Chapter 5
Parametric Control to Avoid Bifurcation Based on Maximum Local Lyapunov Exponent

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5.1 Introduction

Discrete-time dynamical systems [2] are widely used for mathematical modeling of various systems. In many cases, desired behavior in nonlinear discrete-time dynamical systems corresponds to stable fixed and periodic points. The values of system parameters can be determined through bifurcation analysis [9, 10, 15] in advance so that desired behavior is produced in a steady state. However, when the parameter values are set far from appropriate values for any reason, the systems may not work correctly owing to undesirable behavior caused by bifurcations of desired behavior, for example, as alternans in the heart model [14].

Control systems to avoid bifurcations can prevent the emergence of undesirable states and keep proper states of dynamical systems. Here, we assume that desired behavior corresponds to a stable periodic point and consider a problem of avoiding its bifurcations in order to construct robust and resilient dynamical systems that are controlled so as not to make bifurcations.

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Bifurcations of stable periodic points occur when their degree of stability (stability index) defined in Chap. 2 becomes one, i.e., their bifurcations can be avoided by suppressing the stability index below one. However, as described in Chap. 3, the optimization of the stability index has a difficulty because the stability index is not differentiable with respect to system parameters in general.

In this chapter, by using the maximum Lyapunov exponent (MLE) [11, 13, 16] that is related to the stability index [4, 5, 11, 12], we present a parametric controller that can avoid bifurcations of stable periodic points for unexpected parameter variation [6]. In practice, we substitute the maximum local LE (MLLE) [1, 3] defined in finite time to relieve a difficulty in computation of the MLE. Compared with the stability index, using the MLLE has the following advantages [6]: simple gradient methods can be used to optimize the MLLE, and the calculations of the MLLE and control input to avoid bifurcations can be realized along the passage of time. Experimental results applied to the Hénon map [7] to evaluate whether our parametric controller is effective to avoid bifurcations are also presented.

5.2 Problem Statement

Consider a discrete-time dynamical system described by

$$x(t+1) = f(x(t), p(t), q(t)), \tag{5.1}$$

where t denotes the discrete time, $x \in \mathbb{R}^N$ representing the set of real numbers is the vector of state variables, and $p \in \mathbb{R}^M$ and $q \in \mathbb{R}^L$ are time-variant system parameters. Here, we assume that f is known and differentiable, all states are always observable, and the values of p can be forcibly changed for any reason and are out of control, but q is handleable. We also assume that these parameter values can be changed only at $t = mT$ $(m = 0, 1, 2, \ldots)$ where T represents an interval to get the value of the MLLE and control input to avoid bifurcations, which are defined later.

When all parameter values are constant, fixed and periodic points of f are defined as follows. If a point $x^* \in \mathbb{R}^N$ satisfies $x^* - f(x^*, p, q) = 0$, then x^* is a fixed point of f. In the same way, a periodic point with period n, i.e. an n-periodic point, of f is defined as a point x^* such that $x^* - f^n(x^*, p, q) = 0$ and $x^* - f^k(x^*, p, q) \neq 0$ for $k < n$ where f^n denotes the nth iterate of f. By describing the Jacobian matrix of f as

$$Df(x(t), p, q) = \frac{\partial}{\partial x} f(x(t), p, q), \tag{5.2}$$

we introduce the characteristic equation of an n-periodic point x^* as

$$\chi(x^*, p, q, \mu) = \det\left(\mu I - Df^n(x^*, p, q)\right) = 0, \tag{5.3}$$

where I denotes the $N \times N$ identity matrix; μ is an eigenvalue of $Df^n(x^*, p, q)$ and is called the characteristic multiplier of x^*.

The stability index of an n-periodic point (x^*) is defined by using the maximum modulus of its characteristic multipliers, i.e., it is equivalent to the spectral radius of the Jacobian matrix, $\rho\,(Df^n(x^*, p, q))$, where $\rho(\cdot)$ represents the spectral radius of a matrix. Therefore, a periodic point x^* is stable if and only if $\rho\,(Df^n(x^*, p, q)) < 1$ and bifurcations of a stable periodic point occur when $\rho\,(Df^n(x^*, p, q)) = 1$. The parameter values at which bifurcations occur can be numerically found by using a powerful computing method [9, 15].

We now assume that desired behavior corresponds to a stable periodic point and treat a situation that bifurcations of desired behavior may emerge owing to the forcible variation of p. For the situation, we consider avoiding bifurcations of desired behavior by adjusting the values of q only when the parameter values approach any bifurcation points. Therefore, this problem resembles the problems treated in Chaps. 2, 3 and 4.

5.3 Proposed Method

From the aforementioned assumptions, the values of p and q are constant for the duration of interval T. When an initial value $x(mT)$ at $t = mT$ $(m = 0, 1, 2, \ldots)$ that converges to a stable periodic point and a small perturbation $w(mT) \in \mathbb{R}^N$ to $x(mT)$ are given, the MLLE is defined as

$$\lambda(x(mT), p, q, T) = \frac{1}{T} \sum_{t=mT}^{(m+1)T-1} \ln\|w(t+1)\|, \tag{5.4}$$

where $\|\cdot\|$ represents the Euclidean norm of a vector. The trajectory of $w(t+1)$ is obtained from the linearized system defined by

$$w(t+1) = Df(x(t), p, q) \cdot v(t), \tag{5.5}$$

where $v(t) = w(t)/\|w(t)\|$. This normalization is to relieve a computational difficulty in (5.4). In the following, we simplify the notation of $\lambda(x(mT), p, q, T)$ as λ.

The problem of avoiding bifurcations of stable periodic points can be formulated as the minimization problem of an objective function defined by

$$G(\lambda) = \frac{1}{2}\left(\lambda - H(\lambda)\right)^2, \tag{5.6}$$

where H is a map described as

$$H(\lambda) = \begin{cases} \lambda & \text{if}\, \lambda \le \lambda^*, \\ \lambda^* & \text{otherwise.} \end{cases} \tag{5.7}$$

The user-defined parameter λ^* is set to a negative value close to zero; it is used not only to detect the approach of the values of p and q to any bifurcation points but also a set point to control λ when $\lambda^* < \lambda < 0$. Since λ is the function with respect to q, we can obtain a gradient system of (5.6), i.e. the updating rule of q, as

$$q((m+1)T) - q(mT) = -\eta \frac{\partial G(\lambda)}{\partial q} = -\eta(\lambda - \lambda^*)\frac{\partial \lambda}{\partial q}, \tag{5.8}$$

where η is a positive parameter given by users. The formulas to calculate the values of $\partial\lambda/\partial q$ can be explicitly expressed [6, 8] and their computations can be realized in real time without off-line calculation to find the exact position of stable periodic points to be controlled objects. Note that the formulas we derived [6] can be commonly used in a variety of nonlinear discrete-time dynamical systems. By updating the values of q according to (5.8) only when $\lambda^* < \lambda < 0$, the MLLE can remain a negative value, i.e., bifurcations of stable periodic points can be avoided.

5.4 Experimental Results

To evaluate whether the proposed parametric controller is effective, we carried out several experiments for stable periodic points observed in the Hénon map [7]. The dynamics of the Hénon map is described as

$$x_1(t+1) = 1 + x_2(t) - p(t) \cdot x_1(t)^2, \tag{5.9a}$$
$$x_2(t+1) = q(t) \cdot x_1(t), \tag{5.9b}$$

where x_1 and x_2 are state variables and t is the discrete time. We here assumed that p and q correspond to out-of-control and control parameters, respectively. In the following experiments, we set $T = 500$, $\eta = 0.1$, and $\lambda^* = -0.2$ in (5.4) and (5.8).

Before carrying out experiments, we analyzed bifurcations on fixed and periodic points observed in (5.9). As shown in Fig. 5.1, we found a fixed point, n-periodic

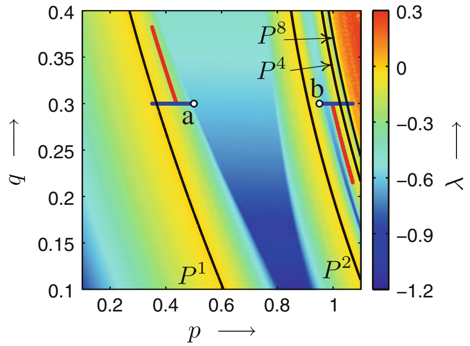

Fig. 5.1 Bifurcation diagram on fixed and periodic points in the Hénon map, MLLE, and *blue horizontal* and *red diagonal curves* corresponding to parameter variation without and with control

points ($n = 2, 4, 8$), and their period-doubling bifurcations where the solid curve with P^n represents the set of bifurcation points of the n-periodic point. The stable fixed point is present in the left-hand-side parameter regions of the curve P^1 and the stable n-periodic point exists in the parameter regions surrounded by the curves of $P^{\frac{n}{2}}$ and P^n. The MLLE on the fixed and periodic points is indicated in color, for example, the color in the parameter regions surrounded by the curves of P^1 and P^2 shows the MLLE on the stable two-periodic point. The relationship between the MLLE and colors is shown in the right bar graph. We note that these analyses are not necessary to avoid bifurcations using our controller, i.e., it was carried out only to demonstrate whether bifurcation points are avoided in space of system parameters.

When we set $(p(0), q(0)) = (0.5, 0.3)$ corresponding to the point "a" in Fig. 5.1 and $(x_1(0), x_2(0)) = (1.43, 0.0)$, the two-periodic point was observed in a steady state. By decreasing the value of p with 0.0015 every T along the blue horizontal line from the initial point "a", the two-periodic point bifurcated on the curve P^1 and instead the fixed point appeared at $t \simeq 93T$ as shown by the blue trajectory of x_1 in Fig. 5.2a. To avoid the period-doubling bifurcation, the proposed controller adjusted the value of q so as to keep $\lambda \simeq \lambda^*$ after $t = 42T$ (Fig. 5.2b). The trajectory of the controlled parameter is also shown as the red diagonal curve branching from the blue horizontal line with the point "a" in Fig. 5.1. Consequently, the stable two-periodic point could be observed for the duration of $0 \leq t \leq 100T$ without bifurcating.

We also analyzed avoiding the period-doubling bifurcation of the stable four-periodic point. The initial values were set to $(x_1(0), x_2(0)) = (1.04, -0.18)$ and $(p(0), q(0)) = (0.95, 0.3)$ corresponding to the point "b" in Fig. 5.1. When we changed the value of p along the blue horizontal line starting from the point "b", we

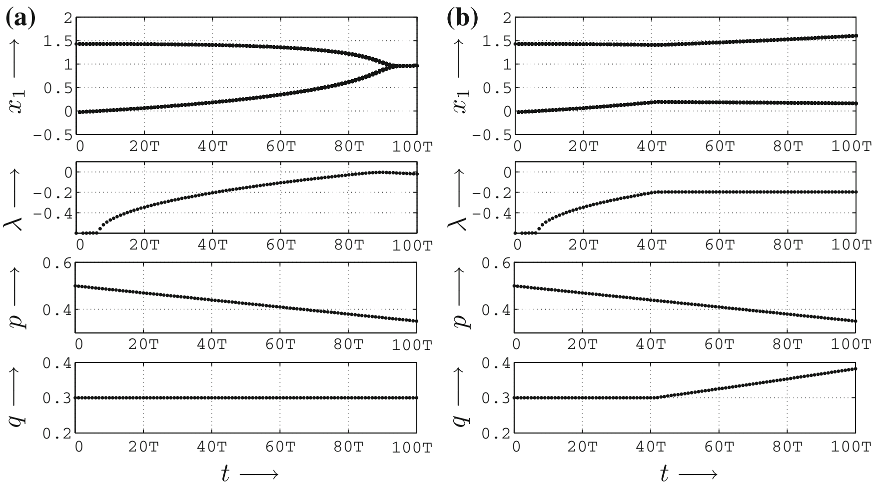

Fig. 5.2 Experimental results of bifurcation avoidance for a two-periodic point in the Hénon map displayed as time series. The *blue* and *red* sequences correspond to the trajectories both without and with control **a** case without control **b** case with control

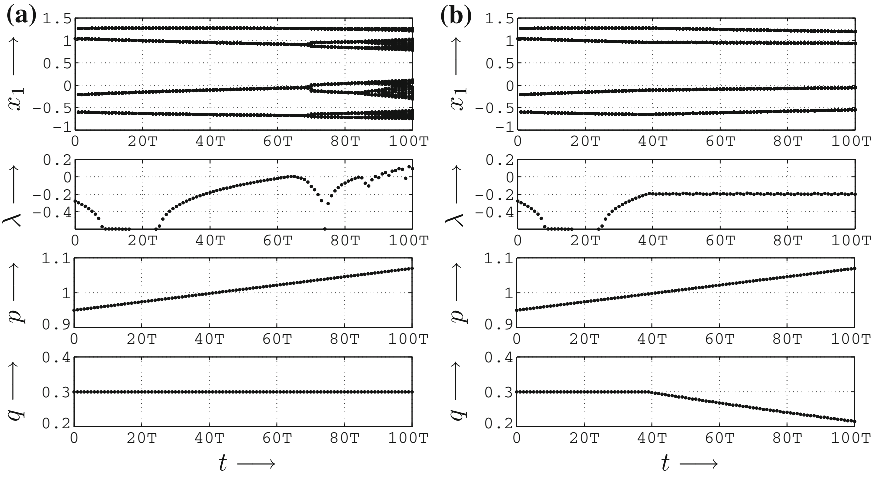

Fig. 5.3 Experimental results of bifurcation avoidance for a four-periodic point in the Hénon map displayed as time series **a** case without control **b** case with control

observed the eight-periodic points and a chaotic state caused by a cascade of period-doubling bifurcations across the curves of P^4 and beyond (Figs. 5.1 and 5.3a). Hence, the stable four-periodic point bifurcated and vanished at $t \simeq 70T$ owing to its period-doubling bifurcation curve P^4. In contrast, the red diagonal curve branching from the blue horizontal line with the point "b" in Fig. 5.1 indicated that the proposed controller was used to avoid the bifurcation curve of P^4. As the results, as shown in Fig. 5.3b, we could observe the four-periodic point for the duration of $0 \leq t \leq 100T$.

5.5 Conclusion

In this chapter, we presented a parametric control system to avoid bifurcations of stable periodic points in nonlinear discrete-time dynamical systems with parameter variation. The parameter updating of our controller is theoretically derived from the minimization of an objective function with respect to the MLLE. The computations of the MLLE and parameter variation to avoid bifurcations can be realized in real time without finding the exact positions of stable periodic points to be controlled objects. Our experimental results showed that the proposed controller effectively worked to avoid bifurcations of stable periodic points in the Hénon map. We note that this control system can be also applied to avoid bifurcations of stable fixed points. Further, the parameter-updating formulas we derived [6] can be widely used to a variety of nonlinear discrete-time dynamical systems.

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