Chapter 2
Robust Bifurcation Analysis Based on Degree of Stability

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2.1 Introduction

Consider nonlinear dynamical systems modelled by parameterised differential and difference equations. When the values of the system parameters vary from those at which the system is currently operated, it can exhibit qualitative changes in behaviour and a steady-state may disappear or become unstable through a bifurcation [1, 2]. Bifurcation analysis is clearly useful and a bifurcation diagram composed of bifurcation sets projected into parameter space displays various nonlinear phenomena in dynamical systems. On the other hand, when considering a steady-state which is closely approaching a bifurcation point due to unexpected factors like environmental changes, major incidents, and aging, self recovery is an important strategy for constructing a robust and resilient system. Traditional bifurcation analysis is not effective for this purpose because the global features of a bifurcation diagram in parameter

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space needs to be computed to enable system behaviour to be totally understood with variations in parameters treated as measurable and directly controlled variables.

We present an approach to analyzing the system parameters based on the idea of characterising a steady-state with the degree of stability as a function of the parameters. The robust bifurcation analysis defined in this chapter provides a direct method for finding the values of the parameters at which the dynamical system acquires a steady-state with a high degree of stability using a method of optimization; this makes it possible to avoid bifurcations caused by the adverse effects of unexpected factors, without having to prepare the bifurcation diagrams that are required in advance for methods using traditional bifurcation analysis. As a result, we can design a system that is robust and resilient to unexpected factors.

The bifurcation control [3–10] also deals with modifications to bifurcation characteristics. It requires a feedback system to be constructed by adding control input, whereas our method uses prescribed parameters to optimize the stability.

In the following, after introducing the theoretical results [11] of robust bifurcation analysis, we will present numerical experiments of continuous-time dynamical systems. An example of representative results obtained in a model of the ventricular muscle cell suggests that our method has a possibility to suppress the alternans and reduce the risk of sudden death.

2.2 System Description and Robust Bifurcation Analysis

This section gives an exact definition of robust bifurcation analysis, which we describe both for equilibria in continuous-time dynamical systems and fixed points in discrete-time dynamical systems.

2.2.1 Continuous-Time Systems

We consider parameterised autonomous differential equations for continuous-time systems described by

$$\frac{dx}{dt} = f(x, \lambda), \quad t \in R, \tag{2.1}$$

where $x(t) = (x_1, x_2, \ldots, x_N)^\top \in R^N$ is a state vector, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_M)^\top \in R^M$ is a measurable and directly controlled parameter vector, and f is assumed to be a C^∞ function for simplicity. Suppose that there exists an equilibrium x^* satisfying

$$f(x^*, \lambda) = 0, \tag{2.2}$$

and it can be expressed locally as a function of the parameters λ, namely $x^*(\lambda)$. The Jacobian matrix or the derivative of f with respect to x at $x = x^*$ denoted by

$$D(x^*(\lambda), \lambda) := \left. \frac{\partial f(x, \lambda)}{\partial x} \right|_{x=x^*(\lambda)} \tag{2.3}$$

gives information to determine the stability of the equilibrium. The eigenvalues μ of D at x^* are obtained by solving the characteristic equation

$$\det(\mu I - D(x^*(\lambda), \lambda)) = 0, \tag{2.4}$$

where I is the identity matrix. We call x^* a hyperbolic equilibrium of the system, if D is hyperbolic, i.e., all the real parts of the eigenvalues of D are different from zero. If all eigenvalues lie in the left-half plane, then the equilibrium is asymptotically stable. A local bifurcation occurs if an equilibrium loses its hyperbolicity due to continuous parameter variations. The generic bifurcations of an equilibrium are the saddle-node or tangent bifurcation and the Hopf bifurcation.

The main objective of bifurcation analysis is to find sets of bifurcation values. The bifurcation sets can be obtained by solving simultaneous equations consisting of the equilibrium equation in (2.2) and the characteristic equation in (2.4) for unknown variables x^* and λ_m for $m = 1, 2, \ldots, M$. Thus, the bifurcation sets for all m's need to be computed to enable system behaviour to be totally understood with parameter variations.

2.2.2 Discrete-time Systems

Next, let us consider parameterised difference equations for discrete-time systems

$$x(k+1) = g(x(k), \lambda), \quad k = 1, 2, \ldots,$$

or equivalently, a map defined by

$$g : R^N \times R^M \to R^N \; ; \; (x, \lambda) \mapsto g(x, \lambda), \tag{2.5}$$

where $x(k)$ and x are state vectors in R^N, $\lambda \in R^M$ are the parameters, and the function g is assumed to be C^∞. Note that, for simplicity, we have used the same symbols for the state variables x, the parameters λ, and others in both continuous-time and discrete-time systems. A discrete-time system can be a Poincaré map to take periodic solutions into consideration in autonomous or periodically forced nonautonomous differential equations. The existence of a fixed point x^* satisfying

$$x^* - g(x^*, \lambda) = 0 \tag{2.6}$$

is assumed. The Jacobian matrix of g at the fixed point x^* is indicated by

$$D(x^*(\lambda), \lambda) := \left.\frac{\partial g(x, \lambda)}{\partial x}\right|_{x=x^*(\lambda)}. \tag{2.7}$$

The roots of the characteristic equation denoted by

$$\det(\mu I - D(x^*(\lambda), \lambda)) = 0 \tag{2.8}$$

become the characteristic multipliers of the fixed point. The fixed point is hyperbolic, if all absolute values of the eigenvalues of D are different from unity. If every characteristic multiplier of the hyperbolic fixed point is located inside the unit circle, then it is asymptotically stable. When the hyperbolicity is destroyed by varying the parameters, a local bifurcation occurs. Generic bifurcations are the tangent, period-doubling, and Neimark-Sacker bifurcations, which correspond to the critical distribution of characteristic multiplier μ such that $\mu = +1$, $\mu = -1$, and $\mu = e^{i\theta}$ with $i = \sqrt{-1}$, respectively. Further, a pitchfork bifurcation can appear in a symmetric system as a degenerate case of the tangent bifurcation.

We can simultaneously solve equations consisting of the fixed point equation in (2.6) and the characteristic equation in (2.8) with a fixed μ depending on the bifurcation conditions to obtain unknown bifurcation sets x^* and λ_m.

2.2.3 Robust Bifurcation Analysis

Let $\mu_{\max}(D)$ be the maximum value of the real parts (or the absolute values) of eigenvalues with respect to the matrix D for a continuous-time system in (2.3) (or a discrete-time system in (2.7)). We denote this as a function of the parameters as follows

$$\rho(\lambda) := \mu_{\max}(D(x^*(\lambda), \lambda)).$$

Consider that the dynamics f or g with parameter values λ^* defines a system after being affected by unexpected factors, and the steady-state $x^*(\lambda^*)$ has a low degree of stability, which means that the value of $\rho(\lambda^*)$ is near the condition of a bifurcation. Then, the purpose of robust bifurcation analysis is to find $\lambda \in R^M$ such that

$$\rho(\lambda) < \rho(\lambda^*)$$

for given $\lambda^* \in R^M$ satisfying

$$\left.\frac{\partial \rho(\lambda)}{\partial \lambda_m}\right|_{\lambda=\lambda^*} \neq 0$$

for some m's in $\{1, 2, \ldots, M\}$. We assume that the unexpected factors do not change during the process.

The corresponding eigenvalues can be used to make a contour along which the real part of an eigenvalue is equal to zero to analyze the stability of an equilibrium observed in a continuous-time system. This should be a curve in the parameter plane defining the boundary of a region in which the equilibrium stably exists. The level curves at a fixed maximum eigenvalue in the real part similarly imply the degree of stability in parameter space. Robust bifurcation analysis provides a method for finding the values of the parameters at which stable equilibrium has a higher degree of stability. By obtaining a set of parameters

$$\Lambda := \arg\min_{\lambda \in R^M} \rho(\lambda) \tag{2.9}$$

subject to $\rho(\lambda) < \rho(\lambda^*)$, then it enables us to design a system that has a steady-state with a high degree of stability. The same argument can be applied to a fixed point observed in a discrete-time system by taking into consideration the absolute values of eigenvalues instead of the real parts.

2.3 Method of Robust Bifurcation Analysis

We present a method for finding the set of parameters for the optimization problem in (2.9), assuming that the characteristic multiplier with maximum absolute value is real in the case of discrete-time systems.

Because the eigenvalues are not differentiable with respect to the parameters at points where they coalesce, we consider the optimization problem below instead of directly solving (2.9):

$$\min_{\lambda \in R^M,\ \nu \geq \rho(\lambda)} J(\lambda, \nu),$$

where

$$J(\lambda, \nu) := \det\left(\nu I - D(x^*(\lambda), \lambda)\right). \tag{2.10}$$

The characteristic polynomial J is non-negative for $\nu \geq \rho(\lambda)$ and the optimization problem is formulated to minimise the maximum absolute value of eigenvalues of $D(x^*(\lambda), \lambda)$ for discrete-time systems and the real parts of eigenvalues of $D(x^*(\lambda), \lambda)$ for continuous-time systems by varying the parameter λ. Note that the method provides a uniform treatment of both continuous-time and discrete-time systems according to the above assumption on discrete-time systems, and that, when the characteristic multiplier with maximum absolute value is negative for discrete-time systems, the problem of optimization should be modified to maximise the minimum real characteristic multiplier (see Sect. 2.4.2 for an example). We use a continuous gradient method to obtain the values of the parameters λ and the supplementary variable ν. The descent flow is given by the initial value problem of the following differential equations:

$$\begin{aligned}\frac{d\lambda}{d\tau} &= -(\nu - \rho(\lambda))\frac{\partial J}{\partial \lambda}^{\mathsf{T}}, \\ \frac{d\nu}{d\tau} &= -(\nu - \rho(\lambda))\frac{\partial J}{\partial \nu}.\end{aligned} \tag{2.11}$$

Here, the solution $(\lambda(\tau), \nu(\tau))$ is a function of the independent variable τ with the initial conditions $\lambda(0) = \lambda^*$ and $\nu(0) > \rho(\lambda^*)$. When $\nu \neq \rho(\lambda)$, the gradient parts of the right hand sides in (2.11) are given by

$$\begin{aligned}\frac{\partial J}{\partial \lambda_m} &= -\mathrm{tr}\left(\mathrm{adj}(\nu I - D)\frac{\partial D}{\partial \lambda_m}\right) \\ &= -J\mathrm{tr}\left((\nu I - D)^{-1}\frac{\partial D}{\partial \lambda_m}\right),\end{aligned} \tag{2.12}$$

$$\begin{aligned}\frac{\partial J}{\partial \nu} &= \mathrm{tr}\left(\mathrm{adj}(\nu I - D)\right) \\ &= J\mathrm{tr}\left((\nu I - D)^{-1}\right),\end{aligned} \tag{2.13}$$

for $m = 1, 2, \ldots, M$, where $\mathrm{tr}(\cdot)$ and $\mathrm{adj}(\cdot)$ correspond to the trace and adjugate. We need the derivative of the (i, j) element of D in (2.12) with respect to the parameter λ_m. For the difference system g, this is expressed by

$$\frac{\partial}{\partial \lambda_m}\frac{\partial g_i}{\partial x_j} = \sum_{n=1}^{N}\frac{\partial^2 g_i}{\partial x_j \partial x_n}\frac{\partial x_n^*}{\partial \lambda_m} + \frac{\partial^2 g_i}{\partial x_j \partial \lambda_m}.$$

Here, $\partial x_n^*/\partial \lambda_m$, $n = 1, 2, \ldots, N$ can be obtained by differentiating the fixed point equation in (2.6) with respect to λ_m. Then, we have

$$\frac{\partial x^*}{\partial \lambda_m} = \left(I - \left.\frac{\partial g}{\partial x}\right|_{x=x^*}\right)^{-1}\left.\frac{\partial g}{\partial \lambda_m}\right|_{x=x^*}.$$

A similar argument can be applied to the differential dynamics f.

Let us present a theoretical result for the behaviour of the solution to the dynamical system in (2.11). Note that the subspace of the state $(\lambda, \nu) \in R^{M+1}$ satisfying $\nu = \rho(\lambda)$ is an equilibrium set of (2.11). Therefore, the trajectories cannot pass through the set, according to the uniqueness of solutions for the initial value problem. This leads to the fact that, if we choose initial values $(\lambda(0), \nu(0))$ with $\nu(0) > \rho(\lambda(0))$, any solution $(\lambda(\tau), \nu(\tau))$ stays in the subspace $\nu > \rho(\lambda)$ for all $\tau > 0$. Under this condition, the derivative of J along the solution to (2.11) is given by

$$\left.\frac{dJ}{d\tau}\right|_{(2.11)} = \begin{pmatrix} \frac{\partial J}{\partial \lambda} & \frac{\partial J}{\partial \nu} \end{pmatrix} \begin{pmatrix} \frac{d\lambda}{d\tau} \\ \frac{d\nu}{d\tau} \end{pmatrix}$$
$$= -(\nu - \rho(\lambda)) \left(\left\| \frac{\partial J}{\partial \lambda} \right\|_2^2 + \left(\frac{\partial J}{\partial \nu} \right)^2 \right)$$
$$< 0,$$

where the last inequality is obtained because $\nabla J \neq 0$ if $\nu > \rho(\lambda)$. Then, we can see that the value of $J(\lambda(\tau), \nu(\tau))$ with $J > 0$, for all τ, monotonically decreases as time passes.

An interior point method [12] is effective when implementing the algorithm to solve the optimization problem.

2.4 Numerical Examples

Here, we present two representative examples (equilibrium points and periodic solutions) of our robust bifurcation analysis for continuous-time systems. An example of a discrete-time system is shown in [11].

2.4.1 Equilibrium Point

The first example involves the following two-dimensional differential equations, known as the BvP (Bonhöffer-van der Pol) equations:

$$\frac{dx_1}{dt} = \frac{3}{2}(x_1 - \frac{1}{3}x_1^3 + x_2), \qquad (2.14)$$
$$\frac{dx_2}{dt} = -\frac{2}{3}(-\lambda_1 + x_1 + \lambda_2 x_2).$$

The result with our method is shown in Fig. 2.1. A contour plot of an eigenvalue with the maximum real part is also presented. The curve T indicates the saddle-node bifurcation of an equilibrium. Equation (2.14) has three equilibria in the gradation region. A pair of stable and unstable equilibria disappears at the saddle-node bifurcation. Here, we try to find the parameter values at which the equilibrium has a high degree of stability. The eigenvalue is -0.04 at the initial parameter values λ^* labeled by a. The eigenvalue becomes -0.47 at the parameter value b after our method was applied. Thus, we can avoid the occurrence of the saddle-node bifurcation and obtain a high degree of stability by automatically changing the parameters when considering

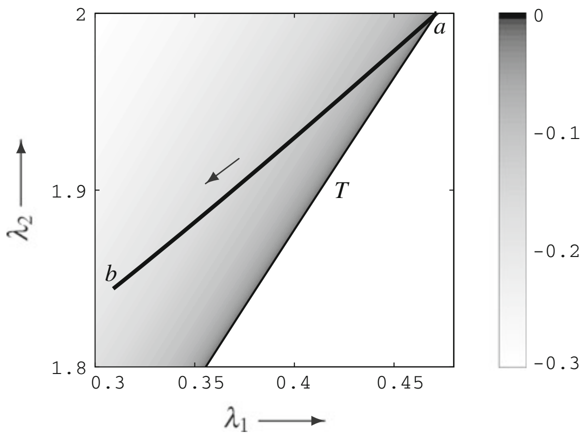


Fig. 2.1 *Bold solid line* indicates the trace of maximum eigenvalue of the equilibrium in BvP equations. Contour plot represents the maximum real part of the eigenvalues, as scaled by the gradation bar. The maximum eigenvalue is equal to 0 on the saddle-node bifurcation *curve* denoted by T

that the situation of the equilibrium with a low degree of stability at the parameter values λ^* near a bifurcation is caused by the effect of unexpected factors.

Figure 2.2 outlines the basins of attraction for the stable equilibrium before and after our method was applied. The phase diagrams in Fig. 2.2a, b correspond to the parameter values a and b in Fig. 2.1. If we put the initial states in the shaded region, we can obtain the targeted equilibrium labeled e in Fig. 2.2 as a steady-state. We can see that our method expanded the basins of attraction by comparing Fig. 2.2a, b.

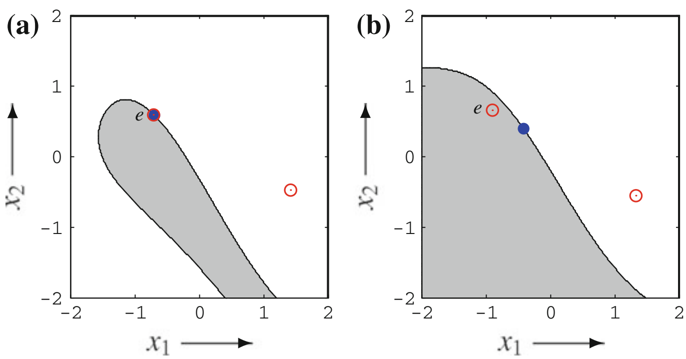


Fig. 2.2 Basins of attraction of equilibria for BvP equations. *Red open* and *blue closed circles* correspond to stable and unstable equilibria. The *red open circle* with label e and the *blue closed circle* are very close in **a** because the parameter value is near the saddle-node bifurcation. The stable manifolds of the saddle-type equilibrium (the *blue closed circle*) form the basin boundary between *shaded* and *unshaded* regions

2.4.2 Periodic Solution

Next, we consider a periodic solution in (2.1). We can reduce it to a fixed point defined by (2.6) by constructing a Poincaré map. Thus, we can use the method described in Sect. 2.3. Here, we study a system of the Luo-Rudy (LR) model [13] with a synaptic input (I_{syn}) described by the following 8-dimensional ordinary differential equations:

$$\begin{aligned} C\frac{dV}{dt} &= -(I_{Na} + I_{Ca} + I_K + I_{K_1} + I_{Kp} + I_b + I_{syn}), \\ \frac{dy}{dt} &= \frac{y_\infty - y}{\tau_y}, \quad (y = m, h, j, d, f, X), \\ \frac{d[Ca]_i}{dt} &= -10^{-4} I_{Ca} + 0.07(10^{-4} - [Ca]_i), \end{aligned} \tag{2.15}$$

where ionic currents are given by

$$\begin{aligned} I_{Na} &= 23m^3hj(V - E_{Na}), \quad I_{Ca} = G_{Ca}df(V - E_{Ca}), \\ I_K &= \overline{G_K}XX_i(V - E_K), \quad I_{K_1} = \overline{G_{K1}}K_{1_\infty}(V - E_{K1}), \\ I_{Kp} &= 0.0183K_p(V - E_{K_p}), \quad I_b = 0.03921(V + 59.87), \\ I_{syn} &= G_{syn}(V_{syn} - V)\frac{\tau_1}{\tau_2 - \tau_1}\left(-\exp\left(-\frac{t'}{\tau_1}\right) + \exp\left(-\frac{t'}{\tau_2}\right)\right). \end{aligned}$$

Here, we define that the time t' is reset at every basic cycle length (BCL) defined by the period of the external synaptic current I_{syn}. Detailed explanation and normal parameter values for the LR model are given in [13].

The LR model is of the ventricular muscle cell. In a previous study, we clarified alternans corresponding to a two-periodic state that appears by a period-doubling bifurcation through changing the value of the parameter BCL. It is well known that the alternans triggers cardiac electrical instability (ventricular arrhythmias) and may cause sudden cardiac death. Thus, suppressing alternans is important for reducing the risk of sudden death. Here, using our proposed algorithm we show the suppression of the alternans.

Figure 2.3 shows a result of maximising the smallest real characteristic multiplier of the LR model in the parameter plane $\lambda = (\text{BCL}, G_{syn})$. Initial parameter values are marked by the closed circle, which is very close to the period-doubling bifurcation. Then, the smallest real characteristic multiplier is -0.998. In Fig. 2.3, two-periodic solutions generated by the period-doubling bifurcation exist in the shaded parameter region. The waveform of the membrane potential after the period-doubling bifurcation is shown in Fig. 2.4. This waveform shows a typical alternans. From this initial point our algorithm changes the values of the parameters to avoid the bifurcation

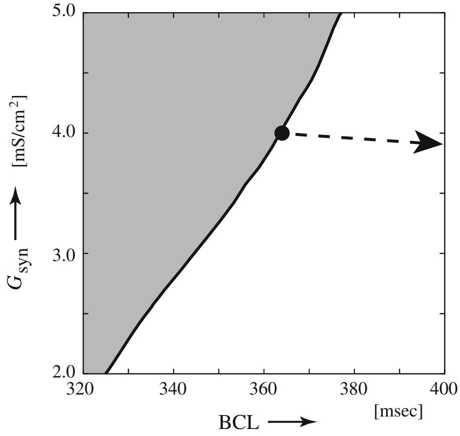
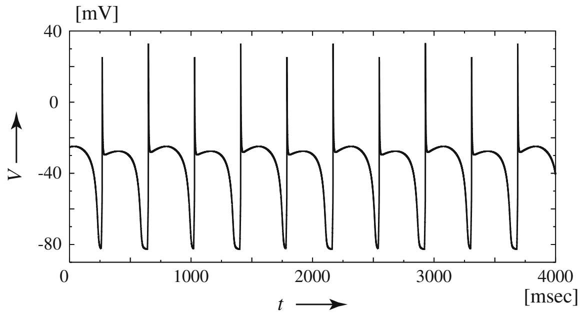


Fig. 2.3 Result of robust bifurcation analysis for the LR model. Initial parameter values are $G_{syn} = 4.0$, BCL = 362 and $\nu = -1.15$. The *solid* and *dashed curves* indicate the period-doubling bifurcation set in the parameter plane (BCL, G_{syn}) and the trace of these parameter values while our control method works. In the *shaded region*, stable two-periodic solutions generated by the period-doubling bifurcation exist

Fig. 2.4 Waveforms of the membrane potentials (alternans). BCL = 376 and $G_{syn} = 5.0$



(the trace is shown by the dashed curve.). After that, the smallest real characteristic multiplier becomes −0.434. Thus, our method can avoid the alternans and reduce the risk of sudden death.

2.5 Conclusion

Traditional bifurcation analysis in parameter space deals with contour or level sets of the eigenvalue for a bifurcation, whereas our robust bifurcation analysis is used for finding parameter sets that cause a gradient decrease in the bifurcating eigenvalue. An automatic trace of the gradient based on our method can effectively construct a robust system that has a steady-state with a high degree of stability.

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