquad (2) \\ &\quad + \sum_{\nu=1}^{3}\{-3\eta_\nu(X)\xi_\nu + \eta_\nu(\xi)\phi_\nu\phi X - \eta(\phi_\nu X)\phi_\nu\xi - \eta(X)\eta_\nu(\xi)\xi_\nu\}, \end{aligned}$$

where h denotes the trace of A, that is, $h = \mathrm{Tr}A$ (see [6, (1.4)]). By inserting $Y = Z = \xi$ into the curvature tensor $R(X, Y)Z$, using the condition of being Hopf and fundamental formulas in [9, Sect. 2], the structure Jacobi operator R_ξ becomes

$$\begin{aligned} R_\xi(X) &= R(X,\xi)\xi \\ &= X - \eta(X)\xi - \sum_{\nu=1}^{3}\Big\{\eta_\nu(X)\xi_\nu - \eta(X)\eta_\nu(\xi)\xi_\nu \\ &\qquad + 3g(\phi_\nu X,\xi)\phi_\nu\xi + \eta_\nu(\xi)\phi_\nu\phi X\Big\} + \alpha AX - \alpha^2\eta(X)\xi \end{aligned} \tag{3}$$

(see [4, Sect. 4]).

Using these Eqs. (1), (2) and (3), we can prove that ξ of M belongs to either $\mathscr{Q}$ or $\mathscr{Q}^\perp$.

Lemma 1. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, satisfying (C-1). If the principal curvature $\alpha = g(A\xi, \xi)$ is constant along the direction of ξ, then ξ belongs to either the distribution $\mathscr{Q}$ or the distribution $\mathscr{Q}^\perp$.*

Now, we shall divide our consideration into two cases that ξ belongs to either $\mathscr{Q}^\perp$ or $\mathscr{Q}$, respectively. Next we further study the case $\xi \in \mathscr{Q}^\perp$. We may put $\xi = \xi_1 \in \mathscr{Q}^\perp$ for convenience.

Lemma 2. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If the Reeb vector field ξ belongs to $\mathscr{Q}^\perp$, then the Ricci tensor S commutes the shape operator A, that is, $SA = AS$.*

Lemma 3. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If the Reeb vector field ξ belongs to $\mathscr{Q}^\perp$, we have the following formulas*

(i) $\phi SX = 2\eta_3(SX)\xi_2 - 2\eta_2(SX)\xi_3 + \phi_1 SX + Rem(X)$ *and*
(ii) $S\phi X = 2\eta_3(X)S\xi_2 - 2\eta_2(X)S\xi_3 + S\phi_1 X + Rem(X)$,

where $Rem(X)$ denotes $4(m+2)\{2\eta_2(X)\xi_3 - 2\eta_3(X)\xi_2 + \phi X - \phi_1 X\}$.

By virtue of Lemmas 2 and 3, we assert the following:

Lemma 4. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ satisfying (1). If $\xi \in \mathscr{Q}^\perp$, we have $A(\phi S - S\phi) = (\phi S - S\phi)A$.*

Lemma 5. *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If M satisfies $A(\phi S - S\phi) = (\phi S - S\phi)A$ and $\xi \in \mathscr{Q}^\perp$, then we have $S\phi = \phi S$.*

Summing up Lemmas 2, 3, 4, 5 and Theorem 2, we conclude that if M is a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ satisfying (1), then M must be of Type (A).

Hereafter, let us check whether the Ricci tensor of a model space of Type (A) satisfies the commuting condition (C-1).

From [2, Proposition 3], we obtain the following

$$SX = \begin{cases} (4m + h\alpha - \alpha^2)\xi & \text{if } X = \xi \in T_\alpha \\ (4m + 6 + h\beta - \beta^2)\xi_\nu & \text{if } X = \xi_\nu \in T_\beta \\ (4m + 6 + h\lambda - \lambda^2)X & \text{if } X \in T_\lambda \\ (4m + 8)X & \text{if } X \in T_\mu, \end{cases}$$

$$R_\xi(X) = \begin{cases} 0 & \text{if } X = \xi \in T_\alpha \\ (\alpha\beta + 2)\xi_\nu & \text{if } X = \xi_\nu \in T_\beta \\ (\alpha\lambda + 2)\phi X & \text{if } X \in T_\lambda \\ 0 & \text{if } X \in T_\mu, \quad \text{and} \end{cases}$$

$$(R_\xi\phi)X = \begin{cases} 0 & \text{if } X = \xi \in T_\alpha \\ (\alpha\beta + 2)\phi\xi_\nu & \text{if } X = \xi_\ell \in T_\beta \\ (\alpha\lambda + 2)\phi X & \text{if } X \in T_\lambda \\ 0 & \text{if } X \in T_\mu. \end{cases}$$

$$S(R_\xi\phi)X = \begin{cases} 0 & \text{if } X = \xi \in T_\alpha \\ (\alpha\beta + 2)(4m + 6 + h\beta - \beta^2)\phi\xi_\nu & \text{if } X = \xi_\ell \in T_\beta \\ (\alpha\lambda + 2)(4m + 6 + h\lambda - \lambda^2)\phi X & \text{if } X \in T_\lambda \\ 0 & \text{if } X \in T_\mu. \end{cases}$$

Combining these three formulas, it follows that

$$(R_\xi\phi)SX - SR_\xi\phi X = \begin{cases} 0 \text{ if } X = \xi \in T_\alpha \\ 0 \text{ if } X = \xi_\ell \in T_\beta \\ 0 \text{ if } X \in T_\lambda \\ 0 \text{ if } X \in T_\mu. \end{cases}$$

When $\xi \in \mathscr{Q}^\perp$, a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ satisfying (C-1) is locally congruent to of Type (A) by virtue of Theorem 2.

Now let us consider our problem for a model space of Type (B) which will be denoted by M_B. In order to do this, let us prove that $(R_\xi\phi)S = SR_\xi\phi$ related to the M_B. On T_xM_B, $x \in M_B$, the Eqs. (2) and (3) are reduced to the following equations, respectively:

$$SX = (4m + 7)X - 3\eta(X)\xi + hAX - A^2X - \sum_{\nu=1}^{3}\{3\eta_\nu(X)\xi_\nu + \eta(\phi_\nu X)\phi_\nu\xi\} \text{ and} \tag{4}$$

$$R_\xi(X) = X - \eta(X)\xi + \alpha AX - \alpha^2\eta(X)\xi - \sum_{\nu=1}^{3}\Big\{\eta_\nu(X)\xi_\nu + 3\eta_\nu(\phi X)\phi_\nu\xi\Big\}. \tag{5}$$

From (4) and (5) and [2, Proposition 2], we obtain the following

$$SX = \begin{cases} (4m + 4 + h\alpha - \alpha^2)\xi & \text{if } X = \xi \in T_\alpha \\ (4m + 4 + h\beta - \beta^2)\xi_\ell & \text{if } X = \xi_\ell \in T_\beta \\ (4m + 8)\phi\xi_\ell & \text{if } X = \phi\xi_\ell \in T_\gamma \\ (4m + 7 + h\lambda - \lambda^2)X & \text{if } X \in T_\lambda \\ (4m + 7 + h\mu - \mu^2)X & \text{if } X \in T_\mu, \end{cases} \tag{6}$$

$$R_\xi(X) = \begin{cases} 0 & \text{if } X = \xi \in T_\alpha \\ \alpha\beta\xi_\ell & \text{if } X = \xi_\ell \in T_\beta \\ 4\phi\xi_\ell & \text{if } X = \phi\xi_\ell \in T_\gamma \\ (1 + \alpha\lambda)\phi X & \text{if } X \in T_\lambda \\ (1 + \alpha\mu)\phi X & \text{if } X \in T_\mu, \quad \text{and} \end{cases} \tag{7}$$

$$(R_\xi\phi)X = \begin{cases} 0 & \text{if } X = \xi \in T_\alpha \\ 4\phi\xi_\ell & \text{if } X = \xi_\ell \in T_\beta \\ -\alpha\beta\xi_\ell & \text{if } X = \phi\xi_\ell \in T_\gamma \\ (1 + \alpha\mu)\phi X & \text{if } X \in T_\lambda \\ (1 + \alpha\lambda)\phi X & \text{if } X \in T_\mu. \end{cases} \tag{8}$$

$$S(R_\xi\phi)X = \begin{cases} 0 & \text{if } X = \xi \in T_\alpha \\ 4(4m + 8)\phi\xi_\ell & \text{if } X = \xi_\ell \in T_\beta \\ -\alpha\beta(4m + 4 + h\beta - \beta^2)\xi_\ell & \text{if } X = \phi\xi_\ell \in T_\gamma \\ (1 + \alpha\mu)(4m + 7 + h\mu - \mu^2)\phi X & \text{if } X \in T_\lambda \\ (1 + \alpha\lambda)(4m + 7 + h\lambda - \lambda^2)\phi X & \text{if } X \in T_\mu. \end{cases} \tag{9}$$

From (6), (7),(8) and (9), it follows that

$$(R_\xi\phi)SX - SR_\xi\phi X = \begin{cases} 0 & \text{if } X = \xi \in T_\alpha \\ 4(h\beta - \beta^2 - 4)\phi\xi_\ell & \text{if } X = \xi_\ell \in T_\beta \\ \alpha\beta(h\beta - \beta^2 - 4)\xi_\ell & \text{if } X = \phi\xi_\ell \in T_\gamma \\ (1 + \alpha\mu)(\lambda - \mu)(h - \lambda - \mu)\phi X & \text{if } X \in T_\lambda \\ (1 + \alpha\lambda)(\mu - \lambda)(h - \lambda - \mu)\phi X & \text{if } X \in T_\mu. \end{cases} \tag{10}$$

By calculation, we have $\lambda + \mu = \beta$ on M_B. From (10), we see that M_B satisfies (C-1), only when $h = \beta$ and $h\beta - \beta^2 - 4 = 0$. This gives rise to contradiction.

Hence summing up these considerations, we give a complete proof of our Theorem 1 in the introduction.

3 Proof of Theorem 5

In this section, by using the notion of normal Jacobi operator $\bar{R}(X,N)N \in T_xM$, $x \in M$ for real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ and geometric quantities in [7] and [8], we give a complete proof of Theorem 5.

From now on, let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying

$$(\bar{R}_N\phi)SX = S(\bar{R}_N\phi)X \tag{11}$$

for any tangent vector field X on M. The normal Jacobi operator $\bar{R}_N$ of M is defined by $\bar{R}_N(X) = \bar{R}(X,N)N$ for any tangent vector $X \in T_xM$, $x \in M$. In [7, Introduction], we have the following equation

$$\begin{aligned}\bar{R}_N(X) =& X + 3\eta(X)\xi + 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\ & - \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi X - \eta_\nu(\xi)\eta(X)\xi_\nu - \eta_\nu(\phi X)\phi_\nu\xi\}.\end{aligned} \tag{12}$$

Lemma 6. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, satisfying* (C-2)*. If the principal curvature $\alpha = g(A\xi,\xi)$ is constant along the direction of ξ, then ξ belongs to either the distribution $\mathfrak{Q}$ or the distribution $\mathfrak{Q}^\perp$.*

Lemma 7. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying* (11)*. If $\xi \in \mathfrak{Q}^\perp$, we have $S\phi = \phi S$.*

In the case of $\xi \in \mathfrak{Q}^\perp$, by using (i) and (ii) in Lemmas 3, and 7, we can easily check that the commuting condition $S\phi = \phi S$ is equivalent condition to $(\bar{R}_N\phi)S = S(\bar{R}_N\phi)$.

Therefore by Lemma 7 and Theorem 2, we can assert that:

Remark 1. Real hypersurfaces of Type (A) in $G_2(\mathbb{C}^{m+2})$ satisfy the condition (C-2).

When $\xi \in \mathfrak{Q}$, a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ satisfying (C-2) is locally congruent to one of Type (B) by virtue of Theorem 3 given in the introduction.

Let us consider our problem for a model space of Type (B) which will be denoted by M_B. In order to do this, let us calculate $(\bar{R}_N\phi)S = S\bar{R}_N\phi$ of M_B. From [2, Proposition 2], we obtain

$$\bar{R}_N(X) = \begin{cases} 4\xi & \text{if } X = \xi \in T_\alpha \\ 4\xi_\ell & \text{if } X = \xi_\ell \in T_\beta \\ 0 & \text{if } X = \phi\xi_\ell \in T_\gamma \\ X & \text{if } X \in T_\lambda \\ X & \text{if } X \in T_\mu, \quad \text{and} \end{cases} \tag{13}$$

$$(\bar{R}_N\phi)X = \begin{cases} 0 & \text{if } X = \xi \in T_\alpha \\ 0 & \text{if } X = \xi_\ell \in T_\beta \\ -4\xi_\ell & \text{if } X = \phi\xi_\ell \in T_\gamma \\ \phi X & \text{if } X \in T_\lambda \\ \phi X & \text{if } X \in T_\mu. \end{cases} \tag{14}$$

From (13) and (14), it follows that

$$(\bar{R}_N\phi)SX - S\bar{R}_N\phi X = \begin{cases} 0 & \text{if } X = \xi \in T_\alpha \\ 0 & \text{if } X = \xi_\ell \in T_\beta \\ 4(h\beta - \beta^2 - 4)\xi_\ell & \text{if } X = \phi\xi_\ell \in T_\gamma \\ (\lambda - \mu)(h - \lambda - \mu)\phi X & \text{if } X \in T_\lambda \\ (\mu - \lambda)(h - \lambda - \mu)\phi X & \text{if } X \in T_\mu. \end{cases}$$

So we see that M_B satisfies (C-2), only when $h = \beta$ and $h\beta - \beta^2 - 4 = 0$. This gives rise to contradiction.

Hence summing up these considerations, we give a complete proof of our Theorem 5 in the introduction.

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Parallelism on Jacobi Operators for Hopf Hypersurfaces in Complex Two-Plane Grassmannians

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Abstract In relation to the generalized Tanaka-Webster connection, we consider a new notion of parallel Jacobi operator for real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ and show results about real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with generalized Tanaka-Webster parallel structure Jacobi operator and normal Jacobi operator.

1 Introduction

Respect to real hypersurfaces with parallel curvature tensor, many differential geometers studied in complex projective spaces or in quaternionic projective spaces [7, 10, 11, 13] and [16]. From another perspective, it is interesting to classify real hypersurfaces in complex two-plane Grassmannians with parallel shape operator, structure Jacobi operator and Ricci tensor (see [3, 4, 12, 14, 17–20] and [21]).

A complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ consists of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. This Riemannian symmetric space is the unique compact irreducible Riemannian manifold being equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J. Then, we could naturally consider two geometric conditions for hypersurfaces M in $G_2(\mathbb{C}^{m+2})$, namely, that the one-dimensional distribution $[\xi] = \text{Span}\{\xi\}$ and the three-dimensional distribution $\mathfrak{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ are both invariant under the shape operator A of M [2], where the *Reeb* vector field ξ is defined by $\xi = -JN$, N denotes a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$ and the *almost contact three-structure* vector fields ξ_ν are defined by $\xi_\nu = -J_\nu N$ $(\nu = 1, 2, 3)$.

By using the result in Alekseevskii [1], Berndt and Suh [2] proved the following :

Theorem 1. *Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

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(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*
(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

When we consider the Reeb vector field ξ in the expression of the curvature tensor R for a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, the structure Jacobi operator R_ξ can be defined as

$$R_\xi(X) = R(X, \xi)\xi,$$

for any tangent vector field X on M.

By using the structure Jacobi operator R_ξ, Jeong, Pérez and Suh considered the notion of *parallel structure Jacobi operator*, that is, $\nabla_X R_\xi = 0$ for any vector field X on M and gave a non-existence theorem (see [4]).

On the other hand, the Reeb vector field ξ is said to be *Hopf* if it is invariant under the shape operator A. The one dimensional foliation of M by the integral manifolds of the Reeb vector field ξ is said to be the *Hopf foliation* of M. We say that M is a *Hopf hypersurface* in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of M is totally geodesic. By the formulas in [5, Sects. 2, 3] it can be easily checked that M is Hopf if and only if the Reeb vector field ξ is Hopf.

Moreover, the authors [6] considered the general notion of $\mathfrak{D}^\perp$*-parallel structure Jacobi operator* defined by $\nabla_{\xi_\nu} R_\xi = 0,\ \nu = 1, 2, 3$, which is weaker than the notion of parallel structure Jacobi operator mentioned above. They gave a non-existence theorem (see [6]).

Now, we consider another one instead of Levi-Civita connection for real hypersurfaces in Kähler manifolds, namely, the *generalized Tanaka-Webster connection* (in short, the *g-Tanaka-Webster* connection) $\hat{\nabla}^{(k)}$ for a non-zero real number k [8]. This new connection $\hat{\nabla}^{(k)}$ is defined by the naturally extended one of Tanno's generalized Tanaka-Webster connection $\hat{\nabla}$ for contact metric manifolds. Actually, Tanno [23] introduced the *generalized Tanaka-Webster* connection $\hat{\nabla}$ for contact Riemannian manifolds by the canonical connection which coincides with the Tanaka-Webster connection if the associated CR-structure is integrable. On the other hand, the original *Tanaka-Webster connection* [22, 24] is given as a unique affine connection on a non-degenerate, pseudo-Hermitian CR manifolds associated with the almost contact structure. In particular, if a real hypersurface in Kähler manifolds satisfies $\phi A + A\phi = 2k\phi (k \neq 0)$, then the g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$ coincides with the Tanaka-Webster connection.

Using this g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$, we consider the new notion of *Reeb-parallel structure Jacobi operator in the generalized Tanaka-Webster connection*, that is, $\hat{\nabla}^{(k)}_\xi R_\xi = 0$. We can give a non-existence theorem as follows:

Theorem 2. *There do not exist any Hopf hypersurfaces in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb-parallel structure Jacobi operator in the generalized Tanaka-Webster connection.*

In addition, we consider other new notion for g-Tanaka-Webster parallelism of the structure Jacobi operator on a real hypersurface M in $G_2(\mathbb{C}^{m+2})$. If the structure Jacobi operator R_ξ of M satisfies $(\hat{\nabla}^{(k)}_X R_\xi)Y = 0$ for any tangent vector fields X and Y in M, then the structure Jacobi operator is said to be *parallel in the generalized Tanaka-Webster connection*. Naturally, we see that this notion of parallel structure Jacobi operator in the g-Tanaka-Webster connection is stronger than Reeb-parallel structure Jacobi operator in the g-Tanaka-Webster connection. Related to this notion, we have the following :

Corollary 1. *There do not exist any Hopf hypersurfaces in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel structure Jacobi operator in the generalized Tanaka-Webster connection.*

Next, motivated by Jeong et al. [6] and Theorem 2, we consider another new notion for g-Tanaka-Webster parallelism of the structure Jacobi operator on a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, when the structure Jacobi operator R_ξ of M satisfies $(\hat{\nabla}^{(k)}_X R_\xi)Y = 0$ for any $X \in \mathfrak{D}^\perp$ and any Y in M. In this case, the structure Jacobi operator is said to be $\mathfrak{D}^\perp$*-parallel in the generalized Tanaka-Webster connection*. Naturally, such a notion of parallelism is a generalized condition that is weaker than usual parallelism of the structure Jacobi operator in the g-Tanaka-Webster connection.

Theorem 3. *Let M be a connected orientable Hopf hypersurface in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If the structure Jacobi operator R_ξ is $\mathfrak{D}^\perp$-parallel in the generalized Tanaka-Webster connection, M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.*

As a prevailing notion, in a Riemannian manifold $(\bar{M}, \bar{g})$, a vector field X along a geodesic γ of $\bar{M}$ is called a *Jacobi field* if it satisfies the following second order Jacobi equation

$$\bar{\nabla}^2_{\dot{\gamma}} X + \bar{R}(X, \dot{\gamma})\dot{\gamma} = 0,$$

where $\dot{\gamma}$ is the vector tangent to γ.

For any tangent vector field X at $x \in \bar{M}$, the Jacobi operator $\bar{R}_X$ is defined by

$$(\bar{R}_X Y)(x) = (\bar{R}(Y, X)X)(x),$$

for any vector field $Y \in T_x\bar{M}$.

Now, let us put a unit normal vector field N to a hypersurface M into the curvature tensor $\bar{R}$ of the ambient space $\bar{M}$. Then for any tangent vector field X on M, the *normal Jacobi operator* $\bar{R}_N$ is defined by

$$\bar{R}_N(X) = \bar{R}(X, N)N.$$

Also, using this g-Tanaka-Webster connection $\hat{\nabla}^{(k)}$, we consider the new notion of *parallel normal Jacobi operator in the generalized Tanaka-Webster connection*, that is, $\hat{\nabla}^{(k)}_X \bar{R}_N = 0$ for any vector field $X \in TM$. We can give a non-existence theorem as follows :

Theorem 4. *There do not exist any Hopf hypersurfaces in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel normal Jacobi operator in the generalized Tanaka-Webster connection.*

2 Proof of Theorem 2

Let us denote by $R(X, Y)Z$ the curvature tensor of M in $G_2(\mathbb{C}^{m+2})$. Then the structure Jacobi operator R_ξ of M in $G_2(\mathbb{C}^{m+2})$ can be defined by $R_\xi X = R(X, \xi)\xi$ for any vector field $X \in T_x M = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$.

In [4] and [6], by using the structure Jacobi operator R_ξ, the authors obtained

$$\begin{aligned}
&(\nabla_X R_\xi)Y \\
&\quad = -g(\phi AX, Y)\xi - \eta(Y)\phi AX \\
&\qquad - \sum_{\nu=1}^{3}\Big[g(\phi_\nu AX, Y)\xi_\nu - 2\eta(Y)\eta_\nu(\phi AX)\xi_\nu + \eta_\nu(Y)\phi_\nu AX \\
&\qquad + 3\Big\{g(\phi_\nu AX, \phi Y)\phi_\nu\xi + \eta(Y)\eta_\nu(AX)\phi_\nu\xi \\
&\qquad + \eta_\nu(\phi Y)\big(\phi_\nu\phi AX - \alpha\eta(X)\xi_\nu\big)\Big\} \\
&\qquad + 4\eta_\nu(\xi)\Big\{\eta_\nu(\phi Y)AX - g(AX, Y)\phi_\nu\xi\Big\} + 2\eta_\nu(\phi AX)\phi_\nu\phi Y\Big] \\
&\qquad + \eta\big((\nabla_X A)\xi\big)AY + \alpha(\nabla_X A)Y - \eta\big((\nabla_X A)Y\big)A\xi \\
&\qquad - g(AY, \phi AX)A\xi - \eta(AY)(\nabla_X A)\xi - \eta(AY)A\phi AX.
\end{aligned} \tag{1}$$

On the other hand, by using the g-Tanaka-Webster connection, we have

$$\begin{aligned}
(\hat{\nabla}^{(k)}_X R_\xi)Y &= \hat{\nabla}^{(k)}_X(R_\xi Y) - R_\xi(\hat{\nabla}^{(k)}_X Y) \\
&= \nabla_X(R_\xi Y) + g(\phi AX, R_\xi Y)\xi - \eta(R_\xi Y)\phi AX - k\eta(X)\phi R_\xi Y \\
&\quad - R_\xi(\nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y).
\end{aligned} \tag{2}$$

From this, together with the fact that M is Hopf, it becomes

$$
\begin{aligned}
&(\hat{\nabla}^{(k)}_X R_\xi)Y \\
&\quad= -\sum_{\nu=1}^{3}\Big[g(\phi_\nu AX, Y)\xi_\nu - \eta(Y)\eta_\nu(\phi AX)\xi_\nu + \eta_\nu(Y)\phi_\nu AX \\
&\qquad + 3\Big\{g(\phi_\nu AX, \phi Y)\phi_\nu\xi + \eta(Y)\eta_\nu(AX)\phi_\nu\xi \\
&\qquad + \eta_\nu(\phi Y)\Big(\phi_\nu\phi AX - \alpha\eta(X)\xi_\nu\Big)\Big\} \\
&\qquad + 4\eta_\nu(\xi)\Big\{\eta_\nu(\phi Y)AX - g(AX, Y)\phi_\nu\xi\Big\} + 2\eta_\nu(\phi AX)\phi_\nu\phi Y \\
&\qquad + \eta_\nu(Y)\eta_\nu(\phi AX)\xi - \eta_\nu(\xi)\eta(Y)\eta_\nu(\phi AX)\xi \\
&\qquad + 3\eta(\phi_\nu Y)g(\phi AX, \phi_\nu\xi)\xi + \eta_\nu(\xi)g(\phi AX, \phi_\nu\phi Y)\xi \\
&\qquad - \eta_\nu(Y)\eta_\nu(\xi)\phi AX + \eta_\nu^2(\xi)\eta(Y)\phi AX - \eta_\nu(\xi)\eta(\phi_\nu\phi Y)\phi AX \\
&\qquad - k\eta(X)\eta_\nu(Y)\phi\xi_\nu - 4k\eta(X)\eta(\phi_\nu Y)\eta_\nu(\xi)\xi - 4k\eta(X)\eta(\phi_\nu Y)\xi_\nu \\
&\qquad + 3\eta(Y)\eta(\phi_\nu\phi AX)\phi_\nu\xi - \eta(Y)\eta_\nu(\xi)\phi_\nu AX + \alpha\eta(X)\eta(Y)\eta_\nu(\xi)\phi_\nu\xi \\
&\qquad + 3k\eta(X)\eta(\phi_\nu\phi Y)\phi_\nu\xi + k\eta(X)\eta(Y)\eta_\nu(\xi)\phi_\nu\xi\Big] \\
&\qquad + \eta((\nabla_X A)\xi)AY + \alpha(\nabla_X A)Y - \alpha\eta((\nabla_X A)Y)\xi \\
&\qquad - \alpha\eta(Y)(\nabla_X A)\xi - \alpha k\eta(X)\phi AY + \alpha k\eta(X)A\phi Y
\end{aligned}
\tag{3}
$$

for any tangent vector fields X and Y on M.

Let us assume that the structure Jacobi operator R_ξ on a Hopf hypersurface M in a complex two-plane Grassmann manifold $G_2(\mathbb{C}^{m+2})$ is *Reeb-parallel* in the g-Tanaka-Webster connection, that is,

$$(\hat{\nabla}^{(k)}_\xi R_\xi)Y = 0 \tag{4}$$

for any tangent vector field Y on M.

Here, it is a main goal to show that the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or orthogonal complement $\mathfrak{D}^\perp$ such that $TM = \mathfrak{D} \oplus \mathfrak{D}^\perp$ in $G_2(\mathbb{C}^{m+2})$ when the structure Jacobi operator is Reeb-parallel in the generalized Tanaka-Webster connection.

From now on, unless otherwise stated in the present section, we may put the Reeb vector field ξ as follows :

$$\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1 \tag{*}$$

for some unit vector fields $X_0 \in \mathfrak{D}$ and $\xi_1 \in \mathfrak{D}^\perp$.

Putting $X = \xi$ in (3) and using the condition (4), we have

$$
\begin{aligned}
0 =&(\hat{\nabla}^{(k)}_{\xi} R_{\xi})Y \\
=&-\sum_{\nu=1}^{3}\Big[\alpha g(\phi_\nu\xi, Y)\xi_\nu + \alpha\eta_\nu(Y)\phi_\nu\xi \\
&+ 3\Big\{\alpha g(\phi_\nu\xi, \phi Y)\phi_\nu\xi + \alpha\eta(Y)\eta_\nu(\xi)\phi_\nu\xi - \alpha\eta_\nu(\phi Y)\xi_\nu\Big\} \\
&+ 4\eta_\nu(\xi)\Big\{\alpha\eta_\nu(\phi Y)\xi - \alpha g(\xi, Y)\phi_\nu\xi\Big\} \\
&- k\eta_\nu(Y)\phi\xi_\nu - 4k\eta(\phi_\nu Y)\eta_\nu(\xi)\xi - 4k\eta(\phi_\nu Y)\xi_\nu \\
&- \alpha\eta(Y)\eta_\nu(\xi)\phi_\nu\xi + \alpha\eta(Y)\eta_\nu(\xi)\phi_\nu\xi \\
&+ 3k\eta(\phi_\nu\phi Y)\phi_\nu\xi + k\eta(Y)\eta_\nu(\xi)\phi_\nu\xi\Big] \\
&+ \eta((\nabla_\xi A)\xi)AY + \alpha(\nabla_\xi A)Y - \alpha\eta((\nabla_\xi A)Y)\xi \\
&- \alpha\eta(Y)(\nabla_\xi A)\xi - \alpha k\phi AY + \alpha k A\phi Y
\end{aligned}
\tag{5}
$$

for any tangent vector field Y on M.

Now, using these facts, we prove the following:

Lemma 1. *Let M be a Hopf hypersurface in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb-parallel structure Jacobi operator in the generalized Tanaka-Webster connection. Then the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

Then by Lemma 1 we shall divide our consideration in two cases depending on the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}^\perp$ or the distribution $\mathfrak{D}$.

First of all, we consider the case $\xi \in \mathfrak{D}^\perp$. Without loss of generality, we may put $\xi = \xi_1$.

Lemma 2. *If the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$, then there does not exist any Hopf hypersurface M in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb-parallel structure Jacobi operator in the generalized Tanaka-Webster connection.*

Next we consider the case $\xi \in \mathfrak{D}$. Using Theorem 1, Lee and Suh [9, Main Theorem] gave a characterization of real hypersurfaces of Type (B) in $G_2(\mathbb{C}^{m+2})$ in terms of the Reeb vector field ξ as follows:

Lemma 3. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with Reeb-parallel structure Jacobi operator in the generalized Tanaka-Webster connection. If the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$, then M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, $m = 2n$.*

From the above two Lemmas 2, 3 and the classification theorem given by Theorem 1 in this paper, we see that M is locally congruent to a model space of Type (B) in Theorem 1 under the assumption of our Theorem 2 given in the introduction.

Hence it remains to check if the structure Jacobi operator R_ξ of real hypersurfaces of Type (B) satisfies the condition (4) for any tangent vector field Y on M or not. To check this, we suppose that M has Reeb-parallel structure Jacobi operator in the g-Tanaka-Webster connection. Using [2, Proposition 2], we know that this gives a contradiction. So we give a complete proof of our Theorem 2 in the introduction.

On the other hand, we consider a new notion which is different from Reeb-parallel structure Jacobi operator in the g-Tanaka-Webster connection. The *parallel structure Jacobi operator in the generalized Tanaka-Webster connection* can be defined in such a way that

$$(\hat{\nabla}^{(k)}_X R_\xi)Y = 0 \tag{6}$$

for any tangent vector fields X and Y on M. From this notion, together with Lemmas 1, 2, 3 and the classification theorem given by Theorem 1 in the introduction, we see that M is locally congruent to a model space of Type (B) in Theorem 1. Hence we can check if the structure Jacobi operator R_ξ of real hypersurfaces of Type (B) satisfies the condition (4) for any tangent vector fields X and Y in M or not. Similarly, we can give a complete proof of our Corollary 1 in the introduction.

3 Proof of Theorem 3

Let us assume that the structure Jacobi operator R_ξ of a Hopf hypersurface M in a complex two-plane Grassmann manifold $G_2(\mathbb{C}^{m+2})$ is $\mathfrak{D}^\perp$-*parallel* in the g-Tanaka-Webster connection, that is,

$$(\hat{\nabla}^{(k)}_X R_\xi)Y = 0 \tag{7}$$

for any $X \in \mathfrak{D}^\perp$ and any tangent vector field Y on M.

Before getting our result, it is an important step to show that the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$ such that $TM = \mathfrak{D} \oplus \mathfrak{D}^\perp$ in $G_2(\mathbb{C}^{m+2})$ when the structure Jacobi operator is $\mathfrak{D}^\perp$-parallel in the g-Tanaka-Webster connection.

Now using the condition (7) and (*), we prove the following :

Lemma 4. *Let M be a Hopf hypersurface in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{D}^\perp$-parallel structure Jacobi operator in the generalized Tanaka-Webster connection. Then the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

Then by Lemma 4 we shall divide our consideration in two cases depending on whether the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^{\perp}$ or the distribution $\mathfrak{D}$.

First of all, we consider the case $\xi \in \mathfrak{D}^{\perp}$. Without loss of generality, we may put $\xi = \xi_1$. Using this notion of $\mathfrak{D}^{\perp}$-parallel structure Jacobi operator in the g-Tanaka-Webster connection, we get the following :

Lemma 5. *If the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^{\perp}$, then there does not exist any Hopf hypersurface M in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{D}^{\perp}$-parallel structure Jacobi operator in the generalized Tanaka-Webster connection.*

Next we consider the other case $\xi \in \mathfrak{D}$. Using Theorem 1, Lee and Suh [9] gave a characterization of real hypersurfaces of Type (B) in $G_2(\mathbb{C}^{m+2})$ in terms of the Reeb vector field ξ as follows :

Theorem 5. *Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, $m = 2n$.*

From Lemma 4, we see that M is locally congruent to a model space of Type (B) in Theorem 1 under the assumption of our Theorem 3 given in the introduction.

Hence it remains to check whether the structure Jacobi operator R_ξ of a real hypersurface of Type (B) satisfies the condition (7) or not. To check this problem, we can use [2, Proposition 2].

Hence, we have given a complete proof of our Theorem 3 in the introduction.

4 Proof of Theorem 4

Let us denote by $\bar{R}(X,Y)Z$ the curvature tensor in $G_2(\mathbb{C}^{m+2})$. Then for M in $G_2(\mathbb{C}^{m+2})$ the normal Jacobi operator $\bar{R}_N$ as an endomorphism of T_xM can be defined by $\bar{R}_N X = \bar{R}(X,N)N$ for any vector field $X \in T_xM = \mathfrak{D} \oplus \mathfrak{D}^{\perp}$, $x \in M$ (see [5]).

In [3] and [5], the derivative of the normal Jacobi operator is written as

$$
\begin{aligned}
(\nabla_X \bar{R}_N)Y = 3g(\phi AX, Y)\xi &+ 3\eta(Y)\phi AX \\
&+ 3\sum_{\nu=1}^{3}\Big\{g(\phi_\nu AX, Y)\xi_\nu + \eta_\nu(Y)\phi_\nu AX\Big\} \\
&- \sum_{\nu=1}^{3}\Big[2\eta_\nu(\phi AX)(\phi_\nu\phi Y - \eta(Y)\xi_\nu) - g(\phi_\nu AX, \phi Y)\phi_\nu\xi \\
&- \eta(Y)\eta_\nu(AX)\phi_\nu\xi - \eta_\nu(\phi Y)(\phi_\nu\phi AX - g(AX,\xi)\xi_\nu)\Big]
\end{aligned}
\tag{8}
$$

for any tangent vector fields X and Y on M.

By using the g-Tanaka-Webster connection, we have

$$\begin{aligned}(\hat{\nabla}^{(k)}_X \bar{R}_N)Y &= \hat{\nabla}^{(k)}_X(\bar{R}_N Y) - \bar{R}_N(\hat{\nabla}^{(k)}_X Y)\\ &= \nabla_X(\bar{R}_N Y) + g(\phi AX, \bar{R}_N Y)\xi - \eta(\bar{R}_N Y)\phi AX - k\eta(X)\phi\bar{R}_N Y\\ &\quad - \bar{R}_N(\nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y).\end{aligned}$$

From this, together with the fact that M is Hopf, it becomes

$$\begin{aligned}&(\hat{\nabla}^{(k)}_X \bar{R}_N)Y\\ &= \sum_{\nu=1}^{3}\Big\{3g(\phi_\nu AX, Y)\xi_\nu + 3\eta_\nu(Y)\phi_\nu AX - 2\eta_\nu(\phi AX)\phi_\nu\phi Y\\ &\quad + 5\eta_\nu(\phi AX)\eta(Y)\xi_\nu + g(\phi_\nu AX, \phi Y)\phi_\nu\xi + \eta_\nu(\phi Y)\phi_\nu\phi AX\\ &\quad - \alpha\eta(X)\eta_\nu(\phi Y)\xi_\nu + 3\eta_\nu(\phi AX)\eta_\nu(Y)\xi - \eta_\nu(\xi)g(\phi AX, \phi_\nu\phi Y)\xi\\ &\quad + \eta_\nu(\xi)\eta_\nu(\phi AX)\eta(Y)\xi - \alpha\eta_\nu(\xi)\eta(X)\eta_\nu(\phi Y)\xi + \eta_\nu(AX)\eta_\nu(\phi Y)\xi\\ &\quad - 4\eta_\nu(\xi)\eta_\nu(Y)\phi AX - 4k\eta(X)\eta_\nu(Y)\phi_\nu\xi + k\eta_\nu(\xi)\eta(X)\phi\phi_\nu\phi Y\\ &\quad - k\eta_\nu(\xi)\eta(X)\eta(Y)\phi_\nu\xi - k\eta_\nu(\xi)\eta(X)\eta_\nu(\phi Y)\xi + 4k\eta(X)\eta_\nu(\phi Y)\xi_\nu\\ &\quad - 4\eta_\nu(\xi)g(\phi AX, Y)\xi_\nu + \eta_\nu(\xi)\eta(Y)\phi_\nu AX + k\eta_\nu(\xi)\eta(X)\phi_\nu Y\Big\}\end{aligned} \tag{9}$$

for any tangent vector fields X and Y on M.

Let us assume that the normal Jacobi operator $\bar{R}_N$ on a Hopf hypersurface M in a complex two-plane Grassmann manifold $G_2(\mathbb{C}^{m+2})$ is *parallel* in the g-Tanaka-Webster connection, that is,

$$(\hat{\nabla}^{(k)}_X \bar{R}_N)Y = 0 \tag{10}$$

for any tangent vector fields X and Y on M.

Here, it is a main goal to show that the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or its orthogonal complement $\mathfrak{D}^\perp$ such that $TM = \mathfrak{D} \oplus \mathfrak{D}^\perp$ in $G_2(\mathbb{C}^{m+2})$ when the normal Jacobi operator is parallel in the g-Tanaka-Webster connection.

After some progress, we just have proved that the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.

First of all, we consider the case $\xi \in \mathfrak{D}^\perp$. Without loss of generality, we may put $\xi = \xi_1$.

Lemma 6. *Let M be a Hopf hypersurface of $G_2(\mathbb{C}^{m+2})$ with parallel normal Jacobi operator in the generalized Tanaka-Webster connection. If the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$, then* $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$

In the case of $\xi \in \mathfrak{D}$, from [9, Main Theorem] we know that M must be locally congruent to a real hypersurface of Type (B) under our assumptions. So, we see that M is locally congruent to a model space of either Type (A) or Type (B) in Theorem 1 under the assumption of our Theorem 4 given in the introduction.

Hence it remains to check whether the normal Jacobi operator $\bar{R}_N$ of real hypersurfaces of Type (A) or Type (B) satisfies the condition (10) for any tangent vector field Y on M or not. Using [2, Propositions 2, 3], we know that this case can not occur.

Hence, we can assert our Theorem 4 in the introduction.

Now, we consider a new notion which is different from parallel normal Jacobi operator in the g-Tanaka-Webster connection.

Let us assume that the normal Jacobi operator $\bar{R}_N$ on Hopf hypersurfaces M in complex two-plane Grassmann manifolds $G_2(\mathbb{C}^{m+2})$ is *Reeb-parallel* in the g-Tanaka-Webster connection defined in such a way that

$$(\hat{\nabla}^{(k)}_{\xi}\bar{R}_N)Y = 0 \tag{11}$$

for any tangent vector field Y on M. From this notion, together with the proof of Theorem 4 we see that the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^{\perp}$. From such a point of view, we will show that the assumption of being Reeb-parallel normal Jacobi operator in the g-Tanaka-Webster connection has no meaning for $\xi \in \mathfrak{D}^{\perp}$.

Proposition 1. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, such that $\xi \in \mathfrak{D}^{\perp}$. Then the normal Jacobi operator $\bar{R}_N$ is Reeb-parallel in the generalized Tanaka-Webster connection.*

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The *-Ricci Tensor of Real Hypersurfaces in Symmetric Spaces of Rank One or Two

George Kaimakamis and Konstantina Panagiotidou

Abstract Complex projective and hyperbolic spaces, i.e. non-flat complex space forms, are symmetric spaces of rank one. Complex two-plane Grassmannians are symmetric spaces of rank two. Let M be a real hypersurface in a symmetric space of rank one or two. Many geometers, such as Berndt, Jeong, Kim, Ortega, Pérez, Santos, Suh, Takagi and others have studied real hypersurfaces in above spaces in terms of their operators and tensor fields. This paper will be divided into two parts. Firstly, results concerning real hypersurfaces in non-flat complex space forms in terms of their *-**Ricci tensor**, S^*, which in case of real hypersurfaces was first studied by Hamada (Real hypersurfaces of complex space forms in terms of Ricci *-tensor. Tokyo J. Math. **25**, 473–483 (2002)), will be presented. More precisely, it will be answered if there exist or not real hypersurfaces, whose *-Ricci tensor is parallel, semi-parallel, i.e. $R \cdot S^* = 0$, or pseudo-parallel, i.e. $R(X, Y) \cdot S^* = L\{(X \wedge Y) \cdot S^*\}$ with $L \neq 0$ (Kaimakamis and Panagiotidou, Parallel *-Ricci tensor of real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$. Taiwan. J. Math., to appear, DOI 10.11650/tjm.18.2014.4271; Kaimakamis and Panagiotidou, Conditions of parallelism of *-Ricci tensor of real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$. Preprint). Secondly, the formula of *-Ricci tensor of real hypersurfaces in complex two-plane Grassmannians will be provided (Panagiotidou, The *-Ricci tensor of real hypersurfaces in complex two-plane Grassmannians, work in progress).

1 Introduction

A *complex space form* is a Kaehler manifold with constant holomorphic sectional curvature c for all the J-invariant planes Π in $T_P M$ at every point $P \in M$. A complete and simply connected complex space form is complex analytically

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isometric to complex projective space $\mathbb{C}P^n$, complex Euclidean space $\mathbb{C}^n$ or complex hyperbolic space $\mathbb{C}H^n$, if $c > 0$, $c = 0$, or $c < 0$ respectively. The notion of non-flat complex space form, denoted by $M_n(c)$ with $c \neq 0$ refers to complex projective or hyperbolic space, when it is not necessary to distinguish them.

Let M be a connected real hypersurface in $M_n(c)$, i.e. a submanifold of $M_n(c)$ with real codimension equal to one. Furthermore, let N be a locally defined unit normal vector field on M and A the shape operator of M with respect to N. The complex structure J of $M_n(c)$ induces on M an *almost contact metric structure* (φ, ξ, η, g). The eigenvalues of the shape operator A are called *principal curvatures* and the eigenvectors are called *principal curvature vectors*. Furthermore, if ξ is principal, i.e. $A\xi = \alpha\xi$, then the real hypersurface M is called *Hopf hypersurface*.

The study of real hypersurfaces in non-flat complex space forms was initiated by Takagi. In [22, 23], he provided the classification of homogeneous real hypersurfaces in complex projective space and divided them into six categories, namely $(A_1), (A_2), (B), (C), (D)$ and (E). All these real hypersurfaces are Hopf hypersurfaces with constant principal curvatures. In case of complex hyperbolic space the problem of studying real hypersurfaces with constant principal curvatures was initiated by Montiel in [10] and completed by Berndt in [1]. In this case real hypersurfaces are divided into two categories, namely (A) and (B), and they are homogeneous. In contrast to the case of complex projective space in complex hyperbolic space, there are homogeneous real hypersurfaces which are non-Hopf. A complete classification of homogeneous real hypersurfaces in complex hyperbolic space was given recently by Berndt and Tamaru in [3]. More information on the problem of real hypersurfaces with constant principal curvatures in non-flat complex space forms can be found in [5].

Many geometers have characterized real hypersurfaces in non-flat complex space forms, when the Ricci tensor, S, of them satisfies certain geometric conditions. Generally, the Ricci tensor S is given by the relation

$$S(X, Y) = trace\{Z \to R(Z, X)Y\},$$

where $X, Y \in TM$. A Riemannian manifold is called *Einstein*, if the Ricci tensor satisfies the relation $S = \rho I$, where ρ is constant.

In case of non-flat complex space forms Cecil and Ryan in [4], Montiel in [10] and Niebergall and Ryan in [13] proved that there are no Einstein real hypersurfaces in $\mathbb{C}P^n$, $n \geq 3$, in $\mathbb{C}H^n$, $n \geq 3$ and in $M_2(c)$ respectively. There are several results concerning real hypesurfaces in non-flat complex space forms, when the Ricci tensor of them satisfy conditions of parallelness. For a review on this results one can look at [12].

Hamada in [6], motivated by Tachibana, who in [21] studied the **-Ricci tensor* of almost Hermitian manifolds, introduced the notion of *-Ricci tensor, S^*, for real hypersurfaces in non-flat complex space forms. The *-Ricci tensor is given by

$$S^*(X, Y) = \frac{1}{2} trace(Z \to R(X, \varphi Y)\varphi Z), \tag{1}$$

where $X, Y \in TM$. Moreover, in [6] the notion of *-*Einstein* real hypersurfaces in non-flat complex space forms is included and a characterization of them is given.

Definition 1. A real hypersurface in $M_n(c)$ is called *-*Einstein*, if the *-Ricci tensor satisfies the relation

$$S^*(X,Y) = \frac{\rho^*}{2(n-1)} g(X,Y),$$

where X,Y are orthogonal to ξ, i.e. belong to the holomorphic distribution.

Recently, Ivey and Ryan in [7] extended the previous classification of *-Einstein real hypersurfaces in $M_n(c)$ and proved equivalent relations between *-Einstein, pseudo-Einstein and pseudo-Ryan for three-dimensional real hypersurfaces.

A *complex two-plane Grassmannian* $G_2(\mathbb{C}^{m+2})$ is the set of all two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. It is equipped with both a Kaehler structure J and a quaternionic Kaehler structure $\mathfrak{J}$, with a canonical local basis $\{J_1, J_2, J_3\}$, which does not contain J.

Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, N a locally defined unit normal vector field and A the shape operator of M with respect to N. The Kaehler structure J of $G_2(\mathbb{C}^{m+2})$ induces on M an almost contact metric structure (φ, ξ, η, g), with $\xi = -JN$. Furthermore, each of J_ν, $\nu = 1, 2, 3$ of the quaternionic structure induces on M an almost contact metric structure $(\varphi_\nu, \xi_\nu, \eta_\nu, g)$, with $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$.

In [2] the study of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ was initiated by Berndt and Suh. More precisely, they proved that the distributions $[\xi]$ and $\mathfrak{D}^\perp$, where $\mathfrak{D}^\perp = Span\{\xi_1, \xi_2, \xi_3\}$ of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ are invariant under the shape operator if and only if M

- (A) is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$,
- (B) or is an open part of a tube around a totally geodesic $\mathbb{Q}P^n$ in $G_2\left(\mathbb{C}^{2m+2}\right)$ and m is even, say $m = 2n$.

In case of complex two-plane Grassmannians Pérez and Suh in [17] proved that there are no Hopf Einstein real hypersurfaces in $G_2(\mathbb{C}^{m+2})$. There are several results concerning real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, whose Ricci tensor satisfy conditions of parallelness, i.e. the covariant derivative of the Ricci tensor with respect to a vector field vanishes, or conditions of invariance, i.e. the Lie derivative of the Ricci tensor with respect to a vector field vanishes or commuting conditions (see [18–20]).

Motivated by the work that so far has been done the following questions raised naturally

1. *Are there real hypersurfaces in $M_n(c)$ whose *-Ricci tensor is parallel or semi-parallel or pseudo-parallel?*
2. *Which is the formula of *-Ricci tensor for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$?*

The aim of this paper is to answer the above questions. More precisely, sketches of proof of the following Theorems will be provided

Theorem 1. *There do not exist real hypersurfaces in* $\mathbb{C}P^2$ *with parallel *-Ricci tensor.*

Theorem 2. *There do not exist real hypersurfaces* M *in* $\mathbb{C}P^2$, *whose *-Ricci tensor is semi-parallel.*

Theorem 3. *Every real hypersurface* M *in* $\mathbb{C}P^2$, *whose *-Ricci tensor is pseudo-parallel is a Hopf hypersurface. More precisely,* M *is locally congruent*

- *either to a geodesic hypersphere of radius r, where* $0 < r < \frac{\pi}{2}$, *and* $L = cot^2(r)$
- *or to a non-homogeneous real hypersurface, which is considered as a tube of radius* $\frac{\pi}{4}$ *over a holomorphic curve and* $L = 1$.

Furthermore, in Sect. 4 of this paper the formula of *-Ricci tensor for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ will be given.

Remark 1. Throughout this paper all manifolds and vector fields are assumed to be of class C^∞ and all manifolds are assumed to be connected.

2 Real Hypersurfaces in Complex Space Forms

In this section basic information about real hypersurfaces is given.

Let M be a real hypersurface without boundary immersed in a non-flat complex space form $(M_n(c)$,G) with complex structure J, N a locally defined unit normal vector field on M. The Riemannian connections $\overline{\nabla}$ in $M_n(c)$ and ∇ in M are related for any vector fields X, Y on M by $\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N$, where g is the Riemannian metric induced from the metric G. The shape operator A of the real hypersurface M with respect to N is given by $\overline{\nabla}_X N = -AX$.

As mentioned in the introduction the complex structure J induces on M an almost contact metric structure (φ, ξ, η, g) which is defined in the following way

- the structure vector field ξ is defined by $\xi = -JN$,
- the structure tensor φ is defined to be the tangential component of J, i.e. $JX = \varphi X + \eta(X)N$ and it is a tensor field of type (1,1),
- the 1-form η is given by $\eta(X) = g(X, \xi) = G(JX, N)$
- and g is the metric induced by G.

Moreover, the following relations are satisfied

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta \circ \varphi = 0, \quad \varphi\xi = 0, \quad \eta(\xi) = 1,$$
$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \varphi Y) = -g(\varphi X, Y),$$
$$\nabla_X \xi = \varphi AX, \quad (\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{2}$$

The Gauss and the Codazzi equations of a real hypersurface in $M_n(c)$ are given respectively by

$$R(X,Y)Z = \frac{c}{4}[g(Y,Z)X - g(X,Z)Y + g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y - 2g(\varphi X,Y)\varphi Z] + g(AY,Z)AX - g(AX,Z)AY, \quad (3)$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X,Y)\xi], \quad (4)$$

where R is the Riemannian curvature tensor on M and X, Y, Z are any tangent vector fields on M.

The tangent space $T_P M$, at every point $P \in M$, can be decomposed as $T_P M = span\{\xi\} \oplus \mathbb{D}$, where $\mathbb{D} = \ker\eta = \{X \in T_P M : \eta(X) = 0\}$ and is called *holomorphic distribution*. Due to the previous decomposition, the vector field $A\xi$ can be written

$$A\xi = \alpha\xi + \beta U, \quad (5)$$

where $\beta = |\varphi\nabla_\xi\xi|$ and $U = -\frac{1}{\beta}\varphi\nabla_\xi\xi \in \ker\eta$, provided that $\beta \neq 0$.

In the study of real hypersurfaces in $M_n(c)$, the following Theorem plays an important role. In case of $\mathbb{C}P^n$ it was proved by Okumura (see [14]) and in case of $\mathbb{C}H^n$ by Montiel and Romero (see [11]).

Theorem 4. *Let M be a real hypersurface of $M_n(c)$, $n \geq 2$. Then $A\varphi = \varphi A$, if and only if M is locally congruent to a homogeneous real hypersurface of type (A). More precisely:*

- *In case of $\mathbb{C}P^n$*

(A_1) *a geodesic hypersphere of radius r, where $0 < r < \frac{\pi}{2}$,*
(A_2) *a tube of radius r over a totally geodesic $\mathbb{C}P^k$, $1 \leq k \leq n-2$, where $0 < r < \frac{\pi}{2}$.*

- *In case of $\mathbb{C}H^n$*

(A_0) *a horosphere,*
(A_1) *a geodesic hypersphere or a tube over a totally geodesic complex hyperbolic hyperplane $\mathbb{C}H^{n-1}$,*
(A_2) *a tube over a totally geodesic $\mathbb{C}H^k$, $1 \leq k \leq n-2$.*

2.1 Three Dimensional Real Hypersurfaces in $\mathbb{C}P^2$ or $\mathbb{C}H^2$

In this section basic relations about three-dimensional real hypersurfaces in $M_2(c)$ are included.

Lemma 1. *Let M be a real hypersurface in $M_2(c)$. Then the following relations hold on M*

$$AU = \gamma U + \delta\varphi U + \beta\xi, \quad A\varphi U = \delta U + \mu\varphi U, \tag{6}$$

$$\nabla_U \xi = -\delta U + \gamma\varphi U, \quad \nabla_{\varphi U}\xi = -\mu U + \delta\varphi U, \quad \nabla_\xi \xi = \beta\varphi U, \tag{7}$$

$$\nabla_U U = \kappa_1 \varphi U + \delta\xi, \quad \nabla_{\varphi U} U = \kappa_2\varphi U + \mu\xi, \quad \nabla_\xi U = \kappa_3\varphi U, \tag{8}$$

$$\nabla_U \varphi U = -\kappa_1 U - \gamma\xi, \quad \nabla_{\varphi U}\varphi U = -\kappa_2 U - \delta\xi, \quad \nabla_\xi\varphi U = -\kappa_3 U - \beta\xi, \tag{9}$$

where $\gamma, \delta, \mu, \kappa_1, \kappa_2, \kappa_3$ are smooth functions on M and $\{U, \varphi U, \xi\}$ is an orthonormal basis of M.

The proof of Lemma 1 is included in [16].

3 Conditions of Parallelness of *-Ricci Tensor of Real Hypesurfaces in Complex Space Forms

The constant holomorphic sectional curvature of $\mathbb{C}P^n$ is $c = 4$ and of $\mathbb{C}H^n$ is $c = -4$ $n \geq 2$. Similar calculations to those of Theorem 2 in [7] imply that the *-Ricci tensor of M is given by

$$S^* X = -[\frac{cn}{2}\varphi^2 X + (\varphi A)^2 X], \quad \text{for } X \in TM. \tag{10}$$

Let M be a three-dimensional real hypersurface in $M_2(c)$, then relation (10) becomes

$$S^* X = -[c\varphi^2 X + (\varphi A)^2 X], \quad \text{for } X \in TM. \tag{11}$$

If M is a non-Hopf hypersurface and $\{U, \varphi U, \xi\}$ is a local orthonormal basis of it, the *-Ricci tensor for $X \in \{U, \varphi U, \xi\}$ due to (6) takes the form

$$S^*\xi = \beta\mu U - \beta\delta\varphi U, \quad S^* U = (c + \gamma\mu - \delta^2)U \quad \text{and} \quad S^*\varphi U = (c + \gamma\mu - \delta^2)\varphi U. \tag{12}$$

If M is a Hopf hypersurface relation $A\xi = \alpha\xi$ holds with α being constant (Theorem 2.1 in [12]). Let $\{W, \varphi W, \xi\}$ be a local orthonormal basis in a neighborhood of a point $P \in M$, such that $AW = \lambda W$ and $A\varphi W = \nu\varphi W$. Then the *-Ricci tensor for $X \in \{W, \varphi W, \xi\}$ becomes

$$S^*\xi = 0, \quad S^* W = (c + \lambda\nu)W \quad \text{and} \quad S^*\varphi W = (c + \lambda\nu)\varphi W. \tag{13}$$

Theorem 5. *There do not exist real hypersurfaces in $\mathbb{C}P^2$ with vanishing *-Ricci tensor.*

Proof. Let M be a real hypersurface in $\mathbb{C}P^2$ with vanishing *-Ricci tensor, i.e. $S^*X = 0$, $X \in TM$. We consider $\mathcal{N}$ the open subset of M such that

$$\mathcal{N} = \{P \in M : \beta \neq 0, \text{ in a neighborhood of } P\}.$$

On $\mathcal{N}$ relation (5) holds. The inner product of $S^*X = 0$ for $X = \xi$ with U and φU due to the first of (12) implies $\mu = \delta = 0$. The second relation of (12) becomes $S^*U = 4U$ and since $S^*U = 0$ leads to a contradiction. Therefore, $\mathcal{N}$ is empty and M is a Hopf hypersurface.

Let $\{W, \varphi W, \xi\}$ be a local orthonormal basis in a neighborhood of a point $P \in M$, such that $AW = \lambda W$ and $A\varphi W = \nu\varphi W$. The following relation holds (Corollary 2.3 [12])

$$\lambda\nu = \frac{\alpha}{2}(\lambda + \nu) + 1. \tag{14}$$

Relation $S^*X = 0$ for $X = W$ because of the second relation of (13) results in

$$4 + \lambda\nu = 0. \tag{15}$$

Substitution of the above relation in (14) implies that $\alpha(\lambda + \nu) = -10$. The last one yields that λ, ν are constant. Therefore, the real hypersurface has either two (type (A)) or three distinct eigenvalues (type (B)).

Case I: the real hypersurface has two constant principal curvature. In this case $\lambda = \nu$ and relation (15) becomes $4 + \lambda^2 = 0$, which is impossible. Therefore, the *-Ricci tensor of real hypersurface of type (A) in $\mathbb{C}P^2$ can not vanish.

Case II: the real hypersurface has three constant principal curvature. In this case $\lambda \neq \nu$ and M is locally congruent to a type (B), i.e. a tube of radius r over complex quadric, where $0 < r < \frac{\pi}{4}$. The eigenvalues are

$$\alpha = 2\cot(2r), \quad \lambda = \cot(r) \quad \text{and} \quad \nu = -\tan(r).$$

Substitution of the eigenvalues in $4 + \lambda\nu = 0$ leads to a contradiction and this completes the proof of the present Theorem. □

3.1 *Real Hypersurfaces in $M_2(c)$ with Parallel *-Ricci Tensor*

In this subsection the results concern real hypersurfaces M in $M_2(c)$, whose *-Ricci tensor is parallel, i.e. $\nabla_X S^* = 0$, for any $X \in TM$. More analytically, the previous relation is written

$$\nabla_X(S^*Y) = S^*(\nabla_X Y), \quad X, Y \in TM. \tag{16}$$

Proposition 1. *Every real hypersurface in $M_2(c)$, whose *-Ricci tensor is parallel, is a Hopf hypersurface.*

Sketch of proof of Proposition 1: Let $\mathcal{N}$ be an open subset of M such that

$$\mathcal{N} = \{P \in M : \beta \neq 0, \text{ in a neighborhood of } P\}.$$

Relation (16) for $X = Y = \xi$, for $X = \varphi U$ and $Y = \xi$ and for $X = \xi$ and $Y = \varphi U$ using relations of Lemma 1 and (12) leads to a contradiction. So $\mathcal{N}$ is empty.

Next we are focused on Hopf real hypersurfaces M in $\mathbb{C}P^2$ equipped with parallel *-Ricci tensor. Let $\{W, \varphi W, \xi\}$ be a local orthonormal basis in a neighborhood of a point $P \in M$, such that $AW = \lambda W$ and $A\varphi W = \nu\varphi W$ and relation (14) is satisfied.

Relation (16) for $X = W$ and $Y = \xi$ and for $X = \varphi W$ and $Y = \xi$ due to (13) implies respectively

$$\lambda(4 + \lambda\nu) = 0 \text{ and } \nu(4 + \lambda\nu) = 0.$$

Suppose that $4 + \lambda\nu \neq 0$. Then the above relations imply $\lambda = \nu = 0$ and substitution of the last ones in (14) implies $c = 0$, which is a contradiction. So $4 + \lambda\nu = 0$ and relation (13) implies that the *-Ricci tensor vanishes. Therefore, due to Theorem 5 the proof of Theorem 1 is completed.

*3.2 Real Hypersurfaces in $M_2(c)$ with Semi-Parallel *-Ricci Tensor*

In this subsection the results concern real hypersurfaces M in $M_2(c)$, whose *-Ricci tensor is semi-parallel, i.e. $R(X, Y) \cdot S^* = 0$. The previous relation due to the fact that R acts as a derivation on the tensor field implies

$$R(X, Y)S^*Z = S^*[R(X, Y)Z] \quad X, Y, Z \in TM. \tag{17}$$

Proposition 2. *Every real hypersurface in $M_2(c)$, whose *-Ricci tensor is semi-parallel, is a Hopf hypersurface.*

Sketch of proof of Proposition 2: Let $\mathcal{N}$ be an opensubset of M such that

$$\mathcal{N} = \{P \in M : \beta \neq 0, \text{ in a neighborhood of } P\}.$$

Relation (17) for $X = Z = U$ and $Y = \varphi U$, for $X = Z = U$ and $Y = \xi$ and for $X = Z = \varphi U$ and $Y = \xi$, using the Gauss equation (3) for the above combinations and relations of Lemma 1 leads to a contradiction. So $\mathcal{N}$ is empty.

Next we are focused on Hopf real hypersurfaces M in $\mathbb{C}P^2$ equipped with semi-parallel *-Ricci tensor. Let $\{W, \varphi W, \xi\}$ be a local orthonormal basis in a neighborhood of a point $P \in M$, such that $AW = \lambda W$ and $A\varphi W = \nu\varphi W$ and relation (14) is satisfied.

Relation (17) for $X = Z = W$ and $Y = \xi$ and for $X = Z = \varphi W$ and $Y = \xi$ implies respectively

$$(4 + \lambda\nu)(1 + \alpha\lambda) = 0 \text{ and } (4 + \lambda\nu)(1 + \alpha\nu) = 0. \tag{18}$$

Combining the above two relations leads to

$$\alpha(\lambda - \nu)(4 + \lambda\nu) = 0.$$

Suppose $4 + \lambda\nu \neq 0$, then $\alpha(\lambda - \nu) = 0$. If $\alpha \neq 0$ then $\lambda = \nu$ and M is locally congruent to a geodesic hypersphere (type (A)), with principal curvatures $\alpha = 2\cot(2r)$ and $\lambda = \cot(r)$. Moreover, the first of (18) yields $\alpha\lambda + 1 = 0$. Substitution in the last relation the eigenvalues of α and λ yields $\cot^2(r) = 0$, which is impossible. So, $\alpha = 0$ and the first of (18) since $4 + \lambda\nu \neq 0$, leads to a contradiction. Therefore, $4 + \lambda\nu = 0$ and relation (13) implies that the *-Ricci tensor vanishes. So, because of Theorem 5 the proof of Theorem 2 is finished.

*3.3 Real Hypersurfaces in $M_2(c)$ with Pseudo-Parallel *-Ricci Tensor*

In this subsection the results concern real hypersurfaces M in $M_2(c)$, whose *-Ricci tensor is pseudo-parallel, i.e. $R(X,Y) \cdot S^* = L\{(X \wedge Y) \cdot S^*\}$. Since R acts as a derivation on the tensor field and the wedge product of vectors is given by $(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y$ the latter yields

$$\begin{aligned} R(X,Y)S^*Z - S^*[R(X,Y)Z] &= L\{(X \wedge Y)S^*Z - S^*[(X \wedge Y)Z]\} \\ R(X,Y)S^*Z - S^*[R(X,Y)Z] &= L\{g(Y,S^*Z)X - g(X,S^*Z)Y \\ &\quad - S^*[g(Y,Z)X - g(X,Z)Y]\}, \end{aligned} \tag{19}$$

where $X, Y, Z \in TM$ and $L \neq 0$ and is a function.

Proposition 3. *Every real hypersurface in $M_2(c)$, whose *-Ricci tensor is pseudo-parallel, is a Hopf hypersurface.*

The proof of Proposition 3 follows the same procedure as the proof of Proposition 2.

Next we are focused on Hopf real hypersurfaces M in $\mathbb{C}P^2$ equipped with pseudo-parallel *-Ricci tensor. Let $\{W, \varphi W, \xi\}$ be a local orthonormal basis in a neighborhood of a point $P \in M$, such that $AW = \lambda W$ and $A\varphi W = \nu\varphi W$ and relation (14) is satisfied.

Relation (19) for $X = Z = W$ and $Y = \xi$ implies $(4 + \lambda\nu)(1 + \alpha\lambda - L) = 0$. Let $\mathscr{M}_1$ be the open subset of M such that

$$\mathscr{M}_1 = \{P \in M : L \neq 1 + \alpha\lambda, \text{ in a neighborhood of } P\}.$$

Then on $\mathscr{M}_1$ we have $4 + \lambda\nu = 0$, which due to (13) results in $S^*X = 0$ for $X \in TM$. Because of Theorem 5 we conclude that $\mathscr{M}_1$ is empty and on M relation $L = 1 + \alpha\lambda$ holds.

Relation (19) for $X = Z = \varphi W$ and $Y = \xi$ implies $(4 + \lambda\nu)(1 + \alpha\nu - L) = 0$. Since $4 + \lambda\nu \neq 0$ for the same reason as above, we conclude that on M relation $L = 1 + \alpha\nu$ holds. Combination of the last one with $L = 1 + \alpha\lambda$ results in

$$\alpha(\lambda - \nu) = 0.$$

The last relation because of Theorem 4 implies that a real hypersurface in $\mathbb{C}P^2$ equipped with pseudo-parallel *-Ricci tensor is locally congruent to a geodesic hypersphere of radius $0 < r < n/2$ or to a non-homogeneous real hypesurface with $\alpha = 0$, which is considered as a tube of radius $r = \frac{\pi}{4}$ over a holomorphic curve.

Conversely, it can be proved that the *-Ricci tensor of the above real hypersurfaces is pseudo-parallel. Moreover, L is constant and this completes proof of Theorem 3.

Remark 2. More details on the above issues are included in [8] and [9].

4 *-Ricci Tensor of Real Hypersurfaces in $G_2(\mathbb{C}^{m+2})$

In this section we introduce the formula of *-Ricci tensor for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$. It is known that the Gauss equation of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ satisfies the relation

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y - 2g(\varphi X,Y)\varphi Z \\ &+ \sum_{\nu=1}^{3}\{g(\varphi_\nu Y,Z)\varphi_\nu X - g(\varphi_\nu X,Z)\varphi_\nu Y - 2g(\varphi_\nu X,Y)\varphi_\nu Z\} \\ &+ \sum_{\nu=1}^{3}\{g(\varphi_\nu\varphi Y,Z)\varphi_\nu\varphi X - g(\varphi_\nu\varphi X,Z)\varphi_\nu\varphi Y\} \\ &- \sum_{\nu=1}^{3}\{\eta(Y)\eta_\nu(Z)\varphi_\nu\varphi X - \eta(X)\eta_\nu(Z)\varphi_\nu\varphi Y\} \end{aligned}$$

$$- \sum_{\nu=1}^{3} \{\eta(X) g(\varphi_\nu \varphi Y, Z) - \eta(Y) g(\varphi_\nu \varphi X, Z)\} \xi_\nu$$
$$+ g(AY, Z) AX - g(AX, Z) AY. \tag{20}$$

The *-Ricci tensor of M in $G_2(\mathbb{C}^{m+2})$ is defined as in the case of real hypersurfaces in $M_n(c)$, i.e. it satisfies relation (1). Let $\{E_i\}_{i=1}^{4m-1}$ be an orthonormal basis of M. Then following similar calculations to those of Theorem 2 in [7] and taking into account the relation (20)

$$\begin{aligned} g(S^* X, Y) = & \frac{1}{2}[-g(\varphi^2 Y, X) + g(\varphi X, \varphi Y) - g(\varphi^3 Y, \varphi X) + g(\varphi^2 X, \varphi^2 Y) \\ & + 2g(\varphi^2 X, Y) Tr\varphi^2 - 2g(\varphi A \varphi A X, Y) \\ & + \sum_{\nu=1}^{3} \{2g(\varphi\varphi_\nu X, \varphi_\nu \varphi Y) + 2g(\varphi\varphi_\nu X, Y) Tr\varphi_\nu\varphi + 2g(\varphi\varphi_\nu\varphi X, \varphi_\nu\varphi^2 Y) \\ & - 2\eta(X) g(\varphi\xi_\nu, \varphi_\nu\varphi^2 Y)\}]. \end{aligned}$$

The above relation taking into account the following relations

$$\begin{aligned} Tr\varphi^2 &= -2(2m-1), \\ Tr\varphi_\nu\varphi &= 2\eta_\nu(\xi), \\ \varphi\varphi_\nu X &= \varphi_\nu\varphi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu, \end{aligned}$$

results in

$$S^* X = -(4m\varphi^2 X + (\varphi A)^2 X) + \sum_{\nu=1}^{3} [(\varphi\varphi_\nu)^2 X + 2\eta_\nu(\xi)\varphi\varphi_\nu X + (\varphi^2\varphi_\nu^2)^2 X]\}.$$

Remark 3. In [15], which is work in progress more details about the *-Ricci tensor of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ are included.

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On Totally Geodesic Surfaces in Symmetric Spaces of Type AI

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Abstract We develop an approach to the classification of nonflat totally geodesic surfaces in Riemannian symmetric spaces of noncompact type. In this paper, we concentrate on the case of symmetric spaces of type AI, and show that such surfaces correspond to certain nilpotent matrices. As applications, we obtain explicit classifications in the cases of rank two and three.

1 Introduction

Totally geodesic submanifolds would be one of the most fundamental objects in submanifold theory. Let $(\overline{M}, g)$ be a Riemannian manifold. A submanifold M of $\overline{M}$ is said to be *totally geodesic* if the second fundamental form of M vanishes at every point, or equivalently, every geodesic in M is also an geodesic in $\overline{M}$. In this paper, we only consider connected and complete totally geodesic submanifolds.

Studies on totally geodesic submanifolds in Riemannian symmetric spaces have a long history. Among others, Chen and Nagano [4, 5] have made substantial contributions. On the other hand, Klein [8–12] recently showed that some of the classifications in [4, 5] are incomplete. It then turns out that a general classification problem of totally geodesic submanifolds is more complicated than it looks. Until now, the full classifications of totally geodesic submanifolds have been known only for irreducible Riemannian symmetric spaces of rank one and two, and a general classification problem remains widely open. For further information, we refer to a survey paper [9] and references therein. We also refer to the recent work by Mashimo [14], who studies totally geodesic surfaces in symmetric spaces of classical type. In fact, his study leads us to the topic of this paper.

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In the study on totally geodesic submanifolds in Riemannian symmetric spaces, Lie triple systems play fundamental roles. In fact, it is well-known that there is a one-to-one correspondence between totally geodesic submanifolds in Riemannian symmetric spaces and Lie triple systems (see Sect. 2 for details). It then follows that there is a one-to-one correspondence between totally geodesic submanifolds in a Riemannian symmetric space of compact type and those in its noncompact dual space. One of the key points of our study is that we focus on the noncompact setting. In the noncompact case, one can use the theory of Iwasawa decompositions, solvable Lie groups, and parabolic subgroups, which do not appear in the compact setting. These tools have played important roles in studying geometry of symmetric spaces of noncompact type, for examples, see [1, 3, 16].

As a first step of the studies on totally geodesic submanifolds in symmetric spaces in the noncompact setting, in this paper, we study totally geodesic surfaces (that is, of dimension two) in the symmetric space

$$\overline{M} = \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n). \tag{1}$$

This symmetric space is of type AI, and has rank $n-1$. Among symmetric spaces of higher rank, the symmetric spaces of type AI would be the most simplest ones. In particular, the Iwasawa decomposition

$$\mathfrak{sl}_n(\mathbb{R}) = \mathfrak{so}(n) \oplus \mathfrak{a} \oplus \mathfrak{n} \tag{2}$$

has a very simple matrix expression. Recall that $\mathfrak{n}$ coincides with the set of upper-triangular matrices with all diagonal entries 0.

In the main theorem of this paper, we show that there is a correspondence between nonflat totally geodesic surfaces in $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$ and matrices $X \in \mathfrak{n}$ satisfying certain conditions. As applications, we obtain explicit classifications of nonflat totally geodesic surfaces in the cases of $n = 3$ and 4. When $n = 3$, totally geodesic submanifolds have already been classified by Klein [11], but our argument gives a very elementary proof, based on direct matrix calculations. When $n = 4$, our classification of nonflat totally geodesic surfaces seems to be a new result.

We here would like to note that the main theorem of this paper seems to be generalizable to nonflat totally geodesic surfaces in an arbitrary Riemannian symmetric space of noncompact type. This kind of generalization and further applications will be discussed in the forthcoming papers.

2 Preliminaries

In this section, we describe some basic notions and facts on Riemannian symmetric spaces and their totally geodesic submanifolds. We mention some general notions in the first subsection. In the second subsection, we give descriptions for the case of type AI in an explicit way.

2.1 General Preliminaries

In this subsection, we recall basic notions on Riemannian symmetric spaces and their totally geodesic submanifolds. We refer to the textbooks [2, 7, 13].

Let $(\overline{M}, g)$ be a Riemannian symmetric space. We denote by G the identity component of the isometry group of $\overline{M}$, and by K the isotropy subgroup at some $o \in \overline{M}$, called the origin. One knows that

$$\overline{M} = G/K. \tag{3}$$

Let us denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. Then one has a canonical decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}. \tag{4}$$

Note that the symmetry at o, denoted by s_o, defines the involution $\theta : \mathfrak{g} \to \mathfrak{g}$ whose (± 1)-eigenspace decomposition coincides with (4).

Definition 1. A linear subspace V in $\mathfrak{p}$ is called a *Lie triple system* if it satisfies $[[X, Y], Z] \in V$ for every $X, Y, Z \in V$.

It is well-known that Lie triple systems correspond to totally geodesic submanifolds. We identify $\mathfrak{p}$ with the tangent space $T_o\overline{M}$ in a natural way.

Theorem 1. *Let V be a linear subspace in $\mathfrak{p}$. Then, V is a Lie triple system if and only if there exists a totally geodesic submanifold M in $\overline{M}$ satisfying $o \in M$ and $T_oM = V$.*

Therefore, the classification problem of totally geodesic submanifolds in $\overline{M}$ can be reduced to the classification problem of Lie triple systems in $\mathfrak{p}$. Nevertheless to say, we are only interested in the classification problem of totally geodesic submanifolds up to isometric congruence.

We also note that we have only to study nonflat totally geodesic submanifolds. This is because, every flat totally geodesic submanifold is contained in a maximal flat, and every maximal flat are isometrically congruent to each other. Recall that a Lie triple system V is abelian (that is, $[V, V] = 0$) if and only if the corresponding totally geodesic submanifold is flat.

2.2 Preliminaries for Type AI

In this subsection, we recall some of the basic facts on $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$. We are not going into a general theory of symmetric spaces of noncompact type, but describe it explicitly in this symmetric space. We refer to [6].

First of all, we describe the Cartan decomposition associated to the above symmetric space. Let us define

$$\mathrm{Sym}_n^0(\mathbb{R}) := \{X \in \mathfrak{sl}_n(\mathbb{R}) \mid X = {}^tX\}. \tag{5}$$

Proposition 1. *The decomposition* $\mathfrak{sl}_n(\mathbb{R}) = \mathfrak{so}(n) \oplus \mathrm{Sym}_n^0(\mathbb{R})$ *is a direct sum as vector spaces.*

The proof is an easy exercise of linear algebra. Note that the above decomposition is nothing but the Cartan decomposition. In fact, it is the (± 1)-eigenspace decomposition with respect to the involution $\theta : \mathfrak{sl}_n(\mathbb{R}) \to \mathfrak{sl}_n(\mathbb{R}) : X \mapsto -{}^tX$. Note that θ is a Lie algebra automorphism, in other words, it satisfies

$${}^t[Y, Z] = -[{}^tY, {}^tZ] \qquad (\forall Y, Z \in \mathfrak{sl}_n(\mathbb{R})). \tag{6}$$

Proposition 2. *Let* $\mathfrak{a}$ *be the set of all diagonal matrices with trace* 0*, and* $\mathfrak{n}$ *be the set of all upper-triangular matrices with all diagonal entries* 0*. Then, one has the following direct sum decomposition as vector spaces*:

$$\mathfrak{sl}_n(\mathbb{R}) = \mathfrak{so}(n) \oplus \mathfrak{a} \oplus \mathfrak{n}. \tag{7}$$

The above decomposition is known as the Iwasawa decomposition. It is easy to see that $\mathfrak{a}$ is an abelian subalgebra, $\mathfrak{n}$ is a nilpotent subalgebra, and $\mathfrak{a} \oplus \mathfrak{n}$ is a solvable subalgebra.

We next recall an $\mathrm{O}(n)$-invariant inner product on $\mathrm{Sym}_n^0(\mathbb{R})$. Note that the orthogonal group $\mathrm{O}(n)$ acts on $\mathrm{Sym}_n^0(\mathbb{R})$ by conjugation, that is, for each $g \in \mathrm{O}(n)$ and $X \in \mathrm{Sym}_n^0(\mathbb{R})$,

$$g.X := gXg^{-1} \in \mathrm{Sym}_n^0(\mathbb{R}). \tag{8}$$

Proposition 3. *Let* $X, Y \in \mathrm{Sym}_n^0(\mathbb{R})$*. Then, the following defines an* $\mathrm{O}(n)$*-invariant inner product on* $\mathrm{Sym}_n^0(\mathbb{R})$:

$$\langle X, Y \rangle := \mathrm{tr}(XY). \tag{9}$$

This inner product gives rise to an $\mathrm{SL}_n(\mathbb{R})$-invariant Riemannian metric on the reductive space $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$, and makes this space symmetric. In this paper, we always assume that this Riemannian metric is equipped.

We have to study Lie triple systems in $\mathrm{Sym}_n^0(\mathbb{R})$ up to congruence. We here define this terminology explicitly.

Definition 2. Two subsets $V, V' \subset \mathrm{Sym}_n^0(\mathbb{R})$ are said to be *congruent* if there exists $g \in \mathrm{O}(n)$ such that $gVg^{-1} = V'$.

If two Lie triple systems are congruent, then the corresponding totally geodesic submanifolds are isometrically congruent. This follows from the fact that $\mathrm{O}(n)$ acts isometrically on $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$. For the structure of the full isometry group of a symmetric space, we refer to Takeuchi [15] and Loos ([13], Chap. VII).

Finally in this subsection, we recall the sectional curvatures. Note that, by the definition of our inner product, one has

$$\langle [Y, Z], W \rangle = \langle Z, [Y, W] \rangle \quad (\forall Y, Z, W \in \mathrm{Sym}_n^0(\mathbb{R})). \tag{10}$$

Proposition 4. *Let Σ be a nonabelian two-dimensional Lie triple system in* $\mathrm{Sym}_n^0(\mathbb{R})$*, and $\{Y_1, Y_2\}$ be an orthonormal basis of Σ. Then one has*

(1) *there exists $a > 0$ such that $[[Y_1, Y_2], Y_2] = aY_1$,*
(2) *the (constant) sectional curvature of the totally geodesic surface corresponding to Σ coincides with $-a$.*

Proof. We show (1). Since Σ is a Lie triple system, there exist $a, b \in \mathbb{R}$ such that

$$[[Y_1, Y_2], Y_2] = aY_1 + bY_2. \tag{11}$$

By taking the inner product with Y_2, one has from (10) that

$$b = \langle aY_1 + bY_2, Y_2 \rangle = \langle [[Y_1, Y_2], Y_2], Y_2 \rangle = \langle [Y_1, Y_2], [Y_2, Y_2] \rangle = 0. \tag{12}$$

By taking the inner product with Y_1, one also has

$$a = \langle aY_1 + bY_2, Y_1 \rangle = \langle [[Y_1, Y_2], Y_2], Y_1 \rangle = ||[Y_1, Y_2]||^2 > 0, \tag{13}$$

since Σ is nonabelian. This completes the proof of the first assertion.

We show (2). It is well-known that the Riemannian curvature tensor R at o is given by

$$R_o(Y, Z)W = -[[Y, Z], W] \quad (\forall Y, Z, W \in \mathrm{Sym}_n^0(\mathbb{R})). \tag{14}$$

Let K be the sectional curvature of the totally geodesic surface corresponding to Σ. It then follows from the first assertion that

$$K = \langle R_o(Y_1, Y_2)Y_2, Y_1 \rangle = -\langle [[Y_1, Y_2], Y_2], Y_1 \rangle = -\langle aY_1, Y_1 \rangle = -a. \tag{15}$$

This completes the proof of the second assertion. □

3 Main Theorem

In this section, we show that every nonabelian two-dimensional Lie triple system in $\mathrm{Sym}_n^0(\mathbb{R})$ corresponds to a certain nilpotent matrix.

Recall that $\mathfrak{n}$ denotes the set of all upper-triangular matrices with all diagonal entries zero. One also needs

$$W := \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in \mathfrak{sl}_n(\mathbb{R}) \mid a_1 \geq \cdots \geq a_n \right\}. \tag{16}$$

Theorem 2. *Assume that $X \in \mathfrak{n} \setminus \{0\}$ satisfies*

(C1) $[X, {}^tX] \in W$,
(C2) *there exists $u > 0$ such that $[[X, {}^tX], X] = uX$.*

Then, $\mathfrak{L}_X := \mathrm{Span}_{\mathbb{R}}\{[{}^tX, X], X + {}^tX\}$ is a nonabelian two-dimensional Lie triple system in $\mathrm{Sym}^0_n(\mathbb{R})$. Conversely, every such Lie triple system can be constructed in this way up to congruence.

Proof (Proof of the First Assertion). Assume that $X \in \mathfrak{n} \setminus \{0\}$ satisfies (C1) and (C2). By (C2), there exists $u > 0$ such that

$$[[X, {}^tX], X] = uX. \tag{17}$$

It then follows from (6) that

$$[[X, {}^tX], {}^tX] = -{}^t[{}^t[X, {}^tX], X] = -{}^t[[X, {}^tX], X] = -u{}^tX. \tag{18}$$

This shows that

$$[\mathfrak{L}_X, \mathfrak{L}_X] = \mathrm{Span}_{\mathbb{R}}\{X - {}^tX\}. \tag{19}$$

One can also see that

$$[[{}^tX, X], X - {}^tX] = -u(X + {}^tX) \in \mathfrak{L}_X, \tag{20}$$

$$[X + {}^tX, X - {}^tX] = -2u[X, {}^tX] \in \mathfrak{L}_X, \tag{21}$$

which show that $\mathfrak{L}_X$ is a Lie triple system. This completes the proof of the first assertion of Theorem 2. □

Proof of the second assertion will be divided into several lemmas. Let Σ be a nonabelian two-dimensional Lie triple system in $\mathrm{Sym}^0_n(\mathbb{R})$.

Let $\{V_1, V_2\}$ be an orthonormal basis of Σ. Since V_1 is a symmetric matrix, an elementary linear algebra shows that there exists $g \in \mathrm{O}(n)$ such that $gV_1g^{-1} \in W$. We will show that $g\Sigma g^{-1} = \mathfrak{L}_X$ for some X. Let us denote by

$$H := gV_1g^{-1}, \quad Y := gV_2g^{-1}. \tag{22}$$

Lemma 1. *There exist* $b, c > 0$ *such that*

$$[H,[H,Y]] = bY, \quad [Y,[Y,H]] = cH. \tag{23}$$

Proof. Note that $\{H, Y\}$ is an orthonormal basis of the Lie triple system $g\Sigma g^{-1}$. Hence, the lemma directly follows from Proposition 4 (1). □

In the next step, we use $[H,[H,Y]] = bY$, and show that Y is an off-diagonal matrix. This determines $X \in \mathfrak{n}$.

Lemma 2. *We use the above notations. Then, there exists* $X \in \mathfrak{n}$ *such that*

$$Y = X + {}^tX, \quad [H,[H,X]] = bX. \tag{24}$$

Proof. Since Y is symmetric, one can decompose $Y = D + X + {}^tX$, where D is diagonal and $X \in \mathfrak{n}$. Since H and D are diagonal, we have

$$[H,Y] = [H, D + X + {}^tX] = [H,X] + [H, {}^tX] = [H,X] - {}^t[H,X]. \tag{25}$$

Hence, Lemma 1 yields that

$$b(D + X + {}^tX) = bY = [H,[H,Y]] = [H,[H,X]] + {}^t[H,[H,X]]. \tag{26}$$

By comparing the diagonal components and the $\mathfrak{n}$-components of the both sides, one has $D = 0$ and $bX = [H,[H,X]]$. This completes the proof. □

In order to show that X satisfies (C1) and (C2), we need the following lemma. It is crucial that $H \in W$.

Lemma 3. *The above* X *satisfies* $[H,X] = b^{1/2}X$.

Proof. Let us denote by $H = \mathrm{diag}(a_1, \ldots, a_n)$ and $X = (x_{ij})$ in terms of matrix elements. Then, a direct calculation shows that

$$[H,X] = ((a_i - a_j)x_{ij}). \tag{27}$$

Thus, we have only to show that

$$(a_k - a_l)x_{kl} = b^{1/2}x_{kl} \quad (\forall k, l). \tag{28}$$

Take any k and l. The case when $x_{kl} = 0$ is obvious. Let us consider the case when $x_{kl} \neq 0$. Lemma 2 yields that

$$((a_i - a_j)^2 x_{ij}) = [H,[H,X]] = bX = (bx_{ij}). \tag{29}$$

This yields $(a_k - a_l)^2 = b$. Note that, since $X \in \mathfrak{n}$, one knows $k < l$. One then has $a_k - a_l \geq 0$, since $H \in W$. We thus have $a_k - a_l = b^{1/2}$. This shows (28), which completes the proof. □

We now use $cH = [Y, [Y, H]]$ of Lemma 1, and complete the proof of the second assertion of Theorem 2. Namely, we have $g\Sigma g^{-1} = \mathfrak{L}_X$.

Proposition 5. *The above X satisfies (C1), (C2), and $g\Sigma g^{-1} = \mathfrak{L}_X$.*

Proof. It follows from Lemma 3 that

$$[Y, H] = [X + {}^tX, H] = [X, H] - {}^t[X, H] = b^{1/2}(-X + {}^tX). \tag{30}$$

Hence, Lemma 1 yields that

$$cH = [Y, [Y, H]] = b^{1/2}[X + {}^tX, -X + {}^tX] = 2b^{1/2}[X, {}^tX]. \tag{31}$$

Since $b, c > 0$ and $H \in W$, this shows that X satisfies (C1). Furthermore, by the above calculation, one can directly see that

$$[[X, {}^tX], X] = c(2b^{1/2})^{-1}[H, X] = (1/2)cX, \tag{32}$$

which shows (C2). Now it is easy to see that

$$\mathfrak{L}_X = \mathrm{Span}_{\mathbb{R}}\{[X, {}^tX], X + {}^tX\} = \mathrm{Span}_{\mathbb{R}}\{H, Y\} = g\Sigma g^{-1}, \tag{33}$$

which completes the proof. □

4 Some Explicit Classifications

We have seen in Theorem 2 that every nonabelian two-dimensional Lie triple system in $\mathrm{Sym}^0_n(\mathbb{R})$ can be constructed from a nilpotent matrix $X \in \mathfrak{n}$ satisfying Conditions (C1) and (C2). In this section, we apply this result to explicit classifications of nonabelian two-dimensional Lie triple systems.

4.1 The Case of $n = 3$

In this subsection, we classify nonabelian two-dimensional Lie triple systems in $\mathrm{Sym}^0_3(\mathbb{R})$. This gives another proof of the classification by Klein [11].

First of all, we study $X \in \mathfrak{n}$ satisfying (C1) and (C2). We denote by E_{ij} the matrix whose (i, j)-component is 1 and others are 0.

Lemma 4. *Let* $X := x_{12}E_{12} + x_{13}E_{13} + x_{23}E_{23} \neq 0$. *If* X *satisfies (C1) and (C2), then it satisfies one of the following conditions*:

(1) $x_{12} = x_{23} = 0$,
(2) $x_{13} = 0$, $x_{12} = \pm x_{23}$.

Proof. Assume that X satisfies (C1) and (C2). A direct calculation yields that

$$[X, {}^tX] = \begin{pmatrix} x_{12}^2 + x_{13}^2 & x_{13}x_{23} & 0 \\ x_{13}x_{23} & -x_{12}^2 + x_{23}^2 & -x_{12}x_{13} \\ 0 & -x_{12}x_{13} & -(x_{13}^2 + x_{23}^2) \end{pmatrix}. \tag{34}$$

Case 1: $x_{13} \neq 0$. Since $[X, {}^tX]$ is diagonal by (C1), one has

$$x_{13}x_{23} = x_{12}x_{13} = 0. \tag{35}$$

We thus have $x_{12} = x_{23} = 0$, and hence X satisfies (1).
Case 2: $x_{13} = 0$. In this case, one knows

$$W \ni [X, {}^tX] = \begin{pmatrix} x_{12}^2 & 0 & 0 \\ 0 & -x_{12}^2 + x_{23}^2 & 0 \\ 0 & 0 & -x_{23}^2 \end{pmatrix}. \tag{36}$$

Then, (C1) yields that $x_{12}^2 \geq -x_{12}^2 + x_{23}^2 \geq -x_{23}^2$. Since $X \neq 0$, we have

$$x_{12} \neq 0, \quad x_{23} \neq 0. \tag{37}$$

A direct calculation shows that

$$[[X, {}^tX], X] = \begin{pmatrix} 0 & (2x_{12}^2 - x_{23}^2)x_{12} & 0 \\ 0 & 0 & (2x_{23}^2 - x_{12}^2)x_{23} \\ 0 & 0 & 0 \end{pmatrix}. \tag{38}$$

Therefore, (C2) yields that $2x_{12}^2 - x_{23}^2 = 2x_{23}^2 - x_{12}^2$. We thus have $x_{12}^2 = x_{23}^2$, and hence X satisfies (2). □

By studying the congruency of $\mathfrak{L}_X$ for each X listed in Lemma 4, we obtain a classification of nonabelian two-dimensional Lie triple systems in $\mathrm{Sym}_3^0(\mathbb{R})$.

Proposition 6. *Every nonabelian two-dimensional Lie triple system in* $\mathrm{Sym}_3^0(\mathbb{R})$ *is congruent to one of the following*:

$$\Sigma_{3,1} := \mathrm{Span}_{\mathbb{R}} \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\},$$

$$\Sigma_{3,2} := \mathrm{Span}_{\mathbb{R}} \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}.$$

Proof. Let Σ be a nonabelian two-dimensional Lie triple system in $\mathrm{Sym}_3^0(\mathbb{R})$. Then, Theorem 2 yields that Σ is congruent to

$$\mathfrak{L}_X := \mathrm{Span}_{\mathbb{R}}\{[X, {}^tX], X + {}^tX\} \tag{39}$$

for some X satisfying (C1) and (C2). Furthermore, X satisfies one of two conditions mentioned in Lemma 4.

Case 1: $x_{12} = x_{23} = 0$. In this case, it is easy to see that $\mathfrak{L}_X = \Sigma_{3,1}$.
Case 2: $x_{13} = 0$, $x_{12} = \pm x_{23}$. In this case, $\mathfrak{L}_X$ coincides with $\Sigma_{3,2}$ or

$$\Sigma_{3,3} := \mathrm{Span}_{\mathbb{R}} \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \right\}. \tag{40}$$

One can easily check that $g := \mathrm{diag}(1, 1, -1)$ satisfies $g\Sigma_{3,2}g^{-1} = \Sigma_{3,3}$. Therefore, $\mathfrak{L}_X$ is congruent to $\Sigma_{3,2}$. □

Finally in this subsection, we show that the above two Lie triple systems are not congruent. This can be seen by the sectional curvatures.

Proposition 7. *The totally geodesic submanifolds corresponding to $\Sigma_{3,1}$ and $\Sigma_{3,2}$ have the sectional curvatures -2 and $-1/2$, respectively. In particular, they are not isometrically congruent to each other.*

Proof. This follows from direct calculations in terms of Proposition 4. □

4.2 The Case of $n = 4$

In this subsection, we classify nonabelian two-dimensional Lie triple systems in $\mathrm{Sym}_4^0(\mathbb{R})$. Note that $\mathrm{SL}_4(\mathbb{R})/\mathrm{SO}(4)$ has rank three, and a classification of totally geodesic surfaces in this space would be a new result.

As in the case of $n = 3$, first of all, we classify $X \in \mathfrak{n}$ satisfying Conditions (C1) and (C2). In this subsection, the double-signs $\pm$ correspond.

Lemma 5. *Let $X = \sum_{1 \leq i < j \leq 4} x_{ij} E_{ij} \neq 0$. If X satisfies (C1) and (C2), then it satisfies one of the following conditions*:

(1) $x_{14} \neq 0$, *and others are* 0,
(2) $x_{13} = \pm x_{24}$, $x_{14} = \mp x_{23}$, *and others are* 0,
(3) $4x_{12}^2 = 4x_{34}^2 = 3x_{23}^2$, *and others are* 0,
(4) $x_{12} = \pm x_{24}$, $x_{13} = \pm x_{34}$, *and others are* 0.

Proof. Assume that X satisfies (C1) and (C2). A direct calculation yields that the diagonal component of $[X, {}^tX]$ is

$$\begin{pmatrix} x_{12}^2 + x_{13}^2 + x_{14}^2 & & & \\ & x_{23}^2 + x_{24}^2 - x_{12}^2 & & \\ & & x_{34}^2 - x_{13}^2 - x_{23}^2 & \\ & & & -x_{14}^2 - x_{24}^2 - x_{34}^2 \end{pmatrix}. \tag{41}$$

It follows from (C1) that the upper-triangular component ($\mathfrak{n}$-component) of $[X, {}^tX]$ is zero. This means

$$\begin{pmatrix} 0 & x_{13}x_{23} + x_{14}x_{24} & x_{14}x_{34} & 0 \\ & 0 & x_{24}x_{34} - x_{12}x_{13} & -x_{12}x_{14} \\ & & 0 & -x_{13}x_{14} - x_{23}x_{24} \\ & & & 0 \end{pmatrix} = 0. \tag{42}$$

We divide the proof into several cases and subcases.

Case 1: $x_{14} \neq 0$. It follows from (42) that

$$x_{34} = x_{12} = 0, \quad x_{13}x_{23} + x_{14}x_{24} = 0, \quad x_{13}x_{14} + x_{23}x_{24} = 0. \tag{43}$$

Subcase 1-(i): $x_{14} \neq 0$, $x_{13} = 0$. In this case, the second equation of (43) yields that $x_{24} = 0$. If $x_{23} = 0$, then X obviously satisfies (1). Hence, let us assume $x_{23} \neq 0$. One knows from (41) that

$$[X, {}^tX] = \begin{pmatrix} x_{14}^2 & & & \\ & x_{23}^2 & & \\ & & -x_{23}^2 & \\ & & & -x_{14}^2 \end{pmatrix}. \tag{44}$$

Thus, a direct calculation shows that

$$[[X, {}^tX], X] = \begin{pmatrix} 0 & 0 & 0 & 2x_{14}^2 x_{14} \\ & 0 & 2x_{23}^2 x_{23} & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}. \tag{45}$$

It follows from (C2) that $x_{14}^2 = x_{23}^2$, and hence X satisfies (2).

Subcase 1-(ii): $x_{14} \neq 0$, $x_{13} \neq 0$. One knows from (41) that

$$[X, {}^tX] = \begin{pmatrix} x_{13}^2 + x_{14}^2 & & & \\ & x_{23}^2 + x_{24}^2 & & \\ & & -x_{13}^2 - x_{23}^2 & \\ & & & -x_{14}^2 - x_{24}^2 \end{pmatrix}. \tag{46}$$

Note that the third equation of (43) yields $x_{23}x_{24} \neq 0$. A direct calculation shows that

$$[[X, {}^tX], X] = \begin{pmatrix} 0 & 0 & (2x_{13}^2 + x_{14}^2 + x_{23}^2)x_{13} & (x_{13}^2 + 2x_{14}^2 + x_{24}^2)x_{14} \\ & 0 & (x_{13}^2 + 2x_{23}^2 + x_{24}^2)x_{23} & (x_{14}^2 + x_{23}^2 + 2x_{24}^2)x_{24} \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}. \tag{47}$$

Thus, it follows from (C2) that

$$2x_{13}^2 + x_{14}^2 + x_{23}^2 = x_{14}^2 + x_{23}^2 + 2x_{24}^2, \quad x_{13}^2 + 2x_{14}^2 + x_{24}^2 = x_{13}^2 + 2x_{23}^2 + x_{24}^2. \tag{48}$$

They show $x_{13}^2 = x_{24}^2$ and $x_{14}^2 = x_{23}^2$. On the other hand, one knows $x_{13}x_{23} + x_{14}x_{24} = 0$ from (43). Thus X satisfies (2).

Case 2: $x_{14} = 0$. It follows from (42) that

$$x_{13}x_{23} = 0, \quad x_{24}x_{34} - x_{12}x_{13} = 0, \quad x_{23}x_{24} = 0. \tag{49}$$

Subcase 2-(i): $x_{14} = 0$, $x_{23} \neq 0$. In this case, (49) yields that $x_{13} = x_{24} = 0$. Hence, one knows

$$W \ni [X, {}^tX] = \begin{pmatrix} x_{12}^2 & & & \\ & x_{23}^2 - x_{12}^2 & & \\ & & x_{34}^2 - x_{23}^2 & \\ & & & -x_{34}^2 \end{pmatrix}. \tag{50}$$

This means $x_{12}^2 \geq x_{23}^2 - x_{12}^2 \geq x_{34}^2 - x_{23}^2 \geq -x_{34}^2$. Since $X \neq 0$, one can see that $x_{12} \neq 0$ and $x_{34} \neq 0$. A direct calculation shows that

$$[[X, {}^tX], X] = \begin{pmatrix} 0 & (2x_{12}^2 - x_{23}^2)x_{12} & 0 & 0 \\ & 0 & (2x_{23}^2 - x_{12}^2 - x_{34}^2)x_{23} & 0 \\ & & 0 & (2x_{34}^2 - x_{23}^2)x_{34} \\ & & & 0 \end{pmatrix}. \tag{51}$$

Then, it follows from (C2) that

$$2x_{12}^2 - x_{23}^2 = 2x_{23}^2 - x_{12}^2 - x_{34}^2 = 2x_{34}^2 - x_{23}^2. \tag{52}$$

We thus have $x_{12}^2 = x_{34}^2$ and $4x_{12}^2 = 3x_{23}^2$, and hence X satisfies (3).

Subcase 2-(ii): $x_{14} = 0, x_{23} = 0, x_{12} = 0, x_{34} = 0$. In this case, one knows

$$W \ni [X, {}^tX] = \begin{pmatrix} x_{13}^2 & & & \\ & x_{24}^2 & & \\ & & -x_{13}^2 & \\ & & & -x_{24}^2 \end{pmatrix}. \quad (53)$$

Since $x_{13}^2 \geq x_{24}^2 \geq -x_{13}^2 \geq -x_{24}^2$, one has $x_{13}^2 = x_{24}^2$. Therefore, X satisfies (2).

Subcase 2-(iii): $x_{14} = 0, x_{23} = 0, x_{12} = 0, x_{34} \neq 0$. The second equation of (49) yields that $x_{24} = 0$. Then one has

$$[X, {}^tX] = \begin{pmatrix} x_{13}^2 & & & \\ & 0 & & \\ & & x_{34}^2 - x_{13}^2 & \\ & & & -x_{34}^2 \end{pmatrix}. \quad (54)$$

Since $X \neq 0$, one has $x_{13} \neq 0$. A direct calculation shows that

$$[[X, {}^tX], X] = \begin{pmatrix} 0 & 0 & (2x_{13}^2 - x_{34}^2)x_{13} & 0 \\ & 0 & 0 & 0 \\ & & 0 & (2x_{34}^2 - x_{13}^2)x_{34} \\ & & & 0 \end{pmatrix}. \quad (55)$$

It follows from (C2) that $2x_{13}^2 - x_{34}^2 = 2x_{34}^2 - x_{13}^2$. This shows $x_{13}^2 = x_{34}^2$, and hence X satisfies (4).

Subcase 2-(iv): $x_{14} = 0, x_{23} = 0, x_{12} \neq 0, x_{13} = 0$. One knows

$$W \ni [X, {}^tX] = \begin{pmatrix} x_{12}^2 & & & \\ & x_{24}^2 - x_{12}^2 & & \\ & & x_{34}^2 & \\ & & & -x_{24}^2 - x_{34}^2 \end{pmatrix}. \quad (56)$$

Since $x_{24}^2 - x_{12}^2 \geq x_{34}^2$ and $X \neq 0$, one has $x_{24} \neq 0$. Thus, the second equation of (49) yields $x_{34} = 0$. A direct calculation shows that

$$[[X, {}^tX], X] = \begin{pmatrix} 0 & (2x_{12}^2 - x_{24}^2)x_{12} & 0 & 0 \\ & 0 & 0 & (2x_{24}^2 - x_{12}^2)x_{24} \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}. \quad (57)$$

It follows from (C2) that $x_{12}^2 = x_{24}^2$, and hence X satisfies (4).

Subcase 2-(v): $x_{14} = 0, x_{23} = 0, x_{12} \neq 0, x_{13} \neq 0$. One knows from (41) that

$$[X, {}^tX] = \begin{pmatrix} x_{12}^2 + x_{13}^2 & & & \\ & x_{24}^2 - x_{12}^2 & & \\ & & x_{34}^2 - x_{13}^2 & \\ & & & -x_{24}^2 - x_{34}^2 \end{pmatrix}. \tag{58}$$

Note that the second equation of (49) yields $x_{24}x_{34} \neq 0$. A direct calculation shows that

$$[[X, {}^tX], X] = \begin{pmatrix} 0 & (2x_{12}^2 + x_{13}^2 - x_{24}^2)x_{12} & (2x_{13}^2 + x_{12}^2 - x_{34}^2)x_{13} & 0 \\ & 0 & 0 & (2x_{24}^2 + x_{34}^2 - x_{12}^2)x_{24} \\ & & 0 & (2x_{34}^2 + x_{24}^2 - x_{13}^2)x_{34} \\ & & & 0 \end{pmatrix}. \tag{59}$$

It follows from (C2) that $x_{12}^2 = x_{24}^2$ and $x_{13}^2 = x_{34}^2$. On the other hand, one knows $x_{24}x_{34} - x_{12}x_{13} = 0$ from (49). Therefore, X satisfies (4). □

Similarly to the case of $n = 3$, we can classify nonabelian two-dimensional Lie triple systems in $\mathrm{Sym}_4^0(\mathbb{R})$.

Proposition 8. *Every nonabelian two-dimensional Lie triple system in* $\mathrm{Sym}_4^0(\mathbb{R})$ *is congruent to one of the following Lie triple systems*:

$$\Sigma_{4,1} := \mathrm{Span}_{\mathbb{R}} \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\},$$

$$\Sigma_{4,2} := \mathrm{Span}_{\mathbb{R}} \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\},$$

$$\Sigma_{4,3} := \mathrm{Span}_{\mathbb{R}} \left\{ \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, \begin{pmatrix} 0 & 3^{1/2} & 0 & 0 \\ 3^{1/2} & 0 & 2 & 0 \\ 0 & 2 & 0 & 3^{1/2} \\ 0 & 0 & 3^{1/2} & 0 \end{pmatrix} \right\},$$

$$\Sigma_{4,4} := \mathrm{Span}_{\mathbb{R}} \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

Proof. Let Σ be a nonabelian two-dimensional Lie triple system in $\mathrm{Sym}^0_4(\mathbb{R})$. Then, Σ is congruent to $\mathfrak{L}_X$ for some X satisfying (C1) and (C2). Furthermore, X satisfies one of four conditions mentioned in Lemma 5. We study them individually.

Case 1: $x_{14} \neq 0$, and others are 0. In this case, it is easy to see that $\mathfrak{L}_X = \Sigma_{4,4}$.

Case 2: $x_{13} = \pm x_{24}$, $x_{14} = \mp x_{23}$, and others are 0. In this case, one has

$$\mathfrak{L}_X = \mathrm{Span}_{\mathbb{R}}\{\mathrm{diag}(1,1,-1,-1), X + {}^tX\}. \tag{60}$$

By applying a conjugation by $\mathrm{diag}(1,-1,1,1) \in \mathrm{O}(4)$ if necessary, we may assume $x_{13} = x_{24}$ and $x_{14} = -x_{23}$ without loss of generality. Denote by

$$X = \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & -b & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{61}$$

Let us put

$$g := \begin{pmatrix} a/(a^2+b^2)^{1/2} & -b/(a^2+b^2)^{1/2} & 0 & 0 \\ b/(a^2+b^2)^{1/2} & a/(a^2+b^2)^{1/2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{O}(4). \tag{62}$$

Then, a direct calculation shows that $g\mathfrak{L}_X g^{-1} = \Sigma_{4,2}$.

Case 3: $4x_{12}^2 = 4x_{34}^2 = 3x_{23}^2$, and others are 0. In this case, one can write

$$X = \begin{pmatrix} 0 & \pm 3^{1/2} & 0 & 0 \\ 0 & 0 & \pm 2 & 0 \\ 0 & 0 & 0 & \pm 3^{1/2} \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{63}$$

A direct calculation shows that

$$\mathfrak{L}_X = \mathrm{Span}_{\mathbb{R}}\{\mathrm{diag}(3,1,-1,-3), X + {}^tX\}. \tag{64}$$

Thus, by applying a conjugation by suitable $\mathrm{diag}(\pm 1, \pm 1, \pm 1, \pm 1) \in \mathrm{O}(4)$, one can see that $\mathfrak{L}_X$ is congruent to $\Sigma_{4,3}$.

Case 4: $x_{12} = \pm x_{24}$, $x_{13} = \pm x_{34}$, and others are 0. In this case, one has

$$\mathfrak{L}_X = \mathrm{Span}_{\mathbb{R}}\{\mathrm{diag}(1,0,0,-1), X + {}^tX\}. \tag{65}$$

As before, we may assume $x_{12} = x_{24}$ and $x_{13} = x_{34}$ without loss of generality.

Denote by

$$X = \begin{pmatrix} 0 & a & b & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{66}$$

Let us put

$$g := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b/(a^2+b^2)^{1/2} & -a/(a^2+b^2)^{1/2} & 0 \\ 0 & a/(a^2+b^2)^{1/2} & b/(a^2+b^2)^{1/2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{O}(4). \tag{67}$$

Then, a direct calculation shows that $g\mathfrak{L}_X g^{-1} = \Sigma_{4,1}$.

We finally see that the totally geodesic surfaces corresponding to the above four Lie triple systems have the different sectional curvatures. The proof follows from direct calculations in terms of Proposition 4.

Proposition 9. *The totally geodesic submanifolds corresponding to $\Sigma_{4,1}$, $\Sigma_{4,2}$, $\Sigma_{4,3}$, and $\Sigma_{4,4}$ have the sectional curvatures $-1/2$, -1, $-1/5$, and -2, respectively. In particular, they are not isometrically congruent to each other.*

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Examples of Non-Kähler Solvmanifolds Admitting Hodge Decomposition

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Abstract We consider some Hodge theoretical properties (formality, hard-Lefschetz property, E_1-degeneration of Frölicher spectral sequence, $\partial\bar{\partial}$-Lemma and their twisted versions) on non-Kähler symplectic and complex manifolds. It is known that if nilmanifolds satisfy formality, hard-Lefschetz property, or $\partial\bar{\partial}$-Lemma, then they are only tori. Hodge theory on solvmanifolds are more complicated. We give non-Kähler solvmanifolds satisfying these properties.

1 Introduction

In [23], studying certain complex parallelizable manifolds, Nakamura gave some remarks on Kodaira-Spencer deformation of non-Kähler compact complex manifolds. Particularly, Nakamura studied compact complex homogeneous space of three-dimensional complex solvable Lie groups. Recently the complex parallelizable manifolds of three-dimensional complex non-nilpotent solvable Lie groups are called Nakamura manifolds. Nakamura manifolds have been giving many important notes on geometry of non-Kähler manifolds. In this report, we focus the Hodge theoretical properties of Nakamura manifolds and Nakamura-like manifolds.

1.1 Nakamura Manifolds

Let $G_1 = \mathbb{C} \ltimes_{\phi} \mathbb{C}^2$ such that

$$\phi(z) = \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix}.$$

Then we have

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Theorem 1. *The following statements hold.*

1. *For any lattice Γ in G_1, $\Gamma\backslash G_1$ does not admit a Kähler structure. (Note that for some lattice Γ, $\Gamma\backslash G_1$ admits a pseudo-Kähler structure. [30])*
2. *For any lattice Γ in G_1, $\Gamma\backslash G_1$ is formal (Definition 3) and hyper-formal (Definition 4).*
3. *G_1 admits a left-invariant symplectic structure ω. For any lattice Γ in G_1, the symplectic solvmanifold $(\Gamma\backslash G_1, \omega)$ is hard-Lefschetz (Definition 5) and hyper-hard-Lefschetz (Definition 6).*
4. *Consider the standard complex structure J on the complex Lie group G_1. For any lattice Γ in G, the Frölicher spectral sequence of the complex parallelizable solvmanifold $\Gamma\backslash G_1$ does not degenerate at the E_1-term but degenerates at the E_2-term. Hence $\Gamma\backslash G_1$ does not satisfy the $\partial\bar{\partial}$-Lemma (Definition 7).*
5. *For some lattice Γ, there exists a holomorphic deformation J_t of J such that the deformed complex solvmanifold $(\Gamma\backslash G_1, J_t)$ satisfy the $\partial\bar{\partial}$-Lemma.*

1.2 Nakamura-Like Manifolds

Let $G_2 = \mathbb{C} \ltimes_\phi \mathbb{C}^2$ such that

$$\phi(z) = \begin{pmatrix} e^{\frac{z+\bar{z}}{2}} & 0 \\ 0 & e^{-\frac{z+\bar{z}}{2}} \end{pmatrix}.$$

Then we have

Theorem 2. *The following statements hold.*

1. *For any lattice Γ in G_2, $\Gamma\backslash G_2$ does not admit a Kähler structure.*
2. *For any lattice Γ in G_2, $\Gamma\backslash G_2$ is formal and hyper-formal.*
3. *G_2 admits a left-invariant symplectic structure ω. For any lattice Γ in G_2, the symplectic solvmanifold $(\Gamma\backslash G_2, \omega)$ satisfies the hard Lefschetz property and hyper hard Lefschetz property.*
4. *G_2 admits a left-invariant complex structure J. For some lattice Γ, complex solvmanifold $(\Gamma\backslash G_2, J)$ satisfy the $\partial\bar{\partial}$-Lemma.*
5. *For any lattice Γ in G_2, $(\Gamma\backslash G_2, J)$ does not admit the hyper-strong-Hodge-decomposition (Definition 8).*

2 Kähler Geometry

2.1 Formality

Definition 1. A *differential graded algebra* (called *DGA*) is a graded $\mathbb{R}$-algebra A^* with the following properties:

(1) A^* is graded commutative, i.e.

$$y \wedge x = (-1)^{p \cdot q} x \wedge y \quad x \in A^p \quad y \in A^q.$$

(2) There is a differential operator $d : A \to A$ of degree one such that $d \circ d = 0$ and

$$d(x \wedge y) = dx \wedge y + (-1)^p x \wedge dy \quad x \in A^p.$$

Let A and B be DGAs. If a morphism of graded algebra $\varphi : A \to B$ satisfies $d \circ \varphi = \varphi \circ d$, we call φ a morphism of DGAs. If a morphism of DGAs induces a cohomology isomorphism, we call it a quasi-isomorphism.

Definition 2. A and B are *weakly equivalent* if there is a finite diagram of DGAs

$$A \leftarrow C_1 \rightarrow C_2 \leftarrow \cdots \leftarrow C_n \rightarrow B$$

such that all the morphisms are quasi-isomorphisms.

Let M be a smooth manifold. The de Rham complex $A^*(M)$ of M is a DGA. The cohomology algebra $H^*(M, \mathbb{R})$ is a DGA with $d = 0$.

Definition 3. A smooth manifold M is *formal* if $A^*(M)$ and $H^*(M, \mathbb{R})$ are weakly equivalent.

We denote by $\mathscr{C}(\pi_1(M))$ the space of characters of $\pi_1(M)$ which can be factored as

$$\pi_1(M) \to H_1(\pi_1(M), \mathbb{Z})/(\text{torsion}) \to \mathbb{C}^*.$$

For $\alpha \in \mathscr{C}(\pi_1(M))$ we consider the flat bundle E_α which corresponds to α, the cochain complex $A^*(M, E_\alpha)$ of the differential forms with values in E_α and the local system cohomology $H^*(M, E_\alpha)$. The cochain complex $A^*(M, E_\alpha)$ is isomorphic to the space $A^*(M) \otimes \mathbb{C}$ with the differential operator $d + \phi$ such that ϕ is closed one-form and satisfies $\alpha(\gamma) = e^{\int_\gamma \phi}$ for $\gamma \in \pi_1(M)$. Let

$$\overline{A}^*(M) = \bigoplus_{\alpha \in \mathscr{C}(\pi_1(M))} A^*(M, E_\alpha)$$

Then by isomorphism $E_{\alpha_1} \otimes E_{\alpha_2} \cong E_{\alpha_1\alpha_2}$, the $\overline{A}^*(M)$ is a differential graded algebra.

Definition 4. M is *hyper-formal* if the differential graded algebra $\overline{A}^*(M)$ is formal.

2.2 Hard Lefschetz Property

Definition 5. Let (M, ω) be a $2n$-dimensional symplectic manifold. We say that (M, ω) is *hard-Lefschetz* if the linear map

$$[\omega^{n-i}]\wedge : H^i(M, \mathbb{R}) \to H^{2n-i}(M, \mathbb{R})$$

is an isomorphism for any $0 \leq i \leq n$.

Definition 6. Let (M, ω) be a $2n$-dimensional symplectic manifold. We say that (M, ω) is *hyper-hard-Lefschetz* if for each $\alpha \in \mathscr{C}(\pi_1(M))$ the linear map

$$[\omega]^{n-i}\wedge : H^i(M, E_\alpha) \to H^{2n-i}(M, E_\alpha)$$

is an isomorphic for any $i \leq n$ where $\dim M = 2n$.

2.3 Hodge Decomposition

Let (M, J) be a compact complex manifold. Consider the double complex $(A^{*,*}(M), \partial, \bar\partial)$ and the Dolbeault cohomology $H^{*,*}_{\bar\partial}(M)$.

Definition 7. We say that (M, J) satisfies $\partial\bar\partial$-Lemma if

$$\ker \partial \cap \ker \bar\partial \cap \operatorname{im} d = \operatorname{im} \partial\bar\partial.$$

We define the *Bott-Chern cohomology*

$$H^{*,*}_{BC} = \frac{\ker \partial \cap \ker \bar\partial}{\operatorname{im} \partial\bar\partial}.$$

Then (M, J) satisfies $\partial\bar\partial$-*Lemma* if and only if (M, J) admits the *strong-Hodge-decomposition* i. e. the canonical maps

$$\mathrm{Tot}^* H^{*,*}_{BC}(M) \to H^*(M),$$
$$H^{*,*}_{BC}(M) \to H^{*,*}_{\bar\partial}(M)$$

are isomorphisms.

Theorem 3 ([3]). *Let (M, J) be a n-dimensional compact complex manifold. Then for every $k \in \mathbb{Z}$ the following inequality holds*

$$\sum_{p+q=k} (\dim H^{p,q}_{BC}(M) + \dim H^{n-p,n-q}_{BC}(M)) \geq 2 \dim H^k(M).$$

Moreover if for every $k \in \mathbb{Z}$ the equality

$$\sum_{p+q=k} (\dim H^{p,q}_{BC}(M) + \dim H^{n-p,n-q}_{BC}(M)) = 2 \dim H^k(M)$$

holds, then (M, J) satisfies $\partial\bar{\partial}$-Lemma.

Suppose (M, J) satisfies the $\partial\bar{\partial}$-Lemma. Then for each $\alpha \in \mathscr{C}(\pi_1(M))$, by $H^1(M) = H^{1,0}_{BC}(M) \oplus H^{0,1}_{BC}(M)$ we have holomorphic 1-forms θ_1, θ_2 such that:

- Denoting

$$\partial_{(\theta_1,\theta_2)} = \partial + \theta_2 + \bar{\theta}_1$$

  and

$$\bar{\partial}_{(\theta_1,\theta_2)} = \partial - \bar{\theta}_2 + \theta_1$$

  $(A^*(M), \partial_{(\theta_1,\theta_2)}, \bar{\partial}_{(\theta_1,\theta_2)})$ is bi-differential cochain complex.
- The cochain complex $A^*(M, E_\alpha)$ is isomorphic to the space $A^*(M) \otimes \mathbb{C}$ with the differential operator $d + \phi$ where $\phi = \theta_1 + \bar{\theta}_1 + \theta_2 - \bar{\theta}_2$.

Define the cohomologies as

$$H^*_{\partial_{(\theta_1,\theta_2)}}(M) = \frac{\ker \partial_{(\theta_1,\theta_2)}}{\operatorname{im} \partial_{(\theta_1,\theta_2)}},$$

$$H^*_{\bar{\partial}_{(\theta_1,\theta_2)}}(M) = \frac{\ker \bar{\partial}_{(\theta_1,\theta_2)}}{\operatorname{im} \bar{\partial}_{(\theta_1,\theta_2)}},$$

$$H^*_{BC_{(\theta_1,\theta_2)}}(M) = \frac{\ker \partial_{(\theta_1,\theta_2)} \cap \ker \bar{\partial}_{(\theta_1,\theta_2)}}{\operatorname{im} \ker \partial_{(\theta_1,\theta_2)} \bar{\partial}_{(\theta_1,\theta_2)}}.$$

Definition 8. Let (M, J) be a compact complex manifold satisfying the $\partial\bar{\partial}$-Lemma. We say that (M, J) admits the *hyper-strong-Hodge-decomposition* if for each $\alpha \in \mathscr{C}(\pi_1(M))$ the canonical maps

$$H^*_{BC_{(\theta_1,\theta_2)}}(M) \to H^*(M, E_\alpha),$$

$$H^*_{BC_{(\theta_1,\theta_2)}}(M) \to H^*_{\partial_{(\theta_1,\theta_2)}}(M),$$

$$H^*_{BC_{(\theta_1,\theta_2)}}(M) \to H^*_{\bar{\partial}_{(\theta_1,\theta_2)}}(M)$$

are isomorphisms.

Theorem 4. *Let (M, J, ω) be a compact Kähler manifold. Then we have:*

1. *(M, J) satisfies the $\partial\bar{\partial}$-Lemma and the* hyper-strong-Hodge-decomposition. [9, 28]
2. *$\partial\bar{\partial}$-Lemma implies the formality and hyper-strong-Hodge-decomposition implies the hyper-formal.* [9, 17, 22, 28] *Hence M is formal and hyper-formal.*
3. *(M, ω) is hard-Lefschetz and hyper-hard-Lefschetz.* [11, 28]

3 Cohomologies of Solvmanifolds

Let G be a simply connected solvable Lie group. A discrete cocompact subgroup of G is called a lattice. In general G does not admits a lattice. We suppose that G admits a lattice Γ. Then a compact homogeneous space $\Gamma\backslash G$ is called a solvmanifold. If G is nilpotent, then $\Gamma\backslash G$ is called a nilmanifold.

3.1 Cohomology of Nilmanifolds

Theorem 5 ([24]). *Let N be a simply connected real nilpotent Lie group and $\mathfrak{n}$ the Lie algebra of N. Suppose N has a lattice Γ. Consider the cochain complex $\bigwedge \mathfrak{n}^*$ of Lie algebra and the canonical inclusion*

$$\bigwedge \mathfrak{n}^* \to A^*(\Gamma\backslash N).$$

Then the inclusion induces a cohomology isomorphism

$$H^*(\mathfrak{n}) \cong H^*(\Gamma\backslash N).$$

Theorem 6 (cf. [26, Theorem 2.4]). *Let N be a simply connected nilpotent Lie group with a lattice Γ. We suppose N admits a left-invariant complex structure J. We consider the DBA $\bigwedge^{*,*} \mathfrak{n}^*$ of the complex valued left-invariant differential forms with the operator $\bar{\partial}$ and the canonical inclusion*

$$\bigwedge^{*,*} \mathfrak{n}^* \to A^{*,*}(\Gamma\backslash N).$$

Then the inclusion induces a cohomology isomorphism

$$H^{*,*}_{\bar{\partial}}(\mathfrak{n}) \cong H^{*,*}_{\bar{\partial}}(\Gamma\backslash N)$$

if (G, J, Γ) meet one of the following conditions

- *The complex manifold $(\Gamma\backslash G, J)$ has the structure of an iterated principal holomorphic torus bundle [7].*
- *J is a small deformation of a rational complex structure i.e. for the rational structure $\mathfrak{g}_{\mathbb{Q}} \subset \mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ induced by a lattice Γ (see [25, Sect. 2]) we have $J(\mathfrak{g}_{\mathbb{Q}}) \subset \mathfrak{g}_{\mathbb{Q}}$ [6].*
- *(G, J) is a complex Lie group [27].*

3.2 de Rham Cohomology of Solvmanifolds

Let G be a simply connected solvable Lie group and $\mathfrak{g}$ the Lie algebra of G with the nilradical $\mathfrak{n}$. Take a subvector space (not necessarily Lie algebra) V of $\mathfrak{g}$ so that $\mathfrak{g} = V \oplus \mathfrak{n}$ as the direct sum of vector spaces and for any $A, B \in V (\mathrm{ad}_A)_s (B) = 0$ where $(ad_A)_s$ is the semi-simple part of ad_A (see [10, Proposition III.1.1]). We define the diagonalizable representation $\mathrm{ad}_s : \mathfrak{g} \to D(\mathfrak{g})$ as $\mathrm{ad}_{sA+X} = (\mathrm{ad}_A)_s$ for $A \in V$ and $X \in \mathfrak{n}$. We denote by $\mathrm{Ad}_s : G \to \mathrm{Aut}(\mathfrak{g})$ the extension of ad_s. Take a basis $X_1, \cdots, X_n$ of $\mathfrak{g} \otimes \mathbb{C}$ such that Ad_s is represented by diagonal matrices. We have $\mathrm{Ad}_{sg} X_i = \alpha_i(g) X_i$ for characters α_i of G. Let $x_1, \ldots, x_n$ be the dual basis of $X_1, \ldots, X_n$.

We suppose G has a lattice Γ. Let

$$\overline{A}^*(\Gamma\backslash G) = \bigoplus_{\alpha} A^*(\Gamma\backslash G, E_\alpha).$$

For a character α_I of G, take a global frame v_{α_I} of the flat bundle E_{α_I} such that $d v_{\alpha_I} = \alpha_I^{-1} d\alpha_I v_{\alpha_I}$.

Theorem 7 ([17, 21, 22]). *The inclusion*

$$\bigwedge \langle x_1 \otimes v_{\alpha_1}, \ldots, x_n \otimes v_{\alpha_n} \rangle \hookrightarrow \overline{A}^*(\Gamma\backslash G)$$

induces a cohomology isomorphism.

Theorem 8. *We consider the sub-DGA A^*_Γ of the de Rham complex $A^*(\Gamma\backslash G) \otimes \mathbb{C}$ which is defined by*

$$A^p_\Gamma = \left\langle \alpha_I x_I \,\middle|\, \begin{array}{c} I \subset \{1, \ldots, n\}, \\ (\alpha_I)_{|\Gamma} = 1 \end{array} \right\rangle.$$

where for a multi-index $I = \{i_1, \ldots, i_p\}$ we write $x_I = x_{i_1} \wedge \cdots \wedge x_{i_p}$, and $\alpha_I = \alpha_{i_1} \cdots \alpha_{i_p}$.

Then the inclusion $A^*_\Gamma \subset A^*(\Gamma\backslash G) \otimes \mathbb{C}$ induces a cohomology isomorphism

$$H^*(A_\Gamma) \cong H^*(\Gamma\backslash G, \mathbb{C}).$$

Let $\bar{\mathfrak{g}} = \mathrm{Im}\,\mathrm{ad}_s \ltimes \mathfrak{g}$ and

$$\mathfrak{u} = \{X - \mathrm{ad}_{sX} \in \bar{\mathfrak{g}} | X \in \mathfrak{g}\}.$$

Then we have $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n} \subset \mathfrak{u}$ and $\mathfrak{u}$ is the nilradical of $\bar{\mathfrak{g}}$ (see [10]). Hence we have $\bar{\mathfrak{g}} = \mathrm{Im}\,\mathrm{ad}_s \ltimes \mathfrak{u}$. It is known that the structure of the Lie algebra $\mathfrak{u}$ is independent of a choice of a subvector space V (see [10, Corollary III.3.6]).

Proposition 1 ([16, 17]). *$\mathfrak{u}$ is Ableian if and only if $\mathfrak{g} = \mathbb{R}^m \ltimes \mathbb{R}_n$ such that ϕ is a semi-simple action.*

We have a differential graded algebra isomorphism

$$\bigwedge \langle x_1 \otimes v_{\alpha_1}, \dots, x_n \otimes v_{\alpha_n} \rangle \cong \bigwedge \mathfrak{u}^* \otimes \mathbb{C}.$$

Theorem 9. *We have a quasi-isomorphism*

$$\bigwedge \mathfrak{u}^* \to \overline{A}^*(\Gamma \backslash G)$$

and hence $\bigwedge \mathfrak{u}^*$ is the Sullivan minimal model ([29]) of $\overline{A}^*(\Gamma\backslash G)$.

3.3 Dolbeault and Bott-Chern cohomologies of Splitting Solvmanifolds

Let G be a semi-direct product $\mathbb{C}^n \ltimes_\phi N$ with a left-invariant complex structure $J = J_{\mathbb{C}} \oplus J_N$ so that:

1. N is a simply connected nilpotent Lie group with a left-invariant complex structure J_N.
   Let $\mathfrak{a}$ and $\mathfrak{n}$ be the Lie algebras of $\mathbb{C}^n$ and N respectively.
2. For any $t \in \mathbb{C}^n$, $\phi(t)$ is a holomorphic automorphism of (N, J_N).
3. ϕ induces a semi-simple action on the Lie algebra $\mathfrak{n}$ of N.
4. G has a lattice Γ. (Then Γ can be written by $\Gamma = \Gamma' \ltimes_\phi \Gamma''$ such that Γ' and Γ'' are lattices of $\mathbb{C}^n$ and N respectively and for any $t \in \Gamma'$ the action $\phi(t)$ preserves Γ''.)
5. The inclusion $\bigwedge^{*,*} \mathfrak{n}^*_{\mathbb{C}} \subset A^{*,*}(\Gamma''\backslash N)$ induces an isomorphism

$$H^{*,*}_{\bar\partial}(\mathfrak{n}) \cong H^{*,*}_{\bar\partial}(\Gamma''\backslash N)$$

   where $\bigwedge^{*,*} \mathfrak{n}^*_{\mathbb{C}}$ is the differential bigraded algebra of the complex valued left-invariant differential forms on the nilmanifold N/Γ''.

Consider the decomposition $\mathfrak{n} \otimes \mathbb{C} = \mathfrak{n}_{1,0} \oplus \mathfrak{n}_{0,1}$ associated with J_N. By the condition (2), this decomposition is a direct sum of $\mathbb{C}^n$-modules. By the condition (3) we have a basis $Y_1, \dots, Y_m$ of $\mathfrak{n}^{1,0}$ such that the action ϕ on $\mathfrak{n}_{1,0}$ is represented by $\phi(t) = \mathrm{diag}(\alpha_1(t), \dots, \alpha_m(t))$. Since Y_j is a left-invariant vector field on N, the vector field $\alpha_j Y_j$ on $\mathbb{C}^n \ltimes_\phi N$ is left-invariant. Hence we have a basis $X_1, \dots, X_n, \alpha_1 Y_1, \dots, \alpha_m Y_m$ of $\mathfrak{g}_{1,0}$. Let $x_1, \dots, x_n, \alpha_1^{-1} y_1, \dots, \alpha_m^{-1} y_m$ be the basis of $\mathfrak{g}_{1,0}^*$ which is dual to $X_1, \dots, X_n, \alpha_1 Y_1, \dots, \alpha_m Y_m$. Then we have

$$\bigwedge^{p,q} \mathfrak{g}_{\mathbb{C}}^* = \bigwedge^{p} \langle x_1, \dots, x_n, \alpha_1^{-1} y_1, \dots, \alpha_m^{-1} y_m \rangle \otimes \bigwedge^{q} \langle \bar{x}_1, \dots, \bar{x}_n, \bar{\alpha}_1^{-1} \bar{y}_1, \dots, \bar{\alpha}_m^{-1} \bar{y}_m \rangle.$$

Let $\alpha : \mathbb{C}^n \to \mathbb{C}^*$ be a character (i.e. a representation on one-dimensional vector space $\mathbb{C}_\alpha$) of $\mathbb{C}^n$. By the projection $\mathbb{C}^n \ltimes_\phi N \to \mathbb{C}^n$, we regard α as a character of G. We consider the holomorphic line bundle $L_\alpha = (G \times \mathbb{C}_\alpha)/\Gamma$ and the Dolbeault complex $(A^{*,*}(\Gamma \backslash G, L_\alpha), \bar{\partial})$ with values in the line bundle L_α. Let $\mathscr{L}$ be the set of isomorphism classes of line bundles over $\Gamma \backslash G$ given by characters of $\mathbb{C}^n$. We consider the direct sum

$$\bigoplus_{L_\beta \in \mathscr{L}} A^{*,*}(\Gamma \backslash G, L_\beta)$$

of Dolbeault complexes. Then by the wedge products and the tensor products, the direct sum $\bigoplus_{L_\beta \in \mathscr{L}} A^{*,*}(\Gamma \backslash G, L_\beta)$ is a DBA.

Lemma 1 ([18, Lemma 2.2]). *Let $\alpha : \mathbb{C}^n \to \mathbb{C}^*$ be a C^∞-character of $\mathbb{C}^n$. There exists a unique unitary character β such that $\alpha\beta^{-1}$ is holomorphic.*

By this lemma, take the unique unitary characters β_i and γ_i on $\mathbb{C}^n$ such that $\alpha_i \beta_i^{-1}$ and $\bar{\alpha}\gamma_i^{-1}$ are holomorphic.

Let $A^{*,*}$ be the subDBA of $\bigoplus_{L_\alpha \in \mathscr{L}} A^{*,*}(\Gamma \backslash G, L_\alpha)$ defined by

$$A^{p,q} = \bigwedge^{p} \langle x_1, \dots, x_n, \alpha_1^{-1} y_1 \otimes v_{\beta_1}, \dots, \alpha_m^{-1} y_m \otimes v_{\beta_m} \rangle$$
$$\bigotimes \bigwedge^{q} \langle \bar{x}_1, \dots, \bar{x}_n, \bar{\alpha}_1^{-1} \bar{y}_1 \otimes v_{\gamma_1}), \dots, \bar{\alpha}_m^{-1} \bar{y}_m \otimes v_{\gamma_m} \rangle.$$

Theorem 10 ([18]). *We have a DBA isomorphism $\iota : \bigwedge^{*,*} (\mathfrak{a} \oplus \mathfrak{n})^* \cong A^{*,*}$ and the inclusion*

$$A^{*,*} \to \bigoplus_{L_\beta \in \mathscr{L}} A^{*,*}(\Gamma \backslash G, L_\beta)$$

induces a cohomology isomorphism.

Theorem 11 ([18, Corollary 4.2]). *Let $B_\Gamma^{*,*} \subset A^{*,*}(\Gamma \backslash G)$ be the differential bigraded subalgebra of $A^{*,*}(\Gamma \backslash G)$ given by*

$$B_\Gamma^{p,q} = \left\langle x_I \wedge \alpha_J^{-1} \beta_J y_J \wedge \bar{x}_K \wedge \bar{\alpha}_L^{-1} \gamma_L \bar{y}_L \,\middle|\, \begin{array}{c} |I| + |J| = p,\ |K| + |L| = q \\ (\beta_J \gamma_L)_{|\Gamma} = 1 \end{array} \right\rangle.$$

Then the inclusion $B_{\Gamma}^{*,*} \subset A^{*,*}(\Gamma\backslash G)$ *induces a cohomology isomorphism*

$$H_{\bar\partial}^{*,*}(B_{\Gamma}^{*,*}) \cong H_{\bar\partial}^{*,*}(\Gamma\backslash G).$$

Let

$$C^{*,*} = B_{\Gamma}^{*,*} + \overline{B_{\Gamma}^{*,*}}.$$

Then the inclusion

$$C^{*,*} \subset A^{*,*}(\Gamma\backslash G)$$

induces an isomorphism

$$H_{BC}^{*,*}(C^{*,*}) \cong H_{BC}^{*,*}(\Gamma\backslash G).$$

3.4 Dolbeault and Bott-Chern Cohomologies of Complex Parallelizable Solvmanifolds

Let G be a simply connected n-dimensional complex solvable Lie group. Consider the Lie algebra $\mathfrak{g}_{1,0}$ (resp. $\mathfrak{g}_{0,1}$) of the left-invariant holomorphic (resp. anti-holomorphic) vector fields on G. Let N be the nilradical of G.We can take a simply connected complex nilpotent subgroup $C \subset G$ such that $G = C \cdot N$ (see [8]).Since C is nilpotent, the map

$$C \ni c \mapsto (\mathrm{Ad}_c)_s \in \mathrm{Aut}(\mathfrak{g}_{1,0})$$

is a homomorphism where $(\mathrm{Ad}_c)_s$ is the semi-simple part of Ad_s.

We have a basis $X_1, \ldots, X_n$ of $\mathfrak{g}_{1,0}$ such that

$$(\mathrm{Ad}_c)_s = \mathrm{diag}(\alpha_1(c), \ldots, \alpha_n(c))$$

for $c \in C$. Let $x_1, \ldots, x_n$ be the basis of $\mathfrak{g}_{1,0}^*$ which is dual to $X_1, \ldots, X_n$.

Theorem 12 ([21, Corollary 6.2 and Its Proof]). *Let* B_{Γ}^{*} *be the subcomplex of* $(A^{0,*}(\Gamma\backslash G), \bar\partial)$ *defined as*

$$B_{\Gamma}^{*} = \left\langle \frac{\bar\alpha_I}{\alpha_I}\bar x_I \,\middle|\, \left(\frac{\bar\alpha_I}{\alpha_I}\right)_{|\Gamma} = 1 \right\rangle.$$

Then the inclusion $B_{\Gamma}^{*} \subset A^{0,*}(\Gamma\backslash G)$ *induces an isomorphism*

$$H^{*}(B_{\Gamma}^{*}) \cong H^{0,*}(\Gamma\backslash G).$$

Let

$$C_{\Gamma}^{*,*} = \bigwedge \mathfrak{g}_{1,0}^{*} \otimes B_{\Gamma}^{*} + \overline{B_{\Gamma}^{*}} \otimes \mathfrak{g}_{0,1}^{*}.$$

Then the inclusion

$$C^{*,*} \subset A^{*,*}(\Gamma\backslash G)$$

induces an isomorphism

$$H_{BS}^{*,*}(C^{*,*}) \cong H_{BC}^{*,*}(\Gamma\backslash G).$$

By this theorem we can prove the following theorem.

Theorem 13 ([19]). *Let G be a simply connected complex solvable Lie-group with a lattice Γ. Then the Frölicher spectral sequence of the complex parallelizable solvmanifold G/Γ degenerates at the E_2-term.*

4 Hodge Theory on Solvmanifolds

Theorem 14 ([12]). *Consider a DGA $\bigwedge \mathfrak{n}^*$ which is the dual of a nilpotent Lie algebra $\mathfrak{n}$. Then $\bigwedge \mathfrak{n}^*$ is formal if and only if $\mathfrak{n}$ is abelian.*

Hence a nilmanifold $\Gamma\backslash N$ is formal if and only if $\Gamma\backslash N$ is a torus.

Theorem 15 ([5]). *Consider a DGA $\bigwedge \mathfrak{n}^*$ which is the cochain complex of the dual of a nilpotent Lie algebra $\mathfrak{n}$. Suppose we have $[\omega] \in H^2(\bigwedge \mathfrak{n}^*)$ such that $[\omega]^n \neq 0$ where $2n = \dim \mathfrak{n}$. Then for any $0 \leq i \leq n$ the linear operator*

$$[\omega]^{n-i}\wedge : H^i(\bigwedge \mathfrak{n}^*) \to H^{2n-i}(\bigwedge \mathfrak{n}^*)$$

is an isomorphism if and only if $\mathfrak{n}$ is Abelian.

Hence a symplectic nilmanifold $\Gamma\backslash N$ is hard Lefschetz if and only if $\Gamma\backslash N$ is a torus.

By Theorem 9 and Proposition 1 we have following theorems.

Theorem 16 ([16]). *A solvmanifold $\Gamma\backslash G$ is formal if $G = \mathbb{R}^m \ltimes_{\phi} \mathbb{R}^n$ such that ϕ is a semi-simple action.*

Theorem 17 ([16]). *A symplectic solvmanifold $(\Gamma\backslash G, \omega)$ is hard-Lefschetz if $G = \mathbb{R}^m \ltimes_{\phi} \mathbb{R}^n$ such that ϕ is a semi-simple action.*

Theorem 18 ([17, 22]). *A solvmanifold $\Gamma\backslash G$ is Hyper-formal if and only if $G = \mathbb{R}^m \ltimes_{\phi} \mathbb{R}^n$ such that ϕ is a semi-simple action.*

Theorem 19 ([17, 22]). *A symplectic solvmanifold $(\Gamma\backslash G, \omega)$ is hyper-hard-Lefschetz if and only if $G = \mathbb{R}^m \ltimes_\phi \mathbb{R}^n$ such that ϕ is a semi-simple action.*

By using Theorem 11, we have the following theorem.

Theorem 20 ([20]). *Let $G = \mathbb{C}^n \ltimes_\phi \mathbb{C}^m$ with a semi-simple action $\phi : \mathbb{C}^n \to GL_m(\mathbb{C})$ (not necessarily holomorphic). Suppose G has a lattice Γ. Then we show that under some conditions on G and Γ, $\Gamma\backslash G$ admits a Hermitian metric such that the space of harmonic forms satisfies the Hodge symmetry and decomposition.*

We notice that Kähler solvmanifolds are completely characterized by hyper-strong-Hodge-decomposition.

Theorem 21 ([22]). *Let $\Gamma\backslash G$ be a $2n$-dimensional solvmanifold. Then the following conditions are equivalent*

1. *$\Gamma\backslash G$ admits a complex structure J and (M, J) satisfies $\partial\bar{\partial}$-Lemma and hyper-strong-Hodge-decomposition.*
2. *$G = \mathbb{R}^{2k} \ltimes_\varphi \mathbb{R}^{2l}$ such that the action $\varphi : \mathbb{R}^{2k} \to \mathrm{Aut}(\mathbb{R}^{2l})$ is semi-simple and for any $x \in \mathbb{R}^{2k}$ the all eigenvalues of $\phi(x)$ are unitary.*
3. *M admits a Kähler structure.*

Remark 1. Equivalence of (2) and (3) in Theorem 21 were already proved by Hasegawa in [13] by using Arapura-Nori ’ s results in [4].

5 Computations on Nakamura Manifolds and Nakamura-Like Manifolds

5.1 Nakamura Manifolds

Let $G_1 = \mathbb{C} \ltimes_\phi \mathbb{C}^2$ such that

$$\phi(x + \sqrt{-1}y) = \begin{pmatrix} e^{x+\sqrt{-1}y} & 0 \\ 0 & e^{-x-\sqrt{-1}y} \end{pmatrix}.$$

For a coordinate $(z_1, z_2, z_3) \in \mathbb{C} \ltimes \mathbb{C}^2$, we have

$$\bigwedge^{p,q} \mathfrak{g}^* = \bigwedge^{p,q} \langle dz_1, e^{-z_1} dz_2, e^{z_1} dz_3 \rangle \otimes \langle d\bar{z}_1, e^{-\bar{z}_1} d\bar{z}_2, e^{\bar{z}_1} d\bar{z}_3 \rangle.$$

We study a lattice $\Gamma = (\mathbb{Z}(a + \sqrt{-1}b) + \mathbb{Z}(c + \sqrt{-1}d)) \ltimes_\phi \Gamma''$ such that Γ'' is a lattice of $\mathbb{C}^2$, $\mathbb{Z}(a + \sqrt{-1}b) + \mathbb{Z}(c + \sqrt{-1}d)$ is a lattice in $\mathbb{C}$ and $\phi(a + \sqrt{-1}b)$ and $\phi(c + \sqrt{-1}d)$ are conjugate to elements of $SL(4, \mathbb{Z})$.

We consider the following two cases

(1-A) $b \notin \pi\mathbb{Z}$ or $c \notin \pi\mathbb{Z}$ (See [14, Example 2.]),
(1-B) $b, d \in \pi\mathbb{Z}$ (See [14, Example 3.]).

We consider the complex parallelizable solvmanifold $(\Gamma\backslash G_1, J)$ for each case. Moreover, in the case (1-B) we consider the deformation $\{J_t\}_{t\in B}$ over a ball $B \subset \mathbb{C}$ given by

$$t\frac{\partial}{\partial z_1} \otimes d\bar{z}_1 \in H^{0,1}(\Gamma\backslash G_1, T^{1,0}\Gamma\backslash G_1).$$

We write the results of computation of cohomologies (see [1,2]). We summarize in Table 1 the results of the computations of the de Rham, Dolbeault and Bott-Chern cohomologies. The table says for example, $\dim H^{2,0}_{BC}(\Gamma\backslash G_1) = 0$ in case (1-B) with the original complex structure J_0.

5.2 Nakamura-Like Manifolds

Let $G_2 = \mathbb{C} \ltimes_\phi \mathbb{C}^2$ such that

$$\phi(z) = \begin{pmatrix} e^{\frac{z+\bar{z}}{2}} & 0 \\ 0 & e^{-\frac{z+\bar{z}}{2}} \end{pmatrix}.$$

Table 1 Summary of the dimensions of the cohomologies of the Nakamura manifold and its deformation

	Case (1-A)			Case (1-B)		J_0		J_t
	dR	$\bar\partial$	BC	dR	$\bar\partial$	BC	$\bar\partial$	BC
(0,0)	1	1	1	1	1	1	1	1
(1,0)	2	3	1	2	3	1	1	1
(0,1)		1	1		3	1	1	1
(2,0)	3	3	3	5	3	3	1	1
(1,1)		3	1		9	7	3	3
(0,2)		1	3		3	3	1	1
(3,0)	4	1	1	8	1	1	1	1
(2,1)		3	3		9	9	3	3
(1,2)		3	3		9	9	3	3
(0,3)		1	1		1	1	1	1
(3,1)	3	3	3	5	1	1	1	1
(2,2)		3	5		9	11	3	3
(1,3)		3	1		3	3	1	1
(3,2)	2	1	3	2	3	5	1	1
(2,3)		3	3		3	5	1	1
(3,3)	1	1	1	1	1	1	1	1

Then for a coordinate $(z_1 = x + \sqrt{-1}y, z_2, z_3) \in \mathbb{C} \ltimes_\phi \mathbb{C}^2$ we have

$$\bigwedge^{p,q} \mathfrak{g}^* = \bigwedge^{p,q} \langle dz_1,\ e^{-x}dz_2, e^x dz_3\rangle \otimes \langle dz_1,\ e^{-x}d\bar{z}_2,\ e^x d\bar{z}_3\rangle.$$

For some $a \in \mathbb{R}$ the matrix $\begin{pmatrix} e^a & 0 \\ 0 & e^{-a} \end{pmatrix}$ is conjugate to an element of $SL(2, \mathbb{Z})$. Hence for any $0 \neq b \in \mathbb{R}$ we have a lattice $\Gamma = (a\mathbb{Z} + b\sqrt{-1}\mathbb{Z}) \ltimes \Gamma''$ such that Γ'' is a lattice of $\mathbb{C}^2$.

Since G_2 is completely solvable, for any lattice Γ the de Rham cohomology $H^*(\Gamma\backslash G_2)$ is isomorphic to the cohomology of Lie algebra of G_2 (see [15]).

We consider the following cases:

(2-A) $b = 2n\pi$ for $n \in \mathbb{Z}$,
(2-B) $b = (2n-1)\pi$ for $n \in \mathbb{Z}$,
(2-C) $b \neq n\pi$ for any $n \in \mathbb{Z}$.

We write the results of computation of cohomologies (see [1]). We summarize in Table 2 the results of the computations of the de Rham, Dolbeault and Bott-Chern cohomologies.

Table 2 The dimensions of the de Rham, Dolbeault, and Bott-Chern cohomologies of the Nakamura-like manifold

		Case (2-A)		Case (2-B)		Case (2-C)	
	dR	$\bar\partial$	BC	$\bar\partial$	BC	$\bar\partial$	BC
(0,0)	1	1	1	1	1	1	1
(1,0)	2	3	1	1	1	1	1
(0,1)		3	1	1	1	1	1
(2,0)	5	3	3	1	1	1	1
(1,1)		9	7	5	3	3	3
(0,2)		3	3	1	1	1	1
(3,0)	8	1	1	1	1	1	1
(2,1)		9	9	5	5	3	3
(1,2)		9	9	5	5	3	3
(0,3)		1	1	1	1	1	1
(3,1)	5	3	3	1	1	1	1
(2,2)		9	11	5	7	3	3
(1,3)		3	3	1	1	1	1
(3,2)	2	3	5	1	1	1	1
(2,3)		3	5	1	1	1	1
(3,3)	1	1	1	1	1	1	1

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Some Characterizations of Real Hypersurfaces in Complex Hyperbolic Two-Plane Grassmannians

Hyunjin Lee and Young Jin Suh

Abstract A main objective in submanifold geometry is the classification of homogeneous hypersurfaces. Homogeneous hypersurfaces arise as principal orbits of cohomogeneity one actions, and so their classification is equivalent to the classification of cohomogeneity one actions up to orbit equivalence. Actually, the classification of cohomogeneity one actions in irreducible simply connected Riemannian symmetric spaces of rank 2 of noncompact type was obtained by J. Berndt and Y.J. Suh (for complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2{\cdot}U_m)$, (Berndt and Suh, Int. J. Math. **23**, 1250103 (35pages), 2012)). From this classification, in (Suh, Adv. Appl. Math. **50**, 645–659, 2013) Suh classified real hypersurfaces with isometric Reeb flow in $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 2$. Each one can be described as a tube over a totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ or a horosphere whose center at infinity is singular. By using this result, we want to give another characterization for these model spaces by the Reeb invariant shape operator, that is, $\mathcal{L}_\xi A = 0$.

1 Introduction

Let us consider our motivation for this paper as follows: *For a given almost Hermitian manifold $\bar{M}$, classify all orientable real hypersurfaces M in $\bar{M}$ for which the Reeb flow is isometric.* The almost Hermitian structure on $\bar{M}$ induces an almost contact metric structure on M. The corresponding unit tangent vector field on M is the *Reeb* vector field, and its flow is said to be the *Reeb flow* on M.

A classical example is the anti-de Sitter sphere H_1^{2m-1} in $\mathbb{C}^m$, where the orbits of the Reeb flow induce the Hopf foliation on H_1^{2m-1} with principal S^1-bundle by

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time-like totally geodesic fibers. It is well known that H_1^{2m-1} is a principal S^1-bundle over a complex hyperbolic space $\mathbb{C}H^m$ with projection $\pi : H_1^{2m+1} \to \mathbb{C}H^m$. Moreover, in a paper due to Montiel and Romero [7] it was proved that the second fundamental tensor A' of a Lorentzian hypersurface in H_1^{2m-1} is parallel if and only if the hypersurface M in $\mathbb{C}H^m$ has *isometric Reeb flow*, that is, $\phi A = A\phi$, where $\pi^* A = A'$, $\pi^* A$ is called a pullback of the shape operator A for a hypersurface in $\mathbb{C}H^m$ by the projection π and ϕ denotes the structure tensor induced from the Kähler structure J of $\mathbb{C}H^m$.

The classification of all real hypersurfaces in complex projective space $\mathbb{C}P^m$ with isometric Reeb flow has been obtained by Okumura [8]. The corresponding classification in complex hyperbolic space $\mathbb{C}H^m$ is due to Montiel and Romero [7] and in quaternionic projective space $\mathbb{H}P^m$ is due to Martinez and Pérez [6] respectively.

Let us denote by $SU_{2,m}/S(U_2{\cdot}U_m)$ the *complex hyperbolic two-plane Grassmannian* which consists of all complex two-dimensional linear subspaces in indefinite complex Euclidean space $\mathbb{C}_2^{m+2}$, where $SU_{2,m}$ denotes the set of $(m+2)\times(m+2)$ indefinite special unitary matrices, U_2 and U_m the set of 2×2 and $m\times m$-unitary matrices respectively. Then it is known that $SU_{2,m}/S(U_2{\cdot}U_m)$ has both a Kähler structure J and a quaternionic Kähler structure $\{J_1, J_2, J_3\}$.

Now let us introduce a paper due to Suh [9] for the classification of all real hypersurfaces with isometric Reeb flow in complex hyperbolic two-plane Grassmann manifolds $SU_{2,m}/S(U_2{\cdot}U_m)$ as follows:

Theorem 1. *Let M be a connected orientable real hypersurface in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$. Then the Reeb flow on M is isometric if and only if M is an open part of a tube around some totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ or a horosphere whose center at infinity is singular.*

A tube around $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ is a principal orbit of the isometric action of the maximal compact subgroup $SU_{1,m+1}$ of SU_{m+2}, and the orbits of the Reeb flow corresponding to the orbits of the action of U_1. The action of $SU_{1,m+1}$ has two kinds of singular orbits. One is a totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ and the other is a totally geodesic $\mathbb{C}H^m$ in $SU_{2,m}/S(U_2{\cdot}U_m)$.

When the shape operator A of M in $SU_{2,m}/S(U_2{\cdot}U_m)$ is Lie-parallel along the direction of Reeb vector field ξ, that is, $\mathscr{L}_\xi A = 0$, we say that the shape operator A is *Reeb invariant*. In this article, we introduce a classification of real hypersurfaces in $SU_{2,m}/S(U_2{\cdot}U_m)$ with Reeb invariant shape operator as follows (see Lee, Kim and Suh [5]):

Theorem 2. *Let M be a connected orientable real hypersurface in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$. Then the shape operator on M is Reeb invariant if and only if M is an open part of a tube around some totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ or a horosphere whose center at infinity is singular.*

A remarkable consequence of Theorem 2 is that a connected complete real hypersurface in $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$ with isometric Reeb flow is homogeneous. This was also true in compact complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, which could be identified with symmetric space of compact type $SU_{m+2}/S(U_2 \cdot U_m)$ (see Berndt and Suh [1]).

Using the result of Theorem 2, we have the following two corollaries related to the invariancy of shape operator.

Corollary 1. *There does not exist any connected orientable real hypersurface in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$, with $\mathscr{F}$-invariant shape operator.*

Corollary 2. *There does not exist any connected orientable real hypersurface in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2 \cdot U_m)$, $m \geq 3$, with invariant shape operator.*

In previous corollary, if the shape operator A of M in $SU_{2,m}/S(U_2 \cdot U_m)$ satisfies the property of $\mathscr{L}_X A = 0$ on a distribution $\mathscr{F}$ defined by $\mathscr{F} = \mathscr{C}^{\perp} \cup \mathscr{Q}^{\perp}$ (or for all tangent vector fields X on M, resp.), then it is said to be $\mathscr{F}$*-invariant* (or *invariant*, resp.).

This paper is organized as follows. In Sect. 2 we summarize some basic facts about the Riemannian geometry of $SU_{2,m}/S(U_2 \cdot U_m)$. In Sect. 3 we get some basic geometric equations for real hypersurfaces in $SU_{2,m}/S(U_2 \cdot U_m)$. In Sect. 4 we study real hypersurfaces in $SU_{2,m}/S(U_2 \cdot U_m)$ with Reeb invariant shape operator and prove Theorem 2. Lastly, we give some proof for Corollary 1 and Corollary 2 using the proof of Theorem 2 given in Sect. 4.

2 Complex Hyperbolic Two-Plane Grassmannians

In this section we summarize basic material about the complex hyperbolic two-plane Grassmann manifold $SU_{2,m}/S(U_2 \cdot U_m)$, for details we refer to [1–4] and [9].

The Riemannian symmetric space $SU_{2,m}/S(U_2 \cdot U_m)$, which consists of all complex two-dimensional linear subspaces in indefinite complex Euclidean space $\mathbb{C}_2^{m+2}$, becomes a connected, simply connected, irreducible Riemannian symmetric space of noncompact type and with rank two. Let $G = SU_{2,m}$ and $K = S(U_2 \cdot U_m)$, and denote by $\mathfrak{g}$ and $\mathfrak{k}$ the corresponding Lie algebra of the Lie group G and K respectively. Let B be the Killing form of $\mathfrak{g}$ and denote by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B. The resulting decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. The Cartan involution $\theta \in \mathrm{Aut}(\mathfrak{g})$ on $\mathfrak{su}_{2,m}$ is given by $\theta(A) = I_{2,m} A I_{2,m}$, where

$$I_{2,m} = \begin{pmatrix} -I_2 & 0_{2,m} \\ 0_{m,2} & I_m \end{pmatrix}$$

I_2 and I_m denotes the identity (2×2)-matrix and $(m \times m)$-matrix respectively. Then $< X, Y >= -B(X, \theta Y)$ becomes a positive definite $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{g}$. Its restriction to $\mathfrak{p}$ induces a metric g on $SU_{2,m}/S(U_2{\cdot}U_m)$, which is also known as the Killing metric on $SU_{2,m}/S(U_2{\cdot}U_m)$. Throughout this paper we consider $SU_{2,m}/S(U_2{\cdot}U_m)$ together with this particular Riemannian metric g.

The Lie algebra $\mathfrak{k}$ decomposes orthogonally into $\mathfrak{k} = \mathfrak{su}_2 \oplus \mathfrak{su}_m \oplus \mathfrak{u}_1$, where $\mathfrak{u}_1$ is the one-dimensional center of $\mathfrak{k}$. The adjoint action of $\mathfrak{su}_2$ on $\mathfrak{p}$ induces the quaternionic Kähler structure $\mathfrak{J}$ on $SU_{2,m}/S(U_2{\cdot}U_m)$, and the adjoint action of

$$Z = \begin{pmatrix} \frac{mi}{m+2} I_2 & 0_{2,m} \\ 0_{m,2} & \frac{-2i}{m+2} I_m \end{pmatrix} \in \mathfrak{u}_1$$

induces the Kähler structure J on $SU_{2,m}/S(U_2 \cdot U_m)$. By construction, J commutes with each almost Hermitian structure J_ν in $\mathfrak{J}$ for $\nu = 1, 2, 3$. Recall that a canonical local basis J_1, J_2, J_3 of a quaternionic Kähler structure $\mathfrak{J}$ consists of three almost Hermitian structures J_1, J_2, J_3 in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu$, where the index ν is to be taken modulo 3. The tensor field JJ_ν, which is locally defined on $SU_{2,m}/S(U_2{\cdot}U_m)$, is selfadjoint and satisfies $(JJ_\nu)^2 = I$ and $\mathrm{tr}(JJ_\nu) = 0$, where I is the identity transformation. For a nonzero tangent vector X we define $\mathbb{R}X = \{\lambda X | \lambda \in \mathbb{R}\}$, $\mathbb{C}X = \mathbb{R}X \oplus \mathbb{R}JX$, and $\mathbb{H}X = \mathbb{R}X \oplus \mathfrak{J}X$.

We identify the tangent space $T_oSU_{2,m}/S(U_2{\cdot}U_m)$ of $SU_{2,m}/S(U_2{\cdot}U_m)$ at o with $\mathfrak{p}$ in the usual way. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Since $SU_{2,m}/S(U_2{\cdot}U_m)$ has rank two, the dimension of any such subspace is two. Every nonzero tangent vector $X \in T_oSU_{2,m}/S(U_2{\cdot}U_m) \cong \mathfrak{p}$ is contained in some maximal abelian subspace of $\mathfrak{p}$. Generically this subspace is uniquely determined by X, in which case X is called regular. If there exists more than one maximal abelian subspace of $\mathfrak{p}$ containing X, then X is called singular. There is a simple and useful characterization of the singular tangent vectors: A nonzero tangent vector $X \in \mathfrak{p}$ is singular if and only if $JX \in \mathfrak{J}X$ or $JX \perp \mathfrak{J}X$.

Up to scaling there exists a unique $S(U_2{\cdot}U_m)$-invariant Riemannian metric g on $SU_{2,m}/S(U_2{\cdot}U_m)$. Equipped with this metric $SU_{2,m}/S(U_2{\cdot}U_m)$ is a Riemannian symmetric space of rank two which is both Kähler and quaternionic Kähler. For computational reasons we normalize g such that the minimal sectional curvature of $(SU_{2,m}/S(U_2{\cdot}U_m), g)$ is -4. The sectional curvature K of the noncompact symmetric space $SU_{2,m}/S(U_2{\cdot}U_m)$ equipped with the Killing metric g is bounded by $-4 \leq K \leq 0$. The sectional curvature -4 is obtained for all 2-planes $\mathbb{C}X$ when X is a non-zero vector with $JX \in \mathfrak{J}X$.

When $m = 1$, $G_2^*(\mathbb{C}^3) = SU_{1,2}/S(U_1{\cdot}U_2)$ is isometric to the two-dimensional complex hyperbolic space $\mathbb{C}H^2$ with constant holomorphic sectional curvature -4. When $m = 2$, we note that the isomorphism $SO(4, 2) \simeq SU(2, 2)$ yields an isometry between $G_2^*(\mathbb{C}^4) = SU_{2,2}/S(U_2{\cdot}U_2)$ and the indefinite real Grassmann manifold $G_2^*(\mathbb{R}_2^6)$ of oriented two-dimensional linear subspaces of an indefinite Euclidean space $\mathbb{R}_2^6$. For this reason we assume $m \geq 3$ from now on, although many of the subsequent results also hold for $m = 1, 2$.

The Riemannian curvature tensor $\bar{R}$ of $SU_{2,m}/S(U_2{\cdot}U_m)$ is locally given by

$$\begin{aligned}\bar{R}(X,Y)Z = -\frac{1}{2}\Big[& g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ & - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + \sum_{\nu=1}^{3}\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\} \\ & + \sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\}\Big],\end{aligned} \tag{1}$$

where J_1, J_2, J_3 is any canonical local basis of $\mathfrak{J}$.

Recall that a maximal flat in a Riemannian symmetric space $\bar{M}$ is a connected complete flat totally geodesic submanifold of maximal dimension. A non-zero tangent vector X of $\bar{M}$ is singular if X is tangent to more than one maximal flat in $\bar{M}$, otherwise X is regular. The singular tangent vectors of $SU_{2,m}/S(U_2{\cdot}U_m)$ are precisely the eigenvectors and the asymptotic vectors of the self-adjoint endomorphisms JJ_1, where J_1 is any almost Hermitian structure in $\mathfrak{J}$. In other words, a tangent vector X to $SU_{2,m}/S(U_2{\cdot}U_m)$ is singular if and only if $JX \in \mathfrak{J}X$ or $JX \perp \mathfrak{J}X$.

Now we want to focus on a singular vector X of type $JX \in \mathfrak{J}X$. In this paper, we will have to compute explicitly Jacobi vector fields along geodesics whose tangent vectors are all singular of type $JX \in \mathfrak{J}X$. For this we need the eigenvalues and eigenspaces of the Jacobi operator $\bar{R}_X := \bar{R}(.,X)X$. Let X be a singular unit vector tangent to $SU_{2,m}/S(U_2{\cdot}U_m)$ of type $JX \in \mathfrak{J}X$. Then there exists an almost Hermitian structure J_1 in $\mathfrak{J}$ such that $JX = J_1X$ and the eigenvalues, eigenspaces and multiplicities of $\bar{R}_X$ are respectively given by Table 1 where $\mathbb{R}X$, $\mathbb{C}X$ and $\mathbb{H}X$ denotes the real, complex and quaternionic span of X, respectively, and $\mathbb{C}^{\perp}X$ the orthogonal complement of $\mathbb{C}X$ in $\mathbb{H}X$.

Table 1 The Case: $JX \in \mathfrak{J}X$—Eigenvalues, eigenspaces and multiplicities of $\bar{R}_X$

Eigenvalues	Eigenspace	Multiplicity
0	$\mathbb{R}X \oplus \{Y \mid Y \perp \mathbb{H}X,\ JY = -J_1Y\}$	$2m-1$
-1	$(\mathbb{H}X \ominus \mathbb{C}X) \oplus \{Y \mid Y \perp \mathbb{H}X,\ JY = J_1Y\}$	$2m$
-4	$\mathbb{R}JX$	1

The maximal totally geodesic submanifolds of $SU_{2,m}/S(U_2{\cdot}U_m)$ are

$$SU_{2,m-1}/S(U_2{\cdot}U_{m-1}),\ \mathbb{C}H^m,\ \mathbb{C}H^k\times\mathbb{C}H^{m-k}\ (1\le k\le[m/2]),\ G_2^*(\mathbb{R}^{m+2})$$

and $\mathbb{H}H^n$ (if $m=2n$). The first three are complex submanifolds and the other two are real submanifolds with respect to the Kähler structure J. The tangent spaces of the totally geodesic $\mathbb{C}H^m$ are precisely the maximal linear subspaces of the form $\{X|JX=J_1X\}$ for some fixed almost Hermitian structure $J_1\in\mathfrak{J}$.

3 Real Hypersurfaces in $SU_{2,m}/S(U_2{\cdot}U_m)$

Let M be a real hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Levi-Civita covariant derivative of (M,g). We denote by $\mathscr{C}$ and $\mathscr{Q}$ the maximal complex and quaternionic subbundle of the tangent bundle TM of M, respectively. Now let us put

$$JX=\phi X+\eta(X)N,\quad J_\nu X=\phi_\nu X+\eta_\nu(X)N \tag{2}$$

for any tangent vector field X of a real hypersurface M in $SU_{2,m}/S(U_2{\cdot}U_m)$, where ϕX denotes the tangential component of JX and N a unit normal vector field of M in $SU_{2,m}/S(U_2{\cdot}U_m)$. From the Kähler structure J of $SU_{2,m}/S(U_2{\cdot}U_m)$ there exists an almost contact metric structure (ϕ,ξ,η,g) induced on M such that

$$\phi^2X=-X+\eta(X)\xi,\ \eta(\xi)=1,\ \phi\xi=0,\ \eta(X)=g(X,\xi) \tag{3}$$

for any vector field X on M and $\xi=-JN$.

If M is orientable, then the vector field ξ is globally defined and said to be the induced *Reeb vector field* on M. Furthermore, let J_1,J_2,J_3 be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces a local almost contact metric structure $(\phi_\nu,\xi_\nu,\eta_\nu,g)$, $\nu=1,2,3$, on M. Locally, $\mathscr{C}$ is the orthogonal complement in TM of the real span of ξ, and $\mathscr{Q}$ the orthogonal complement in TM of the real span of $\{\xi_1,\xi_2,\xi_3\}$.

Furthermore, let $\{J_1,J_2,J_3\}$ be a canonical local basis of $\mathfrak{J}$. Then the quaternionic Kähler structure $\mathfrak{J}$ of $SU_{2,m}/S(U_2{\cdot}U_m)$, together with the condition

$$J_\nu J_{\nu+1}=J_{\nu+2}=-J_{\nu+1}J_\nu$$

in Sect. 2, induces the almost contact metric 3-structure $(\phi_\nu,\xi_\nu,\eta_\nu,g)$, $\nu=1,2,3$, on M as follows:

$$\begin{cases}\phi_\nu^2X=-X+\eta_\nu(X)\xi_\nu, \quad \phi_\nu\xi_\nu=0, \quad \eta_\nu(\xi_\nu)=1\\ \phi_{\nu+1}\xi_\nu=-\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1}=\xi_{\nu+2},\\ \phi_\nu\phi_{\nu+1}X=\phi_{\nu+2}X+\eta_{\nu+1}(X)\xi_\nu,\\ \phi_{\nu+1}\phi_\nu X=-\phi_{\nu+2}X+\eta_\nu(X)\xi_{\nu+1}\end{cases} \tag{4}$$

for any vector field X tangent to M. The tangential and normal components of the commuting identity $JJ_\nu X = J_\nu JX$ give

$$\phi\phi_\nu X - \phi_\nu\phi X = \eta_\nu(X)\xi - \eta(X)\xi_\nu \quad \text{and} \quad \eta_\nu(\phi X) = \eta(\phi_\nu X). \tag{5}$$

The last equation implies $\phi_\nu\xi = \phi\xi_\nu$. The tangential and normal components of $J_\nu J_{\nu+1}X = J_{\nu+2}X = -J_{\nu+1}J_\nu X$ give

$$\phi_\nu\phi_{\nu+1}X - \eta_{\nu+1}(X)\xi_\nu = \phi_{\nu+2}X = -\phi_{\nu+1}\phi_\nu X + \eta_\nu(X)\xi_{\nu+1} \tag{6}$$

and

$$\eta_\nu(\phi_{\nu+1}X) = \eta_{\nu+2}(X) = -\eta_{\nu+1}(\phi_\nu X), \tag{7}$$

respectively.

Moreover, putting $X = \xi_\nu$ and $X = \xi_{\nu+1}$ into the first of these two equations yields $\phi_{\nu+2}\xi_\nu = \xi_{\nu+1}$ and $\phi_{\nu+2}\xi_{\nu+1} = -\xi_\nu$ respectively. From the Kähler condition $(\bar\nabla_X J) = 0$ and the quaternionic Kähler condition $(\bar\nabla_X J_\nu) = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2}$, $\nu = 1, 2, 3$, together with Gauss and Weingarten formulas, we obtain respectively

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{8}$$

and

$$\begin{cases} (\nabla_X\phi_\nu)Y = -q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX - g(AX,Y)\xi_\nu, \\ \nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX \end{cases} \tag{9}$$

for all tangent vector field X on M.

Finally, using the explicit expression for the Riemannian curvature tensor $\bar R$ of $SU_{2,m}/S(U_2{\cdot}U_m)$ in (1) the Codazzi equation takes the form

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X = -\frac{1}{2}\Big[&\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi \\ &+ \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X,Y)\xi_\nu\} \\ &+ \sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\} \\ &+ \sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\xi_\nu\Big].\end{aligned} \tag{10}$$

Hereafter, unless otherwise stated, we want to use these basic equations mentioned above frequently without referring to them explicitly.

4 Proof of Theorem 2

In order to give a complete proof of Theorem 2 in the introduction, we need the following Key Proposition. Here we omit the proofs. For more detail proofs see [5].

Proposition 1. *Let M be a real hypersurface in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$. If the shape operator A of M satisfies $\mathscr{L}_\xi A = 0$, then it commutes with the structure tensor ϕ.*

Therefore, by virtue of Proposition 4.1 in [9] and Theorem 1 we assert that if a real hypersurface M in $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$, satisfies the assumption in Proposition 1, then M is locally congruent to an open part of a tube around some totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ or a horosphere whose center at infinity is singular.

From now on, let us check whether the tube M_r of radius $r \in \mathbb{R}_+$ around the totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ and a horosphere $\mathscr{H}$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ whose center at infinity is a singular point of type $JX \in \mathfrak{J}X$ satisfy the condition of Reeb invariant shape operator. In fact, the principal curvatures, their eigenspaces and multiplicities of the tube M_r and a horosphere $\mathscr{H}$ are given by Table 2, respectively.

By the property of Lie derivative we obtain

$$(\mathscr{L}_X A)Y = \mathscr{L}_X(AY) - A(\mathscr{L}_X Y) = (\nabla_X A)Y - \nabla_{AY}X + A(\nabla_Y X)$$

for all tangent vector field X and Y.

On the other hand, from Table 2 all principal curvatures on M_r (reps. $\mathscr{H}$) are constant. So, we may put $AY = \sigma Y$ on T_pM_r (resp. $T_p\mathscr{H}$). From this, the Lie derivative of the shape operator A along any direction X becomes

Table 2 Principal curvatures, eigenspaces and multiplicities of M_r and $\mathscr{H}$

Type	Eigenvalues	Eigenspace	Multiplicity
M_r	$2\coth(2r)$	$T_\alpha = \mathscr{C}^\perp$	1
	$\coth(r)$	$T_\beta = \mathscr{C} \ominus \mathscr{Q}$	2
	$\tanh(r)$	$T_{\lambda_1} = E_{-1}$	$2m-2$
	0	$T_{\lambda_2} = E_{+1}$	$2m-2$
$\mathscr{H}$	2	$T_\alpha = \mathscr{C}^\perp$	1
	1	$T_\beta = (\mathscr{C} \ominus \mathscr{Q}) \oplus E_{-1}$	$2m$
	0	$T_\lambda = E_{+1}$	$2m-2$

where $\mathscr{C}$ (resp. $\mathscr{Q}$) is the maximal complex (resp. quaternionic) subbundle of TM. For the case $JN \in \mathfrak{J}N$, on $\mathscr{Q}$ we have $(\phi\phi_1)^2 = I$ and $\mathrm{tr}(\phi\phi_1) = 0$. Let E_{+1} and E_{-1} be the eigenbundles of $\phi\phi_1|_{\mathscr{Q}}$ with respect to the eigenvalues $+1$ and -1, respectively

$$\begin{aligned}(\mathscr{L}_X A)Y &= (\nabla_X A)Y - \nabla_{AY}X + A(\nabla_Y X)\\ &= \sigma(\nabla_X Y) - A(\nabla_X Y) - \sigma(\nabla_Y X) + A(\nabla_Y X)\\ &= (\sigma I - A)[X, Y]\end{aligned} \tag{11}$$

for all tangent vector fields X and Y with $AY = \sigma Y$ on M_r (or $\mathscr{H}$).

In order to check our problem mentioned above, we fix $X = \xi$ in (11), that is,

$$(\mathscr{L}_\xi A)Y = (\sigma I - A)[\xi, Y]. \tag{12}$$

Moreover, from the covariant derivative of $JN = J_1 N$ with respect to a tangent vector field X we see that one form q_μ becomes $q_\mu(X) = 2g(AX, \xi_\mu)$ for $\mu = 2, 3$. Using these facts, we assert the following proposition.

Proposition 2. *The tube M_r of radius $r \in \mathbb{R}_+$ around the totally geodesic complex hyperbolic Grassmannian $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ has Reeb invariant shape operator.*

Proof. As shown in Table 2 above, the tube M_r has four distinct constant principal curvatures. So, to prove this proposition let us check whether the Eq. (12) vanishes with respect to each eigenspace of M_r.

Case 1. $Y \in T_\alpha, \sigma = \alpha$

Since $[\xi, Y] = [\xi, \xi] = 0$, it is clear.

Case 2. $Y \in T_\beta, \sigma = \beta$

Because $T_\beta = \text{Span}\{\xi_2, \xi_3\}$, we first consider the subcase $Y = \xi_2$. From (9), since $JN = J_1 N$ and $q_3(\xi) = 2g(A\xi, \xi_3) = 0$, we get $[\xi, \xi_2] = q_1(\xi)\xi_3 - \alpha\xi_3 + \beta\xi_3$. It implies that $(\mathscr{L}_\xi A)\xi_2$ vanishes.

On the other hand, since $[\xi, \xi_3] = -q_1(\xi)\xi_2 + \alpha\xi_2 - \beta\xi_2$ for $Y = \xi_3$, we obtain the same result by similar calculations.

It implies that the Lie derivative of the shape operator along the direction of Reeb vector field ξ is vanishing on T_β.

Case 3. $Y \in T_{\lambda_1} \oplus T_{\lambda_2} = \mathscr{Q}$

Since $[\xi, Y] = \nabla_\xi Y - \sigma\phi Y$ for $Y \in \mathscr{Q}$, the Eq. (12) can be written as

$$(\mathscr{L}_\xi A)Y = \sigma(\nabla_\xi Y) - \sigma^2\phi Y - A(\nabla_\xi Y) + \sigma A\phi Y \tag{13}$$

for all $Y \in \mathscr{Q}$. Moreover, since $Y \in \mathscr{Q}$, the tangent vector field $\nabla_\xi Y$ also belongs to $\mathscr{Q}$. In fact, we get $g(\nabla_\xi Y, \xi_\nu) = 0$ for each $\nu = 1, 2, 3$, from $g(Y, \xi_\nu) = 0$ and (9).

Subcase 3-(1). $Y \in T_{\lambda_1}, \sigma = \lambda_1$

Since $T_{\lambda_1} = \{Y \in \mathscr{Q} \,|\, \phi Y = \phi_1 Y\}$, $\phi T_{\lambda_1} \subset T_{\lambda_1}$. In addition, we have $\phi(\nabla_\xi Y) = \phi_1(\nabla_\xi Y)$, which implies $\nabla_\xi Y \in T_{\lambda_1}$. Thus, we can assert that the Lie derivative of the shape operator along the direction of Reeb vector field ξ is vanishing on T_{λ_1}.

Subcase 3-(2). $Y \in T_{\lambda_2}, \sigma = \lambda_2$

Since $T_{\lambda_2} = \{Y \in \mathcal{Q} \,|\, \phi Y = -\phi_1 Y\}$, we see that $\phi T_{\lambda_2} \subset T_{\lambda_2}$ and $\nabla_\xi Y \in T_{\lambda_2}$. Actually, the property $\phi Y = -\phi_1 Y$ gives us $\phi(\nabla_\xi Y) = -\phi_1(\nabla_\xi Y)$ together with the Eqs. (8) and (9).

Summing up these cases, we can assert that the shape operator A of M_r becomes Reeb invariant. □

Moreover, we have also the following proposition by virtue of Table 2.

Proposition 3. *A horosphere $\mathcal{H}$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ whose center at infinity is a singular point of type $JX \in \mathfrak{J}X$ satisfies the property of Reeb invariant shape operator.*

5 Proof of Corollaries

From the definitions of three kinds of the invariancy for the shape operator on M given in Sect. 1, namely invariant, $\mathcal{F}$-invariant and Reeb invariant shape operator, we see that the property of Reeb invariant shape operator is the weakest condition among them. Thus from Theorem 2, we assert that *if a real hypersurface M in $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$, has $\mathcal{F}$-invariant (or invariant) shape operator, then M is locally congruent to an open part of a tube around some totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ or a horosphere whose center at infinity is singular*.

Now, let us consider whether a tube M_r of radius r around the totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ and a horosphere $\mathcal{H}$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ whose center at infinity is singular have $\mathcal{F}$-invariant (or invariant) shape operator. In order to investigate this problem, we suppose that M_r and $\mathcal{H}$ have $\mathcal{F}$-invariant (or invariant) shape operator. From this assumption we obtain $(\mathcal{L}_{\xi_2} A)\xi_3 = 0$. On the other hand, as $[\xi_2, \xi_3] = 2\beta\xi_1 - q_1(\xi_2)\xi_2 - q_1(\xi_3)\xi_3$ from (9), the Eq. (11) with respect to $X = \xi_2 \in T_\beta$ and $Y = \xi_3 \in T_\beta$ can be written as

$$(\mathcal{L}_{\xi_2} A)\xi_3 = 2\beta(\beta - \alpha)\xi_1. \tag{14}$$

Since a tube M_r has $\mathcal{F}$-invariant (or invariant) shape operator and $\xi = \xi_1$ is unit, from (14) we get $\beta(\beta - \alpha) = 0$. But from the facts $\alpha = 2\coth(2r)$ and $\beta = \coth r$ given in Table 2 we have a contradiction.

Moreover, on $\mathcal{H}$ since $\alpha = 2$ and $\beta = 1$, we assert that the shape operator A of $\mathcal{H}$ does not satisfy not only the property of $\mathcal{F}$-invariant but also the property of invariant shape operator.

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Heat Content Asymptotics on a Compact Riemannian Manifold with Boundary

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Abstract We review the heat content asymptotics on a compact Riemannian manifold with boundary and with specific heat and initial temperature distributions. Some computation of first a few terms in the asymptotic series are shown given the existence of a complete asymptotic series. This follows from a joint work with M. van den Berg and P. Gilkey.

1 Introduction

The heat conduction and diffusion have been studied for centuries (see [12]), and the heat trace of an operator of Laplace type has been explicitly computed since the mid twentieth century (see [3, 14] and the references within and [10] for recent development). For the heat content, computations with various conditions have been extensively carried out by van den Berg and Gilkey (see [1, 2, 4, 13, 15] for earlier works and [5–8, 11] for the settings with singular data). In this article, we summarise the main theorems in [9] without proofs and review some of the recent results on the heat content on a compact Reimannian manifold with singular data. At the centre of the discussion of these works, the existence of the complete asymptotic series of the heat content for $\downarrow 0$ stands as a fundamental issue, and with this established, the computation of the coefficients which appear in the series relies on the specific examples in which analysis can be done explicitly. To generalise the statements in the cases with doubly singular data, this existence was conjectured in [9], and we shall do so in the first part of the this paper. In the second part, we introduce the results on the existence of the complete asymptotic series on a compact subdomain of a closed manifold.

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1.1 Definitions and Notation

We give basic definitions and notation as in [9]. Let (M, g) be a compact Riemannian manifold of dimension m with or without smooth boundary ∂M. Let V be a smooth vector bundle over M and let $D : C^\infty(V) \to C^\infty(V)$ be a second order partial differential operator on the space of smooth sections to V. Throughout this article, we adopt the Einstein convention and sum over repeated indices. Now we introduce a type of differential operators which we consider here.

Definition 1. We say that D is of *Laplace type* if the leading symbol of D is given by the metric tensor, i.e. locally D has the form:

$$D = -\left\{ g^{\mu\nu} \partial_{x_\mu} \partial_{x_\nu} + A^\sigma \partial_{x_\sigma} + B \right\} .$$

Recall the Bochner formalism from [14] that there exists a unique connection ∇ on V and a unique endomorphism E of V so that $D = D(g, \nabla, E) = -(g^{\mu\nu} \nabla_{\partial_{x_\nu}} \nabla_{\partial_{x_\mu}} + E)$.

Now we impose suitable boundary conditions. Let e_m be the inward unit normal vector field on the boundary. Let $S_{\mathscr{R}}$ be an auxiliary endomorphism of $V|_{\partial M}$ and let ∇ be the connection on V given above. Let $B_{\mathscr{D}}$ and $B_{\mathscr{R}}$ be the *Dirichlet* and the *Robin* boundary operators which are defined, respectively, by setting:

$$B_{\mathscr{D}} f = f|_{\partial M} \quad \text{and} \quad B_{\mathscr{R}} f = (\nabla_{e_m} f + S_{\mathscr{R}} f)|_{\partial M} \quad \text{for} \quad f \in C^\infty(V) .$$

The *Neumann* boundary operator is defined by taking $S_{\mathscr{R}} = 0$. We let D_B denote either the *Dirichlet* or the *Robin* realisation of D depending on whether $B = B_{\mathscr{D}}$ or $B = B_{\mathscr{R}}$.

1.2 The Heat Equation and the Heat Content

Let r be the geodesic distance to the boundary, and let $\mathbf{y} = (y^1, \ldots, y^{m-1})$ be a system of local coordinates near $p \in \partial M$. Then $\mathbf{x} = (y, r)$ for $r \in [0, \epsilon]$ is an adapted system of local coordinates near $p \in M$ for some $\epsilon > 0$. Let ϕ and ρ denote the *initial temperature* and the *specific heat* of M, respectively. For singular behavior of these, assume that they are smooth on the interior of M and that $r^{\alpha_1}\phi$ and $r^{\alpha_2}\rho$ are smooth sections in V and V^*, respectively, near ∂M for $\alpha_1, \alpha_2 \in \mathbb{C}$ satisfying

$$\Re(\alpha_1) < 1, \quad \Re(\alpha_2) < 1, \quad \text{and} \quad \alpha_1 + \alpha_2 \notin \mathbb{Z} . \tag{1}$$

The parameter α_i, $i = 1, 2$, controls the blow up (resp. decay) near the boundary if $\Re(\alpha_i) > 0$ (resp. $\Re(\alpha_i) < 0$). Now the temperature $T := e^{-tD_B}\phi$ is characterised by the following parabolic equation with initial condition:

$$\begin{aligned}
&(\partial_t + D)T = 0, \\
&\lim_{t\downarrow 0} T(\cdot;t) = \phi, \\
&BT(\cdot;t) = 0 \text{ for } t > 0.
\end{aligned} \tag{2}$$

Let $\langle \cdot, \cdot \rangle$ be the *natural pairing* between V and the dual bundle V^*, let dx be the *Riemannian measure* on M, let dy be the Riemannian measure on ∂M, and let ρ be the specific heat of M as given above.

Definition 2. The total *heat content* of the manifold is defined by

$$\beta(\phi, \rho, D, B)(t) := \int_M \langle e^{-tD_B}\phi, \rho \rangle dx \, .$$

There is a smooth *heat kernel* $K = K_{D,B}$ so that $T(x;t) = \int_M K(x, \tilde{x}; t)\phi(\tilde{x})d\tilde{x}$, and

$$\beta(\phi, \rho, D, B)(t) = \int_{M\times M} \langle K(x, \tilde{x}; t)\phi(\tilde{x}), \rho(x)\rangle d\tilde{x} dx \, ,$$

which is well defined for $\phi \in L^1(V)$ and $\rho \in L^1(V^*)$ due to (1).

2 Asymptotic Series of Heat Content

In order to consider integrals which may be divergent, we introduce the *regularisation* of those integrals. Note that the integral $\int_M \langle \phi, \rho \rangle dx$ is divergent if $1 < \Re(\alpha_1 + \alpha_2) < 2$. Though the Riemannian measure is not in general a product near the boundary, one may write $dx = dydr$ on the boundary of M, one can decompose

$$\langle \phi, \rho \rangle dx = \langle \phi^0, \rho^0 \rangle r^{-\alpha_1-\alpha_2} dydr + O(r^{1-\alpha_1-\alpha_2}) \, .$$

For $\Re(\alpha_1 + \alpha_2) < 2$ and $\alpha_1 + \alpha_2 \neq 1$, define

$$\begin{aligned}
\mathscr{I}^g_{Reg}(\phi, \rho) := & \int_{M-\mathscr{C}_\epsilon} \langle \phi, \rho \rangle dx + \int_{\mathscr{C}_\epsilon} \{\langle \phi, \rho \rangle dx - \langle \phi^0, \rho^0 \rangle r^{-\alpha_1-\alpha_2} dydr\} \\
& + \int_{\partial M} \langle \phi^0, \rho^0 \rangle dy \times \varepsilon^{1-\alpha_1-\alpha_2}(1-\alpha_1-\alpha_2)^{-1} \, ,
\end{aligned}$$

where $C_\varepsilon := \{x \in M : r(x) \leq \varepsilon\}$ is a small collared neighbourhood of the boundary. This is clearly independent of ε and agrees with $\int_M \langle \phi, \rho \rangle dx$ if $\Re(\alpha_1 + \alpha_2) < 1$. The regularisation $\mathscr{I}_{\text{Reg}}(\phi, \rho)$ is a meromorphic function of $\alpha_1 + \alpha_2$ with a simple pole at $\alpha_1 + \alpha_2 = 1$. We set

$$\beta^M_n(\phi, \rho, D) := (-1)^n/n! \cdot \mathscr{I}^g_{Reg}\{\langle D^n \phi, \rho \rangle\} \, .$$

If M is a closed manifold, these are the invariants which would appear in the heat content expansion. In this section, we assume that the following conjecture (which extends the discussion of [5, 6]) holds. We refer to [11] where a related result was established when the specific heat ρ is smooth.

Conjecture 1. If (α_1, α_2) satisfy (1), then there is a complete asymptotic series as $t \downarrow 0$ of the form:

$$\beta(\phi, \rho, D, \mathscr{B}_{\mathscr{D}/\mathscr{R}})(t) \sim \sum_{n=0}^{\infty} t^n \beta_n^M(\phi, \rho, D) + \sum_{j=0}^{\infty} t^{(1+j-\alpha_1-\alpha_2)/2} \beta_{j,\alpha_1,\alpha_2}^{\partial M}(\phi, \rho, D, \mathscr{B}_{\mathscr{D}/\mathscr{R}}) .$$

The coefficients $\beta_{j,\alpha_1,\alpha_2}^{\partial M}(\phi, \rho, D, \mathscr{B}_{\mathscr{D}/\mathscr{R}})$ are given by integrals of local invariants over the boundary.

This conjecture has been established in [11] using the calculus of pseudo-differential operators in the special case that $\alpha_2 \in \mathbb{N}$ or that $\alpha_1 \in \mathbb{N}$.

To look at closely how these coefficients $\beta_{j,\alpha_1,\alpha_2}^{\partial M}(\phi, \rho, D, \mathscr{B}_{\mathscr{D}/\mathscr{R}})$ can be written in terms of geometric datum and the singular data, we give the lemma below. Let R_{ijkl} denote the Riemann curvature tensor; with our sign convention, we have that $R_{1221} = +1$ on the unit sphere S^2 in $\mathbb{R}^3$. Let Ric denote the Ricci tensor, let τ denote the scalar curvature, and let L_{ab} denote the second fundamental form. We let indices $\{i, j, k, l\}$ range from 1 to m and index a local orthonormal frame for TM; we let indices $\{a, b, c\}$ range from 1 to $m-1$ and index a local orthonormal frame for $T\partial M$. On the boundary, e_m will always denote the inward unit geodesic normal and ';' will denote the components of the covariant derivative.

Lemma 1. *There exist universal constants $\varepsilon^{\nu}_{\mathscr{D}/\mathscr{R},\alpha_1,\alpha_2}$ so that:*

$$\beta_{0,\alpha_1,\alpha_2}^{\partial M}(\phi, \rho, D, \mathscr{B}_{\mathscr{D}/\mathscr{R}}) = \int_{\partial M} \varepsilon_{\mathscr{D}/\mathscr{R},\alpha_1,\alpha_2} \langle \phi^0, \rho^0 \rangle dy.$$

$$\beta_{1,\alpha_1,\alpha_2}^{\partial M}(\phi, \rho, D, \mathscr{B}_{\mathscr{D}/\mathscr{R}}) = \int_{\partial M} \Big\{ \varepsilon^1_{\mathscr{D}/\mathscr{R},\alpha_1,\alpha_2} \langle \phi^1, \rho^0 \rangle + \varepsilon^2_{\mathscr{D}/\mathscr{R},\alpha_1,\alpha_2} \langle L_{aa}\phi^0, \rho^0 \rangle$$
$$+ \varepsilon^3_{\mathscr{D}/\mathscr{R},\alpha_1,\alpha_2} \langle \phi^0, \rho^1 \rangle + \varepsilon^{15}_{\mathscr{R},\alpha_1,\alpha_2} \langle S_{\mathscr{R}}\phi^0, \rho^0 \rangle \Big\} dy.$$

$$\beta_{2,\alpha_1,\alpha_2}^{\partial M}(\phi, \rho, D, \mathscr{B}_{\mathscr{D}/\mathscr{R}}) = \int_{\partial M} \Big\{ \varepsilon^4_{\mathscr{D}/\mathscr{R},\alpha_1,\alpha_2} \langle \phi^2, \rho^0 \rangle + \varepsilon^5_{\mathscr{D}/\mathscr{R},\alpha_1,\alpha_2} \langle L_{aa}\phi^1, \rho^0 \rangle$$
$$+ \varepsilon^6_{\mathscr{D}/\mathscr{R},\alpha_1,\alpha_2} \langle E\phi^0, \rho^0 \rangle$$
$$+ \varepsilon^7_{\mathscr{D}/\mathscr{R},\alpha_1,\alpha_2} \langle \phi^0, \rho^2 \rangle + \varepsilon^8_{\mathscr{D}/\mathscr{R},\alpha_1,\alpha_2} \langle L_{aa}\phi^0, \rho^1 \rangle$$
$$+ \varepsilon^9_{\mathscr{D}/\mathscr{R},\alpha_1,\alpha_2} \langle \mathrm{Ric}_{mm}\, \phi^0, \rho^0 \rangle + \varepsilon^{10}_{\mathscr{D}/\mathscr{R},\alpha_1,\alpha_2} \langle L_{aa} L_{bb}\phi^0, \rho^0 \rangle$$
$$+ \varepsilon^{11}_{\mathscr{D}/\mathscr{R},\alpha_1,\alpha_2} \langle L_{ab} L_{ab}\phi^0, \rho^0 \rangle + \varepsilon^{12}_{\mathscr{D}/\mathscr{R},\alpha_1,\alpha_2} \langle \phi^0_{;a}, \rho^0_{;a} \rangle + \varepsilon^{13}_{\mathscr{D}/\mathscr{R},\alpha_1,\alpha_2} \langle \tau\phi^0, \rho^0 \rangle$$

$$+ \varepsilon^{14}_{\mathscr{D}/\mathscr{R},\alpha_1,\alpha_2}\langle\phi^1,\rho^1\rangle + \varepsilon^{16}_{\mathscr{R},\alpha_1,\alpha_2}\langle S^2_{\mathscr{R}}\phi^0,\rho^0\rangle + \varepsilon^{17}_{\mathscr{R},\alpha_1,\alpha_2}\langle S_{\mathscr{R}}\phi^1,\rho^0\rangle$$
$$+ \varepsilon^{18}_{\mathscr{R},\alpha_1,\alpha_2}\langle S_{\mathscr{R}}\phi^0,\rho^1\rangle + \varepsilon^{19}_{\mathscr{R},\alpha_1,\alpha_2}\langle S_{\mathscr{R}}L_{aa}\phi^0,\rho^0\rangle\Big\}\,dy.$$

We omit the terms involving $S_{\mathscr{R}}$ for Dirichlet boundary conditions.

A computation on the half-line as a special case leads to identify the constants $\varepsilon_{\mathscr{D}/\mathscr{R},\alpha_1,\alpha_2}$ above. Let

$$\varepsilon_{\mathscr{D}/\mathscr{R}} := \begin{Bmatrix} -1 \text{ if } \mathscr{B} = \mathscr{B}_{\mathscr{D}} \\ +1 \text{ if } \mathscr{B} = \mathscr{B}_{\mathscr{R}} \end{Bmatrix}.$$

Lemma 2. *Suppose that* (α_1,α_2) *satisfy* (1)*, then*

$$\varepsilon_{\mathscr{D}/\mathscr{R},\alpha_1,\alpha_2} := \varepsilon_{\mathscr{D}/\mathscr{R}}\cdot 2^{-\alpha_1-\alpha_2}\pi^{-1/2}\Gamma\left(\frac{2-\alpha_1-\alpha_2}{2}\right)\cdot\frac{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)}{\Gamma(2-\alpha_1-\alpha_2)}$$
$$+2^{-\alpha_1-\alpha_2}\pi^{-1/2}\Gamma\left(\frac{2-\alpha_1-\alpha_2}{2}\right)\Gamma(\alpha_1+\alpha_2-1)\cdot\left(\frac{\Gamma(1-\alpha_1)}{\Gamma(\alpha_2)}+\frac{\Gamma(1-\alpha_2)}{\Gamma(\alpha_1)}\right).$$

Using the functional equations of the Γ function, this leads to

Lemma 3. *We have the recursion relations:*

1. $\varepsilon_{\mathscr{D}/\mathscr{R},\alpha_1-2,\alpha_2} = \dfrac{2(\alpha_1-2)(\alpha_1-1)}{3-\alpha_1-\alpha_2}\varepsilon_{\mathscr{D}/\mathscr{R},\alpha_1,\alpha_2}.$
2. $\varepsilon_{\mathscr{D}/\mathscr{R},\alpha_1,\alpha_2-2} = \dfrac{2(\alpha_2-2)(\alpha_2-1)}{3-\alpha_1-\alpha_2}\varepsilon_{\mathscr{D}/\mathscr{R},\alpha_1,\alpha_2}.$
3. $\varepsilon_{\mathscr{D}/\mathscr{R},\alpha_1-1,\alpha_2-1} = -\dfrac{2(\alpha_1-1)(\alpha_2-1)}{3-\alpha_1-\alpha_2}\varepsilon_{\mathscr{R}/\mathscr{D},\alpha_1,\alpha_2}.$

These recursion relations are important ingredients along with functorial properties of the heat content asymptotics to obtain the following theorems.

2.1 Heat Content Asymptotics for Dirichlet Boundary Conditions

In [5, 11], the computation for the Dirichlet boundary conditions have been done as follows:

Theorem 1. *Suppose that* (α_1,α_2) *satisfy* (1)*, then:*

$$\beta^{\partial M}_{0,\alpha_1,\alpha_2}(\phi,\rho,D,\mathscr{B}_{\mathscr{D}}) = \int_{\partial M}\varepsilon_{\mathscr{D},\alpha_1,\alpha_2}\langle\phi^0,\rho^0\rangle dy,$$

$$\beta^{\partial M}_{1,\alpha_1,\alpha_2}(\phi,\rho,D,\mathscr{B}_{\mathscr{D}}) = \int_{\partial M}\{-\frac{1}{2}(\varepsilon_{\mathscr{D},\alpha_1-1,\alpha_2}+\varepsilon_{\mathscr{D},\alpha_1,\alpha_2-1})L_{aa}\langle\phi^0,\rho^0\rangle$$

$$
\begin{aligned}
&+ \varepsilon_{\mathscr{D},\alpha_1-1,\alpha_2}\langle\phi^1,\rho^0\rangle + \varepsilon_{\mathscr{D},\alpha_1,\alpha_2-1}\langle\phi^0,\rho^1\rangle\}dy,\\
\beta^{\partial M}_{2,\alpha_1,\alpha_2}(\phi,\rho,D,\mathscr{B}_{\mathscr{D}}) &= \int_{\partial M}\{-\frac{1}{2}(\varepsilon_{\mathscr{D},\alpha_1-2,\alpha_2}+\varepsilon_{\mathscr{D},\alpha_1-1,\alpha_2-1})L_{aa}\langle\phi^1,\rho^0\rangle\\
&+ \varepsilon_{\mathscr{D},\alpha_1,\alpha_2}\langle E\phi^0,\rho^0\rangle + \varepsilon_{\mathscr{D},\alpha_1-2,\alpha_2}\langle\phi^2,\rho^0\rangle + \varepsilon_{\mathscr{D},\alpha_1,\alpha_2-2}\langle\phi^0,\rho^2\rangle\\
&- \frac{1}{2}(\varepsilon_{\mathscr{D},\alpha_1-1,\alpha_2-1}+\varepsilon_{\mathscr{D},\alpha_1,\alpha_2-2})L_{aa}\langle\phi^0,\rho^1\rangle\\
&+ (-\frac{1}{4}\varepsilon_{\mathscr{D},\alpha_1-2,\alpha_2} - \frac{1}{4}\varepsilon_{\mathscr{D},\alpha_1,\alpha_2-2} + \frac{1}{2}\varepsilon_{\mathscr{D},\alpha_1,\alpha_2})(L_{ab}L_{ab}+\mathrm{Ric}_{mm})\langle\phi^0,\rho^0\rangle\\
&- \varepsilon_{\mathscr{D},\alpha_1,\alpha_2}\langle\phi^0_{;a},\rho^0_{;a}\rangle + 0\tau\phi^0\rho^0 + \varepsilon_{\mathscr{D},\alpha_1-1,\alpha_2-1}\langle\phi^1,\rho^1\rangle\\
&+(\frac{1}{8}\varepsilon_{\mathscr{D},\alpha_1-2,\alpha_2}+\frac{1}{8}\varepsilon_{\mathscr{D},\alpha_1,\alpha_2-2}+\frac{1}{4}\varepsilon_{\mathscr{D},\alpha_1-1,\alpha_2-1} - \frac{1}{4}\varepsilon_{\mathscr{D},\alpha_1,\alpha_2})L_{aa}L_{bb}\langle\phi^0,\rho^0\rangle\}dy.
\end{aligned}
$$

2.2 Heat Content Asymptotics for Robin Boundary Conditions

Using various functorial properties of the invariants involved, the following theorem was shown in [9]. Note the dependence of (α_1,α_2) in the coefficients of ε.

Theorem 2. *Suppose that* (α_1,α_2) *satisfy* (1), *then:*

$$
\begin{aligned}
\beta^{\partial M}_{0,\alpha_1,\alpha_2}(\phi,\rho,D,\mathscr{B}_{\mathscr{R}}) &= \int_{\partial M}\varepsilon_{\mathscr{R},\alpha_1,\alpha_2}\langle\phi^0,\rho^0\rangle dy.\\
\beta^{\partial M}_{1,\alpha_1,\alpha_2}(\phi,\rho,D,\mathscr{B}_{\mathscr{R}}) &= \int_{\partial M}\Big\{\varepsilon_{\mathscr{R},\alpha_1-1,\alpha_2}\langle\phi^1,\rho^0\rangle + \varepsilon_{\mathscr{R},\alpha_1,\alpha_2-1}\langle\phi^0,\rho^1\rangle\\
&- \frac{1}{2}\{\frac{\alpha_1}{\alpha_1-1}\varepsilon_{\mathscr{R},\alpha_1-1,\alpha_2} + \frac{\alpha_2}{\alpha_2-1}\varepsilon_{\mathscr{R},\alpha_1,\alpha_2-1}\}\langle L_{aa}\phi^0,\rho^0\rangle\\
&+ \{-\frac{1}{\alpha_1-1}\varepsilon_{\mathscr{R},\alpha_1-1,\alpha_2} - \frac{1}{\alpha_2-1}\varepsilon_{\mathscr{R},\alpha_1,\alpha_2-1}\}\langle S_{\mathscr{R}}\phi^0,\rho^0\rangle\Big\}dy.\\
\beta^{\partial M}_{2,\alpha_1,\alpha_2}(\phi,\rho,D,\mathscr{B}_{\mathscr{R}}) &= \int_{\partial M}\Big\{\varepsilon_{\mathscr{R},\alpha_1-2,\alpha_2}\langle\phi^2,\rho^0\rangle + \varepsilon_{\mathscr{R},\alpha_1,\alpha_2-2}\langle\phi^0,\rho^2\rangle\\
&- \frac{1}{2}\{\frac{\alpha_1-1}{\alpha_1-2}\varepsilon_{\mathscr{R},\alpha_1-2,\alpha_2} + \frac{\alpha_2}{\alpha_2-1}\varepsilon_{\mathscr{R},\alpha_1-1,\alpha_2-1}\}\langle L_{aa}\phi^1,\rho^0\rangle\\
&- \frac{1}{2}\{\frac{\alpha_1}{\alpha_1-1}\varepsilon_{\mathscr{R},\alpha_1-1,\alpha_2-1} + \frac{\alpha_2-1}{\alpha_2-2}\varepsilon_{\mathscr{R},\alpha_1,\alpha_2-2}\}\langle L_{aa}\phi^0,\rho^1\rangle\\
&- \frac{1}{2}\frac{\alpha_1^2-2\alpha_1+\alpha_2^2-2\alpha_2+1}{3-\alpha_1-\alpha_2}\varepsilon_{\mathscr{R},\alpha_1,\alpha_2}\langle(\mathrm{Ric}_{mm}+L_{ab}L_{ab})\phi^0,\rho^0\rangle\\
&+ \{\frac{\alpha_1^2+\alpha_2^2-1}{4(3-\alpha_1-\alpha_2)}\varepsilon_{\mathscr{R},\alpha_1,\alpha_2} + \frac{1}{4}\frac{\alpha_1\alpha_2}{(\alpha_1-1)(\alpha_2-1)}\varepsilon_{\mathscr{R},\alpha_1-1,\alpha_2-1}\}\langle L_{aa}L_{bb}\phi^0,\rho^0\rangle
\end{aligned}
$$

$$- \varepsilon_{\mathscr{R},\alpha_1,\alpha_2}\langle \phi^0_{;a}, \rho^0_{;a}\rangle + 0\cdot\langle \tau\phi^0, \rho^0\rangle$$
$$+ \varepsilon_{\mathscr{R},\alpha_1,\alpha_2}\langle E\phi^0, \rho^0\rangle + \varepsilon_{\mathscr{R},\alpha_1-1,\alpha_2-1}\langle \phi^1, \rho^1\rangle$$
$$+ \{\frac{1}{(\alpha_1-1)(\alpha_2-1)}\varepsilon_{\mathscr{R},\alpha_1-1,\alpha_2-1} + \frac{2}{3-\alpha_1-\alpha_2}\varepsilon_{\mathscr{R},\alpha_1,\alpha_2}\}\langle S^2_{\mathscr{R}}\phi^0, \rho^0\rangle$$
$$+ \{-\frac{1}{\alpha_1-2}\varepsilon_{\mathscr{R},\alpha_1-2,\alpha_2} - \frac{1}{\alpha_2-1}\varepsilon_{\mathscr{R},\alpha_1-1,\alpha_2-1}\}\langle S_{\mathscr{R}}\phi^1, \rho^0\rangle$$
$$+ \{-\frac{1}{\alpha_1-1}\varepsilon_{\mathscr{R},\alpha_1-1,\alpha_2-1} - \frac{1}{\alpha_2-2}\varepsilon_{\mathscr{R},\alpha_1,\alpha_2-2}\}\langle S_{\mathscr{R}}\phi^0, \rho^1\rangle$$
$$+ \left\{ \frac{\alpha_1+\alpha_2}{3-\alpha_1-\alpha_2}\varepsilon_{\mathscr{R},\alpha_1,\alpha_2} + \frac{1}{2}\frac{\alpha_1+\alpha_2}{(\alpha_1-1)(\alpha_2-1)}\varepsilon_{\mathscr{R},\alpha_1-1,\alpha_2-1}\right\} \langle S_{\mathscr{R}}L_{aa}\phi^0, \rho^0\rangle\Big\} dy.$$

3 Existence of Heat Content Asymptotic Series

So far the existence of the heat content asymptotics with doubly singular data has been assumed to hold and we have been focused on computing the coefficients of the asymptotic series. However, the recent work [7] by van den Berg and Gilkey showed that this indeed holds for the heat flow of a compact submanifold in a closed manifold which we will describe in this section. Let Ω be a compact subdomain of a closed manifold M with smooth boundary $\partial\Omega$, and let ϕ_Ω and ρ_Ω be the extension of ϕ and ρ to M to be zero on the complement of Ω. Let D_M be an operator of Laplace type on a smooth vector bundle V over M. Then the heat content of Ω in M is given for $t > 0$ by:

$$\beta_\Omega(\phi, \rho, D_M)(t) := \int_M \langle e^{-tD_M}\phi_\Omega, \rho_\Omega\rangle dx$$

Consider the following subset of $\mathbb{C}$ which gives a weaker assumption on (α_1, α_2) than in Sect. 2:

$$\mathscr{O} := \{(\alpha_1, \alpha_2) \in \mathbb{C} : \Re(\alpha_1) < 1,\ \Re(\alpha_2) < 1,\ \alpha_1 + \alpha_2 \neq 1, -1, -3, \dots\}. \quad (3)$$

In [7], the existence theorem was obtained for doubly singular data:

Theorem 3. *Let D_M be an operator of Laplace type on a smooth vector bundle V over a compact Riemannian manifold (M, g) without boundary. Let Ω be a compact subdomain of M with smooth boundary. Let $\phi \in C^\infty(V|_{\mathrm{int}(\Omega)})$ and let $\rho \in C^\infty(V^*|_{\mathrm{int}(\Omega)})$. Let $(\alpha_1, \alpha_2) \in \mathscr{O}$. We assume that $r^{\alpha_1}\phi$ and $r^{\alpha_2}\rho$ are smooth near the boundary of Ω. Let $\beta_\Omega(\phi, \rho, D_M)(t)$ be the heat content of Ω in M. Then there is a complete asymptotic expansion of $\beta_\Omega(\phi, \rho, D_M)(t)$ for small time such that for any positive integer N as $t \downarrow 0$ we have:*

$$\beta_{\Omega}(\phi,\rho,D_M)(t) = \sum_{n=0}^{N} t^n \beta^{\Omega}_{n,\alpha_1,\alpha_2}(\phi,\rho,D_M) + \sum_{j=0}^{N} t^{(1+j-\alpha_1-\alpha_2)/2} \beta^{\partial\Omega}_{j,\alpha_1,\alpha_2}(\phi,\rho,D_M) + O(t^{(N-1)/2}).$$

The coefficient $\beta^{\partial\Omega}_{0,\alpha_1,\alpha_2}(\phi,\rho,D_M)$ *of* $t^{(1+j-\alpha_1-\alpha_2)/2}$ *is given by*

$$\beta^{\partial\Omega}_{0,\alpha_1,\alpha_2}(\phi,\rho,D_M) = c(\alpha_1,\alpha_2)\int_{\partial\Omega}\langle\phi_0,\rho_0\rangle dy,$$

$$\text{where } \; c(\alpha_1,\alpha_2) := 2^{-\alpha_1-\alpha_2}\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{2-\alpha_1-\alpha_2}{2}\right)\Gamma(\alpha_1+\alpha_2-1) \times\left(\frac{\Gamma(1-\alpha_1)}{\Gamma(\alpha_2)}+\frac{\Gamma(1-\alpha_2)}{\Gamma(\alpha_1)}\right).$$

More generally, the coefficients $\beta^{\Omega}_{n,\alpha_1,\alpha_2}(\cdot)$ *are given as regularized integrals of local invariants over the interior of* Ω *that are bilinear in the derivatives of* $\{\phi,\rho\}$ *up to order* $2n$ *with coefficients that depend holomorphically on the parameters* (α_1,α_2)*, that depend smoothly on the* 0*-jets of the metric, and that are polynomial in the derivatives of the total symbol of* D_M *up to order* $2n$*. The coefficients* $\beta^{\partial\Omega}_{j}(\cdot)$ *are given similarly as integrals of local invariants over the boundary* $\partial\Omega$ *where the derivatives of* $\{\phi,\rho\}$ *and of the total symbol of* D_M *are up to order* j*.*

Furthermore, in [7], it was finally shown that for $\alpha_1+\alpha_2 = 1,-1,-3,\ldots$ (as expected in [9]), *log* terms appear in the asymptotic expansion. The following theorem demonstrates this for the heat kernel of $D_{\mathbb{R}} := -\partial_x^2$ on $\mathbb{R}$:

Theorem 4. *Let* $(a,b)\in\mathbb{R}^2$ *with* $\alpha_1<1$ *and* $\alpha_2<1$*. Assume that* $\alpha_1+\alpha_2=1$*. Let* Ξ_i *be smooth monotonically decreasing cut-off functions which are identically 1 near* $x=0$ *and identically* 0 *in an open neighborhood of the interval* $[1/2,1]$*. Then for* $t\downarrow 0$

$$\beta_{[0,1]}(x^{-\alpha_1}\Xi_1, x^{-\alpha_2}\Xi_2, D_{\mathbb{R}})(t) = \beta_0(\alpha_1,\alpha_2,\Xi_1,\Xi_2) - \frac{1}{2}\log(t) + O(t^{\frac{1}{2}}\log(t))$$

where for any $\epsilon>0$ *sufficiently small,*

$$\beta_0(\alpha_1,\alpha_2,\Xi_1,\Xi_2) = \frac{1}{2}\log(\epsilon^2) + \frac{1}{2}\gamma + \log 4(2^{1/2}-1) + \int_{[\epsilon,\frac{1}{2}]}\Xi_1(x)\Xi_2(x)x^{-1}dx + \frac{1}{2}\int_{[0,1]} dq\, q^{-1}\left(\frac{(1+q)^{\alpha_1-1}}{(1-q)^{\alpha_1}} + \frac{(1-q)^{\alpha_1-1}}{(1+q)^{\alpha_1}} - \frac{2}{(1+q^2)^{1/2}}\right),$$

where γ *is the Euler's constant.*

Note that the parameter ϵ serves to regularise the integral and does not contribute to the value of β_0.

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Real Hypersurfaces in Complex Two-Plane Grassmannians with Recurrent Structure Jacobi Operator

Imsoon Jeong, Young Jin Suh, and Changhwa Woo

Abstract In this paper, we introduce a new notion of recurrent structure Jacobi operator, that is, $(\nabla_X R_\xi)Y = \omega(X)R_\xi Y$ for any tangent vector fields X and Y on a real hypersurface M in a complex two-plane Grassmannian, where R_ξ denotes the structure Jacobi operator and ω a certain 1-form on M. Next, we prove that there does not exist any Hopf hypersurface M in a complex two-plane Grassmannian with recurrent structure Jacobi operator.

1 Introduction

In this paper, we will discuss about a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ which is a kind of Hermitian symmetric space with rank 2 of compact type and consists of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. $G_2(\mathbb{C}^{m+2})$ is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J (see Berndt and Suh [2] and [3]).

The almost contact structure vector field ξ defined by $\xi = -JN$ is said to be a *Reeb* vector field, where J denotes an almost Hermitian structure and N denotes a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. Since $G_2(\mathbb{C}^{m+2})$ is a Kähler manifold, the structure J becomes a Kähler structure. The *almost contact 3-structure* vector fields $\{\xi_1, \xi_2, \xi_3\}$ for the three-dimensional distribution $\mathfrak{D}^\perp$ of M in $G_2(\mathbb{C}^{m+2})$ are defined by $\xi_\nu = -J_\nu N$ $(\nu = 1, 2, 3)$, where $\{J_\nu\}_{\nu=1,2,3}$ gives a canonical local basis of a quaternionic Kähler structure $\mathfrak{J}$ and $T_x M = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$, where $\mathfrak{D}$ denotes the orthogonal complement of $\mathfrak{D}^\perp$.

In $G_2(\mathbb{C}^{m+2})$, we consider two natural geometric conditions for real hypersurfaces that $[\xi] = \mathrm{Span}\{\xi\}$ and $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$ are invariant under the shape operator. By using such notions and the results in Alekseevskii [1], Berndt and Suh [2] have proved the following:

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Theorem 1. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^{\perp}$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic quaternionic projective space $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

In the case (A) in Theorem A, a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to be of Type (A). Similarly in the case (B) of Theorem A, we call M a real hypersurface of Type (B). Furthermore, the Reeb vector field ξ is said to be *Hopf* if it is invariant under the shape operator A. The one-dimensional foliation of M by the integral manifolds of the Reeb vector field ξ is said to be a *Hopf foliation* of M. In this case we say that M is a *Hopf hypersurface* in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of M is totally geodesic. By the formulae in Sect. 3 it can be easily checked that M is *Hopf* if and only if the *Reeb vector field* ξ is *Hopf* (see Berndt and Suh [3]).

Now let us introduce a theorem due to Lee and Suh [7] as follows:

Theorem 2. *Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, $m = 2n$, where the distribution $\mathfrak{D}$ denotes the orthogonal complement of $\mathfrak{D}^{\perp}$ in T_xM, $x \in M$.*

On the other hand, we know that Jacobi fields along geodesics of a given Riemannian manifold $(\bar{M}, g)$ satisfy a well-known differential equation. This classical differential equation naturally inspires the so-called Jacobi equation. If $\bar{R}$ denotes the curvature operator of $\bar{M}$ and X any tangent vector field to $\bar{M}$, then the Jacobi operator with respect to X at $x \in \bar{M}$, $\bar{R}_X \in \mathrm{End}(T_x\bar{M})$ can be defined in such a way that

$$(\bar{R}_X Y)(x) = (\bar{R}(Y, X)X)(x)$$

for any $Y \in T_x\bar{M}$, $x \in M$. It becomes a self-adjoint endomorphism of the tangent bundle $T\bar{M}$.

Let us denote by $R(X, Y)Z$ the curvature tensor of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$. Then the structure Jacobi operator R_ξ of M in $G_2(\mathbb{C}^{m+2})$ can be defined by $R_\xi = R(X, \xi)\xi$ for any tangent vector field X to M.

In the paper due to Jeong, Pérez and Suh [6] they proved that there does not exist any connected Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel structure Jacobi operator if the distribution $\mathfrak{D}$ or $\mathfrak{D}^{\perp}$-component of the Reeb vector field is invariant under the shape operator. Moreover, also in [5], they have proved that there does not exist any connected Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{D}^{\perp}$-parallel structure Jacobi operator if the principal curvature $\alpha = g(A\xi, \xi)$ is constant along the direction of the Reeb vector field ξ. As a generalization of this fact, we considered some conditions weaker than parallel structure Jacobi operator and proved that there does not exist any connected Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m\geq 3$, satisfying such conditions. As an example, in a paper due to Jeong, Lee and Suh [4], the authors proved that there does not exist any connected Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m\geq 3$, whose structure Jacobi operator is of Codazzi type.

In this paper, we want to introduce the new notion of *recurrent* structure Jacobi operator which is weaker than *parallel* structure Jacobi operator. Actually, a non-zero tensor field K of type (r,s) on a manifold M is said to be *recurrent* if there exists a 1-form α such that $\nabla K = K \otimes \alpha$. Specifically, recurrent tensor fields can be applied to classify for submanifolds in a complex projective space $\mathbb{C}P^n$ which has constant positive sectional curvature.

From such a view point, Pérez and Santos [8] have defined the notion of *recurrent* structure Jacobi operator in $\mathbb{C}P^n$ defined in such a way that $(\nabla_X R_\xi)Y = \omega(X)R_\xi Y$ for a certain 1-form ω on a real hypersurface M in complex projective space $\mathbb{C}P^n$. Using such a notion, they [8] proved that there does not exist any hypersurface in $\mathbb{C}P^n$ with *recurrent* structure Jacobi operator.

Now, let us consider the notion of $\mathfrak{D}$-*recurrent* structure Jacobi operator R_ξ for a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ defined by $(\nabla_X R_\xi)Y = \omega(X)R_\xi Y$ for any $X \in \mathfrak{D}$ and $Y \in TM$, where ω denotes an 1-form defined on M. Usually, this notion is weaker than *recurrent* structure Jacobi operator. Then in this paper we want to give a non-existence theorem for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}$-*recurrent* structure Jacobi operator as follows:

Main Theorem. There does not exist any connected Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m\geq 3$, with $\mathfrak{D}$-*recurrent* structure Jacobi operator if the distribution $\mathfrak{D}$ or $\mathfrak{D}^\perp$-component of the Reeb vector field is invariant under the shape operator.

As a consequence, we also obtain the following:

Corollary 1. *There does not exist any Hopf hypersurface in* $G_2(\mathbb{C}^{m+2})$*,* $m\geq 3$*, with recurrent structure Jacobi operator if the distribution* $\mathfrak{D}$ *or* $\mathfrak{D}^\perp$*-component of the Reeb vector field is invariant under the shape operator.*

In [6], Jeong, Pérez and Suh proved the following:

Corollary 2. *There does not exist any Hopf hypersurface in* $G_2(\mathbb{C}^{m+2})$*,* $m\geq 3$*, with parallel structure Jacobi operator if the distribution* $\mathfrak{D}$ *or* $\mathfrak{D}^\perp$*-component of the Reeb vector field is invariant under the shape operator.*

2 Riemannian Geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [2] and [3]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2)\times U(m)) \subset G$.

Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K. We get many properties of $G_2(\mathbb{C}^{m+2})$ by investigating the homogeneous space G/K. The real dimension of G/K is $4m$ and it is a compact irreducible manifold. Moreover, we equip it with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. In this paper, we will assume $m \geq 3$.

Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$ which induces Riemannian metric. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight.

As $G_2(\mathbb{C}^{m+2})$ is irreducible, the Lie algebra $\mathfrak{k}$ has the unique direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ denotes the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_ν is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_\nu = J_\nu J$, and JJ_ν is a symmetric endomorphism with $(JJ_\nu)^2 = I$ and $\operatorname{tr}(JJ_\nu) = 0$ for $\nu = 1, 2, 3$.

A canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu$, where the index ν is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\bar{\nabla}_X J_\nu = q_{\nu+2}(X) J_{\nu+1} - q_{\nu+1}(X) J_{\nu+2} \tag{1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}
\bar{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\
& - g(JX,Z)JY - 2g(JX,Y)JZ \\
& + \sum_{\nu=1}^{3} \Big\{ g(J_\nu Y, Z) J_\nu X - g(J_\nu X, Z) J_\nu Y - 2g(J_\nu X, Y) J_\nu Z \Big\} \\
& + \sum_{\nu=1}^{3} \Big\{ g(J_\nu JY, Z) J_\nu JX - g(J_\nu JX, Z) J_\nu JY \Big\},
\end{aligned} \tag{2}$$

where $\{J_1, J_2, J_3\}$ denotes a canonical local basis of $\mathfrak{J}$.

3 Some Fundamental Formulas for Real Hypersurfaces in $G_2(\mathbb{C}^{m+2})$

Now in this section we want to derive the structure Jacobi operator from the curvature tensor of a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ given in (2) and the equation of Gauss. Moreover, in this section we give some basic formulae for a real hypersurface in $G_2(\mathbb{C}^{m+2})$.

Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, that is, a submanifold of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal vector field of M and A the shape operator of M with respect to N. The Kähler structure J of $G_2(\mathbb{C}^{m+2})$ induces on M an almost contact metric structure (ϕ, ξ, η, g). Furthermore, let $J_1,\ J_2,\ J_3$ be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces an almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M.

Now let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N \tag{3}$$

for any tangent vector field X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$. Then the following identities can be proved in a straightforward method and will be used frequently in subsequent calculations:

$$\begin{aligned} &\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2}, \\ &\phi\xi_\nu = \phi_\nu\xi, \quad \eta_\nu(\phi X) = \eta(\phi_\nu X), \\ &\phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu, \\ &\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}. \end{aligned} \tag{4}$$

From this and the formulas (1) and (4) we have that

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X\xi = \phi AX, \tag{5}$$

$$\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{6}$$

$$\begin{aligned} (\nabla_X\phi_\nu)Y = &-q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX \\ &- g(AX, Y)\xi_\nu. \end{aligned} \tag{7}$$

Moreover, from $JJ_\nu = J_\nu J$, $\nu = 1, 2, 3$, it follows that

$$\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu. \tag{8}$$

Then from (2) and the above formulas, the equation of Gauss is given by

$$
\begin{aligned}
R(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y \\
& + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z \\
& + \sum_{\nu=1}^{3}\Big\{g(\phi_\nu Y,Z)\phi_\nu X - g(\phi_\nu X,Z)\phi_\nu Y - 2g(\phi_\nu X,Y)\phi_\nu Z\Big\} \\
& + \sum_{\nu=1}^{3}\Big\{g(\phi_\nu\phi Y,Z)\phi_\nu\phi X - g(\phi_\nu\phi X,Z)\phi_\nu\phi Y\Big\} \\
& - \sum_{\nu=1}^{3}\Big\{\eta(Y)\eta_\nu(Z)\phi_\nu\phi X - \eta(X)\eta_\nu(Z)\phi_\nu\phi Y\Big\} \\
& - \sum_{\nu=1}^{3}\Big\{\eta(X)g(\phi_\nu\phi Y,Z) - \eta(Y)g(\phi_\nu\phi X,Z)\Big\}\xi_\nu \\
& + g(AY,Z)AX - g(AX,Z)AY.
\end{aligned}
\tag{9}
$$

On the other hand, we introduce a lemma due to Berndt and Suh [3] as follows:

Lemma 1. *If M is a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$ with geodesic Reeb flow, then*

$$
\begin{aligned}
&\alpha g((A\phi + \phi A)X,Y) - 2g(A\phi AX,Y) + 2g(\phi X,Y) \\
&= 2\sum_{\nu=1}^{3}\Big\{\eta_\nu(X)\eta_\nu(\phi Y) - \eta_\nu(Y)\eta_\nu(\phi X) - g(\phi_\nu X,Y)\eta_\nu(\xi) \\
&\quad - 2\eta(X)\eta_\nu(\phi Y)\eta_\nu(\xi) + 2\eta(Y)\eta_\nu(\phi X)\eta_\nu(\xi)\Big\}
\end{aligned}
$$

for any tangent vector fields X and Y on M in $G_2(\mathbb{C}^{m+2})$.

4 The $\mathfrak{D}$-recurrent Structure Jacobi Operator

Let us assume that the structure Jacobi operator of a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ is $\mathfrak{D}$-*recurrent*. It is given by $(\nabla_X R_\xi)Y = \omega(X)R_\xi Y$ for any $X \in \mathfrak{D}$, $Y \in TM$ and an 1-form ω on M.

In this section, by using above assumption let us show that the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$ such that $T_x M = \mathfrak{D}\oplus\mathfrak{D}^\perp$ for any point $x \in M$.

If $\omega(X) = 0$, then we get $(\nabla_X R_\xi)Y = 0$. In [6], they gave some non-existence theorems for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with *parallel* structure Jacobi operator, that is, $(\nabla_X R_\xi)Y = 0$ for any tangent vector fields X and Y on M in $G_2(\mathbb{C}^{m+2})$. In this paper, let us assume that $\omega(X) \neq 0$ for any $X \in \mathfrak{D}$. Then it can be easily checked that any Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}$-*recurrent* structure Jacobi operator satisfies the following equation (see [6])

$$\begin{aligned}
&-g(\phi AX, Y)\xi - \eta(Y)\phi AX \\
&\quad - \sum_{\nu=1}^{3} \Big[g(\phi_\nu AX, Y)\xi_\nu - 2\eta(Y)\eta_\nu(\phi AX)\xi_\nu + \eta_\nu(Y)\phi_\nu AX \\
&\quad + 3\big\{ g(\phi_\nu AX, \phi Y)\phi_\nu \xi + \eta(Y)\eta_\nu(AX)\phi_\nu \xi + \eta_\nu(\phi Y)(\phi_\nu \phi AX - \alpha\eta(X)\xi_\nu)\big\} \\
&\quad + 4\eta_\nu(\xi)(\eta_\nu(\phi Y)AX - g(AX, Y)\phi_\nu \xi) + 2\eta_\nu(\phi AX)\phi_\nu \phi Y \Big] \\
&\quad + \eta((\nabla_X A)\xi)AY + \alpha(\nabla_X A)Y - \alpha\eta((\nabla_X A)Y)\xi - \alpha g(AY, \phi AX)\xi \\
&\quad - \alpha\eta(Y)(\nabla_X A)\xi - \alpha\eta(Y)A\phi AX \\
&\quad = \omega(X)\Big[Y - \eta(Y)\xi + \alpha AY - \alpha^2\eta(Y)\xi \\
&\quad - \sum_{\nu=1}^{3} \big\{ \eta_\nu(Y)\xi_\nu - \eta(Y)\eta_\nu(\xi)\xi_\nu + 3\eta_\nu(\phi Y)\phi_\nu \xi + \eta_\nu(\xi)\phi_\nu \phi Y \big\} \Big]
\end{aligned} \tag{10}$$

for any $X \in \mathfrak{D}$, $Y \in TM$.

Let us check whether or not the structure Jacobi operator of real hypersurfaces of Type (A) in Theorem 1 is $\mathfrak{D}$-*recurrent*. In order to do this, we apply Proposition 3 in Berndt and Suh [2].

By putting $Y = \xi$ and applying $\xi = \xi_1$ into (10), we obtain

$$\phi AX + \alpha A\phi AX + \phi_1 AX + \sum_{\nu=1}^{3} \{-\eta_\nu(\phi AX)\xi_\nu + 3\eta_\nu(AX)\phi_\nu \xi\} = 0$$

for any $X \in \mathfrak{D}$. From this, let us consider a unit eigenvector $X \in T_\lambda \subset \mathfrak{D}$. Then we have

$$\lambda\phi X + \alpha\lambda A\phi X + \lambda\phi_1 X + \sum_{\nu=1}^{3} \{-\lambda\eta_\nu(\phi X)\xi_\nu + 3\lambda\eta_\nu(X)\phi_\nu \xi\} = 0.$$

Since $\phi X = \phi_1 X$, we get

$$2\lambda\phi X + \alpha\lambda A\phi X = 0.$$

Since $\phi T_\lambda \subset T_\lambda$ and $AX = \lambda X$, we have $A\phi X = \lambda\phi X$. Thus we get $\lambda(2 + \alpha\lambda)\phi X = 0$. As $\lambda = -\sqrt{2}\tan(\sqrt{2}r)$ with some $r \in (0, \pi/\sqrt{8})$, λ is not zero. Accordingly, we know that

$$\begin{aligned}\alpha\lambda + 2 &= (\sqrt{8}\cot(\sqrt{8}r)(-\sqrt{2}\tan(\sqrt{2}r)) + 2\\ &= 2\tan^2(\sqrt{2}r) > 0.\end{aligned}$$

Thus we have $\phi X = 0$. It gives us a contradiction.

Thus we know that the structure Jacobi operator R_ξ of real hypersurface of Type (A) in $G_2(\mathbb{C}^{m+2})$ is not $\mathfrak{D}$-*recurrent* if ξ belongs to the distribution $\mathfrak{D}^\perp$. If the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$, then there exist no real hypersurfaces of Type (A) in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}$-*recurrent* structure Jacobi operator.

Next, we check for the case $\xi \in \mathfrak{D}$ that the structure Jacobi operator of real hypersurfaces of Type (B) is $\mathfrak{D}$-*recurrent*. In order to do this we use Proposition 2 in Berndt and Suh [2].

By putting $Y = \xi$ and applying $\xi \in \mathfrak{D}$ into (10), we have

$$\phi AX + \alpha A\phi AX + \sum_{\nu=1}^{3}\{-\eta_\nu(\phi AX)\xi_\nu + 3\eta_\nu(AX)\phi_\nu\xi\} = 0.$$

From this, we consider a unit eigenvector $X \in T_\lambda \subset \mathfrak{D}$. Then it follows that

$$\lambda\phi X + \alpha\lambda A\phi X + \sum_{\nu=1}^{3}\{-\lambda\eta_\nu(\phi X)\xi_\nu + 3\lambda\eta_\nu(X)\phi_\nu\xi\} = \lambda\phi X + \alpha\lambda A\phi X = 0.$$

Since $JT_\lambda = T_\mu$ and $X \in T_\lambda$, we know that $\phi X \in T_\mu$. This means that $A\phi X = \mu\phi X$. Naturally we also have

$$\lambda\phi X + \alpha\lambda\mu\phi X = 0.$$

Since $\lambda = \cot(r)$ with some $r \in (0, \pi/4)$, λ is non-zero,

$$1 + \alpha\mu = 1 + (-2\tan(2r))(-\tan(r)) = 1 + 4\frac{\sin^2(r)}{\cos^2(r)} > 1.$$

This implies that $\phi X = 0$, which gives us a contradiction. Accordingly, the structure Jacobi operator R_ξ of real hypersurfaces of Type (B) in $G_2(\mathbb{C}^{m+2})$ can not be $\mathfrak{D}$-*recurrent* if the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$.

5 Proof of the Main Theorem

In this section, in order to give our complete proof of Main Theorem in the introduction, we need the following lemmas:

Lemma 2. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$. If $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ for some unit $X_0 \in \mathfrak{D}$, $\eta(X_0)\eta(\xi_1) \neq 0$ and the distribution $\mathfrak{D}$ or $\mathfrak{D}^{\perp}$-component of the Reeb vector ξ is invariant under the shape operator A of M, then it becomes $AX_0 = \alpha X_0$, $A\xi_1 = \alpha\xi_1$, where the smooth function α denotes $\eta(A\xi)$.*

Proof. Since M is Hopf, that is $A\xi = \alpha\xi$, we have

$$A(\eta(X_0)X_0 + \eta(\xi_1)\xi_1) = \alpha(\eta(X_0)X_0 + \eta(\xi_1)\xi_1). \tag{11}$$

From this, taking the inner product with X_0, we have

$$\eta(X_0)g(AX_0, X_0) + \eta(\xi_1)g(A\xi_1, X_0) = \alpha\eta(X_0).$$

Since $AX_0 = g(AX_0, X_0)X_0$, we have $g(A\xi_1, X_0) = 0$. So we obtain

$$\eta(X_0)g(AX_0, X_0) = \alpha\eta(X_0).$$

Since $\eta(X_0) \neq 0$, we get $\alpha = g(AX_0, X_0)$. By applying the equation above to (11), we have $A\xi_1 = \alpha\xi_1$. □

Lemma 3. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}$-recurrent structure Jacobi operator. If the distribution $\mathfrak{D}$ or $\mathfrak{D}^{\perp}$-component of the Reeb vector field ξ is invariant under the shape operator A of M, then the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^{\perp}$.*

Proof. When the function $\alpha = g(A\xi, \xi)$ identically vanishes, it can be seen directly from Pérez and Suh [9]. So we consider the case that the function α is non-vanishing. Putting $Y = \xi$ in (10), we have

$$\begin{aligned} &\phi AX + \alpha A\phi AX \\ &\quad + \sum_{\nu=1}^{3}\{-\eta_\nu(\phi AX)\xi_\nu + \eta_\nu(\xi)\phi_\nu AX + 3\eta_\nu(AX)\phi_\nu\xi - 4\alpha\eta_\nu(\xi)\eta(X)\phi_\nu\xi\} = 0. \end{aligned} \tag{12}$$

We assume that

$$\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1 \tag{13}$$

for some unit $X_0 \in \mathfrak{D}$ and $\eta(X_0)\eta(\xi_1) \neq 0$. Then it suffices to show that $\eta(X_0) = 0$ or $\eta(\xi_1) = 0$. By applying (13) to (12), we get

$$\phi AX + \alpha A\phi AX + \eta_1(\xi)\phi_1 AX - 4\alpha\eta_1(\xi)\eta(X)\phi_1\xi + \sum_{\nu=1}^{3}\{-\eta_\nu(\phi AX)\xi_\nu + 3\eta_\nu(AX)\phi_\nu\xi\} = 0.$$

From this, by putting $X = X_0$ and $\phi_1\xi = \eta(X_0)\phi_1 X_0$, we have

$$\phi AX_0 + \alpha A\phi AX_0 + \eta_1(\xi)\phi_1 AX_0 - 4\alpha\eta_1(\xi)\eta^2(X_0)\phi_1 X_0 + \sum_{\nu=1}^{3}\{-\eta_\nu(\phi AX_0)\xi_\nu + 3\eta_\nu(AX_0)\phi_\nu\xi\} = 0. \tag{14}$$

By assumption and Lemma 2, we know that

$$AX_0 = \alpha X_0. \tag{15}$$

By applying (15) to (14) and using $\eta_\nu(\phi X_0) = 0$, we have

$$\alpha\phi X_0 + \alpha^2 A\phi X_0 + \alpha\eta_1(\xi)\phi_1 X_0 - 4\alpha\eta_1(\xi)\eta^2(X_0)\phi_1 X_0 = 0. \tag{16}$$

We know the fact

$$\phi X_0 = -\eta(\xi_1)\phi_1 X_0, \tag{17}$$

which is induced by $\phi\xi = 0$. By applying (17) into (16), we obtain

$$-\alpha\eta(\xi_1)\phi_1 X_0 - \alpha^2\eta(\xi_1)A\phi_1 X_0 + \alpha\eta(\xi_1)\phi_1 X_0 - 4\alpha\eta_1(\xi)\eta^2(X_0)\phi_1 X_0 = 0. \tag{18}$$

On the other hand, by Lemma 1, we have

$$\alpha A\phi X + \alpha\phi AX - 2A\phi AX + 2\phi X \tag{19}$$
$$= 2\sum_{\nu=1}^{3}\{-\eta_\nu(X)\phi\xi_\nu - \eta_\nu(\phi X)\xi_\nu - \eta_\nu(\xi)\phi_\nu X + 2\eta(X)\eta_\nu(\xi)\phi\xi_\nu + 2\eta_\nu(\phi X)\eta_\nu(\xi)\xi\}.$$

Substituting (18) into above formula and replacing X by X_0, we have

$$\alpha A\phi X_0 + \alpha\phi AX_0 - 2A\phi AX_0 = -4\eta^2(X_0)\phi X_0.$$

From this, by using $AX_0 = \alpha X_0$, we have

$$A\phi X_0 = \frac{\alpha^2 + 4\eta^2(X_0)}{\alpha}\, \phi X_0.$$

Apply (17) to above equation, we have

$$A\phi_1 X_0 = \frac{\alpha^2 + 4\eta^2(X_0)}{\alpha}\, \phi_1 X_0. \tag{20}$$

From (18) and (20), we get

$$\begin{aligned} &-\alpha^2\eta(\xi_1)\frac{\alpha^2 + 4\eta^2(X_0)}{\alpha}\, \phi_1 X_0 - 4\alpha\eta_1(\xi)\eta^2(X_0)\phi_1 X_0 \\ &= -\alpha\eta(\xi_1)(\alpha^2 + 8\eta^2(X_0))\phi_1 X_0 = 0. \end{aligned}$$

Since $\alpha \neq 0$, $\eta(\xi_1) \neq 0$ and $\alpha^2 + 8\eta^2(X_0) \neq 0$, we get $\phi_1 X_0 = 0$ which makes a contradiction. Accordingly, we get a complete proof of our lemma. □

Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}$-*recurrent* structure Jacobi operator, that is, $(\nabla_X R_\xi)Y = \omega(X)R_\xi Y$ for any $X \in \mathfrak{D}$, $Y \in TM$ and an 1-form ω to TM. By virtue of Lemma 3, we consider the following cases. First we consider the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$.

Lemma 4. *Let M be a Hopf hypersurface of $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}$-recurrent structure Jacobi operator. If the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$, then $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.*

Proof. We may put $\xi = \xi_1$, because $\xi \in \mathfrak{D}^\perp$. By putting $Y = \xi$ into (3.1), we have $(\nabla_X R_\xi)\xi = 0$. So we obtain

$$\begin{aligned} &\phi AX + \alpha A\phi AX \\ &\quad + \sum_{\nu=1}^{3}\{-\eta_\nu(\phi AX)\xi_\nu + \eta_\nu(\xi)\phi_\nu AX + 3\eta_\nu(AX)\phi_\nu\xi - 4\alpha\eta_\nu(\xi)\eta(X)\phi_\nu\xi\} = 0. \end{aligned}$$

On the other hand, in a paper due to Jeong, Pérez and Suh [6], the authors have classified all real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel structure Jacobi operator. Thus by using a similar method given in [6, p.182–183], we can prove that $g(AX, \xi_\nu) = 0$ for $\nu = 1, 2, 3$ and any $X \in \mathfrak{D}$, that is, $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$. This gives a complete proof of our lemma. □

Summing up Lemmas 2, 3, 4 and using Theorems 1 and 2 in the introduction, we know that any connected Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}$-*recurrent* structure Jacobi operator is locally congruent to one either of Type (A) or of

Type (B). But, by using Propositions in [2], it can be easily checked that the structure Jacobi operator R_ξ of any real hypersurfaces of Type (A) or of Type (B) is not $\mathfrak{D}$-*recurrent*. So we complete the proof of our Main Theorem in the introduction.

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The Warped Product Approach to GMGHS Spacetime

Jaedong Choi

Abstract In the framework of Lorentzian multiply warped products we study the Gibbons-Maeda-Garfinkle-Horowitz-Strominger (GMGHS) spacetime and the nonsmooth geodesic motion near hypersurfaces in the interior of the event horizon. We also investigate the geodesics of the GMGHS spacetime with C^0-warping functions.

1 Introduction

The concept of a warped product manifold was introduced to provide a class of complete Riemannian manifolds with negative curvature everywhere [1], and was developed to point out that several of the well-known exact solutions to Einstein field equations are pseudo-Riemannian warped products [2]. Furthermore, certain causal and completeness properties of a spacetime can be determined by the presence of a warped product structure [3], and a general theory Lorentzian multiply warped were applied to discuss the Schwarzschild spacetime in the interior of the event horizon [4–7]. The role of warped products in the study of exact solutions to Einstein's equations are now firmly established to generate interest in other areas of geometry.

On the other hand, there were enormous interests in the spherically symmetric static charged black holes in the four-dimensional heterotic string theory. Gad [8] also studied geodesic and geodesic deviation of the magnetically charged GMGHS black hole solution. By turning antisymmetric tensor gauge fields off, the static charged black hole solution was found by Gibbons, Maeda [9], and by Garfinkle [10] independently. Recently, null geodesics and hidden symmetries in the Sen black hole was investigated by Hioki and Miyamoto [11], which is reduced to the magnetically charged GMGHS black hole in the nonrotating limit. Very recently, Fernando [12] fully investigated null geodesic motions of the same solution both in the Einstein and string frame. However, the studies of these null geodesics solutions are mainly based on the exterior region of the event horizon.

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Moreover, the Lorentzian manifolds with non-smooth metric tensors have been extensively discussed from various view points [4, 13–16]. In a spacetime where the metric tensor is continuous but has a jump in its first and second derivatives across a submanifold in an admissible coordinate system, one can have a curvature tensor containing a Dirac delta function [17]. The support of this distribution may be of three, two, or one dimension or may even consist of a single event. Lichnerowicz's formalism [14] for dealing with such tensors is modified so that one can obtain the Riemannian curvature tensor and Ricci curvature tensor defined in the sense of distributions.

Donald Marolf and Amos Ori [18] argued that late infall-time observers encounter a null shock wave at the location of the would-be outgoing inner horizon. In particular, for spherically symmetric black hole spacetimes we demonstrate that freely-falling observers experience a metric discontinuity across this shock, that is, a gravitational shock-wave. In a Lorentzian multiply warped product spacetime, by exchanging timelike and spacelike coordinates, we are interested in geodesic motion inside of the event horizon.

In this paper we study the magnetically charged GMGHS interior spacetime of the framework of Lorentzian multiply warped products and investigate the geodesic motion near hypersurfaces of this spacetime with C^0-warping functions. We shall use geometrized units, i.e., $G = c = 1$, for notational convenience.

2 Magnetically Charged GMGHS Black Hole in the Framework of Warped Products

Let us now consider the magnetically charged GMGHS black hole in terms of the string metric. The Einstein metric does not change when we go from electric to magnetic charged black hole, but since ϕ changes sign, the string metric does change. We get the GMGHS solution of the Einstein field equations represents the geometry exterior to a spherically symmetric static charged black hole.

In the Schwarzschild coordinates, the line element for the magnetically charged GMGHS black hole metric in the exterior region $r > 2m$ has the following form

$$ds^2 = -\frac{\left(1 - \frac{2m}{r}\right)}{\left(1 - \frac{Q^2}{mr}\right)} dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r}\right)\left(1 - \frac{Q^2}{mr}\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where

$$\frac{Q^2}{m} < r < 2m \quad (2)$$

Here, the parameters m and Q are mass and charge respectively. Note that the metric in the t-r plane is identical to the Schwarzschild case. As in the Schwarzschild spacetime, the magnetically charged GMGHS has an event horizon at $r = 2m$.

On the other hand, the line element for the magnetically charged GMGHS metric for the interior region $r < 2m$ can be described by

$$ds^2 = -\frac{dr^2}{\left(\frac{2m}{r}-1\right)\left(1-\frac{Q^2}{mr}\right)} + \frac{\left(\frac{2m}{r}-1\right)}{\left(1-\frac{Q^2}{mr}\right)}dt^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3}$$

where r and t are now new temporal and spacial variables, respectively. A multiply warped product manifold, denoted by $M = (B \times F_1 \times \ldots \times F_n, g)$, consists of the Riemannian base manifold (B, g_B) and fibers (F_i, g_i) $(i = 1, \ldots, n)$ associated with the Lorentzian metric [4].

In particular, for the specific case of $(B = R,\ g_B = -d\mu^2)$, the magnetically charged GMGHS metric (3) can be rewritten as a multiply warped product $(a, b) \times_{f_1} R \times_{f_2} S^2$ by making use of a function

$$d\mu^2 = \frac{dr^2}{\left(\frac{2m}{r}-1\right)\left(1-\frac{Q^2}{mr}\right)} \tag{4}$$

$$\begin{aligned}\mu &= -\sqrt{(2m-r)(mr-Q^2)} \\ &\quad -\left(m+\frac{Q^2}{2m}\right)\tan^{-1}\left(\frac{2m^2+Q^2-2mr}{2\sqrt{m(2m-r)(mr-Q^2)}}\right) \\ &= F(r)\end{aligned} \tag{5}$$

Since $\mu \to \frac{\pi(2m^2+Q^2)}{4m}$ as $r \to \frac{Q^2}{m}$, choose integration constant $C \equiv \frac{\pi(2m^2+Q^2)}{4m}$. Therefore we have $\mu \to \frac{\pi(2m^2+Q^2)}{4m}$ as $r \to \frac{Q^2}{m}$ and

$$\begin{aligned}\mu &= -\sqrt{(2m-r)(mr-Q^2)} \\ &\quad -\left(m+\frac{Q^2}{2m}\right)\tan^{-1}\left(\frac{2m^2+Q^2-2mr}{2\sqrt{m(2m-r)(mr-Q^2)}}\right) + \frac{\pi(2m^2+Q^2)}{4m} \\ &= F(r)\end{aligned} \tag{6}$$

Notice $\frac{dF}{dr} > 0$ implies F^{-1} is a well-defined function as well as warping functions given by f_1 and f_2 as follows

$$f_1(\mu) = \left(\frac{\frac{2m}{F^{-1}(\mu)}-1}{1-\frac{Q^2}{F^{-1}(\mu)m}}\right)^{1/2},\ f_2(\mu) = F^{-1}(\mu) \tag{7}$$

By using (7) we rewrite (1) as

$$ds^2 = \frac{\left(\frac{2m}{r} - 1\right)}{\left(1 - \frac{Q^2}{mr}\right)} dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r}\right)\left(1 - \frac{Q^2}{mr}\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$
$$= -d\mu^2 + f_1^2(\mu)dt^2 + f_2^2(\mu)(d\theta^2 + \sin^2\theta d\phi^2) \quad (8)$$

The lapse function (4) is well defined in the region $r < r_H(= 2m)$ to rewrite it as a multiply warped product spacetime by defining a new coordinate μ as follows

$$\mu = \int_0^r \frac{dx\ x^{1/2}}{(r_H - x)^{1/2}} = F(r). \quad (9)$$

Setting the integration constant zero as $r \to 0$, we have

$$\mu = 2m \cos^{-1}\left(\frac{r_H - r}{r_H}\right) - [(r_H - r)r]^{1/2}, \quad (10)$$

which has boundary conditions as follows

$$\lim_{r \to r_H} F(r) = (2n - 1)m\pi,\ \lim_{r \to 0} F(r) = 0, \quad (11)$$

for a positive integer n, and $dr/d\mu > 0$ implies that $F^{-1}(\mu)$ is a well-defined function. We can thus rewrite the GMGHS metric (3) with the lapse function (4)

$$ds^2 = -d\mu^2 + \left(\frac{2m}{F^{-1}(\mu)} - 1\right)dr^2 + \left(F^{-1}(\mu)^2 - \alpha F^{-1}(\mu)\right)d\Omega^2$$
$$= -d\mu^2 + f_1(\mu)^2 dr^2 + f_2(\mu)^2 d\Omega^2 \quad (12)$$

by using the warping functions (7).

Thus, in the case of the interior region $r < 2m$, the GMGHS metric has been rewritten as a multiply warped product spacetime having the warping functions in terms of f_1 and f_2. Moreover, we can write down the Ricci curvature on the multiply warped product as

$$
\begin{aligned}
R_{\mu\mu} &= -\frac{f_1''}{f_1} - \frac{2f_2''}{f_2}, \\
R_{tt} &= f_1 f_1'' + \frac{2 f_1 f_1' f_2'}{f_2}, \\
R_{\theta\theta} &= \frac{f_1' f_2 f_2'}{f_1} + f_2'^2 + f_2 f_2'' + 1, \\
R_{\phi\phi} &= \left(\frac{f_1' f_2 f_2'}{f_1} + f_2'^2 + f_2 f_2'' + 1 \right) \sin^2\theta, \\
R_{mn} &= 0, \text{ for } m \neq n,
\end{aligned}
\tag{13}
$$

which have the same form as the Ricci curvature of the multiply warped interior Schwarzschild metric [4]. The only difference with the Schwarzschild metric is the α term in the warping function f_2 in (7).

3 Geodesic Motion Near a Hypersurface with C^0-warping Function

A full understanding of the GMGHS spacetime having an event horizon with an essential singularity at the center and a surface singularity at $r = \alpha$, etc, was recently achieved only comparatively. Also, since the geodesics in the GMGHS spacetime illuminate some basic aspects of universe within the event horizon, we shall include an account of them. In this section, we briefly revisit the GMGHS interior spacetime with two warping functions at a singular point $r = \alpha$ in the hypersurfaces, and we are interested in investigating the geodesic curves of a static spherically symmetric GMGHS spacetime near hypersurfaces.

In local coordinates $\{x^i\}$ the line element corresponding to this metric (3) will be denoted by

$$
dS^2 = g_{ij} dx^i dx^j. \tag{14}
$$

Consider the equations of geodesics in the GMGHS spacetime with affine parameter λ given by

$$
\frac{dx^i}{d\lambda^2} + \Gamma^i_{jk} \frac{dx^j}{d\lambda} \frac{dx^k}{d\lambda} = 0. \tag{15}
$$

Let a geodesic γ be given by $\gamma(\tau) = \Big(\mu(\tau), r(\tau), \theta(\tau), \phi(\tau)\Big)$ of the interior GMGHS spacetime in the case of $r < 2m$ from (3), then the orbits of the geodesics equation are given as follows

$$\frac{d^2\mu}{d\tau^2} + f_1\frac{df_1}{d\mu}\left(\frac{dr}{d\tau}\right)^2 + f_2\frac{df_2}{d\mu}\left(\frac{d\theta}{d\tau}\right)^2 + f_2\frac{df_2}{d\mu}\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2 = 0, \quad (16)$$

$$\frac{d^2r}{d\tau^2} + \frac{2}{f_1}\frac{df_1}{d\tau}\frac{dr}{d\tau} = 0, \quad (17)$$

$$\frac{d^2\theta}{d\tau^2} + \frac{2}{f_2}\frac{df_2}{d\tau}\frac{d\theta}{d\tau} - \sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 = 0, \quad (18)$$

$$\frac{d^2\phi}{d\tau^2} + \frac{2}{f_2}\frac{df_2}{d\tau}\frac{d\phi}{d\tau} + 2\cot\theta\frac{d\theta}{d\tau}\frac{d\phi}{d\tau} = 0 \quad (19)$$

with a following constraint along the geodesic

$$-\left(\frac{d\mu}{d\tau}\right)^2 + f_1^2\left(\frac{dr}{d\tau}\right)^2 + f_2^2\left(\frac{d\theta}{d\tau}\right)^2 + f_2^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2 = \varepsilon. \quad (20)$$

Note that a timelike (nulllike) geodesic is taken as $\varepsilon = -1$ ($\varepsilon = 0$).

Hereafter, without loss of generality, suppose the geodesic

$$\gamma(\tau_0) = \Big(\mu(\tau_0), r(\tau_0), \theta(\tau_0), \phi(\tau_0)\Big) \quad (21)$$

for some τ_0 and the equatorial plane of $\theta = \frac{\pi}{2}$, thus $\frac{d\theta}{d\tau} = 0$. Then, the geodesic equations are reduced to

$$\frac{d^2\mu}{d\tau^2} + f_1\frac{df_1}{d\mu}\left(\frac{dr}{d\tau}\right)^2 + f_2\frac{df_2}{d\mu}\left(\frac{d\phi}{d\tau}\right)^2 = 0, \quad (22)$$

$$\frac{d^2r}{d\tau^2} + \frac{2}{f_1}\frac{df_1}{d\tau}\frac{dr}{d\tau} = 0, \quad (23)$$

$$\frac{d^2\theta}{d\tau^2} = 0, \quad (24)$$

$$\frac{d^2\phi}{d\tau^2} + \frac{2}{f_2}\frac{df_2}{d\tau}\frac{d\phi}{d\tau} = 0 \quad (25)$$

with a constraint

$$-\left(\frac{d\mu}{d\tau}\right)^2 + f_1^2\left(\frac{dr}{d\tau}\right)^2 + f_2^2\left(\frac{d\phi}{d\tau}\right)^2 = \varepsilon. \quad (26)$$

These geodesic equations can be simplified as follows

$$\frac{dr}{d\tau} = \frac{c_1}{f_1^2}, \quad (27)$$

$$\frac{d\phi}{d\tau} = \frac{c_2}{f_2^2}, \tag{28}$$

$$\frac{d^2\theta}{d\tau^2} = 0 \tag{29}$$

with a constraint

$$-\left(\frac{d\mu}{d\tau}\right)^2 + \frac{c_1^2}{f_1^2} + \frac{c_2^2}{f_2^2} = \varepsilon. \tag{30}$$

The constant c_1 represents the total energy per unit rest mass of a particle as measured by a static observer [8,19,20], and c_2 represents the angular momentum in the GMGHS spacetimes. The equations for r and ϕ are obtained from (23) and (25), respectively. Making use of these r, ϕ equations, we can show that (22) is exactly the same as (26) when we take the integration constant as $-\varepsilon/2$.

First of all, we consider the null geodesics in the r-direction, which is defined by the hypersurface Σ_r by taking $d\theta = d\phi = 0$. Then, we have $c_2 = 0$ in (28). Equations (27) and (30) are now reduced to give

$$\frac{d\mu}{dr} = f_1(\mu), \tag{31}$$

Let us consider the geodesic in the ϕ-direction, which lies on the hypersurface Σ_ϕ at $\theta = \frac{\pi}{2}$ with $dr = 0$. Then, we have $c_1 = 0$ in (27). Equations (28) and (30) are reduced to give

$$\frac{d\mu}{d\phi} = f_2(\mu), \tag{32}$$

where $f_2(\mu)$ is given by (7).

Finally, let us find the geodesic in the μ-direction, which is defined by the hypersurface Σ_μ, eliminating μ in (31) and (32), leading to

$$\frac{d\phi}{dr} = \frac{1}{r}\sqrt{\frac{2m-r}{r-\alpha}}. \tag{33}$$

This has a solution as

$$\phi(r) = \sqrt{\frac{2m}{\alpha}}\cot^{-1}\left(\frac{2\sqrt{2m\alpha(2m-r)(r-\alpha)}}{2mr-4m\alpha+r\alpha}\right) + \tan^{-1}\left(\frac{2(m-r)+\alpha}{2\sqrt{(2m-r)(r-\alpha)}}\right). \tag{34}$$

Now, we reconsider the GMGHS spacetime $M = (a,b) \times_{f_1} R \times_{f_2} S^2$ in the framework of the Lorentzian multiply warped products. Let subspace Σ_{x^i} of the GMGHS spacetime M be a regularly embedded x^i-directed hypersurface having coordinate neighborhood $U(p)$ with local coordinates (x^1, x^2, x^3, x^4) such that $\Sigma_{x^i} \cap U = \{(x^1, x^2, x^3, x^4) \in U \mid x^i = p\}$ for all $p \in \Sigma_{x^i}$. For convenience, we say that such a neighborhood U is partitioned by Σ_{x^i}.

A C^0-Lorentzian metric on M is a nondegenerate (0,2) tensor of Lorentzian signature such that:

(1) $g \in C^0$ on $\Sigma_{x^i} \cap U$
(2) $g \in C^\infty$ on $\Sigma_{x^i} \cap U^c$
(3) For all $p \in \Sigma_{x^i} \cap U$, and $U(p)$ partitioned by Σ_{x^i}, $g|_{U_p^+}$ and $g|_{U_p^-}$ has smooth extensions to U. We call Σ_{x^i} a C^0-singular hypersurface of (M, g).

Clearly f is continuous at p in the usual sense if and only if it is both continuous from the right and continuous from left at p:

$$\lim_{\eta\to 0} f(p+\eta) = \lim_{\eta\to 0} f(p-\eta) = f(p) \tag{35}$$

and

$$f(p^+) = \lim_{\eta\to 0} f(p+\eta), \qquad \eta > 0 \tag{36}$$

If, in addition, this limit is equal to the value $f(p)$, which the function f actually assumes at the point p, then f is said to be *continuous from the right at* p. Similarly any point to the left of p can be expressed as $t = p - \eta$ where η is positive. The limit, if it exists, of $f(p-\eta)$ as η tend to 0 through positive value $f(p^-)$ is called *the left-hand limit of* f *as* t *tend to* p^- and usually written as $f(p^-)$. If this limit does exist and is equal to the value $f(p)$ then f is said to be continuous from the left at p.

If f is continuous at a point p, then it must certainly be the case that $f(p)$ is defined. In general a function f is said to have a *removable discontinuity at* p if both the right-hand and left-hand limits of f at p exist and are equal but $f(p)$ is undefined or else has a value different from $f(p^+)$ and $f(p^-)$. The discontinuity disappears on suitably defining (or re-defining) $f(p)$. If the one-sided limits $f(p^+)$ and $f(p^-)$ both exist but are unequal in value, then f is said to have a *jump discontinuity* at p. The number $f(p^+) - f(p^-)$ is then called the *jump* of the function at p.

We consider the geodesic in the μ-direction which is C^0 on a hypersurface $\Sigma_{x^i} \cap U$. If we only have $f_i \in C^0(\Sigma_{x^1} \cap U)$, then f_i' is discontinuous at p, In this case we rewrite $f_i'(t)$ by using the unit function U.

$$\frac{df_i}{d\mu} = f_i'(\mu(\tau)) = f_i'^{+} U(\mu(\tau) - p) + f_i'^{-} U(p - \mu(\tau)) \tag{37}$$

Then we define one sided limits

$$f_i^+(\mu(\tau)) = \lim_{x_m \to \mu(\tau)} f_i(x_m) \tag{38}$$

for $f_i : \Sigma_{x^1} \cap U^c \to R^+$ at $\mu(\tau) \in \Sigma_{x^1}$ and $x_m \in U^+$, similarly

$$f_i^-(\mu(\tau)) = \lim_{x_m \to \mu(\tau)} f_i(x_m) \tag{39}$$

for $x_m \in U^-$,

Now assume U to be a local coordinate neighborhood partitioned by Σ_{x^1} with coordinates $(\mu(\tau_0), r(\tau_0), \theta(\tau_0), \phi(\tau_0))$. Then the orbits of the geodesics equations (22), (23), (25) are reduced to

$$\begin{aligned}\frac{d^2\mu}{d\tau^2} &+ f_1\Big[f_1'^+ U(\mu(\tau) - p) + f_1'^- U(p - \mu(\tau))\Big]\left(\frac{dr}{d\tau}\right)^2 \\ &+ f_2\Big[f_2'^+ u(\mu(\tau) - p) + f_2'^- U(p - \mu(\tau))\Big]\left(\frac{d\phi}{d\tau}\right)^2 = 0, \end{aligned} \tag{40}$$

$$\frac{d^2 r}{d\tau^2} + \frac{2}{f_1}\Big[f_1'^+ U(\mu(\tau) - p) + f_1'^- U(p - \mu(\tau))\Big]\frac{d\mu}{d\tau}\frac{dr}{d\tau} = 0, \tag{41}$$

$$\frac{d^2\theta}{d\tau^2} = 0, \tag{42}$$

$$\frac{d^2\phi}{d\tau^2} + \frac{2}{f_2}\Big[f_2'^+ U(\mu(\tau) - p) + f_2'^- U(p - \mu(\tau))\Big]\frac{d\mu}{d\tau}\frac{d\phi}{d\tau} = 0 \tag{43}$$

Conclusions

In this paper, we have studied the GMGHS interior spacetime associated with a multiply warped product manifold. In the multiply warped product manifold, the GMGHS spacetime has been characterized by two warping functions $f_1(\mu)$ and $f_2(\mu)$, compared with the Schwarzschild spacetime which has the only warping function $f_1(\mu)$.We have also investigated the nonsmooth geodesic motion near hypersurfaces in the interior of the event horizon. We also investigated the geodesic of the GMGHS spacetime with C^0-warping functions.

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Reeb Flow Invariant Ricci Tensors

Jong Taek Cho

Abstract We study symmetries along the Reeb flow on almost contact three-manifolds.

1 Introduction

Let M be a smooth manifold of odd dimension $m = 2n + 1$. Then M is said to be an *almost contact manifold* if its structure group $GL_m\mathbb{R}$ of the linear frame bundle is reducible to $U(n) \times \{1\}$. This is equivalent to the existence of an endomorphism field φ, a vector field ξ and a 1-form η satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,$$

where I denotes the identity transformation. We remark that a compact orientable manifold of odd dimension has a non-vanishing vector field. Moreover, since $U(n) \times \{1\} \subset SO(2n + 1)$, M admits a Riemannian metric g satisfying

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any vector fields X, Y on M. g is called a compatible Riemannian metric to a given almost contact manifold. Such (η, φ, ξ, g) is called an *almost contact metric structure*. The *fundamental 2-form* Φ is defined by $\Phi(X, Y) = g(X, \varphi Y)$. If M satisfies in addition $d\eta = \Phi$, then M is called a *contact metric manifold*, where d is the exterior differential operator.

In case that their automorphism groups have the maximum dimension $(n + 1)^2$, they are classified by the following three classes [19]: (1) Sasakian space forms, that is, complete, simply connected, normal contact Riemannian manifolds of constant holomorphic sectional curvature; (2) $\mathbb{R} \times F(k)$ or $\mathbb{S} \times F(k)$, the product spaces of a line $\mathbb{R}$ or a circle $\mathbb{S}$ and a complex space form $F(k)$; (3) warped product

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spaces $\mathbb{R} \times_\rho \mathbb{C}E^n$ of a real line and a complex Euclidean space with a warping function ρ. Each class has been intensively developed. For the class (1), the contact metric structure including Sasakian structure has been investigated by many authors (cf. [1]). The geometric property of (2) is represented as the so-called *cosymplectic structure*. The trivial products of a real line or a circle and a Kählerian manifold admit such a structure. Extending the model (3), K. Kenmotsu [11] introduced another class, which is expressed (locally) by a warped product space of an open interval and a Kählerian manifold. We call such a manifold M a *Kenmotsu manifold* and its almost contact metric structure is called a *Kenmotsu structure*. It is worth noting that every orientable 2-surface N admits a Kählerian metric. Taking a product metric or a warped product metric with a warping function $\rho = c \exp t$ (c a positive constant) on the product space $\mathbb{R} \times N$, then we have a cosymplectic or a Kenmotsu 3-manifold, respectively. A Sasakian, a cosymplectic, or a Kenmotsu manifold holds the *CR-integrability*, and moreover the *normality*. Without normality we have much broader classes. An almost contact metric manifold $(M; \eta, \varphi, \xi, g)$ is said to be almost cosymplectic if $d\eta = 0$ and $d\Phi = 0$. Such a class was introduced by S. I. Goldberg and K. Yano [9]. The trivial products of an almost Kählerian manifold and a real line or a circle are the simplest examples of such manifolds. Recently, D. Perrone [17] classified all homogeneous almost cosymplectic three manifolds. An almost contact metric manifold M is said to be almost Kenmotsu if $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. The warped products of an almost Kählerian manifold and a real line give examples of almost Kenmotsu manifolds. For further properties and examples of almost Kenmotsu manifolds, we refer to [8].

The class of almost contact manifolds with which we concerned holds the properties $\pounds_\xi \xi = \pounds_\xi \eta = 0$, that is, the Reeb vector field and its associated 1-form are invariant along the Reeb flow. In former works, we studied such a class of almost contact metric three-manifolds whose Ricci operator S is invariant along the Reeb flow ξ. That is, we proved the following theorems.

Theorem A ([3]). *The Ricci operator S of a contact metric three-manifold M is invariant along the Reeb flow ξ, that is, M satisfies $\pounds_\xi S = 0$ if and only if M is Sasakian or locally isometric to a Lie group* SU(2), SL(2, $\mathbb{R}$), $E(2)$ (*the group of rigid motions of Euclid 2-plane*) *with a left invariant contact Riemannian metric respectively.*

Theorem B ([4]). *An almost cosymplectic three-manifold M satisfies $\pounds_\xi S = 0$ if and only if M is cosymplectic or locally isometric to the group $E(1,1)$ of rigid motions of Minkowski 2-space with a left invariant almost cosymplectic structure.*

It should be remarkable that $E(1,1)$ admits also a left invariant contact metric structure. But, for such a contact metric structure $\pounds_\xi S \neq 0$ (see also [7]).

Theorem C ([7]). *An almost Kenmotsu three-manifold M satisfies $\pounds_\xi S = 0$ if and only if M is locally isometric to either a hyperbolic space $\mathbb{H}^3(-1)$ or a non-unimodular Lie group with a left invariant almost Kenmotsu structure.*

The non-unimodular Lie groups in Theorem C are given explicitly by some solvable Lie groups (cf. [5, 10]). Especially, the product space $\mathbb{R} \times \mathbb{H}^2(-4)$ appears among them. For contact metric three-manifolds or almost cosymplectic three-manifolds, we can find that the commutativity condition $S\varphi = \varphi S$ is equivalent to $\pounds_\xi S = 0$ (cf. [3, 4]). But, for almost Kenmotsu three-manifolds this equivalency does not hold any more. Indeed, only a non-unimodular Lie group with left invariant almost Kenmotsu structure which satisfies $S\varphi = \varphi S$ is $\mathbb{H}^3(-1)$ [5]. Moreover, even for Kenmotsu three-manifolds, the commutativity condition $S\varphi = \varphi S$ is not equivalent to $\pounds_\xi S = 0$ ([7]).

In the present paper, we again concentrate on a contact three-manifold $(M; \eta)$. For an associated Riemannian metric g, if the Reeb vector field ξ generates an isometric flow, that is, M satisfies $\pounds_\xi g = 0$, then M is said to be K-contact. We note that a K-contact manifold is already Sasakian in dimension three. There is an interesting intermediate class, the so-called H-contact manifolds, which includes K-contact manifolds. It means that the Reeb vector field is a *harmonic vector field*. (For the detail, see Sect. 2). Other than the Ricci operator S, we have fundamental tensors in contact metric geometry: $\ell = R(\cdot, \xi)\xi$, which is called the *characteristic Jacobi operator*, $h = \frac{1}{2}\pounds_\xi\varphi$, and $\tau = \pounds_\xi g$. Then, we study several equivalent conditions to $\pounds_\xi S = 0$. Namely, we prove

Theorem D. *Let M be an H-contact three-manifold. Then M satisfies $\pounds_\xi \ell = 0$, $\nabla_\xi \ell = 0$, $\nabla_\xi S = 0$, $\ell\varphi = \varphi\ell$, $\nabla_\xi h = 0$, or $\nabla_\xi \tau = 0$, respectively if and only if M is Sasakian or locally isometric to* SU(2) (*or* SO(3)), SL$(2, \mathbb{R})$ (*or* $O(1, 2)$), $E(2)$ *with a left invariant contact Riemannian metric respectively.*

In Theorem D, we cannot omit the H-contact condition. In fact, we have a non-unimodular Lie group with contact left invariant Riemannian metric which satisfies $\pounds_\xi \ell = 0$, $\nabla_\xi \ell = 0$, $\nabla_\xi S = 0$, $\ell\varphi = \varphi\ell$, $\nabla_\xi h = 0$, or $\nabla_\xi \tau = 0$, respectively, but $S\xi \neq \sigma\xi$ (see Remark 2). A non-homogeneous example of such a contact metric three-manifold is given in Example 1.

2 Preliminaries

All manifolds in the present paper are assumed to be connected and of class C^∞. A $(2n+1)$-dimensional manifold M^{2n+1} is said to be a contact manifold if it admits a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. Given a contact form η, we have a unique vector field ξ, which is called the Reeb vector field, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X. It is well-known that there exists a Riemannian metric g and a $(1, 1)$-tensor field φ such that

$$\eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \varphi Y), \quad \varphi^2 X = -X + \eta(X)\xi, \tag{1}$$

where X and Y are vector fields on M. From (1) it follows that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2}$$

A Riemannian manifold M equipped with structure tensors (η, g) satisfying (1) is said to be a contact Riemannian manifold and is denoted by $M = (M; \eta, g)$. Given a contact Riemannian manifold M, we define a $(1,1)$-tensor field h by $h = \frac{1}{2}\pounds_\xi \varphi$, where $\pounds$ denotes Lie differentiation. Then we may observe that h is self-adjoint and satisfies

$$h\xi = 0 \quad \text{and} \quad h\varphi = -\varphi h, \tag{3}$$

$$\nabla_X \xi = -\varphi X - \varphi h X, \tag{4}$$

where ∇ is Levi-Civita connection. From (3) and (4) we see that each trajectory of ξ is a geodesic. We denote by R the Riemannian curvature tensor defined by

$$R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z$$

for all vector fields X, Y, Z. Along a trajectory of ξ, the Jacobi operator $\ell X = R(X,\xi)\xi$ is another symmetric (1,1)-tensor field, that is, $g(\ell X, Y) = g(X, \ell Y)$. We have

$$(\text{trace } \ell) = \rho(\xi,\xi) = 2n - (\text{trace } h^2), \tag{5}$$

$$\nabla_\xi h = \varphi - \varphi\ell - \varphi h^2, \tag{6}$$

where ρ is the Ricci curvature tensor defined by $\rho(X,Y) = g(SX,Y)$.

A contact Riemannian manifold for which ξ is Killing is called a K-contact Riemannian manifold. It is easy to see that a contact Riemannian manifold is K-contact if and only if $h = 0$. Moreover, we compute

$$\begin{aligned}(\pounds_\xi h)X &= \pounds_\xi hX - h\pounds_\xi X\\ &= [\xi, hX] - h[\xi, X]\\ &= (\nabla_\xi h)X - \nabla_{hX}\xi + h\nabla_X \xi.\end{aligned}$$

Then we find that $\pounds_\xi h = 0 \Leftrightarrow \nabla_\xi h = -2\varphi h$ and $\varphi h^2 = 0$. Thus, we have

Proposition 1. *A contact Riemannian manifold is K-contact if and only if $\pounds_\xi h = 0$.*

We put $\tau = \pounds_\xi g$. Then D. Perrone obtained the following result.

Proposition 2 ([14]). *For a contact Riemannian manifold M, the following four conditions are mutually equivalent.*

- $\nabla_\xi h = 0$,
- $\nabla_\xi \tau = 0$,
- $\nabla_\xi \ell = 0$,
- $\ell\varphi = \varphi\ell$.

Let (M, g) be a Riemannian manifold with unit tangent sphere bundle T_1M. We equip the Riemannian metric $\hat{g}/4$ on T_1M. Here is $\hat{g}$ the Sasaki-lift metric on T_1M. Denote by $\mathfrak{X}_1(M)$ the space of all smooth unit vector fields on M. A unit vector field $V \in \mathfrak{X}_1(M)$ is said to be *harmonic* if it is a critical point of the energy functional restricted to $\mathfrak{X}_1(M)$. In particular, a contact Riemannian manifold M is said to be an *H-contact manifold* if its Reeb vector field is harmonic in the above sense. In [16] it was proved that a contact Riemannian manifold M is H-contact if and only if ξ is an eigenvector field of S, that is, $S\xi = \sigma\xi$ for some function σ.

For a contact Riemannian manifold M one may define naturally an almost complex structure J on $M \times \mathbb{R}$;

$$J(X, f\frac{d}{dt}) = (\varphi X - f\xi, \eta(X)\frac{d}{dt}),$$

where X is a vector field tangent to M, t the coordinate of $\mathbb{R}$ and f a function on $M \times \mathbb{R}$. If the almost complex structure J is integrable, M is said to be normal or Sasakian. It is known that M is normal if and only if M satisfies

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0,$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. A Sasakian manifold is characterized by a condition $(\nabla_X\varphi)Y = g(X, Y)\xi - \eta(Y)X$ for all vector fields X and Y on the manifold. For more details about contact Riemannian manifolds, we refer to [1].

3 Reeb Flow Symmetries on Contact Three-Manifolds

First, using (5) and (6) we easily obtain

Lemma 1. *A contact Riemannian manifold is Sasakian if and only if $\ell = I - \eta \otimes \xi$.*

In dimension three, we have

Lemma 2 (cf. [3]). *A Sasakian three-manifold is η-Einstein, that is, $S = (\frac{r}{2} - 1)I + (3 - \frac{r}{2})\eta \otimes \xi$, where r is the scalar curvature with $dr(\xi) = 0$.*

Let $M = (M^3; \eta, g)$ be a three-dimensional contact Riemannian manifold. Then, the curvature tensor R is expressed by

$$\begin{aligned} R(X, Y)Z =& \rho(Y, Z)X - \rho(X, Z)Y + g(Y, Z)SX - g(X, Z)SY \\ & - \frac{r}{2}\{g(Y, Z)X - g(X, Z)Y\} \end{aligned} \tag{7}$$

for all vector fields X, Y, Z. Now we prove

Proposition 3. *Under the assumption* $S\xi = \sigma\xi$, *the following conditions are mutually equivalent:*

(i) $\pounds_\xi \ell = 0$,
(ii) $\nabla_\xi \ell = 0$,
(iii) $\nabla_\xi S = 0$.

Proof. If $h = 0$ on M, then from Lemmas 1 and 2 we see at once that the above three conditions always hold. So, we consider on M the maximal open subset U_1 on which $h \neq 0$ and the maximal open subset U_2 on which h is identically zero. (U_2 is the union of all points p in M such that $h = 0$ in a neighborhood of p). $U_1 \cup U_2$ is open and dense in M. Suppose that M is non-Sasakian. Then U_1 is non-empty and there is a local orthonormal frame field $\{e_1 = e, e_2 = \varphi e, e_3 = \xi\}$ on U_1 such that $h(e_1) = \lambda e_1$, $h(e_2) = -\lambda e_2$ for some positive function λ. We denote $\Gamma_{ijk} = g(\nabla_{e_i} e_j, e_k)$, $\rho_{ij} = \rho(e_i, e_j)$, for $h, i, j, k, l = 1, 2, 3$. Also, from (6) and taking account of (5) and (7), we have

$$\xi\lambda = \rho_{12} \tag{8}$$

and

$$4\lambda\Gamma_{312} = \rho_{22} - \rho_{11}. \tag{9}$$

Assume that $S\xi = \sigma\xi$. Then, from (7) we have

$$\ell(X) = SX + (\sigma - \frac{r}{2})X - (2\sigma - \frac{r}{2})\eta(X)\xi. \tag{10}$$

We first prove (i) $\Longleftrightarrow$ (ii). Suppose that M satisfies $\pounds_\xi \ell = 0$. Then, we compute

$$\begin{aligned} 0 =& \pounds_\xi(\ell X) - \ell(\pounds_\xi X) \\ =& [\xi, \ell X] - \ell[\xi, X]. \end{aligned}$$

From this, using (4) we get an equivalent equation to $\pounds_\xi \ell = 0$:

$$(\nabla_\xi \ell)X = (\ell\varphi - \varphi\ell)X + (\ell\varphi h - \varphi h\ell)X. \tag{11}$$

Since $\nabla_\xi \ell$ is a self-adjoint operator, we get

$$\ell\varphi h - \varphi h\ell = \ell h\varphi - h\varphi\ell.$$

Since $h\varphi = -\varphi h$, it follows that

$$\ell\varphi h = \varphi h\ell. \tag{12}$$

Applying e_1 to (12) and taking an inner product with e_2 (with respect to g), then we get

$$\ell_{11} = \ell_{22} \tag{13}$$

on U_1. And we get from (11) and (12)

$$\nabla_\xi \ell = \ell\varphi - \varphi\ell. \tag{14}$$

Differentiating (12) covariantly along ξ, then we get

$$(\nabla_\xi \ell)\varphi h + \ell\varphi(\nabla_\xi h) = \varphi(\nabla_\xi h)\ell + \varphi h(\nabla_\xi \ell). \tag{15}$$

Use (6) and (14) to obtain

$$h^2\ell + \varphi h\ell\varphi - h\ell = -\ell h - \varphi\ell\varphi h + \ell h^2, \tag{16}$$

and then use (12) again to have

$$h^2\ell - \ell h^2 = 2(h\ell - \ell h). \tag{17}$$

Applying e_1 to (17) and taking an inner product with e_2 (with respect to g), then we get

$$\ell_{12} = \ell_{21} = 0 \tag{18}$$

on U_1. From (13) and (18), we find that $\ell\varphi = \varphi\ell$, and consequently from (14) we get $\nabla_\xi \ell = 0$. Conversely, we assume that $S\xi = \sigma\xi$ and $\nabla_\xi \ell = 0$. From (5) we have $\xi\lambda = 0$, and hence $\rho_{12} = \rho_{21} = 0$, where we have used (8). So, from (10) we get $\ell_{12} = \ell_{21} = 0$. Since $\nabla_\xi \ell = 0$, we have

$$\Gamma_{312}(\ell_{11} - \ell_{22}) = 0. \tag{19}$$

From (9) and (19) we have

$$\ell_{11} = \ell_{22},$$

because $\rho_{11} = \rho_{22}$ implies that $\ell_{11} = \ell_{22}$. Thus, together with (18) we find that $\ell\varphi = \varphi\ell$. Moreover, we see that $\ell\varphi h = \varphi h\ell$ on U_1. After all, we have $\pounds_\xi \ell = 0$.

In the middle of above proof we find that $S\xi = \sigma\xi$ and $\pounds_\xi \ell = 0$ implies $S\varphi = \varphi S$. The converse also holds (cf. [6]). In [3] it was proved that $S\xi = \sigma\xi$ and $\nabla_\xi S = 0$ if and only if $\pounds_\xi S = 0$. And it is equivalent to the condition $S\varphi = \varphi S$. This completes the proof. □

From Propositions 2, 3 and Theorem A, we have

Theorem 1. *Let M be an H-contact three-manifold. Then M satisfies $\pounds_\xi \ell = 0$, $\nabla_\xi \ell = 0$, $\nabla_\xi S = 0$, $\ell\varphi = \varphi\ell$, $\nabla_\xi h = 0$, or $\nabla_\xi \tau = 0$, respectively if and only if M is Sasakian or locally isometric to* SU(2) (*or* SO(3)), SL(2, $\mathbb{R}$) (*or* $O(1,2)$), $E(2)$ *with a left invariant contact Riemannian metric respectively.*

4 Lie Groups and Examples

By a theorem due to K. Sekigawa [18] and the classification due to J. Milnor [13] of three-dimensional Lie groups with a left invariant metric, D. Perrone [15] classified all simply connected homogeneous contact Riemannian 3-manifolds. Recall that M is called unimodular if its left invariant Haar measure is also right invariant. In terms of the Lie algebra $\mathfrak{m}$, M is unimodular if and only if the adjoint transformation ad_X has trace zero for every $X \in \mathfrak{m}$. Then we have

Proposition 4 ([13]). *Let M be a three-dimensional unimodular Lie group with a left invariant contact Riemannian structure, then there exists an orthonormal basis $\{e_1, e_2 = \varphi e_1, e_3 = \xi\} \in \mathfrak{m}$ such that*

$$[e_1, e_2] = 2e_3, \ [e_2, e_3] = c_2 e_1, \ [e_3, e_1] = c_3 e_2. \tag{20}$$

Remark 1 (cf. [2]). In fact, every three-dimensional unimodular Lie group, with only exception of the commutative Lie group $\mathbb{R}^3$, admits a left-invariant contact metric structure. Also, M is K-contact (or Sasakian) if and only if $c_2 = c_3$.

Since $c_1 = 2 > 0$, the possible combinations of the signs of c_1, c_2 and c_3 and the associated Lie groups are indicated in the following table (see [13]);

Signs of c_1, c_2, c_3	Associated Lie group
+, +, +	SU(2) or SO(3)
+, +, −	SL(2, $\mathbb{R}$) or $O(1,2)$
+, +, 0	$E(2)$
+, −, −	SL(2, $\mathbb{R}$) or $O(1,2)$
+, −, 0	$E(1,1)$
+, 0, 0	Heisenberg group

SU(2) : group of 2×2 unitary matrices of determinant 1 ; homeomorphic to the unit 3-sphere.
SO(3) : rotation group of 3-space, isomorphic to SU(2)$/\{\pm I\}$.
SL(2, $\mathbb{R}$) : group of 2×2 real matrices of determinant 1.
$O(1,2)$: Lorentz group consisting of linear transformations preserving the quadratic form $t^2 - x^2 - y^2$. Its identity component is isomorphic to $SL(2,\mathbb{R})/\{\pm I\}$, or to the group of rigid motions of hyperbolic 2-space.
$E(2)$: group of rigid motions of Euclidean 2-space.
$E(1,1)$: group of rigid motions of Minkowski 2-space.

Finally, the Heisenberg group can be described as the group of all 3×3 real matrices of the form

$$\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}.$$

From (20), we obtain

$$S\xi = \left(-\frac{1}{2}(c_3 - c_2)^2 + 2\right)\xi.$$

Moreover, we have (cf. [3])

Proposition 5. *Let M be a three-dimensional unimodular Lie group with left invariant contact Riemannian structure. Suppose that M satisfies $\nabla_\xi S = 0$, $\pounds_\xi \ell = 0$, $\ell\varphi = \varphi\ell$, $\nabla_\xi \ell = 0$, $\nabla_\xi h = 0$, or $\nabla_\xi \tau = 0$, respectively. Then M is isometric to one of the following Lie groups:*

- $\mathrm{SU}(2)$ (*or* $\mathrm{SO}(3)$) *with Sasakian metric or contact Riemannian metric,*
- $\mathrm{SL}(2, \mathbb{R})$ (*or* $O(1, 2)$) *with Sasakian metric or contact Riemannian metric,*
- *Heisenberg group with Sasakian metric,*
- $E(2)$ *with contact Riemannian metric.*

Now, let M be a three-dimensional non-unimodular Lie group with left invariant Riemannian metric. Then we have

Proposition 6 ([13]). *Let M be a three-dimensional non-unimodular Lie group with left invariant contact Riemannian structure. Then there exists an orthonormal basis $\{e_1, e_2 = \varphi e_1, e_3 = \xi\} \in \mathfrak{m}$ such that*

$$[e_1, e_2] = \alpha e_2 + 2e_3, \ [e_2, e_3] = 0, \ [e_3, e_1] = \gamma e_2, \tag{21}$$

where $\alpha \neq 0$. Moreover, M is Sasakian if and only if $\gamma = 0$.

From (21), by a direct computation we find that

$$S\xi = \alpha\gamma e_2 + \left(2 - \frac{\gamma^2}{2}\right)\xi \tag{22}$$

Also, we have

$$\begin{aligned} \ell(e_1) &= \left(\frac{-3\gamma^2 + 4\gamma + 4}{4}\right) e_1 \\ \ell(e_2) &= \frac{(\gamma - 2)^2}{4} e_2. \end{aligned} \tag{23}$$

For example, we compute

$$\begin{aligned}(\pounds_\xi \ell)(e_1) &= \pounds_\xi(\ell e_1) - \ell(\pounds_\xi \ell e_1)\\ &= [\xi, \ell e_1] - \ell[\xi, e_1]\\ &= \left(\frac{-3\gamma^2+4\gamma+4}{4}\right)\gamma e_2 - \frac{(\gamma-2)^2}{4}\gamma e_2 \quad (\because (21) \text{ and } (23))\\ &= \gamma^2(-\gamma+2)e_2\end{aligned}$$

and

$$\begin{aligned}(\pounds_\xi \ell)(e_2) &= \pounds_\xi(\ell e_2) - \ell(\pounds_\xi e_2)\\ &= [\xi, \ell e_2] - \ell[\xi, e_2]\\ &= 0.\end{aligned}$$

Then, since $(\pounds_\xi \ell)(\xi) = 0$, from the above two items, we have that M satisfies $\pounds_\xi \ell = 0$ if and only if either $\gamma = 0$ (Sasakian) or $\gamma = 2$. By similar computations, we have

Proposition 7. *Let M be a three-dimensional non-unimodular Lie group with left invariant contact Riemannian structure. Then M satisfies $\pounds_\xi \ell = 0$, $\ell\varphi = \varphi\ell$, $\nabla_\xi \ell = 0$, $\nabla_\xi S = 0$, $\nabla_\xi h = 0$, or $\nabla_\xi \tau = 0$, respectively if and only if either $\gamma = 0$ $(Sasakian)$ or $\gamma = 2$.*

Remark 2. From Proposition 7, we see that the a non-unimodular Lie group whose Lie algebra structure is given by (21) with $\gamma = 2$ satisfies $\pounds_\xi \ell = 0$, $\ell\varphi = \varphi\ell$, $\nabla_\xi \ell = 0$, $\nabla_\xi S = 0$, $\nabla_\xi h = 0$ and $\nabla_\xi \tau = 0$, but not $S\xi = \sigma\xi$. In fact, $S\xi = 2\alpha e_2$ $(\alpha \neq 0)$.

The following example gives a non-homogeneous contact Riemannian manifold satisfying $\pounds_\xi \ell = 0$, $\ell\varphi = \varphi\ell$, $\nabla_\xi \ell = 0$, $\nabla_\xi S = 0$, $\nabla_\xi h = 0$, and $\nabla_\xi \tau = 0$, but not $S\xi = \sigma\xi$.

Example 1 ([12]). On the Cartesian 3-space $\mathbb{R}^3(x, y, z)$, we define a contact 1-form η by

$$\eta = dx + 2ye^{-z}dz.$$

Next we define a frame field $\{e_1, e_2, e_3\}$ by

$$e_1 = -2y\frac{\partial}{\partial x} + (2x - ye^z)\frac{\partial}{\partial y} + e^z\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial x}.$$

Then we define a Riemannian metric g by the condition $\{e_1, e_2, e_3\}$ is orthonormal with respect to it. One can see that g is an associated metric to η. The Reeb vector field is $\xi = e_3$. As usual, the endomorphism field φ is defined by $\varphi e_1 = e_2$, $\varphi e_2 = -e_1$ and $\varphi e_3 = 0$. Direct computations show that $he_1 = e_1$, $he_2 = -e_2$, and $\ell = 0$. Moreover, we obtain that $\pounds_\xi \ell = 0$, $\ell\varphi = \varphi\ell$, $\nabla_\xi \ell = 0$, $\nabla_\xi h = 0$, and $\nabla_\xi \tau = 0$. The Ricci operator has components $\rho_{23} = \rho_{32} = ye^z/2$. All the other components are zero. And then we have $\nabla_\xi S = 0$. But, $S\xi (= ye^z/2\ e_2)$ is not parallel to ξ. Note that M is neither homogeneous nor flat. We may also refer to [6].

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Totally Geodesic Surfaces of Riemannian Symmetric Spaces

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Abstract A submanidfold S of a Riemannian manifold is called a *totally geodesic submanifold* if every geodesic of S is also a geodesic of M. Totally geodesic submanifolds of Riemannian symmetric spaces have long been studied by many mathematicians. We give a classification of non-flat totally geodesic surfaces of the Riemannian symmetric space of type AI, $AIII$ and BDI.

1 Introduction

Let G be a compact simple Lie group and θ be an involutive automorphism of G. We denote by $\mathfrak{g}$ the Lie algebra of G and denote also by θ the differential of θ. Let $\mathfrak{k}$ be the set of all θ-invariant elements of $\mathfrak{g}$ and K be a Lie subgroup of G of which Lie algebra coincides with $\mathfrak{k}$.

Let $\langle , \rangle$ be an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$ and $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$. We extend the restriction of $\langle , \rangle$ on $\mathfrak{p}$ to the G-invariant Riemannian metric on G/K and denote it also by $\langle , \rangle$.

A subspace $\mathfrak{s}$ of $\mathfrak{p}$ is called a *Lie triple system* if it satisfies $[[\mathfrak{s}, \mathfrak{s}]\mathfrak{s}] \subset \mathfrak{s}$. There exits a one-to-one correspondence between the set of totally geodesic submanifold of M through the origin $o = eK$ and the set of Lie triple systems in $\mathfrak{p}$ [1].

Important constructions and classification results of totally geodesic submanifolds in Riemannian symmetric spaces are summarized in an expository article by S. Klein [2].

In [3] the author classified non-flat totally geodesic surfaces in irreducible Riemannian symmetric spaces where G is $SU(n)$, $Sp(n)$ or $SO(n)$. The main tool used in [3] is the representation theory of $SU(2)$. The aim of this article is to introduce the outline of the contents of [3].

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2 Irreducible Representation of $SU(2)$

In this section, we review real and complex irreducible representations of $SU(2)$.

Let H, X, Y be a basis of the complexification of the Lie algebra $\mathfrak{su}(2)$ of $SU(2)$ satisfying

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H. \tag{1}$$

2.1 Complex Irreducible Representations

If we denote by V_d the set of polynomial functions on $\mathbb{C}^2$ and by ρ_d the contragradient action of $SU(2)$ on V_d, then (V_d, ρ_d) is a complex irreducible representation of $SU(2)$. On the other hand, every finite dimensional complex irreducible representation of $SU(2)$ is equivalent to (V_d, ρ_d) for some positive integer d.

The next proposition plays an important role in our classification.

Proposition 1. *Let (V, ρ) be a $(d+1)$-dimensional complex irreducible representation of $SU(2)$ and $\langle\, ,\, \rangle$ be an $SU(2)$-invariant Hermitian inner product on V. If we put λ the largest eigenvalue of $\rho(H)$ and $v_0 \in V$ be a corresponding eigen vector, then we have $\lambda = d$ and $\rho(Y)^i(v_0)$ is an eigen vector of $\rho(H)$ corresponding to the eigenvalue $(\lambda - 2i)$.*

Let ε_i $(0 \leq i \leq d)$ be arbitrary complex numbers with $|\varepsilon_i| = 1$, and put $v_i = \dfrac{\varepsilon_i}{|\rho(Y)^i v_0|}\rho(Y)^i v_0$ $(0 \leq i \leq d)$. Then $v_0, v_1, \cdots, v_d$ is an orthonormal basis of V_d and the matrix representations of $\rho(H)$, $\rho(X)$, $\rho(Y)$ with respect to $v_0, \cdots, v_d$ are as follows

$$\rho(H) = \begin{bmatrix} d & 0 & \cdots & 0 \\ 0 & d-2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -d \end{bmatrix}, \quad \rho(X) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ c_1 & 0 & \cdots & 0 & 0 \\ 0 & c_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & c_d & 0 \end{bmatrix},$$

$$\rho(Y) = \begin{bmatrix} 0 & c_1' & 0 & \cdots & 0 \\ 0 & 0 & c_2' & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_d' \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad \text{where} \quad c_i' = \overline{c_i}, \quad |c_i| = \sqrt{i(d-i+1)}.$$

2.2 Real Irreducible Representations

Let (V, ρ) be a complex representation of $SU(2)$ and $v_1, \cdots, v_N$ be a basis of V. We denote by $\overline{V}$ the complex vector space, which is V itself as an additive group and the scalar multiplication is defined by $c * x = \overline{c}\, x$ $(c \in \mathbb{C},\ x \in V)$. Define the action $\overline{\rho}$ of $SU(2)$ on $\overline{V}$ so that

$$\overline{\rho}\left(\sum z_i * v_i\right) = \sum z_i * \rho(v_i).$$

The representation $(\overline{V}, \overline{\rho})$ is called the *conjugate* representation of (V, ρ).

A complex irreducible representation (V, ρ) of G is said to be a *self-conjugate* representation if there exists a conjugate-linear automorphism $\hat{j} : V \to V$ which commute with $\rho(g)$ for any $g \in SU(2)$. A conjugate-linear automorphism commuting with ρ is called a *structure map* of (V, ρ).

Let (V, ρ) be a self-conjugate representation and $\hat{j}$ be a structure map. By Schur's lemma, $\hat{j}^2 = c$ for some constant. It is known that the constant c is a real number and (V, ρ) is said to be of *index* 1 (resp. -1) if $c > 0$ (resp. $c < 0$).

Each complex irreducible representation (V_d, ρ_d) of $SU(2)$ is a self-conjugate representation and its index is equal to $(-1)^d$. If d is an even integer, the subspace of V_d invariant under the structure map $\hat{j}$ is a real irreducible representation of $SU(2)$. If d is an odd integer, V_d (viewed as a real representation by restricting the coefficient field from $\mathbb{C}$ to $\mathbb{R}$) is also a Real irreducible representation and V_d admits a structure of vector space over the field of quaternions.

3 Classification

The standard orthonormal basis of $\mathbb{R}^N$ or $\mathbb{C}^N$ will be denote by $e_1, \cdots, e_N$. We denote by G_{ij} $(i \neq j)$ the skew-symmetric endomorphism satisfying

$$G_{ij}(e_j) = e_i, \quad G_{ij}(e_i) = -e_j, \quad G_{ij}(e_k) = 0 \quad (k \neq i, j),$$

and by S_{ij} the symmetric endomorphism

$$S_{ij}(e_j) = e_i, \quad S_{ij}(e_i) = e_j, \quad S_{ij}(e_k) = 0 \quad (k \neq i, j).$$

3.1 $AI : SU(n)/SO(n)$

We denote by τ the conjugation on $\mathbb{C}^N$ with respect to $\mathbb{R}^N$ and denote by θ the involutive automorphism on $SU(N)$ defined by $\theta(g) = \tau \circ g \circ \tau$ $(g \in SU(n))$.

Theorem 1. *Let M be a non-flat totally geodesic surface of $SU(n)/SO(n)$ and U be the set of all elements in $SU(n)$ leaving M invariant.*

(i) There exists an orthogonal direct sum decomposition of $\mathbb{C}^n$ by τ-invariant and U-invariant subspaces.

(ii) Let X_2, X_3 be a basis of the Lie triple system corresponding to M with

$$[[X_2, X_3], X_2] = 4\, X_3, \quad [[X_2, X_3], X_3] = -4\, X_2.$$

Assume that $\mathbb{C}^n$ is U-invariant. There exists an element $g = [u_1, \cdots, u_n] \in SO(n)$ such that

$$\mathrm{Ad}(g) X_2 = \sqrt{-1} \sum_{i=1}^{n} (n - 2\, i)\, E_{i,i} \tag{2}$$

$$\mathrm{Ad}(g) X_3 = -\sqrt{-1} \left[\sum_{i=1}^{n-2} \sqrt{i(n-i)} S_{i,i+1} + \varepsilon \sqrt{n-1} S_{n-1,n} \right] \tag{3}$$

where

$$\varepsilon = \begin{cases} 1 & \text{if } n \equiv 1 \ (\mathrm{mod} 2), \\ \pm 1 & \text{if } n \equiv 0 \ (\mathrm{mod} 2). \end{cases}$$

Proof. We omit the proof of (i) and assume that the action of U on $\mathbb{C}^n$ is irreducible.

Note that $\mathfrak{k} = \{X : \theta(X) = X\} = \mathrm{Skew}(n; \mathbb{R})$ and $\mathfrak{p} = \{X : \theta(X) = -X\} = \sqrt{-1}\, \mathrm{Sym}(n; \mathbb{R})$.

If we put $a_1 \geq a_2 \geq \cdots \geq a_n$ the set of eigenvalues of $H = -\sqrt{-1}\, X_2 \in \mathrm{Sym}(n; \mathbb{R})$, then by the action of $\mathrm{Ad}(SO(n))$ we may assume that $H = \mathrm{Diag}(a_1, a_2, \cdots, a_n)$.

If we put

$$H = [X_2, X_3], \quad X = \frac{1}{2}(\sqrt{-1}\, X_3 + X_1), \quad Y = \frac{1}{2}(\sqrt{-1}\, X_3 - X_1),$$

we have

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

Since a_i are weights of the complex irreducible representation of U we have

$$a_1 - a_2 = a_2 - a_3 = \cdots = a_{n-1} - a_n = 2.$$

Put $n = d + 1$ and $v_0 = e_1$. Since each eigenspace (the weight space) of H is one-dimensional there exists ε_i $(1 \le i \le d)$ such that $e_i = \dfrac{\varepsilon_i}{|H^i v_0|} H^i v_0$. Thus the matrix H, X and Y are of the form given in the Proposition 1. We can choose unit complex numbers ε_i' $(0 \le i \le d)$ such that by a change of basis $\{e_i\} \to \{\varepsilon_i' e_i\}$ all the components of X, Y in the Proposition 1 are changed to real numbers. We omit further detail. □

3.2 $AIII : SU(p+q)/S(U(p) \times U(q))$

We denote by I_n the unit matrix of order n and put $I_{p,q} = \begin{bmatrix} I_p & O \\ O & -I_q \end{bmatrix}$.

Theorem 2. *Let M be a non-flat totally geodesic surface of $SU(p+q)/S(U(p) \times U(q))$ and U be the set of all elements of $SU(p+q)$ which leave M invariant.*

(i) There exists an orthogonal direct sum decomposition of $\mathbb{C}^{p+q}$ by $I_{p,q}$-invariant, U-irreducible subspaces.

(ii) If V is an $I_{p,q}$-invariant, U-irreducible subspace of $\mathbb{C}^{p+q}$, then we have

$$\left|\dim\{v \in V : I_{p,q}(v) = v\} - \dim\{v \in V : I_{p,q}(v) = -v\}\right| \le 1.$$

(iii) Assume that the action of $SU(2)$ on $\mathbb{C}^{p+q}$ is irreducible. Let X_2, X_3 be a basis of the Lie triple system corresponding to M with

$$[[X_2, X_3], X_2] = 4\,X_3, \quad [[X_2, X_3], X_3] = -4\,X_2.$$

There exists an element $g = [u_1, \cdots, u_{p+q}] \in S(U(p) \times U(q))$ such that

$$\begin{aligned} \mathrm{Ad}(g)X_2 = &\sum_{i=1}^{q} \sqrt{(2i-1)(p+q+1-2i)}\, G_{i,p+i} \\ &+ \sum_{i=1}^{p-1} \sqrt{2i(p+q-2i)}\, G_{p+i,i+1} \end{aligned} \tag{4}$$

$$\begin{aligned} \mathrm{Ad}(g)X_3 = \sqrt{-1}\, &\left[\sum_{i=1}^{q} \sqrt{(2i-1)(p+q+1-2i)}\, S_{p+i,i} \right. \\ &\left. + \sum_{i=1}^{p-1} \sqrt{2i(p+q-2i)}\, S_{i+1,p+i} \right] \end{aligned} \tag{5}$$

Proof. We omit the proof of (i).

Assume that the action of U on $\mathbb{C}^{p+q}$ is irreducible.

Take a basis X_1, X_2, X_3 of the Lie algebra $\mathfrak{u}$ of U which satisfy

$$I_{p,q} \circ X_1 = X_1 \circ I_{p,q}, \quad I_{p,q} \circ X_i = -X_i \circ I_{p,q} \quad (i = 2, 3),$$
$$[X_1, X_2] = 2X_3, \quad [X_2, X_3] = 2X_1, \quad [X_3, X_1] = 2X_2,$$

and put

$$H = -\sqrt{-1}X_1,\ X = \frac{1}{2}(X_2 - \sqrt{-1}X_3),\ Y = -\frac{1}{2}(X_2 + \sqrt{-1}X_3) = {}^t\overline{X}.$$

Since H is a Hermitian matrix, there exists an element $g \in S(U(p) \times U(q))$ such that

$$\mathrm{Ad}(g)H = \mathrm{diag}(a_1, \cdots, a_p; b_1, \cdots, b_q)$$

where $a_1 > \cdots > a_p$ and $b_1 > \cdots > b_q$ holds. We denote by ξ_i the i-th column vector of g. The set of eigenvalues of H coincides with the set of weights of the $(p+q)$-dimensional complex irreducible representation of $SU(p+q))$, namely we have

$$\{a_1, \cdots, a_p, b_1, \cdots, b_q\} = \{p+q-1,\ p+q-2,\ \cdots,\ 1-p-q\}.$$

We assume that $a_1 > b_1$ holds.

- We have $a_1 = p + q - 1$ and $I_{p,q}\xi_1 = \xi_1$, $H \cdot \xi_1 = (p + q - 1)\,\xi_1$ hold.
- From $I_{p,q} \circ Y = -Y \circ I_{p,q}$, we have $I_{p,q}(Y \cdot \xi_1) = -Y \cdot \xi_1$ and from $[H, Y] = -2Y$ we have $H(Y \cdot \xi_1) = (p + q - 3)\, Y \cdot \xi_1$. Thus we have $b_1 = p + q - 3$ and there exists a complex number γ_i with

$$Y \cdot \xi_1 = \gamma_1\, \xi_{p+1}, \quad |\gamma_1| = \sqrt{p+q-1}.$$

- Similarly we have

$$Y \cdot \xi_{p+1} = \gamma_2\, \xi_2, \quad |\gamma_2| = \sqrt{2(p+q-2)}$$

etc.

Finally we have $p - q = 0, 1$ and the matrix representation of Y with respect to the basis $\xi_1, \cdots, \xi_p, \xi_{p+1}, \cdots, \xi_{p+q}$ is

$$\mathrm{Ad}(g)Y = \sum_{i=1}^{q} \gamma_{2i-1}\, E_{p+i,i} + \sum_{i=1}^{p-1} \gamma_{2i}\, E_{i+1,p+i}.$$

Let ε_i $(1 \leq i \leq p+q)$ be unit complex numbers and put $g = (\varepsilon_1 \xi_1, \cdots, \varepsilon_{p+q} \xi_{p+q})$. We can choose ε_i so that the all of the coefficients γ_{2i} and γ_{2i-1} in the representation of $\mathrm{Ad}(g)Y$ above are positive real numbers. From

$$X_2 = {}^t\overline{Y} - Y, \quad X_3 = \sqrt{-1}\left({}^t\overline{Y} + Y\right)$$

we obtain (4) and (5). □

3.3 BDI : $SO(p+q)/S(O(p) \times O(q))$

Let θ be the involutive automorphism on $G = SO(p+q)$ defined by

$$\theta(g) = I_{p,q} \circ g \circ I_{p,q}$$

and put

$$K = \{g \in SO(p+q) : \theta(g) = g\} = S(O(p) \times O(q)).$$

We can classify totally geodesic surfaces of $SO(p+q)/S(O(p) \times O(q))$ by similar argument to that on $SU(p+q)/S(U(p) \times U(q))$. But, since there are two types of real irreducible representations of $SU(2)$, the classification result is divided into two cases; (iii) and (iv) in the following theorem. Since it is troublesome to give the representation matrix of the action of $\mathfrak{su}(2)$ on the odd-dimensional real irreducible representation ((iii) in the following theorem), we give only the result without proof.

Theorem 3. *Let M be a non-flat totally geodesic surface of $SO(p+q)/S(O(p) \times O(q))$ and U be the set of all elements in $SO(p+q)$ leaving M invariant.*

(i) There exists an orthogonal direct sum decomposition of $\mathbb{R}^{p+q}$ by $I_{p,q}$-invariant and U-irreducible subspaces.

(ii) For each $I_{p,q}$-invariant, U-irreducible subspace V of $\mathbb{R}^{p+q}$, we have

$$\left|\dim\{v \in V : I_{p,q}(v) = v\} - \dim\{v \in V : I_{p,q}(v) = -v\}\right| \leq 1.$$

(iii) Assume that the action of U on $\mathbb{R}^{p+q}$ is irreducible and $p = q+1 \geq 3$.

We denote by p' the integer part of $p/2$ and by q' the integer part of $q/2$. There exists an element $g \in S(O(p) \times O(q))$ such that

$$\mathrm{Ad}(g)X_2 = -\sum_{i=1}^{q'} \sqrt{(2i-1)(p+q+1-2i)}\left(G_{p+2i-1,2i-1} + G_{p+2i,2i}\right)$$

$$+\sum_{i=1}^{p'-1}\sqrt{2i(p+q-2i)}\left(G_{p+2i-1,2i+1}+G_{p+2i,2i+2}\right)$$

$$+\begin{cases}(-\sqrt{2})\sqrt{p\,q}\,G_{p+q,q} & (\text{if } p=0 \mod 2)\\ \sqrt{2}\sqrt{p\,q}\,G_{p+q-1,p} & (\text{if } p=1 \mod 2)\end{cases} \tag{6}$$

$$\mathrm{Ad}(g)X_3=\sum_{i=1}^{q'}\sqrt{(2i-1)(p+q+1-2i)}\left(G_{p+2i,2i-1}-G_{p+2i-1,2i}\right)$$

$$+\sum_{i=1}^{p'-1}\sqrt{2i(p+q-2i)}\left(G_{p+2i,2i+1}-G_{p+2i-1,2i+2}\right)$$

$$+\begin{cases}(-\sqrt{2})\sqrt{p\,q}\,G_{p+q,p} & (\text{if } p=0 \mod 2)\\ \sqrt{2}\sqrt{p\,q}\,G_{p+q,p} & (\text{if } p=1 \mod 2)\end{cases} \tag{7}$$

(iv) Assume that the action of U on $\mathbb{R}^{p+q}$ is irreducible and $p=q$. Then p is an even integer, say $p=2p'$, and there exists an element $g\in S(O(p)\times O(q))$ such that

$$\mathrm{Ad}(g)X_2=\sum_{i=1}^{p'-1}\sqrt{2i(p-2i)}\left(G_{p+i,i+1}+G_{p+p'+i,p'+i+1}\right)$$

$$-\sum_{i=1}^{p'}\sqrt{(2i-1)(p+1-2i)}\left(G_{p+i,i}+G_{p+p'+i,p'+i}\right) \tag{8}$$

$$\mathrm{Ad}(g)X_3=\sum_{i=1}^{p'-1}\sqrt{2i(p-2i)}\left(G_{p+p'+i,i+1}-G_{p+i,p'+i+1}\right)$$

$$+\sum_{i=1}^{p'}\sqrt{(2i-1)(p+1-2i)}\left(G_{p+p'+i,i}-G_{p+i,p'+i}\right) \tag{9}$$

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The Geometry on Hyper-Kähler Manifolds of Type A_∞

Kota Hattori

Abstract Hyper-Kähler manifolds of type A_∞ are noncompact complete Ricci-flat Kähler manifolds of complex dimension 2, constructed by Anderson, Kronheimer, LeBrun (Commun. Math. Phys., **125**, 637–642, 1989) and Goto (Geom. Funct. Anal., **4**(4), 424–454, 1994). We review the asymptotic behavior, the holomorphic symplectic structures and period maps on these manifolds.

1 Introduction

Hyper-Kähler manifolds of type A_∞ were first constructed by Anderson, Kronheimer and LeBrun in [1], as the first example of complete Ricci-flat Kähler manifolds with infinite topological type. Here, infinite topological type means that their homology groups are infinitely generated. The construction in [1] is due to Gibbons-Hawking ansatz, and Goto [5] has constructed these manifolds in another way, using hyper-Kähler quotient construction. Some of the topological and geometric properties of hyper-Kähler manifolds of type A_∞ were studied well in the above papers. In this article, we focus on the volume growth of the hyper-Kähler metrics, the holomorphic symplectic structures, and the period maps.

The construction of hyper-Kähler manifolds of type A_∞ is similar to that of ALE spaces of type A_k, where k is a nonnegative integer. Moreover, their topological properties and complex geometric properties are also similar to type A_k. For example, both of the ALE spaces of type A_k and the hyper-Kähler manifolds of type A_∞ have the parameter naturally given by the construction. We review that they correspond to the cohomology classes of three Kähler forms along [8].

On the other hand, one of the essentially different properties between them appears in their asymptotic behaviors. In fact, the volume growth of ALE spaces is Euclidean, but that of hyper-Kähler manifolds of type A_∞ are less than Euclidean volume growth, which is a main result of [7].

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Moreover, we will review the independence of the volume growth of hyper-Kähler metrics and the complex structures. More precisely, we review the result in [9] to the effect that the volume growth of the hyper-Kähler metric of type A_∞ can be deformed preserving the complex structure.

2 Hyper-Kähler Manifolds of Type A_∞

2.1 Hyper-Kähler Quotient Construction

In this section, we review shortly the construction of hyper-Kähler manifolds of type A_∞ along [5]. For more details, see [1, 5] or review in Section 2 of [7].

First of all, hyper-Kähler manifolds are defined as follows.

Definition 1. Let (X, g) be a Riemannian manifold of dimension $4n$ with three integrable complex structures I_1, I_2, I_3, and g be a hermitian metric with respect to each I_i. Then (X, g, I_1, I_2, I_3) is a hyper-Kähler manifold if (I_1, I_2, I_3) satisfying the relations $I_1^2 = I_2^2 = I_3^2 = I_1 I_2 I_3 = -1$ and each $\omega_i := g(I_i\cdot, \cdot)$ being closed.

Denote by $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k = \mathbb{C} \oplus \mathbb{C}j$ the quaternion and denote by $\mathrm{Im}\mathbb{H} = \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ its Imaginary part. Then an $\mathrm{Im}\mathbb{H}$-valued 2-form $\omega := i\omega_1 + j\omega_2 + k\omega_3 \in \Omega^2(X) \otimes \mathrm{Im}\mathbb{H}$ characterizes the hyper-Kähler structure (g, I_1, I_2, I_3). Accordingly, we call ω the hyper-Kähler structure on X instead of (g, I_1, I_2, I_3).

Now we construct hyper-Kähler quotient method introduced in [9]. Put

$$(\mathrm{Im}\mathbb{H})_0^{\mathbb{N}} := \{\lambda = (\lambda_n)_{n\in\mathbb{N}} \in (\mathrm{Im}\mathbb{H})^{\mathbb{N}};\ \sum_{n\in\mathbb{N}} \frac{1}{1 + |\lambda_n|} < +\infty\},$$

where $\mathbb{N}$ is the set of positive integers. Here, we denote by $S^{\mathbb{N}}$ the set of all maps from $\mathbb{N}$ to a set S.

Let

$$M_{\mathbb{N}} := \{v \in \mathbb{H}^{\mathbb{N}};\ \|v\|_{\mathbb{N}}^2 < +\infty\},$$

where

$$\langle u, v\rangle_{\mathbb{N}} := \sum_{n\in\mathbb{N}} u_n \bar{v}_n, \quad \|v\|_{\mathbb{N}}^2 := \langle v, v\rangle_{\mathbb{N}}$$

for $u, v \in \mathbb{H}^{\mathbb{N}}$. Here, the quaternionic conjugate of v_n is denoted by $\bar{v}_n$.

For each $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{N}}$, $\Lambda \in \mathbb{H}^{\mathbb{N}}$ can be taken so that $\Lambda_n i \overline{\Lambda}_n = \lambda_n$. Put

$$M_\Lambda := \Lambda + M_{\mathbb{N}} = \{\Lambda + v;\ v \in M_{\mathbb{N}}\},$$

$$G_\lambda := \{g \in (S^1)^{\mathbb{N}};\ \sum_{n\in\mathbb{N}} (1 + |\lambda_n|)|1 - g_n|^2 < +\infty,\ \prod_{n\in\mathbb{N}} g_n = 1\}.$$

Here, $\prod_{n\in\mathbb{N}} g_n$ always converges by the condition

$$\sum_{n\in\mathbb{N}} \frac{1}{1+|\lambda_n|} < +\infty.$$

Then G_λ is an infinite dimensional Lie group, and G_λ acts on M_Λ by $xg := (x_n g_n)_{n\in\mathbb{N}}$ for $x \in M_\Lambda$, $g \in G_\lambda$.

Now G_λ acts on

$$N_\Lambda = \{x \in M_\Lambda;\ x_n i \overline{x}_n - \lambda_n = x_m i \overline{x}_m - \lambda_m \text{ for all } n, m \in \mathbb{N}\}$$

and we obtain the quotient space N_Λ/G_λ which is called the hyper-Kähler quotient. Here, N_Λ corresponds to the level set of the hyper-Kähler moment map.

Definition 2. $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{N}}$ is generic if $\lambda_n - \lambda_m \neq 0$ for all distinct $n, m \in \mathbb{N}$.

Theorem 1 ([5]). *If $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{N}}$ is generic, then N_Λ/G_λ is a smooth manifold of real dimension* 4*, and the hyper-Kähler structure on M_Λ induces a hyper-Kähler structure ω_λ on N_Λ/G_λ.*

Although the hyper-Kähler quotient N_Λ/G_λ seems to depend on the choice of $\Lambda \in \mathbb{H}^{\mathbb{N}}$, the induced hyper-Kähler structure on N_Λ/G_λ depends only on λ by the argument of Section 2 of [7]. Accordingly we may put

$$\begin{aligned} X(\lambda) &:= N_\Lambda/G_\lambda \\ &= \{x \in M_\Lambda; x_n i \bar{x}_n - \lambda_n \text{ is independent of } n \in \mathbb{N}\}/G_\lambda, \end{aligned}$$

and call it a hyper-Kähler manifold of type A_∞

If $\mathbb{N}$ is replaced by a finite set in the above construction, $(X(\lambda), \omega_\lambda)$ becomes an ALE hyper-Kähler manifold of type A_k [4].

2.2 S^1-actions and Moment Maps

An S^1-action on $X(\lambda)$ preserving the hyper-Kähler structure is defined as follows. (See also [5].) Let $[x] \in N_\Lambda/G_\lambda$ be the equivalence class represented by $x \in N_\Lambda$. Take $m \in \mathbb{N}$ arbitrarily and let

$$[x]g := [x_m g, (x_n)_{n\in\mathbb{N}\setminus\{m\}}]$$

for $x = (x_m, (x_n)_{n\in\mathbb{N}\setminus\{m\}}) \in N_\Lambda$ and $g \in S^1$. This definition does not depend on the choice of $m \in \mathbb{N}$. Then we obtain the hyper-Kähler moment map

$$\mu_\lambda([x]) := x_n i \bar{x}_n - \lambda_n \in \mathrm{Im}\mathbb{H}.$$

The right hand side is independent of the choice of $n \in \mathbb{N}$ since x is an element of N_Λ.

We have a principal S^1-bundle $\mu_\lambda\big|_{X(\lambda)^*} : X(\lambda)^* \to Y(\lambda)$, where

$$X(\lambda)^* := \{[x] \in X(\lambda);\ x_n \neq 0 \text{ for all } n \in \mathbb{N}\},$$
$$Y(\lambda) := \mathrm{Im}\mathbb{H} \backslash \{-\lambda_n;\ n \in \mathbb{N}\}.$$

By the Gibbons-Hawking construction [1], we can check easily that $X(\lambda)$ and $X(\lambda')$ are isomorphic as hyper-Kähler manifolds if λ and λ' satisfy one of the following conditions; (i) $\lambda'_n - \lambda_n \in \mathrm{Im}\mathbb{H}$ is independent of n, (ii) $\lambda'_n = \lambda_{a(n)}$ for some bijective maps $a : \mathbb{N} \to \mathbb{N}$, (iii) $\lambda = -\lambda'$.

3 The Volume Growth

Here we focus on the Riemannian geometric aspects of $X(\lambda)$, especially their volume growth.

For a Riemannian manifold (X, g), denote by $V_g(p, r)$ the volume of the geodesic ball of radius $r > 0$ centered at $p \in X$. By the volume comparison theorem [2, 6], we can deduce that

$$\lim_{r\to\infty} \frac{V_g(p_0, r)}{V_g(p_1, r)} = 1$$

for any Ricci flat manifold (X, g) and any $p_0, p_1 \in X$. Thus the volume growth of g is the invariant for Ricci flat manifolds.

Theorem 2 ([7]). *For each* $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{N}}$ *and* $p_0 \in X(\lambda)$*, the function* $V_{g_\lambda}(p_0, r)$ *satisfies*

$$0 < \liminf_{r\to+\infty} \frac{V_{g_\lambda}(p_0, r)}{r^2 \tau_\lambda^{-1}(r^2)} \le \limsup_{r\to+\infty} \frac{V_{g_\lambda}(p_0, r)}{r^2 \tau_\lambda^{-1}(r^2)} < +\infty,$$

where the function $\tau_\lambda : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ *is defined by*

$$\tau_\lambda(R) := \sum_{n\in\mathbb{N}} \frac{R^2}{R + |\lambda_n|}$$

for $R \ge 0$*. Moreover, we have*

$$\lim_{r\to+\infty} \frac{V_{g_\lambda}(p_0, r)}{r^4} = 0, \qquad \lim_{r\to+\infty} \frac{V_{g_\lambda}(p_0, r)}{r^3} = +\infty.$$

Next we see some examples computed in [7].

Example 1. Fix $\gamma > 1$ and put $\lambda_n^\gamma := i \cdot n^\gamma \in \mathrm{Im}\mathbb{H}$. Then there exist positive constants $A, B > 0$ such that

$$Ar^{4-\frac{2}{\gamma+1}} \leq V_{g_{\lambda^\gamma}}((p_0, r) \leq Br^{4-\frac{2}{\gamma+1}}.$$

Example 2. Put $\lambda_n := i \cdot e^n \in \mathrm{Im}\mathbb{H}$. Then there exist positive constants $A, B > 0$ such that

$$A\frac{r^4}{\log r} \leq V_{g_\lambda}(p_0, r) \leq B\frac{r^4}{\log r}$$

for any $\alpha < 4$.

4 Period Maps

4.1 Holomorphic Curves

In this subsection, we see that there are several compact minimal submanifolds in $X(\lambda)$ following [8].

Definition 3. (i) Let X be a complex manifold of dimension $2n$ and $\omega_\mathbb{C}$ be a holomorphic 2-form on X. Then $(X, \omega_\mathbb{C})$ is called a holomorphic symplectic manifold if $d\omega_\mathbb{C} = 0$ and $\omega_\mathbb{C}^n$ is nowhere vanishing. (ii) An n dimensional complex submanifold L of a holomorphic symplectic manifold $(X, \omega_\mathbb{C})$ is holomorphic Lagrangian submanifold if $\omega_\mathbb{C}|_L = 0$.

Let (X, ω) be a hyper-Kähler manifold of real dimension $4n$. For each $y \in \mathrm{Im}\mathbb{H}$ with $|y| = 1$, $\mathrm{Im}\mathbb{H}$ is decomposed into y-component and its orthogonal complement. Then we denote by $\omega_y \in \Omega^2(X)$ the y-component of $\omega \in \Omega^2(X) \otimes \mathrm{Im}\mathbb{H}$. Let I_y be the complex structure corresponding to the Kähler form ω_y.

Let $\eta = (\eta_1\ \eta_2\ \eta_3) \in SO(3)$, where $\langle \eta_1, \eta_2, \eta_3 \rangle$ is an orthonormal basis of $\mathbb{R}^3$. Then η gives the orthogonal decomposition $\mathrm{Im}\mathbb{H} = \mathbb{R}^3 = \mathbb{R}\eta_1 \oplus \mathbb{R}\eta_2 \oplus \mathbb{R}\eta_3$, and the hyper-Kähler structure $\omega \in \Omega^2(X) \otimes \mathrm{Im}\mathbb{H}$ can be written as $\omega = \eta_1\omega_{\eta_1} + \eta_2\omega_{\eta_2} + \eta_3\omega_{\eta_3}$ for every $\eta \in SO(3)$. Now we regard (X, I_{η_1}) as a complex manifold. Then a holomorphic symplectic structure on X is given by $\omega_{\eta\mathbb{C}} := \omega_{\eta_2} + i\omega_{\eta_3}$.

Proposition 1. *Let (X, ω) be a hyper-Kähler manifold and take $\eta \in SO(3)$. Then each holomorphic Lagrangian submanifold $L \subset X$ with respect to $\omega_{\eta\mathbb{C}}$ gives the minimum volume in their homology class.*

Proof. The pair of a Kähler form ω_{η_3} and a holomorphic volume form $(\omega_{\eta_1} + i\omega_{\eta_2})^n$ gives the Calabi-Yau structure on (X, I_{η_3}). Here, n is the half of the complex dimension of X. Now, assume that $L \subset X$ is a holomorphic Lagrangian submanifold with respect to $\omega_{\eta\mathbb{C}}$. Then $\omega_{\eta_2}|_L = \omega_{\eta_3}|_L = 0$, hence L is lagrangian with respect to ω_{η_3}. Since $\mathrm{Im}(\omega_{\eta_1} + i\omega_{\eta_2})^n$ is the multiplication of ω_{η_2} and some differential forms, we also have $\mathrm{Im}(\omega_{\eta_1} + i\omega_{\eta_2})^n|_L = 0$, which means L is a special Lagrangian submanifold. The volume minimizing property of special Lagrangian submanifolds [11] gives the assertion. □

Take a generic $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{N}}$ and consider the hyper-Kähler manifold $(X(\lambda), \omega_\lambda)$ as constructed in Sect. 2. Put

$$[a, b] := \{ta + (1-t)b \in \mathrm{Im}\mathbb{H};\ 0 \le t \le 1\},$$
$$(a, b] := \{ta + (1-t)b \in \mathrm{Im}\mathbb{H};\ 0 \le t < 1\},$$
$$[a, b) := \{ta + (1-t)b \in \mathrm{Im}\mathbb{H};\ 0 < t \le 1\},$$
$$(a, b) := \{ta + (1-t)b \in \mathrm{Im}\mathbb{H};\ 0 < t < 1\}$$

for $a, b \in \mathrm{Im}\mathbb{H}$.

Proposition 2. *Let $n, m \in \mathbb{N}$ satisfy $n \ne m$ and $(-\lambda_n, -\lambda_m) \subset Y(\lambda)$. The inverse image $\mu_\lambda^{-1}([-\lambda_n, -\lambda_m]) \cong \mathbb{C}P^1$ is a complex submanifold of $X(\lambda)$ with respect to I_y and gives the minimum volume in its homology class, where $y := \frac{\lambda_n - \lambda_m}{|\lambda_n - \lambda_m|}$.*

Proof. Let $\eta \in SO(3)$ satisfies $\eta i = y$. If we write $\mu_\lambda = (\mu_{\lambda,1}, \mu_{\lambda,2}, \mu_{\lambda,3})$ with respect to the decomposition $\mathrm{Im}\mathbb{H} = \mathbb{R}\eta_1 \oplus \mathbb{R}\eta_2 \oplus \mathbb{R}\eta_3$, then $\mu_{\lambda,2}$ and $\mu_{\lambda,3}$ are constant on $\mu_\lambda^{-1}([-\lambda_n, -\lambda_m])$. Hence we have $d\mu_{\lambda,\alpha}|_{\mu_\lambda^{-1}([-\lambda_n, -\lambda_m])} = 0$ for $\alpha = 2, 3$, which gives $\omega_{\lambda,\eta\mathbb{C}}|_{\mu_\lambda^{-1}([-\lambda_n, -\lambda_m])} = 0$. □

4.2 Topology

In this subsection we review the construction of the deformation retracts of $X(\lambda)$ following [3, 5]. See also [8]. In the case of toric hyper-Kähler varieties, the deformation retracts are constructed in [3].

For $(-\lambda_n, -\lambda_m) \subset Y(\lambda)$, the orientation of $\mu_\lambda^{-1}([-\lambda_n, -\lambda_m])$ is determined as follows. By taking a smooth section $(-\lambda_n, -\lambda_m) \to \mu_\lambda^{-1}((-\lambda_n, -\lambda_m))$ of μ_λ, a coordinate (s, t) on $\mu_\lambda^{-1}((-\lambda_n, -\lambda_m))$ is naturally given where $t \in \mathbb{R}/2\pi\mathbb{Z}$ is the parameter of S^1-action and a function $s : \mu_\lambda^{-1}((-\lambda_n, -\lambda_m)) \to \mathbb{R}$ is given by

$$s(p) := \frac{\lambda_n + \mu_\lambda(p)}{\lambda_n - \lambda_m}$$

for $p \in \mu_\lambda^{-1}((-\lambda_n, -\lambda_m))$. Then the orientation of $\mu_\lambda^{-1}([-\lambda_n, -\lambda_m])$ is given by $ds \wedge dt$. Therefore, $\mu_\lambda^{-1}([-\lambda_n, -\lambda_m])$ and $\mu_\lambda^{-1}([-\lambda_m, -\lambda_n])$ are same as manifolds but have opposite orientations.

For $n, m, l \in \mathbb{N}$, $\mu_\lambda^{-1}([-\lambda_n, -\lambda_m]) \cup \mu_\lambda^{-1}([-\lambda_m, -\lambda_l])$ and $\mu_\lambda^{-1}([-\lambda_n, -\lambda_l])$ determines the same homology class since the boundary of $\mu_\lambda^{-1}(\triangle_{n,m,l})$ is given by $\mu_\lambda^{-1}([-\lambda_n, -\lambda_m]) \cup \mu_\lambda^{-1}([-\lambda_m, -\lambda_l]) \cup \mu_\lambda^{-1}([-\lambda_l, -\lambda_n])$, where

$$\triangle_{n,m,l} := \{-\alpha\lambda_n - \beta\lambda_m - \gamma\lambda_l \in \mathrm{Im}\mathbb{H};\ \alpha + \beta + \gamma = 1,\ \alpha, \beta, \gamma \geq 0\}.$$

We denote by $C_{n,m}$ the homology class determined by $\mu_\lambda^{-1}([-\lambda_n, -\lambda_m])$. Then the above observation implies

$$C_{n,m} + C_{m,l} + C_{l,n} = C_{n,m} + C_{m,n} = 0$$

for $n, m, l \in \mathbb{N}$.

If $n, m, l, h \in \mathbb{N}$ satisfies $n \neq h$, $n \neq m$ and $l \neq h$ then the intersection number $C_{n,m} \cdot C_{l,h}$ is given by

$$C_{n,m} \cdot C_{l,h} = \begin{cases} 1 \ (m = l) \\ 0 \ (m \neq l) \end{cases}$$

and $C_{n,m} \cdot C_{n,m} = -2$.

Since the subset of $(\mathrm{Im}\mathbb{H})_0^{\mathbb{N}}$ consisting of generic elements is connected in $(\mathrm{Im}\mathbb{H})_0^{\mathbb{N}}$, the topological structure of $X(\lambda)$ does not depend on λ. Consequently, it suffices to study $X(\hat{\lambda})$ for investigating the topology of $X(\lambda)$, where $\hat{\lambda}$ is the special one defined by $\hat{\lambda}_n := (n^2, 0, 0) \in \mathrm{Im}\mathbb{H}$.

Proposition 3. *There exists a deformation retract of* $\mu_{\hat{\lambda}}^{-1}(\bigcup_{n\in\mathbb{N}}[-\hat{\lambda}_n, -\hat{\lambda}_{n+1}]) \subset X(\hat{\lambda})$.

Proof. There is a deformation retract

$$F : \mathrm{Im}\mathbb{H} \times [0, 1] \to \mathrm{Im}\mathbb{H}$$

which satisfy $F(\cdot, 0) = id_{\mathrm{Im}\mathbb{H}}$, $F(\mathrm{Im}\mathbb{H}, 1) = \bigcup_{n\in\mathbb{N}}[-\hat{\lambda}_n, -\hat{\lambda}_{n+1}]$ and $F(\zeta, 1) = \zeta$ for $\zeta \in \bigcup_{n\in\mathbb{N}}[-\hat{\lambda}_n, -\hat{\lambda}_{n+1}]$. Then we have the horizontal lift $\tilde{F} : X(\hat{\lambda}) \times [0, 1] \to X(\hat{\lambda})$ of F by using the S^1-connection on $X(\hat{\lambda})^*$ naturally induced from the hyper-Kähler metric on $X(\hat{\lambda})^*$. The map $\tilde{F}$ is a deformation retract as we expect. □

Corollary 1. *The second homology group* $H_2(X(\lambda), \mathbb{Z})$ *is generated by* $\{C_{n,m};\ n, m \in \mathbb{N}\}$.

Thus we obtain the followings.

Theorem 3. *Let* $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{N}}$ *be generic. Then* $H_2(X(\lambda), \mathbb{Z})$ *is a free* $\mathbb{Z}$*-module generated by* $\{C_{n,m};\ n, m \in \mathbb{N}\}$ *with relations*

$$C_{n,m} + C_{m,l} + C_{l,n} = 0, C_{n,m} + C_{m,n} = 0$$

for all $n, m, l \in \mathbb{N}$*. Moreover the intersection form on* $H_2(X(\lambda), \mathbb{Z})$ *is given by*

$$C_{n,m} \cdot C_{l,h} = \begin{cases} 1 \ (m = l) \\ 0 \ (m \neq l) \end{cases}$$

and $C_{n,m} \cdot C_{n,m} = -2$ *for* $n, m, l, h \in \mathbb{N}$ *taken to be* $n \neq h$*,* $n \neq m$ *and* $l \neq h$*.*

4.3 Period Maps

Let $[\omega_\lambda] \in H^2(X(\lambda), \mathbb{R}) \otimes \mathrm{Im}\mathbb{H}$ be the cohomology class of ω_λ. In this subsection we compute $[\omega_\lambda]$, that is, compute the value of $\langle [\omega_\lambda], C_{n,m}\rangle := \int_{C_{n,m}} \omega_\lambda \in \mathrm{Im}\mathbb{H}$ for all $n, m \in \mathbb{N}$ along [8]. In the case of finite topological type of toric hyper-Kähler varieties, the period maps are computed in [12].

Theorem 4. *Let* $\lambda \in (\mathrm{Im}\mathbb{H})_0^{\mathbb{N}}$ *be generic. Then*

$$\langle [\omega_\lambda], C_{n,m}\rangle = \lambda_n - \lambda_m$$

for all $n, m \in \mathbb{N}$*.*

Proof. Take a smooth path $\gamma : [0, 1] \to \mathrm{Im}\mathbb{H}$ such that $\gamma(0) = -\lambda_n$, $\gamma(1) = -\lambda_m$ and $\gamma(s) \in Y(\lambda)$ for $s \in (0, 1)$. Since the homology class represented by $\mu_\lambda^{-1}(\gamma([0, 1]))$ is $C_{n,m}$, we have

$$\langle [\omega_\lambda], C_{n,m}\rangle = \int_{\mu_\lambda^{-1}(\gamma([0,1]))} \omega_\lambda.$$

Take the local coordinate $(t, \mu_{\lambda,1}, \mu_{\lambda,2}, \mu_{\lambda,3})$ of an open subset of $X(\lambda)^*$, where $\mu_\lambda = (\mu_{\lambda,1}, \mu_{\lambda,2}, \mu_{\lambda,3})$ and t is the coordinate of S^1-action. Then the local coordinate (s, t) on $\mu_\lambda^{-1}(\gamma([0, 1]))$ is given by $(t, \mu_{\lambda,1} \circ \gamma(s), \mu_{\lambda,2} \circ \gamma(s), \mu_{\lambda,3} \circ \gamma(s))$. By using this, we can see that

$$\omega_{\lambda,\alpha} = \gamma'_\alpha(s) \frac{1}{2\pi} ds \wedge dt$$

for $\alpha = 1, 2, 3$, where $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s)) \in \mathrm{Im}\mathbb{H} = \mathbb{R}^3$. Hence we have

$$\begin{aligned} \int_{\mu_\lambda^{-1}(\gamma([0,1]))} \omega_{\lambda,\alpha} &= \int_{\mu_\lambda^{-1}(\gamma([0,1]))} \gamma'_\alpha(s) \frac{1}{2\pi} ds \wedge dt \\ &= \int_0^{2\pi} \frac{1}{2\pi} dt \int_0^1 \gamma'_\alpha(s) ds \\ &= \gamma_\alpha(1) - \gamma_\alpha(0) = \lambda_{n,\alpha} - \lambda_{m,\alpha}. \end{aligned}$$

□

5 Holomorphic Symplectic Structures

In this section we regard a hyper-Kähler manifold (X, g, I_1, I_2, I_3) as a complex manifold by I_1. Then the holomorphic 2-form $\omega_{\mathbb{C}} = \omega_2 + \sqrt{-1}\omega_3$ is called the holomorphic symplectic structure, and the cohomology class $[\omega_2 + \sqrt{-1}\omega_3]$ is called the holomorphic symplectic class.

Let λ^γ as in Example 1 of Sect. 3. Then, we can see that the holomorphic symplectic class $[\omega_{\lambda^\gamma,\mathbb{C}}]$ is independent of γ by Theorem 4.

Theorem 5 ([9]). *The holomorphic symplectic structures $\omega_{\lambda^\gamma,\mathbb{C}}$ are independent of γ. In particular, $X(\lambda^\gamma)$ and $X(\lambda^{\hat{\gamma}})$ are biholomorphic for all $\gamma, \hat{\gamma} > 1$.*

Since the function $4 - \frac{2}{\gamma+1}$ gives one-to-one correspondence between open intervals $(1, \infty)$ and $(3, 4)$, we have the following conclusion by combining Theorems 2 with 5.

Theorem 6. *Let $\alpha \in (3, 4)$. Then there is a complex manifold X and the family of Ricci-flat Kähler metrics $\{g_\alpha\}_{3<\alpha<4}$ whose volume growth satisfies*

$$Ar^\alpha \leq V_{g_\alpha}(p_0, r) \leq Br^\alpha$$

for some positive constants A, B.

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The Fixed Point Set of a Holomorphic Isometry and the Intersection of Two Real Forms in the Complex Grassmann Manifold

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Abstract We study the fixed point set of a holomorphic isometry of the complex Grassmann manifold and the intersection of two real forms which are congruent to the real Grassmann manifold. Furthermore, we investigate the relation between them.

1 Introduction

Let M be a compact Riemannian symmetric space. A subset $A \subset M$ is called an *antipodal set* if $s_x(y) = y$ for any $x, y \in A$, where s_x denotes the geodesic symmetry at x. The maximal cardinality of antipodal sets of M is called the *2-number* of M denoted by $\#_2 M$. An antipodal set $A \subset M$ with $\#A = \#_2 M$ is called *great*, where $\#A$ denotes the cardinality of A. These notions were introduced by Chen-Nagano [1].

Let M be a Hermitian symmetric space of compact type and let σ be an involutive anti-holomorphic isometry of M. Then the fixed point set of σ is called a *real form* of M.

In the previous papers [6, 7] and [8] the second and the third authors proved that the intersection of two real forms L_1 and L_2 in a Hermitian symmetric space M of compact type is an antipodal set if $L_1 \cap L_2$ is discrete by making use of Chen-Nagano theory. They also proved that the intersection is a great antipodal set if L_1

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and L_2 are congruent. Here L_1 and L_2 are congruent if there is an element g in the identity component of the group of all holomorphic isometries of M such that $gL_1 = L_2$.

Let $F(g, X)$ denote the fixed point set $F(g, X) = \{x \in X \mid g(x) = x\}$ for a set X and a bijection $g : X \to X$. The following lemma is easily seen.

Lemma 1. *Let $g_1, g_2 : X \to X$ be bijections of a set X. Then we have*

$$F(g_1, X) \cap F(g_2, X) = F(g_2^{-1} g_1, X) \cap F(g_i, X) \quad (i = 1, 2).$$

In particular, we have

$$F(g_1, X) \cap F(g_2, X) \subset F(g_2^{-1} g_1, X).$$

Let $L_1 = F(\sigma_1, M)$ and $L_2 = F(\sigma_2, M)$ be real forms defined by involutive anti-holomorphic isometries σ_1 and σ_2 of M respectively. Since σ_1 and σ_2 are anti-holomorphic, $\sigma_2^{-1}\sigma_1$ is holomorphic. According to Lemma 1, if $F(\sigma_2^{-1}\sigma_1, X)$ is discrete, $L_1 \cap L_2$ is discrete and if $F(\sigma_2^{-1}\sigma_1, X)$ is an antipodal set, $L_1 \cap L_2$ is an antipodal set. From this viewpoint we investigate the necessary and sufficient condition that the fixed point set of a holomorphic isometry of M is discrete and the relation between the intersection $L_1 \cap L_2$ and $F(\sigma_2^{-1}\sigma_1, X)$ in [4].

In this article, we consider the case where a Hermitian symmetric space of compact type is the complex Grassmann manifold. We firstly investigate the fixed point set of a holomorphic isometry of the complex Grassmann manifold which belongs to the identity component of the group of all holomorphic isometries in Sect. 2. In Sect. 3 we investigate the intersection of two real forms which are congruent to the real Grassmann manifold which is naturally embedded in the complex Grassmann manifold which is treated in [5]. In Sect. 4 we refer to the relation between the intersection of such real forms and the fixed point set of a holomorphic isometry. We showed these results in more general situation in [4]. We investigated the relation between the fixed point set of a holomorphic isometry in a Hermitian symmetric space of compact type and the intersection of two real forms in a Hermitian symmetric space of compact type where we made use of symmetric triads introduced by the first author in [3], which gives an alternative proof of the fact that the intersection of two real forms is an antipodal set. These are referred in Sect. 5.

2 The Fixed Point Set of a Holomorphic Isometry of the Complex Grassmann Manifold

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and let $G_k(\mathbb{K}^n)$ be the Grassmann manifold which is the set of k-dimensional $\mathbb{K}$-subspaces in $\mathbb{K}^n$. It is known that the complex Grassmann manifold $G_k(\mathbb{C}^n)$ is a Hermitian symmetric space of compact type. The natural action of $U(n)$ on $\mathbb{C}^n$ induces an action of $U(n)$ on $G_k(\mathbb{C}^n)$, which coincides with that of $SU(n)$. Let $A(G_k(\mathbb{C}^n))$ denote the group of holomorphic isometries of $G_k(\mathbb{C}^n)$ and let $A_0(G_k(\mathbb{C}^n))$ denote the identity component of $A(G_k(\mathbb{C}^n))$. Then $A_0(G_k(\mathbb{C}^n))$ coincides with the action of $U(n)$ on $G_k(\mathbb{C}^n)$.

We show the necessary and sufficient condition that $F(g, G_k(\mathbb{C}^n))$ is discrete for $g \in U(n)$ in two ways. In one way we use linear algebra and in another way we use a root system.

Let $W_1, \ldots, W_s$ be $\mathbb{K}$-subspaces of $\mathbb{K}^n$ which satisfy $\mathbb{K}^n = W_1 \oplus \cdots \oplus W_s$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$. For positive integers $k, k_1, \ldots, k_s$ satisfying $k = k_1 + \cdots + k_s$ we have

$$G_{k_1}(W_1) \times \cdots \times G_{k_s}(W_s) = \{U_1 \oplus \cdots \oplus U_s \in G_k(\mathbb{K}^n) \mid U_a \in G_{k_a}(W_a)\ (1 \le a \le s)\}.$$

Lemma 2. *For $g \in U(n)$, let V_a $(1 \le a \le s)$ be the eigenspace of g with eigenvalue α_a, where $\alpha_a \neq \alpha_b$ if $a \neq b$ $(1 \le a, b \le s)$. Then,*

$$F(g, G_k(\mathbb{C}^n)) = \bigcup_{\substack{k_1 + \cdots + k_s = k \\ 0 \le k_a \le \dim V_a\ (1 \le a \le s)}} G_{k_1}(V_1) \times \cdots \times G_{k_s}(V_s).$$

Proof. Let $V \in G_{k_1}(V_1) \times \cdots \times G_{k_s}(V_s)$, then V can be represented as

$$V = U_1 \oplus \cdots \oplus U_s, \qquad U_a \in G_{k_a}(V_a)\ \ (1 \le a \le s).$$

Then we have

$$\begin{aligned} gV &= gU_1 \oplus \cdots \oplus gU_s \\ &= U_1 \oplus \cdots \oplus U_s = V, \end{aligned}$$

hence $V \in F(g, G_k(\mathbb{C}^n))$. Conversely, let $V \in F(g, G_k(\mathbb{C}^n))$. Then, $gV = V$ and we have

$$V = V \cap V_1 \oplus \cdots \oplus V \cap V_s.$$

If we set $k_a = \dim(V \cap V_a)$ $(1 \le a \le s)$, then

$$V \in G_{k_1}(V_1) \times \cdots \times G_{k_s}(V_s).$$

□

The former part of the following theorem follows from Lemma 2 immediately and the latter part of it follows from Lemma 2 and the proof of Proposition 6.1 in [1].

Theorem 1. *For $g \in U(n)$, $F(g, G_k(\mathbb{C}^n))$ is discrete if and only if the multiplicity of each eigenvalue of g is equal to 1. In this case*

$$F(g, G_k(\mathbb{C}^n)) = \{\langle v_{i_1}, \dots, v_{i_k}\rangle_{\mathbb{C}} \mid 1 \le i_1 < \cdots < i_k \le n\}$$

is a great antipodal set of $G_k(\mathbb{C}^n)$, where v_i $(1 \le i \le n)$ is a unit vector of each eigenspace of g.

We set $G = SU(n)$, then G is a connected compact semisimple Lie group and its Lie algebra $\mathfrak{g}$ is identified with

$$\mathfrak{su}(n) = \{X \in M(n, \mathbb{C}) \mid X = -{}^t\bar{X},\ \mathrm{tr}(X) = 0\},$$

where $M(n, \mathbb{C})$ denotes the set of $n \times n$ complex matrices and $\mathrm{tr}(X)$ denotes the trace of X. We take an $\mathrm{Ad}(G)$-invariant inner product $\langle X, Y\rangle = -\mathrm{tr}(XY)$ on $\mathfrak{su}(n)$. We take $J \in \mathfrak{su}(n)$ as

$$J = \sqrt{-1}\begin{bmatrix} (1-\frac{k}{n})1_k & 0 \\ 0 & -\frac{k}{n}1_{n-k} \end{bmatrix},$$

where 1_k (resp. 1_{n-k}) denotes the identity matrix of degree k (resp. $n-k$). J satisfies $(\mathrm{ad}J)^3 = -\mathrm{ad}J$. Then the adjoint orbit $M := \mathrm{Ad}(G)J \subset \mathfrak{g}$ through J is described as $SU(n)/S(U(k) \times U(n-k))$ which is $G_k(\mathbb{C}^n)$. We set $\mathfrak{k} = \mathrm{Ker}(\mathrm{ad}J)$ and $\mathfrak{m} = \mathrm{Im}(\mathrm{ad}J)$. Then

$$\mathfrak{k} = \left\{ \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \middle| \ X = -{}^t\bar{X}, Y = -{}^t\bar{Y}, \mathrm{tr}(X) + \mathrm{tr}(Y) = 0 \right\},$$

$$\mathfrak{m} = \left\{ \begin{bmatrix} 0 & Z \\ -{}^t\bar{Z} & 0 \end{bmatrix} \middle| \ Z \in M(k, n-k, \mathbb{C}) \right\},$$

where $M(k, n-k, \mathbb{C})$ denotes the set of $k \times (n-k)$ complex matrices. Then we have an orthogonal direct sum decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. The automorphism $e^{\pi \mathrm{ad}J}$ of $\mathfrak{g}$ is involutive and $\mathfrak{k}$ (resp. $\mathfrak{m}$) is the $(+1)$ (resp. (-1))–eigenspace of $e^{\pi \mathrm{ad}J}$. The action of $\mathrm{ad}J$ on $\mathfrak{m}$ defines a complex structure on $\mathfrak{m}$ which can be identified with the tangent space of M at J, hence it defines a complex structure on M. The natural action of $SU(n)$ on M gives holomorphic isometries of M.

We take a maximal abelian subspace $\mathfrak{t} \subset \mathfrak{k}$ as

$$\mathfrak{t} = \left\{ \sqrt{-1}\mathrm{diag}(t_1, \dots, t_n) \middle| \ t_i \in \mathbb{R}, \sum_{i=1}^{n} t_i = 0 \right\},$$

where $\mathrm{diag}(t_1, \dots, t_n)$ denotes a diagonal matrix

$$\mathrm{diag}(t_1, \dots, t_n) = \begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{bmatrix}.$$

We denote by E_{ij} the $n \times n$ matrix whose (p, q)-element is 1 if $(p, q) = (i, j)$ and is 0 otherwise. We define $\alpha_{ij} \in \mathfrak{t}$ $(1 \le i, j \le n, i \ne j)$ as

$$\alpha_{ij} := \sqrt{-1}(E_{ii} - E_{jj}) \in \mathfrak{t}$$

and we set

$$\mathfrak{g}_{\alpha_{ij}} := \mathbb{C}E_{ij}.$$

Then α_{ij} is a root of $\mathfrak{g}$ with respect to $\mathfrak{t}$ and $\mathfrak{g}_{\alpha_{ij}}$ is the root space corresponding to α_{ij}.

We define

$$F_{ij} := E_{ij} - E_{ji},$$
$$G_{ij} := \sqrt{-1}(E_{ij} + E_{ji}).$$

for $1 \le i < j \le n$. Then we have

$$\mathfrak{g} = \mathfrak{t} + \sum_{1 \le i < j \le n} (\mathbb{R}F_{ij} + \mathbb{R}G_{ij}).$$

We set

$$T = \exp \mathfrak{t} = \left\{ \mathrm{diag}(e^{\sqrt{-1}t_1}, \dots, e^{\sqrt{-1}t_n}) \,\middle|\, t_i \in \mathbb{R}, \sum_{i=1}^{n} t_i = 0 \right\},$$

which is a maximal torus of $SU(n)$.

Since $SU(n)$ is a normal subgroup of $U(n)$, $\mathrm{Ad}(U(n))$ acts on $\mathfrak{su}(n)$ which induces holomorphic isometries of $M = \mathrm{Ad}(SU(n))J$. The identity component of the group of all holomorphic isometries of M is isomorphic to $\mathrm{Ad}(U(n))$.

Lemma 3. *For $a = \exp H \in T$ with $H = \sqrt{-1}\mathrm{diag}(t_1, \dots, t_n) \in \mathfrak{t}$ we have*

$$F(\mathrm{Ad}(a), \mathfrak{g}) = \mathfrak{t} + \sum_{\substack{1 \le i < j \le n \\ t_i - t_j \in 2\pi\mathbb{Z}}} (\mathbb{R}F_{ij} + \mathbb{R}G_{ij}).$$

Theorem 2. *Let* $M = \mathrm{Ad}(SU(n))J \subset \mathfrak{su}(n)$ *be an expression of* $G_k(\mathbb{C}^n)$ *as an adjoint orbit. The fixed point set* $F(g, M)$ *for* $g \in SU(n)$ *is discrete if and only if there is* $g_1 \in SU(n)$ *and* $a \in T$ *such that* $g = g_1 a g_1^{-1}$ *and*

$$a = \exp\sqrt{-1}\mathrm{diag}(t_1, \ldots, t_n), \quad t_i - t_j \notin 2\pi\mathbb{Z} \quad (1 \leq i < j \leq n).$$

In the case $F(g, G_k(\mathbb{C}^n))$ *is a great antipodal set of* $G_k(\mathbb{C}^n)$*.*

3 The Intersection of Two Real Grassmann Manifolds in the Complex Grassmann Manifold

Let $\tau : \mathbb{C}^n \to \mathbb{C}^n$ be an involutive transformation defined by $\tau(z) = \bar{z}$ $(z \in \mathbb{C}^n)$. We also denote by τ the involutive isometry of $G_k(\mathbb{C}^n)$ induced by the map. Then the fixed point set $F(\tau) := F(\tau, G_k(\mathbb{C}^n))$ is $G_k(\mathbb{R}^n)$ naturally embedded in $G_k(\mathbb{C}^n)$.

Let $u \in U(n)$ and we have

$$uG_k(\mathbb{R}^n) = G_k(u\mathbb{R}^n) = uF(\tau) = F(u\tau u^{-1}).$$

Lemma 4 ([5]). *For any* $u \in U(n)$ *there exists* $z_i \in U(1)$ $(1 \leq i \leq n)$ *and positively oriented orthogonal basis* $v_1, \ldots, v_n$ *and* $w_1, \ldots, w_n$ *of* $\mathbb{R}^n$ *which satisfy*

$$uw_i = z_i v_i \quad (1 \leq i \leq n), \qquad \det u = z_1 \cdots z_n.$$

We define an equivalent relation $\sim$ *on* $\{1, \ldots, n\}$ *such that* $i \sim j$ *if* $z_i = \pm z_j$ *and*

$$\{1, \ldots, n\} = N_1 \cup \cdots \cup N_s$$

is the decomposition to the equivalent classes with respect to $\sim$*. If unit vectors* $v, w \in \mathbb{R}^n$ *and* $z \in \mathbb{C}$ *satisfy* $uw = zv$*, then there exists* $1 \leq a \leq s$ *which satisfy*

$$v \in \bigoplus_{i \in N_a} \langle v_i \rangle_{\mathbb{R}}, \quad w \in \bigoplus_{i \in N_a} \langle w_i \rangle_{\mathbb{R}}, \quad z = \pm z_i \ (i \in N_a).$$

Theorem 3 ([5]). *Under the situation of Lemma 4 and for* $0 \leq k \leq n$ *we have*

$$G_k(\mathbb{R}^n) \cap G_k(u\mathbb{R}^n) = \bigcup_{\substack{k_1 + \cdots + k_s = k \\ 0 \leq k_a \leq \#N_a \ (1 \leq a \leq s)}} G_{k_1}\left(\bigoplus_{i_1 \in N_1} \langle v_{i_1} \rangle_{\mathbb{R}}\right) \times \cdots \times G_{k_s}\left(\bigoplus_{i_s \in N_s} \langle v_{i_s} \rangle_{\mathbb{R}}\right)$$

in $G_k(\mathbb{C}^n)$*. The intersection of* $G_k(\mathbb{R}^n)$ *and* $G_k(u\mathbb{R}^n)$ *is discrete if and only if* $\#N_a = 1$ *for all* a*. In this case these intersect transversally and*

$$G_k(\mathbb{R}^n) \cap G_k(u\mathbb{R}^n) = \{\langle v_{i_1}, \dots, v_{i_k}\rangle_{\mathbb{C}} \mid 1 \le i_1 < \cdots < i_k \le n\},$$

which is a great antipodal set of $G_k(\mathbb{C}^n)$*.*

4 The Intersection and the Fixed Point Set

Lemma 5. *Let* (G, K) *be a compact symmetric pair and let* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ *be the canonical decomposition of the Lie algebra* $\mathfrak{g}$ *of* G *associated to* (G, K)*. Let* $\mathfrak{a} \subset \mathfrak{m}$ *be a maximal abelian subspace and set* $A = \exp \mathfrak{a} \subset G$*. Then we have* $G = KAK$*.*

Proof. Let o be the base point of G/K. Then Ao is a maximal torus in G/K and we have

$$G/K = \bigcup_{k \in K} kAo.$$

Thus for any $go \in G/K$ there exist $k \in K$ and $a \in A$ such that $go = kao$, which implies $(ka)^{-1}g \in K$. So we have $g \in KAK$. □

Theorem 4. *When* $u \in U(n)$*,* $F(\tau) \cap uF(\tau)$ *is discrete if and only if* $F(u\tau u^{-1}\tau^{-1})$ *is discrete. In this case they are equal and great antipodal sets of* $G_k(\mathbb{C}^n)$*.*

Proof. When we apply Lemma 5 to the compact symmetric pair $(U(n), SO(n))$, we obtain the decomposition of u as

$$u = k_1 \mathrm{diag}(z_1, \dots, z_n) k_2,$$

where $k_1, k_2 \in SO(n)$ and $z_1, \dots, z_n \in U(1)$. We define a equivalent relation $\sim$ on $\{1, \dots, n\}$ such that $i \sim j$ if $z_i = \pm z_j$ $(i, j \in \{1, \dots, n\})$. We decompose $\{1, \dots, n\}$ into a disjoint union of equivalent classes N_a $(1 \le a \le s)$ with respect to $\sim$:

$$\{1, \dots, n\} = N_1 \cup \cdots \cup N_s.$$

Then $F(\tau) \cap uF(\tau)$ is discrete if and only if $\#N_a = 1$ for every a $(1 \le a \le s)$ by Theorem 3. Moreover, it is equivalent to that $z_1, \dots, z_n$ are different to each other under multiplying ± 1. It is also equivalent to that $z_1^2, \dots, z_n^2$ are different to each other. When we set $a = \mathrm{diag}(z_1, \dots, z_n)$, $a^{-1} = \bar{a}$. Since $\tau k_j = k_j \tau$ for $j = 1, 2$, we have

$$\begin{aligned} u\tau u^{-1}\tau^{-1} &= k_1 a k_2 \tau k_2^{-1} a^{-1} k_1^{-1} \tau^{-1} \\ &= k_1 a k_2 \tau k_2^{-1} \bar{a} k_1^{-1} \tau^{-1} \\ &= k_1 a \tau \bar{a} \tau^{-1} k_1^{-1} \\ &= k_1 a^2 k_1^{-1}. \end{aligned}$$

Hence $F(u\tau u^{-1}\tau^{-1})$ is discrete if and only if $z_1^2, \dots, z_n^2$ are different to each other. When $F(u\tau u^{-1}\tau^{-1})$ is discrete, it is a great antipodal set by Theorem 1. □

5 Further Results

Let G be a connected compact semisimple Lie group and $\mathfrak{g}$ its Lie algebra. We take an $\mathrm{Ad}(G)$-invariant inner product $\langle , \rangle$ on $\mathfrak{g}$. We take a nonzero element $J \in \mathfrak{g}$ satisfying $(\mathrm{ad}J)^3 = -\mathrm{ad}J$. The adjoint orbit $M = \mathrm{Ad}(G)J \subset \mathfrak{g}$ is a Hermitian symmetric space of compact type with respect to the induced metric from $\langle , \rangle$. For $g \in G$, the action of $\mathrm{Ad}(g)$ on M is holomorphic and isometric. Conversely, every Hermitian symmetric space of compact type is constructed in this manner.

Theorem 5 ([4]). *Under the situation above, the fixed point set $F(\mathrm{Ad}(g), M)$ is discrete if and only if g is a regular element of G. In the case $F(\mathrm{Ad}(g), M)$ is a great antipodal set of M.*

Here we call $g \in G$ a *regular element* if $\dim F(\mathrm{Ad}(g), \mathfrak{g}) = \mathrm{rank}(G)$.

Let $A(M)$ denote the group of all holomorphic isometries of a Hermitian symmetric space M of compact type and let $A_0(M)$ denote its identity component. If $M = \mathrm{Ad}(G)J \subset \mathfrak{g}$ for a connected compact semisimple Lie group G with Lie algebra $\mathfrak{g}$, $A_0(M)$ coincides with $\{\mathrm{Ad}(g)|_M \mid g \in G\}$. Hence Theorem 5 says that if $g \in A_0(M)$, $F(\mathrm{Ad}(g), M)$ is discrete if and only if g is a regular element. For $g \in A(M) - A_0(M)$, we obtain a necessary and sufficient condition that $F(\mathrm{Ad}(g), M)$ is discrete when M is irreducible in [4].

Let $M = \mathrm{Ad}(G)J \subset \mathfrak{g}$ be a Hermitian symmetric space of compact type and let $L = F(\tau, M)$ be a real form of M. Let I_τ be an involutive automorphism of G defined by $I_\tau(x) = \tau x \tau^{-1}$ for $x \in G$. Then $(G, F(I_\tau))$ is a compact symmetric pair, where $F(I_\tau) := F(I_\tau, G)$. By Lemma 5 we have $G = F(I_\tau)AF(I_\tau)$ for a suitable torus A of G. Hence when we consider $L \cap gL$ for $g \in G$, it suffices to consider $L \cap aL$ for $a \in A$.

Theorem 6 ([4]). *$L \cap aL$ is discrete if and only if a is a regular element of G. In this case $L \cap aL = M \cap \mathfrak{a} = W(R)J$, where $A = \exp \mathfrak{a}$, R denotes the restricted root system with respect to $\mathfrak{a}$ and $W(R)$ denotes the Weyl group of R.*

Let $L_1 = F(\tau_1, M)$ and $L_2 = F(\tau_2, M)$ be real forms of M. We assume that there is no element $g \in G$ with $gL_1 = L_2$. Then there is no element $g \in G$ with $g\tau_1 g^{-1} = \tau_2$. In this case we may assume that I_{τ_1} and I_{τ_2} are commutative.

due to the classification of real forms. Then we have two compact symmetric pairs $(G, F(I_{\tau_1}))$ and $(G, F(I_{\tau_2}))$ and their canonical decompositions $\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{p}_1 = \mathfrak{k}_2 \oplus \mathfrak{p}_2$. Since I_{τ_1} and I_{τ_2} are commutative, we have

$$\mathfrak{g} = (\mathfrak{k}_1 \cap \mathfrak{k}_2) \oplus (\mathfrak{p}_1 \cap \mathfrak{p}_2) \oplus (\mathfrak{k}_1 \cap \mathfrak{p}_2) \oplus (\mathfrak{k}_2 \cap \mathfrak{p}_1).$$

We take a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}_1 \cap \mathfrak{p}_2$ so that $J \in \mathfrak{a}$. We have $G = F(I_{\tau_1})(\exp \mathfrak{a})F(I_{\tau_2})$ by [2]. We consider $L_1 \cap gL_2$ for $g \in G$, it suffices to consider $L_1 \cap aL_2$ for $a = \exp H \in \exp \mathfrak{a}$.

A compact symmetric triad $(G, F(I_{\tau_1}), F(I_{\tau_2}))$ induces the symmetric triad $(\tilde{\Sigma}, \Sigma, W)$ [3]. We call $H \in \mathfrak{a}$ a *regular point* if $\langle \lambda, H \rangle \notin \pi\mathbb{Z}$ for each $\lambda \in \Sigma$ and $\langle \alpha, H \rangle \notin \frac{\pi}{2} + \pi\mathbb{Z}$ for each $\alpha \in W$.

Theorem 7 ([4]). *$L_1 \cap aL_2$ is discrete if and only if H is a regular point of $(\tilde{\Sigma}, \Sigma, W)$. In this case $L_1 \cap aL_2 = W(\tilde{\Sigma})J = W(R_1)J \cap \mathfrak{a} = W(R_2)J \cap \mathfrak{a}$. Here $W(\tilde{\Sigma})$ is the Weyl group of a root system $\tilde{\Sigma}$, R_i denotes the restricted root system of $(G, F(I_{\tau_i}))$ with respect to $\mathfrak{a}_i$, a maximal abelian subspace in $\mathfrak{p}_i$ including $\mathfrak{a}$ and $W(R_i)$ denotes the Weyl group of R_i for $i = 1, 2$.*

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Canonical Forms Under Certain Actions on the Classical Compact Simple Lie Groups

Osamu Ikawa

Abstract A maximal torus of a compact connected Lie group can be seen as a canonical form of adjoint action since any two maximal tori can be transformed each other by an inner automorphism. A. Kollross defined a σ-action on a compact Lie group which is a generalization of the adjoint action. Since a σ-action is hyperpolar, it has a canonical form called a section. In this paper we study the structure of the orbit space of a σ-action and properties of each orbit, such as minimal, austere and totally geodesic, using symmetric triads introduced by the author, when σ is an involution of outer type on the compact simple Lie groups of classical type. As an application, we investigate the fixed point set of a holomorphic isometry of an irreducible Hermitian symmetric space of compact type which does not belong to the identity component of the group of holomorphic isometries.

1 σ-actions

Let G be a compact connected Lie group and σ an automorphism of G. Kollross defined a σ-action of G on itself by $g \cdot x = gx\sigma(x)^{-1}$ for $g, x \in G$. Clearly $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ for $g_1, g_2, x \in G$. If σ is identity, then σ-action is nothing but the adjoint action. We define two involutions θ_1 and θ_2 on $G \times G$ by

$$\theta_1(g, h) = (\sigma^{-1}(h), \sigma(g)), \qquad \theta_2(g, h) = (h, g).$$

The fixed point set $F(\theta_1, G \times G)$ of G is given by $F(\theta_1, G \times G) = \{(g, \sigma(g)) \mid g \in G\}$. Here for a set X and a map $\phi : X \to X$ we define $F(\phi, X) = \{x \in X \mid \phi(x) = x\}$. We use the notation throughout of the paper. Two involutions θ_1 and θ_2 commute each other if and only if $\sigma^2 = 1$. If we set $\Delta G = F(\theta_2, G \times G)$ then $(G \times G, F(\theta_1, G \times G))$ and $(G \times G, \Delta G)$ are compact symmetric pairs. Hence we can define a Hermann action of $F(\theta_1, G \times G)$ on $(G \times G)/\Delta G$ as follows:

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$$(g, \sigma(g))((a, b)\Delta G) = (ga, \sigma(g)b)\Delta G.$$

If we identify G with $(G \times G)/\Delta G$ in a natural manner then

$$(g, \sigma(g)) \cdot x = gx\sigma(g)^{-1}.$$

Hence σ-action is a kind of Hermann action. Since Hermann action is hyperpolar, so is σ-action. In order to study a section of the σ-action, denote by $\mathfrak{g}$ the Lie algebra of G. We define a closed subgroup K_σ of G by $K_\sigma = F(\sigma, G)$, then the Lie algebra $\mathfrak{k}_\sigma$ is given by $\mathfrak{k}_\sigma = F(\sigma, \mathfrak{g})$. Set $\mathfrak{m}_i = F(-\theta_i, \mathfrak{g} \times \mathfrak{g})$. Then $\mathfrak{m}_1 \cap \mathfrak{m}_2 = \{(X, -X) \mid X \in \mathfrak{k}_\sigma\}$. Take a maximal torus A of K_σ and denote by $\mathfrak{a}$ the Lie algebra of A. If we set $\hat{\mathfrak{a}} = \{(H, -H) \mid H \in \mathfrak{a}\}$ then $\hat{\mathfrak{a}}$ is a maximal abelian subspace of $\mathfrak{m}_1 \cap \mathfrak{m}_2$. Thus $\exp \hat{\mathfrak{a}} = \{(a, a^{-1}) \mid a \in A\}$ is a section of the σ-action [2]:

$$(G \times G)/\Delta = \bigcup_{g \in G} (g, \sigma(g)) \exp \hat{\mathfrak{a}}.$$

If we identify G with $(G \times G)/\Delta G$ then

$$G = \bigcup_{g \in G} gA\sigma(g)^{-1}.$$

Hence A can be considered as a canonical form for the σ-action. We call $\dim A = \mathrm{rank}(K_\sigma)$ the cohomogeneity of the σ-action. The following equivalent relation was introduced by T. Matsuki.

Definition 1 ([6]). Let (θ_1, θ_2) and (θ_1', θ_2') be two pairs of involutions on $G \times G$. Then (θ_1, θ_2) and (θ_1', θ_2') are *equivalent* if there exist an automorphism $\rho \in \mathrm{Aut}(G \times G)$ of $G \times G$ and $(a, b) \in G \times G$ such that

$$\theta_1' = \tau_{(a,b)} \rho \theta_1 \rho^{-1} \tau_{(a,b)}^{-1}, \quad \theta_2' = \rho \theta_2 \rho^{-1},$$

where $\tau_{(a,b)}$ denote the inner automorphism of $G \times G$ defined by (a, b). In this case we write $(\theta_1, \theta_2) \sim (\theta_1', \theta_2')$.

The relation $(\theta_1, \theta_2) \sim (\theta_1', \theta_2')$ means that the action of $F(\theta_1, G \times G)$ on $G \times G/F(\theta_2, G \times G)$ is essentially the same as that of $F(\theta_1', G \times G)$ on $G \times G/F(\theta_2', G \times G)$. Since the theory of adjoint action on a compact connected Lie group is well-known, we are interested in a σ-action which is essentially different from the adjoint action. In order to investigate such a σ-action we prepare the following lemma.

Lemma 1 (cf. [5]). *Let G be a compact connected Lie group and σ an automorphism of G. Set $\theta_1(g, h) = (\sigma^{-1}(h), \sigma(g))$ and $\theta_2(g, h) = (h, g)$. Then $(\theta_1, \theta_2) \sim (\theta_2, \theta_2)$ holds if and only if σ is an inner automorphism.*

Proof. Assume that $(\theta_1, \theta_2) \sim (\theta_2, \theta_2)$. Then there exist $(a, b) \in G$ and $\rho \in \mathrm{Aut}(G \times G)$ such that

$$\theta_1 = \tau_{(a,b)} \rho \theta_2 \rho^{-1} \tau_{(a,b)}^{-1}, \quad \theta_2 = \rho \theta_2 \rho^{-1}.$$

Hence $\theta_1 = \tau_{(a,b)} \theta_2 \tau_{(a,b)}^{-1}$. Applying (g, h) to the both sides of the equation we have

$$(\sigma^{-1}(h), \sigma(g)) = (ab^{-1}hba^{-1}, ba^{-1}gab^{-1}).$$

Hence $\sigma(g) = ba^{-1}gab^{-1}$, which is of inner type.

Conversely we assume that σ is of inner type. Then there exists $x \in G$ such that $\sigma = \tau_x$. Set $\rho = 1$. Then $\theta_2 = \rho\theta_2\rho^{-1}$ and $\tau_{(x,1)} \rho \theta_1 \rho^{-1} \tau_{(x,1)}^{-1} = \theta_2$. Hence $(\theta_1, \theta_2) \sim (\theta_2, \theta_2)$. □

By the lemma above we are interested in σ-actions in the case where σ is of outer type. The following lemma means that σ-action is essentially the same as $\tau_a \sigma \tau_a^{-1}$-action for $a \in G$.

Lemma 2. *Let G be a compact connected Lie group. For $\sigma \in \mathrm{Aut}(G)$ and $a \in G$ we define $\sigma' \in \mathrm{Aut}(G)$ by $\sigma' = \tau_a \sigma \tau_a^{-1}$. Set $\theta_1(g, h) = (\sigma^{-1}(h), \sigma(g))$, $\theta_1'(g, h) = (\sigma'^{-1}(h), \sigma'(g))$ and $\theta_2(g, h) = (h, g)$. Then $(\theta_1, \theta_2) \sim (\theta_1', \theta_2)$.*

Proof. It is clear from $\theta_1' = \tau_{(a,a)} \theta_1 \tau_{(a,a)}^{-1}$. □

When σ is an involution of a compact connected simple Lie group, we can determine which σ is of outer type by the following proposition.

Proposition 1. *Let G be a compact connected simple Lie group and σ be an involution of G. Then σ is of outer type if and only if* $\mathrm{rank}(G) > \mathrm{rank}(K_\sigma)$.

Proof. In general $\mathrm{rank}(G) \geq \mathrm{rank}(K_\sigma)$ holds. If σ is of inner type then $\mathrm{rank}(G) = \mathrm{rank}(K_\sigma)$. Hence if $\mathrm{rank}(G) > \mathrm{rank}(K_\sigma)$ then σ is of outer type. The converse follows from the list below. □

In the table below we can directly verify the type of σ when G is a classical group. When $G = E_6$ and σ is an outer automorphism, then the cohomogeneity of σ-action is equal to 4 [5]. Hence when $(G, K_\sigma) = (E_6, SU(6) \cdot SU(2))$, then σ is of inner type. When $(G, K_\sigma) = (E_6, SO(10) \cdot S^1), (E_7, E_6 \cdot S^1)$, then σ is of inner type since the coset manifold G/K_σ is an irreducible Hermitian symmetric space of compact type. When $G = E_7, E_8, F_4$ and G_2, then σ is of inner type since the Dynkin diagram of G has no nontrivial symmetry.

When σ is an outer automorphism whose order is finite and greater than two on a compact connected simple Lie group G, then $\mathfrak{g} = \mathfrak{so}(8), \mathrm{ord}(\sigma) = 3$ and $\mathfrak{k}_\sigma = \mathfrak{g}_2$ by the classification of automorphisms on a compact simple Lie algebra (Table 1).

Table 1 The type of σ for (G, K_σ)

(G, K_σ)	rank(G)	rank(K_σ)	σ
$(SU(n), SO(n))(n \geq 3)$	$n-1$	$[n/2]$	Outer
$(SU(2n), Sp(n))$	$2n-1$	n	Outer
$(E_6, Sp(4))$	6	4	Outer
(E_6, F_4)	6	4	Outer
$(SO(p+q), SO(p) \times SO(q))$	$[(p+q)/2]$	$[p/2]+[q/2]$	Outer$\Leftrightarrow$ p, q:odd
$(SU(p+q), SU(p) \times U(q))$	$p+q-1$	$p+q-1$	Inner
$(SO(2n), U(n))$	n	n	Inner
$(Sp(n), U(n))$	n	n	Inner
$(Sp(p+q), Sp(p) \times Sp(q))$	$p+q$	$p+q$	Inner
$(E_6, SU(6) \cdot SU(2))$	6	6	Inner
$(E_6, SO(10) \cdot S^1)$	6	6	Inner
$(E_7, SU(8))$	7	7	Inner
$(E_7, SO(12) \cdot SU(2))$	7	7	Inner
$(E_7, E_6 \cdot S^1)$	7	7	Inner
$(E_8, SO(16))$	8	8	Inner
$(E_8, E_7 \cdot SU(2))$	8	8	Inner
$(F_4, Sp(3) \cdot SU(2))$	4	4	Inner
$(F_4, SO(9))$	4	4	Inner
$(G_2, SU(2) \cdot SU(2))$	2	2	Inner

2 σ-actions and Symmetric Triads

In this section we denote by G a compact connected Lie group with Lie algebra $\mathfrak{g}$ and by σ an automorphism of G. Take an invariant metric $\langle\ ,\ \rangle$ on G. Denote by $\mathfrak{a}$ a maximal abelian subalgebra of $\mathfrak{k}_\sigma$.

2.1 General Case

For $\alpha \in \mathfrak{a}$ we define a subspace $\mathfrak{g}(\mathfrak{a}, \alpha)$ of $\mathfrak{g}^{\mathbb{C}}$, the complexification of $\mathfrak{g}$, by

$$\mathfrak{g}(\mathfrak{a}, \alpha) = \{X \in \mathfrak{g}^{\mathbb{C}} \mid [H, X] = \sqrt{-1}\langle \alpha, H \rangle X \quad (H \in \mathfrak{a})\}.$$

and set $\tilde{\Sigma} = \{\alpha \in \mathfrak{a} - \{0\} \mid \mathfrak{g}(\mathfrak{a}, \alpha) \neq \{0\}\}$. Then we have a direct sum decomposition of $\mathfrak{g}^{\mathbb{C}}$:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}(\mathfrak{a}, 0) \oplus \sum_{\alpha \in \tilde{\Sigma}} \mathfrak{g}(\mathfrak{a}, \alpha). \tag{1}$$

Denote by $^{-}$ the conjugation of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{g}$. If $\alpha \in \tilde{\Sigma}$ then $-\alpha \in \tilde{\Sigma}$ since $\overline{\mathfrak{g}(\mathfrak{a}, \alpha)} = \mathfrak{g}(\mathfrak{a}, -\alpha)$. The following lemma is clear from the definition of $\mathfrak{g}(\mathfrak{a}, \alpha)$.

Lemma 3. *(1)* $[\mathfrak{g}(\mathfrak{a}, \alpha), \mathfrak{g}(\mathfrak{a}, \beta)] \subset \mathfrak{g}(\mathfrak{a}, \alpha + \beta)$.
(2) $\mathfrak{g}(\mathfrak{a}, \alpha)$ *is* σ*-invariant.*

Lemma 4. *If we denote by* $\mathfrak{z}$ *the center of* $\mathfrak{g}$ *then* $\mathrm{span}(\tilde{\Sigma}) = \mathfrak{z}^{\perp} \cap \mathfrak{a}$.

Proof. For $H \in \mathfrak{a}$ we have

$$H \in \mathfrak{z} \Leftrightarrow [H, \mathfrak{g}^{\mathbb{C}}]{=}0 \Leftrightarrow [H, \mathfrak{g}(\mathfrak{g}, \alpha)]{=}0 \quad (\alpha \in \tilde{\Sigma}) \Leftrightarrow \langle \tilde{\Sigma}, H \rangle{=}0 \Leftrightarrow H \in \tilde{\Sigma}^{\perp}.$$

Here we used (1). □

Denote by Σ the root system of $\mathfrak{k}_{\sigma}$ with respect to $\mathfrak{a}$. Then Σ is a reduced root system. Denote by $m(\lambda)$ the multiplicity of $\lambda \in \Sigma$. Then $m(\lambda) = 2$.

2.2 In the Case When σ Is of Finite Order

In the sequel we assume that the order s of σ is finite. We define a subgroup of $U(1)$ by $\{\epsilon_1 = 1, \epsilon_2, \cdots, \epsilon_s\} = \{\epsilon \in U(1) \mid \epsilon^s = 1\}$. We define a subspace $\mathfrak{g}(\mathfrak{a}, \alpha, \epsilon_j)$ of $\mathfrak{g}(\mathfrak{a}, \alpha)$ by

$$\mathfrak{g}(\mathfrak{a}, \alpha, \epsilon_j) = \{X \in \mathfrak{g}(\mathfrak{a}, \alpha) \mid \sigma X = \epsilon_j X\}.$$

In particular $\mathfrak{g}(\mathfrak{a}, 0, 1) = \mathfrak{a}^{\mathbb{C}}$. By Lemma 3(2) we have

$$\mathfrak{g}(\mathfrak{a}, \alpha) = \sum_{j=1}^{s} \mathfrak{g}(\mathfrak{a}, \alpha, \epsilon_j).$$

The following lemma is clear from the definition of $\mathfrak{g}(\mathfrak{a}, \alpha, \epsilon_j)$.

Lemma 5. *(1)* $\mathfrak{g}(\mathfrak{a}, \alpha, \epsilon_j)$ *is* σ*-invariant.*
(2) $\overline{\mathfrak{g}(\mathfrak{a}, \alpha, \epsilon_j)} = \mathfrak{g}(\mathfrak{a}, -\alpha, \epsilon_j^{-1})$.
(3) $[\mathfrak{g}(\mathfrak{a}, \alpha, \epsilon_i), \mathfrak{g}(\mathfrak{a}, \beta, \epsilon_j)] \subset \mathfrak{g}(\mathfrak{a}, \alpha + \beta, \epsilon_i \epsilon_j)$.

Lemma 6. $\tilde{\Sigma}$ *is a root system of* $\mathfrak{z}^{\perp} \cap \mathfrak{a}$.

Proof. By Lemma 4 $\tilde{\Sigma}$ spans $\mathfrak{z}^{\perp} \cap \mathfrak{a}$. For any $\alpha \in \tilde{\Sigma}$ there exists ϵ_j such that $\mathfrak{g}(\mathfrak{a}, \alpha, \epsilon_j) \neq \{0\}$. By Lemma 5(2) if X is in $\mathfrak{g}(\mathfrak{a}, \alpha, \epsilon_j) - \{0\}$ then $\overline{X} \in \mathfrak{g}(\mathfrak{a}, -\alpha, \epsilon_j^{-1})$. By Lemma 5(3) we have $[X, \overline{X}] \in \mathfrak{g}(\mathfrak{a}, 0, 1) = \mathfrak{a}^{\mathbb{C}}$. Since $\overline{[X, \overline{X}]} = -[X, \overline{X}]$, we have $[X, \overline{X}] \in \sqrt{-1}\mathfrak{a}$. Since for any $H \in \mathfrak{a}$,

$$\langle H, [X, \overline{X}] \rangle = \langle [H, X], \overline{X} \rangle = \sqrt{-1} \langle \alpha, H \rangle \langle X, \overline{X} \rangle,$$

we have $[X, \overline{X}] = \sqrt{-1}\langle X, \overline{X}\rangle\alpha$. Hence we have the following isomorphism:

$$\mathfrak{sl}(2, \mathbb{C}) = \mathbb{C}\alpha \oplus \mathbb{C}X \oplus \mathbb{C}\overline{X}.$$

Let β be in $\tilde{\Sigma}$. Then $\mathfrak{sl}(2, \mathbb{C})$ acts on

$$\sum_{n\in\mathbb{Z}} \mathfrak{g}(\beta + n\alpha) = \sum_{n=r}^{s} \mathfrak{g}(\mathfrak{a}, \beta + n\alpha).$$

Since $[X, \overline{X}] = \sqrt{-1}\langle X, \overline{X}\rangle\alpha$ we have $\mathrm{tr}(\alpha) = 0$. Thus

$$0 = \mathrm{tr}(\alpha) = \sqrt{-1}\sum_{n=r}^{s}\langle \alpha, \beta + n\alpha\rangle = \sqrt{-1}(s - r + 1)(\langle \alpha, \beta\rangle + \frac{1}{2}(s + r)\|\alpha\|^2),$$

which implies that

$$-\frac{2\langle \alpha, \beta\rangle}{\|\alpha\|^2} = s + r \in \mathbb{Z}.$$

Hence α-series containing β is of the form $\beta + n\alpha$ $(p \le n \le q)$, and

$$p \le p + q = -\frac{2\langle \alpha, \beta\rangle}{\|\alpha\|^2} \le q.$$

Hence $s_\alpha\beta := \beta - \frac{2\langle \alpha, \beta\rangle}{\|\alpha\|^2}\alpha \in \tilde{\Sigma}$. □

We decompose the root system $\tilde{\Sigma}$ into some irreducible root systems $\tilde{\Sigma}_i$ and write $\tilde{\Sigma} = \tilde{\Sigma}_1 \cup \cdots \cup \tilde{\Sigma}_r$. Denote by $\mathfrak{g}_i^{\mathbb{C}}$ the subalgebra of $\mathfrak{g}^{\mathbb{C}}$ generated by $\sum_{\alpha\in\tilde{\Sigma}_i} \mathfrak{g}(\mathfrak{a}, \alpha)$. Then

$$\mathfrak{g}_i^{\mathbb{C}} \subset \mathfrak{g}(\mathfrak{a}, 0) \oplus \sum_{\alpha\in\tilde{\Sigma}_i} \mathfrak{g}(\mathfrak{a}, \alpha).$$

Lemma 7. *$\mathfrak{g}_i^{\mathbb{C}}$ is an ideal of $\mathfrak{g}^{\mathbb{C}}$, which is not equal to $\{0\}$. When $i \neq j$, then $[\mathfrak{g}_i^{\mathbb{C}}, \mathfrak{g}_j^{\mathbb{C}}] = \{0\}$. In particular if $\mathfrak{g}$ is simple then $\tilde{\Sigma}$ is an irreducible root system of $\mathfrak{a}$.*

Proof. It is clear that $\mathfrak{g}_i^{\mathbb{C}} \neq \{0\}$ by the definition of $\mathfrak{g}_i^{\mathbb{C}}$. By (1) we have

$$\begin{aligned}[\mathfrak{g}^{\mathbb{C}}, \sum_{\beta\in\tilde{\Sigma}_i} \mathfrak{g}(\mathfrak{a}, \alpha)] &= [\mathfrak{g}(\mathfrak{a}, 0) \oplus \sum_{\alpha\in\tilde{\Sigma}} \mathfrak{g}(\mathfrak{a}, \alpha), \sum_{\alpha\in\tilde{\Sigma}_i} \mathfrak{g}(\mathfrak{a}, \alpha)] \\ &\subset \sum_{\beta\in\tilde{\Sigma}_i} \mathfrak{g}(\mathfrak{a}, \beta) \oplus \sum_{\alpha,\beta\in\tilde{\Sigma}_i} [\mathfrak{g}(\mathfrak{a}, \alpha), \mathfrak{g}(\mathfrak{a}, \beta)]) \subset \mathfrak{g}_i^{\mathbb{C}}.\end{aligned}$$

By Jacobi identity $\mathfrak{g}_i^{\mathbb{C}}$ is an ideal of $\mathfrak{g}^{\mathbb{C}}$. When $i \neq j$ then

$$[\sum_{\alpha \in \tilde{\Sigma}_i} \mathfrak{g}(\mathfrak{a}, \alpha), \sum_{\beta \in \tilde{\Sigma}_j} \mathfrak{g}(\mathfrak{a}, \beta)] = \{0\}.$$

By Jacobi identity $[\mathfrak{g}_i^{\mathbb{C}}, \sum_{\beta \in \tilde{\Sigma}_j} \mathfrak{g}(\mathfrak{a}, \beta)] = \{0\}$. Using Jacobi identity again we have $[\mathfrak{g}_i^{\mathbb{C}}, \mathfrak{g}_j^{\mathbb{C}}] = \{0\}$. □

2.3 In the Case When $\sigma^2 = 1$

In this subsection we assume that $\sigma^2 = 1$. If we set $\mathfrak{m}_\sigma = F(-\sigma, \mathfrak{g})$, then $\mathfrak{g} = \mathfrak{k}_\sigma \oplus \mathfrak{m}_\sigma$. Since $\mathfrak{a} \subset \mathfrak{k}_\sigma$, we have $[\mathfrak{a}, \mathfrak{m}_\sigma] \subset \mathfrak{m}_\sigma$. Define subspaces $V(\mathfrak{m}_\sigma)$ and $V^\perp(\mathfrak{m}_\sigma)$ of $\mathfrak{m}_\sigma$ by

$$V(\mathfrak{m}_\sigma) = \{X \in \mathfrak{m}_\sigma \mid [\mathfrak{a}, X] = \{0\}\}, \quad V^\perp(\mathfrak{m}_\sigma) = \{X \in \mathfrak{m}_\sigma \mid X \perp V(\mathfrak{m}_\sigma)\}.$$

For $\alpha \in \mathfrak{a}$ define a subspace $V_\alpha^\perp(\mathfrak{m}_\sigma)$ of $V^\perp(\mathfrak{m}_\sigma)$ by

$$V_\alpha^\perp(\mathfrak{m}_\sigma) = \{X \in V^\perp(\mathfrak{m}_\sigma) \mid (\mathrm{ad}H)^2 X = -\langle \alpha, H\rangle^2 X\},$$

and set $W = \{\alpha \in \mathfrak{a} \mid V_\alpha^\perp(\mathfrak{m}_\sigma) \neq \{0\}\}$. Then W is invariant under the multiplication by -1 since $V_{-\alpha}^\perp(\mathfrak{m}_\sigma) = V_\alpha^\perp(\mathfrak{m}_\sigma)$. For $\alpha \in W$ set $n(\alpha) = \dim V_\alpha^\perp(\mathfrak{m}_\sigma)$, which we call the multiplicity of α. Clearly we have $\tilde{\Sigma} = \Sigma \cup W$.

Lemma 8. *For $\alpha \in W$, $V_\alpha^\perp(\mathfrak{m}_\sigma)$ is an $\mathfrak{a}$-invariant subspace, and $n(\alpha)$ is even. W is invariant under the action of the Weyl group $W(\Sigma)$ of Σ. For $s \in W(\Sigma)$ and $\alpha \in W$, we have $s(V_\alpha^\perp(\mathfrak{m}_\sigma)) = V_{s\alpha}^\perp(\mathfrak{m}_\sigma)$ and $n(s\alpha) = n(\alpha)$.*

Proof. Since $[\mathfrak{k}_\sigma, \mathfrak{m}_\sigma] \subset \mathfrak{m}_\sigma$ we have $[\mathfrak{a}, V_\alpha^\perp(\mathfrak{m}_\sigma)] \subset [\mathfrak{k}_\sigma, \mathfrak{m}_\sigma] \subset \mathfrak{m}_\sigma$. Since $\langle\ ,\ \rangle$ is an invariant metric we have

$$\langle V(\mathfrak{m}_\sigma), [\mathfrak{a}, V_\alpha^\perp(\mathfrak{m}_\sigma)]\rangle = \langle [V(\mathfrak{m}_\sigma), \mathfrak{a}], V_\alpha^\perp(\mathfrak{m}_\sigma)\rangle = \{0\}.$$

Thus $[\mathfrak{a}, V_\alpha^\perp(\mathfrak{m}_\sigma)] \subset V^\perp(\mathfrak{m}_\sigma)$. For $H \in \mathfrak{a}$ and $X \in V_\alpha^\perp(\mathfrak{m}_\sigma)$,

$$(\mathrm{ad}H)^2[H, X] = [H, (\mathrm{ad}H)^2 X] = -\langle \alpha, H\rangle^2 [H, X].$$

Hence $V_\alpha^\perp(\mathfrak{m}_\sigma)$ is $\mathfrak{a}$-invariant.

For $X \in V_\alpha^\perp(\mathfrak{m}_\sigma) - \{0\}$ take $H \in \mathfrak{a}$ such that $[H, X] \neq 0$. We show that $\{X, [H, X]\}$ is linearly independent. Assume that $aX + b[H, X] = 0$. Clearly $a = 0$ if and only if $b = 0$. Applying $\mathrm{ad}H$ to $aX + b[H, X] = 0$ we have $a[H, X] - \langle \alpha, H\rangle^2 bX = 0$. If a were not equal to 0, set $c = b/a$. Then we would have

$$X + c[H, X] = 0, \qquad -\langle \alpha, H\rangle^2 cX + [H, X] = 0.$$

The equations would imply $(1+\langle\alpha, H\rangle^2 c^2)X = 0$, which would be a contradiction. Hence $a = b = 0$. Thus $\{X, [H, X]\}$ is linearly independent. Since $\mathbb{R}H \oplus \mathbb{R}[H, X]$ is a two-dimensional $\mathfrak{a}$-invariant subspace of $V_\alpha^\perp(\mathfrak{m}_\sigma)$, the orthogonal complement of $\mathbb{R}H \oplus \mathbb{R}[H, X]$ in $V_\alpha^\perp(\mathfrak{m}_\sigma)$ is also $\mathfrak{a}$-invariant. Hence $n(\alpha)$ is even.

For $s \in W(\Sigma)$ there exists $k \in N_{K_\sigma}(\mathfrak{a})$, the normalizer of $\mathfrak{a}$ in K, such that $s = \mathrm{Ad}(k)$ on $\mathfrak{a}$. Hence $\mathfrak{m}_\sigma$ is invariant under the action of $W(\Sigma)$. For $X \in V(\mathfrak{m}_\sigma)$ we have $[\mathfrak{a}, sX] = \mathrm{Ad}(k)[\mathfrak{a}, X] = \{0\}$. Hence $V(\mathfrak{m}_\sigma)$ is also $W(\Sigma)$-invariant. Since the action of $W(\Sigma)$ is orthogonal, $V^\perp(\mathfrak{m}_\sigma)$ is also $W(\Sigma)$-invariant. For $X \in V_\alpha^\perp(\mathfrak{m}_\sigma)$ and $H \in \mathfrak{a}$ we have

$$[H, [H, sX]] = s[s^{-1}H, [s^{-1}H, X]] = -s\langle\alpha, s^{-1}H\rangle^2 X = -\langle s\alpha, H\rangle^2 sX.$$

Hence $s(V_\alpha^\perp(\mathfrak{m}_\sigma)) = V_{s\alpha}^\perp(\mathfrak{m}_\sigma)$. Thus we get $n(s\alpha) = n(\alpha)$. □

The type of $(\tilde{\Sigma}, \Sigma, W)$ and their multiplicities does not depend on the choice of a maximal torus A in K_σ. The involutions σ and $\tau_g \sigma \tau_g^{-1}$ for $g \in G$ determine $(\tilde{\Sigma}, \Sigma, W)$ of the same type and the same multiplicities (Table 2).

Theorem 1. *Let G be a compact connected simple Lie group of classical type and σ an involution of outer type. Then $(\tilde{\Sigma}, \Sigma, W)$ is given by the table below. In particular $(\tilde{\Sigma}, \Sigma, W)$ satisfies the condition of symmetric triad of $\mathfrak{a}$ and Σ is a reduced root system of $\mathfrak{a}$. The multiplicities are given by $m(\lambda) = n(\alpha) = 2$ for any $\lambda \in \Sigma$ and $\alpha \in W$.*

Here we used the following notations (Table 3).

The proof is given by a direct computation as follows. In the examples below we use the following notations unless otherwise stated.

Table 2 The type of $(\tilde{\Sigma}, \Sigma, W)$ for (G, K_σ)

	(G, K_σ)	$(\tilde{\Sigma}, \Sigma, W)$
(1)	$(SU(2m), SO(2m))$ $(m \geq 2)$	(I'-C_m)
(2)	$(SU(2m+1), SO(2m+1))$ $(m \geq 1)$	(II-BC_m)
(3)	$(SU(2m), Sp(m))$	(I-C_m)
(4)	$(SO(2m+2n+2), S(O(2m+1) \times O(2n+1)))$	(I'-B_{m+n})

Table 3 The type of symmetric triad $(\tilde{\Sigma}, \Sigma, W)$

$(\tilde{\Sigma}, \Sigma, W)$	$\tilde{\Sigma}$	Σ	W
(I'-C_m)	C_m	D_m	C_m
(II-BC_m)	BC_m	B_m	BC_m
(I-C_m)	C_m	C_m	C_m
(I'-B_{m+n})	B_{m+n}	$B_m \cup B_n$	$(\tilde{\Sigma} - \Sigma) \cup \{\pm e_i\}$

$$H(x_1, \cdots, x_m) = \mathrm{diag}(H(x_1), \cdots, H(x_m)) = \begin{pmatrix} H(x_1) & & \\ & \ddots & \\ & & H(x_m) \end{pmatrix}, \ H(x) = \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}.$$

Example 1. In the case when $(G, K_\sigma) = (SU(2m), SO(2m))$:

Define an involution σ on G by $\sigma(g) = \bar{g}$. Then a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{k}_\sigma$ is given by $\mathfrak{a} = \{H(x_1, \cdots, x_m) \mid x_1, \cdots, x_m \in \mathbb{R}\}$. Define $e_i \in \mathfrak{a}$ by $\langle H(x_1, \cdots, x_m), e_i \rangle = x_i$. Then we can verify (1) in Theorem 1. □

Example 2. In the case when $(G, K_\sigma) = (SU(2m+1), SO(2m+1))$:

Define an involution σ on G by $\sigma(g) = \bar{g}$. Then a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{k}_\sigma$ is given by $\mathfrak{a} = \{H(x_1, \cdots, x_m) \mid x_1, \cdots, x_m \in \mathbb{R}\}$, where

$$H(x_1, \cdots, x_m) = \mathrm{diag}(H(x_1), \cdots, H(x_m), 0).$$

Define $e_i \in \mathfrak{a}$ by $\langle H(x_1, \cdots, x_m), e_i \rangle = x_i$. Then we can verify (2) in Theorem 1. □

Example 3. In the case when $(G, K_\sigma) = (SU(2m), Sp(m))$ we have already verified (3) in Theorem 1 [4].

Example 4. In the case when $(G, K_\sigma) = (SO(2m+2n+2), S(O(2m+1) \times O(2n+1)))$:

Define an involution σ on G by

$$\sigma(g) = I_{2m+1,2n+1}\, g\, I_{2m+1,2n+1}^{-1} \quad \text{where} \quad I_{2m+1,2n+1} = \mathrm{diag}(-1_{2m+1}, 1_{2n+1}).$$

Then a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{k}_\sigma$ is given by

$$\mathfrak{a} = \{H(x_1, \cdots, x_m; y_1, \cdots, y_n) \mid x_1, \cdots, x_m, y_1, \cdots, y_n \in \mathbb{R}\},$$

where $H = H(x_1, \cdots, x_m; y_1, \cdots, y_n) = \mathrm{diag}(H(x_1, \cdots, x_m), 0, H(y_1, \cdots, y_n), 0)$. Define $\{e_i\}_{1 \le i \le m+n}$ by $\langle e_i, H \rangle = x_i$ $(1 \le i \le m)$ and $\langle e_{m+j}, H \rangle = y_j$ $(1 \le j \le n)$. Then we can verify (4) in Theorem 1. □

This completes the proof of Theorem 1.

Combining the theorem above and the results in [3] we can describe the orbit space of a σ-action in the theorem above. We can stratify the orbit space and see that each strata has a unique minimal orbit. The notion of austere submanifold was introduced by Harvey-Lawson [1], which is a minimal submanifold whose second fundamental form has a certain symmetry. We also see which orbit of such a σ-action is austere or totally geodesic. We explain another application, which is a joint work with M. S. Tanaka and H. Tasaki [4]. Let $M = G_m(\mathbb{C}^{2m})$ be a complex Grassmann manifold consisting of all complex m-dimensional subspaces in $\mathbb{C}^{2m}$. Denote by $A(M)$ the group of all holomorphic isometries of M and by $A_0(M)$ its identity component. For $\varphi \in A(M) - A_0(M)$ we can see the necessary and sufficient condition for $F(\varphi, M)$ to be discrete, and we can describe $F(\varphi, M)$ when it is discrete using the theorem above in the case where $(G, K_\sigma) = (SU(2m), Sp(m))$.

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Elementary Deformations and the HyperKähler-Quaternionic Kähler Correspondence

Oscar Macia and Andrew Swann

Abstract The hyperKähler-quaternionic Kähler correspondence constructs quaternionic Kähler metrics from hyperKähler metrics with a rotating circle symmetry. We discuss how this may be interpreted as a combination of the twist construction with the concept of elementary deformation, surveying results of our forthcoming paper. We outline how this leads to a uniqueness statement for the above correspondence and indicate how basic examples of c-map constructions may be realised in this context.

1 Introduction

The twist construction was introduced in [16, 17] as a geometric construction that reproduces T-duality arguments in the physicists literature for geometries with torsion. It has proved successful in constructing compact simply connected examples of a number of classes of non-Kähler geometries. However, elsewhere in the physics literature string theory dualities are used to construct metrics of special holonomy. In particular, the c-map construction of Cecotti et al. [6] produces quaternionic Kähler metrics from projective special Kähler manifolds. An intermediate stage in this construction is a passage from hyperKähler manifolds of a given dimension to quaternionic Kähler manifolds of the same dimension.

HyperKähler and quaternionic Kähler metrics are two of the infinite families of geometries in the holonomy classification of Berger [4, 5]. They are both Einstein geometries and there are many open questions about their structure and classification. In 2008, Haydys [9] showed how to each quaternionic Kähler manifold with circle action one may associate hyperKähler manifolds with a symmetry that

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fixes only one of the complex structures. He also provided a description of how to invert that construction. Later Hitchin [11] gave a twister interpretation of this construction along the lines of [12], and [2, 3] provided expressions in arbitrary signature. The metric constructions here all have the flavour of making a conformal change, but with two different factors along and orthogonal to directions determined by a symmetry.

The purpose of this note is to describe the results of [13], where we show that the twist construction can be used to interpret this so-called hyperKähler-quaternionic Kähler correspondence and to prove that there is only one degree of freedom this construction. We then indicate how the computational framework of the twist construction may be applied to understand some of the basic examples of the c-map.

2 Twist Constructions

The twist construction [16, 17] associates to a manifold with a circle action a new space of the same dimension with a distinguished vector field.

Suppose M is manifold of dimension n. Let X be a vector field on M that generates a circle action. A twist W of M is specified as a quotient $W = P/\langle X'\rangle$ of a principal S^1-bundle $P \to M$ by a lift X' of X. It thus fits in to a double fibration

$$M \xleftarrow{\pi_M} P \xrightarrow{\pi_W} W.$$

If $H^2(M,\mathbb{Z})$ has no torsion, the bundle P is specified by the curvature form F of a connection one-form $\theta \in \Omega^1(P)$, given by $\pi_M^* F = d\theta$. We let $\mathscr{H} = \ker\theta$ be the corresponding horizontal distribution on P. Lifts X' of X that preserve θ and the principal vector field Y are given by

$$X' = X^\theta + (\pi_M^* a)Y,$$

where $X^\theta \in \mathscr{H}$ is the horizontal lift of X with respect to θ and $a \in C^\infty(M)$ is a Hamiltonian function satisfying

$$da = -X \lrcorner F. \tag{1}$$

This requires that F is preserved by X. The *twist* $W := P/\langle X'\rangle$ then admits a circle action generated by $(\pi_W)_* Y$.

This essentials of this set-up are specified by the *twist data* (M, X, F, a) with $X \in \mathfrak{X}(M)$ generating a circle action, $F \in \Omega^2_{\mathbb{Z}}(M)^X$ an X-invariant closed two-form with integral periods and a satisfying (1).

Provided a is non-zero, invariant tensors on M may be transferred to W as follows. Note that at $p \in P$, the projections π_M and π_W induce isomorphisms $T_{\pi_M(p)}M \cong \mathscr{H}_p \cong T_{\pi_W(p)}W$. Thus given $p \in \pi_M^{-1}(q)$, a tensor α_q at $q \in M$

induces a tensor $(\alpha_W)_{q'}$ at $q' = \pi_W(p) \in W$. The tensor α_W is well-defined if α is preserved by X. We then say that α and α_W are $\mathscr{H}$-related and write

$$\alpha \sim_{\mathscr{H}} \alpha_W.$$

The two most important computational facts for $\mathscr{H}$-related tensors are:

Property 1. for $\alpha \in \Omega^p(M)^X$ an invariant p-form, the exterior differential of α_W is given by

$$d\alpha_W \sim_{\mathscr{H}} d_W\alpha := d\alpha - \frac{1}{a}F \wedge X \lrcorner \alpha. \tag{2}$$

Property 2. for an invariant complex structure I on M that is $\mathscr{H}$-related to an almost complex structure I_W on W, we have

$$I_W \text{ is integrable if and only if } F \text{ is of type } (1,1) \text{ for } I.$$

Recall that $F \in \Omega^2(M)$ is of type $(1,1)$ if $F(IA, IB) = F(A, B)$ for all $A, B \in TM$.

These facts show that geometric properties of the twist are determined by the twist data.

Example 1. A basic example of the twist construction is provided by $M = \mathbb{C}P(n) \times T^2$. This is a Kähler manifold as a product. Suppose X generates one of the circle factors of $T^2 = S^1 \times S^1$. Taking F to be the Fubini-Study two-form on $\mathbb{C}P(n)$, we have $X \lrcorner F = 0$, so can take $a \equiv 1$. Then $P = S^{2n+1} \times T^2$ and the twist is $W = S^{2n+1} \times S^1$. As F is type $(1,1)$ we have that W is a complex manifold. However W is compact and $b_2(W) = 0$, so W can not be Kähler.

3 Elementary Deformations of HyperKähler Metrics

As formula (2) indicates, the twist of a closed differential form is rarely closed. In a given geometric situation it is therefore interesting to adjust the geometric data before performing a twist.

We wish to work with hyperKähler manifolds. These are (pseudo-)Riemannian manifolds (M, g) with almost complex structures I, J and K such that

1. $IJ = K = -JI$,
2. g is Hermitian with respect to I, J and K,
3. the two-forms $\omega_I = g(I\cdot, \cdot)$, ω_J and ω_K are closed:

$$d\omega_I = 0 = d\omega_J = d\omega_K.$$

By Hitchin [10] the last condition implies that I, J and K are integrable. The restricted holonomy is then a subgroup of $Sp(n)$, where $\dim M = 4n$, and the metric is Ricci-flat. The triples (g, I, ω_I), etc., are then Kähler structures on M.

Let X be a symmetry of a hyperKähler structure (M, g, I, J, K), but which we mean that X is an isometry that preserves the linear span $\langle I, J, K\rangle$ of $I, J, K \in \mathrm{End}(TM)$. The vector field X induces four one-forms on M given by

$$\alpha_0 = g(X, \cdot), \quad \alpha_I = I\alpha_0 = -\alpha(I\cdot), \quad \alpha_J = J\alpha_0, \quad \alpha_K = K\alpha_0.$$

We then define

$$g_\alpha = \alpha_0^2 + \alpha_I^2 + \alpha_J^2 + \alpha_K^2.$$

When X is not null, g_α is positive semi-definite proportional to the restriction of g to $\mathbb{H}X = \langle X, IX, JX, KX\rangle$.

Definition 1. An *elementary deformation* of a hyperKähler metric g with respect to a symmetry X is a metric of the form

$$g^N = fg + hg_\alpha$$

with f and h smooth functions on M.

This is thus more general than a conformal change of g.

As I, J and K are parallel, we have that X acts as a linear transformation on $\mathbb{R}^3 = \langle I, J, K\rangle$. It preserves the algebraic relations, so acts as an element of $\mathfrak{so}(3)$. As $\mathfrak{so}(3)$ has rank one, it follows that the action is either trivial or conjugate a circle action fixing I and mapping J to K. By relabelling the complex structures and normalising X we may thus assume in this latter case that

$$L_X I = 0 \quad \text{and} \quad L_X J = K. \tag{3}$$

An isometry X satisfying (3) will be called *rotating*.

For a rotating symmetry, we have $d\alpha_I = 0$, $d\alpha_J = \omega_K$ and $d\alpha_0 = G - \omega_I$, where $G \in \Omega^2(M)$ is a two-form that is of type $(1,1)$ for I, J and K. As α_I is closed, we may pass to a cover of M and write $\alpha_I = d\mu$ for a smooth map $\mu\colon M \to \mathbb{R}$. The function μ is a Kähler moment map for X with respect to (g, ω_I).

4 The HyperKähler-Quaternionic Kähler Correspondence

Suppose (M, g, I, J, K) is hyperKähler with rotating symmetry X with Kähler moment map μ. Then X does not preserve ω_J or ω_K, but the four-form

$$\Omega = \omega_I^2 + \omega_J^2 + \omega_K^2 \tag{4}$$

is invariant and closed.

If W is manifold of dimension at least 8 with a four-form Ω^W pointwise linearly equivalent to (4), then W has an almost quaternion-Hermitian structure $(g_W, \mathscr{G})$, where $\mathscr{G} \subset \mathrm{End}(TW)$ is a three-dimensional subbundle with a local basis (I_W, J_W, K_W) of almost complex structures for which g_W is Hermitian and with $I_W J_W = K_W = -J_W I_W$. Such a structure is *quaternionic Kähler* if Ω^W is parallel with respect to the Levi-Civita connection of g_W. It follows that g_W is Einstein [5, 14]. If $\dim W \geqslant 12$, then to obtain quaternionic Kähler it is sufficient that $d\Omega^W = 0$ [15]. For $\dim W = 8$, one can work with the local two-forms $\omega^W = (\omega_I^W, \omega_J^W, \omega_K^W)$ and quarternionic Kähler is then equivalent to the existence of a local connection form $A \in \Omega^1(\mathfrak{so}(3))$ such that $d\omega^W = A \wedge \omega^W$.

The behaviour of the exterior derivative under the twist is given by (2), so from the above remarks we may determine whether a twist is quaternionic Kähler by working on M.

Theorem 1 ([13]). *Let (M, g, I, J, K) be a hyperKähler manifold with non-null rotating symmetry X and Kähler moment map μ. If $\dim M \geqslant 8$ then, up to homothety, the only twists of elementary deformations $g^N = fg + hg_\alpha$ of g that are quaternionic Kähler have*

$$g^N = \frac{1}{(\mu - c)^2} g_\alpha - \frac{1}{\mu - c} g \tag{5}$$

for some constant c. Furthermore the corresponding twist data is given by

$$F = kG = k(d\alpha_0 + \omega_I), \quad a = k(g(X, X) - \mu + c),$$

for some constant k.

The method of proof is first to impose the quaternionic Kähler condition on as arbitrary twist of Ω^N, the four-form associated to g^N via I, J and K, and to decompose these equations in type components relative to $\mathbb{H}X$ and its orthogonal complement. From this one deduces that f and h are functions of μ and that $h = f'$. Then we consider the equation $da = -X \lrcorner F$ and determine the twist function a. Finally, we investigate the condition $dF = 0$, which provides an ordinary differential equation for f.

From the theorem, it follows that the constructions provided in [2, 9, 11] of quaternionic Kähler metrics from hyperKähler metrics with rotating circle symmetry agree.

Example 2. We consider $\mathbb{H}^{p,q} = \mathbb{R}^{4p,4q}$, $n = p + q$, with its flat hyperKähler metric

$$g = \sum_{i=1}^{n} \varepsilon_i (dx_i^2 + dy_i^2 + du_i^2 + dv_i^2)$$

with $\varepsilon_i = +1$, for $i \leqslant p$, and $\varepsilon_i = -1$, for $i > p$, and Kähler two-forms

$$\omega_I = \sum_{i=1}^n \varepsilon_i (dx_i \wedge dy_i - du_i \wedge dv_i), \quad \omega_J = \sum_{i=1}^n \varepsilon_i (du_i \wedge dx_i + dv_i \wedge dy_i),$$

$$\omega_K = \sum_{i=1}^n \varepsilon_i (du_i \wedge dy_i - dv_i \wedge dx_i).$$

If X is a rotating circle symmetry then it is an element of $\mathfrak{sp}(p,q) + \mathfrak{u}(1)$, but lies in a maximal compact subgroup, so is conjugate to

$$X = \sum_{i=1}^n \big(\frac{1}{2} - \lambda_i\big)\big(y_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial y_i}\big) - \big(\frac{1}{2} + \lambda_i\big)\big(v_i \frac{\partial}{\partial u_i} - u_i \frac{\partial}{\partial v_i}\big)$$

for some $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. For X to be non-null, we must have $\sum_{i=1}^n \varepsilon_i \lambda_i^2 \neq 0$. This vector field has $d\alpha_0 = d(g(X, \cdot)) = G - \omega_I$ with

$$G = 2 \sum_{i=1}^n \varepsilon_i \lambda_i (dx_i \wedge dy_i + du_i \wedge dv_i)$$

so $G = d\beta$, where $\beta = \sum_{i=1}^n \varepsilon_i \lambda_i (-y_i dx_i + x_i dy_i - v_i du_i + u_i dv_i)$.

The twisting form F is equal to a multiple of $G = d\beta$, so is exact and the twist bundle is trivial $P = \mathbb{H}^n \times S^1$. Let us take $F = G$. The connection one-form may be written as $\theta = \beta + d\tau$, where $\partial/\partial\tau$ generates the principal S^1-action. The horizontal lift X^θ of X to P is then

$$X^\theta = X - \beta(X) \frac{\partial}{\partial \tau}.$$

Direct calculation shows that $d(\beta(X)) = -X \lrcorner F$, so the twist function is $a = \beta(X) + c$ and the twist is the quotient of P by $X' = X + c\frac{\partial}{\partial\tau}$. Thus the twist is

$$W = (\mathbb{H} \times S^1)/\big\langle X + c \frac{\partial}{\partial \tau} \big\rangle.$$

This will be an orbifold if λ_i and c are integers. It is smooth when they are pairwise co-prime.

The theorem says that W is equipped with a quaternionic Kähler metric $\mathscr{H}$-related to g^N in Eq. (5), whenever this is non-degenerate. The function μ is given by

$$\mu = \frac{1}{2} \sum_{i=1}^n \big(\frac{1}{2} - \lambda_i\big)(x_i^2 + y_i^2) + \big(\frac{1}{2} + \lambda_i\big)(u_i^2 + v_i^2).$$

The metric g^N has two contributions to its signature. On the quaternionic span $\mathbb{H}X$ of X, the sign is that of $(\|X\|^2-\mu+c)\|X\|^2/(\mu-c)^2$, orthogonal to $\mathbb{H}X$ the original metric is multiplied by $-\|X\|^2/(\mu-c)$. Thus up to overall sign g^N has quaternionic signature that is either $(p+1, q-1)$, (p, q) or $(p-1, q+1)$. It is degenerate on the sets $(\|X\|^2=0)$, i.e., where X is null, and on $(\|X\|^2-\mu+c=0)$, which is the set where the twist function a vanishes. The metric may also blow-up on $(\mu=c)$.

5 Application to the c-map

The c-map is a construction introduced by Cecotti et al. [6]. It starts with a so-called projective special Kähler manifold of dimension $2n$ and produces a quaternionic Kähler manifold of dimension $4n+4$. Explicit local expressions for the resulting metrics were provided by Ferrara and Sabharwal [7]. Recently Alekseevsky et al. [1, 3] have shown that the hyperKähler-quaterionic Kähler correspondence reproduces the quaternionic Kähler metrics of the c-map. In particular, this means that one may obtain all the known examples homogeneous (positive definite) quaternionic Kähler of negative scalar curvature, and their work is also beginning to produce new examples of complete quaternionic Kähler metrics.

Given the wide generality of the twist construction, it is useful to understand how such homogeneous examples may arise. To be concrete let us consider the real hyperbolic space $\mathbb{R}H(2)$ as a solvable Lie group with Kähler metric of constant curvature. This has a global basis $\{a, b\}$ of one forms, such that $da=0$ and $db=-\lambda a\wedge b$, for some constant λ depending on the scalar curvature. For this to be a projective special Kähler manifold, we need to consider a certain cone metric and show that it admits a flat symplectic connection of special Kähler type, as described by Freed [8].

Let C_0 be a circle bundle over $\mathbb{R}H(2)$ with connection one-form φ whose curvature is $2a\wedge b$. Pulling a and b back to $C=\mathbb{R}_{>0}\times C_0$, the cone geometry is

$$g_C=t^2(a^2+b^2-\varphi^2)-dt^2, \quad \omega_C=t^2 a\wedge b-t\varphi\wedge dt,$$

a Kähler metric of signature $(2, 2)$. It has a symmetry X generated by the principal action on C_0.

Locally, one can show that this admits a special Kähler connection if and only if λ^2 is 4 or $4/3$. In case $\lambda^2=4$, the special connection agrees with the Levi-Civita connection of g_c. In both cases, using the cotangent trivialisation $(\hat{a}, \hat{b}, \hat{\varphi}, \hat{\psi})=(ta, tb, t\varphi, dt)$, one may construct a hyperKäher metric of signature $(4, 4)$ on $H=T^*C$ of the form $g_H=\hat{a}^2+\hat{b}^2-\hat{\varphi}^2-\hat{\psi}^2+\hat{A}^2+\hat{B}^2-\hat{\Phi}^2-\hat{\Psi}^2$. Indeed the flat connection gives $TH=V^*\oplus V$, with $V\cong TM$. This is the rigid c-map, see Freed [8]. The Kähler forms ω_J and ω_K are just the real and imaginary parts of the standard complex symplectic two-form on $H=T^*C$.

Horizontally lifting the symmetry of X of C to $H = T^*C$ using the flat connection, one obtains a rotating symmetry $\tilde{X}$ of the hyperKähler structure. Note that the symmetry X does not preserve the flat connection, and it rotates the quadruple $\tilde{\delta} = (\hat{A}, \hat{B}, \hat{\Phi}, \hat{\Psi})$. The twist data for this lifted action is given by the curvature form

$$F = -\hat{a} \wedge \hat{b} + \hat{\varphi} \wedge \hat{\psi} - \hat{A} \wedge \hat{B} + \hat{\Phi} \wedge \hat{\Psi}$$

and twist function $-t^2/2 + c$. The curvature form is exact, and so we may proceed much as in Example 2.

In particular, we have a coordinate τ on S^1 in $P = H \times S^1$. With $c = 0$ the twist is then diffeomorphic to $(H/\langle\tilde{X}\rangle) \times S^1$. We may use τ to define a new quadruple $\delta = \tilde{\delta}\exp(\mathbf{i}\tau)$, where $\mathbf{i} = \operatorname{diag}(\mathbf{i}_2, \mathbf{i}_2)$ with $\mathbf{i}_2 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Now using (2) one may show that the structure functions of the coframe $\mathscr{H}$-related to $(\hat{a}, \hat{b}, \hat{\varphi}, \hat{\psi}, \delta_1, \delta_2, \delta_3, \delta_4)$ are constants, so these define a dual basis for a Lie algebra. The metric g^N is seen to be positive definite, complete and has constant coefficients in this coframe, so the resulting quaternionic Kähler metric on W is complete. It follows that the universal cover of W is a Lie group G and that the metric on W pulls back to a left-invariant metric. We have $W = G/\mathbb{Z}$ and knowing the structure constants we may identify G as the solvable Lie groups that act transitively on the non-compact symmetric spaces $\mathrm{Gr}_2(\mathbb{C}^{2,2})$ for $\lambda^2 = 4$ or $G_2^*/SO(4)$ for $\lambda^2 = 4/3$. This provides a global verification of the main example of Ferrara and Sabharwal [7].

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A Classification of Ricci Solitons as (k, μ)-Contact Metrics

Amalendu Ghosh and Ramesh Sharma

Abstract If a non-Sasakian (k, μ)-contact metric g is a non-trivial Ricci soliton on a $(2n+1)$-dimensional smooth manifold M, then (M, g) is locally a three-dimensional Gaussian soliton, or a gradient shrinking rigid Ricci soliton on the trivial sphere bundle $S^n(4) \times E^{n+1}$, or a non-gradient expanding Ricci soliton with $k = 0, \mu = 4$. The last case occurs on a Lie group with a left invariant metric, especially for dimension 3, on Sol^3 regarded also as the group $E(1,1)$ of rigid motions of the Minkowski 2-space.

1 Introduction

A Ricci soliton is a generalization of the Einstein metric, defined on a Riemannian manifold M with metric g satisfying the equation

$$\pounds_V g + 2Ric + 2\lambda g = 0 \tag{1}$$

where V is a vector field on M, Ric is the Ricci tensor of g, λ is a constant and $\pounds_V$ denotes the Lie-derivative operator along V. The Ricci soliton is said to be shrinking, steady, and expanding according as $\lambda < 0, \lambda = 0$, and $\lambda > 0$ respectively. Actually, Ricci solitons represent generalized fixed point of the Hamilton's Ricci flow [12]: $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$, viewed as a dynamical system on the space of Riemannian metrics modulo diffeomorphisms and scalings. A Ricci soliton is said to be a gradient Ricci soliton, if it is the sum of a Killing vector field and the gradient of a smooth function f on M. A compact Ricci soliton is known to be gradient (Perelman [13]).

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A Ricci soliton is said to be trivial if V is either zero or Killing on M. Non-trivial compact Ricci solitons may exist only in dimensions ≥ 4 and must have non-constant positive scalar curvature. Ricci solitons are also of interest to physicists who refer to them as quasi-Einstein metrics (for example, see Friedan [8]). For details on Ricci soliton we refer to Chow et al. [6].

Generalizing the trivial Ricci solitons, Petersen and Wylie [14] introduced the notion of rigidity of gradient Ricci solitons as follows.

Definition 1. A gradient Ricci soliton is said to be rigid if it is isometric to a quotient of $N \times E^k$ where N is an Einstein manifold and the potential function $f = -\frac{\lambda}{2}|x|^2$ on the Euclidean factor E^k. That is, (M, g) is isometric to $N \times_\Gamma E^k$, where Γ acts freely on N and by orthogonal transformations on E^k. For $k = n$, i.e. $M = E^n$, it is just the Gaussian soliton $f = -\frac{\lambda}{2}|x|^2$.

Let us recall the recent studies of Ricci solitons within the frame-work of contact geometry. Gradient Ricci solitons were studied by Sharma [15] as a K-contact metric, by Ghosh et al. [11] as a (k, μ)-contact metric and by Cho and Sharma [7] for homogeneous H-contact manifold. For the non-gradient case, Sharma and Ghosh [16] proved that a three-dimensional Sasakian metric which is a non-trivial Ricci soliton, is homothetic to the standard Sasakian metric on nil^3. Subsequently, this result was generalized by Ghosh and Sharma [9] for η-Einstein K-contact manifold. Recently, the aforementioned result in [16] on three-dimensional Sasakian manifold has been extended by Ghosh and Sharma [10] for all dimensions. It is proved that "If a $(2n+1)$-dimensional Sasakian metric is a non-trivial Ricci soliton, then it is null η-Einstein (transverse Calabi-Yau) and expanding". We now recall the following result from [11]: "if a non-Sasakian (k, μ)-contact metric is a gradient Ricci soliton, then in dimension 3 it is flat and in higher dimensions it is locally isometric to the trivial sphere bundle $S^n(4) \times E^{n+1}$". The goal of this paper is to generalize this result by waiving the gradient condition and obtain the following classification result.

Theorem 1. *If a non-Sasakian (k, μ)-contact metric g is a non-trivial Ricci soliton on a $(2n+1)$-dimensional smooth manifold M, then (M, g) is locally a three-dimensional Gaussian soliton, or a gradient shrinking rigid Ricci soliton on the trivial sphere bundle $S^n(4) \times E^{n+1}$, or a non-gradient expanding Ricci soliton with $k = 0, \mu = 4$. The last case occurs on a Lie group with a left invariant metric, especially for dimension 3, on Sol^3 regarded also as the group $E(1, 1)$ of rigid motions of the Minkowski 2-space.*

As the standard contact metric g on the unit tangent bundle $T_1M(c)$ over a space of constant curvature c is a (k, μ)-contact metric with $k = c(2-c)$, $\mu = -2c$, with $c = 1$ corresponding to Sasakian case (see [3]), Theorem 1 shows that $(T_1M(c), g)$ would be a non-Sasakian Ricci soliton if and only if $c = 0$. Hence, we have the following corollary to the above theorem.

Corollary 1. *The standard contact metric g on the unit tangent bundle $T_1M(c)$ over a space of constant curvature c is a non-Sasakian Ricci soliton if and only if $(T_1M(c), g)$ is the trivial sphere bundle $S^n(4) \times E^{n+1}$.*

2 Preliminaries

A $(2n+1)$-dimensional smooth manifold is said to be contact if it has a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ on M. Given a contact 1-form η there always exists a unique vector field ξ such that $(d\eta)(\xi, X) = 0$ and $\eta(\xi) = 1$. Polarization of $d\eta$ on the contact subbundle D (defined by $\eta = 0$), yields a Riemannian metric g and a (1,1)-tensor field φ such that

$$(d\eta)(X, Y) = g(X, \varphi Y), \tag{2}$$

$$\eta(X) = g(X, \xi), \tag{3}$$

$$\varphi^2 = -I + \eta \otimes \xi. \tag{4}$$

The metric g is called an associated metric of η and (φ, η, ξ, g) a contact metric structure. We now recall [2] the self-adjoint operators $h = \frac{1}{2}\pounds_\xi \varphi$ and $l = R(., \xi)\xi$ which annihilate ξ. The tensors h, $h\varphi$ are trace-free and $h\varphi = -\varphi h$. The following formulas hold on a contact metric manifold (see [2]).

$$\nabla_X \xi = -\varphi X - \varphi h X, \tag{5}$$

$$Trl = Ric(\xi, \xi) = 2n - |h|^2, \tag{6}$$

where ∇, R, Ric and Q denote respectively, the Riemannian connection, curvature tensor, Ricci tensor and Ricci operator of g. A contact metric manifold is said to be K-contact if ξ is Killing with respect to g, equivalently, $h = 0$. Moreover, the contact metric structure on M is said to be Sasakian if the almost Kaehler structure on the cone manifold $(M \times R^+, r^2 g + dr^2)$ over M, is Kaehler (see Boyer and Galicki [5]). For details about contact metric manifolds we also refer to Blair [2].

By a (k, μ)-contact manifold we mean a contact metric manifold satisfying the nullity condition (see Blair et al. [3])

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{7}$$

for some real numbers k and μ. This can be equivalently expressed as

$$R(X, \xi)Y = k\,[\eta(Y)X - g(X, Y)\xi] + \mu\,[\eta(Y)hX - g(hX, Y)\xi]\,. \tag{8}$$

This class of manifolds is preserved under a D-homothetic deformation (Tanno [17]): $\bar{\eta} = a\eta, \bar{\xi} = \frac{1}{a}\xi, \bar{\varphi} = \varphi, \bar{g} = ag + a(a-1)\eta \otimes \eta$ for a positive constant a, and includes Sasakian manifolds (for which $k = 1$ and $h = 0$) and the trivial sphere bundle $S^n(4) \times E^{n+1}$ (for which $k = \mu = 0$, a result of Blair [1]). For (k, μ)-contact manifolds, we know [3] that

$$Ric(X,\xi) = 2nkg(X,\xi), \tag{9}$$

$$h^2 = (1-k)(I - \eta \otimes \xi), \quad |h|^2 = 2n(1-k). \tag{10}$$

This shows that $k \leq 1$, and equality holds when $k = 1, h = 0$, i.e. M is Sasakian. For the non-Sasakian case, i.e. $k < 1$, the (k,μ)-nullity condition determines the curvature of M completely. Using this fact Boeckx [4] proved that a non-Sasakian (k,μ)-contact manifold is locally homogeneous and hence analytic. For non-Sasakian (k,μ)-contact manifolds, the following formulas hold [3]:

$$QX = [2(n-1) - n\mu]X + [2(n-1) + \mu]hX \\ +[2(1-n) + n(2k+\mu)]\eta(X)\xi, \tag{11}$$

$$(\nabla_X h)Y = ((1-k)g(X,\varphi Y) - g(X,\varphi hY))\xi \\ - \eta(Y)((1-k)\varphi X + \varphi hX) - \mu\eta(X)\varphi hY, \tag{12}$$

$$(\nabla_X \varphi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX). \tag{13}$$

where X, Y denote arbitrary vector fields on M. Moreover, the scalar curvature is constant and given by

$$r = 2n[2(n-1) + k - n\mu]. \tag{14}$$

3 Proof of Theorem 1

By hypothesis, we have $k < 1$. Using (1) in the commutation formula (see Yano [18], p.23)

$$(\pounds_V \nabla_X g - \nabla_X \pounds_V g - \nabla_{[V,X]} g)(Y,Z) = -g((\pounds_V \nabla)(X,Y),Z) \\ - g((\pounds_V \nabla)(X,Z),Y),$$

and by a straightforward combinatorial computation we have

$$g((\pounds_V \nabla)(X,Y),Z) = (\nabla_Z Ric)(X,Y) - (\nabla_X Ric)(Y,Z) - (\nabla_Y Ric)(X,Z).$$

The use of equations (11), (12) and (5) in the above equation provides

$$(\pounds_V \nabla)(X,Y) = (4nk + 4\mu - \mu^2)g(\varphi hX, Y)\xi \\ + \mu[2(n-1) + \mu]\{\eta(X)\varphi hY + \eta(Y)\varphi hX\} \\ + 2[\mu + 2k - \mu k + n\mu]\{\eta(X)\varphi Y + \eta(Y)\varphi X\}.$$

Using this equation in the formula (see Eq. (5.16) on p. 23 of Yano [18]):

$$(\pounds_V R)(X, Y, Z) = (\nabla_X(\pounds_V \nabla))(Y, Z) - (\nabla_Y(\pounds_V \nabla))(X, Z)$$

and subsequently, substituting ξ for Z, we get

$$\begin{aligned}(\pounds_V R)(X, Y)\xi &= \mu[2(n-1)+\mu][(\nabla_Y h\varphi)X - (\nabla_X h\varphi)Y] \\ &+4[\mu + 2k - \mu k + n\mu][\eta(X)(Y + hY) - \eta(Y)(X + hX)] \\ &+\mu[2(n-1)+\mu][\eta(Y)(hX + h^2X) - \eta(X)(hY + h^2Y)]. \end{aligned} \tag{15}$$

Now, using (12) and (13) we find

$$\begin{aligned}(\nabla_Y h\varphi)X - (\nabla_X h\varphi)Y &= (k-1)\{\eta(X)Y - \eta(Y)X\} \\ &+ (\mu - 1)\{\eta(X)hY - \eta(Y)hX\}.\end{aligned}$$

Using this equation and (10) in (15) yields

$$\begin{aligned}(\pounds_V R)(X, Y)\xi &= \mu[2(n-1)+\mu][2(k-1)\{\eta(X)Y - \eta(Y)X\} \\ &+(\mu-2)\{\eta(X)hY - \eta(Y)hX\}] \\ &+4[\mu + 2k - \mu k + n\mu][\eta(X)(Y + hY) - \eta(Y)(X + hX)]. \end{aligned} \tag{16}$$

Substituting ξ for Y in the foregoing equation we have

$$\begin{aligned}(\pounds_V R)(X, \xi)\xi &= \mu[2(n-1)+\mu][2(k-1)\{\eta(X)\xi - X\} \\ &-(\mu-2)hX] + 4[\mu + 2k - \mu k + n\mu][\eta(X)\xi - (X + hX)]. \end{aligned} \tag{17}$$

Contracting it at X gives

$$(\pounds_V Ric)(\xi, \xi) = -4n\mu(k-1)[2(n-1)+\mu] - 8n[\mu + 2k - \mu k + n\mu]. \tag{18}$$

On the other hand, equations (1) (6) and (10) imply

$$\eta(\pounds_V \xi) = g(\pounds_V \xi, \xi) = \lambda + 2nk. \tag{19}$$

Lie-differentiating the property $Ric(\xi, \xi) = 2nk$ [which follows from (9) by substituting $X = \xi$] along V, and using equations (9) and (19) we get

$$(\pounds_V Ric)(\xi, \xi) = -2Ric(\pounds_V \xi, \xi) = -4nk(\lambda + 2nk).$$

Using this in (18) implies that

$$\mu(k-1)[2(n-1)+\mu] + 2[\mu + 2k - \mu k + n\mu] = k(\lambda + 2nk). \tag{20}$$

Next, substituting ξ for Y in (7) we have

$$R(X,\xi)\xi = k(X - \eta(X)\xi) + \mu hX.$$

Its Lie derivative along V gives

$$(\pounds_V R)(X,\xi)\xi + R(X,\pounds_V\xi)\xi + R(X,\xi)\pounds_V\xi = -k\{(\pounds_V\eta)(X)\}\xi \\ -k\eta(X)\pounds_V\xi + \mu(\pounds_V h)X. \tag{21}$$

Now, taking the Lie derivative of the relation $\eta(X) = g(\xi, X)$ along V, using equations (1) and (9) gives

$$(\pounds_V\eta)(X) = g(\pounds_V\xi, X) - 2(2nk + \lambda)\eta(X). \tag{22}$$

Using this equation, (7) and (8) in (21) and then comparing with (17) we obtain

$$A(hX) - \mu\eta(X)h\pounds_V\xi - \mu g(h\pounds_V\xi, X)\xi = \mu(\pounds_V h)X \tag{23}$$

where we set

$$A = 2\mu(2nk + \lambda) - \mu(2n - 2 + \mu)(\mu - 2) - 4(\mu + 2k - \mu k + n\mu).$$

Now, on one hand we substitute hX for X in (23) getting one equation, and on the other hand we apply h on (23) getting another equation. Adding the resulting equations we find

$$2Ah^2X - \mu[\eta(X)h\pounds_V\xi + g(h\pounds_V\xi, X)\xi] = \mu[(\pounds_V h)hX + h(\pounds_V h)X]. \tag{24}$$

Also, the Lie derivative of the first equation of (10) along V gives

$$(\pounds_V h)hX + h(\pounds_V h)X = (k - 1)[(\pounds_V\eta)(X)\xi + \eta(X)\pounds_V\xi].$$

Using this in (24), contracting the resulting equation at X, and using the second equation of (10), we obtain $A = 0$, i.e.

$$\mu(2n - 2 + \mu)(\mu - 2) + 4(\mu + 2k - \mu k + n\mu) = 2\mu(2nk + \lambda).$$

The use of (20) changes the above equation to the following form

$$\mu(\mu + 2n - 2)(\mu - 2k) = 2(2nk + \lambda)(\mu - k). \tag{25}$$

We now recall the following equation of evolution for scalar curvature of a Ricci soliton [15]

$$\pounds_V r = \Delta r + 2\lambda r + 2|Q|^2$$

where Δ denotes $div(grad)$. As per equation (14), r is constant, and hence the above equation reduces to

$$\lambda r + |Ric|^2 = 0.$$

Computing $|Ric|^2$ from (11), using it and (14) in the above equation gives

$$(n^2 + 1 - k)\mu^2 + [4(1-n)(n+k-1) - n\lambda]\mu \\ = 4(n-1)^2(k-2) - 2nk^2 + \lambda(2 - 2n - k). \tag{26}$$

Let us go back to equation (16), contract it at X and use (20) to get

$$(\pounds_V Ric)(Y, \xi) = -4nk(2nk + \lambda).$$

Taking the Lie derivative of equation (9) and using the above equation we find $Ric(X, \pounds_V \xi) = 2nk\pounds_V \xi$, i.e. $Q\pounds_V \xi = 2nk\pounds_V \xi$. The use of this in (11) shows

$$(\mu + 2n - 2)h\pounds_V \xi = (n\mu - 2n + 2 + 2nk)[\pounds_V \xi - (2nk + \lambda)\xi]. \tag{27}$$

First we consider the case (**I**) when V is a contact vector field, i.e. $\pounds_V \xi = \sigma\xi$ for a smooth function σ on M. In view of equation (19), this is equivalent to

$$\pounds_V \xi = (2nk + \lambda)\xi. \tag{28}$$

The Lie derivative of the relation $\eta(X) = g(\xi, X)$ along V and the use of equations (1) and (9) provides

$$\pounds_V \eta = -(2nk + \lambda)\eta. \tag{29}$$

Using this in the contact metric property (2) and equation (1), and noting the commutativity of Lie derivation and exterior derivation we get

$$\pounds_V \varphi = (\lambda - 2nk)\varphi + 2Q\varphi. \tag{30}$$

Lie differentiating the almost contact structure (4) along V and using (28) and (29) we get $\pounds_V \varphi^2 = 0$. Using this and (30) gives

$$0 = (\pounds_V \varphi)\varphi + \varphi(\pounds_V \varphi) = 2(\lambda - 2nk)\varphi^2 + 2Q\varphi^2 + 2\varphi Q\varphi.$$

Applying φ to this equation we obtain $Q\varphi + \varphi Q = (2nk - \lambda)\varphi$. The use of (11) in the preceding equation yields the relation:

$$2(2n - 2 - n\mu) = 2nk - \lambda. \tag{31}$$

The other remaining case (**II**) is that $\pounds_V\xi-(2nk+\lambda)\xi \neq 0$ on an open neighborhood $\mathscr{U}$ of some point $p \in M$. This case has two sub-cases:

(a) $\mu = 2-2n$. For this sub-case, equation (27) implies $n\mu - 2n + 2 + 2nk = 0$. Hence we get $nk + 1 - n^2 = 0$ which gives $k = n - \frac{1}{n}$ which is possible only if $n = 1$, because $k < 1$. Thus, in this sub-case, $k = \mu = 0$ and λ is free.

(b) $\mu \neq 2-2n$. In this sub-case, we have

$$h(\pounds_V\xi - (2nk+\lambda)\xi) = f(\pounds_V\xi - (2nk+\lambda)\xi)$$

where $f = \frac{n\mu-2n+2+2nk}{\mu+2n-2}$. This implies that

$$h^2(\pounds_V\xi - (2nk+\lambda)\xi) = f^2(\pounds_V\xi - (2nk+\lambda)\xi).$$

Using the first equation of (10) in the above equation and noting that $\pounds_V\xi - (2nk+\lambda)\xi$ is orthogonal to ξ, we conclude that $f^2 = 1-k$. Hence this subcase ends up with

$$(n\mu - 2n + 2 + 2nk)^2 = (1-k)(\mu + 2n - 2)^2. \tag{32}$$

For cases (**I**) and (**II**), we ran equations (31) and (32) respectively, along with equations (20), (25) and (26) on the software **MATLAB** and found the following solutions: (i) $k = \mu = 0$ and (ii) $k = 0, \mu = 4$. Locally, solution (i) represents the flat Gaussian gradient soliton in dimension 3, and the shrinking gradient rigid soliton $S^n(4) \times E^{n+1}$ with $\lambda = 4(1-n)$ in higher dimensions. Solution (ii) represents an expanding non-gradient Ricci soliton with $\lambda = 4(n+1)$ and occurs on a Lie group with a left invariant metric, especially for dimension 3, on Sol^3 also regarded as the group $E(1,1)$ of rigid motions of the Minkowski 2-space, as explained in the next section. This completes the proof.

4 An Example of the Last Case of Theorem 1

Following Boeckx [4], we begin with a $(2n+1)$-dimensional Lie algebra with basis $(\xi, X_1, \ldots, X_n, Y_1, \ldots, Y_n)$ and Lie bracket defined by

$$\begin{aligned}
&[\xi, X_i] = 0, [\xi, Y_i] = 2X_i, [X_i, X_j] = 0, [Y_2, Y_i] = 2Y_i (i \neq 2),\\
&[Y_i, Y_j] = 0 (i, j \neq 2), [X_1, Y_1] = 2(\xi - X_2), [X_1, Y_i] = 0 (i \geq 2),\\
&[X_2, Y_1] = 2X_1, [X_2, Y_2] = 2\xi,\\
&[X_2, Y_i] = 2X_i, [X_i, Y_1] = [X_i, Y_2] = 0 (i \geq 3),\\
&[X_i, Y_j] = 2\delta_{ij}(\xi - X_2)(i, j \geq 3).
\end{aligned}$$

Let us define a left-invariant contact metric structure with metric g on the associated Lie group G as follows: $(\xi, X_1, \dots, X_n, Y_1, \dots Y_n)$ is g-orthogonal, ξ is the characteristic vector field, the contact 1-form η is the metric dual of ξ and the (1,1)-tensor field φ is determined by $\varphi(\xi) = 0, \varphi(X_i) = Y_i, \varphi(Y_i) = -X_i$. Upon a lengthy computation, it turns out that $(G, \xi, \eta, \varphi, g)$ is a (k, μ)-contact metric manifold with $k = 0, \mu = 4$.

For dimension 3 ($n = 1$), we provide an explicit example of a Ricci soliton satisfying the last case of Theorem 1. Setting $X = e, Y = \varphi e$ we obtain the Lie algebra: $[\xi, e] = 0, [\xi, \varphi e] = 2e, [e, \varphi e] = 2\xi$ which corresponds to the unimodular Lie group $E(1,1)$ of rigid motions of the Minkowski 2-space, and which is also regarded as the three-dimensional solvable Lie group $Sol^3 = R \rtimes R^2$ such that the action of $u \in R$ sends $(v, w) \in R^2$ to $(e^u v, e^{-u} w)$. As sol^3 is diffeomorphic to R^3, following the construction procedure given on p. 37 of [6], we define the frame field $F_1 = 2\partial_1, F_2 = 2(e^{-x_1}\partial_2 + e^{x_1}\partial_3), F_3 = 2(e^{-x_1}\partial_2 - e^{x_1}\partial_3)$, where $\partial_i = \frac{\partial}{\partial x^i}$ and x^i are standard coordinates on R^3. A direct calculation shows that $[F_1, F_2] = -2F_3, [F_2, F_3] = 0, [F_3, F_1] = 2F_2$ which are identical to the Lie algebra equations for $(\varphi e, e, \xi)$. So, we can take $\varphi e = F_1, e = \frac{1}{\sqrt{2}}F_2, \xi = \frac{1}{\sqrt{2}}F_3$. The dual frame of $(\varphi e, e, \xi)$ is $(\theta^1 = \frac{1}{2}dx_1, \theta^2 = \frac{1}{2\sqrt{2}}(e^{x_1}dx_2 + e^{-x_1}dx_3), \theta^3 = \frac{1}{2\sqrt{2}}(e^{x_1}dx_2 - e^{-x_1}dx_3)$. Using this, we define the left invariant metric $g = \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2 + \theta^3 \otimes \theta^3$. Its Ricci tensor turns out to be $Ric = -8\theta^1 \otimes \theta^1$. For $t \in R$, we consider the vector field $V = 4t[-F_1 - e^{-x_1}x_3F_2 + e^{-x_1}x_3F_3] + 4(1-t)[F_1 - e^{x_1}x_2F_2 - e^{x_1}x_2F_3]$, and compute the Lie derivative of g along V. This provides $\pounds_V g = -16(\theta^2 \otimes \theta^2 + \theta^3 \otimes \theta^3)$. Hence we obtain the Ricci soliton equation $\pounds_V g + 2Ric + 16g = 0$ with $\lambda = 8$. Thus, it is expanding and is non-gradient. Also, as indicated before, g is a (k, μ)-contact metric with $k = 0, \mu = 4$.

Remarks. 1. The above example presents the known non-gradient expanding Ricci soliton on Sol^3 (given on p. 37 of [6]) as a (k, μ)-contact metric with $k = 0, \mu = 4$.

2. If we assume the Ricci soliton complete, then as a local isometry between complete manifolds must be a covering map, the first and second cases in the conclusion of Theorem 1 would have the Ricci soliton isometric to the quotients of the three-dimensional Gaussian soliton and $S^n(4) \times E^{n+1}$, respectively.

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Real Hypersurfaces in Kähler Manifolds

Yong Seung Cho

Abstract We consider real hypersurfaces of compact Kähler manifolds and show that real hypersurfaces of Kähler manifolds induced by Morse functions have contact structures. As examples we consider preimages of regular values of Morse functions on complex projective spaces, and cosymplectic real hypersurfaces of the products of Kähler manifolds and torus.

1 Introduction

Let (M, h, J) be a Hermitian complex manifold with a complex structure J and a Hermitian metric h. The Hermitian metric $h = g + i\Omega$ on the tangent bundle TM of the M is a Kähler metric if the fundamental 2-form Ω is closed. In this case, Ω is called the Kähler form of h, and the complex manifold (M, h, Ω, J) is called a Kähler manifold. Suppose that ∇ is the Levi-Civita connection of the metric $g = \mathrm{Re}(h)$ of the real part of h on a Hermitian complex manifold (M, h, Ω, J). Then $d\Omega = 0$ if and only if $\nabla\Omega = 0$ if and only if $\nabla J = 0$ if and only if (M, h, Ω, J) is Kähler [14]. By Hodge, all the even dimensional Betti numbers of a compact Kähler manifold (M, Ω) are nonzero and the odd dimensional Betti numbers are even. By Calabi and Eckmann [3], the product $S^{2n+1} \times S^{2m+1}$, $n, m > 0$, is a complex manifold but not a Kähler manifold.

Let N be a real $(2n-1)$-dimensional smooth manifold. We will follow the notations and definitions of references [1,2,7,10,15]. Consider an almost cocomplex structure on N defined by a smooth $(1,1)$-type tensor field φ, a smooth 1-form η, and a smooth vector field ξ on N such that for each point $x \in N$,

$$\varphi_x{}^2 = -I + \eta_x \otimes \xi_x, \quad \eta_x(\xi_x) = 1,$$

where $I : T_xN \to T_xN$ is the identity map.

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A Riemannian manifold N with a metric g and an almost cocomplex structure (φ, η, ξ) such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad X, Y \in TN,$$

is called an almost contact metric manifold.

The fundamental 2-form Φ of an almost contact metric manifold $(N, g, \varphi, \eta, \xi)$ is defined by

$$\Phi(X, Y) = g(\varphi X, Y), \quad X, Y \in \Gamma(TN).$$

The Nijenhuis tensor of the tensor φ is the $(1, 2)$-type tensor field N_φ defined by

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] - [X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y], \quad X, Y \in \Gamma(TN),$$

where $[X, Y]$ is the Lie bracket of X and Y.

An almost cocomplex structure φ on N is said to be integrable if the tensor field $N_\varphi = 0$, and is said to be normal if $N_\varphi + 2d\eta \otimes \xi = 0$.

Definition 1. An almost contact metric manifold $(N, g, \Phi, \varphi, \eta, \xi)$ is said to be

1. cosymplectic if $d\Phi = 0 = d\eta$,
2. contact if $\Phi = d\eta$,
3. coKähler if N is integrable and cosymplectic,
4. Sasakian if N is normal and contact.

Let (M, h, Ω, J) be a compact Kähler manifold of complex dimension n. Let $f : M \to \mathbb{R}$ be a Morse function. Its Hamiltonian vector field $X : M \to TM$ is given by $\Omega(X, \cdot) = df(\cdot)$ and the gradient vector field ∇f of f is given by $g(\nabla f, Y) = df(Y),\ Y \in TM$, where $g = \mathrm{Re}(h)$ is the real part of h.

Lemma 1. *Let $f : M \to \mathbb{R}$ be a Morse function on a Kähler manifold (M, h, Ω, J).*

1. *If $r \in \mathbb{R}$ is a regular value of f, then the preimage $M_1 = f^{-1}(r)$ is an almost contact manifold.*
2. *The Hamiltonian vector field X of f is tangent to the real hypersurface M_1 of M.*
3. *The gradient vector field ∇f of f is orthogonal to the Hamiltonian vector field X.*
4. *$JX = \nabla f$.*

We investigate the contact structure on the real hypersurface M_1 of M, and the Gromov-Witten type invariants and the quantum type cohomologies of M_1.

Let (N, h_1, Ω_1, J_1) be a compact Kähler manifold of complex dimensional $(n-1)$, and $(S^1 \times S^1, h_2, \Omega_2, J_2)$ be the torus of complex dimension 1. The product $M = N \times S^1 \times S^1$ has a natural Kähler structure $h = h_1 \oplus h_2,\ \Omega = \Omega_1 \oplus \Omega_2$ and $J = J_1 \oplus J_2$ with complex dimension n. Then the real hypersurface $M_1 = N \times S^1$ has a natural cosymplectic structure $(g, \Phi, \varphi, \eta, \xi)$ induced from the Kähler structure of the product (M, h, Ω, J). In some other paper we will investigate the Gromov-Witten invariants and the quantum cohomologies of the Kähler manifold $M = N \times S^1 \times S^1$, and the ones on the cosymplectic manifold $M_1 = N \times S^1$, and the relations between them.

2 Real Hypersurfaces of Kähler Manifolds

Let (M, h, Ω, J) be a Kähler manifold of complex dimension n. Let $f : M \to \mathbb{R}$ be a Morse function. Denote by $X : M \to TM$ the Hamiltonian vector field defined by $\Omega(X, \cdot) = df(\cdot)$.

The solution $x(t)$ of the Hamiltonian differential equation: $\dot{x}(t) = X(x(t))$ defines a Kählerian diffeomorphism

$$\phi : M \to M \quad \text{with } \phi^* \Omega = \Omega.$$

We split the Kähler metric h into

$$h = g + i\Omega, \quad \text{and } \Omega(\cdot, \cdot) = g(J\cdot, \cdot).$$

Locally the gradient vector field ∇f of f with respect to the metric g is given by

$$\nabla f = \sum_{\mu} \Big(\frac{\partial f}{\partial x_\mu} \frac{\partial}{\partial x_\mu} + \frac{\partial f}{\partial y_\mu} \frac{\partial}{\partial y_\mu} \Big),$$

where the local coordinates $z = (z_1, \cdots, z_n) = (x_1, y_1, \cdots, x_n, y_n)$.

The Hamiltonian vector field X is given by

$$X = -J\nabla f = \sum_{\mu} \Big(-\frac{\partial f}{\partial x_\mu} \frac{\partial}{\partial y_\mu} + \frac{\partial f}{\partial y_\mu} \frac{\partial}{\partial x_\mu} \Big).$$

Suppose that a value $t \in \mathrm{Im}(f)$ of the Morse function $f : M \to \mathbb{R}$ is a regular value. Then the preimage

$$f^{-1}(t) := M_1 \subset M$$

is a $(2n-1)$-dimensional real hypersurface of M. For each point $x \in M_1 \subset M$, the tangent space $T_x M$ is decomposed into an $(n-1)$-dimensional complex space $\mathscr{D}_x$ and a one-dimensional complex line $\langle X(x), J(x)X(x) \rangle$, i.e., $T_x M = \mathscr{D}_x \oplus \langle X(x), J(x)X(x) \rangle$.

Since $df_x : T_x M \to \mathbb{R}$ is surjective, $T_x M_1 = \mathrm{Ker}(df_x)$, and $df_x(X(x)) = \Omega(X(x), X(x)) = 0$, and the vector $X(x)$ is a tangent vector of M_1 at x. The vector $J(x)X(x) = \nabla f(x)$ is the gradient vector.

Lemma 2. *Under the above notations*

1. *The tangent bundle $TM_1 = \mathscr{D} \oplus \langle X \rangle \to M_1$ is decomposed by an $(n-1)$-dimensional complex vector bundle on M_1, $\mathscr{D} \to M_1$, and a trivial real line bundle $\langle X \rangle \to M_1$, where $\langle X \rangle$ is the real line bundle generated by the vector field X on M_1.*
2. *The total Chern class $i^*(c.(TM)) = c.(i^*\mathscr{D}) \in H^*(M_1)$, where $i : M_1 \to M$ is the inclusion.*

Proof. By definition $\mathscr{D}(x) = \{V \in T_xM \mid g(V, X(x)) = 0 = g(V, J(x)X(x))\}$.

For each $V \in \mathscr{D}(x)$, $g(J(x)V, X(x)) = g(-V, J(x)X(x)) = -g(V, J(x)X(x)) = 0$ and $g(J(x)V, J(x)X(x)) = g(V, X(x)) = 0$. Thus $J(x)V \in \mathscr{D}(x)$, and

$$J(x) : \mathscr{D}(x) \longrightarrow \mathscr{D}(x) \quad \text{and} \quad J(x)^2 = -I.$$

and

$$\begin{aligned} df_x(V) = \Phi(X(x), V) &= g(X(x), J(x)V) \\ &= g(J(x)X(x), -V) \\ &= -g(J(x)X(x), V) = 0, \end{aligned}$$

for each element $V \in \mathscr{D}(x)$.

Thus $\mathscr{D}(x) \subset T_xM_1$, and $T_xM_1 = \mathscr{D}(x) \oplus \langle X(x) \rangle$. Since t is a regular value, for each $x \in M_1 = f^{-1}(t)$ $df_x \neq 0$, $\nabla f(x) \neq 0$, and $X(x) = -J(x)\nabla f(x) \neq 0$, the vector field X on M_1 generates a real line bundle $\langle X \rangle \to M_1$.

Therefore we have the required decomposition of the tangent bundle $TM_1 = \mathscr{D} \oplus \langle X \rangle$.

(2) $i^*(c.(TM)) = c.(i^*TM) = c.(i^*\mathscr{D}) \cdot c.(i^*\langle X, JX \rangle) = c.(i^*\mathscr{D})$, since the complex line bundle generated by X and JX $\langle X, JX \rangle \to M_1$ is trivial.

Note that in general the distribution bundle $\mathscr{D} \to M_1$ is not involutive.

For the $(2n-1)$-dimensional hypersurface $M_1 \subset M$, we consider the tangent bundle $TM_1 = \mathscr{D} \oplus \langle X \rangle \to M_1$. Define a unit vector field ξ on M_1 by $\xi = \frac{X}{|X|}$ and its dual 1-form η on M_1. Locally the Hamiltonian vector field is of the form

$$X = \sum_{\mu} \Big(-\frac{\partial f}{\partial x_\mu} \frac{\partial}{\partial y_\mu} + \frac{\partial f}{\partial y_\mu} \frac{\partial}{\partial x_\mu} \Big),$$

the dual 1-form of ξ is of the form

$$\eta = \frac{1}{|X|} \sum_{\mu} \Big(-\frac{\partial f}{\partial x_\mu} dy_\mu + \frac{\partial f}{\partial y_\mu} dx_\mu \Big).$$

Then $\eta(\xi) = \eta\big(\frac{X}{|X|}\big) = \frac{|X|^2}{|X|^2} = 1$.

We define a (1, 1)-type tensor field on φ by

$$\varphi : TM_1 \longrightarrow TM_1, \quad \varphi = \begin{cases} J & \text{on } \mathscr{D} \\ 0 & \text{on } \langle X \rangle \end{cases}.$$

Then for each $V = V_1 + \lambda X \in \mathscr{D} \oplus \langle X \rangle$, $V_1 \in \mathscr{D}$,

$$\begin{aligned} \varphi^2(V_1 + \lambda X) &= J^2(V_1) = -V_1 \\ (-I + \eta \otimes \xi)(V_1 + \lambda X) &= -V_1 - \lambda X + \eta(\lambda X)\xi \\ &= -V_1 - \lambda X + \lambda |X| \xi \\ &= -V_1 - \lambda X + \lambda X \\ &= -V_1. \end{aligned}$$

Thus $\varphi^2 = -I + \eta \otimes \xi$ on M_1.

For every element $V_1 + \lambda_1 \xi,\ V_2 + \lambda_2 \xi \in TM_1 = \mathscr{D} \oplus \langle X \rangle$,

$$\begin{aligned} g(V_1 + \lambda_1 \xi, V_2 + \lambda_2 \xi) &= g(V_1, V_2) + g(V_1, \lambda_2 \xi) + g(\lambda_1 \xi, V_2) + g(\lambda_1 \xi, \lambda_2 \xi) \\ &= g(V_1, V_2) + \lambda_1 \lambda_2 \\ &= g(\varphi(V_1 + \lambda_1 \xi), \varphi(V_2 + \lambda_2 \xi)) + \eta(V_1 + \lambda_1 \xi)\eta(V_2 + \lambda_2 \xi). \end{aligned}$$

Thus $g(\cdot, \cdot) = g(\varphi \cdot, \varphi \cdot) + \eta \otimes \eta$.

Lemma 3. *Under the above definitions on ξ, η, and φ on M_1, the $(M_1, g, \varphi, \xi, \eta)$ is an almost contact metric manifold of dimension $(2n - 1)$.*

Suppose that M is a compact Kähler manifold. Let $f : M \to \mathbb{R}$ be a Morse function, $M^a := f^{-1}((-\infty, a])$ and $t \in \mathbb{R}$ be a regular value of f, $M_1 = f^{-1}(t) \neq \emptyset$. Suppose that the set $f^{-1}[t - \varepsilon, t + \varepsilon]$ is compact and contains no critical points of f. Then $M^{t-\varepsilon}$ is diffeomorphic to $M^{t+\varepsilon}$. Moreover $M^{t-\varepsilon}$ is a deformation retract of $M^{t+\varepsilon}$, and there is a diffeomorphism $F : N \equiv M_1 \times [t - \varepsilon, t + \varepsilon] \to f^{-1}[t - \varepsilon, t + \varepsilon] \subset M$.

Let $N' := \frac{M_1 \times [t-\varepsilon, t+\varepsilon]}{M_1 \times \{t-\varepsilon, t+\varepsilon\}}$, $g : N' \to [t - \varepsilon, t + \varepsilon]$ be the map induced by f and $\rho : N \to N'$ be the quotient map. Then $g \circ \rho = f \circ F$.

Lemma 4. *1. The map $g : N' \to \mathbb{R}$ is a Morse function with only 2 critical points.*
2. The N' is homeomorphic to the suspension of M_1.
3. $H_k(N') = \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ H_{k-1}(M_1) & \text{if } k > 0. \end{cases}$ □

Suppose that our Morse function $f : M \to \mathbb{R}$ has a restriction $f : f^{-1}[t - \varepsilon, t + \varepsilon] \simeq M_1 \times [t - \varepsilon, t + \varepsilon] \to [t - \varepsilon, t + \varepsilon] \subset \mathbb{R}$ of the form $f(x_1, y_1, \cdots, x_n, y_n) = {x_1}^2 + {y_1}^2 + \cdots + {x_n}^2 + {y_n}^2$. Then the gradient vector field ∇f with respect to the metric $g = \sum_{i=1}^n (dx_i \otimes dx_i + dy_i \otimes dy_i)$ is

$$\nabla f = \sum_i \left(2x_i \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial y_i}\right),$$

and its Hamiltonian vector field X is

$$X = -J\nabla f = \sum_i \left(- 2x_i \frac{\partial}{\partial y_i} + 2y_i \frac{\partial}{\partial x_i}\right).$$

On a neighbourhood of M_1, the vector field ξ is chosen to be

$$\xi = -\frac{X}{|X|} = \frac{2}{|X|} \sum_i \left(x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i}\right),$$

and its dual 1-form is

$$\eta = \frac{2}{|X|} \sum_i (x_i dy_i - y_i dx_i).$$

Indeed,

$$\eta(\xi) = \frac{2}{|X|} \sum_i (x_i dy_i - y_i dx_i) \cdot \frac{2}{|X|} \sum_i \left(x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i}\right)$$
$$= \frac{4}{|X|^2} \sum_i (x_i^2 + y_i^2) = \frac{|X|^2}{|X|^2} = 1.$$

While on M_1, $f(M_1) = f(f^{-1}(t)) = t = \sum_{i=1}^n (x_i^2 + y_i^2)$ and

$$|X| = \left[4 \sum_{i=1}^n (x_i^2 + y_i^2)\right]^{\frac{1}{2}} = 2 \cdot \left(\sum_{i=1}^n (x_i^2 + y_i^2)\right)^{\frac{1}{2}} = 2\sqrt{t}.$$

The exterior derivative of η is

$$d\eta = d\left(\frac{2}{|X|} \sum_{i=1} (x_i dy_i - y_i dx_i) = \frac{2}{2\sqrt{t}} \sum_{i=1}^n dx_i \wedge dy_i - dy_i \wedge dx_i)\right)$$
$$= \frac{2}{\sqrt{t}} \sum_{i=1}^{n-1} (dx_i \wedge dy_i)$$
$$= \frac{2}{\sqrt{t}} \Phi,$$

since $dy_n = 0$ by choosing y_n coordinate on the trajectory of the gradient vector field ∇f, where Φ is the fundamental 2-form on M_1.

Theorem 1. *Let (M, h, J, Ω) be a Kähler manifold of complex dimension n. If $f : M \to \mathbb{R}$ is a Morse function, and $t \in \mathbb{R}$ is a regular value of f, then $(f^{-1}(t) := M_1, \varphi, g, \eta, \xi, \Phi)$ is a contact real hypersurface of dimension $(2n-1)$ of M, where $\varphi,\ g,\ \eta,\ \xi$, and Φ are defined as above.* □

3 Contact Manifolds as Real Hypersurfaces in Complex Projective Spaces

Let $(\mathbb{CP}^n, h, J, \omega)$ be the complex projective space of dimension n, where $\mathbb{CP}^n = S^{2n+1}/S^1$. Let $c_0 < c_1 < \cdots < c_n$ be distinct real numbers [13].

Define $f : \mathbb{CP}^n \to \mathbb{R}$ by

$$f([z_0, \cdots, z_n]) = c_0|z_0|^2 + \cdots + c_n|z_n|.$$

Let $U_i := \{[z_0, \cdots, z_n] \mid z_i \neq 0\},\ i = 0, \cdots, n$. Then the U_i are open in $\mathbb{CP}^n$ and $U_i \simeq \mathbb{C}^n,\ \mathbb{CP}^n = \cup_{i=0}^n U_i \simeq U_i \cup \mathbb{CP}^{n-1}$.

On $U_0 = \{[z_0, \cdots, z_n] \in \mathbb{CP}^n \mid z_0 \neq 0\}$, since $\frac{|z_0|}{z_0} \in S^1$, $[z_0, \cdots, z_n] = [|z_0|, \frac{|z_0|}{z_0} z_1, \cdots, \frac{|z_0|}{z_0} z_n]$.

Let $\frac{|z_0|}{z_0} z_j = x_j + i y_j,\ j = 1, \cdots, n$. Then $(x_1, y_1, \cdots, x_n, y_n)$ is a coordinate system on U_0, and $f(x_1, y_1, \cdots, x_n, y_n) = c_0 + \sum_{j=1}^n (c_j - c_0)(x_j^2 + y_j^2)$.

Since $df = 2\sum_{j=1}^n (c_j - c_0)(x_j dx_j + y_j dy_j)$,

$$df = 0 \text{ on } U_0 \text{ if and only if } x_1 = y_1 = \cdots = x_n = y_n = 0.$$

Thus the point $P_0 = [1, 0, \cdots, 0] \in U_0$ is the critical point of f on U_0.
Similarly, the points $P_0, \cdots, P_n$ are critical points of f on $\mathbb{CP}^n$, where $P_i = [0, \cdots, 0, 1, 0, \cdots, 0]$, and the indices $\mathrm{ind}_f(P_i) = 2i,\ i = 0, \cdots, n$.
The gradient vector field ∇f is given by

$$\nabla f = 2\sum_{j=1}^n (c_j - c_0)\left(x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}\right).$$

The Hamiltonian vector field X_f is given by

$$X_f = -J\nabla f = 2\sum_{j=1}^n (c_j - c_0)\left(-x_j \frac{\partial}{\partial y_j} + y_j \frac{\partial}{\partial x_j}\right).$$

In fact, our map $f : \mathbb{CP}^n \to [c_0, c_n]$ has the minimum c_n, and $c_0, \cdots, c_n$ are the critical values of f.

For a regular value $t \in [c_0, c_n] - \{c_0, \cdots, c_n\}$, the preimage $f^{-1}(t) := M_t$ is a $(2n-1)$-dimensional real hypersurface of $\mathbb{CP}^n$.

Around the critical point P_i, the image of f is

$$f([z_0, \cdots, z_n]) = c_0|z_0|^2 + \cdots + c_n|z_n|^2$$
$$= c_i + (c_0 - c_i)|z_0|^2 + \cdots + (c_{i-1} - c_i)|z_{i-1}|^2 + (c_{i+1} - c_i)|z_{i+1}|^2 + \cdots + (c_n - c_i)|z_n|^2.$$

The inverse image of the critical value c_i is

$$f^{-1}(c_i) = \{[z_0, \cdots, z_n] \in \mathbb{CP}^n |\ (c_0 - c_i)|z_0|^2 + \cdots + (c_{i-1} - c_i)|z_{i-1}|^2$$
$$+ (c_{i+1} - c_i)|z_{i+1}|^2 + \cdots + (c_n - c_i)|z_n|^2 = 0\}.$$

Thus $[z_0, \cdots, z_n] \in f^{-1}(c_i)$ if and only if
$(c_i - c_0)|z_0|^2 + \cdots + (c_i - c_{i-1})|z_{i-1}|^2 = (c_{i+1} - c_i)|z_{i+1}|^2 + \cdots + (c_n - c_i)|z_n|^2.$

The dual 1-form of the Hamiltonian vector field is locally at near P_i,

$$\eta = \sum_{j=1}^{n} \frac{1}{2(c_j - c_i)}(-x_j dy_j + y_j dx_j) \text{ and } \eta(X_f) = 1.$$

And the derivative of η is

$$d\eta = \sum_{j=1}^{n} \frac{1}{2(c_j - c_i)}(-dx_j \wedge dy_j + dy_j \wedge dx_j)$$
$$= -\sum_{j=1}^{n} \frac{1}{c_j - c_i} dx_j \wedge dy_j.$$

By scaling and changing coordinates, we have the standard 2-form

$$\widetilde{d\eta} = \sum_{j=1}^{n} d\tilde{x}_j \wedge d\tilde{y}_j.$$

Lemma 5. *Let* $f : \mathbb{CP}^n \to \mathbb{R}$ *be* $f([z_0, \cdots, z_n]) = c_0|z_0|^2 + \cdots + c_n|z_n|^2$, $c_0 < c_1 < \cdots < c_n$, $|z_0|^2 + \cdots + |z_n|^2 = 1$.

1. *If* $f(P_i) = c_i$, *then* $\{P_0, \cdots, P_n\}$ *are the critical points and* $\{c_0, \cdots, c_n\}$ *are the critical values of* f.

2. $f^{-1}(t) = M_t$, $t \in [c_0, c_n] - \{c_0, \cdots, c_n\}$, *is a* $(2n-1)$*-dimensional contact hypersurface which is diffeomorphic to the standard* $(2n-1)$*-sphere* S^{2n-1}.
3. $f^{-1}(c_0)$ *and* $f^{-1}(c_n)$ *are points and* $f^{-1}(c_i)$ *is diffeomorphic to the wedge of two* S^{2n-1}, *i.e.,* $S^{2n-1} \vee S^{2n-1}$.

Let $M_t = f^{-1}(t)$ be the $(2n-1)$-dimensional real hypersurface of the complex n-dimensional projective space $(\mathbb{CP}^n, h, J, \omega)$, where t is a regular value of f in $[c_0, c_n]$. Since $df \neq 0$ on M_t, the gradient vector field ∇f of f does not vanish on M_t and the Hamiltonian vector field $X_t = -J\nabla f$ does not vanish on M_t. The restriction of the tangent bundle $T\mathbb{CP}^n|_{M_t} = TM_t \oplus \langle \nabla f \rangle = \mathscr{D}_t \oplus \langle X_t, \nabla f \rangle$, and $TM_t = \mathscr{D}_t \oplus \langle X_t \rangle \to M_t$.

Define a $(1,1)$-type tensor $\varphi : TM_t \to TM_t$ by $\varphi = J$ on $\mathscr{D}$ and $\varphi(X_t) = 0$. Then the metric g_t on M_t is $g_t(X, Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y)$, where g is the metric on $\mathbb{CP}^n$ given by $g(X, Y) = \omega(X, JY)$. The fundamental 2-form on M_t is given by $\Phi(X, Y) = g_t(\varphi X, Y)$. Then $d\Phi = 0$.

Theorem 2. *If* $t \in [c_0, c_n]$ *is a regular value of* $f : \mathbb{CP}^n \to \mathbb{R}$,

1. $(M_t, g_t, \varphi, \eta, X_t, \Phi)$ *is a contact* $(2n-1)$*-dimensional hypersurface of* $\mathbb{CP}^n$.
2. $\mathscr{D} \to M_t$ *is a complex* $(n-1)$*-dimensional vector bundle.*

Lemma 6. *Under above definitions and notations,*

1. $\eta(X_t) = 1$, $\varphi(X_t) = 0$,
2. $\varphi^2 = -I + \eta \otimes \xi$,
3. $d\Phi = 0$,
4. $g_t(\varphi X, \varphi Y) = g_t(X, Y)$ *on* $\mathscr{D}$,
5. *on* $\mathscr{D}$, $\Phi(X, Y) = \Phi(\varphi X, \varphi Y)$,
6. $\Phi(\varphi X, X) = g_t(X, X)$ *on* $\mathscr{D}$.

Proof. (3) Since on $\mathscr{D}$, $\Phi = \omega$ and $d\omega = 0$, $d\Phi = 0$.

Theorem 3. *Let* $M_t = f^{-1}(t)$.

1. *If* $t = c_0$ *or* c_n, *then* M_t *is a point and* $H^k(M_t; \mathbb{R}) = \begin{cases} \mathbb{R} & k = 0, \\ 0 & k \neq 0 \end{cases}$.
2. *If* $t \in \{c_1, c_2, \cdots, c_{n-1}\}$, *then* $M_t \simeq S^{2n-1} \vee S^{2n-1}$, *and*
$$H^k(M_t; \mathbb{R}) = \begin{cases} \mathbb{R} & k = 0, \\ \mathbb{R}^2 & k = 2n-1, \\ 0 & k \neq 0,\ 2n-1 \end{cases}.$$
3. *If* $t \in [c_0, c_n] - \{c_0, \cdots, c_n\}$, *then* $M_t \simeq S^{2n-1}$, *and*
$$H^k(M_t) = \begin{cases} \mathbb{R} & k = 0 \text{ or } k = 2n-1 \\ 0 & \textit{otherwise} \end{cases}.$$

4 Cosymplectic Manifolds as Real Hypersurfaces in Kähler Manifolds

Let (M_1, ω_1, J_1) be a $2(n-1)$-dimensional compact Kähler manifold, and $(S^1 \times S^1 = M_2, \omega_2, J_2)$ be the standard torus. Let $(M = M_1 \times M_2, \omega = \omega_1 + \omega_2, J = J_1 + J_2)$ be the product of the M_1 and M_2. Then (M, ω, J) is a Kähler manifold and $(N = M_1 \times S^1, \varphi := \pi_1^* \omega_1, \eta = \pi_1^* d\theta, \xi = \pi_2^*(\frac{d}{dt}))$ is a $(2n-1)$-dimensional compact cosymplectic manifold in the $2n$-dimensional compact Kähler manifold (M, ω, J).

Theorem 4. *Under the above construction,*

1. *The* $(N = M_1 \times S^1, \varphi, \eta, \xi)$ *is a cosymplectic real hypersurface of the Kähler manifold* (M, ω, J).
2. *The distribution bundle* $\mathcal{D} = \pi_1^* TM_1 \to N$ *is the pullback of the tangent bundle* $TM_1 \to M_1$ *by the projection* $\pi_1 : N \to M_1$.

As in Theorem 1 if $f_1 : M_1 \to \mathbb{R}$ is a Morse function on a Kähler manifold M_1 of real dimension $2n-2$, and $t \in \mathbb{R}$ is a regular value of f_1, then the inverse image $f^{-1}(t) = N$ is a contact real hypersurface of M_1.

Corollary 1. *In this case the product* $N \times M_1$ *is a contact real hypersurface of the Kähler manifold* $M = M_1 \times M_2$ *of real dimension* $2n-1$, *where* M_2 *is an elliptic curve.*

Note that many authors in [4, 5, 9, 11, 12, 16] worked Gromov-Witten invariants and quantum cohomologies on symplectic manifolds. In [6–8] we studied Gromov-Witten type invariants and quantum cohomologies on almost contact metric manifolds.

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L^p-Spectral Gap and Gromov-Hausdorff Convergence

Shouhei Honda

Abstract This paper is an announcement of recent results given by the author which gives the continuity of L^p-spectral gaps with respect to the Gromov-Hausdorff topology and applications.

1 Introduction

We say that a pair (X, υ) is *a compact metric measure space* if X is a compact metric space and υ is a Borel probability measure on X. A compact metric measure space (X, υ) is said to be *an n-dimensional compact smooth metric measure space* if X is an n-dimensional closed Riemannian manifold and υ is the canonical Riemannian probability measure on X.

Let $n \in \mathbf{N}, K \in \mathbf{R}, d > 0$ and let $M(n, K, d)$ be the set of n-dimensional compact smooth metric measure spaces (X, υ) with $\mathrm{Ric}_X \geq K(n-1)$ and $\mathrm{diam}\, X \leq d$, where Ric_X is the Ricci curvature of X, and $\mathrm{diam}\, X$ is the diameter of X. let $\overline{M(n, K, d)}$ be the Gromov-Hausdorff compactification of $M(n, K, d)$, i.e., every $(X, \upsilon) \in \overline{M(n, K, d)}$ is the measured Gromov-Hausdorff limit of a sequence of $(X_i, \upsilon_i) \in M(n, K, d)$.

In [8, 9], Gromov and Fukaya proved that $\overline{M(n, K, d)}$ is compact with respect to the measured Gromov-Hausdorff topology. Fukaya also conjectured in [8] the following:

Conjecture 1 (Fukaya [8]). There exists the 'canonical' Laplacian on every $(X, \upsilon) \in \overline{M(n, K, d)}$ such that the k-th eigenvalues is continuous on $\overline{M(n, K, d)}$ for every k.

In [7], Cheeger and Colding proved the conjecture via the structure theory of Gromov-Hausdorff limit spaces given by themselves [4–7], i.e.,

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Theorem 1 (Cheeger and Colding[7]). *Conjecture 1 is true.*

In this paper, we introduce a generalization of Theorem 1 for the first eigenvalues to the nonlinear case given in [13].

We discuss the first eigenvalues only. It is known that the first eigenvalue $\lambda_{1,2}(X)$ of the Laplacian on $(X, \upsilon) \in \overline{M(n, K, d)}$ has the following expression as the L^2-spectral gap:

$$\lambda_{1,2}(X) = \inf_f \frac{|||\nabla f|||^2_{L^2(X)}}{\inf_{c \in \mathbf{R}} ||f - c||^2_{L^2(X)}},$$

where the infimum runs over nonconstant Lipschitz functions f on X and

$$|\nabla f|(x) := \lim_{r \to 0} \left(\sup_{y \in B_r(x)} \frac{|f(x) - f(y)|}{d_X(x, y)} \right).$$

The main target of this paper is the following L^p-spectral gap:

$$\lambda_{1,p}(X) := \inf_f \frac{|||\nabla f|||^p_{L^p(X)}}{\inf_{c \in \mathbf{R}} ||f - c||^p_{L^p(X)}}$$

for $1 \le p < \infty$. If X is a single point, then we put $\lambda_{1,p}(X) := \infty$. It is known that if (X, υ) is an n-dimensional compact smooth metric measure space, then $\lambda_{1,p}(X)$ is equal to the first positive eigenvalue of the following p-Laplacian on X:

$$-\mathrm{div}\left(|\nabla f|^{p-2} \nabla f\right) = \lambda |f|^{p-2} f$$

if $p > 1$, and $\lambda_{1,1}(X)$ is equal to Cheeger's (isoperimetric) constant $h(X)$ of X defined in [2], i.e.,

$$h(X) = \inf_\Omega \frac{H^{n-1}(\partial \Omega)}{H^n(\Omega)},$$

where the infimum runs over open subset Ω of X having the smooth boundary $\partial \Omega$ with $H^n(\Omega) \le H^n(X)/2$, and H^n is the n-dimensional Hausdorff measure.

Let F be the function from $\overline{M(n, K, d)} \times [1, \infty]$ to $(0, \infty]$ defined by $F((X, \upsilon), p) = 2(\mathrm{diam}\, X)^{-1}$ if $p = \infty$, and $F((X, \upsilon), p) = \left(\lambda_{1,p}(X)\right)^{1/p}$ if $1 < p < \infty$.

The main result of [14] is the following:

Theorem 2 ([14]). *We have the following:*

1. *F is upper semicontinuous on $\overline{M(n, K, d)} \times [1, \infty]$.*
2. *F is continuous on $\overline{M(n, K, d)} \times (1, \infty]$.*
3. *F is continuous on $\{(X, \upsilon)\} \times [1, \infty]$ for every $(X, \upsilon) \in \overline{M(n, K, d)}$.*

Theorem 2 for $p = 2$ corresponds to Theorem 1 for the first eigenvalues.

The organization of this paper is as follows. In Sect. 2, we will introduce key steps of the proof of Theorem 2. In Sect. 3, we will discuss applications.

2 Key Steps

We introduce here two key steps in order to prove (2) of Theorem 2. The first key step is the following Rellich type compactness with respect to the Gromov-Hausdorff topology given in [13] by the author. Note that Theorem 3 corresponds to the classical Rellich compactness if $p_i \equiv p$ and $(X_i, \upsilon_i) \equiv (X, \upsilon)$.

Theorem 3 ([13]). *Let $(X_i, \upsilon_i) \to (X_\infty, \upsilon_\infty)$ in $\overline{M(n, K, d)}$ with $\mathrm{diam}\, X_\infty > 0$, and let $p_i \to p_\infty$ in $(1, \infty)$. Then for every sequence $\{f_i\}_{i<\infty}$ of Sobolev functions $f_i \in H_{1,p_i}(X_i)$ with $\sup_{i<\infty} ||f_i||_{H_{1,p_i}} < \infty$, there exist a subsequence $\{f_{i(j)}\}_j$ and a Sobolev function $f_\infty \in H_{1,p_\infty}(X_\infty)$ such that $f_{i(j)}$ $\{L^{p_{i(j)}}\}_j$-converges strongly to f_∞ on X_∞ and that $\nabla f_{i(j)}$ $\{L^{p_{i(j)}}\}_j$-converges weakly to ∇f_∞ on X_∞. In particular, we have*

$$\liminf_{j\to\infty} ||\nabla f_{i(j)}||_{L^{p_{i(j)}}} \geq ||\nabla f_\infty||_{L^{p_\infty}}.$$

See [3, 11, 19] for the definitions and the fundamental properties of the Sobolev space $H_{1,p}(X)$. See also [13] for $\{L^{p_{i(j)}}\}_j$-convergence. It is useful that we mention that if $p_i \equiv p$, then the notion of $\{L^{p_{i(j)}}\}_j$-convergence coincides with that of L^p-convergence.

Since we can reprove Conjecture 1 easily via Theorem 3, we introduce it:

Proof of Theorem 1 via Theorem 3. Let $(X_i, \upsilon_i) \to (X_\infty, \upsilon_\infty)$ in $\overline{M(n, K, d)}$ and let $k \in \mathbf{N}$. We now only prove

$$\lim_{i\to\infty} \lambda_{k,2}(X_i) = \lambda_{k,2}(X_\infty)$$

under the assumption $\mathrm{diam}\, X_\infty > 0$, where $\lambda_{k,2}$ is the k-th eigenvalue of the Laplacian.

First we assume $k = 1$. Let $f_\infty \in H_{1,2}(X_\infty)$ be a $\lambda_{1,2}(X_\infty)$-eigenfunction with $||f_\infty||_{L^2} = 1$, i.e.,

$$\lambda_{1,2}(X_\infty) = ||\nabla f_\infty||_{L^2}^2 \text{ and } \int_{X_\infty} f_\infty d\upsilon_\infty = 0.$$

Then there exists a sequence $\{f_i\}_{i<\infty}$ of Sobolev functions $f_i \in H_{1,2}(X_i)$ such that $f_i, \nabla f_i$ L^2-converge strongly to $f_\infty, \nabla f_\infty$ on X_∞, respectively (see [12, Theorem 4.2]). Thus we have

$$\limsup_{i\to\infty} \lambda_{1,2}(X_i) \le \limsup_{i\to\infty} \frac{|||\nabla f_i|||^2_{L^2}}{\inf_{c\in\mathbf{R}} ||f_i - c||^2_{L^2}} = ||\nabla f_\infty||^2_{L^2} = \lambda_{1,2}(X_\infty).$$

On the other hand, let $\{f_i\}_{i<\infty}$ be a sequence of $\lambda_{1,2}(X_i)$-eigenfunction $f_i \in H_{1,2}(X_i)$ with $||f_i||_{L^2} \equiv 1$. Then Theorem 3 yields that there exist a subsequence $\{f_{i(j)}\}_j$ and a Sobolev function $f_\infty \in H_{1,2}(X_\infty)$ such that $f_{i(j)}$ L^2-converges strongly to f_∞ on X_∞ and that $\nabla f_{i(j)}$ L^2-converges weakly to ∇f_∞ on X_∞. Without loss of generality, we can assume $\liminf_{j\to\infty} \lambda_{1,2}(X_{i(j)}) = \liminf_{i\to\infty} \lambda_{1,2}(X_i)$. Thus we have

$$\liminf_{i\to\infty} \lambda_{1,2}(X_i) = \liminf_{j\to\infty} \frac{|||\nabla f_{i(j)}|||^2_{L^2}}{\inf_{c\in\mathbf{R}} ||f_{i(j)} - c||^2_{L^2}} \ge \frac{|||\nabla f_\infty|||^2_{L^2}}{\inf_{c\in\mathbf{R}} ||f_\infty - c||^2_{L^2}} \ge \lambda_{1,2}(X_\infty).$$

Therefore we have the assertion for $k = 1$. Similarly, we have the assertion for $k \ge 2$ by using min-max principle.

An argument similar to that above gives a part of Theorem 2: F is continuous on $\overline{M(n,K,d)} \times (1,\infty)$.

The second key step is Grosjean's argument in the proof of the following:

Theorem 4 (Grosjean [10]). *We have*

$$\lim_{p\to\infty} \left(\lambda_{1,p}(M)\right)^{1/p} = \frac{2}{\operatorname{diam} M}$$

for every closed Riemannian manifold M.

Grosjean proved Theorem 4 by using the classical Rellich compactness. We can finish the proof of (2) of Theorem 2 by Grosjean's argument with Theorem 3. See [14] for the proof of the remained part of Theorem 2.

3 Applications

We introduce here applications of Theorem 2 without the proofs. The first application is the following isoperimetric inequality for $\lambda_{1,p}$:

Theorem 5 ([14]). *Let M be an n-dimensional closed Riemannian manifold with*

$$(\operatorname{diam} M)^2 \operatorname{Ric}_M \ge K(n-1).$$

Then, we have

$$\left(\lambda_{1,p}(M)\right)^{1/p} \overset{n,K}{\asymp} h(M) \overset{n,K}{\asymp} (\operatorname{diam} M)^{-1}$$

for every $1 < p < \infty$*, where for positive numbers* $a, b \in \mathbf{R}_{>0}$*,* $a \overset{n,K}{\asymp} b$ *means that there exists a positive number* $C := C(n, K) > 1$ *depending only on* n *and* K *such that* $C^{-1}b \leq a \leq Cb$ *holds.*

Note that Theorem 5 for $p = 2$ yields a Cheeger and Buser type inequality [1, 2]:

$$0 < C_1(n, K) \leq \frac{(\lambda_{1,2}(M))^{1/2}}{h(M)} \leq C_2(n, K) < \infty.$$

There are many important works on lower bounds of $\lambda_{1,p}$. See for instance [15–18, 20–22]. It is important that Theorem 5 gives a two-sided bound which is *independent* of the exponent p.

The following is a direct consequence of Theorem 5.

Corollary 1 ([14]). *Let* M *be as in Theorem 5. Then we have*

$$0 < C_1(n, K) \leq \frac{\left(\lambda_{1,p}(M)\right)^{1/p}}{\left(\lambda_{1,q}(M)\right)^{1/q}} \leq C_2(n, K) < \infty$$

for any $p, q \in (1, \infty)$.

Corollary 1 implies that if $(\lambda_{1,p}(M))^{1/p}$ is small (or big) for some p, then $(\lambda_{1,q}(M))^{1/q}$ is also small (or big) for every q, quantitatively. Note that Corollary 1 holds on Gromov-Hausdorff limit spaces. See [14] for the detail.

The second application is a quantitative version of Grosjean's result. It follows from a standard compactness argument with Theorem 2.

Theorem 6 ([14]). *Let* $n \in \mathbf{N}, K \in \mathbf{R}$ *and let* $\epsilon > 0$*. Then there exists* $p := p(n, K, \epsilon) > 1$ *such that*

$$\left| \operatorname{diam} M \left(\lambda_{1,q}(M)\right)^{1/q} - 2 \right| < \epsilon$$

holds for every $p \leq q < \infty$ *and every* n*-dimensional closed Riemannian manifold* M *with* $(\operatorname{diam} M)^2 \operatorname{Ric}_M \geq K(n-1)$.

The third application is the following. Note that this is a generalization of Lévy-Gromov's isoperimetric inequality ($p = 1$), Lichnerowictz-Obata's theorem ($p = 2$), and Meyers's diameter theorem ($p = \infty$) to the case of limit spaces and general p. We use the following notation for simplicity:

$$(\lambda_{1,\infty}(X))^{1/\infty} := \frac{2}{\operatorname{diam} X}.$$

Theorem 7 ([14]). *Let* (X, υ) *be the Gromov-Hausdorff limit space of a sequence of* $(X_i, \upsilon_i) \in \overline{M(n, 1, \pi)}$*. Then we have*

$$\left(\lambda_{1,p}(X)\right)^{1/p} \geq \left(\lambda_{1,p}(\mathbf{S}^n)\right)^{1/p}$$

for every $1 \leq p \leq \infty$. *Moreover, the following three conditions are equivalent:*

1. $\left(\lambda_{1,p}(X)\right)^{1/p} = \left(\lambda_{1,p}(\mathbf{S}^n)\right)^{1/p}$ *holds for some* $1 < p \leq \infty$.
2. $\left(\lambda_{1,p}(X)\right)^{1/p} = \left(\lambda_{1,p}(\mathbf{S}^n)\right)^{1/p}$ *holds for every* $1 < p \leq \infty$.
3. $\mathrm{diam}\, X = \pi$.

Note that Matei proved Theorem 7 for $(X, \upsilon) \in M(n, 1, \pi)$ in [16] and that Cheeger-Colding proved that in Theorem 7, if $\mathrm{diam}\, X = \pi$, then X is isometric to the spherical suspension $\mathbf{S}^0 * Y$ of a compact metric space Y with $\mathrm{diam}\, Y \leq \pi$, where

$$\mathbf{S}^0 * Y := ([0, \pi] \times Y) \, / \, (\{0, \pi\} \times Y)$$

and the distance $d_{S^0 * Y}$ is defined by

$$d_{\mathbf{S}^0 * Y}((t_1, y_1), (t_2, y_2)) := \arccos\left(\cos t_1 \cos t_2 + \sin t_1 \sin t_2 \cos d_Y(y_1, y_2)\right).$$

See [4].

A standard compactness argument with Theorem 7 yields the following corollary:

Corollary 2. *Let* $n \in \mathbf{N}, K \in \mathbf{R}, 1 < p \leq \infty$ *and let* $\epsilon > 0$. *Then there exists* $\delta := \delta(n, K, p, \epsilon) > 0$ *such that if an* n*-dimensional compact Riemannian manifold* M *with* $\mathrm{Ric}_M \geq n - 1$ *and*

$$\left|\left(\lambda_{1,q}(X)\right)^{1/q} - \left(\lambda_{1,q}(\mathbf{S}^n)\right)^{1/q}\right| < \delta$$

for some $q \in [p, \infty]$, *then*

$$\left|(\lambda_{1,r}(X))^{1/r} - (\lambda_{1,r}(\mathbf{S}^n))^{1/r}\right| < \epsilon$$

for every $r \in [p, \infty]$. *In particular,*

$$d_{GH}(M, \mathbf{S}^0 * X) < \epsilon$$

for some compact metric space X *with* $\mathrm{diam}\, X \leq \pi$, *where* d_{GH} *is the Gromov-Hausdorff distance.*

We now recall Cheeger's inequality on a closed Riemannian manifold M:

$$\frac{h(M)^2}{4} \leq \lambda_{1,2}(M).$$

See [2].

Since we can prove that Cheeger's inequality holds on limit spaces by using Theorem 2, we end this section by introducing it:

Matei proved in [17] that

$$p\left(\lambda_{1,p}(M)\right)^{1/p} \le q\left(\lambda_{1,q}(M)\right)^{1/q}$$

holds for every closed Riemannian manifold M and any $1 < p \le q < \infty$. Thus by taking Gromov-Hausdorff limits, (2) of Theorem 2 yields that

$$p\left(\lambda_{1,p}(X)\right)^{1/p} \le 2\left(\lambda_{1,2}(X)\right)^{1/2}$$

holds for every $(X, \upsilon) \in \overline{M(n, K, d)}$ and every $1 < p < 2$. By letting $p \to 1$, (3) of Theorem 2 gives Cheeger's inequality on limit spaces:

$$h(X) \le 2\left(\lambda_{1,2}(X)\right)^{1/2}$$

for every $(X, \upsilon) \in \overline{M(n, K, d)}$.

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Riemannian Questions with a Fundamental Differential System

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Abstract We introduce the reader to a fundamental exterior differential system of Riemannian geometry which arises naturally with every oriented Riemannian $n+1$-manifold M. Such system is related to the well-known metric almost contact structure on the unit tangent sphere bundle SM, so we endeavor to include the theory in the field of contact systems. Our EDS is already known in dimensions 2 and 3, where it was used by Griffiths in applications to mechanical problems and Lagrangian systems. It is also known in any dimension but just for flat Euclidean space. Having found the Lagrangian forms $\alpha_i \in \Omega^n$, $0 \le i \le n$, we are led to the associated functionals $\mathscr{F}_i(N) = \int_N \alpha_i$, on the set of hypersurfaces $N \subset M$, and to their Poincaré-Cartan forms. A particular functional relates to scalar curvature and thus we are confronted with an interesting new equation.

1 Geometric Structures and the Fundamental Differential System

1.1 The Manifold SM

Let M be any smooth oriented $n+1$-dimensional Riemannian manifold. Our study is centred on the geometry of the tangent bundle TM as an oriented Riemannian $2n+2$-manifold, endowed with the well-known Sasaki metric. Let $\pi : TM \longrightarrow M$ denote the canonical projection. The vector bundle $V := \ker \mathrm{d}\pi \simeq \pi^\star TM \longrightarrow TM$ agrees fibrewise with the tangent bundle to the fibres of TM. Moreover the tangent bundle of TM splits as $TTM = H \oplus V$, where H is a sub-vector bundle depending on ∇, the Levi-Civita connection. Clearly the *horizontal* sub-bundle H is also isomorphic to $\pi^* TM$ through the map $\mathrm{d}\pi$. We thus define an endomorphism

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$$B : TTM \longrightarrow TTM \tag{1}$$

transforming H in V and vanishing on the *vertical* sub-bundle V. This is used by many authors perhaps not giving it so much importance. Partly because one simply recurs to lifts of the same vector on M to either horizontal or vertical parts.

There also exists a connection independent vector field ξ over TM defined by $\xi_u = u$, or maybe more precisely $\xi_u = \pi^\star u$, $\forall u \in TM$, turning explicit the vertical lift. Henceforth, there exists a unique horizontal ∇-dependent vector field, formally, $B^{\mathrm{ad}}\xi \in H$, such that $B(B^{\mathrm{ad}}\xi) = \xi$. That field is the *geodesic spray* of the connection, cf. [17]. One can see easily that $\pi^\star\nabla_w\xi = w^v$, one reason being that $H = \ker(\pi^\star\nabla.\xi)$.

The manifold TM also inherits a linear connection, denoted ∇^*, which is just

$$\pi^*\nabla \oplus \pi^\star\nabla$$

preserving the canonical splitting $TTM = H \oplus V \simeq \pi^*TM \oplus \pi^\star TM$. We observe then that the connecting endomorphism B is parallel for such ∇^*. The torsion of ∇^* is given by $\pi^*T^\nabla(v, w) \oplus \mathscr{R}^\xi(v, w)$, $\forall v, w \in TTM$, where the vertical part is $\mathscr{R}^\xi(v, w) = R^{\pi^\star\nabla}(v, w)\xi = \pi^\star R^\nabla(v, w)\xi$.

Now we come forward with the metric tensor of M. The Sasaki metric $\langle\ ,\ \rangle$ on TM is given naturally by the pull-back of the metric on M both to H and V, cf. [18]. The parallel *mirror* morphism $B_| : H \to V$ is then metric-preserving. Now B^{ad} really denotes the adjoint endomorphism of B and the map $J = B - B^{\mathrm{ad}}$ is the Sasaki almost complex structure on TM.

Any frame in H extended with its mirror in V clearly determines an orientation on the manifold TM. We convention to adopt the order 'first H, then V', which is a relevant issue when dim M is odd.

Let us suppose ∇ is the Levi-Civita connection and consider the radius 1 tangent sphere bundle

$$SM = \{u \in TM :\ \|u\| = 1\}. \tag{2}$$

∇^* is a metric connection and so, differentiating $\langle\xi, \xi\rangle = 1$, we deduce $TSM = \xi^\perp$. Since the manifold TM is orientable, SM is also always orientable—the restriction of ξ being a unit *outward normal*. By the Gram-Schmidt process and the orthogonal group action, for any $u \in SM$ we may find a local horizontal orthonormal frame $e_0, e_1, \ldots, e_n$ on a neighbourhood of u in SM and such that $e_0 = B^{\mathrm{ad}}\xi$ or, equivalently, $e_0 = u \in H$.

With the dual horizontal coframing, clearly the identity $\pi^*\mathrm{vol}_M = e^0 \wedge e^1 \wedge \cdots \wedge e^n$ is satisfied. Adding the *mirror* subset $\{\xi^\flat, e^{n+1}, \ldots, e^{2n}\}$, with $e^{n+i} = e^i \circ B^{\mathrm{ad}}$, $\forall i \geq 1$ (equivalently $e^{n+i}(e_j) = e^i(e_{j+n}) = 0$, $e^{n+i}(e_{j+n}) = e^i(e_j) = \delta^i_j$, $\forall i, j$), we find the volume form of TM:

$$\mathrm{Vol}_{TM} = e^0 \wedge e^1 \wedge \cdots \wedge e^n \wedge \xi^\flat \wedge e^{n+1} \wedge \cdots \wedge e^{(2n)} = (-1)^{n+1}\,\xi^\flat \wedge \mathrm{vol} \wedge \alpha. \tag{3}$$

We use $\mathrm{vol} = \pi^*\mathrm{vol}_M$; whereas α denotes the n-form on TM which is defined as the interior product of ξ with the vertical pull-back of the volume form of M. Hence, choosing appropriately $\pm\xi$ as unit normal direction, the canonical orientation of the Riemannian submanifold SM, given by $\pm\xi \lrcorner \mathrm{Vol}_{TM}$, agrees with $\mathrm{vol}\wedge\alpha = e^{01\cdots(2n)}$. A direct orthonormal frame as the one introduced previously is said to be *adapted*.

1.2 Further Metric Properties

The submanifold SM admits a metric linear connection $\nabla^\star$. For any vector fields y, z on SM, the covariant derivative $\nabla^*_y z$ is well-defined and, admitting y, z perpendicular to ξ, we just have to add a correction term:

$$\nabla^{\star}{}_y z = \nabla^*_y z - \langle \nabla^*_y z, \xi\rangle\xi \;=\; \nabla^*_y z + \langle y^v, z^v\rangle\xi. \tag{4}$$

Since $\langle \mathscr{R}^\xi(y,z), \xi\rangle = 0$, then a torsion-free connection D is easy to find as $D_y z = \nabla^{\star}{}_y z - \frac{1}{2}\mathscr{R}^\xi(y,z)$. This connection is most useful for some computations, but ceases to be metric. For the Levi-Civita connection we must add to D another term, A, given by:

$$\langle A_y z, w\rangle = \frac{1}{2}(\langle \mathscr{R}^\xi(y,w), z\rangle + \langle \mathscr{R}^\xi(z,w), y\rangle). \tag{5}$$

Details on metric connections on SM are described in [2, 3].

We have found in [6] the conditions for natural maps to become isometries between tangent sphere bundles of different radius, including weighted Sasaki metric and conformal variation of the metric on the base manifold M when $\dim M \geq 3$. Notice the induced horizontal subspaces on SM are not fixed on the same conformal class on M. We do not explore here these results with the weights and radius, which are all aloud to be pullbacks of functions on M.

Just with the Sasaki metric we have a particular, new result which may catch the readers' attention to those theorems. Consider the constant norm $s > 0$ sphere bundle $S_s M = sSM$ and let $M = M^\pm_R$ denote the space-form with metric g of constant sectional curvature $\pm 1/R^2$, where $R > 0$.

Proposition 1. *Let g^S denote the Sasaki metric on the tangent bundle induced from the metric g on $M^\pm_R$. Then $(S_s M^\pm_R, g^S)$ is isometric to $(S_s M^\pm_1, (R^2 g)^S)$.*

Proof. We use the map F defined in [6, Section 2.6] and then apply twice corollary 2.2 from the same article, so the notation now is also from there:

$$(S_s M^\pm_R, g^S) \simeq (S_{\frac{s}{R}} M^\pm_1, g^{R^2,R^2}) \simeq (S_1 M^\pm_1, (R^2 g)^{1,s^2}) \simeq (S_s M^\pm_1, (R^2 g)^S)\ .$$

We recall the notation, $g^{f_1,f_2} = f_1\pi^* g \oplus f_2\pi^\star g,\ \ g^S = g^{1,1}$. ☐

We have also computed in [2] the scalar curvature of the metrics above. For the weighted metric with f_1, f_2 constant, we have

$$\mathrm{Scal}_{(S_sM_R, g^{f_1,f_2})} = \pm\frac{n(n+1)}{f_1R^2} - \frac{f_2}{4f_1^2}\frac{s^2}{R^4}2n + \frac{(n-1)n}{f_2s^2}, \tag{6}$$

which is a positive (negative) constant for small (large) s, although we do not have an Einstein metric. The value of these results from [2, 6] has only recently been understood. Of course it is fun to verify the isometric invariance of our formulas.

1.3 The Contact Structure

We denote by θ the 1-form on SM defined by

$$\theta = (B^{\mathrm{ad}}\xi)^\flat = \langle \xi, B\,\cdot\rangle = e^0. \tag{7}$$

Tashiro discovered in the 1960s that θ defines a metric contact structure, cf. [10]. In our adapted frame we find $\mathrm{d}\theta = e^{(1+n)1} + \cdots + e^{(2n)n}$. In other words, $\forall v, w \in TSM$, $\mathrm{d}\theta(v,w) = \langle v, Bw\rangle - \langle w, Bv\rangle$.

Now we present the set of natural n-forms $\alpha_0, \alpha_1, \ldots, \alpha_n$ existing always on SM. Together with θ they consist of the fundamental differential system we have announced. But we begin with the low dimension cases before a general definition.

In case $n = 1$ we have a global coframing of SM with θ and two 1-forms $\alpha_0 = e^2$ and $\alpha_1 = e^1$, which are global forms. The following formulas were probably already known (to Cartan?), where k denotes the Gauss curvature of M:

$$\mathrm{d}\theta = \alpha_0 \wedge \alpha_1 \qquad \mathrm{d}\alpha_0 = k\,\alpha_1 \wedge \theta \qquad \mathrm{d}\alpha_1 = \theta \wedge \alpha_0. \tag{8}$$

For the case $n = 2$, $\alpha_0 = e^{34}$, $\alpha_1 = e^{14} + e^{32}$, $\alpha_2 = e^{12}$, or the case $n = 3$, $\alpha_0 = e^{456}$, $\alpha_1 = e^{156} + e^{264} + e^{345}$, $\alpha_2 = e^{126} + e^{234} + e^{315}$, $\alpha_3 = e^{123}$, we do not have any special example or easier way of computing the exterior derivatives other than that which we use in [7] with the connections ∇^*, D above—except in case $n = 3$ and flat metric coordinates, as shown in [1], because the 3-sphere is parallelizable and so we may explicit an adapted frame (just as with $n = 1$).

Finally we define the $n + 1$ natural n-forms on SM. First, for $0 \le i \le n$, let

$$n_i = \frac{1}{i!(n-i)!}. \tag{9}$$

Continuing with the adapted frame introduced earlier, we then define:

$$\alpha_0 = \alpha = \xi\lrcorner(\pi^\star\mathrm{vol}_M) = e^{(n+1)} \wedge \cdots \wedge e^{(2n)} \tag{10}$$

where $\pi^\star \mathrm{vol}_{_M}$ is the vertical pull-back of the volume form of M. Now for each i we write, $\forall v_1,\ldots,v_n \in TSM$,

$$\alpha_i(v_1,\ldots,v_n) = n_i \sum_{\sigma\in S_n} \mathrm{sg}(\sigma)\,\alpha(Bv_{\sigma_1},\ldots,Bv_{\sigma_i},v_{\sigma_{i+1}},\ldots,v_{\sigma_n}). \tag{11}$$

We remark that $\alpha_n = e^{1\ldots n}$, which justifies the introduction of the weight n_i. For convenience we define $\alpha_{-1} = \alpha_{n+1} = 0$.

Only a scarce number of references have used the exterior differential system of θ and the α_i, yet not rising them to the level of a field of study. It seems the n-forms have only been considered as an auxiliary tool in the solution of very few mechanical systems problems. First for 2 or 3 dimensional base space in Griffiths' book [13]. Then in [11, p. 152] with emphasis on a three-dimensional metric and an algebraic problem. The same being true regarding later articles in [14], as well as in [15].

Regarding the n-dimensional case, we suppose to be correct in saying it appears for the first time, though only for the Euclidean base space, in [12, p. 32]. To the best of our knowledge, the definition in full generality (11) is introduced first by the author in [7].

Our differential system is original for we do not have any other reference for the following formulas deduced in [7]. On a manifold with constant sectional curvature k we have

$$\begin{aligned}
\mathrm{d}\alpha_0 &= \theta\wedge(\qquad\qquad -k\,\alpha_1)\\
\mathrm{d}\alpha_1 &= \theta\wedge(\qquad n\,\alpha_0 - 2k\,\alpha_2)\\
\mathrm{d}\alpha_2 &= \theta\wedge((n-1)\,\alpha_1 - 3k\,\alpha_3)\\
&\;\;\vdots\\
\mathrm{d}\alpha_{n-1} &= \theta\wedge(2\,\alpha_{n-2} - nk\,\alpha_n)\\
\mathrm{d}\alpha_n &= \theta\wedge\alpha_{n-1}
\end{aligned} \tag{12}$$

or simply $\mathrm{d}\alpha_i = \theta\wedge((n-i+1)\,\alpha_{i-1} - k(i+1)\,\alpha_{i+1}),\ \forall i = 0,\ldots,n$. The particular case of formula (12) with sectional curvature $k = 0$ is already known, as we referred.

1.4 Some Structural Relations

The proofs of the following are quite easy, cf. [7]. For any $0 \le i \le n$ we have:

$$\begin{aligned}
*\,(\mathrm{d}\theta)^i &= (-1)^{\frac{n(n+1)}{2}}\,\frac{i!}{(n-i)!}\,\theta\wedge(\mathrm{d}\theta)^{n-i}\\
*\alpha_i &= (-1)^i\,\theta\wedge\alpha_{n-i}.
\end{aligned} \tag{13}$$

Also $\alpha_i \wedge d\theta = 0$ and $\alpha_i \wedge \alpha_j = 0,\ \forall j \neq n-i$. Of course $*$ denotes the Hodge star-operator on SM, which satisfies $** = 1$ on Λ^*_{SM}. In our notation,

$$R_{ijkl} = \langle \nabla_{e_i}\nabla_{e_j}e_k - \nabla_{e_j}\nabla_{e_i}e_k - \nabla_{[e_i,e_j]}e_k, e_l\rangle\ . \tag{14}$$

Theorem 1 (1st-order structure equations, [7]). *We have*

$$d\alpha_i = (n-i+1)\,\theta\wedge\alpha_{i-1} + \mathscr{R}^\xi\alpha_i \tag{15}$$

where

$$\mathscr{R}^\xi\alpha_i = \sum_{0\leq j<q\leq n}\sum_{p=1}^{n} R_{jq0p}\, e^{jq}\wedge e_{p+n}\lrcorner\alpha_i\,. \tag{16}$$

This theorem is proved with the tools of connection theory introduced in the first section. We do not have any other method which could ease the computations.

Defining $r = \mathrm{Ric}\,(\xi,\xi) = \sum_{j=1}^n R_{j00j}$ as a smooth function on SM determined by the Ricci curvature of M, we have after computations [7]

$$d\alpha_n = \theta\wedge\alpha_{n-1} \qquad\qquad d\alpha_{n-1} = 2\theta\wedge\alpha_{n-2} - r\,\mathrm{vol}, \tag{17}$$

i.e. $\mathscr{R}^\xi\alpha_n = 0$ and $\mathscr{R}^\xi\alpha_{n-1} = -r\theta\wedge\alpha_n$. Then clearly

$$d(\mathscr{R}^\xi\alpha_i) = (n-i+1)\theta\wedge\mathscr{R}^\xi\alpha_{i-1} \qquad\qquad d\theta\wedge\mathscr{R}^\xi\alpha_i = 0. \tag{18}$$

Proposition 2. *The differential forms θ, α_0 and α_1 are always coclosed. Moreover, for all* $0\leq i\leq n$*,*

$$d(i*\alpha_i + (-1)^i\mathscr{R}^\xi\alpha_{n-i+1}) = 0. \tag{19}$$

Proof. One just applies (13) and (18):

$$d i*\alpha_i = i(-1)^{i+1}\theta\wedge d\alpha_{n-i} = i(-1)^{i+1}\theta\wedge\mathscr{R}^\xi\alpha_{n-i} = (-1)^{i+1}d(\mathscr{R}^\xi\alpha_{n-i+1})$$

Clearly, $\mathscr{R}^\xi\alpha_{n+1} = 0$ and it is true $d*\alpha_n = d\mathrm{vol} = 0$. □

No further assumptions on M are required, so we believe there are good reasons to refer to the d-closed differential ideal $\mathscr{I} = \mathrm{span}\{\theta,\alpha_0,\ldots,\alpha_n\}$ as a *fundamental* object of any oriented Riemannian $n+1$-manifold.

It is quite interesting to consider the case of constant sectional curvature k in any dimension. The Riemann curvature tensor is $R_{ijpq} = k(\delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jq})$, so one may prove that $\mathscr{R}^\xi\alpha_i = -k(i+1)\,\theta\wedge\alpha_{i+1}$, cf. (12).

1.5 Gwistor Space and Problems for Calibrated Geometries

The author's discovery of the exterior differential system $\mathscr{I}$ came after and with that of a natural G_2 structure on SM for M of dimension 4.

In [8, 9] it is proved that the total space of the radius 1 tangent sphere bundle $SM \longrightarrow M$ of any given oriented Riemannian 4-manifold M carries a natural G_2-structure. The space is now called G_2-*twistor* or *gwistor space*. Its fundamental structure 3-form is

$$\phi_1 = \theta \wedge \mathrm{d}\theta + \alpha_2 - \alpha_0 \,. \tag{20}$$

Gwistor space is being studied as a subject of its own importance. It has had several developments in [1, 4, 5, 8, 9] in relation with G_2 geometry. We know that ϕ_1 is never closed and it is coclosed if and only if M is Einstein. The G_2 structure $\phi_2 = \theta \wedge \mathrm{d}\theta + \alpha_3 - \alpha_1$ is more restrictive. There is a *circle* of G_2 structures on SM within ϕ_1 and ϕ_2 compatible with the Sasaki metric.

An important open problem in linear algebra is to find the conditions for which a linear combination $\varphi = \sum_{i=0}^{n} b_i\alpha_i + c\theta^{\varepsilon} \wedge (\mathrm{d}\theta)^{[\frac{n}{2}]}$, with $b_i, c \in C^{\infty}_{SM}$, $\varepsilon = 0, 1$, becomes a calibration. Recall a *calibration* is a closed p-form φ such that $\varphi_{|V} \leq \mathrm{vol}_V$ for every oriented tangent p-plane V, cf. [16]. One expects all b_i, c to be constant, yet we are unable to eliminate other possibility.

For even $n = \dim M - 1$ we have an obvious φ of degree n. For $n = 1$ the question may be solved easily recurring to (8). For $n = 2$ and 3 we have a complete linear algebra classification of the calibrations in [16, Theorems 4.3.2 and 4.3.4]. In case $n = 3$, we recover gwistor space.

The following result is quite interesting. Let $\rho = \xi \lrcorner \pi^{\star}\mathrm{Ric}$, the vertical lift of the Ricci tensor. With an adapted frame, we deduce

$$\rho = \sum_{a,b=1}^{n} R_{ab0a}\, e^{b+n}. \tag{21}$$

We have the following theorem giving a reduction of the degree of a differential equation.

Theorem 2 ([7]). *In any dimension we have* $\mathrm{d} * \alpha_2 = \rho \wedge \mathrm{vol}$. *Henceforth, the metric on* M *is Einstein if and only if* $\delta\alpha_2 = 0$.

2 Geometric Applications

2.1 Recalling Euler-Lagrange Systems

We wish to study the Euler-Lagrange system (SM, θ, φ), where φ is a calibration, in applications to Riemannian geometry. We start further above recalling the theory of contact systems from [12]. In this section we assume (S, θ) is any given contact manifold, not necessarily metric, of dimension $2n + 1$.

The *contact differential ideal* $\mathscr{I}$ is defined as the d-closed ideal generated by $\theta \in \Omega^1_S$. A generalisation of the famous Darboux Theorem assures that locally S is the 1-jet manifold $J^1(\mathbb{R}^n)$ of Euclidean flat space, with (Pfaff) coordinates (z, x^i, p_i) and contact form $\theta = \mathrm{d}z - \sum_{i=1}^n p_i\,\mathrm{d}x^i$. The submanifold N given by $z = 0, p_i = 0$ satisfies $\theta_{|N} = 0$. That is also the case with any submanifold $\{(z(x), x^i, \partial_i z)\}$ where z is a C^1 function on the x^i.

An *integral* submanifold of S consists of a submanifold N together with an immersion $f : N \to S$ such that $f^*\theta = 0$. Then of course $f^*\mathscr{I} = 0$. A *Legendre submanifold* is a C^1-differentiable integral n-dimensional submanifold N. The Legendre submanifolds which appear as the graph of a function on N in the Pfaff coordinates are called *transverse*. Equivalently, N is transverse if and only if $f^*(\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n) \neq 0$.

Any form $\Lambda \in \Omega^n_S$ is called a *Lagrangian*. An equivalence relation is immediately associated with equivalence class $\Lambda + \mathscr{I}^n + \mathrm{d}\Omega^{n-1}$, where Λ is a representative and $\mathscr{I}^n = \mathscr{I} \cap \Omega^n$.

An algebraic identity relation deduced in [12] carries over to the whole contact manifold as:

$$\mathscr{I}^k = \Omega^k, \qquad \forall\, k > n. \tag{22}$$

Hence there exist two forms α, β on S such that $\mathrm{d}\Lambda = \theta \wedge \alpha + \mathrm{d}\theta \wedge \beta = \theta \wedge (\alpha + \mathrm{d}\beta) + \mathrm{d}(\theta \wedge \beta)$. By [12, Theorem 1.1] there exists a unique global exact form Π such that $\Pi \wedge \theta = 0$ and $\Pi \equiv \mathrm{d}\Lambda$ in $\bar{H}^{n+1}(\Omega^*/\mathscr{I}, \mathrm{d})$. The *Poincaré-Cartan* form is $\Pi = \mathrm{d}(\Lambda - \theta \wedge \beta) = \theta \wedge (\alpha + \mathrm{d}\beta)$. The form $\Psi = \alpha + \mathrm{d}\beta$ turns out to be of great importance.

Now one wishes to find the critical points of a functional on the set of smooth, compact Legendre submanifolds $N \hookrightarrow S$, possibly with boundary, defined by:

$$\mathscr{F}_\Lambda(N) = \int_N f^*\Lambda. \tag{23}$$

Note that Λ clearly induces the same functional on its class for Legendre submanifolds without boundary.

Suppose we have a variation of Legendre submanifolds *with fixed boundary*, i.e. suppose there is a curve of smooth maps $f_t : N \to S$ which defines a Legendre submanifold N_t for each t and $\partial(N_t) = \partial(N_0)$. Differentiating $\mathscr{F}_\Lambda(N_t)$, cf. [12], leads to the conclusion that

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\mathscr{F}_\Lambda(N_t)=0 \qquad \text{if and only if} \qquad f^*\Psi=0. \tag{24}$$

A Legendre submanifold satisfying (24) is called a *stationary* Legendre submanifold. The exterior differential system algebraically generated by $\theta, \mathrm{d}\theta, \Psi$ is called the *Euler-Lagrange system* of (S,θ,Λ); its Poincaré-Cartan form Π is said to be non-degenerate if it has no other degree 1 factors besides the multiples of θ.

In order to determine conditions on the stationary submanifolds of $\mathscr{F}_\Lambda$ one proceeds as follows: find the Poincaré-Cartan n-form, transform it into the product $\theta\wedge\Psi$ and then do the analysis of $f^*\Psi=0$, i.e. study the Euler-Lagrange equation.

2.2 On the Unit Tangent Sphere Bundle

Let us admit again an oriented $n+1$-dimensional Riemannian manifold M together with its unit tangent sphere bundle $SM\stackrel{\pi}{\longrightarrow}M$. Now we let $f:N\to M$ be a compact oriented isometric immersed hypersurface.

Then we have also a smooth lift $\hat{f}:N\to SM$ of f, the unique unit normal $\nu\in T_{f(x)}M$ chosen according to the orientations of N and M. Note that $\hat{f}$ is also defined on ∂N. It is easy to see that we have the decomposition into horizontal plus vertical:

$$\mathrm{d}\hat{f}(w)=(\mathrm{d}f(w))^h+(f^*\nabla)_w f^*\nu\ ,\quad \forall w\in T_xN. \tag{25}$$

Indeed, at each point $x\in N$ the vertical part is $\nabla^*_{\mathrm{d}\hat{f}(w)}\xi=(\hat{f}^*\nabla^*)_w\hat{f}^*\xi$, where ξ is the canonical vertical vector field on SM. Clearly, $(\hat{f}^*\xi)_x=\hat{f}(x)=\nu_{f(x)}=(f^*\nu)_x$ and $\hat{f}^*\pi^*=f^*$. By definition of $\hat{f}$ we clearly have that $\hat{f}:N\longrightarrow SM$ defines a Legendre submanifold of the natural contact structure, $\hat{f}^*\theta=0$, and that it is a transverse submanifold.

A smooth Legendre submanifold Y is locally the lift $N\to Y\subset SM$ of an oriented smooth n-submanifold $N\hookrightarrow M$ if and only if $e^{1\cdots n}{}_{|Y}\neq 0$, i.e. precisely when Y is transverse. We are thus going to assume throughout such *open* condition on submanifolds, defined by the top differential form: $\alpha_{n|Y}\neq 0$.

Let us consider an adapted direct orthonormal coframe $e^0,e^1,\ldots,e^n,e^{n+1},\ldots,e^{2n}$ locally defined on SM. Then it may not be tangent to $N\subset SM$. Yet we have also a direct orthonormal coframe $e^1,\ldots,e^n$ for N (we use the same letters for the pull-back). Now, from (25), for any $1\leq j\leq n$ we have

$$\hat{f}^*e^j=e^j \qquad \text{and} \qquad \hat{f}^*e^{j+n}=-\sum_{k=1}^{n}A^j_ke^k \tag{26}$$

with A the second fundamental form of N. We recall, $A = -\nabla\nu : TN \to TN$ is a symmetric endomorphism; the associated tensor $H = \frac{1}{n}(\mathrm{Tr}\,A)\nu$ is the mean curvature vector field.

We now consider the n-forms α_i, which give in their own right interesting Lagrangian systems on the contact manifold SM. We wish to study the functionals $\mathscr{F}_i = \mathscr{F}_{\alpha_i}$ on the set of compact immersed hypersurfaces of M with fixed boundary.

Let $\sigma_i(A)$ denote the elementary symmetric polynomial of degree i in the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A. Then we have that

$$\hat{f}^*\alpha_{n-i} = (-1)^i \sigma_i(A)\,\mathrm{vol}_N. \tag{27}$$

We do not know a more simple proof for the following result (only for Euclidean base space it is in [12] with the same method), than by using (24) on $\mathscr{F}_n$ and the Poincaré-Cartan form given by $\mathrm{d}\alpha_n = \theta \wedge \alpha_{n-1}$.

Theorem 3 (Classical theorem, [7]). *Let N be a compact isometrically immersed hypersurface in the Riemannian manifold M. Then, $\forall v \in \Gamma_0(N, f^*TM)$,*

$$\delta\mathrm{vol}(N)(v) = -\int_N n\langle v, H\rangle\,\mathrm{vol}_N. \tag{28}$$

In particular, N is minimal for the volume functional within all compact hypersurfaces with fixed boundary ∂N if and only if $H = 0$.

As used previously, one deduces $\hat{f}^*\alpha_{n-1} = -n\langle H, \nu\rangle\,\mathrm{vol}_N = -n\|H\|\,\mathrm{vol}_N$, hence the functional $\mathscr{F}_{n-1}$ corresponds with

$$\mathscr{F}_{n-1}(N) = -n\int_N \|H\|\,\mathrm{vol}_N, \tag{29}$$

i.e. the integral of the mean curvature on immersed submanifolds $N \subset M$.

Theorem 4 ([7]). *Suppose the Riemannian manifold M has dimension $n+1 > 2$. Then a compact isometric immersed hypersurface $f : N \to M$ with fixed boundary is stationary for the mean curvature functional $\mathscr{F}_{n-1}$ if and only if*

$$\mathrm{Scal}^N = \mathrm{Scal}^M - r_\nu \tag{30}$$

where $r_\nu = \mathrm{Ric}\,(\nu,\nu)$ is induced from the Ricci tensor of M and Scal *denotes scalar curvature functions.*

In particular, if M is an Einstein manifold, say where $\mathrm{Ric} = cg$ *with c a constant, then N has stationary mean curvature volume if and only if N has constant scalar curvature* $\mathrm{Scal}^N = nc$.

For an Einstein metric on the ambient manifold M, a formula in the last proof shows that $\mathscr{F}_{n-2}$ leads to an Euler-Lagrange equation essentially on the scalar curvature of N.

Theorem 5 ([7]). *Let M be a Riemannian manifold of dimension $n+1>2$ and constant sectional curvature k. Then a compact hypersurface N is a critical point of the scalar curvature functional $\int_N \mathrm{Scal}^N \mathrm{vol}_N$ with fixed boundary if and only if the eigenvalues $\lambda_1,\ldots,\lambda_n$ of A satisfy (assume $\lambda_3=0$ for $n=2$)*

$$6\sum_{j_1<j_2<j_3}\lambda_{j_1}\lambda_{j_2}\lambda_{j_3}+k(n-1)(n-2)(\lambda_1+\cdots+\lambda_n)=0. \tag{31}$$

In other words, $6\sigma_3(A)+kn(n-1)(n-2)\|H\|=0$.

The case $n=2$ is always satisfied and invariant of the ambient manifold—that is partly the theorem of Gauss-Bonnet.

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Lagrangian Intersection Theory and Hamiltonian Volume Minimizing Problem

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Abstract In this article, we first describe antipodal sets and the structure of intersections of two real forms in complex flag manifolds. In particular, in the complex flag manifold consisting of sequences of complex subspaces in a complex vector space we investigate the real form consisting of sequences of quaternionic subspaces. Moreover, we discuss applications to the Hamiltonian volume minimizing problem.

1 Introduction

The intersection of two real forms in a Hermitian symmetric space of compact type is an antipodal set by the results of Tasaki [11] and Tanaka and Tasaki [8–10]. An orbit of the adjoint action of a connected compact semisimple Lie group is called a complex flag manifold, which admits an invariant Kähler structure. Furthermore, any simply-connected compact homogeneous Kähler manifold is a complex flag manifold. Therefore a complex flag manifold is a generalization of a Hermitian symmetric space of compact type. An antipodal set in a compact Riemannian symmetric space was introduced by Chen and Nagano [2]. Sánchez [7] extended this notion in complex flag manifolds and Berndt et al. [1] investigated some properties of antipodal sets of complex flag manifolds and their cardinalities.

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In Sect. 2, we briefly review the definition of k-symmetric structures on a complex flag manifold for $k \geq k_0$, where k_0 is an integer greater than 1 which is determined dependently on the complex flag manifold and show that the fixed point set of the k-symmetry is independent of k. In Sect. 3, we completely describe, in the complex flag manifold consisting of sequences of complex subspaces in a complex vector space, the intersection of real forms consisting of sequences of quaternionic subspaces. In Sect. 4, we discuss applications to the Hamiltonian volume minimizing problem.

2 Antipodal Sets of Complex Flag Manifolds

In this section we recall antipodal sets of complex flag manifolds (see [4] for details).

Let G be a connected compact semisimple Lie group and $\mathfrak{g}$ be its Lie algebra. We take a nonzero element H in $\mathfrak{g}$ and consider its orbit $\mathrm{Ad}(G)H \subset \mathfrak{g}$ under the adjoint action of G. We denote the stabilizer at H of G by G_H, and its Lie algebra by $\mathfrak{g}_H$. Then the orbit $\mathrm{Ad}(G)H$ is diffeomorphic to G/G_H. It is known that $\mathrm{Ad}(G)H$ has an invariant Kähler structure under the action of G. $\mathrm{Ad}(G)H$ is called a *(generalized) complex flag manifold*.

We take a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ containing H. For α in the dual space $\mathfrak{t}^*$ of $\mathfrak{t}$ we define $\mathfrak{g}_\alpha$ by

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}^{\mathbb{C}} \mid [T, X] = \sqrt{-1}\alpha(T)X \text{ for } T \in \mathfrak{t}\},$$

where $\mathfrak{g}^{\mathbb{C}}$ denotes the complexification of $\mathfrak{g}$, and the *root system* Δ of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{t}$ by $\Delta = \{\alpha \in \mathfrak{t}^* - \{0\} \mid \mathfrak{g}_\alpha \neq \{0\}\}$. Then we have the *root space decomposition* of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{t}$:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

The Lie subalgebra $\mathfrak{g}_H$ is given by

$$\mathfrak{g}_H = \mathfrak{t} + \mathfrak{g} \cap \sum_{\substack{\alpha \in \Delta \\ \alpha(H)=0}} \mathfrak{g}_\alpha.$$

We can decompose $\mathfrak{g}$ to a direct sum of simple ideals, which implies a disjoint union $\Delta = \Delta_1 \cup \cdots \cup \Delta_r$, where each Δ_i is an irreducible root system. For each i $(1 \leq i \leq r)$, we can take fundamental roots $\alpha_{i,j}$ $(1 \leq j \leq p_i)$ which satisfy $\alpha_{i,j}(H) \geq 0$. Their union

$$\{\alpha_{i,j} \mid 1 \leq i \leq r,\ 1 \leq j \leq p_i\}$$

is a fundamental root system of Δ. By reordering $\alpha_{i,j}$

$$\{\alpha_{i,j} \mid 1 \leq i \leq r,\ q_i + 1 \leq j \leq p_i\}$$

is a fundamental root system of $\{\alpha \in \Delta \mid \alpha(H) = 0\}$. Let δ_i is the highest root of Δ_i with respect to $\{\alpha_{i,j}\}$. We can describe δ_i by

$$\delta_i = \sum_{j=1}^{p_i} m_{i,j}\alpha_{i,j},$$

where each $m_{i,j}$ is a positive integer. We define

$$k_0 = \max_{1\leq i\leq r} \left\{ 1 + \sum_{j=1}^{q_i} m_{i,j} \right\}.$$

Since

$$\{\alpha_{i,j} \mid 1 \leq i \leq r,\ 1 \leq j \leq p_i\}$$

is a basis of $\mathfrak{t}^*$, we can take its dual basis

$$\{H_{i,j} \mid 1 \leq i \leq r,\ 1 \leq j \leq p_i\}$$

of $\mathfrak{t}$. We define

$$Z = \sum_{i=1}^{r} \sum_{j=1}^{q_i} H_{i,j} \in \mathfrak{t}.$$

By the definitions of $H_{i,j}$ and Z we have $[Z, \mathfrak{g}_H] = \{0\}$. For any integer $k \geq k_0$ we define

$$g_k = \exp \frac{2\pi}{k} Z \in \exp \mathfrak{t} \subset G_H.$$

We define a diffeomorphism θ_k of $\mathrm{Ad}(G)H$ by

$$\theta_k(x) = \mathrm{Ad}(g_k)x \quad (x \in \mathrm{Ad}(G)H).$$

It follows from the definition of g_k that $(\theta_k)^k = 1$. Hence θ_k defines a structure of generalized symmetric space on $\mathrm{Ad}(G)H$ and θ_k is the k-symmetry at H. We note that $\mathrm{Ad}(G)H$ can be a Hermitian symmetric space if and only if $k_0 = 2$.

We can verify that the eigenspace of $\mathrm{Ad}(g_k)$ corresponding to 1 is equal to $\mathfrak{g}_H$. Thus the fixed point set $F(\theta_k, \mathrm{Ad}(G)H)$ of θ_k is equal to $\mathrm{Ad}(G)H \cap \mathfrak{g}_H$, which is independent of the choice of $k \geq k_0$. For $x \in \mathrm{Ad}(G)H$ we have that $\theta_k(x) = x$ if and only if $[H, x] = 0$. We denote by s_H the symmetry θ_k at H and by s_x the symmetry of $\mathrm{Ad}(G)H$ at x. Then we have the following theorem.

Theorem 1 ([4]). *The fixed point set $F(s_x, \mathrm{Ad}(G)H)$ of s_x is independent of the choice of $k \geq k_0$ and is equal to*

$$\{y \in \mathrm{Ad}(G)H \mid [x, y] = 0\}$$

for any x in $\mathrm{Ad}(G)H$. Any maximal antipodal set of $\mathrm{Ad}(G)H$ is equal to $\mathrm{Ad}(G)H \cap \mathfrak{t}'$ for a maximal abelian subalgebra $\mathfrak{t}'$ of $\mathfrak{g}$, which is an orbit of the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{t}'$, and maximal antipodal sets of $\mathrm{Ad}(G)H$ are conjugate to each other under the adjoint action of G.

Let V be an n-dimensional vector space over $\mathbb{K}(= \mathbb{R}, \mathbb{C}$ or $\mathbb{H})$. For positive integers $n_1, \ldots, n_r$ which satisfy $n_1 + \cdots + n_r < n$ we define

$$F^{\mathbb{K}}_{n_1,\ldots,n_r}(V) = \left\{ (V_1, \ldots, V_r) \;\middle|\; \begin{array}{l} V_j \text{ is a } \mathbb{K}\text{-subspace of } V, \\ \dim V_j = n_1 + \cdots + n_j, \\ V_1 \subset V_2 \subset \cdots \subset V_r \subset V \end{array} \right\},$$

which was originally called a flag manifold. Let $v_1, \ldots, v_n$ be a unitary basis of $\mathbb{C}^n$. The set

$$\begin{aligned}
\{(\langle v_{i_1}, \ldots, v_{i_{n_1}} \rangle_{\mathbb{C}}, \langle v_{i_1}, \ldots, v_{i_{n_1+n_2}} \rangle_{\mathbb{C}}, \ldots, \langle v_{i_1}, \ldots, v_{i_{n_1+\cdots+n_r}} \rangle_{\mathbb{C}}) \\
\mid 1 \leq i_1 < \cdots < i_{n_1} \leq n,\ 1 \leq i_{n_1+1} < \cdots < i_{n_1+n_2} \leq n, \ldots, \\
1 \leq i_{n_1+\cdots+n_{r-1}+1} < \cdots < i_{n_1+\cdots+n_r} \leq n, \\
\#\{i_1, \ldots, i_{n_1+\cdots+n_r}\} = n_1 + \cdots + n_r\},
\end{aligned}$$

is a maximal antipodal set of $F^{\mathbb{C}}_{n_1,\ldots,n_r}(\mathbb{C}^n)$ (see [4]). Its cardinality is equal to

$$\frac{n!}{n_1! n_2! \cdots n_{r+1}!} = \binom{n}{n_1, n_2, \ldots, n_{r+1}}.$$

3 The Intersection of Real Flag Manifolds in a Complex Flag Manifold

In [4] we completely described the intersection of two real flag manifolds which are congruent to $F^{\mathbb{R}}_{n_1,\ldots,n_r}(\mathbb{R}^n)$ in the complex flag manifold $F^{\mathbb{C}}_{n_1,\ldots,n_r}(\mathbb{C}^n)$, and gave a necessary and sufficient condition that the intersection is discrete. And we showed that if the intersection is discrete it is a maximal antipodal set of $F^{\mathbb{C}}_{n_1,\dots,n_r}(\mathbb{C}^n)$. In this section we investigate the intersection of two real forms which are congruent to $F^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n)$ in the complex flag manifold $F^{\mathbb{C}}_{2n_1,\dots,2n_r}(\mathbb{C}^{2n})$.

In $\mathbb{C}^{2n}$ we define i, j, k by

$$iv = \sqrt{-1}v, \quad jv = J\bar{v}, \quad kv = ijv \quad (v \in \mathbb{C}^{2n}),$$

where

$$J = \begin{bmatrix} O & 1_n \\ -1_n & O \end{bmatrix}.$$

Then $\mathbb{C}^{2n}$ can be identified with the quaternionic vector space $\mathbb{H}^n$. The real linear map $j : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ induces an antiholomorphic involution of $F^{\mathbb{C}}_{2n_1,\dots,2n_r}(\mathbb{C}^{2n})$, and its fixed point set is $F^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n)$. Hence $F^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n)$ is a real form of $F^{\mathbb{C}}_{2n_1,\dots,2n_r}(\mathbb{C}^{2n})$.

According to the conjugacy of maximal tori of the compact symmetric pair $(U(2n), Sp(n))$ we have

$$U(2n) = Sp(n)\, A\, Sp(n),$$

where

$$A = \left\{ \begin{bmatrix} X & O \\ O & X \end{bmatrix} \;\middle|\; X = \begin{bmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{bmatrix}, z_1, \dots, z_n \in U(1) \right\}.$$

For $u \in U(2n)$ we can express $u = k_1 a k_2$ where $k_1, k_2 \in Sp(n)$ and $a \in A$. Then we have

$$F^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n) \cap uF^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n) = k_1(F^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n) \cap aF^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n)).$$

Therefore it suffices to study $F^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n) \cap aF^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n)$.

For $a \in A$ we have the eigenspace decomposition

$$\mathbb{C}^{2n} = W_1 \oplus \cdots \oplus W_s \tag{1}$$

of a^2. We note that each eigenspace W_i is a quaternionic subspace of $\mathbb{H}^n \cong \mathbb{C}^{2n}$. For positive integers $k, k_1, \dots, k_s$ satisfying $k = k_1 + \cdots + k_s$, we denote

$$F^{\mathbb{H}}_{k_1}(W_1) \times \cdots \times F^{\mathbb{H}}_{k_s}(W_s) = \{x_1 \oplus \cdots \oplus x_s \in F^{\mathbb{H}}_k(\mathbb{H}^n) \mid x_i \in F^{\mathbb{H}}_{k_i}(W_i)\ (1 \le i \le s)\}.$$

We also regard this as a subset of $F_{2k}^{\mathbb{C}}(\mathbb{C}^{2n})$. The following theorem is a refinement of Theorem 6.3 in Tanaka and Tasaki [8]. The actions of $U(2n)$ on $F_{2k}^{\mathbb{C}}(\mathbb{C}^{2n})$ and $F_{2n_1,\dots,2n_r}^{\mathbb{C}}(\mathbb{C}^{2n})$ coincide with those of $SU(2n)$, so we use $U(2n)$ instead of $SU(2n)$.

Theorem 2. *For* $0 < k \leq n$ *we have*

$$F_k^{\mathbb{H}}(\mathbb{H}^n) \cap aF_k^{\mathbb{H}}(\mathbb{H}^n) = \bigcup_{\substack{k_1+\cdots+k_s=k \\ 0\leq k_i \leq \dim_{\mathbb{H}}(W_i)(1\leq i\leq s)}} F_{k_1}^{\mathbb{H}}(W_1) \times \cdots \times F_{k_s}^{\mathbb{H}}(W_s)$$

in $F_{2k}^{\mathbb{C}}(\mathbb{C}^{2n})$. *The intersection of* $F_k^{\mathbb{H}}(\mathbb{H}^n)$ *and* $aF_k^{\mathbb{H}}(\mathbb{H}^n)$ *is discrete if and only if* $\dim_{\mathbb{H}} W_i = 1$ *for all* i. *In this case these intersect transversally and*

$$F_k^{\mathbb{H}}(\mathbb{H}^n) \cap aF_k^{\mathbb{H}}(\mathbb{H}^n) = \{W_{i_1} \oplus \cdots \oplus W_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\},$$

which is an antipodal set of $F_{2k}^{\mathbb{C}}(\mathbb{C}^{2n})$, *and a maximal antipodal set of* $F_k^{\mathbb{H}}(\mathbb{H}^n)$.

Proof.

$$F_k^{\mathbb{H}}(\mathbb{H}^n) \cap aF_k^{\mathbb{H}}(\mathbb{H}^n) = \{V \in F_{2k}^{\mathbb{C}}(\mathbb{C}^{2n}) \mid V \in F_k^{\mathbb{H}}(\mathbb{H}^n), a^{-1}V \in F_k^{\mathbb{H}}(\mathbb{H}^n)\}$$

The first condition is equivalent to $J\overline{V} = V$. From this the second condition is

$$J(\overline{a^{-1}V}) = a^{-1}V \iff aJa\overline{V} = V \iff a^2J\overline{V} = V \iff a^2V = V.$$

Therefore

$$F_k^{\mathbb{H}}(\mathbb{H}^n) \cap aF_k^{\mathbb{H}}(\mathbb{H}^n) = \{V \in F_{2k}^{\mathbb{C}}(\mathbb{C}^{2n}) \mid J\overline{V} = V, a^2V = V\}.$$

To describe this intersection, we use the following lemma.

Lemma 1. *Let* U *be a finite dimensional complex vector space and* $f : U \to U$ *a complex linear map which has eigenspace decomposition* $U = U_1 \oplus \cdots \oplus U_s$. *Then a complex vector subspace* $V \subset U$ *satisfies* $f(V) = V$ *if and only if*

$$V = V \cap U_1 \oplus \cdots \oplus V \cap U_s.$$

This follows from elementary linear algebra. We apply this lemma to a^2 and its eigenspace decomposition $\mathbb{C}^{2n} = W_1 \oplus \cdots \oplus W_s$. Then we have

$$F_k^{\mathbb{H}}(\mathbb{H}^n) \cap aF_k^{\mathbb{H}}(\mathbb{H}^n) = \{V \in F_k^{\mathbb{H}}(\mathbb{H}^n) \mid V = V \cap W_1 \oplus \cdots \oplus V \cap W_s\}.$$

Here $V \cap W_i$ is a quaternionic vector subspace of W_i for each i. Consequently

$$F_k^{\mathbb{H}}(\mathbb{H}^n) \cap aF_k^{\mathbb{H}}(\mathbb{H}^n) = \bigcup_{\substack{k_1+\cdots+k_s=k \\ 0\leq k_i \leq \dim_{\mathbb{H}}(W_i)(1\leq i\leq s)}} F_{k_1}^{\mathbb{H}}(W_1) \times \cdots \times F_{k_s}^{\mathbb{H}}(W_s).$$

The latter half of the statement of the theorem follows from this description immediately.

Theorem 3. *For $n_1, \dots, n_r$ which satisfy $n_1 + \cdots + n_r < n$ we have*

$$\begin{aligned} &F^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n) \cap aF^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n) \\ &= \{(V_1, \dots, V_r) \in F^{\mathbb{C}}_{2n_1,\dots,2n_r}(\mathbb{C}^{2n}) \mid \\ &\qquad V_j \in F^{\mathbb{H}}_{n_1+\cdots+n_j}(\mathbb{H}^n) \cap aF^{\mathbb{H}}_{n_1+\cdots+n_j}(\mathbb{H}^n)\ (1 \le j \le r)\} \end{aligned}$$

in $F^{\mathbb{C}}_{2n_1,\dots,2n_r}(\mathbb{C}^{2n})$. The intersection of $F^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n)$ and $aF^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n)$ is discrete if and only if $\dim_{\mathbb{H}} W_i = 1$ for all i. In this case these intersect transversally and

$$\begin{aligned} &F^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n) \cap aF^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n) \\ &= \{(W_{i_1} \oplus \cdots \oplus W_{i_{n_1}}, W_{i_1} \oplus \cdots \oplus W_{i_{n_1+n_2}}, \dots, W_{i_1} \oplus \cdots \oplus W_{i_{n_1+\cdots+n_r}}) \\ &\quad \mid 1 \le i_1 < \cdots < i_{n_1} \le n,\ 1 \le i_{n_1+1} < \cdots < i_{n_1+n_2} \le n, \dots, \\ &\quad\ 1 \le i_{n_1+\cdots+n_{r-1}+1} < \cdots < i_{n_1+\cdots+n_r} \le n, \\ &\quad\ \#\{i_1, \dots, i_{n_1+\cdots+n_r}\} = n_1 + \cdots + n_r\}, \end{aligned}$$

which is an antipodal set of $F^{\mathbb{C}}_{2n_1,\dots,2n_r}(\mathbb{C}^{2n})$.

Proof. Since

$$\begin{aligned} &F^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n) \\ &= \{(V_1, \dots, V_r) \in F^{\mathbb{C}}_{2n_1,\dots,2n_r}(\mathbb{C}^{2n}) \mid V_j \in F^{\mathbb{H}}_{n_1+\cdots+n_j}(\mathbb{H}^n)\ (1 \le j \le r)\}, \\ &aF^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n) \\ &= \{(V_1, \dots, V_r) \in F^{\mathbb{C}}_{2n_1,\dots,2n_r}(\mathbb{C}^{2n}) \mid V_j \in aF^{\mathbb{H}}_{n_1+\cdots+n_j}(\mathbb{H}^n)\ (1 \le j \le r)\}, \end{aligned}$$

we have

$$\begin{aligned} &F^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n) \cap aF^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n) \\ &= \{(V_1, \dots, V_r) \in F^{\mathbb{C}}_{2n_1,\dots,2n_r}(\mathbb{C}^{2n}) \mid \\ &\qquad V_j \in F^{\mathbb{H}}_{n_1+\cdots+n_j}(\mathbb{H}^n) \cap aF^{\mathbb{H}}_{n_1+\cdots+n_j}(\mathbb{H}^n)\ (1 \le j \le r)\}. \end{aligned}$$

If $\dim_{\mathbb{H}} W_i = 1$ for all i, then

$$F^{\mathbb{H}}_k(\mathbb{H}^n) \cap aF^{\mathbb{H}}_k(\mathbb{H}^n) = \{W_{i_1} \oplus \cdots \oplus W_{i_k} \mid 1 \le i_1 < \cdots < i_k \le n\},$$

$$\begin{aligned} &F^{\mathbb{H}}_{n_1+\cdots+n_j}(\mathbb{H}^n) \cap aF^{\mathbb{H}}_{n_1+\cdots+n_j}(\mathbb{H}^n) \\ &= \{W_{i_1} \oplus \cdots \oplus W_{i_{n_1+\cdots+n_j}} \mid 1 \le i_1 < \cdots < i_{n_1+\cdots+n_j} \le n\} \end{aligned}$$

for each j. Hence we have

$$
\begin{aligned}
&F^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n) \cap aF^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n)\\
&= \{(W_{i_1} \oplus \cdots \oplus W_{i_{n_1}}, W_{i_1} \oplus \cdots \oplus W_{i_{n_1+n_2}}, \dots, W_{i_1} \oplus \cdots \oplus W_{i_{n_1+\cdots+n_r}})\\
&\quad \mid 1 \le i_1 < \cdots < i_{n_1} \le n,\ 1 \le i_{n_1+1} < \cdots < i_{n_1+n_2} \le n, \dots,\\
&\quad\ 1 \le i_{n_1+\cdots+n_{r-1}+1} < \cdots < i_{n_1+\cdots+n_r} \le n,\\
&\quad\ \#\{i_1, \dots, i_{n_1+\cdots+n_r}\} = n_1 + \cdots + n_r\},
\end{aligned}
$$

which is discrete.

Finally we shall show that the intersection of $F^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n)$ and $aF^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n)$ is not discrete if $\dim_{\mathbb{H}} W_i \ge 2$ for some i. Without loss of generality we can suppose $\dim_{\mathbb{H}} W_1 \ge 2$. We take a quaternionic unitary basis $v_1, \dots, v_n$ of $\mathbb{C}^{2n} \cong \mathbb{H}^n$ which is compatible with the decomposition (1). We suppose $v_1, v_2 \in W_1$. Hence we have

$$
\begin{aligned}
&\{\langle l, v_3, \dots, v_{n_1+\cdots+n_j+1}\rangle_{\mathbb{H}} \mid 0 \ne l \in \langle v_1, v_2\rangle_{\mathbb{H}}\}\\
&\subset F^{\mathbb{H}}_{n_1+\cdots+n_j}(\mathbb{H}^n) \cap aF^{\mathbb{H}}_{n_1+\cdots+n_j}(\mathbb{H}^n)
\end{aligned}
$$

for each j, and by the description of $F^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n) \cap aF^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n)$

$$
\begin{aligned}
&\{(\langle l, v_3, \dots, v_{n_1+1}\rangle_{\mathbb{H}}, \langle l, v_3, \dots, v_{n_1+n_2+1}\rangle_{\mathbb{H}}, \dots,\\
&\qquad\qquad \langle l, v_3, \dots, v_{n_1+\cdots+n_r+1}\rangle_{\mathbb{H}}) \mid 0 \ne l \in \langle v_1, v_2\rangle_{\mathbb{H}}\}\\
&\subset F^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n) \cap aF^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n),
\end{aligned}
$$

which means that $F^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n) \cap aF^{\mathbb{H}}_{n_1,\dots,n_r}(\mathbb{H}^n)$ is not discrete.

4 Application to the Hamiltonian Volume Minimizing Problem

In 1990, Oh [6] posed a variational problem concerning volumes of compact Lagrangian submanifolds in Kähler manifolds under Hamiltonian deformations. Kleiner and Oh gave the first non-trivial example, namely, they showed that the real form $\mathbb{R}P^n$ in $\mathbb{C}P^n$ has the least volume under Hamiltonian deformations. Using the structure of the intersection $L_1 \cap L_2$ of two real forms L_1, L_2 of a Hermitian symmetric space of compact type we can calculate Lagrangian Floer homology $HF(L_1, L_2)$ and obtain a generalization of the Arnold-Givental inequality. We apply it and a Crofton type formula due to Lê [5] to real forms in the complex hyperquadric $Q_n(\mathbb{C})$.

The complex hyperquadric is defined as

$$Q_n(\mathbb{C}) = \{[z] \in \mathbb{C}P^{n+1} \mid z_0^2 + z_1^2 + \cdots + z_{n+1}^2 = 0\}$$

equipped with the Kähler structure induced by the Fubini-Study Kähler form ω_{FS} on $\mathbb{C}P^{n+1}$. A real form of $Q_n(\mathbb{C})$ is congruent to

$$\begin{aligned} S^{k,n-k} &= \{[x] \in \mathbb{R}P^{n+1} \mid x_0^2 + x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_{n+1}^2 = 0\} \\ &\cong (S^k \times S^{n-k})/\mathbb{Z}_2 \quad (0 \leq k \leq [n/2]). \end{aligned}$$

Theorem 4 ([3]). *In $Q_n(\mathbb{C})$, we have $\mathrm{vol}(\phi S^{k,n-k}) \geq \mathrm{vol}(S^n)$ for any Hamiltonian diffeomorphism $\phi \in \mathrm{Ham}(Q_n(\mathbb{C}))$. In particular, the real form $S^{0,n} \cong S^n$ of $Q_n(\mathbb{C})$ is Hamiltonian volume minimizing.*

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An $SL_2(\mathbb{C})$ Topological Invariant of Knots

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Abstract In this note, we show that the $SL_2(\mathbb{C})$ algebro-geometric invariant defined in Li and Wang (Int. J. Math. 22(9):1209–1230, 2011) for knots is indeed an $SL_2(\mathbb{C})$ topological invariant. The main ingredient is our short geometric proof of the coincidence of the algebro-geometric multiplicity and topological multiplicity of the intersection of curves on a smooth surface.

1 Introduction

It is natural to extend the $SU(2)$ Casson invariant of knots and 3-manifolds to other compact or noncompact groups, after many successful applications (see [1]). One would like to have the $SL_2(\mathbb{C})$ Casson type invariant for hyperbolic knots and hyperbolic 3-manifolds. There are some attempts in this direction, see [5, 6, 9]. The difficulties lie in the fact that the character variety of a knot group is lack of understanding. By using algebro-geometric method, the first author and Q. Wang defined in [10] an $SL_2(\mathbb{C})$ algebro-geometric invariant of hyperbolic knots.

Let us briefly recall the $SL_2(\mathbb{C})$ algebro-geometric invariant $\lambda(p,q)$ for the manifolds obtained by the (p,q)-Dehn surgery along the knot complement. Let X_0 be the irreducible component of the character variety of a hyperbolic knot K in S^3 which contains the character of the discrete faithful representation associated to the hyperbolic structure of K. Roughly speaking, the invariant $\lambda(p,q)$ counts the algebraic intersection multiplicity of X_0 with another affine curve in the character variety of the boundary from the (p,q)-Dehn surgery. It does not require the non-sufficiently large condition in [5] and the number $\lambda(K_{p/q})$ in [5] is defined over all the components of $X(M)$ and the intersection is taken in a different space (see also [10, Remark 3.4]).

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In [7, Section 4], Fulton and MacPherson showed that the algebraic intersection multiplicity agrees with the topological intersection multiplicity using the machinery of deformation to normal cone. For the intersection of two curves in a smooth surface, we give a simple proof of the coincidence of these multiplicities, without using deformation or perturbation. Applying this into the construction in [10, Definition 3.2], we show that the invariant $\lambda(p, q)$ defined by the algebro-geometric method is indeed a topological invariant of the hyperbolic knot and the resulting 3-manifold from Dehn surgery.

The paper is organized as follows. In Sect. 2 we recall the algebro-geometric invariant $\lambda(p, q)$. In Sect. 3 we show that the algebraic intersection multiplicity agrees with the topological intersection multiplicity for the curve case, and our main result Theorem 1 follows.

2 $SL_2(\mathbb{C})$ Algebro-Geometric Invariant of Knots

In this section, we recall the $SL_2(\mathbb{C})$ algebro-geometric invariant of knots defined in [10] and use same notations therein.

Let K be a hyperbolic knot in S^3. Then its complement $M_K = S^3 - N_K$ is a hyperbolic 3-manifold with finite volume, where N_K is the open tubular neighborhood of K in S^3. There is a discrete faithful representation $\rho_0 \in R(M_K) = \mathrm{Hom}(\pi_1(M_K), SL_2(\mathbb{C}))$ corresponding to its hyperbolic structure. Let R_0 be an irreducible component of $R(M_K)$ containing ρ_0. Let $X_0 = t(R_0)$, where $t : R(M_K) \longrightarrow X(M_K)$ (similar for $t : R(\partial M_K) \longrightarrow X(\partial M_K)$) is the canonical surjective morphism which sends a representation to its character. The natural homomorphism $i : \pi_1(\partial M_K) \longrightarrow \pi_1(M_K)$ induces the restriction maps $r : R(M_K) \longrightarrow R(\partial M_K)$ and $r : X(M_K) \longrightarrow X(\partial M_K)$. They are affine algebraic sets over the complex numbers $\mathbb{C}$. By Culler et al. [4, Proposition 1.1.1], $X_0 \subset X(M_K)$ is an irreducible affine curve.

Since $\pi_1(\partial M_K) = \mathbb{Z} \oplus \mathbb{Z}$ with the fixed meridian μ and longitude λ as its generators, we can identify $R(\partial M_K)$ with the set $\{(A, B) | A, B \in SL_2(\mathbb{C}), AB = BA\}$. Let R_D be the subvariety of $R(\partial M_K)$ consisting of the diagonal representations. For $\rho \in R_D$, we have an isomorphism $p : R_D \to \mathbb{C}^* \times \mathbb{C}^*$ defined by $p(\rho) = (m, l)$ with

$$\rho(\mu) = \begin{bmatrix} m & 0 \\ 0 & m^{-1} \end{bmatrix}, \quad \rho(\lambda) = \begin{bmatrix} l & 0 \\ 0 & l^{-1} \end{bmatrix}.$$

Define a morphism $t : R(\partial M_K) \longrightarrow \mathbb{C}^3$ by $t(\rho) = (\sigma(\rho(\mu)), \sigma(\rho(\lambda)), \sigma(\rho(\mu\lambda)))$ with σ the trace. By the proof of [3, Proposition 1.4.1], $\chi \in X(\partial M_K)$ is determined by its values on μ, λ and $\mu\lambda$. Then $X(\partial M_K) = t(R(\partial M_K))$ is a surface in $\mathbb{C}^3$ given by Li and Wang [10, Proposition 3.1],

$$x^2 + y^2 + z^2 - xyz - 4 = 0. \tag{1}$$

Let t_D be the restriction of $t : R(\partial M_K) \longrightarrow X(\partial M_K)$ to R_D. Then $t_D : R_D = \mathbb{C}^* \times \mathbb{C}^* \to X(\partial M_K)$ is a finite morphism by Li and Wang [10, Proposition 3.2] which is given explicitly by, for $(m,l) \in \mathbb{C}^* \times \mathbb{C}^*$,

$$t_D(m,l) = (m + m^{-1}, l + l^{-1}, ml + m^{-1}l^{-1}).$$

It is easy to check t_D is 2 : 1 branching over four points $(\pm 1, \pm 1)$.

Let $I_\gamma : X(M_K) \to \mathbb{C}$ be defined by $I_\gamma(\chi) = \chi(\gamma)$. The set of functions I_γ with $\gamma \in \pi_1(M_K)$ generates the affine coordinate ring of $X(M_K)$. By Culler et al. [4, Proposition 1.1.1], for every nonzero $\gamma \in \pi_1(\partial M_K)$, the function I_γ is non-constant on X_0. Hence $r(X_0) \subset X(\partial M_K)$ has dimension 1. Set $Y_0 = \overline{r(X_0)}$, the Zariski closure of $r(X_0)$ in $X(\partial M_K)$. Then Y_0 is an irreducible affine curve. Denote by D_0 the inverse image $t_D^{-1}(Y_0)$. Now we have the following diagram:

$$\begin{array}{ccc} R(M_K) \supset R_0 & & D_0 = t_D^{-1}(Y_0) \subset R_D \subset R(\partial M_K) \\ \downarrow t & & \downarrow t_D \\ X(M_K) \supset X_0 = t(R_0) & \xrightarrow{\;r\;} & Y_0 = \overline{r(X_0)} \subset X(\partial M_K). \end{array}$$

By Li and Wang [10, Proposition 3.3], the inverse image $D_0 \subset \mathbb{C}^* \times \mathbb{C}^*$ is an equi-dimensional affine algebraic set with at most two irreducible components of dimension 1, the image of each irreducible component of D_0 under t_D is the whole Y_0.

Let $A_0(m,l)$ be the defining equation of the closure of the affine curve D_0 in $\mathbb{C} \times \mathbb{C}$. It has no repeated factors and it is a factor of the A-polynomial of the knot K defined in [2]. Denote by $\widetilde{X_0}$ (resp. $\widetilde{Y_0}$) a smooth projective model of the affine curve X_0 (resp. Y_0). The restriction morphism $r : X_0 \to Y_0$ induces an isomorphism $\tilde{r} : \widetilde{X_0} \to \widetilde{Y_0}$ by Li and Wang [10, Lemma 3.4].

Let $\gamma = p\mu + q\lambda \in H_1(\partial M_K; \mathbb{Z})$ be a non-zero primitive element with p, q coprime. Define a regular function $f_\gamma = I_\gamma^2 - 4$ on X_0. It is nonconstant and gives rises to a morphism from $\widetilde{X_0}$ to $\mathbb{CP}^1$ which is still denoted by f_γ. Since $\gamma \in H_1(\partial M_K; \mathbb{Z})$, we can think of I_γ as a regular function on Y_0 in $X(\partial M_K)$. Define on Y_0 the function $f'_\gamma = I_\gamma^2 - 4$. Similarly, it induces a non-constant morphism from $\widetilde{Y_0}$ to $\mathbb{CP}^1$, denoted also by f'_γ. Then $f_\gamma = f'_\gamma \circ \tilde{r}$.

Define $Z_\gamma = \{\chi \in X_0 | \chi(\gamma) = \pm 2\}$ the set of zeros of the function f_γ on X_0. If $\chi \in Z_\gamma$, then there exists a representation $\rho \in R_0$ such that the trace $\sigma(\varrho(\gamma)) = \pm 2$ and its character $\chi_\rho = \chi$. Then either $\rho(\gamma) = \pm I$ or $\rho(\gamma) \neq \pm I$ and $\sigma(\rho(\alpha)) = \pm 2$ for all $\alpha \in \pi_1(\partial M_K)$ by Li and Wang [10, Lemma 3.5]. Given an irreducible representation $\rho \in R_0 \subset R(M_K)$, assume that both $\rho(\mu)$ and $\rho(\lambda)$ are parabolic. Up to conjugation, we have

$$\rho(\mu) = \pm \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}; \;\; \rho(\lambda) = \pm \begin{bmatrix} 1 & t(\rho) \\ 0 & 1 \end{bmatrix},$$

where $t(\rho)$ is a complex number. It is conjectured in [10] that if $\rho(\mu)$ and $\rho(\lambda)$ are parabolic, then $t(\rho) \notin \mathbb{Q}$ for an irreducible $SL_2(\mathbb{C})$-representation of a hyperbolic knot.

Let $\chi_\rho \in Z_\gamma$ be the character of an irreducible representation ρ. It was shown in [10, Proposition 3.4] that the trace $\sigma(\rho(\mu)) \neq \pm 2$ is equivalent to $\rho(\gamma) = \pm I$ provided the conjecture mentioned above is true and $p, q \neq 0$. Moreover, for the character $\chi \in Z_\gamma$ of a reducible representation ρ, we have $\chi(\mu) \neq \pm 2$ and $\rho(\gamma) = \pm I$.

Let $E(p,q)$ be the reducible curve $m^p l^q = \pm 1$ in $\mathbb{C}^* \times \mathbb{C}^*$ for p, q coprime integers. Then the image $t_D(E(p,q))$ is a curve in $X(\partial M_K)$. We know $r(X_0)$ is an irreducible curve in $X(\partial M_K)$. They do not have common irreducible component because the traces of characters of X_0 are not constant. Hence $t_D(E(p,q)) \cap r(X_0)$ is finite. We see that the set $t_D(E(p,q)) \cap r(X_0)$ consists of possible characters in X_0 which can also be the characters of $K(p/q)$ in $PSL_2(\mathbb{C})$, where $K(p/q)$ denotes the closed 3-manifold obtained from M_K by Dehn surgery along $\gamma = p\mu + q\lambda \in H_1(\partial M_K; \mathbb{Z})$. The following definition should be thought of as the *algebro-geometric* invariant for (p,q)-Dehn surgery of M_K.

Definition 1. (i)

$$b(p,q) = \sum_{\chi \in t_D(E(p,q)) \cap r(X_0)} n_\chi,$$

where n_χ is the intersection multiplicity at χ in $X(\partial M_K)$ (Definition 3.1 of [10]).

(ii)

$$\lambda(p,q) = \sum_{\chi \in S(p,q)} n_\chi,$$

where $S(p,q) = \{\chi \in t_D(E(p,q)) \cap r(X_0) | \chi(\mu) \neq \pm 2\} \subset X(\partial M_K)$ (Definition 3.2 of [10]).

The integer $b(p,q)$ is a well-defined invariant of the 3-manifold $K(p/q)$ resulting from (p,q)-Dehn surgery on the hyperbolic knot complement M_K. It depends on the knot K and the surgery coefficient p/q, and it is always positive by Li and Wang [10, Theorem 3.1]. The set $S(p,q)$ contains all possible reducible characters. Hence the number $\lambda(p,q)$ counts both irreducible and reducible characters. The quantity $\lambda(p,q)$ is a well-defined *algebro-geometric* $SL_2(\mathbb{C})$ Casson–Walker type invariant of $K(p/q)$ (see [13]). It depends on the knot K and the surgery coefficient p/q by Li and Wang [10, Theorem 3.2]. By definition, $\lambda(p,q) \leq b(p,q)$ for any coprime p, q and a hyperbolic knot in S^3.

3 An $SL_2(\mathbb{C})$ Casson Type Topological Invariant of Hyperbolic Knots

In this section we prove the following main result.

Theorem 1. *The number $\lambda(p,q)$ is a topological invariant of $SL_2(\mathbb{C})$ Casson type invariant of the 3-manifold $K(p/q)$ resulting from (p,q)-Dehn surgery on the hyperbolic knot complement M_K.*

Remark 1. (i) Note that the number $\lambda(p,q)$ counts both irreducible and reducible characters. It can be seen as a Casson–Walker type invariant (see [13]) for hyperbolic knots and its resulting 3-manifolds under the Dehn surgery.

(ii) For two analytic subvarieties of complementary dimension meeting in a finite set of points on a compact complex manifold, Griffiths and Harris showed that the topological intersection number is the algebraic geometric intersection number in [8, Chapter 0, Section 4]. Our intersection points are in the variety $X(\partial M_K)$ which is not a manifold.

According to the definition, the number $\lambda(p,q)$ is an algebro-geometric invariant, which counts algebraic intersection multiplicity of curves in a surface. To prove the theorem, we need to show that the multiplicity is topological. Although this fact is well known in general without a proof, we give it a short geometric proof, which is new to our knowledge. Note that Fulton and MacPherson [7] outlines their arguments that if two subvarieties intersect properly, then the algebraic intersection multiplicity is topological. There is no detailed proof of this result at the present, up to our knowledge.

To begin with, we recall the following definition of intersection multiplicities.

Definition 2. Let C_1 and C_2 be two distinct irreducible affine curves in $\mathbb{C}^2$ defined by equations $f_i(x,y) = 0$ $(i = 1, 2)$. Let $P \in C_1 \cap C_2$.

(i) The algebro-geometric intersection multiplicity $(C_1 \cdot C_2)_P$ of C_1 and C_2 at P is defined to be the length of the $\mathscr{O}_P$-module $\mathscr{O}_P/(f_1, f_2)\mathscr{O}_P$, where $\mathscr{O}_P$ is the local ring of $\mathbb{C}^2$ at P.

(ii) Let $U \subset \mathbb{R}^4$ be a local coordinate chart of P with the induced orientation from an identification $\mathbb{R}^4 = \mathbb{C}^2$(so that U contains no other intersection points of C_1 and C_2). Let $S_i = C_i \cap U$. The topological intersection multiplicity m_P is the local degree of a map ψ at $P \times P$, where $\psi : S_1 \times S_2 \longrightarrow \mathbb{R}^4$ is defined by $\psi(u_1, u_2) = u_1 - u_2$. Here $u_1 - u_2$ is the standard subtraction of vectors in $\mathbb{R}^4$.

Remark 2. 1. For the algebro-geometric intersection multiplicity of high dimensional varieties, one should use Serre's intersection multiplicity (see [12, pp. 106–110]) instead, which involves the length of Tor_i modules. In the curve case, we simply have $\mathrm{Tor}_i = 0$ for $i > 0$.

2. The definition of topological intersection multiplicity is taken from [7]. The local degree of a map $f : X \longrightarrow Y$ at a point $a \in X$ is defined by the homomorphism

$$H_n(X, X - a) \longrightarrow H_n(Y, Y - f(a))$$

with $n = \dim X = \dim Y$.

It is convenient to rephrase the algebro-geometric multiplicity $(C_1 \cdot C_2)_P$ in the following way. Let V be a Zariski open neighborhood of P in $\mathbb{C}^2$ so that P is the only intersection point of C_1 and C_2 in V. This is possible because $C_1 \cap C_2$ is a finite set. Let $D_i = C_i \cap V$. Define a map

$$\phi : D_1 \times D_2 \longrightarrow \mathbb{C}^2 \tag{2}$$

by $\phi(u_1, u_2) = u_1 - u_2$. Then the intersection scheme $D_1 \cap D_2$ is naturally isomorphic to $\phi^{-1}(0)$ with its induced scheme structure. Therefore $(C_1 \cdot C_2)_P$ is equal to the length of $\phi^{-1}(0)$.

We remark that if we take V as a small *analytic* neighborhood, and $D_i' = C_i \cap V$, then in the same way we get a map $\phi_{an} : D_1' \times D_2' \longrightarrow \mathbb{C}^2$. Of course $(C_1 \cdot C_2)_P$ is equal to the length of $\phi_{an}^{-1}(0)$ as well.

For the map $\psi : S_1 \times S_2 \longrightarrow \mathbb{R}^4$ in the above definition, if ϵ is a regular value of ψ, then

$$\deg(\psi, \epsilon) = \sum_{(u_1,u_2)\in\psi^{-1}(\epsilon)} \operatorname{sign}(\det D\psi_{(u_1,u_2)}).$$

Notice that ψ is the restriction of ϕ_{an} to $S_1 \times S_2$, so if ϵ is a regular value and it is close to 0 enough, $\deg(\psi, \epsilon)$ is the topological intersection multiplicity m_P of C_1 and C_2 at P, and it is equal to the length of $\phi_{an}^{-1}(\epsilon)$. The existence of such an ϵ is guaranteed by Sard's theorem.

Proposition 1. *Let C_1 and C_2 be two distinct irreducible algebraic curves in $\mathbb{C}^2$. The algebro-geometric intersection multiplicity of C_1 and C_2 at a point is equal to its topological intersection multiplicity.*

Proof. According to the above discussion, the algebro-geometric intersection multiplicity is equal to the length of $\phi_{an}^{-1}(0)$. For any regular value ϵ of ϕ_{an} which is close to 0, the topological intersection multiplicity is equal to the length of $\phi_{an}^{-1}(\epsilon)$. Therefore, to prove the proposition, we need to show

$$\operatorname{length}(\phi_{an}^{-1}(0)) = \operatorname{length}(\phi_{an}^{-1}(\epsilon)).$$

This equality is in fact a corollary of the following Proposition 2 concerning the flatness of the morphism ϕ. □

Proposition 2. *The map $\phi : D_1 \times D_2 \longrightarrow \mathbb{C}^2$ defined in (2) is a flat morphism.*

We use the following Lemma to prove this Proposition.

Lemma 1 ([11], Proposition 2.5). *Let B be a flat A-algebra and $b \in B$. If the image of b in $B/\mathfrak{m}B$ is not a zero divisor for any maximal ideal $\mathfrak{m}$ of A, then $B/(b)$ is a flat A-algebra.*

Proof (Proof of Proposition 2). Since $D_i \subset C_i$ is Zariski open in C_i, $D_1 \times D_2 \to C_1 \times C_2$ is an open embedding. It is clear that ϕ can be factored as a composition

$$D_1 \times D_2 \mapsto C_1 \times C_2 \mapsto \mathbb{C}^2.$$

Because open embedding is always is flat, and the composition of two flat morphisms is still flat, it suffices to show that the second map in the above sequence is flat. Later we denote this map by ϕ.

Let ι_i be the closed embedding $C_i \to \mathbb{C}^2$. Let $p : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2$ be a map defined by $p(u, v) = u - v$. Then $\phi : C_1 \times C_2 \to \mathbb{C}^2$ can be factored as $\phi = p \circ h$ with $h = \iota_1 \times \iota_2 : C_1 \times C_2 \to \mathbb{C}^2 \times \mathbb{C}^2$.

It is clear that p is flat. Now we express p in terms of coordinates. Let $A = \mathbb{C}[x, y]$ be the affine coordinate ring of $\mathbb{C}^2$. Let $B = A \otimes_{\mathbb{C}} A$. Then B is the affine coordinate ring of $\mathbb{C}^2 \times \mathbb{C}^2$. The map p corresponds to a $\mathbb{C}$-algebra homomorphism $p : A \to B$ defined by

$$p(x) = 1 \otimes x - x \otimes 1, \quad p(y) = 1 \otimes y - y \otimes 1.$$

Next we decompose h further as

$$C_1 \times C_2 \overset{h_2}{\mapsto} C_1 \times \mathbb{C}^2 \overset{h_1}{\mapsto} \mathbb{C}^2 \times \mathbb{C}^2.$$

with $h_1 = \iota_1 \times \mathrm{id}$ and $h_2 = \mathrm{id} \times \iota_2$. Both h_i are closed embedding.

Let $\phi_1 = p \circ h_1 : C_1 \times \mathbb{C}^2 \mapsto \mathbb{C}^2$. We now show that ϕ_1 is flat. Recall that $C_i \subset \mathbb{C}^2$ are defined by polynomials f_i. Let R_1 be the coordinate ring of $C_1 \times \mathbb{C}^2$. Then $R_1 = B/(f_1 \otimes 1)$. Now we apply Lemma 1 to $b = f_1 \otimes 1$. For any maximal ideal $\mathfrak{m} = (x - a, y - b)$ of A, $B/\mathfrak{m}B$ is isomorphic to A as a $\mathbb{C}$-algebra. So $B/\mathfrak{m}B$ is an integral domain. The image of $f_1 \otimes 1$ in $B/\mathfrak{m}B$ is obviously nonzero, which implies that it is not a zero divisor. It follows from Lemma 1 that R_1 is a flat A-algebra.

Finally we let R be the coordinate ring of $C_1 \times C_2$. R is an A-algebra via the map ϕ. We want to show that R is flat over A.

Consider the quotient map $B \mapsto R_1 = B/(f_1 \otimes 1)$. For $1 \otimes f_2 \in B$, let $\overline{1 \otimes f_2}$ be its image in R_1. Then the coordinate ring $R = R_1/(\overline{1 \otimes f_2})$. Notice that for any maximal ideal $\mathfrak{m}$ of A,

$$R_1/\mathfrak{m}R_1 = B/((f_1 \otimes 1) + \mathfrak{m}B) = (B/\mathfrak{m}B)/(f_1 \otimes 1) \cong A/(f_1),$$

where the last equality is an isomorphism as $\mathbb{C}$-algebras. The image of $\overline{1 \otimes f_2}$ in $R_1/\mathfrak{m}R_1$ is equal to $\bar{f_2}$ in $A/(f_1)$. We claim that $\bar{f_2}$ is not a zero divisor in $A/(f_1)$. If not, there is a non zero element $\bar{s} \in A/(f_1)$ so that $\bar{s}\bar{f_2} = 0$. Lifting to A, it says that there exist $t \in A$ so that $sf_2 = tf_1$. Since A is a UFD, and f_1, f_2 are distinct irreducible polynomials, we get $f_1|s$, that is, $\bar{s} = 0$, which contradicts to the assumption that $\bar{s}$ is nonzero. Since R_1 is A-flat, applying Lemma 1, it follows that R is A-flat. □

Remark 3. Proposition 2 and hence Proposition 1 can be generalized to higher dimension without much effort, provided that $\mathrm{Tor}_i = 0$ for $i > 0$.

Proof (Proof of Theorem 1). By identifying $C_1 = D_0$ and $C_2 = E(p,q)$ as two distinct curves in $\mathbb{C}^2$, and using Proposition 1, we obtain the counting from algebro-geometric intersections agrees with the counting from the topological intersections in $\mathbb{C}^2$. By using the identification through an one-to-one and onto regular map t_D in [10] and on $S(p,q)$, we obtain the following: $\lambda(p,q)$ is an topological invariant of $SL_2(\mathbb{C})$ Casson type invariant of the 3-manifold $K(p/q)$ resulting from (p,q)-Dehn surgery on the hyperbolic knot complement M_K. □

For two subvarieties V and W in a nonsingular algebraic variety X, recall that V and W is called to intersect properly if each irreducible component Z_i of $V \cap W$ has dimension $\dim(V)+\dim(W)-\dim(X)$. Hence, the algebraic geometry intersection $V \cdot W$ is uniquely determined by $\sum m(Z_i)Z_i$ with the geometric multiplicity $m(Z_i)$. Fulton and MacPherson in [7] showed that

$$m(Z_i) = \mathrm{length}_{\mathscr{O}_{Z_i}} (\mathscr{O}_V \otimes_{\mathscr{O}_X} \mathscr{O}_W)_{Z_i} + \sum_{j=1}^{\infty} (-1)^j \, \mathrm{length}_{\mathscr{O}_{Z_i}} \left(\mathrm{Tor}_j^{\mathscr{O}_X}(\mathscr{O}_V, \mathscr{O}_W)\right)_{Z_i},$$

and the algebraic intersection multiplicity $m(Z_i)$ is a topological intersection multiplicity. For our special case, $\left(\mathrm{Tor}_j^{\mathscr{O}_X}(\mathscr{O}_V, \mathscr{O}_W)\right)_{Z_i} = 0$ for $j > 0$. It would be interesting to identify $\left(\mathrm{Tor}_j^{\mathscr{O}_X}(\mathscr{O}_V, \mathscr{O}_W)\right)_{Z_i}$ as some type of Walker correction terms in general.

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The Möbius Geometry of Wintgen Ideal Submanifolds

Xiang Ma and Zhenxiao Xie

Abstract Wintgen ideal submanifolds in space forms are those ones attaining equality pointwise in the so-called DDVV inequality which relates the scalar curvature, the mean curvature and the scalar normal curvature. They are Möbius invariant objects. The mean curvature sphere defines a conformal Gauss map into a Grassmann manifold. We show that any Wintgen ideal submanifold of dimension greater than or equal to 3 has a Riemannian submersion structure over a Riemann surface with the fibers being round spheres. Then the conformal Gauss map is shown to be a super-conformal and harmonic map from the underlying Riemann surface. Some of our previous results are surveyed in the final part.

1 Introduction

Geometers are always interested in beautiful shapes. In many cases they arise as the extremal cases of certain geometrical inequalities. In particular, it would be desirable to find some universal inequality, whose equality case include many non-trivial examples. It would be more interesting if such objects are invariant under a suitable transformation group.

For submanifolds in real space forms, such a universal inequality has been found, called the DDVV inequality. The extremal case defines the Wintgen ideal submanifolds. These are invariant object under the Möbius transformations; in particular, the study of them from the viewpoint of Möbius geometry is the focus of this paper.

Recall that given an m-dimensional submanifold M^m immersed in a real space form of dimension $m + p$ with constant sectional curvature c, at any point there holds

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$$\textit{The DDVV inequality: } K \le c + ||H||^2 - K_N. \tag{1}$$

Here $K = \frac{2}{m(m-1)} \sum_{i<j} \langle R(e_i, e_j)e_j, e_i\rangle$ is the normalized scalar curvature with respect to the induced metric on M^m, H is the mean curvature vector, and $K_N = \frac{2}{m(m-1)}||R^{\perp}||$ is the normal scalar curvature.

This remarkable inequality attracts many geometers. It relates the most important intrinsic and extrinsic quantities at one point of a submanifold, and it takes an incredibly general form, without restrictions on the dimension/codimension, or any additional geometrical or topological assumptions. It was first conjectured by De Smet et al. [9] in 1999, and proved by Ge and Tang [12] in 2008. (Lu gave an independent proof in [17].)

After discovering the DDVV inequality, people became interested in the extremal case [5, 8, 9, 17]. Wintgen [20] first proved this inequality for surfaces in $\mathbb{S}^4$, where the equality is attained exactly when the surfaces are *super-conformal*. That means at any point of the surface, the curvature ellipse is a circle, or equivalently, the Hopf differential is an isotropic differential form. According to the suggestion of Chen and other ones [4, 18], we make the following definition.

Definition 1. A submanifold M^m of dimension m and codimension p in a real space form is called a *Wintgen ideal submanifold* if the equality is attained at every point of M^m in the DDVV inequality (1). By the characterization of Ge and Tang in [12], this happens if, and only if, at every point $x \in M$ there exists an orthonormal basis $\{e_1, \cdots, e_m\}$ of the tangent space T_xM^m and an orthonormal basis $\{n_1, \cdots, n_p\}$ of the normal space $T_x^{\perp}M^m$, such that the shape operators $\{A_r, r = 1, \cdots, p\}$ take the form as below:

$$A_1 = \begin{pmatrix} \lambda_1 & \mu_0 & 0 & \cdots & 0 \\ \mu_0 & \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_1 \end{pmatrix}, \; A_2 = \begin{pmatrix} \lambda_2+\mu_0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2-\mu_0 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_2 \end{pmatrix}, \tag{2}$$

$$A_3 = \lambda_3 I_m, \; A_\sigma = 0 \; (\sigma \ge 4),$$

where I_m is the identity matrix of order m.

People have found abundant examples of Wintgen ideal submanifolds [3, 5–8, 13, 21]. It is interesting yet difficult to obtain a complete classification of them.

We emphasize that generally they should be classified up to Möbius transformations, because Wintgen ideal is an Möbius invariant property.[1] This follows directly from (1) and the fact that up to a factor, the traceless part of the second fundamental

[1] It was first noticed by Dajczer and Tojeiro in [8], based on an equivalent formulation of the DDVV inequality in [10].

form is Möbius invariant. So the most suitable framework for the study of Wintgen ideal submanifolds is Möbius geometry. This research program has been carried out by us recently in [14–16, 21] under various additional assumptions. Besides giving a survey of these work, we will also report two new results on general Wintgen ideal submanifolds.

For any submanifold M^m immersed in $\mathbb{S}^{m+p}$, we can define the *mean curvature sphere* at one point $x \in M^m$. It is the unique m-dimensional round sphere tangent to M^m at x which also shares the same mean curvature vector with M^m at x. As a well-known Möbius invariant construction,[2] the characterization above holds true for any other conformal metric of $\mathbb{S}^{m+p}$. Via the light-cone model, this codimension-p sphere corresponds to a space-like p-space $\mathrm{Span}_{\mathbb{R}}\{\xi_1, \cdots, \xi_p\}$ in the Lorentz space $\mathbb{R}^{m+p+2}_1$. We call it *the conformal Gauss map*[3] into the real Grassmannian

$$\Xi = \xi_1 \wedge \cdots \wedge \xi_p \in \mathrm{Gr}(p, \mathbb{R}^{m+p+2}_1).$$

The crucial observation is that the image $\Xi(M^m)$ degenerates to a two-dimensional surface when M^m is Wintgen ideal. Moreover, we have:

Theorem 1. *For a Wintgen ideal submanifold of dimension $m \geq 3$, the conformal Gauss map Ξ factors as a projection map $\pi : M^m \to \overline{M}^2$ (which is a Riemannian submersion up to a constant), and a super-conformal harmonic map from a Riemann surface*

$$\Xi : \overline{M}^2 \to \mathrm{Gr}(p, \mathbb{R}^{m+p+2}_1).$$

In other words, $\Xi(M^m)$ is a super-minimal surface $\overline{M}^2 \subset \mathrm{Gr}(p, \mathbb{R}^{m+p+2}_1)$ (endowed with the induced metric).

This result shows striking similarity with the celebrated characterization of Willmore surfaces by its conformal Gauss map being a harmonic map [2, 11]. Yet it is far more than a parallel generalization. Besides that, it greatly simplifies the study of Wintgen ideal submanifolds by reducing it to surface theory. (See Theorem 3 for stronger result in codimension two.)

As a consequence, these m-dimensional mean curvature spheres is a two-parameter family. We consider their envelope $\hat{M}^m$, which contains M^m as an open subset. The second new result is

Theorem 2. *For a Wintgen ideal submanifold $x : M^m \to \mathbb{S}^{m+p}$ $(m \geq 3)$ and the envelope $\hat{M}^m$, we have the following conclusions:*

1. *There is a fiber bundle structure $S^{m-2} \to \hat{M}^m \to \overline{M}^2$ over a Riemann surface. The fibers are all round spheres of the ambient space.*

[2] The notion of the mean curvature sphere can be traced back to Blaschke [1] in 1920s.

[3] This is an analog to the work of Bryant [2] and Ejiri [11] on Willmore surfaces in $\mathbb{S}^n$.

2. *The projection* $\pi : \hat{M}^m \to \overline{M}^2$ *is a Riemannian submersion up to a constant.*
3. *As a natural extension of* M^m*,* $\hat{M}^m$ *is still a Wintgen ideal submanifold.*

This theorem shows that Wintgen ideal submanifolds have simple and elegant structure. Based on this general picture, we can show that they arise either as cylinders, cones, rotational submanifolds, or Hopf bundles over complex curves in complex projective spaces under various specific assumptions.

This paper is organized as below. In Sect. 2, we will briefly review the submanifold theory in Möbius geometry established by Wang [19]. Section 3 gives the information on the invariants and the structure equations of Wintgen ideal submanifolds. The two results mentioned above are proved separately in Sects. 4 and 5. Finally, we survey some recent results on Wintgen ideal submanifolds based on our joint work with Tongzhu Li and Changping Wang. These include a reduction theorem [14], the characterization of the minimal examples [21], and a classification of Möbius homogeneous examples [16].

2 Submanifold Theory in Möbius Geometry

Here we follow the framework of Wang in [19] except that we take a different canonical lift Y up to a constant.

In the classical light-cone model, the light-like directions in the Lorentz space $\mathbb{R}^{m+p+2}_1$ correspond to points in the round sphere $\mathbb{S}^{m+p}$, and the Lorentz orthogonal group correspond to the conformal transformation group of $\mathbb{S}^{m+p}$. The Lorentz inner product between $Y = (Y_0, Y_1, \cdots, Y_{m+p+1}), Z = (Z_0, Z_1, \cdots, Z_{m+p+1}) \in \mathbb{R}^{m+p+2}_1$ is

$$\langle Y, Z\rangle = -Y_0 Z_0 + Y_1 Z_1 + \cdots + Y_{m+p+1} Z_{m+p+1}.$$

Let $f : M^m \to \mathbb{S}^{m+p} \subset \mathbb{R}^{m+p+1}$ be a submanifold without umbilics. Take $\{e_i | 1 \le i \le m\}$ as the tangent frame with respect to the induced metric $I = \mathrm{d}f \cdot \mathrm{d}f$, and $\{\theta_i\}$ as the dual 1-forms. Let $\{n_r | 1 \le r \le p\}$ be orthonormal frame for the normal bundle. The second fundamental form and the mean curvature of f are

$$II = \sum\nolimits_{ij,r} h^r_{ij} \theta_i \otimes \theta_j n_r,\ H = \frac{1}{m} \sum\nolimits_{j,r} h^r_{jj} n_r = \sum\nolimits_r H^r n_r, \tag{3}$$

respectively. We define the Möbius position vector $Y : M^m \to \mathbb{R}^{m+p+2}_1$ of f by

$$Y = \rho(1, f),\ \rho^2 = \frac{1}{4}\left|II - \frac{1}{m} tr(II) I\right|^2 \tag{4}$$

which is a canonical lift of f. Two submanifolds $f, \bar{f} : M^m \to \mathbb{S}^{m+p}$ are Möbius equivalent if there exists T in the Lorentz group $O(m + p + 1, 1)$ in $\mathbb{R}^{m+p+2}_1$ such that $\bar{Y} = YT$. It follows immediately that

$$\mathrm{g} = \langle \mathrm{d}Y, \mathrm{d}Y \rangle = \rho^2 \mathrm{d}f \cdot \mathrm{d}f \tag{5}$$

is a Möbius invariant, called the Möbius metric of x.

Let Δ be the Laplacian with respect to g. Define

$$N = -\frac{1}{m}\Delta Y - \frac{1}{2m^2}\langle \Delta Y, \Delta Y \rangle Y. \tag{6}$$

Let $\{E_1, \cdots, E_m\}$ be a local orthonormal frame for (M^m, g) with dual 1-forms $\{\omega_1, \cdots, \omega_m\}$. We define the tangent frame $Y_j = E_j(Y)$ and the normal frame

$$\xi_r = (H^r, n_r + H^r f).$$

Then $\{Y, N, Y_j, \xi_r\}$ is a moving frame of $\mathbb{R}^{m+p+2}_1$ along M^m, which is orthonormal except

$$\langle Y, Y \rangle = 0 = \langle N, N \rangle, \ \langle N, Y \rangle = 1.$$

Remark 1. Geometrically, at one point $x \in M^m$, ξ_r (for any given r) corresponds to the unique hypersphere tangent to M^m with normal vector n_r and mean curvature $H^r(x)$. In particular, the spacelike subspace $\mathrm{Span}_{\mathbb{R}}\{\xi_1, \cdots, \xi_p\}$ represents a unique m-dimensional sphere tangent to M^m with the same mean curvature vector $\sum_r H^r n_r$. This well-defined object was naturally named *the mean curvature sphere* of M^m at x, which is well-known to share the same mean curvature at x even when the ambient space is endowed with any other conformal metric.

We fix the range of indices in this section as below: $1 \le i, j, k \le m$; $1 \le r, s \le p$. The structure equations are:

$$\begin{aligned}
\mathrm{d}Y &= \sum\nolimits_i \omega_i Y_i, \\
\mathrm{d}N &= \sum\nolimits_{ij} A_{ij}\omega_i Y_j + \sum\nolimits_{i,r} C^r_i \omega_i \xi_r, \\
\mathrm{d}Y_i &= -\sum\nolimits_j A_{ij}\omega_j Y - \omega_i N + \sum\nolimits_j \omega_{ij} Y_j + \sum\nolimits_{j,r} B^r_{ij}\omega_j \xi_r, \\
\mathrm{d}\xi_r &= -\sum\nolimits_i C^r_i \omega_i Y - \sum\nolimits_{i,j} \omega_i B^r_{ij} Y_j + \sum\nolimits_s \theta_{rs}\xi_s,
\end{aligned} \tag{7}$$

where ω_{ij} are the connection 1-forms of the Möbius metric g; θ_{rs} are the normal connection 1-forms. The tensors

$$\mathbf{A} = \sum\nolimits_{i,j} A_{ij}\omega_i \otimes \omega_j, \ \mathbf{B} = \sum\nolimits_{i,j,r} B^r_{ij}\omega_i \otimes \omega_j \xi_r, \ \Phi = \sum\nolimits_{j,r} C^r_j \omega_j \xi_r \tag{8}$$

are called the Blaschke tensor, the Möbius second fundamental form and the Möbius form of f, respectively [19]. The integrability conditions for the structure equations are given as below:

$$A_{ij,k} - A_{ik,j} = \sum_r (B^r_{ik} C^r_j - B^r_{ij} C^r_k), \tag{9}$$

$$C^r_{i,j} - C^r_{j,i} = \sum_k (B^r_{ik} A_{kj} - B^r_{jk} A_{ki}), \tag{10}$$

$$B^r_{ij,k} - B^r_{ik,j} = \delta_{ij} C^r_k - \delta_{ik} C^r_j, \tag{11}$$

$$R_{ijkl} = \sum_r (B^r_{ik} B^r_{jl} - B^r_{il} B^r_{jk}) + \delta_{ik} A_{jl} + \delta_{jl} A_{ik} - \delta_{il} A_{jk} - \delta_{jk} A_{il}, \tag{12}$$

$$R^{\perp}_{rsij} = \sum_k (B^r_{ik} B^s_{kj} - B^s_{ik} B^r_{kj}). \tag{13}$$

Here the covariant derivatives $A_{ij,k}, B^r_{ij,k}, C^r_{i,j}$ are defined as usual; $R, R^{\perp}$ denote the curvature tensor of g and the normal curvature tensor, respectively. The tensor **B** satisfies the following identities:

$$\sum_j B^r_{jj} = 0, \ \sum_{i,j,r} (B^r_{ij})^2 = 4. \tag{14}$$

All coefficients in the structure equations are determined by $\{\mathrm{g}, \mathbf{B}\}$ and the normal connection $\{\theta_{\alpha\beta}\}$. In particular these are the complete set of Möbius invariants.

3 Invariants of a Wintgen Ideal Submanifold

Let $f : M^m \to \mathbb{S}^{m+p}$ be a Wintgen ideal submanifold ($m \geq 3$). We will always assume that it is umbilic-free unless it is stated otherwise. In terms of the Möbius invariants, that means the existence of a suitable tangent frame $\{E_1, \cdots, E_m\}$ and normal frame $\{\xi_1, \cdots, \xi_p\}$ so that the Möbius second fundamental form is given by

$$B^1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ B^2 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ B^{\alpha} = 0, \ \alpha \geq 3. \tag{15}$$

Remark 2. The reader is warned that the lift Y here is different from [19]. Hence in the formulas below, we have removed the annoying factor $\mu = \sqrt{\frac{m-1}{4m}}$ appearing in [14–16, 21].

Remark 3. The *canonical distribution* $\mathbb{D}_2 = \mathrm{Span}\{E_1, E_2\}$ and the normal subbundle $\mathrm{Span}\{\xi_1, \xi_2\}$ are well-defined if (15) holds and we fix our frame up to rotations

$$(\tilde{E}_1, \tilde{E}_2) = (E_1, E_2) \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \ (\tilde{\xi}_1, \tilde{\xi}_2) = (\xi_1, \xi_2) \begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix}. \tag{16}$$

We will adopt the convention below on the range of indices:

$$1 \le i, j, k, l \le m,\ 3 \le a, b \le m;\ 1 \le r, s \le p,\ 3 \le \alpha, \beta \le p.$$

By definition, we compute the covariant derivatives of B^r_{ij} and obtain

$$B^r_{ab,i} = 0,\ B^\alpha_{1a,i} = B^\alpha_{2a,i} = 0, \tag{17}$$

$$B^1_{12,i} = B^1_{21,i} = 0,\ B^2_{11,i} = B^2_{22,i} = 0, \tag{18}$$

$$\omega_{2a} = \sum_i B^1_{1a,i}\omega_i = -\sum_i B^2_{2a,i}\omega_i,\ \omega_{1a} = \sum_i B^1_{2a,i}\omega_i = \sum_i B^2_{1a,i}\omega_i, \tag{19}$$

$$2\omega_{12} + \theta_{12} = \sum_i \frac{-B^1_{11,i}}{\mu}\omega_i = \sum_i B^1_{22,i}\omega_i = \sum_i B^2_{12,i}\omega_i, \tag{20}$$

$$\theta_{1\alpha} = \sum_i B^\alpha_{12,i}\omega_i,\ \theta_{2\alpha} = \sum_i B^\alpha_{11,i}\omega_i. \tag{21}$$

By (11), $B^r_{ij,k}$ is symmetric for distinctive i, j, k. It follows from (17)∼(20) that

$$\omega_{1a}(E_b) = B^1_{2a,b} = B^1_{ab,2} = 0,\ \omega_{2a}(E_b) = B^1_{1a,b} = B^1_{ab,1} = 0\ (a \ne b);$$
$$\omega_{1a}(E_1) = B^2_{1a,1} = B^1_{2a,1} = B^1_{21,a} = 0,\ \omega_{2a}(E_2) = -B^2_{2a,2} = B^1_{1a,2} = B^1_{21,a} = 0;$$
$$B^2_{1a,2} = \mu\omega_{1a}(E_2) = -\mu\omega_{2a}(E_1) = \mu(2\omega_{12} + \theta_{12})(E_a) = B^1_{2a,2} = B^1_{22,a} = -B^1_{11,a}.$$

Based on these information, we use (11) to compute $C^r_{i,j}$ as below:

$$C^1_1 = B^1_{22,1} - B^1_{21,2} = B^1_{22,1},\qquad C^1_2 = B^1_{11,2} - B^1_{12,1} = B^1_{11,2}, \tag{22}$$
$$C^1_1 = B^1_{aa,1} - B^1_{1a,a} = -B^1_{1a,a},\qquad C^1_2 = B^1_{aa,2} - B^1_{2a,a} = -B^1_{2a,a}, \tag{23}$$
$$C^2_1 = B^2_{aa,1} - B^2_{1a,a} = -B^2_{1a,a},\qquad C^2_2 = B^2_{aa,2} - B^2_{2a,a} = -B^2_{2a,a}, \tag{24}$$
$$C^1_a = B^1_{22,a} - B^1_{2a,2} = 0,\qquad C^2_a = B^2_{11,a} - B^2_{1a,1} = 0, \tag{25}$$
$$C^\alpha_1 = B^\alpha_{aa,1} - B^\alpha_{a1,a} = 0,\qquad C^\alpha_2 = B^\alpha_{aa,2} - B^\alpha_{a2,a} = 0, \tag{26}$$
$$C^\alpha_a = B^\alpha_{11,a} - B^\alpha_{1a,1} = B^\alpha_{11,a},\qquad C^\alpha_a = B^\alpha_{22,a} - B^\alpha_{2a,2} = B^\alpha_{22,a}.\ (\forall\ a, \alpha) \tag{27}$$

Utilizing the fact $\sum_i B^\alpha_{ii,k} = 0$, we deduce from (17) that $C^\alpha_a = 0$. By (18), (19) and (22)–(27), the final result is

$$C^1_1 = -C^2_2 = -\omega_{2a}(e_a),\ C^1_2 = C^2_1 = -\omega_{1a}(e_a), \tag{28}$$
$$C^1_a = C^2_a = 0,\ C^\alpha_i = 0. \tag{29}$$

For similar reasons, (26) and (27) imply

$$\theta_{1\alpha}(E_1) - \theta_{2\alpha}(E_2) = B^{\alpha}_{12,1} - B^{\alpha}_{11,2} = -C^{\alpha}_2 = 0,$$
$$\theta_{1\alpha}(E_2) + \theta_{2\alpha}(E_1) = (B^{\alpha}_{21,2} - B^{\alpha}_{22,1}) + (B^{\alpha}_{22,1} + B^{\alpha}_{11,1}) = -C^{\alpha}_1 = 0.$$

We summarize the most important information on the connection 1-forms as below.

Proposition 1. *For a Wintgen ideal submanifold $M^m, m \geq 3$, denote*

$$L_a = -B^1_{11,a},\ V = C^1_2 = C^2_1,\ U = C^2_2 = -C^1_1,\ S_\alpha = B^{\alpha}_{11,2},\ T_\alpha = B^{\alpha}_{11,1}. \tag{30}$$

We can choose a suitable frame $\{E_3, \cdots, E_m\}$ so that $L_a = -B^1_{11,a} = 0$ when $a \geq 4$ and denote $L \triangleq L_3 = -B^1_{11,3}$. Then

$$\omega_{1a} = L_a\omega_2 - V\omega_a,\ \omega_{2a} = -L_a\omega_1 + U\omega_a; \tag{31}$$

$$2\omega_{12} + \theta_{12} = -U\omega_1 - V\omega_2 + L\omega_3; \tag{32}$$

$$\theta_{1\alpha} = S_\alpha\omega_1 - T_\alpha\omega_2,\ \theta_{2\alpha} = T_\alpha\omega_1 + S_\alpha\omega_2. \tag{33}$$

Before discussing the properties of the conformal Gauss map $\Xi = \xi_1 \wedge \cdots \wedge \xi_p$ in the next section, we notice that the subspace $\mathrm{Span}\{\xi_1, \xi_2\}$ also defines a map into the Grassmannian $\mathrm{Gr}(2, \mathbb{R}^{m+p+2}_1)$. This is also represented by $[\xi_1 - \mathrm{i}\xi_2]$ in a complex quadric

$$\mathbb{Q}^{m+p}_+ = \{[Z] \in \mathbb{C}P^{m+p+1} |\ Z \in \mathbb{R}^{m+4}_1 \otimes \mathbb{C}, \langle Z, Z\rangle = 0, \langle Z, \bar{Z}\rangle > 0\}.$$

We denote $\xi = \xi_1 - \mathrm{i}\xi_2$, and call $[\xi]$ *the second Gauss map* of the Wintgen ideal submanifold. When the codimension $p = 2$, $[\xi]$ is equivalent to the conformal Gauss map Ξ. To understand its geometry, substitute (15), (28), (29) and (33) into the last structure equation of (7). The result is

$$\mathrm{d}(\xi_1 - \mathrm{i}\xi_2) = \mathrm{i}(\omega_1 + \mathrm{i}\omega_2)(\eta_1 + \mathrm{i}\eta_2) + \mathrm{i}\theta_{12}(\xi_1 - \mathrm{i}\xi_2) + (\omega_1 - \mathrm{i}\omega_2)\cdot\sum\nolimits_\alpha (S^\alpha - \mathrm{i}T^\alpha)\xi_\alpha, \tag{34}$$

where

$$\eta_1 = Y_1 + C^1_2 Y = Y_1 + VY,\ \eta_2 = Y_2 + C^1_1 Y = Y_2 - UY. \tag{35}$$

This indicate that the image of $[\xi]$ degenerates to a two-dimensional surface, a property also shared by the conformal Gauss map Ξ.

Differentiate once more, the result would be

$$\mathrm{d}(\eta_1 + \mathrm{i}\eta_2) = (\omega_1 + \mathrm{i}\omega_2)\Big[-\tilde{Y} - FY + \big(\frac{G}{L} - \mathrm{i}L\big)\eta_3\Big] - \mathrm{i}\Omega_{12}(\eta_1 + \mathrm{i}\eta_2) + \mathrm{i}(\omega_1 - \mathrm{i}\omega_2)(\xi_1 - \mathrm{i}\xi_2), \tag{36}$$

where $\Omega_{12} = \langle \mathrm{d}\eta_1, \eta_2 \rangle$ is a connection 1-form,

$$F = A_{11} - C^1_{2,1} + \frac{1}{2}\left(U^2 + V^2 - \left(\frac{G}{L}\right)^2\right), \; G = A_{12} - C^1_{2,2} = (C^1_{1,1} - C^1_{2,2})/2; \tag{37}$$

$$\tilde{Y} = N - VY_1 + UY_2 + \frac{G}{L}Y_3 - \frac{1}{2}\left(U^2 + V^2 + \left(\frac{G}{L}\right)^2\right)Y, \; \eta_3 = Y_3 - \frac{G}{L}Y. \tag{38}$$

Note that we have assumed $L \neq 0$ at here. To prove (36), we have used (10) to compute A_{1j}. We omit the straightforward yet tedious computation at here.

4 The Conformal Gauss Map as a Harmonic Map

Proposition 2. *For an umbilic-free Wintgen ideal submanifold $f : M^m \to \mathbb{S}^{m+p}$ of dimension $m \geq 3$, the following three conclusions hold true:*

1. *The image of the conformal Gauss map $\Xi = \xi_1 \wedge \cdots \wedge \xi_p : M^m \to \mathrm{Gr}(p, \mathbb{R}^{m+p+2}_1)$ is a real two-dimensional surface $\overline{M}^2$.*
2. *The projection $\pi : M^m \to \overline{M}^2$ determined by Ξ is a Riemannian submersion (up to the factor $\sqrt{2}$), where M^m is endowed with the Möbius metric and $\overline{M}^2 \subset \mathrm{Gr}(p, \mathbb{R}^{m+p+2}_1)$ with the induced metric.*
3. *The distribution $\mathbb{D}_2^{\perp} = \mathrm{Span}\{E_3, \cdots, E_m\}$ is integrable. Its integral submanifolds are exactly the fibers of the submersion mentioned above.*

Proof. When $p = 2$, these conclusions and Theorem 1 has been proved in [15]. In the general case when $p \geq 3$, we adopt the convention $3 \leq a \leq m, 3 \leq \alpha \leq p$ on the indices. Then it follows from (7) and Proposition 1 that

$$E_1(\Xi) = -[\eta_2 \wedge \xi_2 \wedge * + \xi_1 \wedge \eta_1 \wedge *], \; (* \triangleq \xi_3 \wedge \xi_4 \wedge \cdots \wedge \xi_p) \tag{39}$$

$$E_2(\Xi) = -[\eta_1 \wedge \xi_2 \wedge * - \xi_1 \wedge \eta_2 \wedge *], \tag{40}$$

$$E_a(\Xi) = 0, \; \forall\, 3 \leq a \leq m. \tag{41}$$

Consequently, the tangent space $\Xi_* T_x \overline{M}^2 \subset T_{\Xi(x)}\mathrm{Gr}(p, \mathbb{R}^{m+p+2}_1)$ is a plane given by

$$\mathrm{Span}\{\eta_2 \wedge \xi_2 \wedge * + \xi_1 \wedge \eta_1 \wedge *, \; \eta_1 \wedge \xi_2 \wedge * - \xi_1 \wedge \eta_2 \wedge *\},$$

and the induced metric is $\mathrm{d}s^2 = 2[(\omega_1)^2 + (\omega_2)^2]$. This proves the first two conclusions. In particular the image of Ξ is a two-dimensional surface $\overline{M}^2$.

As the kernel of the tangent map π_*, $\mathbb{D}_2^{\perp}$, the vertical subspace at every point, is always an integrable distribution whose integral submanifolds are nothing but the fibers of this submersion. Conclusion (3) follows immediately (or by the expressions of $\omega_{1\alpha}, \omega_{2\alpha}$ in (31) and the Frobenius Theorem).

(Proof to Theorem 1). According to Proposition 2, we can regard Ξ as a conformal immersion from the Riemann surface $\overline{M}^2$ to $\mathrm{Gr}(p, \mathbb{R}_1^{m+p+2})$. E_1, E_2 can be viewed as horizontal lift of an orthonormal basis (up to the factor $\sqrt{2}$) of $(T\overline{M}^2, \mathrm{d}s^2)$. The second fundamental form of $\Xi(\overline{M}^2)$ can be read out from a straightforward computation as below using the structure equations:

$$
\begin{aligned}
E_1E_1(\Xi) &= 2\Xi + (\Omega_{12} + \theta_{12})(E_1)\,[\xi_1 \wedge \eta_2 \wedge * - \eta_1 \wedge \xi_2 \wedge *] + 2\eta_1 \wedge \eta_2 \wedge * \\
&\quad - L\eta_3 \wedge \xi_2 \wedge * - \xi_1 \wedge \left(\hat{F}Y + \hat{Y} + \frac{G}{L}\eta_3\right) \wedge * + \xi_1 \wedge \xi_2 \cdots \wedge (S_\alpha\eta_2 + T_\alpha\eta_1) \wedge \cdots \xi_p.
\end{aligned}
$$

In the final expression, the first term is the radial component, the second is the tangential component, and the third term can be ignored because it is not in the tangent space $T_\Xi \mathrm{Gr}(p, \mathbb{R}_1^{m+4})$ at $\Xi = \xi_1 \wedge \cdots \wedge \xi_p$. The last three terms are the normal component. Similarly we compute out

$$
\begin{aligned}
E_2E_2(\Xi) &= 2\Xi + (\Omega_{12} + \theta_{12})(E_2)\,[\eta_2 \wedge \xi_2 \wedge * + \xi_1 \wedge \eta_1 \wedge *] + 2\eta_1 \wedge \eta_2 \wedge * \\
&\quad + L\eta_3 \wedge \xi_2 \wedge * + \xi_1 \wedge \left(\hat{F}Y + \hat{Y} + \frac{G}{L}\eta_3\right) \wedge * + \xi_1 \cdots \wedge (-T_\alpha\eta_1 - S_\alpha\eta_2) \wedge \cdots \xi_p.
\end{aligned}
$$

Thus $(E_1E_1 + E_2E_2)\Xi$ has only radial and tangential components. In other words, the mean curvature vector of the surface $\Xi : \overline{M}^2 \subset \mathrm{Gr}(p, \mathbb{R}_1^{m+4})$ vanishes. In the same manner we derive

$$
\begin{aligned}
E_1E_2(\Xi) &= (\Omega_{12} + \theta_{12})(E_1)\,[\eta_2 \wedge \xi_2 \wedge * + \xi_1 \wedge \eta_1 \wedge *] \\
&\quad + L\xi_1 \wedge \eta_3 \wedge * - \left(\hat{F}Y + \hat{Y} + \frac{G}{L}\eta_3\right) \wedge \xi_2 \wedge * + \xi_1 \cdots \wedge (S_\alpha\eta_1 - T_\alpha\eta_2) \wedge \cdots \xi_p\ .
\end{aligned}
$$

Its normal component has the same squared norm as that of $E_1E_1(\Xi)$ and $E_2E_2(\Xi)$. Thus its curvature ellipse is a circle, which is the characteristic of a *super-conformal* surface. So $\Xi : \overline{M}^2 \subset \mathrm{Gr}(p, \mathbb{R}_1^{m+4})$ is a conformal super-minimal immersion.

5 The Spherical Foliation Structure

This section is devoted to the proof of Theorem 2.

By Theorem 1, the mean curvature spheres $\mathrm{Span}\{\xi_1, \cdots, \xi_p\}$ is a two-parameter family, with the parameter space being $\overline{M}^2$. It is well-known that such a sphere congruence has an envelope $\hat{M}^m$ if and only if $V = \mathrm{Span}\{\xi_r, \mathrm{d}\xi_r : 1 \le r \le p\}$

form a space-like sub-bundle of the trivial bundle $\mathbb{R}_1^{m+p+2} \times \overline{M}^2$. This is satisfied in our situation by (34), with $V = \mathrm{Span}\{\xi_r, \eta_1, \eta_2\}$ being a spacelike sub-bundle of rank $p+2$. In particular, the points of the envelope correspond to the lightlike directions in its orthogonal sub-bundle $V^\perp$ over $\overline{M}^2$. By construction, $\hat{M}^m \supset M^m$; in general we would expect it to be a m-dimensional submanifold (possibly with singularities).

We have noticed that the distribution $\mathbb{D}_2^\perp = \mathrm{Span}\{E_3, \cdots, E_m\}$ is integrable; the integral submanifolds are fibers of the Riemannian submersion mentioned before. We assert that each fiber is contained in a $(m-2)$-dimensional sphere determined by the spacelike subspace V at some point $q \in \overline{M}^2$. This is because of (36), which implies that the subspace V is fixed along any integral submanifold of $\mathbb{D}_2^\perp = \mathrm{Span}\{E_3, \cdots, E_m\}$. In particular, the integration of Y along $\mathbb{D}_2^\perp$ is always contained in $V^\perp$, which implies that any integral submanifold is located on the corresponding $(m-2)$-dimensional sphere. This proves the first conclusion of Theorem 2.

Next we introduce a new moving frame $\{Y, \hat{Y}, \eta_1, \eta_2, \eta_a; \xi_r\}$ along M, which is an orthonormal frame except that $Y, \hat{Y}$ are lightlike with $\langle Y, \hat{Y}\rangle = 1$. They are

$$\eta_1 = Y_1 + VY,\ \eta_2 = Y_2 - UY,\ \eta_a = Y_a + \lambda_a Y. \tag{42}$$

Here $\{\lambda_a\}_{a=3}^m$ are real numbers chosen arbitrarily, depending smoothly on the underlying Riemann surface $\overline{M}^2$. By conclusions in the previous paragraph, $\overline{M}^2$ and $\{\lambda_a\}_{a=3}^m$ give a parametrization of $\hat{M}^m$. When $\{\lambda_a\}_{a=3}^m$ vary arbitrarily, the point corresponding to the lightlike direction

$$\hat{Y} = N - \frac{1}{2}(V^2 + U^2 + \sum\nolimits_a \lambda_a^2)Y - VY_1 + UY_2 + \sum\nolimits_a \lambda_a Y_a \tag{43}$$

will travel around the whole envelope $\hat{M}^m$. Thus we may regard $\hat{Y}$ as a local lift of the parameterized submanifold $\hat{M}^m$, and any property of $\hat{M}^m$ can be obtained from $\hat{Y}$ with arbitrarily given $\{\lambda_a\}_{a=3}^m$. This is the key point in our analysis.

We will focus on the regular subset where $\hat{M}^m$ is immersed. Using the new moving frame (42) and (43), there is a new system of structure equations:

$$\mathrm{d}\xi_1 = -\omega_2\eta_1 - \omega_1\eta_2 + \theta_{12}\xi_2, \tag{44}$$

$$\mathrm{d}\xi_2 = -\omega_1\eta_1 + \omega_2\eta_2 - \theta_{12}\xi_1, \tag{45}$$

$$\mathrm{d}\xi_\alpha = -\theta_{1\alpha}\xi_1 - \theta_{2\alpha}\xi_2 + \sum\nolimits_\beta \theta_{\alpha\beta}\xi_\beta, \tag{46}$$

$$\mathrm{d}\eta_1 = -\hat{\omega}_1 Y - \omega_1\hat{Y} + \sum\nolimits_k \Omega_{1k}\eta_k + \omega_2\xi_1 + \omega_1\xi_2, \tag{47}$$

$$\mathrm{d}\eta_2 = -\hat{\omega}_2 Y - \omega_2\hat{Y} + \sum\nolimits_k \Omega_{2k}\eta_k + \omega_1\xi_1 - \omega_2\xi_2, \tag{48}$$

$$d\eta_a = -\hat{\omega}_a Y - \omega_a \hat{Y} + \sum_k \Omega_{ak}\eta_k, \tag{49}$$

$$dY = \omega Y + \omega_1\eta_1 + \omega_2\eta_2 + \sum_a \omega_a\eta_a, \tag{50}$$

$$d\hat{Y} = -\omega\hat{Y} + \hat{\omega}_1\eta_1 + \hat{\omega}_2\eta_2 + \sum_a \hat{\omega}_a\eta_a. \tag{51}$$

Here $\omega, \omega_k, \hat{\omega}_k, \Omega_{jk}$ are 1-forms locally defined on $\hat{M}^m$ which we don't need to know explicitly.

We claim that the envelope $\hat{M}^m$, viewed as an immersion $[\hat{Y}]$ into the sphere, still has $\mathrm{Span}_{\mathbb{R}}\{\xi_1, \xi_2, \cdots, \xi_p\}$ as its mean curvature sphere.

As a preparation, it is important to notice that there exist some functions $\hat{F}, \hat{G}$ such that

$$\hat{\omega}_1 = \hat{F}\omega_1 + \hat{G}\omega_2,\ \hat{\omega}_2 = -\hat{G}\omega_1 + \hat{F}\omega_2. \tag{52}$$

This follows from (36) and (43) directly (or from the integrability conditions of the system (44)–(51)). Based on this, under the induced metric $\langle d\hat{Y}, d\hat{Y}\rangle = \sum_{j=1}^m \hat{\omega}_j^2$ we take a frame $\{\hat{E}_j\}_{j=1}^m$ so that $\hat{\omega}_i(\hat{E}_j) = (\hat{F}^2 + \hat{G}^2)\delta_{ij}$. Since $\hat{M}^m$ is assumed to be immersed, $\hat{F}^2 + \hat{G}^2 \neq 0$. Modulo the components in $\mathbb{D}_2^{\perp} = \mathrm{Span}\{E_3, \cdots, E_m\}$ one gets

$$\hat{E}_1 \approx \hat{F}\hat{E}_1 + \hat{G}\hat{E}_2,\ \hat{E}_2 \approx -\hat{G}\hat{E}_1 + \hat{F}\hat{E}_2,\ \hat{E}_a \approx 0 \ (\mathrm{mod}\ \mathbb{D}_2^{\perp}). \tag{53}$$

Next we compute the Laplacian $\hat{\Delta}\hat{Y}$. The mean curvature sphere at $\hat{Y}$ is determined by

$$\mathrm{Span}_{\mathbb{R}}\{\hat{Y}, \hat{Y}_j, \sum_{j=1}^m \hat{E}_j\hat{E}_j(\hat{Y})\} = \mathrm{Span}_{\mathbb{R}}\{\hat{Y}, \hat{Y}_j, \hat{\Delta}\hat{Y}\}.$$

To verify our claim, it suffices to show $\langle \sum_{j=1}^m \hat{E}_j\hat{E}_j(\hat{Y}), \xi_r\rangle = 0$. Because $\langle \hat{Y}, \xi_r\rangle = 0 = \langle d\hat{Y}, \xi_r\rangle = \langle \hat{Y}, d\xi_r\rangle$, this is also equivalent to

$$\langle \hat{Y}, \sum_{j=1}^m \hat{E}_j\hat{E}_j(\xi_r)\rangle = 0,\ 1 \le r \le p.$$

This can be checked directly using (53) and (44)–(48). As a consequence, the previous claim is proved.

Finally, for $\hat{Y}$ we take its canonical lift, whose derivatives are clearly combinations of $\hat{Y}, \eta_1, \eta_2, \eta_a$. Its normal frame is just $\{\xi_1, \xi_2\}$ as we have shown. One reads from (44) and (45) that this is still a Wintgen ideal submanifold, which finishes the proof.

6 Special Classes of Wintgen Ideal Submanifolds

This section reviews our recent work on Wintgen ideal submanifolds from a unified viewpoint of the conformal Gauss map Ξ and the fiber bundle structure over $\overline{M}^2$. In the codimension two case we have the following result [15], where Ξ can be identified with the second Gauss map $[\xi]$ from the Riemann surface $\overline{M}^2$. The theorem below is stronger than Theorem 1 by replacing *harmonic map* by *holomorphic map*. It also supplement Theorem 2 by showing the converse is also true.

Theorem 3 ([15]). *The conformal Gauss map $[\xi] = [\xi_1 - i\xi_2] \in \mathbb{Q}_+^{m+2}$ of a Wintgen ideal submanifold of codimension two is a holomorphic and 1-isotropic curve, i.e., with respect to a local complex coordinate z of $\overline{M}^2$, $\xi_{\bar{z}} \parallel \xi$, $\langle \xi_z, \xi_z \rangle = 0$. Conversely, given a holomorphic 1-isotropic curve $[\xi] : \overline{M}^2 \to \mathbb{Q}_+^{m+2} \subset \mathbb{C}P^{m+3}$, the envelope $\hat{M}^m$ of the corresponding two-parameter family spheres is a m-dimensional Wintgen ideal submanifold (at the regular points).*

Remark 4. Dajczer et. al. [8] have shown that codimension two Wintgen ideal submanifolds can always be constructed from Euclidean minimal surfaces. Our description is equivalent to theirs by a complex stereographic projection from $\mathbb{Q}_+^{m+2}$ to the complex space $\mathbb{C}^{m+2} = \mathbb{R}^{m+2} \otimes \mathbb{C}$, which maps holomorphic 1-isotropic curves in one space to holomorphic 1-isotropic curves in the other space.

Consider the canonical distribution $\mathbb{D}_2 = \mathrm{Span}\{E_1, E_2\}$. In the Riemannian submersion structure $\pi : \hat{M}^m \to \overline{M}^2$, it can be viewed as the horizontal lift (at various points) of the tangent plane $T\overline{M}^2$. By Proposition 1, $\mathbb{D}_2$ is integrable if and only if $L = 0$. This is the geometric meaning of the invariant $L = -B^1_{11,3}$ for a Wintgen ideal submanifold. In general we may consider the integrable distribution generated by $\mathbb{D}_2$ with the lowest dimension k and denote it as $\mathbb{D}$. Related with the case $k < m$ we have the following conjecture, which has been proved for $k = 2$ [14] and for $k = 3, 4, 5$ (not published).

Conjecture 1. Let $x : M^m \longrightarrow \mathbb{R}^{m+p}$ be a Wintgen ideal submanifold without umbilic points. If the canonical distribution $\mathbb{D}_2$ generates an integrable distribution $\mathbb{D}$ with dimension $k < m$, then locally x is Möbius equivalent to a cone (res. a cylinder; a rotational submanifold) over a k-dimensional minimal Wintgen ideal submanifold in $\mathbb{S}^{k+p}$ (res. in $\mathbb{R}^{k+p}$; in $\mathbb{H}^{k+p}$.)

In our attempts to prove this *reduction conjecture* for Wintgen ideal submanifolds with a low dimensional ($\dim(\mathbb{D}) = k$ is fixed) integrable distribution $\mathbb{D}$, we notice that it is possible to choose a new frame $\{Y, \hat{Y}, \eta_1, \eta_2, \eta_a\}$ with similar expressions as (42) and (43) (some kind of *gauge transformation*), which helps to find a decomposition of $\mathbb{R}_1^{m+p+2}$ into invariant subspaces [14]. Moreover, the integrability of $\mathbb{D}$ implies that the Lorentz plane bundle $\mathrm{Span}\{Y, \hat{Y}\}$ is flat, i.e., the connection 1-form $\omega = \mathrm{d}Y \cdot \hat{Y}$ is closed. Another conclusion is that the correspondence $[Y] \leftrightarrow [\hat{Y}]$ is a conformal map from $\hat{M}^m$ to itself. We strongly believe that these facts are always true for arbitrary $k \geq 2$.

In all cases we know, ω is a well-defined Möbius invariant whose explicit expression depends on k. For example, when $k = 3$, $\omega = -C_2^1\omega_1 - C_1^1\omega_2 + \frac{E_3(L)}{L}\omega_3$ [21].

A natural question arises: for a fixed k and Wintgen ideal submanifolds of dimension $m = k$ which are *irreducible (i.e., the only integrable distribution containing* $\mathbb{D}_2$ *is the tangent bundle of* M*)*, what is the meaning of $d\omega = 0$? We conjecture the following characterization result, which has been proved for the case $m = 3$, $p = 2$ [21] and the general three-dimensional case (to appear later).

Conjecture 2. For an irreducible Wintgen ideal submanifold M^k of dimension $k \geq 3$, if $d\omega = 0$, then M^k is Möbius equivalent to a minimal Wintgen ideal submanifold in either of the three space forms.

A main difficulty in proving these two conjectures for arbitrary dimension k is that when k changes we have to modify the frame $\{Y, \hat{Y}, \eta_1, \eta_2, \eta_a\}$ as well as the expression ω accordingly, and a unified treatment is still lacking.

Finally, we mention that under the condition of being Möbius homogeneous, Wintgen ideal submanifolds could be classified [16]. It is interesting to note that for a Möbius homogeneous Wintgen ideal submanifold M, the Möbius form must vanish, and M can always be reduced to two or three dimensional minimal examples in the sense of Conjecture 1. Proving these facts are the key steps in obtaining the final classification in [16]. (Later we notice other three-dimensional irreducible examples not contained in our classification results. Thus we will re-write the proof in [16] and fix this problem.)

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Floer Homology for the Gelfand-Cetlin System

Yuichi Nohara and Kazushi Ueda

Abstract The Gelfand-Cetlin system is a completely integrable system on a flag manifold of type A. In contrast to the case of toric moment maps, the Gelfand-Cetlin system has non-torus Lagrangian fibers on some boundary strata of the momentum polytope. In this paper we discuss Lagrangian intersection Floer theory for torus and non-torus Lagrangian fibers of the Gelfand-Cetlin system on the three-dimensional full flag manifold and the Grassmannian of two-planes in a four-dimensional vector space.

1 Introduction

Let $F = GL(n, \mathbb{C})/P$ be a (full or partial) flag manifold. The *Gelfand-Cetlin system* is a completely integrable system

$$\Phi : F \longrightarrow \mathbb{R}^{(\dim_{\mathbb{R}} F)/2}$$

on F, i.e., a set of functionally independent and Poisson commuting functions, which is introduced by Guillemin and Sternberg [11] as a symplectic geometric analogue of Gelfand-Cetlin basis [9]. The image $\Delta = \Phi(F)$ is a convex polytope, which we call the *Gelfand-Cetlin polytope*, and Φ gives a Lagrangian torus fibration structure over the interior $\operatorname{Int}\Delta$ of Δ. Because of non-smoothness of Φ, it has non-torus fibers on some faces of codim ≥ 3. In this paper we study Lagrangian intersection Floer theory for Lagrangian torus and non-torus fiber of the Gelfand-Cetlin system.

Lagrangian intersection Floer theory for torus orbits in a toric manifold has been developed by Fukaya et al. [8]. We recall some of the results which are

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relevant to this paper. Let (X, ω) be a compact toric manifold of $\dim_{\mathbb{C}} X = N$, and $\Phi : X \to \mathbb{R}^N$ be the toric moment map with moment polytope $\Delta = \Phi(X)$. For an interior point $\boldsymbol{u} \in \operatorname{Int} \Delta$, let $L(\boldsymbol{u})$ denote the Lagrangian torus fiber $\Phi^{-1}(\boldsymbol{u})$.

- The potential function $\mathfrak{PO}$ of Lagrangian torus fibers is defined as a function on

$$\bigcup_{\boldsymbol{u} \in \operatorname{Int} \Delta} H^1(L(\boldsymbol{u}); \Lambda_0 / 2\pi\sqrt{-1}\mathbb{Z}) \cong \operatorname{Int} \Delta \times (\Lambda_0 / 2\pi\sqrt{-1}\mathbb{Z})^N,$$

  where Λ_0 is the Novikov ring. In the Fano case, $\mathfrak{PO}$ can be regarded as a Laurent polynomial, and it coincides with the superpotential of the Landau-Ginzburg mirror of X.
- Each critical point of $\mathfrak{PO}$ corresponds to a pair $(L(\boldsymbol{u}), b)$ of a fiber $L(\boldsymbol{u})$ and $b \in H^1(L(\boldsymbol{u}); \Lambda_0 / 2\pi\sqrt{-1}\mathbb{Z})$ with nontrivial Floer homology.
- The quantum cohomology of X is isomorphic to the Jacobi ring $\operatorname{Jac}(\mathfrak{PO})$ of the potential function.

See [8] or [7] for more detail. In particular, the number of critical points of $\mathfrak{PO}$ is equal to the rank of the cohomology group $H^*(X)$ of X, provided that $\mathfrak{PO}$ is a Morse function.

In the case of Gelfand-Cetlin system, Nishinou and we [12] compute the potential function of Lagrangian torus fibers by using a toric degeneration of the flag manifold, and show that it coincides with the superpotential of the Landau-Ginzburg mirror of the flag manifold [1,10]. In contrast to the toric case, the number of critical points of the potential function, and hence the number of Lagrangian torus fibers with nontrivial Floer homology, is smaller than the rank of $H^*(F)$ in general. Eguchi et al. [3] and Rietsch [13] consider a partial compactification of the mirror of F to get as many critical points of the superpotential as rank $H^*(F)$. It is natural to expect that the critical points at "infinity" correspond to Lagrangian fibers on the boundary of the Gelfand-Cetlin polytope. In this paper, we study Floer homology of such non-torus fibers in the three-dimensional flag manifold Fl(3) and the Grassmannian Gr(2, 4) of two-planes in $\mathbb{C}^4$.

This paper is organized as follows. In Sect. 2 we recall the construction of the Gelfand-Cetlin system and see non-torus Lagrangian fibers in Fl(3) and Gr(2, 4). In Sect. 3 we study the potential function for the Gelfand-Cetlin system. The computation of the Floer homologies of non-torus Lagrangian fibers in Fl(3) and Gr(2, 4) is given in Sect. 4.

2 Gelfand-Cetlin System

Fix a sequence $0 = n_0 < n_1 < \cdots < n_r < n_{r+1} = n$ of integers, and set $k_i = n_i - n_{i-1}$ for $i = 1, \dots, r+1$. The flag manifold $F = F(n_1, \dots, n_r, n)$ is defined by

$$F = U(n)/(U(k_1) \times \cdots \times U(k_{r+1})).$$

Let $\mathrm{Fl}(n) := F(1, 2, \ldots, n)$ and $\mathrm{Gr}(k, n) := F(k, n)$ denote the full flag manifold and the Grassmannian of k-planes in $\mathbb{C}^n$, respectively. The dimension of $F(n_1, \ldots, n_r, n)$ is given by

$$N = N(n_1, \ldots, n_r, n) := \dim_{\mathbb{C}} F(n_1, \ldots, n_r, n) = \sum_{i=1}^{r} (n_i - n_{i-1})(n - n_i).$$

We identify the dual $\mathfrak{u}(n)^*$ of the Lie algebra $\mathfrak{u}(n)$ of $U(n)$ with the space $\sqrt{-1}\mathfrak{u}(n)$ of Hermitian matrices by using an invariant inner product. Then F is identified with the adjoint orbit $\mathscr{O}_\lambda \subset \sqrt{-1}\mathfrak{u}(n)$ of a diagonal matrix $\lambda = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$ with

$$\underbrace{\lambda_1 = \cdots = \lambda_{n_1}}_{k_1} > \underbrace{\lambda_{n_1+1} = \cdots = \lambda_{n_2}}_{k_2} > \cdots > \underbrace{\lambda_{n_r+1} = \cdots = \lambda_n}_{k_{r+1}}.$$

Note that $\mathscr{O}_\lambda$ consists of Hermitian matrices with fixed eigenvalues $\lambda_1, \ldots, \lambda_n$. Let ω be the Kostant-Kirillov form on $\mathscr{O}_\lambda$.

For $x \in \mathscr{O}_\lambda$ and $k = 1, \ldots, n-1$, let $x^{(k)}$ denote the upper-left $k \times k$ submatrix of x. Since $x^{(k)}$ is also a Hermitian matrix, it has real eigenvalues $\lambda_1^{(k)}(x) \geq \lambda_2^{(k)}(x) \geq \cdots \geq \lambda_k^{(k)}(x)$. By taking the eigenvalues for all $k = 1, \ldots, n-1$, we obtain a set of $n(n-1)/2$ functions $(\lambda_i^{(k)})_{1 \leq i \leq k \leq n-1}$. Since the eigenvalues satisfy the following inequalities

$$\begin{array}{ccccccccccc}
\lambda_1 & & \lambda_2 & & \lambda_3 & \cdots & \lambda_{n-1} & & \lambda_n \\
 \geq & & \leq \;\; \geq & & \leq & & & \geq & & \leq \\
 & \lambda_1^{(n-1)} & & \lambda_2^{(n-1)} & & & & \lambda_{n-1}^{(n-1)} & \\
 & \geq & & \leq & & & & \leq & \\
 & & \lambda_1^{(n-2)} & & & & \lambda_{n-2}^{(n-2)} & & \\
 & & \geq & & & & \leq & & \\
 & & & \ddots & & \iddots & & & \\
 & & & \geq & & \leq & & & \\
 & & & & \lambda_1^{(1)} & & & &
\end{array}, \tag{1}$$

some of $\lambda_i^{(k)}$ are constant functions if F is not a full flag manifold. It is easy to see that the number of nonconstant $\lambda_i^{(k)}$ coincides with $N = \dim_{\mathbb{C}} F$. The Gelfand-Cetlin system is defined to be the tuple

$$\Phi = (\lambda_i^{(k)})_{i,k} : F(n_1, \ldots, n_r, n) \longrightarrow \mathbb{R}^{N(n_1, \ldots, n_r, n)}$$

of nonconstant $\lambda_i^{(k)}$.

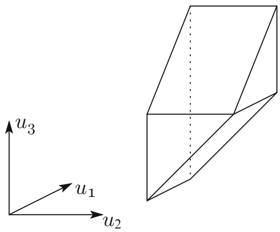

Fig. 1 The Gelfand-Cetlin polytope for Fl(3)

Proposition 2.1 (Guillemin and Sternberg [11]). *The map Φ is a completely integrable system on $(F(n_1, \dots, n_r, n), \omega)$, and the functions $\lambda_i^{(k)}$ are action coordinates. The image $\Delta = \Phi(F)$ is a convex polytope defined by (1), and the fiber $L(\boldsymbol{u}) = \Phi^{-1}(\boldsymbol{u})$ over each interior point $\boldsymbol{u} \in \operatorname{Int} \Delta$ is a Lagrangian torus.*

Example 2.1. The Gelfand-Cetlin polytope for the three-dimensional flag manifold Fl(3) is defined by

$$\begin{array}{ccccccccc} \lambda_1 & & & & \lambda_2 & & & & \lambda_3 \\ & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & \\ & & u_1 & & & & u_2 & & \\ & & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & & & \\ & & & & u_3 & & & & \end{array} .$$

The Gelfand-Cetlin system has a non-torus fiber over the vertex $\boldsymbol{u}_0 = (\lambda_2, \lambda_2, \lambda_2)$, where four edges are intersecting (see Fig. 1). The fiber $L_0 = \Phi^{-1}(\boldsymbol{u}_0)$ is given by

$$L_0 = \left\{ \begin{pmatrix} \lambda_2 & 0 & z_1 \\ 0 & \lambda_2 & z_2 \\ \bar{z}_1 & \bar{z}_2 & \lambda_1 - \lambda_2 + \lambda_3 \end{pmatrix} \in \mathscr{O}_\lambda \,\middle|\, |z_1|^2 + |z_2|^2 = (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3) \right\},$$

which is diffeomorphic to a 3-sphere S^3.

Example 2.2. Next we consider the case of $\operatorname{Gr}(2,4)$. After a translation, we may assume that $\lambda_1 = \lambda_2 = -\lambda_3 = -\lambda_4 = \lambda$ for $\lambda > 0$. Then Δ is given by

$$\begin{array}{ccccccccccccc} \lambda & & & & \lambda & & & & -\lambda & & & & -\lambda \\ & \rotatebox{-45}{$=$} & & \rotatebox{45}{$=$} & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & & \rotatebox{-45}{$=$} & & \rotatebox{45}{$=$} & \\ & & \lambda & & & & u_1 & & & & -\lambda & & \\ & & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & & & \\ & & & & u_2 & & & & u_3 & & & & \\ & & & & & \rotatebox{-45}{$\geq$} & & \rotatebox{45}{$\geq$} & & & & & \\ & & & & & & u_4 & & & & & & \end{array}$$

Figure 2 shows the projection $\Delta \to [-\lambda, \lambda]$, $\boldsymbol{u} = (u_1, u_2, u_3, u_4) \mapsto u_1$.

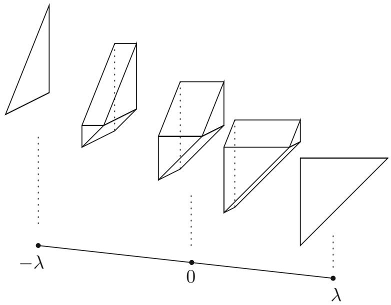

Fig. 2 The Gelfand-Cetlin polytope for Gr(2, 4)

In this case non-torus fibers appear along the edge $u_1 = u_2 = u_3 = u_4$. For $-\lambda < t < \lambda$, the fiber $L_t = \Phi^{-1}(\boldsymbol{u}_t)$ over $\boldsymbol{u}_t = (t, t, t, t)$ is given by

$$L_t = \left\{ \begin{pmatrix} tI_2 & \sqrt{\lambda^2 - t^2}P \\ \sqrt{\lambda^2 - t^2}P^* & (-t)I_2 \end{pmatrix} \in \sqrt{-1}\mathfrak{u}(4) \;\middle|\; P \in U(2) \right\} \cong U(2).$$

3 Potential Functions for Gelfand-Cetlin Systems

Let $\Lambda_0 = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \;\middle|\; a_i \in \mathbb{C},\ \lambda_i \geq 0,\ \lim_{i\to\infty} \lambda_i = \infty \right\}$ be the Novikov ring. The maximal ideal and the quotient field of the local ring Λ_0 will be denoted by Λ_+ and Λ respectively. For a spin and oriented Lagrangian submanifold L in a symplectic manifold (X, ω), one can equip an A_∞-structure

$$\mathfrak{m}_k = \sum_{\beta \in \pi_2(X,L)} T^{\omega(\beta)} \mathfrak{m}_{k,\beta} : H^*(L; \Lambda_0)^{\otimes k} \longrightarrow H^*(L; \Lambda_0)$$

on the cohomology group of L with coefficients in Λ_0 by "counting" pseudo-holomorphic disks [5, Theorem A]. An element b in $H^1(L; \Lambda_+)$ (or $H^1(L; \Lambda_0)$) is called a *weak bounding cochain* if it satisfies the *Maurer-Cartan equation*

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b, \ldots, b) \equiv 0 \mod \mathrm{PD}([L]). \tag{2}$$

The set of weak bounding cochains will be denoted by $\hat{\mathscr{M}}_{\mathrm{weak}}(L)$. For any $b \in \hat{\mathscr{M}}_{\mathrm{weak}}(L)$, one can twist the Floer differential as

$$\mathfrak{m}_1^b(x) = \sum_{k,l} \mathfrak{m}_{k+l+1}(b^{\otimes k} \otimes x \otimes b^{\otimes l}).$$

The Maurer-Cartan equation (2) implies $\mathfrak{m}_1^b \circ \mathfrak{m}_1^b = 0$, and the *Floer homology* of the pair (L, b) is defined by

$$HF((L,b),(L,b);\Lambda_0) = \operatorname{Ker}\mathfrak{m}_1^b / \operatorname{Im}\mathfrak{m}_1^b.$$

The *potential function* $\mathfrak{PO} : \hat{\mathscr{M}}_{\text{weak}}(L) \to \Lambda_0$ is defined by

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b,\dots,b) = \mathfrak{PO}(b) \cdot \mathrm{PD}([L]).$$

Now we consider the Gelfand-Cetlin system $\Phi : F = F(n_1,\dots,n_r,n) \to \Delta$. Take primitive vectors $v_i \in \mathbb{Z}^N$ and $\tau_i \in \mathbb{R}$ so that the Gelfand-Cetlin polytope is given by

$$\Delta = \{\boldsymbol{u} \in \mathbb{R}^N \mid \ell_i(\boldsymbol{u}) = \langle v_i, \boldsymbol{u}\rangle - \tau_i \geq 0,\ i = 1,\dots,m\},$$

where m is the number of codimension one faces of Δ. Note that ℓ_i has the form $\ell_i(u) = u_j - u_k$ or $\ell_i(u) = \pm(u_j - \lambda_k)$. Let $\{\theta_i^{(k)}\}_{i,k}$ be the angle coordinates dual to the action coordinates $\{\lambda_i^{(k)}\}_{i,k}$. For each interior point $\boldsymbol{u} \in \operatorname{Int}\Delta$, we will identify $H^1(L(\boldsymbol{u});\Lambda_0)$ with Λ_0^N by

$$\sum_{i,k} x_i^{(k)} d\theta_i^{(k)} \in H^1(L(\boldsymbol{u});\Lambda_0) \longleftrightarrow x = \left(x_i^{(k)}\right)_{i,k} \in \Lambda_0^N.$$

The following theorem is a Gelfand-Cetlin analogue of [2, Section 15] and [6, Proposition 3.2 and Theorem 3.4].

Theorem 3.1 ([12, Theorem 10.1]). *For any interior point $\boldsymbol{u} \in \operatorname{Int}\Delta$, we have an inclusion $H^1(L(\boldsymbol{u});\Lambda_0) \subset \hat{\mathscr{M}}_{\text{weak}}(L(\boldsymbol{u}))$, and the potential function on $H^1(L(\boldsymbol{u});\Lambda_0) \cong \Lambda_0^N$ is given by*

$$\mathfrak{PO}(x) = \sum_{i=1}^{m} e^{\langle v_i, x\rangle} T^{\ell_i(\boldsymbol{u})}.$$

After the coordinate change

$$y_k = e^{x_k} T^{u_k}, \qquad k = 1,\dots,N,$$
$$Q_j = T^{\lambda_{n_j}}, \qquad j = 1,\dots,r+1,$$

the potential function can be regarded as a Laurent polynomial in $y_1,\dots,y_N$ with coefficients in $\mathbb{Q}[Q_1^{\pm 1},\dots,Q_{r+1}^{\pm 1}]$.

Example 3.1. In the case of three-dimensional flag manifold Fl(3), the potential function is given by

$$\begin{aligned}\mathfrak{PO} &= e^{-x_1}T^{-u_1+\lambda_1} + e^{x_1}T^{u_1-\lambda_2} + e^{-x_2}T^{-u_2+\lambda_2} \\ &\quad + e^{x_2}T^{u_2-\lambda_3} + e^{x_1-x_3}T^{u_1-u_3} + e^{-x_2+x_3}T^{-u_2+u_3} \\ &= \frac{Q_1}{y_1} + \frac{y_1}{Q_2} + \frac{Q_2}{y_2} + \frac{y_2}{Q_3} + \frac{y_1}{y_3} + \frac{y_3}{y_2}.\end{aligned}$$

Critical points of $\mathfrak{PO}$ are given by

$$\begin{aligned}y_1 &= y_3^2/y_2, \\ y_2 &= \pm\sqrt{Q_3(y_3+Q_2)}, \\ y_3 &= \sqrt[3]{Q_1Q_2Q_3},\ e^{2\pi\sqrt{-1}/3}\sqrt[3]{Q_1Q_2Q_3},\ e^{4\pi\sqrt{-1}/3}\sqrt[3]{Q_1Q_2Q_3}.\end{aligned}$$

It is easy to see that all critical points are nondegenerate and have the same valuation which lies in the interior of the Gelfand-Cetlin polytope. Hence we have as many critical point as $\dim H^*(\mathrm{Fl}(3)) = 6$ in this case.

Example 3.2. Next we discuss the case of $\mathrm{Gr}(2,4)$, where $\lambda_1 = \lambda_2 > \lambda_3 = \lambda_4$. The potential function is given by

$$\begin{aligned}\mathfrak{PO} &= e^{-x_2}T^{-u_2+\lambda_1} + e^{-x_1+x_2}T^{-u_1+u_2} + e^{x_1-x_3}T^{u_1-u_3} \\ &\quad + e^{x_3}T^{u_3-\lambda_3} + e^{x_2-x_4}T^{u_2-u_4} + e^{-x_3+x_4}T^{-u_3+u_4} \\ &= \frac{Q_1}{y_2} + \frac{y_2}{y_1} + \frac{y_1}{y_3} + \frac{y_3}{Q_3} + \frac{y_2}{y_4} + \frac{y_4}{y_3},\end{aligned}$$

whose critical points are given by

$$y_1 = y_4 = \pm\sqrt{Q_1Q_3}, \quad y_2 = Q_1Q_3/y_3, \quad y_3 = \pm\sqrt{2Q_3y_1}.$$

These four critical points are non-degenerate and have a common valuation in the interior of the Gelfand-Cetlin polytope. Since $\dim H^*(\mathrm{Gr}(2,4)) = 6$, one has less critical point than $\dim H^*(\mathrm{Gr}(2,4))$.

4 Floer Homologies of Non-torus Fibers

In this section we discuss Floer homologies of non-torus Lagrangian fibers in Fl(3) and $\mathrm{Gr}(2,4)$. Detailed proofs of the results in this section will appear elsewhere.

4.1 Floer Homology of Lagrangian S^3 in Fl(3)

Recall that $\pi_2(\mathrm{Fl}(3)) \cong \mathbb{Z}^2$ is generated by one-dimensional Schubert varieties X_1, X_2. Since the fiber L_0 is diffeomorphic to S^3, the exact homotopy sequence yields $\pi_2(\mathrm{Fl}(3), L_0) \cong \pi_2(\mathrm{Fl}(3)) \cong \mathbb{Z}^2$. Let β_1, β_2 be generators of $\pi_2(\mathrm{Fl}(3), L_0)$ corresponding to X_1 and X_2, respectively. The Maslov index and the symplectic area of β_i are given by

$$\mu_{L_0}(\beta_1) = \mu_{L_0}(\beta_2) = 4, \quad \omega(\beta_1) = \lambda_1 - \lambda_2, \quad \omega(\beta_2) = \lambda_2 - \lambda_3.$$

Theorem 4.1. *The Floer homology of L_0 over the Novikov ring Λ_0 is*

$$HF(L_0, L_0; \Lambda_0) \cong \Lambda_0 / T^{\min\{\lambda_1-\lambda_2, \lambda_2-\lambda_3\}} \Lambda_0.$$

Hence the Floer homology over the Novikov field Λ is trivial: $HF(L_0, L_0; \Lambda) = 0$.

Sketch of proof. Since the minimal Maslov number is four, the only nontrivial parts of the Floer differential are

$$\mathfrak{m}_{1,\beta_i} : H^3(L_0; \Lambda_0) \cong H_0(L_0; \Lambda_0) \longrightarrow H^0(L_0; \Lambda_0) \cong H_3(L_0; \Lambda_0)$$

for $i = 1, 2$.

Lemma 4.1. *For each $p_0 \neq p_1 \in L_0$ and β_i, there exists a holomorphic disk $v : (D^2, \partial D^2) \to (\mathrm{Fl}(3), L_0)$ such that $v(1) = p_0$, $v(-1) = p_1$, and $[v] = \beta_i$. Such v is unique up to the action of $\{g \in \mathrm{Aut}(D^2) \mid g(1) = 1,\ g(-1) = -1\}$.*

Let J be the standard complex structure on Fl(3). Since $(\mathrm{Fl}(3), L_0)$ is $SU(2)$-homogeneous in the sense of Evans and Lekili [4, Definition 1.1.1], the result [4, Proposition 3.2.1] implies that any J-holomorphic disk in $(\mathrm{Fl}(3), L_0)$ is Fredholm regular. Hence Lemma 4.1 implies the following.

Lemma 4.2. *The moduli space $\mathscr{M}_2(J, \beta_i)$ of J-holomorphic disks in the class β_i with two boundary marked points is a smooth manifold of dimension 6, and the evaluation map* $\mathrm{ev} = (\mathrm{ev}_0, \mathrm{ev}_1) : \mathscr{M}_2(J, \beta_i) \to L_0 \times L_0$ *is generically one-to-one.*

Then for the generator $[p] \in H_0(L_0; \mathbb{Z})$ we have

$$\mathfrak{m}_{1,\beta_1}([p]) = \mathfrak{m}_{1,\beta_2}([p]) = \mathrm{ev}_{0*}[\mathscr{M}_2(J, \beta_1)_{\mathrm{ev}_1} \times \{p\}] = [L_0],$$

and thus

$$\mathfrak{m}_1([p]) = \sum_{i=1}^{2} T^{\omega(\beta_i)} \mathfrak{m}_{1,\beta_i}([p]) = (T^{\lambda_1-\lambda_2} + T^{\lambda_2-\lambda_3})[L_0],$$

which proves the theorem.

4.2 Floer Homologies of $U(2)$-fibers in $\mathrm{Gr}(2,4)$

Assume that $\lambda_1 = \lambda_2 = -\lambda_3 = -\lambda_4 = \lambda > 0$, and set $L_t = \Phi^{-1}(t,t,t,t)$ as in Example 2.2. Recall that $\pi_2(\mathrm{Gr}(2,4)) \cong \mathbb{Z}$ is generated by a one-dimensional Schubert variety X_1. Since $\pi_1(\mathrm{Gr}(2,4)) = \pi_2(L_t) = 0$ and $\pi_1(L_t) \cong \mathbb{Z}$, the exact sequence

$$0 \longrightarrow \pi_2(\mathrm{Gr}(2,4)) \longrightarrow \pi_2(\mathrm{Gr}(2,4), L_t) \longrightarrow \pi_1(L_t) \longrightarrow 0$$

implies that $\pi_2(\mathrm{Gr}(2,4), L_t) \cong \mathbb{Z}^2$. Let β_1, β_2 be generators of $\pi_2(\mathrm{Gr}(2,4), L_t)$ such that $\beta_1 + \beta_2 = [X_1] \in \pi_2(\mathrm{Gr}(2,4))$. The Maslov index and the symplectic area are given by

$$\mu_{L_t}(\beta_1) = \mu_{L_t}(\beta_2) = 4, \quad \omega(\beta_1) = \lambda + t, \quad \omega(\beta_2) = \lambda - t.$$

Since L_t is diffeomorphic to $U(2) \cong S^1 \times S^3$, we have $H^*(L_t) \cong H^*(S^1) \otimes H^*(S^3)$. Let $e_1 \in H^1(L_t;\mathbb{Z}) \cong H^1(S^1;\mathbb{Z})$ and $e_3 \in H^3(L_t;\mathbb{Z}) \cong H^3(S^3;\mathbb{Z})$ be generators. Since the minimal Maslov number is four, the only nontrivial parts of the Floer differential $\mathfrak{m}_1^b$ are

$$\mathfrak{m}_{1,\beta_i}^b : H^4(L_t;\Lambda_0) \longrightarrow H^1(L_t;\Lambda_0), \quad H^3(L_t;\Lambda_0) \longrightarrow H^0(L_t;\Lambda_0) \cong \Lambda_0$$

for $i = 1, 2$. By a similar argument to the proof of Theorem 4.1, we have the following.

Theorem 4.2. *For $b = xe_1 \in H^1(L_0;\Lambda_0/2\pi\sqrt{-1}\mathbb{Z}) \cong \Lambda_0/2\pi\sqrt{-1}\mathbb{Z}$, the Floer differential $\mathfrak{m}_1^b$ is given by*

$$\mathfrak{m}_1^b(e_3) = e^x T^{\lambda+t} + e^{-x} T^{\lambda-t},$$
$$\mathfrak{m}_1^b(e_1 \otimes e_3) = (e^x T^{\lambda+t} + e^{-x} T^{\lambda-t}) e_1.$$

Hence the Floer homologies of (L_t, b) are

$$HF((L_t,b),(L_t,b);\Lambda_0) \cong \begin{cases} H^*(L_0;\Lambda_0) & \text{if } t=0 \text{ and } x = \pm\pi\sqrt{-1}/2, \\ (\Lambda_0/T^{\min\{\lambda-t,\lambda+t\}}\Lambda_0)^2 & \text{otherwise}, \end{cases}$$

$$HF((L_t,b),(L_t,b);\Lambda) \cong \begin{cases} H^*(L_0;\Lambda) & \text{if } t = 0 \text{ and } x = \pm\pi\sqrt{-1}/2, \\ 0 & \text{otherwise}. \end{cases}$$

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The Regularized Mean Curvature Flow for Invariant Hypersurfaces in a Hilbert Space

Naoyuki Koike

Abstract In this note, I state some results for the regularized mean curvature flow starting from invariant hypersurfaces in a Hilbert space equipped with an isometric almost free Hilbert Lie group action whose orbits are minimal regularizable submanifolds. First we derive the evolution equations for some geometric quantities along this flow. Some of the evolution equations are described by using the O'Neill fundamental tensor of the orbit map of the Hilbert Lie group action, where we note that the O'Neill fundamental tensor implies the obstruction for the integrability of the horizontal distribution of the orbit map. Next, by using the evolution equations, we derive some results for this flow. Furthermore, we derive some results for the mean curvature flow starting from compact Riemannian suborbifolds in the orbit space (which is a Riemannian orbifold) of the Hilbert Lie group action.

1 Introduction

Hamilton [3] proved the existenceness and the uniqueness (in short time) of solutions satisfying any initial condition of a weakly parabolic equation for sections of a finite dimensional vector bundle. The Ricci flow equation for Riemannian metrics on a fixed compact manifold M is a weakly parabolic equation, where we note that the Riemannian metrics are sections of the $(0,2)$-tensor bundle $T^{(0,2)}M$ of M. Let f_t $(0 \leq t < T)$ be a C^∞-family of immersions of M into the m-dimensional Euclidean space $\mathbb{R}^m$. Define a map $F : M \times [0,T) \to \mathbb{R}^m$ by $F(x,t) := f_t(x)$ $((x,t) \in M \times [0,T))$. The mean curvature flow equation is described as

$$\frac{\partial F}{\partial t} = \triangle_t f_t,$$

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where $\triangle_t$ is the Laplacian operator of the metric g_t on M induced from the Euclidean metric of $\mathbb{R}^m$ by f_t. Here we note that $\triangle_t f_t$ is equal to the mean curvature vector of f_t. This evolution equation also is a weakly parabolic equation, where we note that the immersions f_t's are regarded as sections of the trivial bundle $M \times \mathbb{R}^m$ over M under the identification of f_t and its graph immersion $\mathrm{id}_M \times f : M \to M \times \mathbb{R}^m$ (id_M : the identity map of M). Hence we can apply the Hamilton's result to this evolution equation and hence can show the existenceness and the uniqueness (in short time) of solution of this evolution equation satisfying any initial condition. In this paper, we consider the case where the ambient space is a (separable infinite dimensional) Hilbert space V. Let M be a Hilbert manifold and f_t $(0 \le t < T)$ be a C^∞-family of immersions of M into V. Assume that f_t is regularizable, where "regularizability" means that f_t is proper Fredholm and that, for each normal vector v of M, the regularized trace $\mathrm{Tr}_r\, A^t_v$ of the shape operator A^t_v of f_t and the trace $\mathrm{Tr}\,(A^t_v)^2$ of $(A^t_v)^2$ exist. Denote by H_t the regularized mean curvature vector of f_t. See the next section about the definitions of $\mathrm{Tr}_r\, A^t_v$ and H_t. Define a map $F : M \times [0, T) \to V$ as above in terms of f_t's. We call f_t's $(0 \le t < T)$ the *regularized mean curvature flow* if the following evolution equation holds:

$$\frac{\partial F}{\partial t} = \triangle^r_t f_t. \tag{1}$$

Here $\triangle^r_t f_t$ is defined as the vector field along f_t satisfying

$$\langle \triangle^r_t f_t, v\rangle := \mathrm{Tr}_r \langle (\nabla^t df_t)(\cdot,\cdot), v\rangle^\sharp \quad (\forall\, v \in V),$$

where ∇^t is the Riemannian connection of the metric g_t on M induced from the metric $\langle\ ,\ \rangle$ of V by f_t, $\langle (\nabla^t df_t)(\cdot,\cdot), v\rangle^\sharp$ is the $(1,1)$-tensor field on M defined by $g_t(\langle (\nabla^t df_t)(\cdot,\cdot), v\rangle^\sharp(X), Y) = \langle (\nabla^t df_t)(X, Y), v\rangle$ $(X, Y \in TM)$ and $\mathrm{Tr}_r(\cdot)$ is the regularized trace of $(\cdot)$. Note that $\triangle^r_t f_t$ is equal to H_t. Also, we note that the regularized mean curvature flow was used in the investigation of the mean curvature flow starting from equifocal submanifolds in a symmetric space of compact type (see [7]). In general, the existenceness and the uniqueness (in short time) of solutions of this evolution equation satisfying any initial condition has not been shown yet. For we cannot apply the Hamilton's result to this evolution equation because it is regarded as the evolution equation for sections of the *infinite* dimensional vector bundle $M \times V$ over M. However we can show the existenceness and the uniqueness (in short time) of solutions of this evolution equation in special case. In this paper, we consider a isometric almost free action of a Hilbert Lie group G on a Hilbert space V whose orbits are regularized minimal, that is, they are regularizable submanifold and their regularized mean curvature vectors vanish, where "almost free" means that the isotropy group of the action at each point is discrete. Let $M(\subset V)$ be a G-invariant submanifold in V. Assume that the image of M by the orbit map of the G-action is compact. Let f be the inclusion map of M into V. We first show that the regularized mean curvature flow starting from M exists uniquely in short time (see Proposition 4.1). In particular, we consider the case where M is a hypersurface. The first purpose of this paper is to obtain the evolution equations for various geometrical quantities along the regularized mean curvature flow starting from G-invariant hypersurfaces (see Sect. 4). The second purpose is to prove a horizontally strongly convexity preservability theorem for the regularized mean curvature flow starting from the above invariant hypersurface by using the evolution equations in Sect. 4 and using the maximum principle (see Sect. 5). From this theorem, we derive the strongly convexity preservability theorem for the mean curvature flow starting from compact Riemannian suborbifolds in the orbit space V/G (which is a Riemannian orbifold) (see Sect. 6).

2 The Regularized Mean Curvature Flow

Let f_t $(0 \le t < T)$ be a one-parameter C^∞-family of immersions of a manifold M into a (finite dimensional) Riemannian manifold N, where T is a positive constant or $T = \infty$. Denote by H_t the mean curvature vector of f_t. Define a map $F : M \times [0, T) \to N$ by $F(x, t) = f_t(x)$ $((x, t) \in M \times [0, T))$. If, for each $t \in [0, T)$, $\frac{\partial F}{\partial t} = H_t$ holds, then f_t $(0 \le t < T)$ is called a *mean curvature flow*.

Let f be an immersion of an (infinite dimensional) Hilbert manifold M into a Hilbert space V and A the shape tensor of f. If $\mathrm{codim}\, M < \infty$, if the differential of the normal exponential map $\exp^\perp$ of f at each point of M is a Fredholm operator and if the restriction $\exp^\perp$ to the unit normal ball bundle of f is proper, then M is called a *proper Fredholm submanifold*. Then each shape operator A_v is a compact operator. In 1989, this notion was introduced by Terng [11]. Furthermore, if, for each normal vector v of f, the regularized trace $\mathrm{Tr}_r\, A_v$ and $\mathrm{Tr}\, A_v^2$ exist, then M is called *regularizable submanifold*, where $\mathrm{Tr}_r\, A_v$ is defined by $\mathrm{Tr}_r\, A_v := \sum_{i=1}^{\infty}(\mu_i^+ + \mu_i^-)$ $(\mu_1^- \le \mu_2^- \le \cdots \le 0 \le \cdots \le \mu_2^+ \le \mu_1^+$: the spectrum of $A_v)$. In 2006, this notion was introduced by Heintze–Liu–Olmos [4]. In this paper, we then call f *regularizable immersion*. If f is a regularizable immersion, then the *regularized mean curvature vector* H of f is defined by $\langle H, v\rangle = \mathrm{Tr}_r\, A_v$ $(\forall\, v \in T^\perp M)$, where $\langle\ ,\ \rangle$ is the inner product of V and $T^\perp M$ is the normal bundle of f. If $H = 0$, then f is said to be *minimal*. In particular, if f is of codimension one, then we call the norm $||H||$ of H the *regularized mean curvature function* of f.

Let f_t $(0 \le t < T)$ be a C^∞-family of regularizable immersions of M into V. Denote by H_t the regularized mean curvature vector of f_t. Define a map $F : M \times [0, T) \to V$ by $F(x, t) := f_t(x)$ $((x, t) \in M \times [0, T))$. If $\frac{\partial F}{\partial t} = H_t$ holds, then we call f_t $(0 \le t < T)$ the *regularized mean curvature flow*. It has not been known whether the regularized mean curvature flow starting from any regularizable hypersurface exists uniquely in short time. However its existence and uniqueness (in short time) is shown in a special case (see Proposition 4.1).

3 The Mean Curvature Flow in Riemannian Orbifolds

In this section, we shall define the notion of the mean curvature flow starting from a suborbifold in a Riemannian orbifold. First we recall the notions of a Riemannian orbifold and a suborbifold following to [1, 2, 10, 13]. Let M be a paracompact Hausdorff space and $(U, \phi, \tilde{U}/\Gamma)$ a triple satisfying the following conditions:

(i) U is an open set of M,
(ii) $\hat{U}$ is an open set of $\mathbb{R}^n$ and Γ is a finite subgroup of the C^k-diffeomorphism group $\mathrm{Diff}^k(\hat{U})$ of $\hat{U}$,
(iii) ϕ is a homeomorphism of U onto $\hat{U}/\Gamma$.

Such a triple $(U, \phi, \hat{U}/\Gamma)$ is called an *n-dimensional orbifold chart*. Let $\mathscr{O} := \{(U_\lambda, \phi_\lambda, \hat{U}/\Gamma_\lambda) \mid \lambda \in \Lambda\}$ be a family of n-dimensional orbifold charts of M satisfying the following conditions:

(O1) $\{U_\lambda \mid \lambda \in \Lambda\}$ is an open covering of M,
(O2) For any $\lambda, \mu \in \Lambda$ with $U_\lambda \cap U_\mu \neq \emptyset$ and any $x \in U_\lambda \cap U_\mu$, there exists an n-dimensional orbifold chart $(W, \psi, \hat{W}/\Gamma')$ about x such that C^k-embeddings $\rho_\lambda : \hat{W} \hookrightarrow \hat{U}_\lambda$ and $\rho_\mu : \hat{W} \hookrightarrow \hat{U}_\mu$ satisfying $\phi_\lambda^{-1} \circ \pi_{\Gamma_\lambda} \circ \rho_\lambda = \psi^{-1} \circ \pi_{\Gamma'}$ and $\phi_\mu^{-1} \circ \pi_{\Gamma_\mu} \circ \rho_\mu = \psi^{-1} \circ \pi_{\Gamma'}$, where π_{Γ_λ}, π_{Γ_μ} and $\pi_{\Gamma'}$ are the orbit maps of Γ_λ, Γ_μ and Γ', respectively.

Such a family $\mathscr{O}$ is called an *n-dimensional C^k-orbifold atlas* of M and the pair $(M, \mathscr{O})$ is called an *n-dimensional C^k-orbifold*. Let $(U_\lambda, \phi_\lambda, \hat{U}_\lambda/\Gamma_\lambda)$ be an n-dimensional orbifold chart around $x \in M$. Then the group $(\Gamma_\lambda)_{\hat{x}} := \{b \in \Gamma_\lambda \mid b(\hat{x}) = \hat{x}\}$ is unique for x up to the conjugation, where $\hat{x}$ is a point of $\hat{U}_\lambda$ with $(\phi_\lambda^{-1} \circ \pi_{\Gamma_\lambda})(\hat{x}) = x$. Denote by $(\Gamma_\lambda)_x$ the conjugate class of this group $(\Gamma_\lambda)_{\hat{x}}$, This conjugate class is called the *local group at x*. If the local group at x is not trivial, then x is called a *singular point* of $(M, \mathscr{O})$. Denote by $\mathrm{Sing}(M, \mathscr{O})$ (or $\mathrm{Sing}(M)$) the set of all singular points of $(M, \mathscr{O})$. This set $\mathrm{Sing}(M, \mathscr{O})$ is called the *singular set* of $(M, \mathscr{O})$.

Let $(M, \mathscr{O}_M)$ and $(N, \mathscr{O}_N)$ be orbifolds, and f a map from M to N. If, for each $x \in M$ and each pair of an orbifold chart $(U_\lambda, \phi_\lambda, \hat{U}_\lambda/\Gamma_\lambda)$ of $(M, \mathscr{O}_M)$ around x and an orbifold chart $(V_\mu, \psi_\mu, \hat{V}_\mu/\Gamma'_\mu)$ of $(N, \mathscr{O}_N)$ around $f(x)$ ($f(U_\lambda) \subset V_\mu$), there exists a C^k-map $\hat{f}_{\lambda,\mu} : \hat{U}_\lambda \to \hat{V}_\mu$ with $f \circ \phi_\lambda^{-1} \circ \pi_{\Gamma_\lambda} = \psi_\mu^{-1} \circ \pi_{\Gamma'_\mu} \circ \hat{f}_{\lambda,\mu}$, then f is called a *C^k-orbimap* (or simply a *C^k-map*). Also $\hat{f}_{\lambda,\mu}$ is called a *local lift* of f with respect to $(U_\lambda, \phi_\lambda, \hat{U}_\lambda/\Gamma_\lambda)$ and $(V_\mu, \psi_\mu, \hat{V}_\mu/\Gamma'_\mu)$. Furthermore, if each local lift $\hat{f}_{\lambda,\mu}$ is an immersion, then f is called a *C^k-orbiimmersion* (or simply a *C^k-immersion*) and $(M, \mathscr{O}_M)$ is called a *C^k-(immersed) suborbifold* in $(N, \mathscr{O}_N, g)$. Similarly, if each local lift $\hat{f}_{\lambda,\mu}$ is a submersion, then f is called a *C^k-orbisubmersion*.

Now we shall define the notion of the mean curvature flow starting from a C^∞-suborbifold in a C^∞-Riemannian orbifold. Let f_t $(0 \le t < T)$ be a C^∞-family of

C^∞-orbiimmersions of a C^∞-orbifold $(M, \mathscr{O}_M)$ into a C^∞-Riemannian orbifold $(N, \mathscr{O}_N, g)$. Assume that, for each $(x_0, t_0) \in M \times [0, T)$ and each pair of an orbifold chart $(U_\lambda, \phi_\lambda, \hat{U}_\lambda/\Gamma_\lambda)$ of $(M, \mathscr{O}_M)$ around x_0 and an orbifold chart $(V_\mu, \phi_\mu, \hat{V}_\mu/\Gamma'_\mu)$ of $(N, \mathscr{O}_N)$ around $f_{t_0}(x_0)$ such that $f_t(U_\lambda) \subset V_\mu$ for any $t \in [t_0, t_0 + \varepsilon)$ (ε : a sufficiently small positive number), there exists local lifts $(\hat{f}_t)_{\lambda,\mu} : \hat{U}_\lambda \to \hat{V}_\mu$ of f_t ($t \in [t_0, t_0 + \varepsilon)$) such that they give the mean curvature flow in $(\hat{V}_\mu, \hat{g}_\mu)$, where $\hat{g}_\mu$ is the local lift of g to $\hat{V}_\mu$. Then we call f_t $(0 \le t < T)$ the *mean curvature flow* in $(N, \mathscr{O}_N, g)$.

Theorem 3.1 ([8]). *For any C^∞-orbiimmersion f of a compact C^∞-orbifold into a C^∞-Riemannian orbifold, the mean curvature flow starting from f exists uniquely in short time.*

Proof. Let f be a C^∞-orbiimmersion of an n-dimensional compact C^∞-orbifold $(M, \mathscr{O}_M)$ into an $(n+r)$-dimensional C^∞-Riemmannian orbifold $(N, \mathscr{O}_N, g)$. Fix $x_0 \in M$. Take an orbifold chart $(U_\lambda, \phi_\lambda, \hat{U}_\lambda/\Gamma_\lambda)$ of $(M, \mathscr{O}_M)$ around x_0 and an orbifold chart $(V_\mu, \psi_\mu, \hat{V}_\mu/\Gamma'_\mu)$ of $(N, \mathscr{O}_N)$ around $f(x_0)$ such that $f(U_\lambda) \subset V_\mu$ and that $\hat{U}_\lambda$ is relative compact. Also, let $\hat{f}_{\lambda,\mu} : \hat{U}_\lambda \hookrightarrow \hat{V}_\mu$ be a local lift of f and $\hat{g}_\mu$ a local lift of g (to $\hat{V}_\mu$). Since $\hat{U}_\lambda$ is relative compact, there exists the mean curvature flow $(\hat{f}_{\lambda,\mu})_t : \hat{U}_\lambda \hookrightarrow (\hat{V}_\mu, \hat{g}_\mu)$ $(0 \le t < T)$ starting from $\hat{f}_{\lambda,\mu} : \hat{U}_\lambda \hookrightarrow (\hat{V}_\mu, \hat{g}_\mu)$. Since $\hat{f}_{\lambda,\mu}$ is projetable to $f|_{U_\lambda}$ and $\hat{g}_\mu$ is Γ'_μ-invariant, $(\hat{f}_{\lambda,\mu})_t$ $(0 \le t < T)$ also are projectable to maps of U_λ into V_μ. Denote by $(f_{\lambda,\mu})_t$'s these maps of U_λ into V_μ. It is clear that $(f_{\lambda,\mu})_t$ $(0 \le t < T)$ is the mean curvature flow starting from $f|_{U_\lambda}$. Hence, it follows from the arbitrariness of x_0 and the compactness of M that the mean curvature flow starting from f exists uniquely in short time (Figs. 1 and 2). □

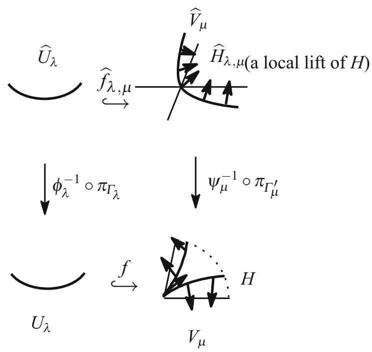

Fig. 1 This is an example of the mean curvature orbivector of an orbiimmersion

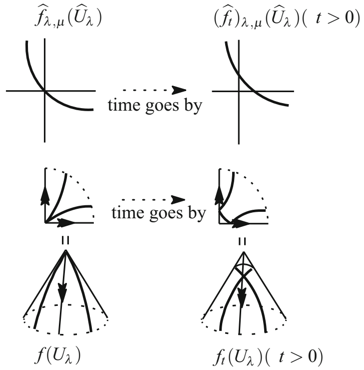

Fig. 2 This is an example of the mean curvature flow starting from an orbiimmersion

4 Evolution Equations

Let $G \curvearrowright V$ be an isometric almost free action with minimal regularizable orbit of a Hilbert Lie group G on a Hilbert space V equipped with an inner product $\langle\ ,\ \rangle$. The orbit space V/G is a (finite dimensional) C^∞-orbifold. Let $\phi : V \to V/G$ be the orbit map and set $N := V/G$. Here we give an example of such an isometric almost free action of a Hilbert Lie group.

Example. Let G be a compact semi-simple Lie group, K a closed subgroup of G and Γ a discrete subgroup of G. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. Assume that a reductive decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ exists. Let B be the Killing form of $\mathfrak{g}$. Give G the bi-invariant metric induced from $-B$. Let $H^0([0,1],\mathfrak{g})$ be the Hilbert space of all paths in the Lie algebra $\mathfrak{g}$ of G which are L^2-integrable with respect to $-B$. Also, let $H^1([0,1],G)$ the Hilbert Lie group of all paths in G which are of class H^1 with respect to g. This group $H^1([0,1],G)$ acts on $H^0([0,1],\mathfrak{g})$ isometrically and transitively as a gauge action:

$$(a * u)(t) = \mathrm{Ad}_G(a(t))(u(t)) - (R_{a(t)})_*^{-1}(a'(t))$$
$$(a \in H^1([0,1],G),\ u \in H^0([0,1],\mathfrak{g})),$$

(see [11, 12]), where Ad_G is the adjoint representation of G and $R_{a(t)}$ is the right translation by $a(t)$ and a' is the weak derivative of a. Set $P(G,\Gamma \times K) := \{a \in H^1([0,1],G) \mid (a(0),a(1)) \in \Gamma \times K\}$. The group $P(G,\Gamma \times K)$ acts on $H^0([0,1],\mathfrak{g})$ almost freely and isometrically, and the orbit space of this action is diffeomorphic to the orbifold $\Gamma \setminus G \,/\, K$. Furthermore, each orbit of this action is regularizable and minimal.

Give N the Riemannian orbimetric such that ϕ is a Riemannian orbisubmersion. Let $f : M \hookrightarrow V$ be a G-invariant submanifold immersion such that $(\phi \circ f)(M)$ is compact. For this immersion f, we can take an orbiimmesion $\overline{f}$ of a compact orbifold $\overline{M}$ into N and an orbisubmersion $\phi_M : M \to \overline{M}$ with $\phi \circ f = \overline{f} \circ \phi_M$. Let $\overline{f}_t$ $(0 \leq t < T)$ be the mean curvature flow starting from $\overline{f}$. The existenceness and the uniqueness of this flow in short time is assured by Theorem 3.1. Define a map $\overline{F} : \overline{M} \times [0,T) \to N$ by $\overline{F}(x,t) := \overline{f}_t(x)$ $((x,t) \in \overline{M} \times [0,T))$. Denote by H

the regularized mean curvature vector of f and $\overline{H}$ the mean curvature vector of $\overline{f}$. Since ϕ has minimal regularizable fibres, H is the horizontal lift of $\overline{H}$. Take $x \in \overline{M}$ and $u \in \phi_M^{-1}(x)$. Define a curve $c_x : [0,T) \to N$ by $c_x(t) := \overline{f}_t(x)$ and let $(c_x)_u^L : [0,T) \to V$ be the horizontal lift of c_x starting from $f(u)$. Define an immersion $f_t : M \hookrightarrow V$ by $f_t(u) = (c_x)_u^L(t)$ $(u \in \tilde{M})$ and a map $F : M \times [0,T) \to V$ by $F(u,t) = f_t(u)$ $((u,t) \in M \times [0,T))$.

Proposition 4.1 ([8]). *The flow f_t $(0 \leq t < T)$ is the regularized mean curvature flow starting from f.*

Proof. Denote by $\overline{H}_t$ the mean curvature vector of $\overline{f}_t$ and H_t the regularized mean curvature vector of f_t. Take any $(u,t) \in M \times [0,T)$. Set $x := \phi_M(u)$. It is clear that $\phi \circ f_t = \overline{f}_t \circ \phi_M$. Hence, since each fibre of ϕ is regularizable and minimal, $(H_t)_u$ coincides with one of the horizontal lifts of $(\overline{H}_t)_x$ to $f_t(u)$. On the other hand, from the definition of F, we have $\frac{\partial F}{\partial t}(u,t) = ((c_x)_u^L)'(t)$, which is one of the horizontal lifts of $(\overline{H}_t)_x$ to $f_t(u)$. These facts together with $\frac{\partial F}{\partial t}(u,0) = H_u$ implies that $\frac{\partial F}{\partial t}(u,t) = (H_t)_u$. Thus it follows from the arbitrariness of (u,t) that f_t $(0 \leq t < T)$ is the regularized mean curvature flow starting from f. This completes the proof. □

Assume that the codimension of f is equal to one. Denote by $\tilde{\mathscr{H}}$ (resp. $\tilde{\mathscr{V}}$) the horizontal (resp. vertical) distribution of ϕ. Denote by $\mathrm{pr}_{\tilde{\mathscr{H}}}$ (resp. $\mathrm{pr}_{\tilde{\mathscr{V}}}$) the orthogonal projection of TV onto $\tilde{\mathscr{H}}$ (resp. $\tilde{\mathscr{V}}$). For simplicity, for $X \in TV$, we denote $\mathrm{pr}_{\tilde{\mathscr{H}}}(X)$ (resp. $\mathrm{pr}_{\tilde{\mathscr{V}}}(X)$) by $X_{\tilde{\mathscr{H}}}$ (resp. $X_{\tilde{\mathscr{V}}}$). Define a distribution $\mathscr{H}_t$ on M by $f_{t*}((\mathscr{H}_t)_u) = f_{t*}(T_uM) \cap \tilde{\mathscr{H}}_{f_t(u)}$ $(u \in M)$ and a distribution $\mathscr{V}_t$ on M by $f_{t*}((\mathscr{V}_t)_u) = \tilde{\mathscr{V}}_{f_t(u)}$ $(u \in M)$. Note that $\mathscr{V}_t$ is independent of the choice of $t \in [0,T)$. Denote by g_t, h_t, A_t, H_t and ξ_t the induced metric, the second fundamental form, the shape tensor and the regularized mean curvature vector and the unit normal vector field of f_t, respectively. The group G acts on M through f_t. Since $\phi : V \to V/G$ is a G-orbibundle and $\tilde{\mathscr{H}}$ is a connection of the orbibundle, it follows from Proposition 4.1 that this action $G \curvearrowright M$ is independent of the choice of $t \in [0,T)$. It is clear that quantities g_t, h_t, A_t and H_t are G-invariant. Also, let ∇^t be the Riemannian connection of g_t. Let π_M be the projection of $M \times [0,T)$ onto M. For a vector bundle E over M, denote by $\pi_M^* E$ the induced bundle of E by π_M. Also denote by $\Gamma(E)$ the space of all sections of E. Define a section g of $\pi_M^*(T^{(0,2)}M)$ by $g(u,t) = (g_t)_u$ $((u,t) \in M \times [0,T))$, where $T^{(0,2)}M$ is the $(0,2)$-tensor bundle of M. Similarly, we define a section h of $\pi_M^*(T^{(0,2)}M)$, a section A of $\pi_M^*(T^{(1,1)}M)$, sections H and ξ of the induced bundle F^*TV of TV by F. We regard H and ξ as V-valued functions over $M \times [0,T)$ under the identification of $T_{F(u,t)}V$'s $((u,t) \in M \times [0,T))$ and V. Define a subbundle $\mathscr{H}$ (resp. $\mathscr{V}$) of $\pi_M^* TM$ by $\mathscr{H}_{(u,t)} := (\mathscr{H}_t)_u$ (resp. $\mathscr{V}_{(u,t)} := (\mathscr{V}_t)_u$). Denote by $\mathrm{pr}_{\mathscr{H}}$ (resp. $\mathrm{pr}_{\mathscr{V}}$) the orthogonal projection of $\pi_M^*(TM)$ onto $\mathscr{H}$ (resp. $\mathscr{V}$). For simplicity, for $X \in \pi_M^*(TM)$, we denote $\mathrm{pr}_{\mathscr{H}}(X)$ (resp. $\mathrm{pr}_{\mathscr{V}}(X)$) by $X_{\mathscr{H}}$ (resp. $X_{\mathscr{V}}$). The bundle $\pi_M^*(TM)$ is regarded as a subbundle of $T(M \times [0,T))$. For a section B of

$\pi_M^*(T^{(r,s)}M)$, we define $\frac{\partial B}{\partial t}$ by $\left(\frac{\partial B}{\partial t}\right)_{(u,t)} := \frac{dB_{(u,t)}}{dt}$, where the right-hand side of this relation is the derivative of the vector-valued function $t \mapsto B_{(u,t)}\ (\in T_u^{(r,s)}M)$. Also, we define a section $B_{\mathscr{H}}$ of $\pi_M^*(T^{(r,s)}M)$ by

$$B_{\mathscr{H}} = (\underbrace{\mathrm{pr}_{\mathscr{H}} \otimes \cdots \otimes \mathrm{pr}_{\mathscr{H}}}_{(r-\text{times})}) \circ B \circ (\underbrace{\mathrm{pr}_{\mathscr{H}} \otimes \cdots \otimes \mathrm{pr}_{\mathscr{H}}}_{(s-\text{times})}).$$

The restriction of $B_{\mathscr{H}}$ to $\mathscr{H} \times \cdots \times \mathscr{H}$ (s-times) is regarded as a section of the (r,s)-tensor bundle $\mathscr{H}^{(r,s)}$ of $\mathscr{H}$. This restriction also is denoted by the same symbol $B_{\mathscr{H}}$. For a tangent vector field X on M (or an open set U of M), we define a section $\bar{X}$ of π_M^*TM (or $\pi_M^*TM|_U$) by $\bar{X}_{(u,t)} := X_u$ ($(u,t) \in M \times [0,T)$). Denote by $\tilde{\nabla}$ the Riemannian connection of V. Define a connection ∇ of π_M^*TM by

$$(\nabla_X Y)_{(\cdot,t)} := \nabla^t_X Y_{(\cdot,t)} \text{ and } \nabla_{\frac{\partial}{\partial t}} Y := \frac{dY_{(u,\cdot)}}{dt}$$

for $X \in T_{(u,t)}(M \times \{t\})$ and $Y \in \Gamma(\pi_M^*TM)$, where $\frac{dY_{(u,t)}}{dt}$ is the derivative of the vector-valued function $t \mapsto Y_{(u,t)}\ (\in T_uM)$. Define a connection $\nabla^{\mathscr{H}}$ of $\mathscr{H}$ by $\nabla^{\mathscr{H}}_X Y := (\nabla_X Y)_{\mathscr{H}}$ for $X \in T(M \times [0,T))$ and $Y \in \Gamma(\mathscr{H})$. Similarly, define a connection $\nabla^{\mathscr{V}}$ of $\mathscr{V}$ by $\nabla^{\mathscr{V}}_X Y := (\nabla_X Y)_{\mathscr{V}}$ for $X \in T(M \times [0,T))$ and $Y \in \Gamma(\mathscr{V})$. Now we shall derive the evolution equations for some geometric quantities. First we derive the following evolution equation for $g_{\mathscr{H}}$.

Lemma 4.2 ([8]). *The sections $(g_{\mathscr{H}})_t$'s of $\pi_M^*(T^{(0,2)}M)$ satisfy the following evolution equation:*

$$\frac{\partial g_{\mathscr{H}}}{\partial t} = -2||H||h_{\mathscr{H}},$$

where $||H|| := \sqrt{\langle H, H\rangle}$.

Proof. Take $X, Y \in \Gamma(TM)$. We have

$$\begin{aligned}
&\frac{\partial g_{\mathscr{H}}}{\partial t}(\bar{X},\bar{Y}) = \frac{\partial}{\partial t} g_{\mathscr{H}}(\bar{X},\bar{Y}) = \frac{\partial}{\partial t} g(\bar{X}_{\mathscr{H}},\bar{Y}_{\mathscr{H}}) = \frac{\partial}{\partial t}\langle F_*\bar{X}_{\mathscr{H}}, F_*\bar{Y}_{\mathscr{H}}\rangle \\
&= \left\langle \frac{\partial}{\partial t}(\bar{X}_{\mathscr{H}}F), \bar{Y}_{\mathscr{H}}F\right\rangle + \left\langle \bar{X}_{\mathscr{H}}F, \frac{\partial}{\partial t}(\bar{Y}_{\mathscr{H}}F)\right\rangle \\
&= \left\langle \bar{X}_{\mathscr{H}}\left(\frac{\partial F}{\partial t}\right) + \left[\frac{\partial}{\partial t}, \bar{X}_{\mathscr{H}}\right]F, \bar{Y}_{\mathscr{H}}F\right\rangle + \left\langle \bar{X}_{\mathscr{H}}F, \bar{Y}_{\mathscr{H}}\left(\frac{\partial F}{\partial t}\right) + \left[\frac{\partial}{\partial t}, \bar{Y}_{\mathscr{H}}\right]F\right\rangle \\
&= \langle \bar{X}_{\mathscr{H}}(||H||\xi), \bar{Y}_{\mathscr{H}}F\rangle + \langle \bar{X}_{\mathscr{H}}F, \bar{Y}_{\mathscr{H}}(||H||\xi)\rangle \\
&= -||H||g(A\bar{X}_{\mathscr{H}},\bar{Y}_{\mathscr{H}}) - ||H||g(\bar{X}_{\mathscr{H}}, A\bar{Y}_{\mathscr{H}}) = -2||H||h_{\mathscr{H}}(\bar{X},\bar{Y}),
\end{aligned}$$

where we use $\left[\frac{\partial}{\partial t}, \bar{X}_{\mathscr{H}}\right] \in \mathscr{V}$ and $\left[\frac{\partial}{\partial t}, \bar{Y}_{\mathscr{H}}\right] \in \mathscr{V}$. Thus we obtain the desired evolution equation. □

Next we derive the following evolution equation for ξ.

Lemma 4.3 ([8]). *The unit normal vector fields ξ_t's satisfy the following evolution equation:*

$$\frac{\partial \xi}{\partial t} = -F_*(\mathrm{grad}_g ||H||),$$

where $\mathrm{grad}_g(||H||)$ *is the element of* $\pi_M^*(TM)$ *such that* $d||H||(X) = g(\mathrm{grad}_g ||H||, X)$ *for any* $X \in \pi_M^*(TM)$.

Proof. Since $\langle \xi, \xi \rangle = 1$, we have $\langle \frac{\partial \xi}{\partial t}, \xi \rangle = 0$. Hence $\frac{\partial \xi}{\partial t}$ is tangent to $f_t(M)$. Take any $(u_0, t_0) \in M \times [0, T)$. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal base of $T_{u_0}M$ with respect to $g_{(u_0,t_0)}$. By the Fourier expanding $\frac{\partial \xi}{\partial t}|_{t=t_0}$, we have

$$\begin{aligned}
&\left(\frac{\partial \xi}{\partial t}\right)_{(u_0,t_0)} = \sum \left\langle \left(\frac{\partial \xi}{\partial t}\right)_{(u_0,t_0)}, f_{t_0 *}(\bar{e}_i|_{t=t_0}) \right\rangle f_{t_0 *}(\bar{e}_i|_{t=t_0})\\
&= -\sum \left\langle \xi_{(u_0,t_0)}, \left.\frac{\partial f_{t*}(\bar{e}_i)}{\partial t}\right|_{t=t_0} \right\rangle f_{t_0 *}(\bar{e}_i|_{t=t_0}) = -\sum \left\langle \xi_{(u_0,t_0)}, \left.\frac{\partial}{\partial t}(\bar{e}_i F)\right|_{t=t_0} \right\rangle f_{t_0 *}(\bar{e}_i|_{t=t_0})\\
&= -\sum \left\langle \xi_{(u_0,t_0)}, \bar{e}_i \left(\left.\frac{\partial F}{\partial t}\right|_{t=t_0}\right) \right\rangle f_{t_0 *}(\bar{e}_i|_{t=t_0}) = -\sum \langle \xi_{(u_0,t_0)}, (\bar{e}_i H)|_{t=t_0} \rangle f_{t_0 *}(\bar{e}_i|_{t=t_0})\\
&= -\sum (\bar{e}_i ||H||)|_{t=t_0} f_{t_0 *}(\bar{e}_i|_{t=t_0}) = -\sum g_{(u_0,t_0)}(\mathrm{grad}_{g_{(u_0,t_0)}} ||H_{(u_0,t_0)}||, \bar{e}_i|_{t=t_0}) f_{t_0 *}(\bar{e}_i|_{t=t_0})\\
&= -f_{t_0 *}(\mathrm{grad}_{g_{(u_0,t_0)}} ||H_{(u_0,t_0)}||) = -(F_*(\mathrm{grad}_g ||H||))_{(u_0,t_0)},
\end{aligned}$$

where we use $\left[\frac{\partial}{\partial t}, \bar{e}_i\right] = 0$. Here we note that $\sum (\cdot)_i$ means $\lim_{k\to\infty} \sum_{i \in S_k} (\cdot)_i$ as $S_k := \{i \mid |(\cdot)_i| > \frac{1}{k}\}$ $(k \in \mathbb{N})$. This completes the proof. □

Let S_t $(0 \le t < T)$ be a C^∞-family of a (r, s)-tensor fields on M and S a section of $\pi_M^*(T^{(r,s)}M)$ defined by $S_{(u,t)} := (S_t)_u$. We define a section $\triangle_{\mathscr{H}} S$ of $\pi_M^*(T^{(r,s)}M)$ by

$$(\triangle_{\mathscr{H}} S)_{(u,t)} := \sum_{i=1}^{n} \nabla_{e_i} \nabla_{e_i} S,$$

where ∇ is the connection of $\pi_M^*(T^{(r,s)}M)$ (or $\pi_M^*(T^{(r,s+1)}M)$) induced from ∇ and $\{e_1, \cdots, e_n\}$ is an orthonormal base of $\mathscr{H}_{(u,t)}$ with respect to $(g_{\mathscr{H}})_{(u,t)}$. Also, we define a section $\bar{\triangle}_{\mathscr{H}} S_{\mathscr{H}}$ of $\mathscr{H}^{(r,s)}$ by

$$(\bar{\triangle}_{\mathscr{H}} S_{\mathscr{H}})_{(u,t)} := \sum_{i=1}^{n} \nabla^{\mathscr{H}}_{e_i} \nabla^{\mathscr{H}}_{e_i} S_{\mathscr{H}},$$

where $\nabla^{\mathscr{H}}$ is the connection of $\mathscr{H}^{(r,s)}$ (or $\mathscr{H}^{(r,s+1)}$) induced from $\nabla^{\mathscr{H}}$ and $\{e_1, \cdots, e_n\}$ is as above. Let $\mathscr{A}^\phi$ be the section of $T^*V \otimes T^*V \otimes TV$ defined by

$$\mathscr{A}^\phi_X Y := (\tilde{\nabla}_{X_{\tilde{\mathscr{H}}}} Y_{\tilde{\mathscr{H}}})_{\tilde{\mathscr{V}}} + (\tilde{\nabla}_{X_{\tilde{\mathscr{H}}}} Y_{\tilde{\mathscr{V}}})_{\tilde{\mathscr{H}}} \quad (X, Y \in TV).$$

Also, let $\mathscr{T}^\phi$ be the section of $T^*V \otimes T^*V \otimes TV$ defined by

$$\mathscr{T}^\phi_X Y := (\tilde{\nabla}_{X_{\tilde{\mathscr{V}}}} Y_{\tilde{\mathscr{H}}})_{\tilde{\mathscr{V}}} + (\tilde{\nabla}_{X_{\tilde{\mathscr{V}}}} Y_{\tilde{\mathscr{V}}})_{\tilde{\mathscr{H}}} \quad (X, Y \in TV).$$

Also, let $\mathscr{A}_t$ be the section of $T^*M \otimes T^*M \otimes TM$ defined by

$$(\mathscr{A}_t)_X Y := (\nabla^t_{X_{\mathscr{H}_t}} Y_{\mathscr{H}_t})_{\mathscr{V}_t} + (\nabla^t_{X_{\mathscr{H}_t}} Y_{\mathscr{V}_t})_{\mathscr{H}_t} \quad (X, Y \in TM).$$

Also let $\mathscr{A}$ be the section of $\pi_M^*(T^*M \otimes T^*M \otimes TM)$ defined in terms of $\mathscr{A}_t$'s ($t \in [0, T)$). Also, let $\mathscr{T}_t$ be the section of $T^*M \otimes T^*M \otimes TM$ defined by

$$(\mathscr{T}_t)_X Y := (\nabla^t_{X_{\mathscr{V}_t}} Y_{\mathscr{V}_t})_{\mathscr{H}_t} + (\nabla^t_{X_{\mathscr{V}_t}} Y_{\mathscr{H}_t})_{\mathscr{V}_t} \quad (X, Y \in TM).$$

Also let $\mathscr{T}$ be the section of $\pi_M^*(T^*M \otimes T^*M \otimes TM)$ defined in terms of $\mathscr{T}_t$'s ($t \in [0, T)$). These sections are the analogues of the so-called O'Neill fundamental tensor introduced in [9] for Riemannian submersions between finite dimensional Riemannian manifolds. Clearly we have

$$F_*(\mathscr{A}_X Y) = \mathscr{A}^\phi_{F_* X} F_* Y$$

for $X, Y \in \mathscr{H}$ and

$$F_*(\mathscr{T}_W X) = \mathscr{T}^\phi_{F_* W} F_* X$$

for $X \in \mathscr{H}$ and $W \in \mathscr{V}$. Let E be a vector bundle over M. For a section S of $\pi_M^*(T^{(0,r)}M \otimes E)$, we define $\mathrm{Tr}^\bullet_{g_{\mathscr{H}}} S(\cdots, \overset{j}{\bullet}, \cdots, \overset{k}{\bullet}, \cdots)$ by

$$(\mathrm{Tr}^\bullet_{g_{\mathscr{H}}} S(\cdots, \overset{j}{\bullet}, \cdots, \overset{k}{\bullet}, \cdots))_{(u,t)} = \sum_{i=1}^n S_{(u,t)}(\cdots, \overset{j}{e_i}, \cdots, \overset{k}{e_i}, \cdots)$$

($(u, t) \in M \times [0, T)$), where $\{e_1, \cdots, e_n\}$ is an orthonormal base of $\mathscr{H}_{(u,t)}$ with respect to $(g_{\mathscr{H}})_{(u,t)}$, $S(\cdots, \overset{j}{\bullet}, \cdots, \overset{k}{\bullet}, \cdots)$ means that $\bullet$ is entried into the j-th component and the k-th component of S and $S_{(u,t)}(\cdots, \overset{j}{e_i}, \cdots, \overset{k}{e_i}, \cdots)$ means that e_i is entried into the j-th component and the k-th component of $S_{(u,t)}$.

Then we have the following relation.

Lemma 4.4 ([8]). *Let S be a section of $\pi_M^*(T^{(0,2)}M)$ which is symmetric with respect to g. Then we have*

$$\begin{aligned}(\triangle_{\mathscr{H}} S)_{\mathscr{H}}(X, Y) = (\triangle^{\mathscr{H}}_{\mathscr{H}} S_{\mathscr{H}})(X, Y) \\ -2\mathrm{Tr}^\bullet_{g_{\mathscr{H}}}((\nabla_\bullet S)(\mathscr{A}_\bullet X, Y)) - 2\mathrm{Tr}^\bullet_{g_{\mathscr{H}}}((\nabla_\bullet S)(\mathscr{A}_\bullet Y, X)) \\ -\mathrm{Tr}^\bullet_{g_{\mathscr{H}}} S(\mathscr{A}_\bullet(\mathscr{A}_\bullet X), Y) - \mathrm{Tr}^\bullet_{g_{\mathscr{H}}} S(\mathscr{A}_\bullet(\mathscr{A}_\bullet Y), X) \\ -\mathrm{Tr}^\bullet_{g_{\mathscr{H}}} S((\nabla_\bullet \mathscr{A})_\bullet X, Y) - \mathrm{Tr}^\bullet_{g_{\mathscr{H}}} S((\nabla_\bullet \mathscr{A})_\bullet Y, X) \\ -2\mathrm{Tr}^\bullet_{g_{\mathscr{H}}} S(\mathscr{A}_\bullet X, \mathscr{A}_\bullet Y)\end{aligned}$$

for $X, Y \in \mathscr{H}$, where ∇ is the connection of $\pi_M^*(T^{(1,2)}M)$ induced from ∇.

Proof. Take any $(u_0, t_0) \in M \times [0, T)$. Let $\{e_1, \cdots, e_n\}$ be an orthonormal base of $\mathscr{H}_{(u_0,t_0)}$ with respect to $(g_{\mathscr{H}})_{(u_0,t_0)}$. Take any $X, Y \in \mathscr{H}_{(u_0,t_0)}$. Let $\tilde{X}$ be a section of $\mathscr{H}$ on a neighborhood of (u_0, t_0) with $\tilde{X}_{(u_0,t_0)} = X$ and $(\nabla^{\mathscr{H}} X)_{(u_0,t_0)} = 0$. Similarly we define $\tilde{Y}$ and $\tilde{e}_i$. Let $W = X, Y$ or e_i. Then, it follows from $(\nabla^{\mathscr{H}}_{e_i} \tilde{W})_{(u_0,t_0)} = 0$, $(\nabla_{e_i} \tilde{W})_{(u_0,t_0)} = \mathscr{A}_{e_i} W$ and the skew-symmetricness of $\mathscr{A}|_{\mathscr{H}\times\mathscr{H}}$ that

$$
\begin{aligned}
&(\triangle_{\mathscr{H}} S)_{\mathscr{H}}(X, Y) = \sum_{i=1}^{n} (\nabla_{e_i} \nabla_{e_i} S)(X, Y) \\
&= \sum_{i=1}^{n} (\nabla^{\mathscr{H}}_{e_i} \nabla^{\mathscr{H}}_{e_i} S_{\mathscr{H}})(X, Y) - 2 \sum_{i=1}^{n} ((\nabla_{e_i} S)(\mathscr{A}_{e_i} X, Y) + (\nabla_{e_i} S)(\mathscr{A}_{e_i} Y, X)) \\
&\quad - \sum_{i=1}^{n} (S(\mathscr{A}_{e_i}(\mathscr{A}_{e_i} X), Y) + S(\mathscr{A}_{e_i}(\mathscr{A}_{e_i} Y), X)) - 2 \sum_{i=1}^{n} S(\mathscr{A}_{e_i} X, \mathscr{A}_{e_i} Y) \\
&\quad - \sum_{i=1}^{n} (S((\nabla_{e_i} \mathscr{A})_{e_i} X, Y) + S((\nabla_{e_i} \mathscr{A})_{e_i} Y, X)).
\end{aligned}
$$

The right-hand side of this relation is equal to the right-hand side of the relation in the statement. This completes the proof. □

Also we have the following Simons-type identity.

Lemma 4.5 ([8]). *We have*

$$\triangle_{\mathscr{H}} h = \nabla d||H|| + ||H||(A^2)_\sharp - (\mathrm{Tr}\,(A^2)_{\mathscr{H}})h,$$

where $(A^2)_\sharp$ is the element of $\Gamma(\pi_M^* T^{(0,2)} M)$ defined by $(A^2)_\sharp(X, Y) := g(A^2 X, Y)$ $(X, Y \in \pi_M^* TM)$.

Proof. Take $X, Y, Z, W \in \pi_M^*(TM)$. Since the ambient space V is flat, it follows from the Ricci's identity, the Gauss equation and the Codazzi equation that

$$
\begin{aligned}
&(\nabla_X \nabla_Y h)(Z, W) - (\nabla_Z \nabla_W h)(X, Y) = (\nabla_X \nabla_Z h)(Y, W) - (\nabla_Z \nabla_X h)(Y, W) \\
&= h(X, Y)h(AZ, W) - h(Z, Y)h(AX, W) + h(X, W)h(AZ, Y) - h(Z, W)h(AX, Y).
\end{aligned}
$$

By using this relation, we obtain the desired relation. □

Note. In the sequel, we omit the notation F_* for simplicity.

Define a section $\mathscr{R}$ of $\pi_M^*(\mathscr{H}^{(0,2)})$ by

$$
\begin{aligned}
\mathscr{R}(X, Y) := {} & \mathrm{Tr}^\bullet_{g_{\mathscr{H}}} h(\mathscr{A}_\bullet(\mathscr{A}_\bullet X), Y) + \mathrm{Tr}^\bullet_{g_{\mathscr{H}}} h(\mathscr{A}_\bullet(\mathscr{A}_\bullet Y), X) \\
& + \mathrm{Tr}^\bullet_{g_{\mathscr{H}}} h((\nabla_\bullet \mathscr{A})_\bullet X, Y) + \mathrm{Tr}^\bullet_{g_{\mathscr{H}}} h((\nabla_\bullet \mathscr{A})_\bullet Y, X) \\
& + 2\mathrm{Tr}^\bullet_{g_{\mathscr{H}}} (\nabla_\bullet h)(\mathscr{A}_\bullet X, Y) + 2\mathrm{Tr}^\bullet_{g_{\mathscr{H}}} (\nabla_\bullet h)(\mathscr{A}_\bullet Y, X) \\
& + 2\mathrm{Tr}^\bullet_{g_{\mathscr{H}}} h(\mathscr{A}_\bullet X, \mathscr{A}_\bullet Y) \qquad (X, Y \in \mathscr{H}).
\end{aligned}
$$

From Lemmas 4.3, 4.4 and 4.5, we derive the following evolution equation for $(h_{\mathscr{H}})_t$)s.

Theorem 4.6 ([8]). *The sections* $(h_{\mathscr{H}})_t$*'s of* $\pi_M^*(T^{(0,2)}M)$ *satisfies the following evolution equation:*

$$\frac{\partial h_{\mathscr{H}}}{\partial t}(X,Y) = (\triangle^{\mathscr{H}}_{\mathscr{H}} h_{\mathscr{H}})(X,Y) - 2||H||((A_{\mathscr{H}})^2)_\sharp(X,Y) - 2||H||((\mathscr{A}_\xi^\phi)^2)_\sharp(X,Y) \\ + \mathrm{Tr}\left((A_{\mathscr{H}})^2 - ((\mathscr{A}_\xi^\phi)^2)_{\mathscr{H}}\right) h_{\mathscr{H}}(X,Y) - \mathscr{R}(X,Y)$$

for $X, Y \in \mathscr{H}$.

Proof. Take $X, Y \in \mathscr{H}_{(u,t)}$. Easily we have

$$AX = A_{\mathscr{H}}X + \mathscr{A}_\xi^\phi X \quad \text{and} \quad (A^2)_{\mathscr{H}}X = (A_{\mathscr{H}})^2 X - (\mathscr{A}_\xi^\phi)^2 X, \tag{2}$$

where we use

$$\left(\tilde{\nabla}_W \xi\right)_{\tilde{\mathscr{H}}} = \left(\tilde{\nabla}_\xi W + [W,\xi]\right)_{\tilde{\mathscr{H}}} = \left(\tilde{\nabla}_\xi W\right)_{\tilde{\mathscr{H}}} = \mathscr{A}_\xi W$$

for $W \in \Gamma(\tilde{\mathscr{V}})$ because of $[W,\xi] \in \Gamma(\tilde{\mathscr{V}})$. Also, since $\left[\frac{\partial}{\partial t}, \bar{X}_{\mathscr{H}}\right] \in \mathscr{V}$, we have

$$\left[\frac{\partial}{\partial t}, \bar{X}_{\mathscr{H}}\right] = 2||H||\mathscr{A}_\xi^\phi \bar{X}_{\mathscr{H}}. \tag{3}$$

From Lemmas 4.3, (2) and (3), we have

$$\begin{aligned}
&\frac{\partial h_{\mathscr{H}}}{\partial t}(X,Y) = \frac{\partial}{\partial t}(h_{\mathscr{H}}(\bar{X},\bar{Y})) = \frac{\partial}{\partial t}\langle \xi, \bar{X}_{\mathscr{H}}(\bar{Y}_{\mathscr{H}}F)\rangle \\
&= \langle \frac{\partial \xi}{\partial t}, \bar{X}_{\mathscr{H}}(\bar{Y}_{\mathscr{H}}F)\rangle + \langle \xi, \frac{\partial}{\partial t}\left(\bar{X}_{\mathscr{H}}(\bar{Y}_{\mathscr{H}}F)\right)\rangle \\
&= -\langle F_*(\mathrm{grad}_g||H||), \tilde{\nabla}_X F_* \bar{Y}_{\mathscr{H}}\rangle + \langle \xi, X\left(\bar{Y}_{\mathscr{H}}\left(\frac{\partial F}{\partial t}\right)\right)\rangle \\
&\quad + \langle \xi, X([\frac{\partial}{\partial t}, \bar{Y}_{\mathscr{H}}]F)\rangle + \langle \xi, [\frac{\partial}{\partial t}, \bar{X}_{\mathscr{H}}](\bar{Y}_{\mathscr{H}}F)\rangle \\
&= -g(\mathrm{grad}_g||H||, \nabla_X \bar{Y}_{\mathscr{H}}) + X(\bar{Y}_{\mathscr{H}}||H||) - ||H||\langle \xi, \tilde{\nabla}_X F_*(A(\bar{Y}_{\mathscr{H}}))\rangle \\
&\quad + \langle \xi, \tilde{\nabla}_X F_*([\frac{\partial}{\partial t}, \bar{Y}_{\mathscr{H}}])\rangle + \langle \xi, \tilde{\nabla}_{[\frac{\partial}{\partial t}, \bar{X}_{\mathscr{H}}]} F_* \bar{Y}_{\mathscr{H}}\rangle \\
&= (\nabla d||H||)(X,Y) - ||H||h_{\mathscr{H}}(X, A_{\mathscr{H}}Y) + ||H||h(X, \mathscr{A}_\xi^\phi Y) + 2||H||h(\mathscr{A}_\xi^\phi X, Y) \\
&= (\nabla d||H||)(X,Y) - ||H||g_{\mathscr{H}}((A_{\mathscr{H}})^2 X, Y) - 3||H||g((\mathscr{A}_\xi^\phi)^2 X, Y)
\end{aligned}$$

From this relation and the Simons-type identity in Lemma 4.5, we have

$$\begin{aligned}\frac{\partial h_{\mathscr{H}}}{\partial t} = & \triangle_{\mathscr{H}} h - 2||H||((A_{\mathscr{H}})^2)_\sharp - 2||H||((\mathscr{A}^\phi_\xi)^2)_\sharp \\ & + \mathrm{Tr}\left((A_{\mathscr{H}})^2 - ((\mathscr{A}^\phi_\xi)^2)_{\mathscr{H}}\right) h_{\mathscr{H}}.\end{aligned} \tag{4}$$

Substituting the relation in Lemma 4.4 into (4), we obtain the desired relation. □

From Lemma 4.2, we derive the following relation.

Lemma 4.7 ([8]). *Let X and Y be local sections of $\mathscr{H}$ such that $g(X,Y)$ is constant. Then we have* $g(\nabla_{\frac{\partial}{\partial t}} X, Y) + g(X, \nabla_{\frac{\partial}{\partial t}} Y) = 2||H||h(X,Y)$.

Next we prepare the following lemma for $\mathscr{R}$.

Lemma 4.8 ([8]). *For $X, Y \in \mathscr{H}$, we have*

$$\begin{aligned}\mathscr{R}(X,Y) = & 2\mathrm{Tr}^\bullet_{g_{\mathscr{H}}}\left(\langle(\mathscr{A}^\phi_\bullet X, \mathscr{A}^\phi_\bullet(A_{\mathscr{H}}Y)\rangle + \langle(\mathscr{A}^\phi_\bullet Y, \mathscr{A}^\phi_\bullet(A_{\mathscr{H}}X)\rangle\right) \\ & + 2\mathrm{Tr}^\bullet_{g_{\mathscr{H}}}\left(\langle(\mathscr{A}^\phi_\bullet X, \mathscr{A}^\phi_Y(A_{\mathscr{H}}\bullet)\rangle + \langle(\mathscr{A}^\phi_\bullet Y, \mathscr{A}^\phi_X(A_{\mathscr{H}}\bullet)\rangle\right) \\ & + 2\mathrm{Tr}^\bullet_{g_{\mathscr{H}}}\left(\langle(\tilde{\nabla}_\bullet \mathscr{A}^\phi)_\xi Y, \mathscr{A}^\phi_\bullet X\rangle + \langle(\tilde{\nabla}_\bullet \mathscr{A}^\phi)_\xi X, \mathscr{A}^\phi_\bullet Y\rangle\right) \\ & + \mathrm{Tr}^\bullet_{g_{\mathscr{H}}}\left(\langle(\tilde{\nabla}_\bullet \mathscr{A}^\phi)_\bullet X, \mathscr{A}^\phi_\xi Y\rangle + \langle(\tilde{\nabla}_\bullet \mathscr{A}^\phi)_\bullet Y, \mathscr{A}^\phi_\xi X\rangle\right) \\ & + 2\mathrm{Tr}^\bullet_{g_{\mathscr{H}}}\langle\mathscr{T}^\phi_{\mathscr{A}^\phi_\bullet X}\xi, \mathscr{A}^\phi_\bullet Y\rangle,\end{aligned}$$

where we omit F_*.

Also, we prepare the following lemma.

Lemma 4.9 ([8]). *For $X, Y, Z \in \mathscr{H}$, we have*

$$2\langle\mathscr{T}^\phi_{\mathscr{A}^\phi_X Y}\xi, \mathscr{A}^\phi_X Z\rangle = -\langle\mathscr{A}^\phi_X Z, (\tilde{\nabla}_X \mathscr{A}^\phi)_\xi Y\rangle + \langle\mathscr{A}^\phi_X Z, (\tilde{\nabla}_Y \mathscr{A}^\phi)_\xi X\rangle.$$

From the relations in Lemmas 4.8 and 4.9, we obtain the following relations directly.

Lemma 4.10 ([8]). *For $X \in \mathscr{H}$, we have*

$$\begin{aligned}\mathscr{R}(X,X) = & 4\mathrm{Tr}^\bullet_{g_{\mathscr{H}}}\langle\mathscr{A}^\phi_\bullet X, \mathscr{A}^\phi_\bullet(A_{\mathscr{H}}X)\rangle + 4\mathrm{Tr}^\bullet_{g_{\mathscr{H}}}\langle\mathscr{A}^\phi_\bullet X, \mathscr{A}^\phi_X(A_{\mathscr{H}}\bullet)\rangle \\ & + 3\mathrm{Tr}^\bullet_{g_{\mathscr{H}}}\langle(\tilde{\nabla}_\bullet \mathscr{A}^\phi)_\xi X, \mathscr{A}^\phi_\bullet X\rangle + 2\mathrm{Tr}^\bullet_{g_{\mathscr{H}}}\langle(\tilde{\nabla}_\bullet \mathscr{A}^\phi)_\bullet X, \mathscr{A}^\phi_\xi X\rangle \\ & + \mathrm{Tr}^\bullet_{g_{\mathscr{H}}}\langle\mathscr{A}^\phi_\bullet X, (\tilde{\nabla}_X \mathscr{A}^\phi)_\xi \bullet\rangle\end{aligned}$$

and hence $\mathrm{Tr}^\bullet_{g_{\mathscr{H}}}\mathscr{R}(\bullet,\bullet) = 0$.

By using Theorem 4.6 and Lemmas 4.7 and 4.10, we can show the following evolution equation for $||H_t||$'s.

Corollary 4.11 ([8]). *The norms $||H_t||$'s of H_t satisfy the following evolution equation:*

$$\frac{\partial ||H||}{\partial t} = \triangle_{\mathscr{H}} ||H|| + ||H|| \mathrm{Tr}(A_{\mathscr{H}})^2 - 3||H|| \mathrm{Tr}((\mathscr{A}_{\xi}^{\phi})^2)_{\mathscr{H}}.$$

Remark 4.1. From the evolution equations obtained in this section, the evolution equations for the corresponding geometric quantities of $\overline{f}_t(: \overline{M} \hookrightarrow V/G)$ are derived, respectively. In the case where the G-action is free and hence V/G is a (complete) Riemannian manifold, the above derived evolution equations coincide with the evolution equations for the corresponding geometric quantities along the mean curvature flow in a complete Riemannian manifold which were given by Huisken [6] (see [5] also). That is, the discussion in this section give a new proof of the evolution equations in [6] in the case where the ambient complete Riemannian manifold occurs as V/G. In the proof of [6], one need to take local coordinates of the ambient space to derive the evolution equations. On the other hand, in our proof, one need not take local coordinates of the ambient space because the ambient space is a Hilbert space. This is an advantage of our proof.

5 Horizontally Strongly Convexity Preservability Theorem

Let $G \curvearrowright V$ be an isometric almost free action with minimal regularizable orbit of a Hilbert Lie group G on a Hilbert space V equipped with an inner product $\langle\ ,\ \rangle$ and $\phi : V \to V/G$ the orbit map. Denote by $\tilde{\nabla}$ the Riemannian connection of V. Set $n := \dim V/G - 1$. Let $M(\subset V)$ be a G-invariant hypersurface in V such that $\phi(M)$ is compact. Let f be an inclusion map of M into V and f_t $(0 \leq t < T)$ the regularized mean curvature flow starting from f. We use the notations in Sect. 4. In the sequel, we omit the notation f_{t*} for simplicity. Set

$$L := \max_{(X_1,\cdots,X_5)\in\tilde{\mathscr{H}}_1^5} |\langle \mathscr{A}_{X_1}^{\phi}((\tilde{\nabla}_{X_2}\mathscr{A}^{\phi})_{X_3}X_4),\ X_5\rangle|,$$

where $\tilde{\mathscr{H}}_1 := \{X \in \tilde{\mathscr{H}} \mid ||X|| = 1\}$. Assume that $L < \infty$. Note that $L < \infty$ in the case where V/G is compact. Then we obtain the following horizontally strongly convexity preservability theorem by using the evolution equations stated in Sect. 4 and the maximum principle.

Theorem 5.1 ([8]). *If M satisfies $||H_0||^2(h_{\mathscr{H}})_{(\cdot,0)} > 2n^2L(g_{\mathscr{H}})_{(\cdot,0)}$, then $T < \infty$ holds and $||H_t||^2(h_{\mathscr{H}})_{(\cdot,t)} > 2n^2L(g_{\mathscr{H}})_{(\cdot,t)}$ holds for all $t \in [0,T)$.*

6 Strongly Convex Preservability Theorem in the Orbit Space

Let V, G and ϕ be as in the previous section. Set $N := V/G$ and $n := \dim V/G - 1$. Denote by g_N and R_N the Riemannian orbimetric and the curvature orbitensor of N. Also, ∇^N the Riemannian connection of $g_N|_{N\setminus\mathrm{Sing}(N)}$. Since the Riemannian manifold $(N \setminus \mathrm{Sing}(N), g_N|_{N\setminus\mathrm{Sing}(N)})$ is locally homogeneous, the norm $||\nabla^N R_N||$ of $\nabla^N R_N$ (with respect to g_N) is constant over $N \setminus \mathrm{Sing}(N)$. Set $L_N := ||\nabla^N R_N||$. Assume that $L_N < \infty$. Let $\overline{M}$ be a compact suborbifold of codimension one in N immersed by $\overline{f}$ and $\overline{f}_t$ $(t \in [0, T))$ the mean curvature flow starting from $\overline{f}$. Denote by $\overline{g}_t, \overline{h}_t, \overline{A}_t$ and $\overline{H}_t$ be the induced orbimetric, the second fundamental orbiform, the shape orbitensor and the mean curvature orbifunction of $\overline{f}_t$, respectively, and $\overline{\xi}_t$ the unit normal vector field of $\overline{f}_t|_{\overline{M}\setminus\mathrm{Sing}(\overline{M})}$.

From Theorem 5.1, we obtain the following strongly convexity preservability theorem for compact suborbifolds in N.

Theorem 6.1 ([8]). *If $\overline{f}$ satisfies $||\overline{H}_0||^2\overline{h}_0 > n^2 L_N \overline{g}_0$, then $T < \infty$ holds and $||\overline{H}_t||^2\overline{h}_t > n^2 L_N \overline{g}_t$ holds for all $t \in [0, T)$.*

Proof. Set $M := \{(x, u) \in \overline{M} \times V \mid \overline{f}(x) = \phi(u)\}$ and define $f : M \to V$ by $f(x, u) = u$ $((x, u) \in M)$. It is clear that f is an immersion. Denote by H_0 the regularized mean curvature vector of f. Define a curve $c_x : [0, T) \to N$ by $c_x(t) := \overline{f}_t(x)$ $(t \in [0, T))$ and let $(c_x)^L_u$ be the horizontal lift of c_x starting from u, where $u \in \phi^{-1}(f(x))$. Define an immersion $f_t : M \hookrightarrow V$ by $f_t(x, u) := (c_x)^L_u(t)$ $((x, u) \in M)$. Then f_t $(t \in [0, T))$ is the regularized mean curvature flow starting from f (see the proof of Proposition 4.1). Denote by g_t, h_t, A^t and H_t the induced metric, the second fundamental form, the shape tensor and the mean curvature vector of f_t, respectively. By the assumption, $\overline{f}_0$ satisfies $||\overline{H}_0||^2\overline{h}_0 > n^2 L_N \overline{g}_0$. Also, we can show $L_N = 2L$ by long calculation, where L is as in the previous section. From these facts, we can show that f_0 satisfies $||H_0||^2(h_{\mathscr{H}})_0 > 2n^2 L(g_{\mathscr{H}})_0$. Hence, it follows from Theorem 5.1 that f_t $(t \in [0, T))$ satisfies $||H_t||^2(h_{\mathscr{H}})_t > 2n^2 L(g_{\mathscr{H}})_t$. Furthermore, it follows from this fact that $\overline{f}_t$ $(t \in [0, T))$ satisfies $||\overline{H}_t||^2\overline{h}_t > n^2 L_N \overline{g}_t$. □

Remark 6.1. In the case where the G-action is free and hence N is a (complete) Riemannian manifold, Theorem 6.1 implies the strongly convexity preservability theorem by Huisken (see [6, Theorem 4.2]).

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Harmonic Maps into Grassmannians

Yasuyuki Nagatomo

Abstract A harmonic map from a Riemannian manifold into a Grassmannian manifold is characterized by a vector bundle, a space of sections of the bundle and a Laplace operator [10]. This characterization can be considered a generalization of a theorem of Takahashi [11]. We apply our main result which generalizes a theorem of Do Carmo and Wallach [4] to describe moduli spaces of special classes of harmonic maps from compact reductive Riemannian homogeneous spaces into Grassmannians. As an application, we give an alternative proof of the theorem of Bando and Ohnita [1] which states the rigidity of the minimal immersion of the complex projective line into complex projective spaces. Moreover, a similar method yields rigidity of holomorphic isometric embeddings between complex projective spaces, which is part of Calabi's result [2]. Finally, we give a description of moduli spaces of holomorphic isometric embeddings of the projective line into quadrics [9].

1 Introduction

Let us recall Theorem of Takahashi [11], because one of the main theorems in this note concerns with a generalization of Theorem of Takahashi. Denote the standard co-ordinates of $\mathbf{R}^N$ by $\mathbf{x} := (x_1, \cdots, x_N)$ and let S^{N-1} denote the unit sphere in $\mathbf{R}^N$. Then (a version of) Theorem of Takahashi asserts

Theorem 1 ([11]). *A map $f : M \to S^{N-1}$ is a harmonic map if and only if there exists a function $h : M \to \mathbf{R}$ such that $\Delta \mathbf{x} = h\mathbf{x}$, where Δ is the Laplace operator of (M, g) and $\mathbf{x}$ stands for the pull-back functions by f. Under these conditions, we have $h = |df|^2$.*

We denote by $Gr_p(W)$ a real or complex Grassmannian with a standard metric of Fubini–Study type, where W is a real or complex vector space with a scalar product. Since the tautological bundle is a subbundle of the trivial bundle $\underline{W} = Gr_p(W) \times W \to Gr_p(W)$, we have a quotient bundle which is called the universal

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quotient bundle. The scalar product on W endows the universal quotient bundle with a fibre metric and a connection.

When S^{N-1} is identified with the real Grassmannian of oriented hyperplanes in $\mathbf{R}^N$, functions $x_1, \cdots, x_N$ are regarded as sections of the universal quotient bundle. Hence we can reformulate Theorem of Takahashi from the viewpoint of vector bundles. Then we have a generalization of Theorem 1 (Theorems 2 and 3).

We apply Theorem 3 to obtain a generalization of the theory of Do Carmo and Wallach [4] in Sect. 4. In their work, they apply Theorem of Takahashi to classify minimal immersions of spheres into spheres. We are concerned with harmonic maps from a compact reductive Riemannian homogeneous space M with isometry group G into a Grassmannian. We fix a homogeneous vector bundle with a canonical connection over M. If an eigenspace of the Laplace operator acting on sections globally generates the bundle, then we induce a G-equivariant map from M into a Grassmannian, which is called standard map. We give a sufficient condition for a standard map being harmonic (Lemma 4). We will use Theorem 3 to obtain a classification of a special class of harmonic maps (Theorem 4).

As an application, we give another proof of Theorem of Bando–Ohnita [1], which states the rigidity of minimal immersion of the complex projective line into complex projective spaces. Moreover, a similar method yields rigidity of holomorphic isometric embeddings between complex projective spaces, which is part of Calabi's result [2].

In the final section, we give a description of moduli spaces of holomorphic isometric embeddings of the projective line into quadrics [9].

More details on the results described in this note can be found in [9] and [10].

2 Preliminaries

2.1 A Harmonic Map

Let M and N be Riemannian manifolds and $f : M \to N$ a map. *The energy density* $e(f) : M \to \mathbf{R}$ of f is defined as $e(f)(x) := |df|^2$. Then, *the tension field* $\tau(f)$ of f is defined to be $\tau(f) :=$ trace ∇df which is a section of the pull-back bundle $f^*TN \to M$ of the tangent bundle $TN \to N$.

Definition 1 ([5]). A map $f : M \to N$ is called a *harmonic map* if $\tau(f) \equiv 0$.

2.2 Geometry of Grassmannians

Let W be a real (oriented) or complex N-dimensional vector space and $Gr_p(W)$ a Grassmannian of (oriented) p-planes in W. The tautological vector bundle is denoted by $S \to Gr_p(W)$. By definition, we have a bundle injection $i_S : S \to \underline{W}$, where $\underline{W} \to Gr_p(W)$ is a trivial bundle of fibre W. Then, the quotient vector bundle $Q \to Gr_p(W)$ with a natural projection $\pi_Q : \underline{W} \to Q$ is called the *universal quotient bundle*. By the natural projection π_Q, W can also be regarded as a subspace of $\Gamma(Q)$ which is the space of sections of $Q \to Gr_p(W)$. The (holomorphic) tangent bundle $T \to Gr_p(W)$ is identified with $S^* \otimes Q$.

Next, we fix a scalar product (an inner product or a Hermitian inner product) on W. It gives orthogonal projections and so, we obtain two bundle homomorphisms: $\pi_S : \underline{W} \to S$, and $i_Q : Q \to \underline{W}$. Then the vector bundles $S, Q \to Gr_p(W)$ are equipped with fibre metrics, respectively.

A section t of $Q \to Gr_p(W)$ is regarded as a W-valued function $i_Q(t)$. Then the differential $di_Q(t)$ can be decomposed into two components:

$$di_Q(t) = \pi_S di_Q(t) + \pi_Q di_Q(t).$$

Indeed, $\pi_Q di_Q(t)$ is a connection denoted by $\nabla^Q t$, which is the so-called canonical connection. The other term $\pi_S di_Q(t)$ denoted by Kt is called *the second fundamental form* in the sense of Kobayashi [7], which turns out to be a 1-form with values in $\mathrm{Hom}(Q, S) \cong Q^* \otimes S$.

In a similar way, a connection denoted by ∇^S is defined on $S \to Gr_p(W)$ and we define the second fundamental form $H := \pi_Q di_S$, which is a 1-form with values in $\mathrm{Hom}(S, Q) \cong S^* \otimes Q$.

The Levi–Civita connection is also induced from connections ∇^S and ∇^Q.

3 Harmonic Maps into Grassmannians

If $f : M \to Gr_p(W)$ is a smooth map, then we pull back a fiber metric and a connection on $Q \to Gr_p(W)$ to obtain a fibre metric g_V and a connection ∇^V on the pull-back bundle $f^*Q \to M$, which is denoted by $V \to M$.

The second fundamental forms are also pulled back and denoted by the same symbols $H \in \Gamma(f^*T^* \otimes f^*S^* \otimes V)$ and $K \in \Gamma(f^*T^* \otimes V^* \otimes f^*S)$.

We still have a bundle epimorphism $\pi_V : \underline{W} \to V$, where $\underline{W}$ denotes a trivial bundle $M \times W \to M$. However, note that we have only a linear map $W \to \Gamma(f^*Q)$, because it may not be an injection. Even if the linear map is not injective, we shall also call W a space of sections of $V \to M$.

We assume that M is a Riemannian manifold with metric g, and let $\{e_i\}_{i=1,2,\cdots m}$ be an orthonormal frame field of M. Then, we use the Riemannian structure on M and the pull-back connection on $V \to M$ to define the Laplace operator $\Delta^V = \Delta = -\sum_{i=1}^n \nabla^V_{e_i} \left(\nabla^V\right)(e_i)$ acting on sections of $V \to M$. We will also introduce a bundle homomorphism $A \in \Gamma$ (End V) defined as the trace of the composition of the second fundamental forms:

$$A := \sum_{i=1}^m H_{e_i} K_{e_i},$$

We call $A \in \Gamma(\mathrm{End}\,V)$ the *mean curvature operator* of $f : M \to Gr_p(W)$.

Lemma 1. *The mean curvature operator A is a non-positive symmetric (or Hermitian) operator. The energy density $e(f)$ is equal to* $-\mathrm{trace}\,A$.

For $t \in \Gamma(V)$, Z_t denotes the zero set of t: $Z_t = \{x \in M \mid t(x) = 0\}$.

Definition 2. A space of sections W of a vector bundle $V \to M$ has the *zero property* for the Laplacian if $Z_t \subset Z_{\Delta t}$ for an arbitrary $t \in W$.

Every eigenspace of the Laplacian has the zero property.

Theorem 2. *Let (M, g) be an m-dimensional Riemannian manifold and $f : M \to Gr_p(W)$ a smooth map. We fix a scalar product $(\cdot, \cdot)$ on W, which gives a Riemannian structure on $Gr_p(W)$.*

Then, the following two conditions are equivalent:

1. *$f : M \to Gr_p(W)$ is a harmonic map.*
2. *W has the zero property for the Laplacian.*

Under these conditions, we have for an arbitrary $t \in W$, $\Delta t = -At$ and $e(f) = -\mathrm{trace}\,A$, where A is the mean curvature operator of f.

Theorem 3. *Under the same assumption as in Theorem 2, we have that the following two conditions are equivalent:*

1. *$f : M \to Gr_p(W)$ is a harmonic map and there exists a function $h(x)$ such that $A_x = -h(x)Id_V$ for an arbitrary $x \in M$.*
2. *There exists a function h on M such that $\Delta t = ht$ for an arbitrary $t \in W$.*

Moreover, under the above conditions, we have $e(f) = qh$, where $q = rank\, Q$.

4 A Generalization of Theory of Do Carmo and Wallach

In this section, we give a generalization of Do Carmo–Wallach theory.

Definition 3. Let $V \to M$ be a vector bundle and W a space of sections of $V \to M$. We define an evaluation homomorphism $ev : \underline{W} \to V$ in such a way that $ev(t)(x) := t(x) \in V_x$ for $t \in W$ and $x \in M$. The vector bundle $V \to M$ is said to be *globally generated by* W if the evaluation homomorphism $ev : \underline{W} \to V$ is surjective.

Definition 4. Let $V \to M$ be a real or complex vector bundle of rank q which is globally generated by W of dimension N. If the real vector bundle $V \to M$ has an orientation, we also fix an orientation on W. Then we have a map $f : M \to Gr_p(W)$, where $Gr_p(W)$ is a real (oriented) or complex Grassmannian according to the co-efficient field of $V \to M$ and $p = N - q$. The map f is defined by

$$f(x) := \mathrm{Ker}\, ev_x = \{t \in W \mid t(x) = 0\}.$$

We call $f : M \to Gr_p(W)$ the *induced map by* $(V \to M, W)$, or the *induced map by* W, if the vector bundle $V \to M$ is specified.

From the definition of the induced map $f : M \to Gr_p(W)$, the vector bundle $V \to M$ can be naturally identified with $f^*Q \to M$. Conversely, if $f : M \to Gr_p(W)$ is a smooth map, then $f : M \to Gr_p(W)$ can be recognized as the induced map by $(f^*Q \to M, W)$.

Let $M = G/K_0$ be a compact reductive Riemannian homogeneous space with decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where G is a compact Lie group and K_0 is a closed subgroup of G.

Let V_0 be a q-dimensional real or complex K_0-representation space with a K_0-invariant scalar product. We can construct a homogeneous vector bundle $V \to M$, $V := G \times_{K_0} V_0$ with an invariant fibre metric g_V induced by the scalar product on V_0. Moreover $V \to M$ has a canonical connection ∇ with respect to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. (This means that the horizontal distribution is defined as $\{L_g \mathfrak{m} \mid g \in G\}$ on the principal fibre bundle $G \to M$, where L_g is the left translation.)

A Lie group G naturally acts on the space of sections $\Gamma(V)$ of $V \to M$, which has a G-invariant L_2-inner product.

Using the Levi–Civita connection and ∇, we can decompose the space of sections of $V \to M$ into the eigenspaces of the Laplacian:

$$\Gamma(V) = \oplus_\mu W_\mu, \quad W_\mu := \{t \in \Gamma(V) \mid \Delta t = \mu t\}.$$

It is well-known that W_μ is a finite dimensional G-representation space with a G-invariant scalar product inherited from the L_2-inner product.

Lemma 2. *Let W be a G-submodule of W_μ. If W globally generates $V \to G/K_0$, then V_0 can be regarded as a subspace of W.*

4.1 Standard Maps

Suppose that an eigenspace W_μ globally generates $V \to M$. Then we define the induced map $f_0 : M \to Gr_p(W_\mu)$ by W_μ, where $p = N - q$, $N = \dim W_\mu$, which is called *the standard map* by W_μ.

In general, W_μ is not irreducible as G-representation. Let W be a G-submodule of W_μ and suppose that W globally generates $V \to G/K_0$. Then the induced map by W is also called the standard map by W.

Since $V_0 \subset W$ by Lemma 2, we have the orthogonal complement of V_0 denoted by U_0. Then the induced map $f_0 : M \to Gr_p(W)$ is expressed as $f_0([g]) = gU_0 \subset W$, which is G-equivariant.

Suppose that we have a standard map by W. Next, we consider the pull-back connection ∇^V. Then we have

Lemma 3. *The pull-back connection ∇^V is the canonical connection if and only if* $\mathfrak{m}V_0 \subset U_0$.

Lemma 4. *If a G-module (ϱ, W) satisfies the condition $\mathfrak{m}V_0 \subset U_0$, then the standard map $f_0 : M \to Gr_p(W)$ is harmonic and we have $e(f_0) = q\mu$, and $A = -\mu Id_V$.*

4.2 A Generalization of Do Carmo–Wallach Theory

Let G be a compact Lie group and W a real or complex representation of G with an invariant scalar product $(\cdot,\cdot)_W$. We denote by $\mathrm{H}(W)$ the set of symmetric or Hermitian endomorphisms of W depending on W being a real or complex vector space. We equip $\mathrm{H}(W)$ with an inner product $(\cdot,\cdot)_H$; $(A, B)_H := \mathrm{trace}\, AB$, for$A, B \in \mathrm{H}(W)$. It is easily seen that $(\cdot,\cdot)_H$ is G-invariant. We define a symmetric or Hermitian operator $H(u, v)$ for $u, v \in W$ as

$$H(u,v) := \frac{1}{2}\left\{u \otimes (\cdot, v)_W + v \otimes (\cdot, u)_W\right\}.$$

If U and V are subspaces of W, we define a real subspace $H(U, V) \subset \mathrm{H}(W)$ spanned by $H(u, v)$ where $u \in U$ and $v \in V$. In a similar fashion, $GH(U, V)$ denotes the subspace of $H(W)$ spanned by $gH(u, v)$, where $g \in G$.

In this section, $\mathbf{K}$ denotes either $\mathbf{R}$ or $\mathbf{C}$. Symmetric operators are also called Hermitian operators, for simplicity.

Definition 5. Let $f : M \to Gr_p(\mathbf{K}^m)$ be a map and we regard $\mathbf{K}^m$ as a space of sections of $f^*Q \to M$. Then the map $f : M \to Gr_p(\mathbf{K}^m)$ is called a *full map* if the linear map $\mathbf{K}^m \to \Gamma(f^*Q)$ is injective.

When $\mathbf{K}^m$ has a scalar product, two equivalence relations of maps are given.

Definition 6. Let f_1 and $f_2 : M \to Gr_p(\mathbf{K}^m)$ be maps. Then f_1 is called *image equivalent* to f_2, if there exists an isometry ϕ of $Gr_p(\mathbf{K}^m)$ such that $f_2 = \phi \circ f_1$.

Notice that an isometry ϕ of $Gr_p(\mathbf{K}^m)$ gives a bundle isomorphism of $Q \to Gr_p(\mathbf{K}^m)$ denoted by $\tilde{\phi}$ which covers ϕ. If we have a map $f : M \to Gr_p(\mathbf{K}^m)$, then $\tilde{\phi}$ induces a bundle isomorphism from $f^*Q \to M$ to $f^*\tilde{\phi}Q \to M$ denoted by the same symbol, which is the pull-back bundle of the quotient bundle by $\phi \circ f$.

Definition 7. Let $V \to M$ be a vector bundle and f a map from M into $Gr_p(\mathbf{K}^m)$ such that $f^*Q \to M$ is isomorphic to $V \to M$. We consider a pair (f, ϕ), where $\phi : V \to f^*Q$ is a bundle isomorphism. Then such pairs (f_i, ϕ_i),$(i = 1, 2)$ are called *gauge equivalent*, if there exists an isometry ϕ of $Gr_p(\mathbf{K}^m)$ such that $f_2 = \phi \circ f_1$ and $\phi_2 = \tilde{\phi} \circ \phi_1$.

By definition, gauge equivalence yields image equivalence of maps.

Theorem 4 ([10]). *Let G/K_0 be a compact reductive Riemannian homogeneous space with decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. We fix a homogeneous vector bundle $V = G \times_{K_0} V_0 \to G/K_0$ of rank q with an invariant metric and the canonical connection.*

Let $f : G/K_0 \to Gr_p(\mathbf{K}^m)$ be a full harmonic map satisfying the following two conditions.

(i) *The pull-back bundle $f^*Q \to M$ with the pull-back metric and connection is gauge equivalent to $V \to G/K_0$ with the invariant metric and the canonical connection. (Hence, $q = m - p$.)*
(ii) *The mean curvature operator $A \in \Gamma(\mathrm{End}\, V)$ of f is expressed as $-\mu Id_V$ with some real positive number μ, and so $e(f) = \mu q$.*

Then there exist an eigenspace $W \subset \Gamma(V)$ with an eigenvalue μ of the Laplacian equipped with L_2-scalar product $(\cdot,\cdot)_W$ and a semi-positive Hermitian endomorphism $T \in \mathrm{End}\,(W)$. We regard W as $\mathfrak{g}$-representation (ϱ, W). The pair (W, T) satisfies the following four conditions.

(I) *The vector space $\mathbf{K}^m$ is a subspace of W with the inclusion $\iota : \mathbf{K}^m \to W$ and $V \to G/K_0$ is globally generated by $\mathbf{K}^m$.*
(II) *As a subspace, $\mathbf{K}^m = \mathrm{Ker}\, T^\perp$ and the restriction of T on $\mathbf{K}^m$ is a positive Hermitian transform.*
(III) *The endomorphism T satisfies*

$$\left(T^2 - Id_W, GH(V_0, V_0)\right)_H = 0, \ \left(T^2, GH(\varrho(\mathfrak{m})V_0, V_0)\right)_H = 0. \tag{1}$$

(IV) *The endomorphism T gives an embedding of $Gr_p(\mathbf{K}^m)$ into $Gr_{p'}(W)$, where $p' = p + \dim \mathrm{Ker}\, T$ and also gives a bundle isomorphism $\phi : V \to f^*Q$.*

Then, $f : G/K_0 \to Gr_p(\mathbf{K}^m)$ can be expressed as

$$f\left([g]\right) = \left(\iota^* T \iota\right)^{-1}\left(f_0\left([g]\right) \cap \mathrm{Ker}\, T^\perp\right), \tag{2}$$

where ι^* denotes the adjoint operator of ι under the induced scalar product on $\mathbf{K}^m$ from $(\cdot,\cdot)_W$ on W and f_0 is the standard map by W. Such two pairs (f_i, ϕ_i), $(i = 1, 2)$ are gauge equivalent if and only if $\iota_1^* T_1 \iota_1 = \iota_2^* T_2 \iota_2$, where (T_i, ι_i) correspond to f_i $(i = 1, 2)$ under the expression in (2), respectively.

Conversely, suppose that a vector space $\mathbf{K}^m$, an eigenspace $W \subset \Gamma(V)$ with an eigenvalue μ and a semi-positive Hermitian endomorphism $T \in \mathrm{End}\,(W)$ satisfy conditions (I), (II) and (III). Then we have a unique embedding of $Gr_p(\mathbf{K}^m)$ into $Gr_{p'}(W)$ and a map $f : G/K_0 \to Gr_p(\mathbf{K}^m)$ defined as (2) is a full harmonic map into $Gr_p(\mathbf{K}^m)$ satisfying conditions (i) and (ii) with a bundle isomorphism $V \cong f^*Q$.

Remark 1. When the sphere S^{N-1} is regarded as an oriented Grassmannian of hyperplanes $Gr_{N-1}(\mathbf{R}^N)$, a map $f : M \to Gr_{N-1}(\mathbf{R}^N)$ gives a trivialization of $f^*Q \to M$. Hence when the target is the sphere, we can drop condition (i) in Theorem 4.

Remark 2. When the target is a symmetric space of rank 1, the quotient bundle is also of rank 1. Hence condition (ii) in Theorem 4 is equivalent to the condition that f has constant energy density.

Remark 3. The role of condition (IV) in Theorem 4 should be emphasized. Whenever we consider a full harmonic map from G/K_0 into $Gr_p(\mathbf{K}^m)$ with conditions (i) and (ii), we also have an embedding of $Gr_p(\mathbf{K}^m)$ into $Gr_{p'}(W)$ and a bundle isomorphism $V \to f^*Q$. Hence we can consider the moduli space $\mathscr{M}$ of those *full* maps into $Gr_{p'}(W)$ by *gauge equivalence*. Moreover Theorem 4 interprets the compactification $\overline{\mathscr{M}}$ of $\mathscr{M}$. Under an appropriate assumption, we can deduce that $\mathscr{M}$ is a bounded *open* convex body in a G-submodule of $\mathrm{H}(W)$ with topology induced by L_2-metric. Then each boundary point (W, T) of $\overline{\mathscr{M}}$ corresponds to an *embedding* of $Gr_p(\mathrm{Ker}\, T^{\perp})$ into $Gr_{p'}(W)$ and a *full* map into $Gr_p(\mathrm{Ker}\, T^{\perp})$ with a bundle isomorphism under the compactification in the topology (see Theorem 8).

If we consider the image equivalence relation, then we also need to take the action of the centralizer of the holonomy group on the pull-back bundle into account. This requires a case-by-case consideration, and so we give examples in Sects. 5 and 6.

5 Harmonic Maps into Complex Projective Spaces

We introduce two theorems which are proved in independent ways. A unified proof can be given in the light of Theorem 4.

Theorem 5 ([1]). *Let $f : \mathbf{C}P^1 \to \mathbf{C}P^n$ be a full harmonic map with constant energy density. Then f is an* SU(2)*-equivariant map, in other words, it is a standard map up to gauge equivalence.*

Theorem 6 ([2]). *Let $f : \mathbf{C}P^m \to \mathbf{C}P^n$ be a full holomorphic map with constant energy density. Then f is an* $\mathrm{SU}(m+1)$*-equivariant map, in other words, it is a standard map up to gauge equivalence.*

When we regard $\mathbf{C}P^n$ as a complex Grassmannian $Gr_n(\mathbf{C}^{n+1})$ in both cases, the pull-back bundle has a holomorphic vector bundle structure induced by the pull-back connection. Since $\mathbf{C}P^m$ is Fano, the holomorphic line bundle structure is unique. Then we can easily construct a gauge transformation satisfying condition (i) in Theorem 4, because the pull-back bundle is of rank 1 and the compatible connection with fibre metric and holomorphicity is unique. Combined with Remarks after Theorem 4, we can apply Theorem 4 to get both results.

Toth gives a conception of *polynomial harmonic map* between complex projective spaces [12]. In the definition of polynomial harmonic map, Toth makes use of the Hopf fibration to get *polynomial maps* and implicitly requires condition (i) in Theorem 4 as *horizontality*. Theorem 2 implies that the former condition is not needed to develop the theory.

Lemma 5 ([10]). *Let $f : \mathbf{C}P^m \to \mathbf{C}P^n$ $(m \geqq 2)$ be a harmonic map with constant energy density. Then f is a polynomial harmonic map in the sense of Toth if and only if f satisfies the condition (i) in Theorem 4.*

Toth gives an estimate of the dimension of the moduli space by image equivalence relation. Since the pull-back bundle is holonomy irreducible, Schur's lemma implies that the moduli by image equivalence is the same as the moduli by gauge equivalence of maps. Consequently, we can apply Toth's result to get an estimate the dimension of the moduli space by gauge equivalence.

6 Holomorphic Isometric Embeddings of the Projective Line into Quadrics

Though research on harmonic maps from the projective line into quadrics has been pursued before from various viewpoints (for example, [3, 6, 8] and [13]), we would like to apply Theorem 4 to give a description of the moduli.

A complex quadric of $\mathbf{C}P^{n+1}$ is now realized as a real oriented Grassmannian $Gr_n(\mathbf{R}^{n+2})$. Then the quotient bundle has a holomorphic bundle structure. However we regard the quotient bundle as a real vector bundle of rank 2, when applying Theorem 4. Note that the curvature form of the canonical connection is the fundamental 2-form ω_Q on $Gr_n(\mathbf{R}^{n+2})$ up to a multiple constant. Denote by ω_0 the fundamental 2-form on $\mathbf{C}P^1$.

Definition 8. Let $f : \mathbf{C}P^1 \to Gr_n(\mathbf{R}^{n+2})$ be a holomorphic embedding. Then f is called an isometric embedding of degree k if $f^*\omega_Q = k\omega_0$ (and so, k must be a positive integer).

Lemma 6 ([9]). *Let $f : \mathbf{C}P^1 \to Gr_n(\mathbf{R}^{n+2})$ be a holomorphic embedding. Then f is an isometric embedding of degree k if and only if f satisfies the condition (i) in Theorem 4. Under these conditions, the condition (ii) in Theorem 4 is automatically satisfied.*

If the degree of f is equal to k, then the pull-back of the quotient bundle is regarded as the holomorphic line bundle of degree k on $\mathbf{C}P^1$. The uniqueness of the Einstein–Hermitian connection yields the result.

We also use the Einstein–Hermitian connection to obtain that any holomorphic section of $\mathcal{O}(k) \to \mathbf{C}P^1$ is an eigensection. Since $\mathbf{R}^{n+2}$ is a real subspace of $H^0\left(\mathbf{C}P^1, \mathcal{O}(k)\right)$, it follows from Theorem 3 that the mean curvature operator is proportional to the identity with a constant multiple.

Hence we can apply Theorem 4 to obtain the moduli space $\mathcal{M}_k$ of holomorphic isometric embeddings of degree k by gauge equivalence of maps. Using Lemma 4, the standard map by $H^0\left(\mathbf{C}P^1, \mathcal{O}(k)\right)$ of dimension $k+1$ is a holomorphic isometric embedding of degree k. When k is even, say $2l$, $H^0\left(\mathbf{C}P^1, \mathcal{O}(2l)\right)$ has an invariant real subspace denoted by W of real dimension $2l + 1$. Since W also globally generates $\mathscr{O}(2l) \to \mathbf{C}P^1$, we have a standard map by W which turns out to be a holomorphic isometric embedding of degree $2l$ by Lemma 4.

Theorem 7 ([9]). *Let* $f : \mathbf{C}P^1 \to Gr_{2k-1}(\mathbf{R}^{2k+1})$ *be a holomorphic embedding of degree* $2k$*. Then* f *is the standard map by* W *up to gauge equivalence.*

To state Theorem 8, we denote by $S^{2l}\mathbf{C}^2$ the symmetric power of the standard representation $\mathbf{C}^2$ of SU(2). We apply Theorem 4 to obtain

Theorem 8 ([9]). *Let* $\mathscr{M}_k$ *be the moduli space of holomorphic isometric embeddings of degree* k *of the complex projective line into* $Gr_{2k}(\mathbf{R}^{2k+2})$ *by gauge equivalence of maps. Then* $\mathscr{M}_k$ *can be regarded as a compact convex body in* $\bigoplus_{l=1}^{k \geqq 2l} S^{2k-4l}\mathbf{C}^2$*. The interior points of the moduli correspond to full maps and the boundary points of* $\mathscr{M}_k$ *correspond to maps whose images are included in specified totally geodesic submanifolds* $Gr_p(\mathbf{R}^{p+2})$ *of* $Gr_{2k}(\mathbf{R}^{2k+2})$*, where* $p < 2k$*. Each totally geodesic submanifold* $Gr_p(\mathbf{R}^{p+2})$ *is specified as the common zero set of some sections of* $Q \to Gr_{2k}(\mathbf{R}^{2k+2})$*, which belongs to* $\mathbf{R}^{2k+2}$*.*

We can show that the centralizer of the holonomy group S^1 acts on $\mathscr{M}_k$ with a scalar multiplication. Hence we have

Theorem 9 ([9]). *Let* $\mathbf{M}_k$ *be the moduli space of holomorphic isometric embeddings of degree* k *of the complex projective line into* $Gr_{2k}(\mathbf{R}^{2k+2})$ *by image equivalence of maps. Then* $\mathbf{M}_k = \mathscr{M}_k/S^1$*.*

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Conformal Transformations Between Complete Product Riemannian Manifolds

Byung Hak Kim

Abstract Nagano proved that if the non-homothetic conformal transformation between complete Riemannian manifolds with parallel Ricci tensor is admitted, then the manifolds are irreducible and isometric to a sphere. From this result and other reasons, it is natural to ask for the problem that does there exist globally a non-homothetic conformal transformation between complete product Riemannian manifolds? In this talk, we introduce and consider about this question and related topics.

1 Introduction

As it well known that the conformal transformation on the Riemannian manifold does not change the angle between two vectors at a point and characterized by a change of a Riemannian metric. A conformal transformation in an Einstein manifold is concircular transformation [1]. By use of this fact, Yano and Nagano [6] proved that if a complete Einstein manifold admits a global one-parameter group of non-isometric conformal transformations, then the manifold is isometric to a sphere. Tashiro [1, 5] studied structures of complete Riemannian manifolds admitting a concircular scalar field and devoted to a study of infinitesimal conformal transformations in complete product Riemannian manifolds. In 1967, Tashiro and Miyashita [2] proved that if a complete reducible Riemannian manifold admits a complete non-isometric conformal vector field, then the manifold is locally Euclidean and the vector field is homothetic.

From these facts and other reasons, Tashiro conjectured that there were no global non-isometric conformal transformation between complete product Riemannian manifolds which are not Euclidean. In other words, is a conformal transformation between complete product Riemannian manifolds necessarily isometric?

In this point of a view, I am going to give survey report and results related to this conjecture with the works of Tashiro [3–5].

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2 Theorems and an Example

Let (M, g, F) and (M^*, g^*, G) be Riemannian manifolds of dimension $n \geqq 3$ with metric g and g^* and product structure F and G respectively. Under a diffeomorphism f of M to M^*, the image of a quantity on M^* to M by the induced map f^* will be denoted by the same character as the original. Greek indices run from 1 to n.

The product structures F and G are by definition (1,1)-tensor fields (F_λ^κ) and (G_λ^κ) different from the unit tensor I and satisfying $F^2 = I$ and $G^2 = I$. The structures F and G are said to be *commutative* with one another under a diffeomorphism $f : M \to M^*$ if $FG = GF$.

Let M and M^* be products $M = M_1 \times M_2$ and $M^* = M_1^* \times M_2^*$ respectively. The dimensions of the parts M_1, M_2, M_1^* and M_2^* are denoted by n_1, n_2, n_1^* and n_2^* respectively, $n_1 + n_2 = n_1^* + n_2^* = n$. Latin indices will run on the following ranges

$$h, i, j, k = 1, 2, \cdots, n_1;$$

$$p, q, r, s = n_1 + 1, \cdots, n.$$

A conformal diffeomorphism $f : M \to M^*$ is characterized by a metric change

$$g^*_{\mu\lambda} = \frac{1}{\rho^2} g_{\mu\lambda}$$

where ρ is a positive valued scalar field said to be *associated* with f.

Theorem 1. *There is no global conformal diffeomorphism between complete product Riemannian manifolds M and M^* under which the product structures F and G are not commutative in an open subset of M.*

The following example is a global conformal diffeomorphism under which the product structures F and G are commutative.

Example 1. Let S be a unit circle and T^3 a 3-dimensional torus $S \times S \times S$ of three copies of S. Denote by x, y, z arc lengths modulo 2π of the copies. Take a positive valued function $\rho(y)$ with period 2π, for example $\rho = \sin y + 2$. Consider two Riemannian manifolds M and M^* on the same underlying manifold T^3 with the metrics

$$ds^2 = \{\rho(y)\}^2 dx^2 + dy^2 + dx^2,$$

$$ds^{*2} = dx^2 + \frac{1}{\{\rho(y)\}^2}(dy^2 + dz^2)$$

respectively. These are conformally related with $\rho(y)$ as associated scalar field. The first manifold M is the product $M_1 \times M_2$ of the 2-dimensional manifold M_1 with metric $\rho^2 dx^2 + dy^2$ and the circle $M_2 = S$. The second M^* is the product

$M_1^* \times M_2^*$ of the circle $M_1^* = S$ and M_2^* with metric $(dy^2 + dz^2)/\rho^2$ on T^2. These manifolds are compact and consequently complete. The product structures F and G are given by

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad G = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively, and commutative with one another.

Theorem 2. *Let product Riemannian manifolds M and M^* be complete. If there is a global non-homothetic conformal diffeomorphism f of M onto M^*, then the underlying manifold of M and M^* is the product $N_1 \times N_0 \times N_2$ of three complete Riemannian manifolds, and ρ depends on one part only, say N_0.*

Denoting their metric forms by ds_1^2, ds_0^2 and ds_2^2, then

(i) M is the product $M_1 \times N_2$, where M_1 is irreducible, and the metric form of M is written as

$$\rho^2 ds_1^2 + ds_0^2 + ds_2^2 \tag{1}$$

on the underlying manifold $N_1 \times N_0 \times N_2$, and

(ii) M^* is the product $N_1 \times M_2^*$, where M_2^* is irreducible, and the metric form of M^* is written as

$$ds_1^2 + \rho^{-2}(ds_0^2 + ds_2^2) \tag{2}$$

on the same underlying manifold $N_1 \times N_0 \times N_2$

3 Almost Product Riemannian Structure and Separate Coordinate System

Conditions for (M, g, F) and (M^*, g^*, G) to be almost product Riemannian structures are

$$g_{\nu\mu} F_\lambda^\nu F_\kappa^\mu = g_{\lambda\kappa}, \quad g^*_{\nu\mu} G_\lambda^\nu G_\kappa^\mu = g^*_{\lambda\kappa} \tag{3}$$

and integrability conditions for them to be product Riemannian ones are

$$\nabla_\mu F_\lambda^\kappa = 0, \quad \nabla^*_\mu G_\lambda^\kappa = 0 \tag{4}$$

The covariant tensors $F_{\lambda\kappa} = F_\lambda^\omega g_{\omega\kappa}$ and $G^*_{\lambda\kappa} = G_\lambda^\omega g^*_{\omega\kappa}$ are symmetric and the conditions (4) are equivalent to

$$\nabla_\mu F_{\lambda\kappa} = 0, \quad \nabla^*_\mu G^*_{\lambda\kappa} = 0 \tag{5}$$

We put $\rho_\lambda = \nabla_\lambda \rho$ and denote the gradient vector field by $Y = (\rho^\kappa)$. Then the Christoffel symbol is transformed by the formula

$$\left\{ \begin{matrix} \kappa \\ \mu\lambda \end{matrix} \right\}^* = \left\{ \begin{matrix} \kappa \\ \mu\lambda \end{matrix} \right\}^* - \frac{1}{\rho}(\delta^\kappa_\mu \rho_\lambda + \delta^\kappa_\lambda \rho_\mu - g_{\mu\lambda}\rho^\kappa) \tag{6}$$

Under a conformal diffeomorphism f, we have the equations

$$G^\lambda_\mu G^\kappa_\lambda = \delta^\kappa_\mu, \quad g_{\nu\mu} G^\nu_\lambda G^\mu_\kappa = g_{\lambda\kappa}$$

which mean that G constitutes an almost product Riemannian structure together with g on M but not necessarily integrable. The covariant tensor $G_{\mu\lambda} = G^\kappa_\mu g_{\lambda\kappa}$ is symmetric. Substituting $G^*_{\mu\lambda} = G_{\mu\lambda}/\rho^2$ into the second Eq. (5) and (6), we can obtain the differential equation

$$\nabla_\mu G_{\lambda\kappa} = -\frac{1}{\rho}(G_{\mu\lambda}\rho_\kappa + G_{\mu\kappa}\rho_\lambda - g_{\mu\lambda}G_{\kappa\omega}\rho^\omega - g_{\mu\kappa}G_{\lambda\omega}\rho^\omega), \tag{7}$$

and applying Ricci's formula to this equation, we have

$$\begin{aligned} &\rho(K^\omega_{\nu\mu\lambda} G_{\omega\kappa} + K^\omega_{\nu\mu\kappa} G_{\omega\lambda}) \\ &\quad = G_{\mu\lambda}\nabla_\nu \rho_\kappa + G_{\mu\kappa}\nabla_\nu\rho_\lambda - G_{\nu\lambda}\nabla_\mu\rho_\kappa - G_{\nu\kappa}\nabla_\mu\rho_\lambda \\ &\qquad - g_{\mu\lambda}[(\nabla_\nu\rho^\omega)G_{\kappa\omega} - \frac{\Phi}{\rho}G_{\nu\kappa}] - g_{\mu\kappa}[(\nabla_\nu\rho^\omega)G_{\lambda\omega} - \frac{\Phi}{\rho}G_{\nu\lambda}] \\ &\qquad + g_{\nu\lambda}[(\nabla_\mu\rho^\omega)G_{\kappa\omega} - \frac{\Phi}{\rho}G_{\mu\kappa}] + g_{\nu\kappa}[(\nabla_\mu\rho^\omega)G_{\lambda\omega} - \frac{\Phi}{\rho}G_{\mu\lambda}] \end{aligned} \tag{8}$$

where $K^\kappa_{\nu\mu\lambda}$ is the curvature tensor of M and $\Phi = |Y|^2 = \rho_\kappa\rho^\kappa$.

These equations play important roles and an efficient method is to use freely equations referred with a separate coordinate system.

In the product Riemannian manifold $M = M_1 \times M_2$, there is locally a separate coordinate system (x^h, x^p) such that the metric form of M is expressed as

$$ds^2 = g_{ji}(x^h)dx^j dx^i + g_{rq}(x^p)dx^r dx^q$$

The product structure F has components

$$\begin{pmatrix} \delta^h_i & 0 \\ 0 & -\delta^p_q \end{pmatrix}$$

in such a coordinate system to within signature. The Christoffel symbol $\left\{ \begin{matrix} \kappa \\ \mu\lambda \end{matrix} \right\}$ and the curvature tensor $K^\kappa_{\nu\mu\lambda}$ have pure components only. The covariant differentiations ∇_i along M_1 and ∇_q along M_2 are commutative.

If F and G are commutative under a diffeomorphism, then we have $FGF = G$ which is equivalent to the purity of G with respect to F, i.e., all the hybrid components G_q^h, G_i^p and G_{qi} in a separate coordinate system vanish.

Lemma 1. *A conformal diffeomorphism f of (M, g, F) to (M^*, g^*, G) is a homothety if and only if G constitutes a product structure together with g on M, that is,*

$$\nabla_\mu G_{\lambda\kappa} = 0. \tag{9}$$

Then the structures F and G are commutative under f.

Proof. If f is a homothety, then ρ is a constant. Substituting $\rho_\lambda = 0$ into (7), we have (9). Conversely if (9) is satisfied, then contracting the Eq. (7) put equal to 0 with ρ^κ, we have

$$\Phi G_{\mu\lambda} - g_{\mu\lambda} G_{\kappa\omega}\rho^\kappa\rho^\omega = \rho_\mu G_{\lambda\omega}\rho^\omega - \rho_\lambda G_{\mu\omega}\rho^\omega.$$

Since the left hand side is symmetric in λ and μ and the right hand side is skew-symmetric, both of the sides are equal to 0. By account of $G \neq \pm I$, we can see that $\rho_\lambda = 0$ and ρ is a constant.

Then M is decomposed into the product of a number of irreducible parts. Taking account of (9) on each part and $G^2 = I$, we can see that G is a diagonal matrix having ± 1 as diagonal components. Hence we have $FG = GF$.

We put the parts $Y_1 = (\rho^\kappa)$ and $Y_2 = (\rho^p)$ of $Y = (\rho^\kappa)$ belonging to M_1 and M_2 respectively. If ρ is independent of points of M_2, then $Y_2 = (\rho^p)$ vanishes identically.

When we refer an equation to a separate coordinate system and restrict indices to the parts, for example, $\kappa = \lambda = i, \mu = j, \nu = p$ in (8), we indicate $(\kappa, \lambda, \mu, \nu) = (i, i, j, p)$.

Lemma 2. *If F and G are commutative under a non-homothetic conformal diffeomorphism f, then ρ is a function on either of M_1 or M_2 only.*

Proof. Since F and G are commutative, the hybrid components G_{pi} in M all vanish. The Eq. (7) splits into the equations

$$\begin{cases} \nabla_j G_{ih} = -\rho^{-1}(G_{ji}\rho_h + G_{jh}\rho_i - g_{ji}G_{hk}\rho^k - g_{jh}G_{ik}\rho^k), \\ \nabla_q G_{ji} = 0, \\ \nabla_j G_{pi} = -\rho^{-1}(G_{ji}\rho_p - g_{ji}G_{pr}\rho^r) = 0, \\ \nabla_q G_{pi} = -\rho^{-1}(G_{qp}\rho_i - g_{qp}G_{ih}\rho^h) = 0, \\ \nabla_j G_{qp} = 0, \\ \nabla_r G_{qp} = -\rho^{-1}(G_{rq}\rho_p + G_{rp}\rho_q - g_{rq}G_{ps}\rho^s - g_{rp}G_{qs}\rho^s). \end{cases} \tag{10}$$

The second and fifth equations of (10) are reduced to $\partial_q G_{ji} = 0$ and $\partial_j G_{qp} = 0$, which mean that the part (G_{ji}) depends on M_1 only and $(G_q p)$ depends on M_2 only.

If there are points P nd Q in U such that $\rho_i(P) \neq 0$ and $\rho_q(Q) \neq 0$, then the third equation of (10) implies that G_1 is proportional to I_1 and hence $G_1 = \pm I_1$ in U because of $G^2 = I$ and the independence of G_1 on M_2. Similarly from the fourth equation of (10), we have $G_2 = \pm I_2$. Since G is different from I, we have $G = \pm F$ and hence f is a homothety by Lemma 1. This contradicts the non-homothety of f. Therefore ρ should be dependent of one part only.

A converse to Lemma 2 is the following.

Lemma 3. *If ρ depends on one part only, say M_1, but not a constant, then G is commutative with F under f, or ρ satisfies the equation*

$$\nabla_j \rho_i = c^2 \rho g_{ji} \tag{11}$$

where c is a positive constant.

Proof. By the assumption, $\rho_p = 0$ and $\nabla_p \rho_\lambda = 0$. We suppose G is not commutative with F, and a hybrid component G_{qi} does not vanish. Putting $(\kappa, \lambda, \mu, \nu) = (h, i, j, p)$ in (8), we have

$$G_{qi}(\nabla_j \rho_h - \frac{\Phi}{\rho} g_{jh}) + G_{qh}(\nabla_j \rho_i - \frac{\Phi}{\rho} g_{ji}) = 0.$$

The expression $G_{qi}(\nabla_j \rho_h - \frac{\Phi}{\rho} g_{jh})$ is symmetric in h and j.

On the other hand, the equation above means this expression is skew-symmetric in h and i. Hence the expression vanishes, and we can see that

$$\nabla_j \rho_i = \frac{\Phi}{\rho} g_{ji}. \tag{12}$$

Contracting (12) with $2\rho^i$, integrating and putting $\Phi = \rho_i \rho^i = c^2 \rho^2$, we can obtain the Eq. (11).

We prepare Lemmas 4 and 5, but we omit the proof.

Lemma 4. *Suppose that $Y_1 = (\rho^h)$ and $Y_2 = (\rho^p)$ do not vanish in an open subset U. Then the square Φ of the length of $|Y|^2 = \rho_\kappa \rho^\kappa$ is decomposable in U, i.e. a sum*

$$\Phi = \rho_\kappa \rho^\kappa = \Phi_1 + \Phi_2$$

of functions $\Phi_1(x^h)$ and $\Phi_2(x^p)$, and we obtain the equations

$$\begin{cases} \nabla_k \nabla_j \nabla_i \Phi_1 = k(2g_{ji} \nabla_k \Phi_1 + g_{kj} \nabla_i \Phi_1 + g_{ki} \nabla_j \Phi_1), \\ \nabla_r \nabla_q \nabla_p \Phi_2 = -k(2g_{qp} \nabla_r \Phi_2 + g_{rq} \nabla_p \Phi_2 + g_{rp} \nabla_q \Phi_2) \end{cases} \tag{13}$$

in U. The functions Φ_1 and Φ_2 may be replaced with Φ.

Lemma 5. *Let M and M^* be complete Riemannian manifolds and f a diffeomorphism of M onto M^*. If the length s of a differentiable curve Γ in M is extendable to the infinity, then so is the length s^* of the image $\Gamma^* = f(\Gamma)$ in M^*.*

4 Proof of Theorem 1

There are three following cases to be considered. Case (1) in Lemma 3 occurs the Eq. (11). Case (2) $k = 0$ and Case (3) $k \neq 0$ in the Eq. (13)

Let us prove Case (3) here. Along any geodesic curve with arc-length s and lying in the part M_1, we have

$$(\Phi_1(s) - k\rho^2(s))'' = k(\Phi_1 - k\rho^2) + b - k\Phi_2(P) \tag{14}$$

and

$$(\Phi_1)''' = 4k(\Phi_1)', \tag{15}$$

prime$'$ indicating derivatives in s. We put $k = c^2$ or $k = -c^2$ according as $k > 0$ or $k < 0$. The general solution of (15) is given by

$$\Phi_1 = \begin{cases} Ae^{2cs} + Be^{-2cs} + C & (k = c^2) \\ A\cos 2cs + B\sin 2cs + C & (k = -c^2) \end{cases} \tag{16}$$

and consequently, by means of (14), ρ^2 is given in the form

$$\rho^2 = \begin{cases} \dfrac{1}{c^2}(Ae^{2cs} + Be^{-2cs}) + A_1e^{cs} + B_1e^{-cs} + C_1 \\ \dfrac{1}{c^2}(A\cos 2cs + B\sin 2cs) + A_1\cos cs + B_1\sin cs + C_1, \end{cases} \tag{17}$$

where A, B, C and so on are arbitrary constants.

We may suppose $k = c^2$ in U. Take the part $M_1(P)$ through a point $P \in U$. If $M_1(P)$ contains a relatively open subset where $Y_1 = (\rho^h) = 0$ and $\Phi_1 = 0$, then, by considering the solution of (15) along a geodesic connecting P with a point in the subset, we can see that Y_1 should identically vanish. This is a contradiction.

By similar arguments, in the case where Φ_1 is given by the second equation of (16) somewhere in $M_1(P)$, we can lead a contradiction. Hence the expressions the first equation of (16) and the first equation of (17) are valid in the whole of any geodesic in $M_1(P)$.

At least one of A, B, A_1, B_1 in (17) is different from zero. If, for example, $A \neq 0$, then A should be positive and we put $A = a^2, a > 0$. Take a value s_0 so large that the inequality

$$\rho > \frac{a}{2c}e^{cs}$$

hold on for $s > s_0$. Let Γ^* be the image $f(\Gamma)$, s^* the arc length of Γ^* and s_0^* the value corresponding to s_0. Then s^* is related to s by the differential equation

$$\frac{ds^*}{ds} = \frac{1}{\rho} < \frac{2c}{a} e^{-cs}.$$

The length s^* of the image Γ^* is bounded as s tends to the infinity. This is a contradiction by Lemma 5. Hence, in Case (3), there is no global conformal diffeomorphism.
Case (1) and (2) can be proved by similar arguments.

5 Proof of Theorem 2

Notice that Lemma 2 is globally valid. Therefore the product structure F and G are commutative everywhere and ρ may be assumed as a function of M_1 only. We have $G_2 = \pm I_2$ as seen in the proof of Lemma 2. We choose $G_2 = I_2$, then G_{qp} in a separate coordinate system in M.

Let $M_1(P)$ be the part M_1 passing through any point P of M and M_1' the image $f(M_1(P))$ by f. If we denote by $\overline{ds_1}^2$ and ds_2^2 the metric forms of M_1 and M_2 respectively, then the induced metric form of M_1' in M^* is identical with $\rho^{-2}\overline{ds_1^2}$. The part $M_1(P)$ is simply connected and so is the image M_1'. Since M^* is complete, the closed submanifold M_1' is also complete.

Since the first of Eq. (10) leads to the integrability condition $\nabla_j^* G_{ih} = 0$ of $G_1 = (G_i^h)$ in M_1' and we have chosen $G_2 = I_2$, the structure G_1 on M_1' can be written in the form $G_1 = -I_1$ or

$$G_1 = \begin{pmatrix} \text{-I} & 0 \\ 0 & \text{I} \end{pmatrix} \tag{18}$$

If $G_1 = -I_1$, we have $G = \pm F$ and hence f is homothety by Lemma 1. Thus G_1 must be of the form (18) on M_1', and M_1' is a product manifold $N_1 \times N_0$ of two complete Rimannian manifolds N_1 and N_0. The fourth equation of (10) implies $G_i^h \rho_h = \rho_i$. By means of (18), ρ depends on N_0 only. If we denote the metric forms of N_1 and N_0 by ds_1^2 and ds_0^2 respectively, then the metric form of M_1' is $ds_1^2 + ds_0^2$ and the underlying manifold of M_1 is $N_1 \times N_0$. Therefore the metric form $\overline{ds_1}^2$ of M_1 is written as $\overline{ds_1}^2 = \rho^2(ds_1^2 + ds_0^2)$. Putting $N_2 = M_2$ and rewriting ds_0^2 for $\rho^2 ds_0^2$, we see that the underlying manifold of M is the product $N_1 \times N_0 \times N_2$ and the metric form is given by (1). The metric form of M^* is then expressed as (2) on the same underlying manifold.

If M_1 is reducible and a Riemannian product $M' \times M''$ of two manifolds M' and M'', then we consider M as the product $M' \times (M'' \times M_2)$ in place of $M_1 \times M_2$. If ρ

depends on both the parts M' and M'', then, by means of Theorem 1, there is no global non-homothetic conformal diffeomorphism. Consequently ρ should depend on one part only. Hence M_1 is considered to be irreducible. Similarly M_2^* is an irreducible part of M^*.

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Orbifold Holomorphic Discs and Crepant Resolutions

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Abstract This is a note of a lecture at the conference, "Real and Complex submanifolds". We survey the definition and properties of orbifold holomorphic discs and an application to crepant resolution conjecture

1 Holomorphic Discs

Holomorphic discs with Lagrangian boundary condition has played pivotal role in symplectic geometry and mirror symmetry in the last two decades. It has been discovered that quantum geometric invariants of a space, such as Lagrangian Floer homology and Fukaya category, are based on the moduli space of holomorphic discs, and also has deep connection to quantum cohomology. Moreover, via mirror symmetry, it is also connected to the theory of coherent sheaves, singularity theory, and matrix factorizations. The purpose of this lecture is to give an elementary introduction to the some of the ideas of author's recent works [4, 9] with Poddar, Chan, Lau and Tseng to the non-experts of this field, focusing on the simplest example.

Let (M, ω) be a $2n$-dimensional symplectic manifold. We will be mainly interested in (possibly non-compact) toric manifolds and orbifolds. An n-dimensional submanifold L is called Lagrangian if $\omega|_{TL} \equiv 0$. Easiest example is a circle S^1 in $(\mathbb{C}, dx \wedge dy)$, or their products $(S^1)^n \subset \mathbb{C}^n$. Toric manifolds or orbifolds come with moment maps $\mu : M \to \mathbb{R}^n$, whose images are given by polytopes P_M, and one can check that $\mu^{-1}(p)$ for an interior point $p \in P_M$ becomes a Lagrangian torus fiber.

Holomorphic discs are maps

$$u : (D^2 \partial D^2) \to (M, L).$$

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For $(\mathbb{C}, S^1)$, holomorphic discs are given by a product of biholomorphic maps of D^2, or Blaschke factors $e^{i\theta}\dfrac{z-\alpha}{1-\overline{\alpha}z}$ for $\alpha \in D^2$. The number of such factors can be measured by the winding number of ∂D^2 under u, denoted as $\frac{\mu}{2}$. In general Maslov index μ of a holomorphic disc is defined to be the winding number of Lagrangian sub-bundle in the trivialization of u^*TM over ∂D^2. Clearly the dimension of the space of holomorphic discs is $n + \mu$ for this example $(\mathbb{C}, S^1)$ with $n = 1$, since a choice of θ contributes $n = 1$ dimension, and the choice of α in each factor contributes total of μ-dimensional family. In general, we consider the (compactified) moduli space of holomorphic discs $\mathcal{M}(M, L, \beta)$ of a homotopy class $\beta \in \pi_2(M, L)$, whose expected dimension is $n + \mu(\beta) - 3$, where -3 is from the equivalence relations by $Aut(D^2)$. Let us also consider $k + 1$ marked points $z_0, \cdots, z_k$ on ∂D^2 in a counter-clockwise order to define $\mathcal{M}_{k+1}(M, L, \beta)$. Whether the actual dimension of the moduli space equals the expected dimension depends on the transversality of the (linearized) $\overline{\partial}$ equation for each element, which is called Fredholm-regularity. In general, one may considers J-holomorphic discs for a generic almost complex structure J of M instead of the standard complex structure J_0 or perturbed $\overline{\partial}$ equation $\overline{\partial}u = \nu$, to have a better chance for Fredholm-regularity.

When $\mu = 2$, the expected dimension of $\mathcal{M}_1(M, L, \beta)$ is of dimension n, and hence one can define an one-point open Gromov–Witten number $o_1(L, \beta)$, by considering the intersection of the evaluation map $ev_0 : \mathcal{M}_1(M, L, \beta) \to L$ with a generic point $p \in L$. To determine whether $o_1(L, \beta)$ is an invariant of (M, L), we need to consider the compactification of holomorphic discs, i.e. whether the moduli space $\mathcal{M}_1(M, L, \beta)$ is a cycle (without boundary) or a chain (with boundary).

One of the key property of (J)-holomorphic maps is that the L^2-energy $\int_{D^2} |du|^2$ equals symplectic energy $\int_{D^2} u^*\omega$ which is topological. Gromov explained that the compactification of such maps could only have a bubbling off phenomenon, when L^2-energy locally concentrate to a point or escape to the boundary in the limiting sequence, which is called Gromov-compactness. In the former case, one can re-capture such information by a tree of holomorphic sphere bubbles by rescaling, and in the latter by holomorphic disc bubbles. The latter is a codimension one phenomenon, and hence in general the moduli space of holomorphic discs do have a boundary. As the total Maslov index μ (sum of indices of such bubbles) is preserved, we can deduce that if Maslov index of non-constant holomorphic discs (or spheres) are positive, then $\mu = 2$ moduli space cannot bubble off, and hence becomes a cycle. Hence one point open Gromov–Witten invariants are well-defined for such Lagrangians.

In fact, for a semi-Fano toric manifold, where it is allowed to have index 0 holomorphic spheres (which are included in toric divisors), the stable holomorphic disc of Maslov index two is given with a basic disc component of homotopy class β_0 which intersects in its interior, a (tree) of index 0 holomorphic spheres (of class $m\alpha$ for $m \in \mathbb{N}$). Since, a basic disc component cannot bubble off and spheres bubblings are of codimension two or more, the moduli space of such discs gives a virtual cycle of dimension n, hence defines an open Gromov–Witten invariant $o_1(L, \beta + m\alpha)$ for each $m \in \mathbb{N}$.

It is natural to consider the coefficient ring, called Novikov field

$$\Lambda = \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid, a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}, \lim_{i\to\infty} \lambda_i = \infty \right\}$$

We may consider the space of such Lagrangian torus with a $U(1)$-holonomy (of a flat complex line bundle on it), which is denoted as $\{(L, \nabla)\}$. Then, such one-point open Gromov–Witten invariants may define a function on such space

$$W(L, \nabla) = \sum_{\beta \in \pi_2(M,L)} o_1(L, \beta) T^{\omega(\beta)} Hol_{\nabla}(\partial\beta)$$

where $Hol_{\nabla}(\partial\beta)$ is the holonomy of the line bundle along $\partial\beta$. This may be defined more canonically, using algebraic formalism of bounding cochains (see [14]).

Strominger–Yau–Zaslow approach views mirror symmetry as a phenomenon between two dual torus fibrations. If the Lagrangian L's form a torus bundle M, then the space $\check{M}$ of such $\{(L, \nabla)\}$ may be naturally identified with space of dual torus bundle. This may be applied to the toric case, by considering toric manifold (orbifolds) away from toric divisors as explained in [1].

Hence such one-point open Gromov–Witten invariants define a function on the mirror space, called mirror Landau–Ginzburg super-potential. Such relation were conjectured by Hori–Vafa [17], and verified in the toric Fano case by Cho–Oh [8], in more generality, Fukaya–Oh–Ohta–Ono [15], Chan–Lau–Leung–Tseng [5] and so on.

Holomorphic discs in toric manifolds (resp. orbifolds) with boundary on Lagrangian torus fiber are completely classified by Oh and the author [8] (resp. Poddar and the author [9]). It was shown that there exist unique holomorphic disc of Maslov index two (up to T^n-action and $Aut(D^2)$) corresponding to each facet. Hence, it gives rise to a leading order potential $W_0(L, \nabla)$, which can be read off combinatorially from the moment polytope P_M. The terms in $W - W_0$ are rather difficult to compute. Usually, W_0 is called Hori–Vafa potential (also known previously by Givental's work), and its relation to W, which is sometimes called quantum corrections, seem to contain essential informations such as inverse mirror maps (see [5] for more details).

2 Orbi-Discs

For symplectic orbifold (M, ω), it is natural to consider a holomorphic map from a Riemann surface Σ with orbifold singularity. Such a notion has been defined by Chen and Ruan [6, 7] who developed an orbifold Gromov–Witten theory, but also defined new cohomology theory of orbifolds, which are called Chen–Ruan orbifold cohomology.

Let us first explain what is an orbifold version of Riemann surface. A non-trivial finite group action on $D^2 \subset \mathbb{C}$ which preserves the origin and a complex structure, is nothing but a cyclic group action of $\mathbb{Z}/n$ by rotation. Consider the quotient map $\pi : D^2 \to D^2/\sim$ of $\mathbb{Z}/n$ action, where the latter is again topologically a disc, and the map π is n to 1. Note also that the map $D^2 \to D^2$ given by $z \to z^n$ is also n to 1 map to a disc. It is convenient to identify these two maps, and consider $D^2/\sim$ as D^2. In this sense, orbifold Riemann surface is defined to be a Riemann surface Σ, with finite set of points $z_1^+, \ldots, z_k^+ \in \Sigma$, each of which has a disc neighborhood $D(z_i^+)$ and its branch cover $\widetilde{D(z_i^+)} \to D(z_i^+)$, locally given by $br : z \to z^{m_i}$ for some positive integer m_i. If $m_i = 1$, it is a smooth marked point. We are particularly interested in an orbifold disc, which is D^2 with a finite set of interior points $z_1^+, \ldots, z_k^+$ with a branch covering structure as above.

Let X be a symplectic orbifold. Then, a holomorphic map from an orbifold Riemann surface $(\Sigma, \mathbf{z}^+, \mathbf{m})$ to X is given by a continuous map $u : \Sigma \to X$, holomorphic away from $\mathbf{z}^+$ and has a local lift (which is an equivariant map) $\widetilde{D(z_i^+)} \to \tilde{U}$ near z_i^+, for the uniformizing chart $\tilde{U} \to U$ of $u(z_i^+)$. It is further assumed that the relevant local group homomorphism is injective (called "representable") and is assumed to be a good map (such local lifts are compatible with each other in a specific sense, which we refer to [6]). Let us just remark that the representability is essential in the sense that otherwise, there will be infinitely many types of holomorphic maps from possibly more and more singular Riemann surfaces. The good map condition is essential when dealing with constant maps, and maps which degenerate into singular loci (it is proved in [6] that if the inverse image of the non-singular part of X is open, dense and connected then such a map is good).

As an example, let us consider an orbifold $X = [D^2/(\mathbb{Z}/n)]$, and consider orbifold discs to X. We may write $\mathbb{Z}/n$ as $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}$ with $\frac{n}{n} = 0$, where $\frac{k}{n}$ act on D^2 by a multiplication of $exp(\frac{2\pi k\sqrt{-1}}{n})$, which has a unique fixed point 0 for each $k \neq 0$. A twisted sector of X is given by a fixed point set $X_{\frac{k}{n}} = \{0\}$ for each $k \neq 0$.

An orbifold holomorphic disc (D^2, z_1^+, m_1) to X is given as follows. First, we can set $z_1^+ = 0$ by $Aut(D^2)$. To be representable, we need to map $z_1^+ = 0$ to the unique singular point 0 of X, and we need an injective homomorphism from $\mathbb{Z}/m_1$ to $\mathbb{Z}/n$, which implies that n is a multiple of m_1. Hence, we set $n = m_1 \cdot q$. Consider $z \in \widetilde{D^2}$ which maps to D^2 by $z \to z^q$, which defines a desired $\mathbb{Z}/m_1$ equivariant map. The generator of $\mathbb{Z}/m_1$ maps to the element $\frac{q}{n}$, and such an orbifold holomorphic disc meets the twisted sector $X_{\frac{q}{n}}$. In fact, a map $z \to z^{kn+q}$ for any integer $k \geq 0$ also defines an orbifold holomorphic disc which meets the twisted sector $X_{\frac{q}{n}}$. Let us also remark that when $m_1 = 1$, the domain disc is in fact smooth, and we have $q = n$. Namely, any map of the type z^{kn} for $k \geq 1$ defines a smooth disc to X, and one can see that even when the domain do not have orbifold singularity, the holomorphic disc can pass through the singular loci (when it has the right multiplicity).

To discuss the expected dimension of orbifold discs, we need a notion of Maslov index, which has been defined by Shin and the author [10], by introducing a Chern–Weil version of the Maslov index. Namely, for an orbifold map u, we consider its pull-back orbi-bundle u^*TX, which may not be trivial bundle. We consider an unitary connection ∇ which preserved the Lagrangian boundary condition on along ∂D^2, and define its Maslov index to be

$$\mu_{CW}(u) = \frac{\sqrt{-1}}{\pi}\int_{D^2} tr(F_\nabla)$$

for the curvature F_∇. One can see that such an index is rather topological, once we fix the twisted sectors X_{ν_i} that it intersects with at z_i^+'s.

Then, the consider the moduli space of (compactified) orbifold holomorphic discs $u : (D^2, \mathbf{z}, \mathbf{m}) \to (X, L)$ of homotopy class $\beta \in \pi_2(X, L)$, whose i-th orbifold marked point meets the twisted sector X_{ν_i}, with one boundary marked point z_0, has an expected dimension

$$n + \mu_{CW}(\beta) + 1 + 2k - 3 - \sum_{i=1}^{k} 2\iota(\nu_i),$$

where $\iota(\nu_i)$ is the Chen–Ruan degree shifting number of X_{ν_i}.

Here, $\iota(\nu_i)$ is roughly given by the sum of exponents of the diagonalized normal direction to the fixed loci. Namely, if $\nu_i \in \mathbb{Z}/n$ acts on $\mathbb{C}^n$ by

$$diag(exp(\frac{2\pi a_1\sqrt{-1}}{n}), \ldots, exp(\frac{2\pi a_s\sqrt{-1}}{n}))$$

for $0 \leq a_1, \cdots, a_s < 2\pi$, then the degree shifting number $\iota(\nu_i) = \sum_{j=1}^{s} \frac{a_j}{n}$.

Hence, when

$$2k - 2 + \mu_{CW}(\beta) = 2\sum_{i=1}^{k} \iota(\nu_i), \tag{1}$$

the expected dimension becomes n, and hence we can define an orbifold open Gromov–Witten number $o(L, \beta, \nu_1, \cdots, \nu_k)$, by taking an intersection with a generic point on L. Note that when $\nu = \nu_i$ for all i with $\iota(\nu) = 1$ and when $\mu_{CW}(\beta) = 2$, the above condition is satisfied for any k, and hence gives rise to infinitely many such open Gromov–Witten numbers.

As before, one may ask whether this number becomes an invariant, by considering its compactification as considered in [6]. It is similar to the case of a holomorphic discs, but in this case, there are two things that one has to be careful. One is that nodal point (of the sphere bubble) can have an orbifold structure. And the other is that even constant orbifold spheres may not be Fredholm-regular (and needs virtual technique to perturb them).

In general toric situations, one can use T^n-equivariant perturbation to define such open Gromov–Witten invariants, as the possible codimension one boundary stratum has dimension $n-1$. Such stratum cannot exist since T^n-equivariance implies that the dimension is at least n.

But like in semi-Fano toric manifolds, if there exist a unique disc component, which cannot bubble off, then such property can be used to show that the moduli space has a virtual cycle.

Let us finish this section, by considering the case of $X := [\mathbb{C}^2/(\mathbb{Z}/2)]$, where $\frac{1}{2}$ of $\mathbb{Z}/2$ acts on $\mathbb{C}^2$ by $(z_1, z_2) \to (-z_1, -z_2)$. For the twisted sector $X_{\frac{1}{2}} = \{(0,0)\}$, degree shifting number is $\iota(\frac{1}{2}) = 1$. Consider the Lagrangian torus $L = U(1)^2 \subset \mathbb{C}^2$, and an orbi-disc u with one orbifold point z_1^+ whose boundary lies on $L/(\mathbb{Z}/2)$ (which is still smooth). It is obtained from the $\mathbb{Z}/2$-equivariant lift $\tilde{u} : D^2 \to \mathbb{C}^2$, given by $z \mapsto (z, z)$, modulo $(S^1)^2$ rotation action.

Note that the Maslov index of $\tilde{u}$ is 4. Using Proposition 6.7 of [10], we know that the orbi-disc u has Maslov index 2, which is obtained by from the Maslov index of $\tilde{u}$ divided by the degree of the branch covering map. Hence, if we denote the homotopy class of this orbi-disc by β, one sees that the dimension condition (1) is satisfied for the orbi-discs of class β, with an arbitrary number of twisted sector $\frac{1}{2}$ insertions. Namely, we have open Gromov–Witten invariants $o(L, \beta, \frac{1}{2}, \cdots, \frac{1}{2})$, starting from $o(L, \beta, \frac{1}{2}) = 1$. One can check that there do not exist orbi-discs of class β with $k \geq 2$ twisted sector insertions, except stable discs with one disc component of class β, together with constant orbifold sphere bubble trees attached to the nodal point $\frac{1}{2}$ of the disc β.

3 Lagrangian Floer Superpotential and Crepant Resolutions

From the above example, it is natural to define the following generating function of open orbifold Gromov–Witten invariants in this example of $X = [\mathbb{C}^2/(\mathbb{Z}/2)]$:

$$W(L, \nabla)^{orb} = \sum_{k=0}^{\infty} \frac{1}{k!} o(L, \beta, \underbrace{\frac{1}{2}, \cdots, \frac{1}{2}}_{k}) T^{\omega(\beta)} Hol_{\nabla}(\partial\beta).$$

In fact, one can include an additional parameter as follows. Recall that Chen–Ruan orbifold cohomology, as a module, is given by the cohomology of its inertia orbifold, and hence has additional contribution from the twisted sectors. In this case, we have the contribution from $H^0(X_{\frac{1}{2}})$, regarded as an element of $H^2_{CR}(X)$, after degree shifting $2\iota(\frac{1}{2}) = 2$. To include the Kähler parameter $\tau_{\frac{1}{2}}$ for this cohomology class, we consider $\tau_{tw} = \tau_{\frac{1}{2}} \frac{1}{2} \in H^2_{CR}(X)$ to define the generating function.

The following definition of Lagrangian Floer superpotential has been proposed in the general setting in Definition 19 [4] (see also [9]).

Definition 1. Lagrangian Floer superpotential of an orbifold χ is the function $W_\chi : \check{\chi} \to \mathbb{C}$ defined by

$$W_\chi^{orb} = \sum_{\beta \in \pi_2(\chi, L)} \sum_{k \geq 0} \frac{1}{k!} o(L, \beta, \tau_{tw}, \ldots, \tau_{tw}) exp(-\int_\beta \omega) Hol_\nabla(\partial\beta).$$

Here τ_{tw} is defined using Kähler parameters for all twisted sectors ν's with $\iota(\nu) \leq 1$. Note that the above potential depends on two different kind of Kähler parameters, one is the τ_{tw}, and the other is the standard Kähler parameters in $H^2(X)$, which is implicit in the expression $exp(-\int_\beta \omega)$. We will use q to denote both parameters.

Now, an orbifold χ may admit a crepant resolution Y. The crepant resolution conjecture, which tries to relate quantum cohomology (or in general, Gromov–Witten invariants) of χ and Y, has been actively investigated in the last ten years, and much evidence has been found. Ruan [18] is the first person to gave the conjecture that $QH^*(Y)$ should be isomorphic to $QH^*(\chi)$ after analytic continuation of the quantum parameters, and specialization of some of the parameters, which was further generalized by Bryan–Graber [2], Coates–Iritani–Tseng [12], Coates–Ruan [11] and so on.

In [4], we have formulated the following different version of crepant resolution conjecture, called *open crepant resolution conjecture*. Namely, for a toric orbifold χ and its toric crepant resolution Y, we have claimed that their Lagrangian Floer superpotentials $W_\chi(q)$ and $W_Y(Q)$ are related by analytic continuation and change of variables.

This conjecture, compared to the previous ones, has the following noble aspect. As we have explained, the Lagrangian Floer superpotentials are made from the counting of holomorphic (orbi)-discs. Hence, the mysterious change of variables in the process of crepant resolution relations, can be explained as a relation between disc-counting invariants of χ and Y, providing geometric understanding of the phenomenon.

Such conjecture has been proved for the weighted projective spaces $\chi = \mathbb{P}(1, \ldots, 1, n)$ whose crepant resolution is $Y = \mathbb{P}(K_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}})$ in Theorem 3 [4].

Let us consider case of $n = 2$, namely $X = \mathbb{P}(1, 1, 2)$, which has the previous example $[\mathbb{C}^2/(\mathbb{Z}/2)]$, as an open coordinate chart. Its crepant resolution Y is nothing but the Hirzebruch surface $\mathbb{F}_2$ obtained by blowing up the singular point.

Let us first consider the Lagrangian Floer potential of $\mathbb{F}_2$. Note that $\mathbb{F}_2$ has an exceptional curve D (with self intersection number (-2)) which is obtained from the blowup. The curve D as a holomorphic sphere, has a Chern number 0: such a Chern number is given by the intersection number with union of toric divisors, and D meets two divisors of fiber class, and self-intersect D, which gives the intersection number 0. In particular, $\mathbb{F}_2$ is semi-Fano.

We can define the Lagrangian Floer potential by counting holomorphic discs as described in Sect. 1. Toric fan of $\mathbb{F}_2$ can be chosen so that it has the following one dimensional generators,

$$v_1 = (1,0), v_2 = (0,1), v_3 = (-1,-2), v_4 = (0,-1)$$

First, by classification of holomorphic discs [8], we have unique holomorphic discs of class β_i corresponding to v_i for each $i = 1,2,3,4$. From this, we have the following leading order potential

$$W_{\mathbb{F}_2,0} = z_1 + z_2 + \frac{Q_1 Q_2^2}{z_1 z_2^2} + \frac{Q_2}{z_2}.$$

Here, Q_1 is the Kähler parameter corresponding to the size of D, whereas Q_2 to that of the fiber class. The exceptional divisor D meets the disc β_4, and theoretically, there could be stable holomorphic discs of class $\beta_4 + mD$ for any positive integer m, but it has been computed by Auroux [1], Chan [3], Fukaya–Oh–Ohta–Ono [16] that it is non-trivial only if $m = 1$, whose invariant equal to one.

Hence, the total potential can be written as

$$W_{\mathbb{F}_2} = z_1 + z_2 + \frac{Q_1 Q_2^2}{z_1 z_2^2} + \frac{Q_2 + Q_1 Q_2}{z_2}$$

Now, toric Fan of $\mathbb{P}(1,1,2)$ can be chosen so that it has the following one dimensional generators.

$$b_1 = (1,0), b_2 = (0,1), b_3 = (-1,-2).$$

Hence, by classification of holomorphic discs [9], we have the following potential which just records smooth holomorphic discs

$$W_{\mathbb{P}(1,1,2)} = z_1 + z_2 + \frac{q}{z_1 z_2^2}.$$

To observe the relation of Kähler parameters between $QH^*(\mathbb{F}_2)$ and $QH^*(\mathbb{P}(1,1,2))$, we can directly compare two potentials obtained by counting smooth holomorphic discs

$$W_{\mathbb{F}_2} = W_{\mathbb{P}(1,1,2)},$$

which produces identities

$$Q_1 Q_2^2 = q, Q_2(1 + Q_1) = 0.$$

Hence, we obtain

$$Q_1 = -1, Q_2 = \sqrt{-1}q^{\frac{1}{2}}.$$

The first identification, specialization to the root of unity, was conjectured by Ruan [18] to identify quantum cohomology of the orbifold and its crepant resolution, and we find a geometric explanation of $Q_1 = -1$ in this way. Note that the number of parameters are different between these cases.

In general, if we consider Chen–Ruan quantum cohomology ring of orbifold, then the number of parameters are expected to be the same. To observe the relation of such Kähler parameters between $QH^*(\mathbb{F}_2)$ and $QH^*_{CR}(\mathbb{P}(1,1,2))$, we need to set

$$W_{\mathbb{F}_2} = W^{orb}_{\mathbb{P}(1,1,2)} \tag{2}$$

where the latter is obtained by counting also orbi-discs.

For this, we need to compute the open orbifold Gromov–Witten invariants

$$o(L, \beta, \underbrace{\frac{1}{2}, \cdots, \frac{1}{2}}_{k})$$

for any $k \geq 0$. Such invariants has been computed in [4] by first putting a cap to the orbi-disc to identify it with an orbi-sphere (which is possible since the relevant orbi-disc class has a unique disc component), and compute the number of such orbi-spheres (with one point intersection condition) by using orbifold mirror theorem of Coates–Corti–Iritani–Tseng [13], which identifies so called I-function and J-function upon mirror map. The outcome of such computation shows that we have

$$o(L, \beta, \underbrace{\frac{1}{2}, \cdots, \frac{1}{2}}_{2k+1}) = (-1)^k \frac{1}{(2k+1)!2^{2k}}, \quad o(L, \beta, \underbrace{\frac{1}{2}, \cdots, \frac{1}{2}}_{2k}) = 0.$$

Hence, the superpotential $W^{orb}_{\mathbb{P}(1,1,2)}$ is given as

$$W^{orb}_{\mathbb{P}(1,1,2)} = z_1 + z_2 + \frac{q}{z_1 z_2^2} + \frac{2q^{1/2}\sin(\frac{\tau}{2})}{z_2}.$$

Hence the identification with orbifold potential (2) implies the relation between Kähler parameters

$$Q_1 = e^{-\sqrt{-1}(\pi-\tau)}, \quad Q_2 = q^{\frac{1}{2}} e^{\sqrt{-1}(\pi-\tau)/2}.$$

Note that the region where q, τ are small is different from where Q_1, Q_2 are small, and it illustrates that one need an analytic continuation in general.

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On the Hamiltonian Minimality of Normal Bundles

Toru Kajigaya

Abstract A Hamiltonian minimal (shortly, H-minimal) Lagrangian submanifold in a Kähler manifold is a critical point of the volume functional under all compactly supported Hamiltonian deformations. We show that any normal bundle of a principal orbit of the adjoint representation of a compact simple Lie group G in the Lie algebra $\mathfrak{g}$ of G is an H-minimal Lagrangian submanifold in the tangent bundle $T\mathfrak{g}$ which is naturally regarded as $\mathbb{C}^m$. Moreover, we specify these orbits with this property in the class of full irreducible isoparametric submanifolds in the Euclidean space.

1 Introduction

A Lagrangian submanifold L is an m-dimensional submanifold in a $2m$-dimensional symplectic manifold (M, ω) on which the pull-back of the symplectic form ω vanishes. When M is a Kähler manifold, extrinsic properties of Lagrangian submanifolds have been studied by many authors. Since the Lagrangian property is preserved by Hamiltonian flows, namely, flows generated by Hamiltonian vector fields on M, it is natural to consider the variational problem under the Hamiltonian constraint. A Lagrangian submanifold which attains an extremal of the volume functional under Hamiltonian deformations is called *Hamiltonian minimal* (shortly, H-minimal). This was first investigated by Oh [26], where he gave some basic examples. Many more examples have been constructed in Kähler manifolds by various methods (see Sect. 2).

In this note, we review some basic results of H-minimal Lagrangian submanifolds in a general Kähler manifold (Sect. 2). Furthermore, we focus on constructions of H-minimal Lagrangian submanifolds in the complex Euclidean space $\mathbb{C}^m$ (Sect. 3.1). In particular, we give a new family of non-compact, complete H-minimal Lagrangian submanifolds in the complex Euclidean space $\mathbb{C}^m$ (Sect. 3.2 through 3.4).

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Let N^n be a submanifold in $\mathbb{R}^{n+k}$. Our examples are given by the normal bundle νN of N in $T\mathbb{R}^{n+k} \simeq \mathbb{C}^{n+k}$. It is known that the normal bundle νN is a Lagrangian submanifold in $T\mathbb{R}^{n+k}$. Harvey–Lawson [10] first showed that νN is a minimal Lagrangian submanifold if and only if N is an austere submanifold, namely, the set of principal curvatures of N with respect to any unit normal vector is invariant under the multiplication by -1. In their context, the condition that a Lagrangian submanifold is minimal is equivalent to that it is a special Lagrangian submanifold of some phase. Hence, one can construct examples of special Lagrangian submanifold in $\mathbb{C}^{n+k}$ from austere submanifolds. On the other hand, in [18], the author proves that any normal bundle over the principal orbit of the adjoint action of a compact semi-simple Lie group G is a non-minimal, H-minimal Lagrangian submanifold. Such an orbit is called the complex flag manifold or regular Kähler C-space. Moreover, we show that this property characterizes regular C-spaces among the class of full and irreducible isoparametric submanifolds in the Euclidean space (Sect. 3.3, Theorem 1). In Sect. 3.4, we review a proof of this result which is given in [18].

2 Hamiltonian Minimal Lagrangian Submanifolds

Let $\iota : L \to M$ be a Lagrangian immersion into a Kähler manifold (M, ω, J), where ω is the Kähler form and J is the complex structure on M. An infinitesimal deformation $\iota_t : L \times (-\varepsilon, \varepsilon) \to M$ of ι is called a *Hamiltonian deformation* if $\alpha_{\tilde{V}_t} \in \Omega^1(L)$ is an exact form for $t \in (-\varepsilon, \varepsilon)$, namely, $\alpha_{V_t} = df_t$ for some functions $f_t \in C_0^\infty(L)$, where $\tilde{V}_t := d\iota_t/dt$ is the variational vector field of ι_t. Define the *mean curvature form* of ι by $\alpha_H := \iota^*(\omega(H, \cdot))$, where H is the mean curvature vector of ι. A Lagrangian immersion ι is called *minimal* if $\alpha_H = 0$, or equivalently $H = 0$. When M is Kähler–Einstein, the mean curvature form α_H is a closed 1-from by the result of Dazord [9], and hence, it defines a real cohomology class $[\alpha_H] \in H^1(L, \mathbb{R})$. It is known that any Hamiltonian isotopy preserves $[\alpha_H]$, namely, under any global Hamiltonian isotopy $\{\iota_t\}_{0 \le t \le 1}$ of $\iota = \iota_0$, the 1-forms α_{H_t} on L represent the same cohomology class, where α_{H_t} is the mean curvature form of ι_t (see [27]). In particular, for a Lagrangian immersion ι into a Kähler–Einstein manifold M, if there exist a minimal Lagrangian immersion in its Hamiltonian isotopy class, then $[\alpha_H] = 0$. Therefore, there exist an obstruction for the existence of *minimal* Lagrangian submanifold in the Hamiltonian isotopy class (see also [30]).

A Lagrangian immersion ι is called *Hamiltonian minimal* (shortly, *H-minimal*) if it is a critical point of the volume functional under all compactly supported Hamiltonian deformations. It is known that ι is H-minimal if and only if the mean curvature form $\alpha_H \in \Omega^1(L)$ satisfies the equation $\delta\alpha_H = 0$, where δ is the codifferential acting on $\Omega^1(L)$ (see [26]). When M is Kähler–Einstein, the maximum principle implies that if $\iota : L \to M$ is a non-minimal, H-minimal immersion of a compact manifold L into M, then $H^1(L, \mathbb{R}) \neq 0$ ([26]).

Example 1. (1) Any *minimal* Lagrangian immersion is H-minimal. Thus, the notion of H-minimality is an extension of minimal submanifold.

(2) Any Lagrangian immersion with the *parallel mean curvature vector* (i.e., $\nabla^{\perp} H = 0$) is H-minimal.
(3) A curve with constant geodesic curvature in a Riemann surface.
(4) Any compact extrinsically homogeneous Lagrangian submanifold in a Kähler manifold.

An H-minimal Lagrangian immersion $\iota : L^n \to M^{2n}$ is *Hamiltonian stable (or H-stable)* if the second variation of the volume functional of the immersion is non-negative for all Hamiltonian deformations $\{\iota_t\}_t$. In [26], Oh derived the second variation under a Hamiltonian deformation for a compact Lagrangian submanifold in a Kähler manifold as follows:

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathrm{Vol}(\iota_t(L)) = \int_L \left\{ |\Delta f|^2 - \overline{\mathrm{Ric}}(\nabla f) - 2g(B(\nabla f, \nabla f), H) + g(J\nabla f, H)^2 \right\} dv_L,$$

where $\alpha_{V_0} = df$, $\overline{\mathrm{Ric}}$ is the Ricci curvature of M, and B is the second fundamental form of L. When M is Kähler–Einstein, and L is a compact minimal Lagrangian submanifold, it turns out that the H-stability is equivalent to $\lambda_1 \geq c$, where λ_1 is the first eigenvalue of Δ acting on $C^{\infty}(L)$ and c is the Einstein constant of M. In particular, any compact minimal Lagrangian submanifold in a Kähler–Einstein manifold with non-positive Ricci curvature is H-stable.

Example 2. The following examples are H-stable.

(1) Einstein real forms (i.e., the fixed point sets of an anti-holomorphic involution on M) in a Hermitian symmetric space of compact type [25].
(2) The standard tori $T^m = S^1(r_1) \times \cdots S^1(r_m)$ in $\mathbb{C}^m$ [26].
(3) Lagrangian submanifolds with parallel second fundamental form in $\mathbb{C}^m$ or $\mathbb{C}P^m$ [1,2].

For more examples of H-stable Lagrangian submanifold, we refer to [20,21] and a survey article by Ohnita [28].

A diffeomorphism ϕ on M is called a *Hamiltonian diffeomorphism* of M if ϕ satisfies the following conditions:

(i) ϕ is *symplectic*, namely, $\phi^*\omega = \omega$.
(ii) ϕ is represented by the flow $\{\phi_t\}_{t\in[0,1]}$ of a time dependent Hamiltonian vector field $\{X_{F_t}\}$ on M, namely, $d/dt(\phi_t(x)) = X_{F_t}(\phi_t(x))$ with $\phi_0 = Id_M$ and $\phi_1 = \phi$, where $\omega(X_{F_t}, \cdot) = dF_t$ for $F_t \in C_0^{\infty}(M)$.

We denote the set of all Hamiltonian diffeomorphisms by $\mathrm{Ham}(M, \omega)$. A Lagrangian submanifold L in M is called *Hamiltonian volume minimizing* (or shortly, *H.V.M.* Lagrangian submanifold) if L is a volume minimizer of any Hamiltonian diffeomorphism, namely, L satisfies the inequality $\mathrm{Vol}(\phi(L)) \geq \mathrm{Vol}(L)$ for any $\phi \in \mathrm{Ham}(M, \omega)$. By definition, it follows that an H.V.M. Lagrangian submanifold is necessarily H-minimal and H-stable. We know only a few examples of H.V.M. Lagrangian submanifolds.

Example 3. (1) The totally geodesic $\mathbb{R}P^n$ in $\mathbb{C}P^n$ (Kleiner–Oh, cf. [25]).
(2) The product of two equators $S^1 \times S^1$ in $S^2 \times S^2$ (Iriyeh–Ono–Sakai, [16]).
(3) The totally geodesic S^n in $Q_n(\mathbb{C})$ (cf. Iriyeh–Sakai–Tasaki, [17]).

Note that all known examples of H.V.M. Lagrangians belong to Einstein real forms in a Hermitian symmetric space. Based on these examples, Oh posed the following conjecture:

Conjecture 1. Let L be a real form, i.e., the fixed point sets of an anti-holomorphic involution of a Kähler–Einstein manifold M. If L is Einstein, then L is H.V.M.

More generally, we consider the following problem:

Problem 1. Construct and classify H-minimal Lagrangian submanifolds, H-stable Lagrangian submanifolds and H.V.M. Lagrangian submanifolds in a specific Kähler manifold.

3 Hamiltonian Minimality of Normal Bundles in $T\mathbb{R}^{n+k}$

3.1 H-Minimal Lagrangian Submanifolds in $\mathbb{C}^m$

Let L be an oriented Lagrangian submanifold in the complex Euclidean space $\mathbb{C}^m$. The *Lagrangian angle* of L is an S^1-valued function $\theta : L \to S^1 = \mathbb{R}/2\pi\mathbb{Z}$ on L defined by

$$e^{\sqrt{-1}\theta(p)} = dz_1 \wedge \ldots \wedge dz_m(e_1, \ldots, e_m)(p),$$

where $z_i = x_i + \sqrt{-1}y_i$ and $\{e_1, \ldots, e_m\}$ is an oriented orthonormal basis of L. Then one can show that the mean curvature form α_H of L satisfies the relation

$$\alpha_H = -d\theta. \tag{1}$$

Recall that a Lagrangian submanifold L in $\mathbb{C}^m$ is *special Lagrangian* with phase $e^{\sqrt{-1}\theta}$ if L is calibrated by the calibration $\mathrm{Re}(e^{-\sqrt{-1}\theta}\Omega)$, where $\Omega = dz_1 \wedge \ldots \wedge dz_m$. A special Lagrangian submanifold is automatically volume minimizing in its homology class.

Proposition 1. *For an oriented, connected Lagrangian submanifold L in $\mathbb{C}^m$, we have (i) ι is special Lagrangian if and only if θ is constant, and (ii) ι is H-minimal if and only if θ is harmonic (as a S^1-valued function), namely, $\Delta\theta = 0$.*

The minimality of a Lagrangian submanifold L in $\mathbb{C}^m$ is equivalent to that L is a special Lagrangian submanifold of some phase (see Proposition 2.17 in [10]).

We also note that there exist no compact minimal submanifolds in the (complex) Euclidean space.

On the other hand, Oh [26] pointed out that the standard tori $T^m = S^1(r_1) \times \cdots \times S^1(r_m)$ are H-minimal. Generalizing Oh's results [26], Dong [8] showed that the pre-image of an H-minimal Lagrangian submanifold in the complex projective space $\mathbb{C}P^{m-1}$ via the Hopf fibration $\pi : S^{2m-1} \to \mathbb{C}P^{m-1}$ is H-minimal Lagrangian in $\mathbb{C}^m$. We note that there are some known H-minimal Lagrangian submanifolds in $\mathbb{C}P^{m-1}$. For instance, any compact, extrinsically homogeneous Lagrangian submanifolds in $\mathbb{C}P^{m-1}$ are H-minimal, and Bedulli and Gori [4] gives the complete classification of Lagrangian orbits which are obtained by a simple Lie group of isometries acting on $\mathbb{C}P^{m-1}$. On the other hand, Anciaux and Castro [3] gave examples of H-minimal Lagrangian immersions of manifolds with various topology by taking a product of a Lagrangian surface and Legendrian immersions in odd-dimensional unit spheres. Note that these examples are compact and contained in a sphere. In [18], we give a new family of non-compact, complete H-minimal Lagrangian submanifolds in $\mathbb{C}^m$, which is described in the following subsections. For more examples in $\mathbb{C}^m$ (and $\mathbb{C}P^{m-1}$), we refer to [1–3, 11, 12] and references therein.

3.2 Normal Bundles in $T\mathbb{R}^{n+k}$

Let $\mathbb{R}^{n+k}$ be the Euclidean space with the standard flat metric $\langle , \rangle$. Denote the tangent bundle of $\mathbb{R}^{n+k}$ by $T\mathbb{R}^{n+k}$. Since $T\mathbb{R}^{n+k}$ is trivial, it is identified with the direct sum $\mathbb{R}^{n+k} \oplus \mathbb{R}^{n+k}$ on which we can define the flat metric $g(,)$ induced from $\langle , \rangle$. Moreover, we define the complex structure J by $J(X, Y) = (-Y, X)$ for $(X, Y) \in T_p\mathbb{R}^{n+k} \oplus T_u\mathbb{R}^{n+k}$ where $(p, u) \in \mathbb{R}^{n+k} \oplus \mathbb{R}^{n+k}$. By this identification, we regard $T\mathbb{R}^{n+k}$ as the complex Euclidean space $\mathbb{C}^{n+k}$ with the standard Kähler form $\omega := g(J\cdot, \cdot)$. Let $\iota : N^n \to \mathbb{R}^{n+k}$ be an isometric embedding of an n-dimensional smooth manifold into $\mathbb{R}^{n+k}$. In the following, we always identify N with its image under ι, and call it a submanifold in $\mathbb{R}^{n+k}$. Define the *normal bundle* of N by $\nu N := \{(p, u) \in T\mathbb{R}^{n+k};\ p \in N, u \perp T_pN\}$. This is an $(n + k)$-dimensional submanifold in $T\mathbb{R}^{n+k}$. Moreover, one can check that νN is Lagrangian in $T\mathbb{R}^{n+k}$ with respect to the standard symplectic form.

We denote the Levi–Civita connections on $\mathbb{R}^{n+k}$ and $T\mathbb{R}^{n+k}$ by ∇ and $\tilde{\nabla}$, respectively. For a normal vector $u \in \nu_pN$ at $p \in N$, the *shape operator* $A^u \in \mathrm{End}(T_pN)$ is defined by $A^u(X) := -(\nabla_X u)^\top$ for $X \in T_pN$, where $\top$ denotes the tangent component of the vector. Since A^u is represented by a symmetric matrix, the eigenvalues of A^u are real, and we denote it by $\kappa_i(p, u)$ for $i = 1, \ldots, n$. If u is an unit normal vector, these eigenvalues are called the *principal curvatures* of N with respect to the normal direction u.

Lemma 1 ([18]). *Let N^n be an oriented submanifold in $\mathbb{R}^{n+k}$. Then the Lagrangian angle of the normal bundle νN in $T\mathbb{R}^{n+k} \simeq \mathbb{C}^{n+k}$ is given by*

$$\theta(p,u) = -\sum_{i=1}^{n} \mathrm{Arctan}\kappa_i(p,u) + \frac{k\pi}{2} \pmod{2\pi}, \tag{2}$$

where $\mathrm{Arctan}\kappa_i(p,u)$ *denotes the principal value of* $\arctan \kappa_i(p,u)$.

By the relation (1) and (2), the mean curvature form of the normal bundle can be written by

$$\alpha_H = d\Big(\sum_{i=1}^{n} \arctan \kappa_i\Big). \tag{3}$$

For convenience, we put $\tilde{\theta} := \sum_{i=1}^{n} \arctan \kappa_i$.

Remark 1. We remark that a similar formula of (3) has been obtained by Palmer [33] in a different situation.

The following necessary and sufficient conditions for the minimality of normal bundles in $\mathbb{C}^{n+k}$ was first given by Harvey–Lawson [10]:

Proposition 2 (Theorem 3.11 in [10]). *Let* N^n *be a connected submanifold in* $\mathbb{R}^{n+k}$. *Then the normal bundle* νN *is a minimal Lagrangian submanifold in* $T\mathbb{R}^{n+k} \simeq \mathbb{C}^{n+k}$ *if and only if* N *is austere, namely, the set of principal curvatures* $\{\kappa_i(p,u)\}_{i=1}^{n}$ *is invariant under the multiplication by* -1.

By using this result, one can produce examples of special Lagrangian submanifolds in $\mathbb{C}^{n+k}$ from austere submanifolds in $\mathbb{R}^{n+k}$.

By the explicit formulation of the Lagrangian angle of νN given in Lemma 2.1, we improve Harvey–Lawson's result a bit as follows:

Proposition 3 ([18]). *Let* N^n *be a submanifold in* $\mathbb{R}^{n+k}$. *If the mean curvature vector of the normal bundle* νN *is parallel in* $T\mathbb{R}^{n+k} \simeq \mathbb{C}^{n+k}$, *then* νN *is minimal.*

By Proposition 2 and 3, we obtain the following.

Corollary 1. *Let* N^n *be a submanifold in* $\mathbb{R}^{n+k}$. *Then the following three are equivalent: (i)* N *is austere, (ii) the normal bundle* νN *is minimal in* $T\mathbb{R}^{n+k} \simeq \mathbb{C}^{n+k}$, *(iii)* νN *has parallel mean curvature vector.*

In the following, we investigate the H-minimality of a Lagrangian submanifold in the complex Euclidean space $\mathbb{C}^{n+k}$ obtained as the normal bundle of a submanifold N^n in $\mathbb{R}^{n+k}$. By Lemma 1, the H-minimality of the normal bundle νN in $\mathbb{C}^{n+k}$ is equivalent to

$$\Delta\tilde{\theta} = 0, \text{ where } \tilde{\theta} := \sum_{i=1}^{n} \arctan\kappa_i. \tag{4}$$

We recall that there are no non-minimal, H-minimal Lagrangian normal bundles in $\mathbb{C}^{n+k}$ with parallel mean curvature vector by Corollary 1.

Besides, one can show that the normal bundle of the Riemannian product $N_1 \times N_2 \to \mathbb{R}^{n_1+k_1} \times \mathbb{R}^{n_2+k_2}$ of two embeddings $N_i \to \mathbb{R}^{n_i+k_i}$ $(i = 1, 2)$ is H-minimal if and only if each of νN_i is H-minimal. Thus, in the following, our concern is always irreducible ones.

3.3 Normal Bundles over Isoparametric Submanifolds

In [18], we classify isoparametric submanifolds in $\mathbb{R}^{n+k}$ with H-minimal normal bundles. Before describing the main result, we briefly review the isoparametric submanifolds in $\mathbb{R}^{n+k}$ (For more details, refer to [5, 39] and references therein).

Let N^n be a submanifold in $\mathbb{R}^{n+k}$ of an arbitrary codimension k. There are several ways to define the notion of isoparametric submanifolds (see [39]). In this article, we consider the following two conditions.

(i) For any parallel normal vector field $u(t)$ along a piece-wise smooth curve $c(t)$ on N, the shape operator $A^{u(t)}$ has constant eigenvalues.
(ii) The normal bundle of N is flat, namely, $R^{\perp} = 0$, where $R^{\perp}$ denotes the curvature tensor with respect to the normal connection of N.

If N satisfies the condition (i), we say N *has constant principal curvatures*. If N satisfies both conditions, we call N an *isoparametric submanifold*. It is known that any non-compact complete isoparametric submanifold is a product of compact isoparametric submanifolds and the Euclidean space (see [37]). Since the Euclidean factor is obviously austere, we may assume that an isoparametric submanifold N is compact for our purpose.

In the following, we consider an isoparametric submanifold N^n in $\mathbb{R}^{n+k}$.

The isoparametric hypersurfaces in $\mathbb{R}^{n+1}$ are classified by Somigliana [35] for $n = 3$, and Segre [34] for the general dimension. We denote the number of distinct principal curvatures by g. Then g is at most two, and an isoparametric hypersurface in $\mathbb{R}^{n+1}$ is one of the following:

$g = 1$: An affine hyperplane $\mathbb{R}^n$ or a hypersphere $S^n(r)$, where $r > 0$.
$g = 2$: A spherical cylinder $\mathbb{R}^k \times S^{n-k}(r)$, i.e., a tube around an affine plane $\mathbb{R}^k$, where $r > 0$.

The codimension two isoparametric submanifolds in $\mathbb{R}^{n+2}$ are known as isoparametric hypersurfaces in the unit sphere $S^{n+1}(1)$. One of large subclasses of these hypersurfaces are extrinsically homogeneous hypersurfaces in $S^{n+1}(1)$ and these are classified by Hsiang–Lawson [10]. This result asserts that all homogeneous hypersurfaces in $S^{n+1}(1)$ are obtained by principal orbits of s-representations of symmetric spaces of rank 2, where the s-representation is the isotropy representation of a symmetric space U/K (see Sect. 3.4). Other classes includes infinitely many non-homogeneous examples due to Ozeki–Takeuchi and Ferus–Karcher–Münzner. These are the so called isoparametric hypersurfaces of *OT-FKM type* (for more details, refer to monographs [6, 39] and references therein). The classification of isoparametric hypersurfaces in $S^{n+1}(1)$ has not been completed yet. Let N be an isoparametric hypersurface in the unit sphere $S^{n+1}(1)$, and ν the unit normal vector field on N. We denote the distinct principal curvatures of N with respect to ν by $\kappa_i = \cot\theta_i$ with $0 < \theta_1 < \cdots < \theta_g < \pi$, and these multiplicities by m_i for $i = 1, \ldots, g$, respectively. Then, Münzner showed the following ([23]):

$$\theta_i = \theta_1 + \frac{i-1}{g}\pi, \text{ for } i = 1, \ldots, g, \tag{5}$$

$$m_i = m_{i+2}, \text{ modulo } g \text{ indexing.} \tag{6}$$

In particular, $0 < \theta_1 < \pi/g$, and the multiplicities are same if g is odd. Münzner also proved that the number of distinct principal curvatures g is equal to $1, 2, 3, 4$ or 6 [24].

On the other hand, Thorbergsson [38] proved that any full, irreducible, isoparametric submanifold in $\mathbb{R}^{n+k}$ with $k \geq 3$ is extrinsically homogeneous (see also Olmos [29]). Moreover, combining it with the results of Dadok [7] and Palais-Terng [32], they are principal orbits of an s-representation, namely, an isotropy orbit of semi-simple symmetric space U/K.

Let us describe the main results in [18]. For the H-minimality of normal bundles of isoparametric submanifolds, we prove the following:

Theorem 1 ([18]). *Let N be a full, irreducible isoparametric submanifold in the Euclidean space $\mathbb{R}^{n+k}$. Then the normal bundle νN is H-minimal in $T\mathbb{R}^{n+k} \simeq \mathbb{C}^{n+k}$ if and only if N is a principal orbit of the adjoint action of a compact simple Lie group G.*

In particular, we obtain:

Corollary 2 ([18]). *Let G be a compact, connected, semi-simple Lie group, $\mathfrak{g}$ the Lie algebra of G, and $N^n = \mathrm{Ad}(G)w$ a principal orbit of the adjoint action of G on $\mathfrak{g} \simeq \mathbb{R}^{n+k}$ through $w \in \mathfrak{g}$. Then the normal bundle νN of N is an H-minimal Lagrangian submanifold in the tangent bundle $T\mathfrak{g} \simeq \mathbb{C}^{n+k}$.*

The principal orbit N is diffeomorphic to G/T, where T is a maximal torus of G, and N is called a *complex flag manifold* or *regular Kähler C-space*. Since $N = \mathrm{Ad}(G)w$ is compact, N is never austere in $\mathbb{R}^{n+k}$, and hence, νN is not minimal. Moreover, it does not have parallel mean curvature vector (see Proposition 3). We also note that the normal bundle of $N = \mathrm{Ad}(G)w$ is always trivial, namely, νN is homeomorphic to $N \times \mathbb{R}^k$.

3.4 Outline of a Proof of Theorem 1

The strategy of the proof of Theorem 1 in [18] is as follows. When N is an isoparametric submanifold, the differential equation (4) on νN is expressed in terms of the eigenvalues of the shape operators of N. If the codimension of isoparametric submanifold is equal to 1, by using the classification results, we specify submanifolds with (4). The full and irreducible (or compact) one is the hypersphere. Then we have the following:

Proposition 4. *The normal bundle of the n-dimensional hypersphere $N^n = S^n(r)$ with radius $r > 0$ in $\mathbb{R}^{n+1}$ is H-minimal if and only if $n = 2$.*

When the codimension is 2, they are isoparametric hypersurfaces in the sphere, and the known examples consist of principal orbits of s-representations and non-homogeneous ones. By applying the relations (5) and (6) to (4), we obtain the following crucial lemma:

Lemma 2. *Let N^n be an isoparametric hypersurface in the unit sphere $S^{n+1}(1) \subset \mathbb{R}^{n+2}$. Suppose that the normal bundle νN of N as a submanifold in $\mathbb{R}^{n+2}$ is H-minimal in $\mathbb{C}^{n+2} \simeq T\mathbb{R}^{n+2}$. Then the multiplicities of the distinct principal curvatures in $\{\kappa_i\}_{i=1}^n$ are all equal to 2.*

In particular, it turns out that N is a homogeneous hypersurface. In fact, Cartan proved this for $g \leq 3$, and Ozeki–Takeuchi for the case $(g, m) = (4, 2)$ [31] . The remaining case $(g, m) = (6, 2)$ was settled by R. Miyaoka [22], where m is the same multiplicity. Therefore, together with the results of Hsiang–Lawson [14] and the fact that isoparametric submanifolds of codimension grater than three are homogeneous (Thorbergsson [38]), it is sufficient to consider the normal bundle of principal orbits of s-representations.

The eigenvalues of the shape operators of these orbits are given by the restricted root systems of associated symmetric spaces. Let (U, K) be a Riemannian symmetric pair of compact type, where U is a compact, connected real semi-simple Lie group and K a closed subgroup of U such that there exist an involutive automorphism σ of U so that $\mathrm{Fix}(\sigma, U)^0 \subset K \subset \mathrm{Fix}(\sigma, U)$, where $\mathrm{Fix}(\sigma, U) := \{g \in U;\ \sigma(g) = g\}$ and $\mathrm{Fix}(\sigma, U)^0$ is the identity component of $\mathrm{Fix}(\sigma, U)$. Denote the Lie algebra of U and K by $\mathfrak{u}$ and $\mathfrak{k}$, respectively. Let $(\mathfrak{u}, \sigma)$ be the orthogonal symmetric Lie algebra which corresponds to (U, K), namely, σ is an involution on $\mathfrak{u}$ such that the $+1$-eigenspace coincides with $\mathfrak{k}$ and $\mathfrak{k}$ is a compactly embedded Lie algebra in $\mathfrak{u}$.

We take an inner product $\langle , \rangle$ of $\mathfrak{u}$ which is invariant under σ and $\mathrm{Ad}(U)$ on $\mathfrak{u}$. Then we have the orthogonal decomposition $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$. Since the subspace $\mathfrak{p}$ is invariant under $\mathrm{Ad}(K)|_{\mathfrak{p}}$, K acts on $\mathfrak{p}$ as an orthogonal transformation. We call this action of K the *s-representation* of the symmetric space U/K.

Choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. For an 1-form λ on $\mathfrak{a}$, set

$$\mathfrak{k}_\lambda := \{X \in \mathfrak{k};\ (\mathrm{ad}H)^2 X = -\lambda(H)^2 X \text{ for all } H \in \mathfrak{a}\},$$

$$\mathfrak{p}_\lambda := \{X \in \mathfrak{p};\ (\mathrm{ad}H)^2 X = -\lambda(H)^2 X \text{ for all } H \in \mathfrak{a}\}.$$

Then $\mathfrak{p}_{-\lambda} = \mathfrak{p}_\lambda$, $\mathfrak{k}_{-\lambda} = \mathfrak{k}_\lambda$, $\mathfrak{p}_0 = \mathfrak{a}$, and $\mathfrak{k}_0$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. A non-zero 1-form λ is called a *(restricted) root* of $(\mathfrak{u}, \sigma)$ with respect to $\mathfrak{a}$ if $\mathfrak{p}_\lambda \neq \{0\}$. We denote the set of all roots of $(\mathfrak{u}, \sigma)$ by R, and call R the *restricted root system* on $\mathfrak{a}$. We take a basis of the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$ and define the lexicographic ordering on $\mathfrak{a}^*$ with respect to the basis. We call a root $\lambda \in R$ a *positive root* if $\lambda > 0$, and put $R_+ := \{\lambda \in R;\ \lambda > 0\}$. Then we have decompositions

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\lambda \in R_+} \mathfrak{k}_\lambda, \quad \mathfrak{p} = \mathfrak{a} + \sum_{\lambda \in R_+} \mathfrak{p}_\lambda. \tag{7}$$

These are orthogonal direct sums with respect to $\langle , \rangle$. We put $m_\lambda := \dim_{\mathbb{R}} \mathfrak{p}_\lambda$, and call it the *multiplicity* of $\lambda \in R_+$.

Let us consider orbits of the s-representation. Since any s-representation is polar (see [5]) and the section is given by $\mathfrak{a}$, it is sufficient to consider the orbits through a point $w \in \mathfrak{a}$. The point w is called a *regular* element if $\lambda(w) \neq 0$ for any $\lambda \in R$ (otherwise, it is called *singular*). We note that regular orbits are orbits of maximal dimension [36]. Since the isotropy action does not have any exceptional orbit, an orbit is regular if and only if it is principal.

When w is a regular element, we have the following [36] (See also [15]):

(i) The tangent space $T_w N_w$ and the normal space $\nu_w N_w$ of N_w at w in $\mathfrak{p}$ are given by

$$T_w N_w = \sum_{\lambda \in R_+} \mathfrak{p}_\lambda, \quad \nu_w N_w = \mathfrak{a}.$$

In particular, $\mathrm{codim} N_w = \dim \mathfrak{a}$.

(ii) The shape operator A^u of N_w in $\mathfrak{p}$ in the direction $u \in \nu_w N_w$ satisfies

$$A^u(X_\lambda) = -\frac{\lambda(u)}{\lambda(w)} X_\lambda \text{ for } X_\lambda \in \mathfrak{p}_\lambda \text{ and } \lambda \in R_+.$$

By using these, we characterize the H-minimality of normal bundles over the principal orbits of s-representations as follows: For the root system R, we set

$$r := \{\lambda \in R;\ \lambda/2 \notin R\}, \text{ and } r_+ := r \cap R_+.$$

Then r is a reduced root system, namely, if two roots $\lambda, \mu \in r$ are proportional, then $\mu = \pm\lambda$. We also set $l_\lambda := m_\lambda + m_{2\lambda}$, where $m_{2\lambda} = 0$ unless $2\lambda \in r$. By using an argument of the reduced root system, we prove the following.

Proposition 5. *Let $N^n = N_w$ be a regular orbit of an s-representation through an element $w \in \mathfrak{p} \simeq \mathbb{R}^{n+k}$. Then the normal bundle νN is H-minimal in $T\mathfrak{p} \simeq \mathbb{C}^{n+k}$ if and only if $l_\lambda = 2$ for all $\lambda \in r_+$ (In fact, this is equivalent to $m_\lambda = 2$ for all $\lambda \in R_+$).*

On the other hand, we have the following characterization of symmetric spaces of Type II due to Loos [19]:

Proposition 6 (cf. [19]). *Let* $(\mathfrak{u}, \sigma)$ *be an effective, irreducible orthogonal symmetric Lie algebra of compact type and* $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ *the* ± 1*-eigenspace decomposition with respect to* σ. *Then the following statements are equivalent:*

(i) For the restricted root system R *of* $(\mathfrak{u}, \sigma)$, $m_\lambda = 2$ *for all* $\lambda \in R_+$.
(ii) The dual $\mathfrak{u}^* := \mathfrak{k} + \sqrt{-1}\mathfrak{p}$ *of* $\mathfrak{u}$ *has a complex structure (i.e., there exist a complex structure* J *on* $\mathfrak{u}$ *such that* $J[X, Y] = [X, JY]$ *for any* $X, Y \in \mathfrak{u}$*).*
(iii) $(\mathfrak{u}, \sigma)$ *is isomorphic to an irreducible orthogonal symmetric Lie algebra of Type II (in the sense of [13]).*

The compact Lie group G is regarded as a symmetric space of the Riemannian symmetric pair $(G \times G, \Delta G)$, where $\Delta G = \{(g, g) \in G \times G; g \in G\} \simeq G$, and the isotropy representation is equivalent to the adjoint representation of G. Since the associated globally symmetric space with $(\mathfrak{u}, \sigma)$ of Type II is a compact, connected simple Lie group G, the assertion of Theorem 1 follows from Proposition 5 and 6.

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Construction of Coassociative Submanifolds

Kotaro Kawai

Abstract The notion of coassociative submanifolds is defined as the special class of the minimal submanifolds in G_2-manifolds. In this talk, we introduce the method of [5] to construct coassociative submanifolds by using the symmetry of the Lie group action. As an application, we give explicit examples in the 7-dimensional Euclidean space and in the anti-self-dual bundle over the 4-sphere.

1 Introduction

A G_2-manifold is a Riemannian 7-manifold whose holonomy group is contained in the exceptional Lie group G_2. This is characterized by a closed and coclosed 3-form. This characterization is useful for the study of submanifolds in a G_2-manifold.

Definition 1. Define a 3-form φ_0 on $\mathbb{R}^7$ by

$$\varphi_0 = dx^{123} + dx^1(dx^{45} - dx^{67}) + dx^2(dx^{46} - dx^{57}) + dx^3(dx^{47} - dx^{56}),$$

where $(x^1, \cdots, x^7)$ is the standard coordinate of $\mathbb{R}^7$ and wedge signs are omitted. The stabilizer of φ_0 is the exceptional Lie group G_2:

$$G_2 = \{g \in GL(7, \mathbb{R}); g^*\varphi_0 = \varphi_0\}.$$

This is a 14-dimensional compact simply-connected semisimple Lie group.

The Lie group G_2 also fixes the standard metric g_0, the orientation on $\mathbb{R}^7$ and the Hodge dual $*\varphi_0$. They are uniquely determined by φ_0 via

$$-6g_0(v_1, v_2)\mathrm{vol}_{g_0} = i(v_1)\varphi_0 \wedge i(v_2)\varphi_0 \wedge \varphi_0, \tag{1}$$

where vol_{g_0} is the volume form of g_0, $i(\cdot)$ is the interior product, and $v_i \in T(\mathbb{R}^7)$.

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Remark 1. Identifying $\mathbb{R}^7$ with the anti-self-dual bundle $\Lambda^2_-\mathbb{R}^4 \cong \mathbb{R}^4 \oplus \mathbb{R}^3$ over $\mathbb{R}^4$, we have

$$\varphi = d\tau + \mathrm{vol}_{\mathscr{V}},$$

where $\tau \in \Omega^2(\Lambda^2_-\mathbb{R}^4)$ is a tautological 2-form and $\mathrm{vol}_{\mathscr{V}}$ is a volume 3-form on the vertical fibers.

Definition 2. Let (Y, g) be a 7-dimensional Riemannian manifold and φ a 3-form on Y satisfying $d\varphi = 0$ and $d * \varphi = 0$. A triple (Y, φ, g) is called a *G_2-manifold* if for each $y \in Y$, there exists an oriented isomorphism between T_yY and $\mathbb{R}^7$ identifying φ_y with φ_0 and g is the metric induced from (1).

Lemma 1 ([2]). *A G_2-manifold (Y, φ, g) satisfies Hol(g) $\subset G_2$.*

Definition 3. An oriented 4-submanifold $L \subset Y$ is called *coassociative* if $\varphi|_{TL} = 0$.

Remark 2. The 3-form φ and its Hodge dual $*\varphi$ are calibrations on Y, and the corresponding calibrated submanifolds are called associative and coassociative submanifolds. The latter are characterized in terms of the vanishing of the form as above. This implies that they behave like special Lagrangian submanifolds in Calabi–Yau manifolds.

2 Construction of Examples with Symmetries

To construct a coassociative submanifold L, we suppose that L is preserved by an action of a Lie group G. As is well known in [5], if G acts with cohomogeneity one on L, then the P.D.E. of the coassociative condition reduces to a first-order O.D.E. on the orbit space. We give a summary based on [4].

Proposition 1. *Let (Y, φ, g) be a G_2-manifold. Suppose that a Lie group G with the Lie algebra $\mathfrak{g}$ acts on Y satisfying the following conditions:*

- *$g^*\varphi = r(g) \cdot \varphi$ for a smooth function $r : G \to \mathbb{R}_{>0}$.*
- *The dimension of the generic orbit of G is equal to 3.*

Then we can construct coassociative submanifolds in the following way.

1. *Find a subset $\Sigma \subset Y$ such that*
   - *$G \cdot \Sigma = \{g \cdot x \in Y; g \in G, x \in \Sigma\} = Y$,*
   - *$T_x\Sigma \cap T_x(G\text{-orbit}) = \{0\}$ for each $x \in \Sigma$, where $T_x(G\text{-orbit})$ is the tangent space to the G-orbit at x.*
2. *Find a path $c : I \to \Sigma$ for an open interval $I \subset \mathbb{R}$ satisfying*

$$\varphi(v_1^*, v_2^*, v_3^*)|_c = 0, \ \ \varphi(v_i^*, v_j^*, \dot{c})|_c = 0,$$

for $v_i \in \mathfrak{g}$, *where* $\dot{c} = \frac{dc}{dt}$ *and* v^* *is the vector field on* Y *generated by* $v \in \mathfrak{g}$.
3. *Then* $L := G \cdot \mathrm{Image}(c)$ *is a* G*-invariant coassociative submanifold in* Y.

Remark 3. The subset Σ is usually the union of different dimensional submanifolds. The subset Σ is regarded as (the covering space of) the orbit space. By this method, we easily see whether the resulting submanifold contains singular orbits.

3 Coassociative Submanifolds in $\mathbb{R}^7$

3.1 The Case $G = \mathrm{SU}(2)$

The group SU(2) acts on $\mathbb{R}^4 \cong \mathbb{C}^2$ canonically and this action lifts to $\Lambda^2_- \mathbb{R}^4 = \mathbb{R}^7$. This action preserves φ_0 on $\mathbb{R}^7$. The orbit space Σ of the SU(2)-action is described as follows:

$$\begin{aligned} \Sigma &= \Sigma_1 \sqcup \Sigma_2 \sqcup \Sigma_3, \\ \Sigma_1 &= \{(y^1, 0, 0, 0, a^1, a^2, a^3) \in \mathbb{R}^7; y^1 > 0, a^i \in \mathbb{R}\}, \\ \Sigma_2 &= \left\{(0, 0, 0, 0, a^1, a^2, a^3) \in \mathbb{R}^7; \sum_{i=1}^{3} |a^i|^2 > 0\right\}, \ \Sigma_3 = \{0\}, \end{aligned}$$

Then we have $\mathrm{SU}(2) \cdot \Sigma = \mathbb{R}^7$. The orbit topology is described as follows:

$$\mathrm{SU}(2) \cdot x \cong \begin{cases} S^3 \ (x \in \Sigma_1), \\ S^2 \ (x \in \Sigma_2), \\ \ * \ \ (x \in \Sigma_3). \end{cases}$$

Take the basis $\{X_1, X_2, X_3\}$ of the Lie algebra $\mathfrak{su}(2)$ of SU(2) satisfying $[X_j, X_{j+1}] = X_{j+2} (j \in \mathbb{Z}/3)$. We may find a path $c : I \to \Sigma$ satisfying

$$\varphi(X_1^*, X_2^*, X_3^*)|_c = 0, \ \varphi(X_i^*, X_j^*, \dot{c})|_c = 0 \ \ (1 \le i, j \le 3)$$

Solving these, we see that c is of the form

$$\left\{((y^1, 0, 0, 0), r\mathbf{v}) \in \mathbb{R}^7; r(4r^2 - 5(y^1)^2)^2 = C, r \ge 0\right\}$$

($\mathbf{v} \in S^2 \subset \mathbb{R}^3$, $C \ge 0$) which gives the example of Harvey and Lawson [3].

Theorem 1 (Harvey and Lawson [3]). *For any* $\mathbf{v} \in S^2 \subset \mathbb{R}^3$, $C \ge 0$,

$$M_C := \mathrm{SU}(2) \cdot \left\{((y^1, 0, 0, 0), r\mathbf{v}) \in \mathbb{R}^7; r(4r^2 - 5(y^1)^2)^2 = C, r \ge 0\right\}$$

is an SU(2)*-invariant coassociative submanifold in* $\mathbb{R}^7$.

For $C > 0$, M_C has two connected components $M_C^{\pm}$ defined by

$$M_C^{\pm} := M_C \cap \mathrm{SU}(2) \cdot \left\{((y^1, 0, 0, 0), r\mathbf{v}) \in \mathbb{R}^7; \pm(4r^2 - 5(y^1)^2) > 0\right\}$$

and M_C^{+} (resp. M_C^{-}) is diffeomorphic to the tautological line bundle $\mathscr{O}_{\mathbb{C}P^1}(-1)$ over $\mathbb{C}P^1 \cong S^2$ (resp. $S^3 \times \mathbb{R}$). For $C = 0$, we have

$$M_0 = M_0^0 \sqcup M_0^{'},\ M_0^0 = \mathrm{SU}(2) \cdot \left\{(y^1, 0, 0, 0, 0, 0, 0); y^1 \geq 0\right\},$$
$$M_0^{'} = \mathrm{SU}(2) \cdot \left\{y^1 \cdot \left((1, 0, 0, 0), \tfrac{\sqrt{5}}{2}\mathbf{v}\right) \in \mathbb{R}^7; y^1 > 0\right\},$$

where M_0^0 is a flat $\mathbb{R}^4$ and $M_0^{'}$ is the cone on the graph of the Hopf fibration $S^3 \to S^2$ and isomorphic to $S^3 \times \mathbb{R}$. Moreover, all the coassociative submanifolds invariant under this SU(2)*-action are given in this way.*

3.2 The Case $G = T^2 \times \mathbb{R}_{>0}$

Define the $T^2 \times \mathbb{R}_{>0}$-action on $\mathbb{R}^7$ by

$$(e^{i\theta}, e^{i\psi}, R) \cdot (z^1, z^2, a^1, w) = (Re^{i\theta} z^1, Re^{i\psi} z^2, Ra^1, Re^{i(\psi-\theta)} w),$$

where $(e^{i\theta}, e^{i\psi}, R) \in T^2 \times \mathbb{R}_{>0}$, $(z^1, z^2, a^1, w) \in \mathbb{C}^2 \oplus \mathbb{R} \oplus \mathbb{C} = \mathbb{R}^7$. The orbit space Σ of the $T^2 \times \mathbb{R}_{>0}$-action is described as follows:

$$\Sigma = \Sigma_1 \sqcup \Sigma_2 \sqcup \Sigma_3 \sqcup \Sigma_4,$$
$$\Sigma_1 = \{(y^1, 0, y^3, 0, a^1, a^2, a^3) \in S^6; y^1, y^3 \geq 0, |y^1|^2 + |y^3|^2 > 0\},$$
$$\Sigma_2 = \{(y^1, 0, y^3, 0, a^1, a^2, 0) \in S^6; \#\{x = 0; x \in \{y^1, y^3, a^2\}\} = 2\},$$
$$\Sigma_3 = \{(0, 0, 0, 0, 1, 0, 0)\},\ \Sigma_4 = \{0\}.$$

Then we have $T^2 \times \mathbb{R}_{>0} \cdot \Sigma = \mathbb{R}^7$. The orbit topology is described as follows:

$$T^2 \times \mathbb{R}_{>0} \cdot x \cong \begin{cases} T^2 \times \mathbb{R}_{>0} & (x \in \Sigma_1), \\ S^1 \times \mathbb{R}_{>0} & (x \in \Sigma_2), \\ \mathbb{R}_{>0} & (x \in \Sigma_3), \\ * & (x \in \Sigma_4). \end{cases}$$

Take the orthonormal basis $\{X_1, X_2\}$ of the Lie algebra $\mathfrak{t}^2$ of T^2. We find a path $c : I \to \Sigma$ satisfying

$$\varphi(X_1^*, X_2^*, r\tfrac{\partial}{\partial r})|_c = 0,\ \varphi(X_1^*, X_2^*, \dot{c})|_c = 0,\ \varphi(X_1^*, r\tfrac{\partial}{\partial r}, \dot{c})|_c = 0,\ \varphi(X_2^*, r\tfrac{\partial}{\partial r}, \dot{c})|_c = 0.$$

to obtain the next result.

Theorem 2 ([6]). *Let $\alpha, \gamma : I \to (0, \pi/2)$, and $\beta : I \to \mathbb{R}$ be smooth functions on a small open interval $I \subset \mathbb{R}$ satisfying*

$$\frac{d}{dt}\log(\sin\gamma) = -\frac{2\tan\beta \cdot \tan(2\alpha - \beta)\cdot\dot{\beta}}{\tan(2\alpha-\beta) + 3\tan\beta},$$

$$\frac{d}{dt}\log(\tan\gamma) = -\tan(2\alpha-\beta)\cdot(\dot{\alpha}+\dot{\beta}),$$

where we denote $\dot{\alpha} = \frac{d\alpha}{dt}$, etc. Then the subset M of $\mathbb{C}^2 \oplus \mathbb{R} \oplus \mathbb{C} \cong \mathbb{R}^7$ defined by

$$M = \big\{\big(Re^{i\theta}\cos\gamma(t)\cdot\cos\alpha(t),\, Re^{i\psi}\cos\gamma(t)\cdot\sin\alpha(t), \\ R\sin\gamma(t)\cdot\cos\beta(t),\, Re^{i(\psi-\theta)}\sin\gamma(t)\cdot\sin\beta(t)\big)\,;\, R>0, \theta, \psi \in \mathbb{R}, t \in I\big\}$$

is a T^2-invariant coassociative cone in $\mathbb{R}^7$ which is diffeomorphic to $T^2 \times \mathbb{R}_{>0} \times I$.

4 Coassociative Submanifolds in $\Lambda^2_- S^4$

4.1 G_2-Structure on $\Lambda^2_- S^4$

We introduce the complete metric $g_\lambda (\lambda > 0)$ on the bundle $\Lambda^2_- S^4$ of anti-self-dual 2-forms on the 4-sphere S^4 obtained by Bryant and Salamon [1].

Since $\Lambda^2_- S^4$ has a connection induced by the Levi Civita connection on S^4, the tangent space $T_\omega(\Lambda^2_- S^4)$ has a canonical splitting $T_\omega(\Lambda^2_- S^4) \cong \mathscr{H}_\omega \oplus \mathscr{V}_\omega$ into horizontal and vertical subspaces for each $\omega \in \Lambda^2_- S^4$.

Proposition 2 (Bryant and Salamon [1]). *For $\lambda > 0$, define the 3-form $\varphi_\lambda \in \Omega^3(\Lambda^2_- S^4)$ and the metric g_λ on $\Lambda^2_- S^4$ as*

$$\varphi_\lambda = 2s_\lambda d\tau + \frac{1}{s_\lambda^3}\mathrm{vol}_{\mathscr{V}},\ g_\lambda = 2s_\lambda^2 g_{\mathscr{H}} + \frac{1}{s_\lambda^2} g_{\mathscr{V}},$$

where $s_\lambda = (\lambda + r^2)^{1/4}$, r is the distance function measured by the fiber metric induced by that on S^4, τ is a tautological 2-form and $\mathrm{vol}_{\mathscr{V}}$ is the volume form of $g_{\mathscr{V}}$ on the vertical fiber.

Then for each $\lambda > 0$, $(\Lambda^2_- S^4, \varphi_\lambda, g_\lambda)$ is a G_2-manifold with $\mathrm{Hol}(g_\lambda) = G_2$ and the metric g_λ is complete.

Remark 4. For $\lambda = 0$, the metric g_0 is a cone metric on $\Lambda^2_- S^4 - \{0 - \text{section}\} \cong \mathbb{C}P^3 \times \mathbb{R}_{>0}$. The metric $g_{\mathbb{C}P^3}$ is not the standard metric, but a 3-symmetric Einstein, non-Kähler metric. The metric g_0 is not complete because of the singularity at 0, while it satisfies $\mathrm{Hol}(g_0) = G_2$.

4.2 The Case $G = \mathrm{SU}(2)$

Since $S^4 \subset \mathbb{R}^5 = \mathbb{C}^2 \oplus \mathbb{R}$, SU(2) acts on S^4 in the canonical way and it lifts to $\Lambda^2_- S^4$. This action preserves the G_2-structure φ_λ for each $\lambda > 0$.

Let $(y^1, \cdots, y^4)$ be the local coordinates of $S^4 - \{x^5 = 1\}$ obtained by the stereographic projection, and let (a^1, a^2, a^3) be the fiber coordinate corresponding $(y^1, \cdots, y^4)$. Then the orbit space Σ of the SU(2)-action is described as follows:

$$\Sigma = \Sigma_1 \sqcup \Sigma_2 \sqcup \Sigma_3,$$
$$\Sigma_1 = \{(y^1, 0, 0, 0, a^1, a^2, a^3) \in \mathbb{R}^7; y^1 > 0, a^i \in \mathbb{R}\},$$
$$\Sigma_2 = \Lambda^2_- S^4|_{x^5=-1} - \{0\} \sqcup \Lambda^2_- S^4|_{x^5=1} - \{0\},\ \Sigma_3 = \{x^5 = \pm 1\} \subset S^4,$$

We have $\mathrm{SU}(2) \cdot \Sigma = \mathbb{R}^7$ and the orbit topology is described as follows:

$$\mathrm{SU}(2) \cdot x \cong \begin{cases} S^3 \ (x \in \Sigma_1), \\ S^2 \ (x \in \Sigma_2), \\ \ * \ \ (x \in \Sigma_3). \end{cases}$$

The argument on $\mathbb{R}^7$ applies almost identically on $\Lambda^2_- S^4$ to obtain the following theorem.

Theorem 3 ([6]). *For any $C \in \mathbb{R}$, $\mathbf{v} \in S^2 \subset \mathbb{R}^3$, the set*

$$M_C := \mathrm{SU}(2) \cdot \left\{ ((y^1, 0, 0, 0), r\mathbf{v}); \begin{array}{c} -\int_0^{\sqrt{r}} (\lambda + a^4)^{1/8} da + \frac{(\lambda + r^2)^{1/8} \sqrt{r}}{1 + (y^1)^2} = C, \\ r \geq 0, y^1 \in \mathbb{R} \cup \{\infty\} \end{array} \right\},$$

is an SU(2)*-invariant coassociative submanifold in $\Lambda^2_- S^4$ and we have*

$$M_C \cong \mathcal{O}_{\mathbb{C}P^1}(-1) \ (C \neq 0), \ \ S^4 \sqcup S^3 \times \mathbb{R} \ (C = 0).$$

Moreover, all the coassociative submanifolds invariant under this SU(2)*-action are given in this way.*

4.3 The Case $G = T^2 \times \mathbb{R}_{>0}$

In the same way, we can derive a system of O.D.E.s whose solutions give T^2-invariant coassociative cones in $\Lambda^2_- S^4 - \{\text{0-section}\} \cong \mathbb{C}P^3 \times \mathbb{R}_{>0}$.

We obtain some explicit examples such as $T^* S^2$, where $S^2 \subset S^4$ is totally geodesic, and the rank one vector bundle over a small sphere in S^4.

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Calibrations and Manifolds with Special Holonomy

Selman Akbulut and Sema Salur

Abstract The purpose of this paper is to introduce Harvey–Lawson manifolds and review the construction of certain "mirror dual" Calabi–Yau submanifolds inside a G_2 manifold. More specifically, given a Harvey–Lawson manifold HL, we explain how to assign a pair of tangent bundle valued 2 and 3-forms to a G_2 manifold $(M, HL, \varphi, \Lambda)$, with the calibration 3-form φ and an oriented 2-plane field Λ. As in [3] these forms can then be used to define different complex and symplectic structures on certain 6-dimensional subbundles of $T(M)$. When these bundles are integrated they give mirror CY manifolds (related through HL manifolds).

1 Introduction

Let (M^7, φ) be a G_2 manifold with the calibration 3-form φ. If φ restricts to be the volume form of an oriented 3-dimensional submanifold Y^3, then Y is called an associative submanifold of M. In [3] the authors introduced a notion of mirror duality in any G_2 manifold (M^7, φ) based on the associative/coassociative splitting of its tangent bundle $TM = \mathbb{E} \oplus \mathbb{V}$ by the non-vanishing 2-frame fields provided by [7]. This duality initially depends on the choice of two non-vanishing vector fields, one in $\mathbb{E}$ and the other in $\mathbb{V}$. In this article we give a natural form of this duality where the choice of these vector fields are made more canonical, in the expense of possibly localizing this process to the tubular neighborhood of the 3-skeleton of (M, φ).

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2 Basic Definitions

Let us recall some basic facts about G_2 manifolds (e.g. [2, 4, 5]). Octonions give an 8-dimensional division algebra $\mathbb{O} = \mathbb{H} \oplus l\mathbb{H} = \mathbb{R}^8$ generated by $\langle 1, i, j, k, l, li, lj, lk \rangle$. The imaginary octonions $im\mathbb{O} = \mathbb{R}^7$ is equipped with the cross product operation $\times : \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7$ defined by $u \times v = im(\bar{v}.u)$. The exceptional Lie group G_2 is the linear automorphisms of $im\mathbb{O}$ preserving this cross product. Alternatively:

$$G_2 = \{(u_1, u_2, u_3) \in (\mathbb{R}^7)^3 \mid \langle u_i, u_j \rangle = \delta_{ij},\ \langle u_1 \times u_2, u_3 \rangle = 0\ \}, \tag{1}$$

$$G_2 = \{A \in GL(7, \mathbb{R}) \mid A^* \varphi_0 = \varphi_0\ \} \tag{2}$$

where $\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$ with $e^{ijk} = dx^i \wedge dx^j \wedge dx^k$. We say a 7-manifold M^7 has a G_2 *structure* if there is a 3-form $\varphi \in \Omega^3(M)$ such that at each $p \in M$ the pair $(T_p(M), \varphi(p))$ is (pointwise) isomorphic to $(T_0(\mathbb{R}^7), \varphi_0)$. This condition is equivalent to reducing the tangent frame bundle of M from $GL(7, \mathbb{R})$ to G_2. A manifold with G_2 structure (M, φ) is called a G_2 *manifold* (integrable G_2 structure) if at each point $p \in M$ there is a chart $(U, p) \to (\mathbb{R}^7, 0)$ on which φ equals to φ_0 up to second order term, i.e. on the image of the open set U we can write $\varphi(x) = \varphi_0 + O(|x|^2)$.

One important class of G_2 manifolds are the ones obtained from Calabi–Yau manifolds. Let (X, ω, Ω) be a complex 3-dimensional Calabi–Yau manifold with Kähler form ω and a nowhere vanishing holomorphic 3-form Ω, then $X^6 \times S^1$ has holonomy group $SU(3) \subset G_2$, hence is a G_2 manifold. In this case φ= Re $\Omega + \omega \wedge dt$. Similarly, $X^6 \times \mathbb{R}$ gives a noncompact G_2 manifold.

Definition 1. Let (M, φ) be a G_2 manifold. A 4-dimensional submanifold $X \subset M$ is called *coassociative* if $\varphi|_X = 0$. A 3-dimensional submanifold $Y \subset M$ is called *associative* if $\varphi|_Y \equiv vol(Y)$; this condition is equivalent to the condition $\chi|_Y \equiv 0$, where $\chi \in \Omega^3(M, TM)$ is the tangent bundle valued 3-form defined by the identity:

$$\langle \chi(u, v, w), z \rangle = *\varphi(u, v, w, z). \tag{3}$$

The equivalence of these conditions follows from the 'associator equality' of [5]

$$\varphi(u, v, w)^2 + |\chi(u, v, w)|^2/4 = |u \wedge v \wedge w|^2.$$

Similar to the definition of χ one can define a tangent bundle 2-form, which is just the cross product of M (nevertheless viewing it as a 2-form has its advantages).

Definition 2. Let (M, φ) be a G_2 manifold. Then $\psi \in \Omega^2(M, TM)$ is the tangent bundle valued 2-form defined by the identity:

$$\langle \psi(u, v), w \rangle = \varphi(u, v, w) = \langle u \times v, w \rangle. \tag{4}$$

On a local chart of a G_2 manifold (M, φ), the form φ coincides with the form $\varphi_0 \in \Omega^3(\mathbb{R}^7)$ up to quadratic terms, we can express the tangent valued forms χ and ψ in terms of φ_0 in local coordinates. More generally, if $e_1, \dots e_7$ is any local orthonormal frame and $e^1, \dots, e^7$ is the dual frame, from definitions we get:

$$\begin{aligned}\chi = \; & (e^{256} + e^{247} + e^{346} - e^{357})e_1 + (-e^{156} - e^{147} - e^{345} - e^{367})e_2 \\ & + (e^{157} - e^{146} + e^{245} + e^{267})e_3 + (e^{127} + e^{136} - e^{235} - e^{567})e_4 \\ & + (e^{126} - e^{137} + e^{234} + e^{467})e_5 + (-e^{125} - e^{134} - e^{237} - e^{457})e_6 \\ & + (-e^{124} + e^{135} + e^{236} + e^{456})e_7,\end{aligned}$$

$$\begin{aligned}\psi = \; & (e^{23} + e^{45} + e^{67})e_1 + (e^{46} - e^{57} - e^{13})e_2 + (e^{12} - e^{47} - e^{56})e_3 \\ & + (e^{37} - e^{15} - e^{26})e_4 + (e^{14} + e^{27} + e^{36})e_5 + (e^{24} - e^{17} - e^{35})e_6 \\ & + (e^{16} - e^{25} - e^{34})e_7.\end{aligned}$$

Here are some useful facts :

Lemma 1 ([2]). *To any 3-dimensional submanifold $Y^3 \subset (M, \varphi)$, χ assigns a normal vector field, which vanishes when Y is associative.*

Lemma 2 ([2]). *To any associative manifold $Y^3 \subset (M, \varphi)$ with a non-vanishing oriented 2-plane field, χ defines a complex structure on its normal bundle (notice in particular that any coassociative submanifold $X \subset M$ has an almost complex structure if its normal bundle has a non-vanishing section).*

Proof. Let $L \subset \mathbb{R}^7$ be an associative 3-plane, that is $\varphi_0|_L = vol(L)$. Then for every pair of orthonormal vectors $\{u, v\} \subset L$, the form χ defines a complex structure on the orthogonal 4-plane $L^\perp$, as follows: Define $j : L^\perp \to L^\perp$ by

$$j(X) = \chi(u, v, X). \tag{5}$$

This is well defined i.e. $j(X) \in L^\perp$, because when $w \in L$ we have:

$$\langle \chi(u, v, X), w \rangle = *\varphi_0(u, v, X, w) = - * \varphi_0(u, v, w, X) = \langle \chi(u, v, w), X \rangle = 0.$$

Also $j^2(X) = j(\chi(u, v, X)) = \chi(u, v, \chi(u, v, X)) = -X$. We can check the last equality by taking an orthonormal basis $\{X_j\} \subset L^\perp$ and calculating

$$\begin{aligned}\langle \chi(u, v, \chi(u, v, X_i)), X_j \rangle &= *\varphi_0(u, v, \chi(u, v, X_i), X_j) = - * \varphi_0(u, v, X_j, \chi(u, v, X_i)) \\ &= -\langle \chi(u, v, X_j), \chi(u, v, X_i) \rangle = -\delta_{ij}.\end{aligned}$$

The last equality holds since the map j is orthogonal, and the orthogonality can be seen by polarizing the associator equality, and by noticing $\varphi_0(u, v, X_i) = 0$. Observe that the map j only depends on the oriented 2-plane $\Lambda = < u, v >$ generated by $\{u, v\}$ (i.e. it only depends on the complex structure on Λ). ◻

3 Calabi–Yau's Hypersurfaces in G_2 Manifolds

In [3] authors proposed a notion of *mirror duality* for Calabi–Yau submanifold pairs lying inside of a G_2 manifold (M,φ). This is done first by assigning almost Calabi–Yau structures to hypersurfaces induced by hyperplane distributions. The construction goes as follows. Suppose ξ be a nonvanishing vector filed $\xi \in \Omega^0(M,TM)$, which gives a codimension one integrable distribution $V_\xi := \xi^\perp$ on M. If X_ξ is a leaf of this distribution, then the forms χ and ψ induce a non-degenerate 2-form ω_ξ and an almost complex structure J_ξ on X_ξ as follows:

$$\omega_\xi = \langle \psi, \xi \rangle \quad \text{and} \quad J_\xi(u) = u \times \xi, \tag{6}$$

$$\text{Re } \Omega_\xi = \varphi|_{V_\xi} \quad \text{and} \quad \text{Im } \Omega_\xi = \langle \chi, \xi \rangle, \tag{7}$$

where the inner products, of the vector valued differential forms ψ and χ with vector field ξ, are performed by using their vector part. So $\omega_\xi = \xi \lrcorner\, \varphi$, and $\text{Im } \Omega_\xi = \xi \lrcorner * \varphi$. Call $\Omega_\xi = \text{Re } \Omega_\xi + i \text{ Im } \Omega_\xi$. These induce almost Calabi–Yau structure on X_ξ, analogous to Example 1.

Theorem 1 ([3]). *Let (M,φ) be a G_2 manifold, and ξ be a unit vector field such that $\xi^\perp$ comes from a codimension one foliation on M, then $(X_\xi, \omega_\xi, \Omega_\xi, J_\xi)$ is an almost Calabi–Yau manifold such that $\varphi|_{X_\xi} = Re\ \Omega_\xi$ and $*\varphi|_{X_\xi} = *_3\ \omega_\xi$. Furthermore, if $\mathscr{L}_\xi(\varphi)|_{X_\xi} = 0$ then $d\omega_\xi = 0$, and if $\mathscr{L}_\xi(*\varphi)|_{X_\xi} = 0$ then J_ξ is integrable; when both conditions are satisfied $(X_\xi, \omega_\xi, \Omega_\xi, J_\xi)$ is a Calabi–Yau manifold.*

Here is a brief discussion of [3] with explanation of its terms: Let $\xi^\#$ be the dual 1-form of ξ, and $e_{\xi^\#}$ and $i_\xi = \xi \lrcorner$ denote the exterior and interior product operations on differential forms, then

$$\varphi = e_{\xi^\#} \circ i_\xi(\varphi) + i_\xi \circ e_{\xi^\#}(\varphi) = \omega_\xi \wedge \xi^\# + Re\ \Omega_\xi.$$

This is the decomposition of the form φ with respect to $\xi \oplus \xi^\perp$. The condition that the distribution V_ξ to be integrable is $d\xi^\# \wedge \xi^\# = 0$. Also it is clear from definitions that J_ξ is an almost complex structure on X_ξ, and the 2-form ω_ξ is non-degenerate on X_ξ, because

$$\omega_\xi^3 = (\xi \lrcorner\, \varphi)^3 = \xi \lrcorner\, [\, (\xi \lrcorner\, \varphi) \wedge (\xi \lrcorner\, \varphi) \wedge \varphi\,] = \xi \lrcorner\, (6|\xi|^2 \mu) = 6\mu_\xi$$

where $\mu_\xi = \mu|_{V_\xi}$ is the induced orientation form on V_ξ. For $u, v \in V_\xi$

$$\begin{aligned} \omega_\xi(J_\xi(u), v) &= \omega_\xi(u \times \xi, v) = \langle \psi(u \times \xi, v), \xi \rangle = \varphi(u \times \xi, v, \xi) \\ &= -\varphi(\xi,\ \xi \times u,\ v) = -\langle\, \xi \times (\xi \times u), v\, \rangle \\ &= -\langle\, -|\xi|^2 u + \langle \xi, u \rangle \xi, v\, \rangle = |\xi|^2 \langle u, v \rangle - \langle \xi, u \rangle \langle \xi, v \rangle \\ &= \langle u, v \rangle \end{aligned}$$

implies $\langle J_\xi(u), J_\xi(v)\rangle = -\omega_\xi(u, J_\xi(v)) = \langle u, v\rangle$. By a calculation of J_ξ, one checks that the 3-form Ω_ξ is a $(3,0)$ form, furthermore it is non-vanishing because

$$\begin{aligned}\frac{1}{2i}\ \Omega_\xi \wedge \overline{\Omega}_\xi = Im\ \Omega_\xi \wedge Re\ \Omega_\xi &= (\xi \lrcorner * \varphi) \wedge [\ \xi \lrcorner\ (\xi^\# \wedge \varphi)\ ] \\ &= -\xi \lrcorner\ [\ (\xi \lrcorner * \varphi) \wedge (\xi^\# \wedge \varphi)\ ] \\ &= \xi \lrcorner\ [*(\xi^\# \wedge \varphi) \wedge (\xi^\# \wedge \varphi)\ ] \\ &= |\xi^\# \wedge \varphi|^2\ \xi \lrcorner\ vol(M) \\ &= 4|\xi^\#|^2\ (*\xi^\#) = 4\ vol(X_\xi).\end{aligned}$$

It is easy to see $*Re\ \Omega_\xi = -Im\ \Omega_\xi \wedge \xi^\#$ and $*Im\ \Omega_\xi = Re\ \Omega_\xi \wedge \xi^\#$.

$$*_3 Re\ \Omega_\xi = Im\ \Omega_\xi.$$

Notice that ω_ξ is a symplectic structure on X_ξ when $d\varphi = 0$ and $\mathscr{L}_\xi(\varphi)|_{V_\xi} = 0$, ($\mathscr{L}_\xi$ is the Lie derivative along ξ), since $\omega_\xi = \xi \lrcorner\ \varphi$ and:

$$d\omega_\xi = \mathscr{L}_\xi(\varphi) - \xi \lrcorner\ d\varphi = \mathscr{L}_\xi(\varphi).$$

J_ξ is integrable complex structure if $d^*\varphi = 0$ and $\mathscr{L}_\xi(*\varphi)|_{V_\xi} = 0$ since

$$d(Im\Omega_\xi) = d(\xi \lrcorner * \varphi) = \mathscr{L}_\xi(*\varphi) - \xi \lrcorner\ d(*\varphi) = 0.$$

Also notice that $d\varphi = 0 \implies d(Re\ \Omega_\xi) = d(\varphi|_{X_\xi}) = 0$.

4 HL Manifolds and Mirror Duality in G_2 Manifolds

By [7] any 7-dimensional Riemanninan manifold admits a non-vanishing orthonormal 2-frame field $\Lambda =< u, v >$, in particular (M, φ) admits such a field. Λ gives a section of the bundle of oriented 2-frames $V_2(M) \to M$, and hence gives an associative/coassociative splitting of the tangent bundle $TM = \mathbf{E} \oplus \mathbf{V}$, where $\mathbf{E} = \mathbf{E}_\Lambda =< u, v, u\times v >$ and $\mathbf{V} = \mathbf{V}_\Lambda = \mathbf{E}^\perp$. When there is no danger of confusion we will denote the 2-frame fields and the 2-planes fields which they induce by the same symbol Λ. Also, any unit section ξ of $\mathbf{E} \to M$ induces a complex structure J_ξ on the bundle $\mathbf{V} \to M$ by the cross product $J_\xi(u) = u \times \xi$.

In [3] any two almost Calabi–Yau's X_ξ and $X_{\xi'}$ inside (M, φ) were called *dual* if the defining vector fields ξ and ξ' are chosen from $\mathbf{V}$ and $\mathbf{E}$, respectively. Here we make this correspondence more precise, in particular showing how to choose ξ and ξ' in a more canonical way.

Definition 3. A 3-dimensional submanifold $Y^3 \subset (M, \varphi)$ is called *Harvey–Lawson manifold* (HL in short) if $\varphi|_Y = 0$.

Definition 4 ([2]). Call any orthonormal 3-frame field $\Gamma =< u, v, w >$ on (M, φ), a *G_2-frame field* if $\varphi(u, v, w) =< u \times v, w >= 0$, equivalently w is a unit section of $\mathbf{V}_\Lambda \to X$, with $\Lambda =< u, v >$ (see (1)).

Now pick a nonvanishing 2-frame field $\Lambda =< u, v >$ on M and let $TM = \mathbf{E} \oplus \mathbf{V}$ be the induced splitting with $\mathbf{E} =< u, v, u \times v >$. Let w be a unit section of the bundle $\mathbf{V} \to M$. Such a section w may not exist on whole M, but by obstruction theory it exists on a tubular neighborhood U of the 3-skeleton $M^{(3)}$ of M (which is a complement of some 3-complex $Z \subset M$). So $\varphi(u, v, w) = 0$, and hence $\Gamma =< u, v, w >$ is a G_2 frame field. Next consider the non-vanishing vector fields:

- $R = \chi(u, v, w) = -u \times (v \times w)$,
- $R' = \frac{1}{\sqrt{3}}(u \times v + v \times w + w \times u)$,
- $R'' = \frac{1}{\sqrt{3}}(u + v + w)$.

If the 6-plane fields $R^{\perp}$, $R'^{\perp}$, and $R''^{\perp}$, are integrable we get almost Calabi–Yau manifolds $(X_R, w_R, \Omega_R, J_R)$, $(X_{R'}, w_{R'}, \Omega_{R'}, J_{R'})$, and $(X_{R''}, w_{R''}, \Omega_{R''}, J_{R''})$. Let us use the convention that a, b, c are real numbers, and $[u_1, ..u_n]$ is the distribution generated by the vectors $u_1, .., u_n$.

Lemma 3. *By definitions, the following hold*

(a) $Y := [u, v, w] = [au + bv + cw \mid a + b + c = 0] \oplus [R'']$.
(b) $\mathbb{V} = [u, v, w, R]$*, is a coassociative* 4*-plane field.*
(c) $\mathrm{e} := [u \times v, v \times w, w \times u]$ *is an associative* 3*-plane field.*
(d) $\mathrm{e} \perp \mathbb{V}$.

Theorem 2. *For* $(a, b, c) \in \mathbb{R}^3$ *with* $a + b + c = 0$*, then*

(a) $TX_R = [au + bv + cw, R'', R', a(v \times w) + b(w \times u) + c(u \times v)]$,
$J_R(au + bv + cw) = -a(v \times w) - b(w \times u) - c(u \times v)$,
$J_R(R'') = -R'$,

(b) $TX_{R'} = [au + bv + cw, R'', R, a(v \times w) + b(w \times u) + c(u \times v)]$,
$J_{R'}(au + bv + cw) = -((b - c)u + (c - a)v + (a - b)w)/\sqrt{3}$,
$J_{R'}(a(v \times w) + b(w \times u) + c(u \times v)) =$
$((b - c)(v \times w) + (c - a)(w \times u) + (a - b)(u \times v))/\sqrt{3}$,
$J_{R'}(R'') = R$,

(c) $TX_{R''} = [au + bv + cw, R, R', (a(v \times w) + b(w \times u) + c(u \times v)]$,
$J_{R''}(au + bv + cw) =$
$((b - a)(u \times v) + (c - b)(v \times w) + (a - c)(w \times u))/\sqrt{3}$,
$J_{R''}(R) = R'$,

(d) $\{u, v, w, R, u \times v, v \times w, w \times u\}$ *is an orthonormal frame field.*

Proof. To show (a) by using (4) we calculate:

$$\begin{aligned} R \times u &= \chi(u, v, w) \times u = -[u \times (v \times w)] \times u = u \times [u \times (v \times w)] \\ &= -\chi(u, u, v \times w) - < u, u > (v \times w) + < u, v \times w > u \\ &= -(v \times w) + \varphi(u, v, w)u. \end{aligned}$$

Therefore

$$R \times u = -(v \times w). \tag{8}$$

Similarly, $R \times v = -(w \times u)$ and $R \times w = -(u \times v)$. Therefore we have $J_R(au + bv + cw) = -a(v \times w) - b(w \times u) - c(u \times v)$, and $J_R(R'') = -R'$.

$$\begin{aligned} \sqrt{3}\ R' \times u &= (u \times v + v \times w + w \times u) \times u \\ &= -u \times (u \times v) - u \times (v \times w) - u \times (w \times u) \\ &= < u, u > v - < u, v > u + \ \chi(u, v, w) \\ &\quad + < u, v > w - < u, w > v + < u, w > u - < u, u > w. \end{aligned}$$

Therefore

$$\sqrt{3}\ R' \times u = R + (v - w). \tag{9}$$

Similarly $\sqrt{3}\ R' \times v = R + (w - u)$, and $\sqrt{3}\ R' \times w = \ R + (u - v)$, which implies the first part of (b), and $J_{R'}(R'') = R$.

For the second part of (b) we need the compute the following:

$$\begin{aligned} &\sqrt{3}R' \times [a(v \times w) + b(w \times u) + c(u \times v)] \\ &\quad = (u \times v + v \times w + w \times u) \times [a(v \times w) + b(w \times u) + c(u \times v)]. \end{aligned} \tag{10}$$

For this first by repeatedly using (4) and $\varphi(u, v, w) = 0$ we calculate:

$$\begin{aligned} (v \times u) \times (w \times v) &= -\chi(v \times u, w, v) - < v \times u, w > v + < v \times u, v > w \\ &= -\chi(v \times u, w, v) = -\chi(w, v, v \times u) \\ &= w \times (v \times (v \times u)) + < w, v > (v \times u) - < w, v \times u > v \\ &= w \times (v \times (v \times u)) \\ &= w \times (-\chi(v, v, u) - < v, v > u + < v, u > v) = -(w \times u). \end{aligned}$$

Then by plugging in (9) gives (b). Proof of (c) is similar to (a). □

In particular from the above calculations we can express φ as:

Corollary 1.

$$\begin{aligned}\varphi = {} & u^{\#} \wedge v^{\#} \wedge (u^{\#} \times v^{\#}) + v^{\#} \wedge w^{\#} \wedge (v^{\#} \times w^{\#}) + w^{\#} \wedge u^{\#} \wedge (w^{\#} \times u^{\#}) \\ & + u^{\#} \wedge R^{\#} \wedge (v^{\#} \times w^{\#}) + (v^{\#} \wedge R^{\#}) \wedge (w^{\#} \times u^{\#}) + w^{\#} \wedge R^{\#} \wedge (u^{\#} \times v^{\#}) \\ & - (u^{\#} \times v^{\#}) \wedge (v^{\#} \times w^{\#}) \wedge (w^{\#} \times u^{\#}).\end{aligned}$$

Recall that in an earlier paper we proved the following facts:

Proposition 1 ([3]). *Let $\{\alpha, \beta\}$ be orthonormal vector fields on (M, φ). Then on X_α the following hold*

(i) $Re\ \Omega_\alpha = \omega_\beta \wedge \beta^{\#} + Re\ \Omega_\beta$,
(ii) $Im\ \Omega_\alpha = \alpha \lrcorner (\star\omega_\beta) - (\alpha \lrcorner Im\ \Omega_\beta) \wedge \beta^{\#}$,
(iii) $\omega_\alpha = \alpha \lrcorner Re\ \Omega_\beta + (\alpha \lrcorner \omega_\beta) \wedge \beta^{\#}$.

Proof. Since $Re\ \Omega_\alpha = \varphi|_{X_\alpha}$ (i) follows. Since $Im\ \Omega_\alpha = \alpha \lrcorner * \varphi$ following gives (ii)

$$\begin{aligned}\alpha \lrcorner (\star\omega_\beta) &= \alpha \lrcorner [\, \beta \lrcorner * (\beta \lrcorner \varphi)\,] \\ &= \alpha \lrcorner \beta \lrcorner (\beta^{\#} \wedge *\varphi) \\ &= \alpha \lrcorner * \varphi + \beta^{\#} \wedge (\alpha \lrcorner \beta \lrcorner * \varphi) \\ &= \alpha \lrcorner * \varphi + (\alpha \lrcorner Im\ \Omega_\beta) \wedge \beta^{\#}.\end{aligned}$$

(iii) follows from the following computation

$$\alpha \lrcorner Re\ \Omega_\beta = \alpha \lrcorner \beta \lrcorner (\beta^{\#} \wedge \varphi) = \alpha \lrcorner \varphi + \beta^{\#} \wedge (\alpha \lrcorner \beta \lrcorner \varphi) = \alpha \lrcorner \varphi - (\alpha \lrcorner \omega_\beta) \wedge \beta^{\#}.$$

□

Note that even though the identities of this proposition hold only after restricting the right hand side to X_α, all the individual terms are defined everywhere on (M, φ). Also, from the construction, X_α and X_β inherit vector fields β and α, respectively.

Corollary 2 ([3]). *Let $\{\alpha, \beta\}$ be orthonormal vector fields on (M, φ). Then there are $A_{\alpha\beta} \in \Omega^3(M)$, and $W_{\alpha\beta} \in \Omega^2(M)$ satisfying*

(a) $\varphi|_{X_\alpha} = Re\ \Omega_\alpha$ *and* $\varphi|_{X_\beta} = Re\ \Omega_\beta$,
(b) $A_{\alpha\beta}|_{X_\alpha} = Im\ \Omega_\alpha$ *and* $A_{\alpha\beta}|_{X_\beta} = \alpha \lrcorner (\star\omega_\beta)$,
(c) $W_{\alpha\beta}|_{X_\alpha} = \omega_\alpha$ *and* $W_{\alpha\beta}|_{X_\beta} = \alpha \lrcorner Re\ \Omega_\beta$.

Now we can choose α as R and β as R' of the given HL manifold. That concludes that given a HL submanifold of a G_2 manifold, it will determine a "canonical" mirror pair of Calabi–Yau manifolds (related through the HL manifold) with the complex and symplectic structures given above.

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Sequences of Maximal Antipodal Sets of Oriented Real Grassmann Manifolds

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Abstract We construct two sequences of antipodal sets of the oriented real Grassmann manifolds in a combinatorial way and a sequence of antipodal sets in a different way. We show that they are maximal antipodal sets under certain conditions.

1 Introduction

The author reduced the problem of classifying all maximal antipodal sets in the oriented real Grassmann manifold $\tilde{G}_k(\mathbb{R}^n)$ consisting of k-dimensional oriented subspaces in $\mathbb{R}^n$ to that of classifying all maximal antipodal subsets in the set $P_k(n)$ consisting of subsets of cardinality k in $\{1, \ldots, n\}$ and classified all maximal antipodal subsets of $P_k(n)$ for $k \leq 4$ in Tasaki [2]. A subset S of a Riemannian symmetric space is an *antipodal set*, if $s_x(y) = y$ for any x, y in S, where s_x is a geodesic symmetry with respect to x. The notion of an antipodal set in a Riemannian symmetric space was introduced by Chen and Nagano [1]. According to Theorem 3.1 of [2], any maximal antipodal set of $\tilde{G}_k(\mathbb{R}^n)$ is equal to

$$\{\pm\mathrm{span}\{e_{\alpha(1)}, \ldots, e_{\alpha(k)}\} \mid \alpha \in A\}$$

for an orthonormal basis $\{e_i\}$ of $\mathbb{R}^n$ and a maximal antipodal subset A of $P_k(n)$. Here $\alpha = \{\alpha(1), \ldots, \alpha(k)\}$ and $\pm$ means both orientations of a subspace. The definition of an antipodal subset of $P_k(n)$ is given in Sect. 2.

If k is more than 4, there may be so many maximal antipodal subsets of $P_k(n)$ that it is difficult to classify all of them. In this paper, we construct some sequences of maximal antipodal subsets of $P_k(n)$, which may be usefull for the classification of maximal antipodal subsets of $P_k(n)$,

In Sect. 2 we review the definition of antipodal subsets in $P_k(n)$ and prepare some notation for the sequel sections. In Sect. 3 we define two sequences of antipodal

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subsets in a combinatorial way and prove that they are maximal antipodal subsets under a certain condition. In Sect. 4 we define a sequence of antipodal subsets using even numbers and show which of them are maximal. These are generalizations of the sequences of antipodal subsets defined in [2].

2 Antipodal Subsets

The definition of antipodal subsets in $P_k(n)$ comes from the notion of antipodal sets in Riemannian symmetric spaces. Two elements α and β in $P_k(n)$ are *antipodal*, if the cardinality $\#(\beta - \alpha)$ is even, where $\beta - \alpha = \{i \in \beta \mid i \notin \alpha\}$. A subset A of $P_k(n)$ is *antipodal*, if any α and β in A are antipodal. We denote by $\mathrm{Sym}(n)$ the symmetric group on $\{1, \dots, n\}$. Two subsets X and Y in $P_k(n)$ are *congruent*, if X is transformed to Y by an element of $\mathrm{Sym}(n)$. If $X \subset P_k(n)$ is antipodal, then a subset congruent with X is also antipodal.

In order to describe antipodal subsets we prepare some notation. For a set X we denote by $P_k(X)$ the set consisting of all subsets of cardinality k in X. We simply write $P_k(n)$ instead of $P_k(\{1, \dots, n\})$. When $X = X_1 \cup \cdots \cup X_m$ is a disjoint union, we put

$$A_1 \times \cdots \times A_m = \{\alpha_1 \cup \cdots \cup \alpha_m \mid \alpha_i \in A_i \ (1 \leq i \leq m)\}$$

for subsets A_i of $P_{k_i}(X_i)$. We get

$$A_1 \times \cdots \times A_m \subset P_{k_1 + \cdots + k_m}(X).$$

If any A_i is antipodal in $P_{k_i}(X_i)$, then $A_1 \times \cdots \times A_m$ is antipodal in $P_{k_1 + \cdots + k_m}(X)$.

3 Combinatorial Sequences of Antipodal Subsets

We define two sequences of antipodal subsets as follows:

$$A(2k, 2l) = \{\alpha_1 \cup \cdots \cup \alpha_k \in P_{2k}(2l) \mid \alpha_i \in \{\{1, 2\}, \dots, \{2l - 1, 2l\}\}\},$$
$$A(2k + 1, 2l + 1) = A(2k, 2l) \times \{\{2l + 1\}\}.$$

It follows from the definition that $A(2k, 2l)$ is an antipodal subset of $P_{2k}(2l)$ and that $A(2k+1, 2l+1)$ is an antipodal subset of $P_{2k+1}(2l+1)$. Their cardinalities are

$$\#A(2k, 2l) = \#A(2k + 1, 2l + 1) = \binom{l}{k}.$$

Remark 1. In the case where $k = 1$, the definition of $A(3, 2l + 1)$ is different from that of $A(3, 2l + 1)$ in [2], but they are congruent. According to Theorem 5.1 of [2], $A(3, 2l + 1)$ is a maximal antipodal subset of $P_3(2l + 1)$ and $P_3(2l + 2)$, if $l \geq 4$. Moreover $A(3, 7)$ is not a maximal antipodal subset of $P_3(7)$, as is showed in Sect. 5 of [2], that is, $A(3, 7)$ is included in the set of projective lines in the projective plane $F_2 P^2$ over the binary field F_2 consisting of 0 and 1. The following theorem is a generalization of the phenomenon mentioned above and Proposition 7.1 in [2].

Theorem 1. *If $l \geq 3k-1$, then $A(2k, 2l)$ is a maximal antipodal subset of $P_{2k}(2l)$ and $P_{2k}(2l+1)$. Moreover if $k \geq 2$, then $A(2k+1, 2l+1)$ is a maximal antipodal subset of $P_{2k+1}(2l + 1)$ and $P_{2k+1}(2l + 2)$.*

Proof. Since $A(2k, 2l)$ is a maximal antipodal subset of $P_{2k}(2l)$ and $P_{2k}(2l + 1)$ by Proposition 7.1 in [2], it is sufficient to prove that $A(2k+1, 2l+1)$ is a maximal antipodal subset of $P_{2k+1}(2l + 1)$ and $P_{2k+1}(2l + 2)$.

In order to show that $A(2k + 1, 2l + 1)$ is a maximal antipodal subset of $P_{2k+1}(2l + 1)$ we take $\alpha \in P_{2k+1}(2l + 1)$ which is antipodal with all elements in $A(2k+1, 2l+1)$. If α contains $2l+1$, then $\alpha - \{2l+1\} \in P_{2k}(2l)$ is antipodal with all elements in $A(2k, 2l)$. Since $A(2k, 2l)$ is a maximal antipodal subset of $P_{2k}(2l)$, the element $\alpha - \{2l+1\}$ is contained in $A(2k, 2l)$ and we get $\alpha \in A(2k+1, 2l+1)$. So we suppose that α does not contain $2l + 1$, that is, $\alpha \in P_{2k+1}(2l)$.

We define subsets I_0, I_1 and I_2 of $\{1, \dots, l\}$ by

$$I_j = \{i \mid \#(\{2i - 1, 2i\} \cap \alpha) = j \ (1 \leq i \leq l)\} \quad (j = 0, 1, 2).$$

$\{1, \dots, l\}$ is decomposed to the disjoint union

$$\{1, \dots, l\} = I_0 \cup I_1 \cup I_2.$$

We get

$$2k + 1 = \#\alpha = \#I_1 + 2\#I_2,$$

hence I_1 is not empty.

We divide the argument into two cases of $\#I_1 = 1$ and $\#I_1 \geq 2$. We first consider the case where $\#I_1 = 1$. Since $l \geq 3k - 1$, we have

$$\#(\{1, \dots, l\} - I_1) = l - 1 \geq 3k - 2 \geq k,$$

thus we can take a subset

$$\{j_1, \dots, j_k\} \subset \{1, \dots, l\} - I_1 = I_0 \cup I_2.$$

We take

$$\gamma = \{2j_1 - 1, 2j_1, \dots, 2j_k - 1, 2j_k, 2l + 1\} \in A(2k + 1, 2l + 1).$$

The cardinality $\#(\alpha \cap \gamma)$ is even and α, γ are not antipodal, which is a contradiction. Therefore this case does not happen.

In the case where $\#I_1 \geq 2$, we can take $\{i_1, i_2\} \subset I_1$. Since $\#I_1 \leq \#\alpha = 2k + 1$, we have

$$\#(\{1, \ldots, l\} - I_1) \geq l - (2k + 1) \geq (3k - 1) - (2k + 1) = k - 2 \geq 0.$$

Hence we can take a subset

$$\{j_3, \ldots, j_k\} \subset \{1, \ldots, l\} - I_1 = I_0 \cup I_2.$$

If $k = 2$, the left hand side is empty. We put

$$\delta = \{2i_1 - 1, 2i_1, 2i_2 - 1, 2i_2, 2j_3 - 1, 2j_3, \ldots, 2j_k - 1, 2j_k, 2l + 1\},$$

which is an element in $A(2k+1, 2l+1)$. The cardinality $\#(\alpha \cap \delta)$ is even and α, δ are not antipodal, which is a contradiction. Therefore this case does not happen and any element of $P_{2k+1}(2l + 1)$ which is antipodal with all elements in $A(2k + 1, 2l + 1)$ is contained in $A(2k + 1, 2l + 1)$. This implies that $A(2k + 1, 2l + 1)$ is a maximal antipodal subset of $P_{2k+1}(2l + 1)$.

Next we show that $A(2k + 1, 2l + 1)$ is also a maximal antipodal subset of $P_{2k+1}(2l + 2)$. We take $\alpha \in P_{2k+1}(2l + 2)$ which is antipodal with all elements in $A(2k + 1, 2l + 1)$. We have already showed that $A(2k + 1, 2l + 1)$ is a maximal antipodal subset of $P_{2k+1}(2l + 1)$. So we suppose that α is not contained in $P_{2k+1}(2l + 1)$, which implies that α contains $2l + 2$. We also define subsets I_0, I_1 and I_2 of $\{1, \ldots, l\}$ by

$$I_j = \{i \mid \#(\{2i - 1, 2i\} \cap \alpha) = j \ (1 \leq i \leq l)\} \quad (j = 0, 1, 2).$$

We divide the arguments into three cases of $\#I_1 = 0, 1$ and $\#I_1 \geq 2$. We consider the case where $\#I_1 = 0$. Any $\beta \in A(2k + 1, 2l + 1)$ and α are antipodal, hence α contains $2l + 1$. So we get $\#\alpha = 2 + 2\#I_2$, which is a contradiction. Thus this case does not happen.

We consider the case where $\#I_1 = 1$. Since

$$\#(\{1, \ldots, l\} - I_1) = l - 1 \geq 3k - 2 \geq k,$$

we can take a subset

$$\{j_1, \ldots, j_k\} \subset \{1, \ldots, l\} - I_1 = I_0 \cup I_2.$$

We take an element

$$\gamma = \{2j_1 - 1, 2j_1, \ldots, 2j_k - 1, 2j_k, 2l + 1\} \in A(2k + 1, 2l + 1),$$

which is antipodal with α. So α contains $2l + 1$. We put $I_1 = \{i_1\}$ and take an element

$$\gamma' = \{2i_1 - 1, 2i_1, 2j_2 - 1, 2j_2, \ldots, 2j_k - 1, 2j_k, 2l + 1\} \in A(2k + 1, 2l + 1).$$

The cardinality $\#(\alpha \cap \gamma')$ is even and α, γ' are not antipodal, which is a contradiction. Therefore this case does not happen.

In the case where $\# I_1 \geq 2$, we can take $\{i_1, i_2\} \subset I_1$. Since

$$\# I_1 \leq \#(\alpha - \{2l + 2\}) = 2k,$$

we have

$$\#(\{1, \ldots, l\} - I_1) \geq l - 2k \geq (3k - 1) - 2k = k - 1.$$

Hence we can take a subset

$$\{j_2, \ldots, j_k\} \subset \{1, \ldots, l\} - I_1 = I_0 \cup I_2.$$

We take an element

$$\delta = \{2i_1 - 1, 2i_1, 2i_2 - 1, 2i_2, 2j_3 - 1, 2j_3, \ldots, 2j_k - 1, 2j_k, 2l + 1\},$$

which is an element in $A(2k + 1, 2l + 1)$. Since δ, α are antipodal, α contains $2l + 1$. We take an element

$$\delta' = \{2i_1 - 1, 2i_1, 2j_2 - 1, 2j_2, \ldots, 2j_k - 1, 2j_k, 2l + 1\},$$

which is an element in $A(2k + 1, 2l + 1)$. The cardinality $\#(\alpha \cap \delta')$ is even and α, δ' are not antipodal, which is a contradiction. Therefore this case does not happen and any element of $P_{2k+1}(2l + 2)$ which is antipodal with all elements in $A(2k + 1, 2l + 1)$ is contained in $A(2k + 1, 2l + 1)$. This implies that $A(2k + 1, 2l + 1)$ is a maximal antipodal subset of $P_{2k+1}(2l + 2)$. Therefore we complete the proof of the theorem.

We define

$$a(k, n) = \max\{\# A \mid A \text{ is antipodal in } P_k(n).\}.$$

The existences of $A(2k, 2l)$ and $A(2k + 1, 2l + 1)$ imply the following estimates.

$$(*) \quad a(2k, n) \geq \binom{\left[\frac{n}{2}\right]}{k}, \qquad a(2k + 1, n) \geq \binom{\left[\frac{n-1}{2}\right]}{k}.$$

We can obtain the values of $a(k,n)$ for $k \leq 4$ using the classifications of maximal antipodal subsets of $P_k(n)$ obtained in [2]. The equalities in $(*)$ hold for $a(k,n)$ with $k \leq 4$ and sufficiently large n. The author recently obtained the equality

$$a(5,n) = \binom{\left[\frac{n-1}{2}\right]}{2}$$

for sufficiently large n. Moreover if an antipodal subset A of $P_5(n)$ for such n attains $a(5,n)$, then A is congruent with $A\left(5, 2\left[\dfrac{n-1}{2}\right]+1\right)$. These results will appear in a forthcoming paper.

4 Even Sequences of Antipodal Subsets

For a natural number m we define

$$Ev_{2m} = \{\{\alpha(1), \dots, \alpha(m)\} \mid \alpha(i) \in \{2i-1, 2i\}\ (1 \leq i \leq m),$$
$$\text{the number of even numbers } \alpha(i) \text{ is even}\}.$$

This is a subset of $P_1(\{1,2\}) \times \cdots \times P_1(\{2m-1, 2m\}) \subset P_m(2m)$ and a generalization of Ev_{4m} defined in [2]. In order to prove that Ev_{2m} is an antipodal subset of $P_m(2m)$ we prepare the following lemma.

Lemma 1. *For a natural number m and*

$$\alpha = \{\alpha(1), \dots, \alpha(m)\} \in P_1(\{1,2\}) \times \cdots \times P_1(\{2m-1, 2m\}) \subset P_m(2m)$$

we define

$$\alpha^e = \{i \mid \alpha(i) \text{ is even}\}, \quad \alpha^o = \{i \mid \alpha(i) \text{ is odd}\}.$$

For $\alpha, \beta \in P_1(\{1,2\}) \times \cdots \times P_1(\{2m-1, 2m\})$ *we have*

$$\#(\alpha \cap \beta) = 2\#(\alpha^e \cap \beta^e) + \#\beta^o - \#\alpha^e.$$

Proof. By the definition

$$\alpha^e \cup \alpha^o = \beta^e \cup \beta^o = \{1, \dots, m\}$$

are disjoint unions.

$$\alpha \cap \beta = \{\alpha(i) \mid i \in \alpha^e \cap \beta^e\} \cup \{\alpha(i) \mid i \in \alpha^o \cap \beta^o\}$$

is also a disjoint union, thus we have

$$\begin{aligned}\#(\alpha \cap \beta) &= \#(\alpha^e \cap \beta^e) + \#(\alpha^o \cap \beta^o)\\ &= \#(\alpha^e \cap \beta^e) + (\#\beta^o - \#(\alpha^e \cap \beta^o)).\end{aligned}$$

$\alpha^e = (\alpha^e \cap \beta^e) \cup (\alpha^e \cap \beta^o)$ is a disjoint union, so

$$\#\alpha^e = \#(\alpha^e \cap \beta^e) + \#(\alpha^e \cap \beta^o).$$

Hence

$$\#(\alpha^e \cap \beta^o) = \#\alpha^e - \#(\alpha^e \cap \beta^e)$$

and we obtain

$$\begin{aligned}\#(\alpha \cap \beta) &= \#(\alpha^e \cap \beta^e) + (\#\beta^o - \#(\alpha^e \cap \beta^o))\\ &= 2\#(\alpha^e \cap \beta^e) + \#\beta^o - \#\alpha^e.\end{aligned}$$

Lemma 2. *Ev_{2m} is an antipodal subset of $P_m(2m)$.*

Proof. By Lemma 1, for any $\alpha, \beta \in Ev_{2m}$ we have

$$\begin{aligned}\#(\alpha \cap \beta) &= 2\#(\alpha^e \cap \beta^e) + \#\beta^o - \#\alpha^e \equiv \#\beta^o \pmod 2\\ &= m - \#\beta^e \equiv m \pmod 2.\end{aligned}$$

Therefore α, β are antipodal.

Example 1.

$$Ev_6 = \{\{1,3,5\}, \{2,4,5\}, \{2,3,6\}, \{1,4,6\}\}$$

is transformed by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 1 & 6 & 3 & 4 \end{pmatrix}$$

to

$$B(3,6) = \{\{1,2,3\}, \{1,4,5\}, \{2,4,6\}, \{3,5,6\}\},$$

which was defined in [2] and is a maximal antipodal subset of $P_3(6)$.

Theorem 2. *If $2m \equiv 2, 4, 6 \pmod 8$, then Ev_{2m} is a maximal antipodal subset of $P_m(2m)$. On the other hand, Ev_{8m} is not a maximal antipodal subset of $P_{4m}(8m)$, but $A(4m, 8m) \cup Ev_{8m}$ is a maximal antipodal subset of $P_{4m}(8m)$.*

Proof. We have already proved that $A(4m, 8m) \cup Ev_{8m}$ is a maximal antipodal subset of $P_{4m}(8m)$ in [2]. Hence it is sufficient to prove that Ev_{2m} is a maximal antipodal subset of $P_m(2m)$ in the cases where $2m \equiv 2, 4, 6 (\mathrm{mod} 8)$.

We first consider the cases where $2m \equiv 2, 6 (\mathrm{mod} 8)$ and prove that Ev_{4m+2} is a maximal antipodal subset of $P_{2m+1}(4m + 2)$. We take an element $\alpha \in P_{2m+1}(4m + 2)$ which is antipodal with all elements of Ev_{4m+2}. We take an element

$$\beta = \{2i - 1 \mid 1 \le i \le 2m + 1\} \in Ev_{4m+2}.$$

Since α, β are antipodal, the cardinality of

$$\alpha - \beta = \{\alpha(i) \mid \alpha(i) \text{ is even}\}$$

is even. If $\alpha \in P_1(\{1, 2\}) \times \cdots \times P_1(\{4m + 1, 4m + 2\})$, then $\alpha \in Ev_{4m+2}$. So we consider the case where $\alpha \notin P_1(\{1, 2\}) \times \cdots \times P_1(\{4m + 1, 4m + 2\})$. In this case there exists $1 \le j \le 2m + 1$ which satisfies

$$\{2j - 1, 2j\} \cap \alpha = \emptyset.$$

For $\sigma \in \mathrm{Sym}(2m + 1)$ we define $\tilde{\sigma} \in \mathrm{Sym}(4m + 2)$ by

$$\tilde{\sigma}(2i - 1) = 2\sigma(i) - 1, \quad \tilde{\sigma}(2i) = 2\sigma(i).$$

By this $\mathrm{Sym}(2m + 1)$ acts on $\{1, 2, \ldots, 4m + 2\}$. Ev_{4m+2} is invariant under this action. We take $\tau \in \mathrm{Sym}(2m + 1)$ satisfying $\tau(j) = 2m + 1$. This implies

$$\emptyset = \tilde{\tau}(\{2j - 1, 2j\} \cap \alpha) = \{4m + 1, 4m + 2\} \cap \tilde{\tau}(\alpha),$$

that is, $\tilde{\tau}(\alpha) \in P_{2m+1}(4m)$. Moreover $\tilde{\tau}(\alpha)$ is antipodal with all elements of Ev_{4m+2} and the cardinality of

$$\{\tilde{\tau}(\alpha)(i) \mid \tilde{\tau}(\alpha)(i) \text{ is even}\}$$

is even. We take an element

$$\gamma = \{2i \mid 1 \le i \le 2m\} \cup \{4m + 1\} \in Ev_{4m+2}.$$

$\#(\tilde{\tau}(\alpha) \cap \gamma)$ is even, which contradicts that $\tilde{\tau}(\alpha)$ and γ are antipodal. Therefore $\alpha \in Ev_{4m+2}$. Hence Ev_{4m+2} is a maximal antipodal subset of $P_{2m+1}(4m + 2)$.

We second prove that Ev_{8m+4} is a maximal antipodal subset of $P_{4m+2}(8m + 4)$. We take an element $\alpha \in P_{4m+2}(8m + 4)$ which is antipodal with all elements of Ev_{8m+4}. We take an element

$$\beta = \{2i - 1 \mid 1 \le i \le 4m + 2\} \in Ev_{8m+4}.$$

Since α, β are antipodal, the cardinality of

$$\alpha - \beta = \{\alpha(i) \mid \alpha(i) \text{ is even}\}$$

is even. If $\alpha \in P_1(\{1,2\}) \times \cdots \times P_1(\{8m+3, 8m+4\})$, then $\alpha \in Ev_{8m+4}$. So we consider the case where $\alpha \notin P_1(\{1,2\}) \times \cdots \times P_1(\{8m+3, 8m+4\})$. We put

$$B_j = \{2j-1, 2j\} \quad (1 \leq j \leq 4m+2).$$

There exist $1 \leq j, k \leq 4m+2$ such that

$$B_j \cap \alpha = \emptyset, \quad B_k \subset \alpha.$$

We put

$$B^{eo} = \{B_j \mid B_j \subset \alpha\}, \quad B^o = \{B_j \mid B_j \cap \alpha = \{2j-1\}\},$$
$$B^e = \{B_j \mid B_j \cap \alpha = \{2j\}\}, \quad B^{\emptyset} = \{B_j \mid B_j \cap \alpha = \emptyset\}.$$

Then $B^{eo} \neq \emptyset$, $B^{\emptyset} \neq \emptyset$ and

$$B^{eo} \cup B^o \cup B^e \cup B^{\emptyset} = \{B_j \mid 1 \leq j \leq 4m+2\}$$

is a disjoint union.

$$4m+2 = \#\alpha = 2\#B^{eo} + \#B^o + \#B^e$$

is even, hence $\#B^o + \#B^e$ is also even. If $\#B^o + \#B^e = 0$, then $\#B^{eo} = 2m+1$, which contradicts that $\#\{\alpha(i) \mid \alpha(i) \text{ is even}\}$ is even. Hence $\#B^o + \#B^e \geq 2$. We can take $\gamma \in P_1(B_1) \times \cdots \times P_1(B_{4m+2})$ in the following way. We take an even number in each B_j of B^{eo}, an odd number in each of B^o and an even number in each of B^e, however we change the parity of one in $B^o \cup B^e \neq \emptyset$ in the case where $\#B^{eo} + \#B^o + \#B^e$ is even. We take a number in each of $B^{\emptyset} \neq \emptyset$ such that the numbers of all even numbers we take is even and define γ. Then $\gamma \in Ev_{8m+4}$ and

$$\#(\alpha \cap \gamma) = \#B^{eo} + \#B^o + \#B^e \quad \text{or} \quad \#B^{eo} + \#B^o + \#B^e - 1,$$

which is odd. This contradicts that α and γ are antipodal. Therefore $\alpha \in Ev_{8m+4}$ and Ev_{8m+4} is a maximal antipodal subset of $P_{4m+2}(8m+4)$. Therefore we complete the proof of the theorem.

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