Chapter 11
Feedback Control of Spatial Patterns in Reaction-Diffusion Systems

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11.1 Introduction

There have been plenty of studies on pattern formation such as thermal convection problems, Turing patterns in reaction-diffusion systems, phase transitions in material sciences, and so on (see [1]). There, one can tune a parameter so that the uniform stationary solution loses its stability against perturbations with certain non-zero wavelength. As a result, a spatially non-uniform stationary solution may appear. Thus the local bifurcation analysis is a first step to understand the onset of the pattern formation.

Let us consider an activator-inhibitor system of reaction-diffusion equations as follows.

$$\begin{cases} u_t = D_u \Delta u + f(u, v), \\ v_t = D_v \Delta v + g(u, v), \quad x \in \Omega. \end{cases} \tag{11.1}$$

Here, the typical reaction part of (11.1) is the following:

$$\begin{cases} f(u, v) = u - u^3 - v, \\ g(u, v) = 3u - 2v. \end{cases} \tag{11.2}$$

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This reaction-diffusion system (11.1) has a trivial stationary solution $u = v = 0$ under the Neumann boundary condition:

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad x \in \partial\Omega$$

or the periodic boundary condition. Here, Ω is a bounded interval or a rectangle in $\mathbb{R}$ or $\mathbb{R}^2$. Although the origin $u = v = 0$ is asymptotically stable under the ODE:

$$\begin{cases} \dot{u} = f(u, v), \\ \dot{v} = g(u, v), \end{cases} \tag{11.3}$$

it might be possible to become unstable in the sense of (11.1). This is the so-called Turing instability or diffusion induced instability which is realized by taking the diffusion constants appropriately as we shall review in the following section.

Once we understand the linearized instability mechanism it turns out that non-trivial patterns may appear from the trivial solution by a standard bifurcation analysis. As one can see in Fig. 11.1, solutions of (11.1) may become closer and closer to stripe patterns. Now if the non-linear terms are not symmetric as in (11.2) and moreover they include quadratic terms as

$$\begin{cases} f(u, v) = u - u^3 - au^2 - v, \\ g(u, v) = 3u - 2v, \end{cases} \tag{11.4}$$

the dot-like patterns may appear rather than stripe.

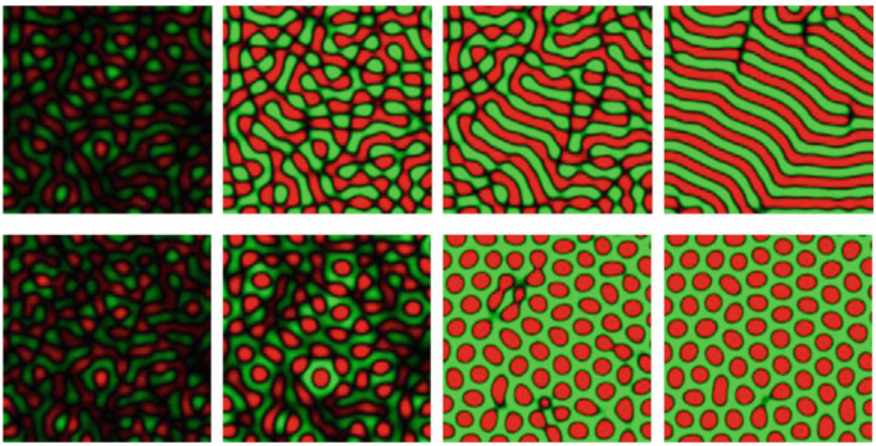

Fig. 11.1 Time sequences of the numerical simulation for (11.1) and (11.4) in a square region with the periodic boundary condition. The coefficient of the quadratic term $a = 0$ and $a = 0.1$ for the *upper* and *lower lines*, respectively. The columns correspond to the time sequences $t = 4, 10, 20, 50$ from the left to right and only the value of u is displayed

In Sect. 11.2, after reviewing the Turing instability from the linearized analysis we introduce the idea which helps us understand the Turing instability mechanism more clearly. In fact we can consider that the inhibitor in the 2-component reaction-diffusion system plays a role of negative global feedback to the activator. By extending this idea we can construct 3-component reaction-diffusion systems which have different types of instabilities.

Since we are considering the bounded interval or a rectangle with Neumann or periodic boundary conditions the problem has the so-called SO(2) symmetry which basically comes from translation invariance. Therefore at the bifurcation point more than one critical modes may interact with each other. As a result, different types of patterns emerge simultaneously. However, the stability of these solutions depends on the nonlinear terms. Now the question is whether we can stabilize each non-trivial pattern or not. Section 11.3 is devoted to partially answer to this question. We formulate and solve a feedback stabilization problem of unstable non-uniform spatial pattern in reaction-diffusion systems. By considering spatial spectrum dynamics, we obtain a finite dimensional approximation that takes over the semi-passivity of the original partial differential equation. By virtue of this property, we can show the diffusive coupling in the spatial frequency domain achieves the desired pattern formation.

11.2 Pattern Formation by Global Feedback

11.2.1 Turing Instability

Turing instability is known to be the fundamental mechanism for the onset of pattern formations. Let us first review the reason why the uniform stationary solution $u = v = 0$ in the reaction-diffusion system (11.1) loses its stability. In fact the linearized stability of the Fourier mode with wavenumber k is characterized by the matrix:

$$M_k = \begin{pmatrix} f_u - k^2 D_u & f_v \\ g_u & g_v - k^2 D_v \end{pmatrix}.$$

Since the fixed point $u = v = 0$ is assumed to be stable in the sense of the ODE (11.3), it turns out that trace$M_0 = f_u + g_v < 0$ and $\det M_0 = f_u g_v - f_v g_u > 0$ hold true. Therefore, M_k has real part positive eigenvalues if and only if $\det M_k < 0$, and M_k has one real positive eigenvalue in this case. Now $\det M_k$ is given by

$$\det M_k = f_u g_v - f_v g_u - (D_v f_u + D_u g_v)k^2 + D_u D_v k^4.$$

Therefore, if $D_v f_u + D_u g_v > 0$ and $(D_v f_u + D_u g_v)^2 - 4(f_u g_v - f_v g_u) D_u D_v > 0$ hold, then $\det M_k < 0$ for some non-zero wavenumber k and, as a result, M_k has a

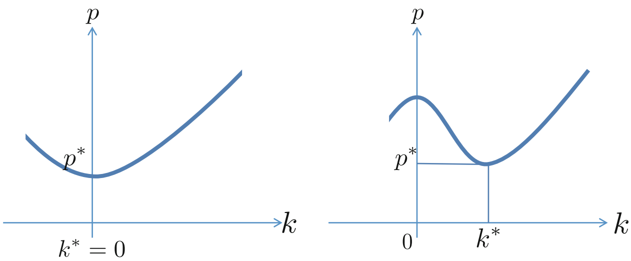

Fig. 11.2 Two types of neutral stability *curves*. The onset of the pattern formation can be observed when the neutral stability *curve* has the shape as in the *right figure*

positive eigenvalue. We have only to take D_v sufficiently large compared to D_u to satisfy these conditions. At the moment of stability change the Fourier mode with a wave-number $k \approx k_c = (\frac{f_u g_v - f_v g_u}{D_u D_v})^{1/4}$ may become unstable.

11.2.2 Interpretation of Turing Instability by Global Feedback

We shall introduce the idea that one can control the onset of pattern formation by a global feedback (see also [2]). Or we can even explain the Turing instability mechanism from the viewpoint of global feedback. Suppose we observe pattern formation from a uniform rest state by changing a parameter p. More precisely, the uniform stationary solution is stable when p is less than a critical value p^* and becomes unstable against certain wavenumber $k^* > 0$ when $p > p^*$. Therefore pattern formation can be observed in the system where the neutral stability curve $p = \phi(k)$ has the property such that $\phi(k)$ attains its minimum at k^* (see Fig. 11.2). Here, the neutral stability curve is the stability boundary in the (k, p)-plane. In other words, it is equivalent to say that the uniform stationary solution is stable against perturbations of wavenumber k when $p < \phi(k)$ and it is unstable when $p > \phi(k)$. Let us consider the following scalar reaction-diffusion equation as a simple example

$$u_t = Du_{xx} + pu, \tag{11.5}$$

where D is a diffusion constant and p is a parameter. Since it is equivalent to

$$\frac{d\tilde{u}}{dt} = (p - Dk^2)\tilde{u}$$

by Fourier transformation, the neutral stability curve is $p - Dk^2 = 0$. Now, it is clear that $\phi(k) = Dk^2$ does not attain its minimum at positive k and the instability for $k = 0$ takes place earlier than any other non-trivial perturbation. Therefore we can not observe pattern formations in scalar reaction-diffusion equations.

Since the scalar reaction-diffusion equation is not sufficient to produce the onset of patterns as we saw above let us consider the following activator-inhibitor type reaction-diffusion equations:

$$\begin{cases} u_t = D_1 u_{xx} + pu - sw, \\ \tau w_t = D_2 w_{xx} + u - w. \end{cases} \tag{11.6}$$

Here, s is assumed to be positive so that species w has negative feedback effect to the species u. On the contrary, u gives positive effects to w. Moreover suppose that the time constant for w is sufficiently small. Let us just plug 0 into τ for simplicity. Then we have the following equations by using the Fourier transformation.

$$\begin{cases} \dfrac{d\tilde{u}}{dt} = (p - D_1 k^2)\tilde{u} - s\tilde{w}, \\ 0 = (-1 - D_2 k^2)\tilde{w} + \tilde{u}. \end{cases} \tag{11.7}$$

Since $\tilde{w}$ is solved as $\tilde{w} = \dfrac{\tilde{u}}{1 + D_2 k^2}$, it turns out that the system (11.7) is equivalent to the following scalar equation:

$$\frac{d\tilde{u}}{dt} = \left(p - D_1 k^2 - \frac{s}{1 + D_2 k^2}\right)\tilde{u}.$$

Therefore the neutral stability curve is given by

$$p = \phi(k) := D_1 k^2 + \frac{s}{1 + D_2 k^2}. \tag{11.8}$$

Moreover we have the following inequality:

$$\begin{aligned} p &= \frac{D_1}{D_2}(1 + D_2 k^2) + \frac{s}{1 + D_2 k^2} - \frac{D_1}{D_2} \\ &\geq 2\sqrt{\frac{D_1 s}{D_2}} - \frac{D_1}{D_2}. \end{aligned}$$

Since the last equality holds true if and only if $(1 + D_2 k^2)^2 = \dfrac{D_2 s}{D_1}$, we have

Theorem 11.1 *The following two conditions are equivalent to each other for the system* (11.6)*:*

- *The neutral stability curve $p = \phi(k)$ attains its minimum at $k = k_0 > 0$.*
- *The constants s, D_1, and D_2 satisfy*

$$\frac{D_2 s}{D_1} > 1. \tag{11.9}$$

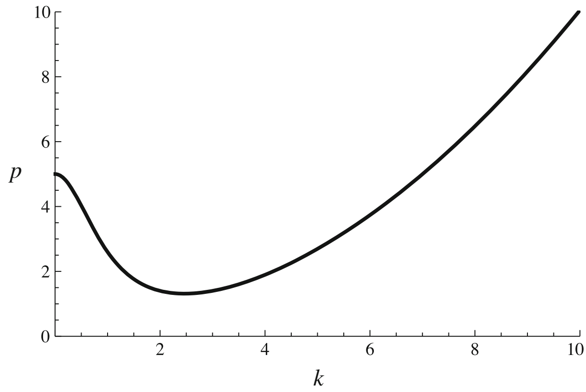

Fig. 11.3 The neutral stability *curve* (11.8). Here, $D_1 = 0.1$, $D_2 = 1$, and $s = 5$

Therefore the neutral stability curve has the desired property when the condition (11.9) is satisfied (see Fig. 11.3).

We can study the bifurcation structure for the system (11.6) with nonlinear terms:

$$\begin{cases} u_t = D_1 u_{xx} + pu - sw + f(u, w) \\ \tau w_t = D_2 w_{xx} + u - w \end{cases} \tag{11.10}$$

on a finite interval $x \in [0, L]$ with periodic boundary condition. Here, $f(u, w)$ denotes higher order terms of u, w. Notice that the bifurcation structure with the Neuman boundary condition is included in the periodic case. By the periodicity we have only to consider countable Fourier modes with the fundamental wavenumber $k_0 = \frac{2\pi}{L}$. We need to take into account wavenumber of the form $k = mk_0$ where $m \in \mathbb{Z}$. It is convenient to draw the neutral stability curves C_m for each mode $m > 0$ in the (k_0, p)-plane:

$$C_m := \{(k_0, p) | p = \phi(mk_0)\}.$$

If the condition (11.9) is satisfied we can conclude that three different neutral stability curves don't intersect at the same point except for $(0, 0)$. This means at most two Fourier modes can be critical at the same time. We call these intersection points degenerate bifurcation points. Moreover, it turns out that there are only degenerate bifurcation points with adjacent modes $n, n+1$ on the first instability (see Fig. 11.4). Therefore the dynamics near the first instability point is governed by the normal form as follows:

Theorem 11.2 *Assume the condition (11.9) is satisfied for (11.6). Suppose the system size* $L > 2\pi\sqrt{\frac{D_1 D_2}{sD_2 - D_1}}$ *(*$k_0 < \sqrt{\frac{sD_2 - D_1}{D_1 D_2}}$*) there is one critical point p where the first instability takes place for n mode (*$n > 0$*). There are the following two cases:*

(I) $m = \pm n$ *are the only critical modes.*
(II) $m = \pm n, \pm n'$ *are the only critical modes.*

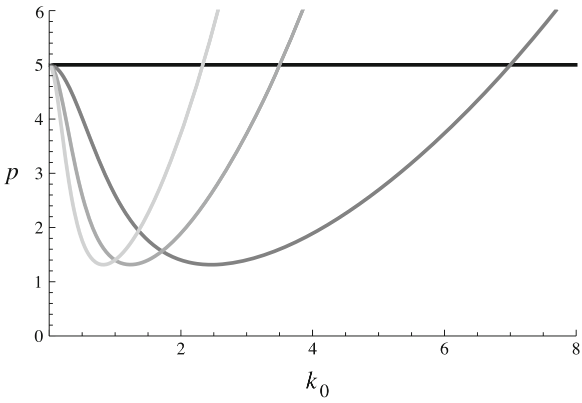

Fig. 11.4 The neutral stability *curves* C_m for $m = 0, 1, 2$ and 3 modes (from *black* to *light gray*). It turns out that more modes become unstable when $k_0 = 2\pi/L$ is small

Here, $n' = n + 1$ *or* $n' = n - 1$. *Dynamics of the solutions to* (11.10) *close to first instability points can be reduced to those on the center manifolds. Moreover they are generically governed by the normal form by taking appropriate real numbers* a, b, c, d.*:*

$$\text{(I)}\quad \dot{\alpha_n} = \lambda\alpha_n + a|\alpha_n|^2\alpha_n + O(4),$$

$$\text{(II)}\quad \begin{cases} \dot{\alpha_n} = \lambda\alpha_n + (a|\alpha_n|^2 + b|\alpha_{n'}|^2)\alpha_n + O(4), \\ \dot{\alpha_{n'}} = \mu\alpha_{n'} + (c|\alpha_n|^2 + d|\alpha_{n'}|^2)\alpha_{n'} + O(4). \end{cases}$$

Here, $\{\alpha_m \in \mathbb{C}; |m| = n\}$ *and* $\{\alpha_m \in \mathbb{C}; |m| = n, n'\}$ *are the critical modes for the cases* (I) *and* (II)*, respectively. Moreover, n is assumed to be larger than 1 in the case* (II).

Proof Since (11.10) is translation invariant the center manifold and its dynamics can be constructed so that they have SO(2) symmetry. Therefore the normal form dose not have quadratic terms.

Remark 11.1 Dynamics close to the degenerate point between 1 and 2 is the exception of the above theorem. Armbruster, Guckenheimer and Holmes [3] studied the interaction of two steady-state bifurcations in a system with O(2)-symmetry, assuming 1:2 resonance for the wavenumbers associated with the critical modes. They found there are rich variety of dynamics in the 1:2 resonance normal form:

$$\begin{aligned} \dot{a_1} &= \overline{a_1}a_2 + a_1(\mu_1 + e_{11}|a_1|^2 + e_{12}|a_2|^2), \\ \dot{a_2} &= ca_1^2 + a_2(\mu_2 + e_{21}|a_1|^2 + e_{22}|a_2|^2) \end{aligned}$$

when the coefficient of the quadratic term c is negative. However, in this case, the two coefficients of quadratic terms have the same sign and, as a result, $c = +1$ which means there are no nontrivial dynamics.

Notice that we solved $(1 - D_2\partial_x^2)w = u$ in the Fourier space. Since the inverse Fourier image of $\dfrac{1}{1 + D_2k^2}$ is $e^{-\frac{|x|}{\sqrt{D_2}}}$, w can be written as

$$w = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{|x-y|}{\sqrt{D_2}}} u(y)dy.$$

Theorem 11.3 *The reaction-diffusion system* (11.6) *is equivalent to the following equation when* $\tau = 0$*:*

$$u_t = D_1 u_{xx} + pu - \frac{s}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{|x-y|}{\sqrt{D_2}}} u(y)dy. \tag{11.11}$$

It should be mentioned that Britton [4] studied the ecological pattern formation also with global feedback. In the population growth model it is natural that intra-specific competition for resources depends not simply on population density at one point in space and time but on a weighted average involving values at all previous points and at all points in space. Therefore he considered

$$u_t = D\Delta u + u\left[p + \alpha u - (1+\alpha)G ** u\right] \tag{11.12}$$

instead of

$$u_t = D\Delta u + u(p - u), \tag{11.13}$$

where, $G ** u$ is a weighted average of u:

$$G ** u := \int_{\mathbb{R}}\int_{-\infty}^{t} G(x-y, t-s)u(y,s)dsdy.$$

Here, G is assumed to be positive, L^1 and moreover,

$$\int_{\mathbb{R}}\int_{-\infty}^{t} G(x,t)dtdx = 1.$$

Notice that the Eq. (11.12) is equivalent to (11.11) when $G(x,t) = \delta(t)\tilde{G}(x)$ with an appropriate $\tilde{G}$. Also (11.12) is equivalent to (11.14) when $G(x,t) = \delta(x)\delta(t)$.

11.2.3 0:1:2-Mode Interaction

We have seen in the previous subsection that the normal form analysis for double degenerate bifurcation points is sufficient to study the Eq. (11.10). On the other hand, we can realize the triple degeneracy if we take another component into account in a three component system as follows:

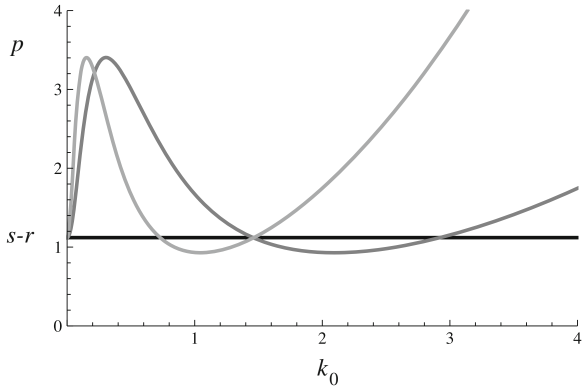

Fig. 11.5 The neutral stability *curves* $C_m = \{(k_0, p) | p = \phi(mk_0)\}$ for $m = 0, 1, 2$ modes (*black*, *gray*, *light gray*, respectively). Here, ϕ is given by (11.15). *Three curves* $C_0,\ C_1,\ C_2$ intersect at the same point for $k_0 \neq 0$ by taking the constants $s,\ t,\ D_1,\ D_2,\ D_3$ appropriately. For instance, $s = 5,\ t = 3.88,\ D_1 = 0.1,\ D_2 = 2.0,\ D_3 = 40.0$ in the figure

$$\begin{aligned} u_t &= D_1 u_{xx} + pu - sv + rw, \\ \tau v_t &= D_2 v_{xx} + u - v, \\ \tau w_t &= D_3 w_{xx} + u - w. \end{aligned} \tag{11.14}$$

Here, τ is again assumed to be small positive number. We also assume negative feedback ($s > 0$) effect from v and positive feedback ($r > 0$) from w. As a consequence of similar discussions in the previous subsection the neutral stability curve for (11.14) is given by

$$p = \phi(k) := D_1 k^2 + \frac{s}{1 + D_2 k^2} - \frac{r}{1 + D_3 k^2}. \tag{11.15}$$

Now, by taking $D_3 \gg D_2 \gg 1$ we can tune the parameters so that the system (11.14) has triple degeneracy of 0:1:2 modes (see Fig. 11.5). The degenerate dynamics by 0:1:2 modes has been studied also by using the normal form analysis. In fact, Smith, Moehlis and Holmes [5] studied the generic quadratic normal form:

$$\begin{aligned} \dot{a_0} &= \mu_0 a_0 + 2(B_1 |a_1|^2 + B_2 |a_2|^2), \\ \dot{a_1} &= \overline{a_1} a_2 + a_1(\mu_1 - B_1 a_0), \\ \dot{a_2} &= -a_1^2 + a_2(\mu_2 - B_2 a_0). \end{aligned}$$

They conclude there are a wide variety of dynamics including heteroclinic cycles. On the other hand, Ogawa and Okuda [6] studied the cubic normal form also with resonance terms under the up-down symmetry:

$$\begin{aligned} \dot{z}_0 &= (\mu_0 + a_1 z_0^2 + a_2 z_1^2 + a_3 z_2^2) z_0 + a_4 z_1^2 z_2, \\ \dot{z}_1 &= (\mu_1 + b_1 z_0^2 + b_2 z_1^2 + b_3 z_2^2) z_1 + b_4 z_0 z_1 z_2, \\ \dot{z}_2 &= (\mu_2 + c_1 z_0^2 + c_2 z_1^2 + c_3 z_2^2) z_2 + c_4 z_0 z_1^2 \end{aligned}$$

and they showed the existence of oscillatory bifurcation from the 1-mode stationary solution not only in the general normal form but also in a particular 3-component reaction-diffusion system.

11.2.4 Wave Instability

Let us introduce pattern formation by an oscillatory instability. We shall start with a 2-dimensional ODE system which has the Hopf bifurcation point:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} p & -p \\ q & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} O(2) \\ O(2) \end{pmatrix}.$$

Here, $q > 1$ is a given constant and we control the parameter p. The linearized matrix $M = \begin{pmatrix} p & -p \\ q & -1 \end{pmatrix}$ has a characteristic polynomial $\lambda^2 - (\mathrm{trace} M)\lambda + \det M$. Therefore M has a pair of purely imaginary eigenvalues when $p = 1$. Now we are interested in the bifurcation in the following reaction-diffusion system:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} D_1 u_{xx} \\ D_2 v_{xx} \end{pmatrix} + \begin{pmatrix} p & -p \\ q & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(u, v) \\ g(u, v) \end{pmatrix}. \tag{11.16}$$

Here, f, g are assumed to consist of higher order terms of u and v. It turns out from the linearized eigenvalue problem about 0 the stability of the trivial solution against perturbation with wavenumber k can be controlled by the matrix

$$M_k = \begin{pmatrix} p - D_1 k^2 & -p \\ q & -1 - D_2 k^2 \end{pmatrix}.$$

Since the oscillatory instability takes place when $\mathrm{trace} M_k = 0$, the neutral stability curve for the Hopf bifurcation is given by $p = 1 + (D_1 + D_2)k^2$. Therefore, we can not observe any stable spatially non-trivial oscillating patten.

Let us introduce the third component w which has the negative feedback effect to the activator u and consider the following three component reaction-diffusion system instead of (11.16):

$$\begin{pmatrix} u_t \\ v_t \\ \tau w_t \end{pmatrix} = \begin{pmatrix} D_1 u_{xx} \\ D_2 v_{xx} \\ D_3 w_{xx} \end{pmatrix} + \begin{pmatrix} p & -p & -s \\ q & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} f(u, v) \\ g(u, v) \\ 0 \end{pmatrix}. \tag{11.17}$$

Here, we also assume $\tau > 0$ is sufficiently small. By a similar argument to (11.7) we can obtain the linearized matrix for the wavenumber k when $\tau = 0$ as follows:

$$A_k = \begin{pmatrix} p - D_1 k^2 - \frac{s}{1+D_3 k^2} & -p \\ q & -1 - D_2 k^2 \end{pmatrix}.$$

Thus, a necessary condition for the Hopf instability is given by

$$\mathrm{trace} A_k = p - 1 - (D_1 + D_2)k^2 - \frac{s}{1 + D_3 k^2} = 0.$$

Let us also call it the neutral stability curve for the Hopf instability

$$p = \phi(k) := 1 + (D_1 + D_2)k^2 + \frac{s}{1 + D_3 k^2}, \tag{11.18}$$

although it is just a necessary condition. Because it has the same form as (11.8) we have the following theorems.

Theorem 11.4 *The following two conditions are equivalent to each other for the system* (11.17)*:*

- *The neutral stability curve* $p = \phi(k)$ *attains its minimum at* $k = k_0 > 0$.
- *The constants* s, D_1 *and* D_2 *satisfy*

$$\frac{s D_3}{D_1 + D_2} > 1. \tag{11.19}$$

Theorem 11.5 *The reaction-diffusion system* (11.17) *is equivalent to the following equations when* $\tau = 0$*:*

$$\begin{aligned} u_t &= D_1 \Delta u + pu - pv + O(2) - \frac{s}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{|x-y|}{\sqrt{D_3}}} u(y) dy, \\ v_t &= D_2 \Delta v + qu - v + O(2). \end{aligned} \tag{11.20}$$

We may have spatio-temporal oscillating patterns if the system (11.17) satisfies the condition (11.19) and this is called "wave instability" (see also [7]).

If we consider the system on a finite interval $(0, L)$ similarly to the above discussion there are three cases in the sense of bifurcation analysis.

Theorem 11.6 *Assume the condition* (11.19) *is satisfied for* (11.17)*. For a given system size* L *(or fundamental wavenumber* k_0*) there is one critical point* p *where the first instability takes place for* n *mode* ($n > 0$)*. There are the following three cases:*

(I) $m = \pm n$ *are the only critical modes.*
(II) $m = 0, \pm n$ *are the only critical modes.*
(III) $m = \pm n, \pm n'$ *are the only critical modes.*

Here, $n' = n + 1$ *or* $n' = n - 1$ *and all the critical modes are of the Hopf type.*

We need to distinguish two critical modes $\{\alpha_m \in \mathbb{C}; |m| = n\}$ in this case since they are the Hopf type critical modes. We introduce the normal form only for the case (I) for the simplicity.

Theorem 11.7 *Under the same setting as* Theorem 11.6 *dynamics of the solutions to* (11.17) *close to first instability points can be reduced to those on the center manifolds. Moreover they are generically governed by the normal form by taking appropriate real numbers a, b in the case* (I)*:*

$$\begin{cases} \dot{\alpha_n} = \mu\alpha_n + (a|\alpha_n|^2 + b|\alpha_{-n}|^2)\alpha_n + O(4), \\ \dot{\alpha_{-n}} = \mu\alpha_{-n} + (b|\alpha_n|^2 + a|\alpha_{-n}|^2)\alpha_{-n} + O(4). \end{cases}$$

Notice that we don't need any non-resonance condition in this theorem in contrast to the standard double Hopf theorem since we have SO(2) symmetry. We can conclude that there are two typical oscillating solutions: rotating wave and standing wave solutions. Stability for both solutions depends on the normal form coefficients a, b. It turns out from simple calculations that $(a, b) = (-3, -6)$ when $f(u, v) = -u^3$, $g(u, v) = 0$. Therefore the rotating wave solution is stable while the standing wave solution is unstable in this case (Fig. 11.6).

Notice that the wave instability criterion depends on s, D_3 and $D_1 + D_2$. Therefore we can realize the situation where both Turing and wave instabilities take place at the same value of p by changing the ratio between D_1 and D_2 without changing $D_1 + D_2$. We draw the neutral stability curves for Turing instability ($\det A_k = 0$) and wave instability ($\mathrm{trace} A_k = 0$) in Fig. 11.7 .

Again it should be mentioned that Gourley and Britton [8] studied the similar 2-component reaction-diffusion system with the global feedback from the population dynamics.

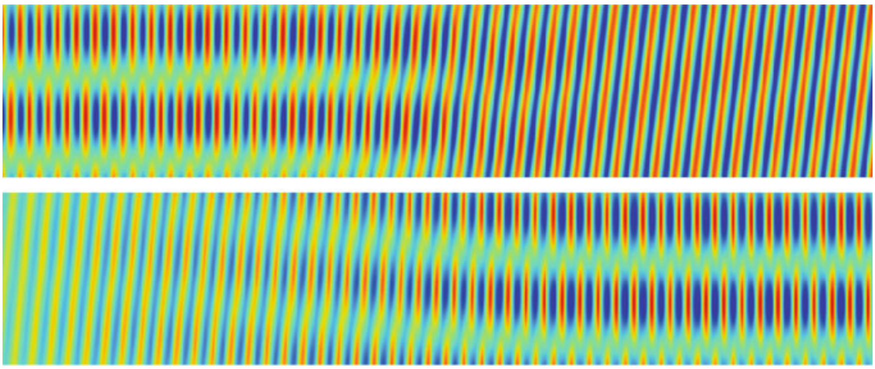

Fig. 11.6 Numerical simulations for (11.17) under the parameters $L = 2\pi$, $p = 2.01$, $q = 1.5$, $s = 2.0$, $D_1 = 0.8$, $D_2 = 0.2$, $f(u, v) = -u^3$, and $g(u, v) = 0$ in the *upper figure*. Therefore the standing wave is unstable and the solution converges to the rotating wave. By taking another nonlinear terms such as $f(u, v) = 0$ and $g(u, v) = uv$

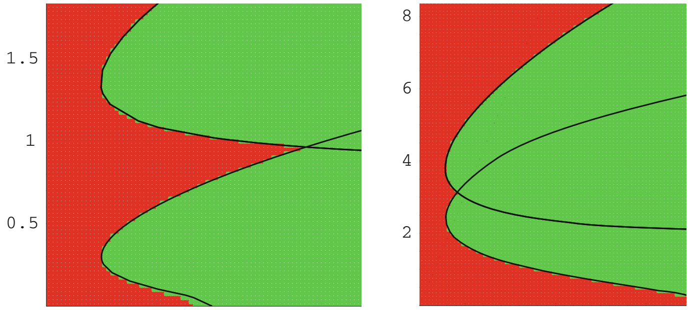

Fig. 11.7 *Two curves* of $\det A_k = 0$ (*upper curve*) and $\mathrm{trace} A_k = 0$ (*lower*) are described in the (k, p)-plane. Parameter sets are $(s, D_1, D_2, D_3, \tau) = (0.6, 0.24, 0.76, 50, 0.1)$ and $(s, D_1, D_2, D_3, \tau) = (20, 0.2, 0.23, 1, 0.01)$ in *left* and *right figures*, respectively

11.2.5 Summary

In this section we introduce the idea that helps us understand the mechanism of the Turing instability from the viewpoint of global feedback. It turns out that not only the Turing instability but also other instabilities relating to pattern formation can be obtained by the global feedback although it might not be described in the system explicitly.

11.3 Selective Stabilization of Turing Patterns

11.3.1 Reaction-Diffusion Systems

In this section, we consider (11.1) with

$$\begin{cases} f(u, v) = a_{11}u - a_{12}v - u^3, \\ g(u, v) = a_{21}u - a_{22}v, \end{cases} \tag{11.21}$$

where the spatial domain $\Omega := [0, L_x] \times [0, L_y]$ with the periodic boundary condition. We denote (Fig. 11.8)

$$z(t, x, y) = \begin{bmatrix} u(t, x, y) \\ v(t, x, y) \end{bmatrix} \in \mathbb{R}^2.$$

This reaction-diffusion system has a trivial equilibrium pattern

$$z_{eq}(x, y) \equiv 0 \text{ on } \Omega. \tag{11.22}$$

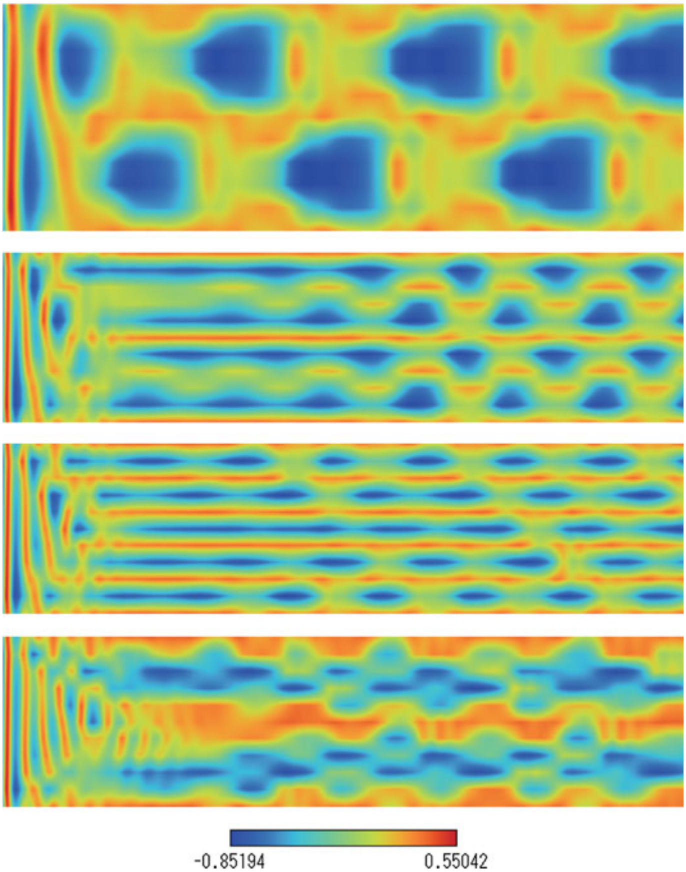

Fig. 11.8 Oscillating patterns observed when both Turing and wave instabilities coexist. *Vertical* and *horizontal axis* denote the interval (0, L) and time, respectively

Even if the reaction term is globally stable, this trivial equilibrium of reaction-diffusion system is not necessarily stable [9].

Consider the spatial Fourier transform

$$z_m(t) := \begin{bmatrix} u_m(t) \\ v_m(t) \end{bmatrix} := \int_\Omega z(t, x, y) p_m(x, y)^* dxdy \in \mathbb{C}^2 \tag{11.23}$$

for *wave number* $m = (m_x, m_y) \in \mathbb{Z}^2$, where

$$p_m(x, y) := \frac{1}{\sqrt{L_x L_y}} e^{2\pi j (\frac{m_x x}{L_x} + \frac{m_y y}{L_y})}$$

and p_m^* is its complex conjugate. This satisfies

$$z(t, x, y) = \sum_{m \in \mathbb{Z}^2} z_m(t) \mathrm{e}^{2\pi j (\frac{m_x x}{L_x} + \frac{m_y y}{L_y})}.$$

Note that

$$z_m = z_{-m}^* \quad \text{for all} \quad m \in \mathbb{Z}^2 \tag{11.24}$$

since the dynamics of our interest is real-valued. Then, it is useful to investigate the dynamics of each spatial component $\{z_m(t)\}_{m \in \mathbb{Z}^2}$ instead of $\{z(t, x, y)\}_{(x,y) \in \Omega}$.

Let us consider the localized dynamics around the trivial pattern z_{eq}. It should be emphasized that each wave number has its *decoupled* local dynamics:

$$\frac{d}{dt} z_m(t) = A_m z_m(t), \tag{11.25}$$

$$A_m := A - s_m D =: \begin{bmatrix} \bar{a}_{11} & -a_{12} \\ a_{21} & -\bar{a}_{22} \end{bmatrix}, \tag{11.26}$$

$$A := \begin{bmatrix} a_{11} & -a_{12} \\ a_{21} & -a_{22} \end{bmatrix}, \quad D := \begin{bmatrix} D_u & 0 \\ 0 & D_v \end{bmatrix}, \tag{11.27}$$

$$s_m := \left(\frac{2\pi m_x}{L_x} \right)^2 + \left(\frac{2\pi m_y}{L_y} \right)^2. \tag{11.28}$$

Note that when $D_u \neq D_v$, stability of A does not necessarily guarantee stability of A_m. In such a case, the corresponding spatial wave p_m *grows* around the z_{eq}. Further discussion on the pattern formation needs to consider the effect of nonlinearity.

11.3.2 Problem Formulation

Let us formulate a stabilization problem of unstable spatial patterns. We define the set of wave numbers for which the local dynamics is unstable: We assume that the finite set $\mathcal{M} \subset \mathbb{Z}^2$ satisfies

1. A_m has at least one eigenvalue in $\mathbb{C}_+$ if $m \in \mathcal{M}$,
2. A_m is stable if $m \notin \pm\mathcal{M} := \{\pm m : m \in \mathcal{M}\}$, and
3. if $m \in \mathcal{M}$, then $-m \notin \mathcal{M}$.

Because $A_m = A_{-m}$ and (11.24), we imposed the condition (3) in order to avoid redundancy. It should be emphasized that $s_{m_1} = s_{m_2}$ can hold for $m_1 \neq m_2$.

Next, for the feedback control problem [10], we assume that we can observe and also manipulate u in a spatially distributed manner. Thus, the controlled reaction-diffusion system is given by

$$\begin{cases} u_t = D_u \Delta u + f(u, v) + \boldsymbol{w}, \\ v_t = D_v \Delta v + g(u, v), \qquad (x, y) \in \Omega. \\ \boldsymbol{w} = W(\boldsymbol{u}), \end{cases} \tag{11.29}$$

Problem 11.1 Let $\theta_m \in \mathbb{R},\ m \in \mathscr{M}$ be given. Under definitions and assumptions above, find a feedback control law $W(\cdot)$ such that

1. $\boldsymbol{z}$ does not diverge,
2. z_m for $m \notin \pm\mathscr{M}$ asymptotically vanishes,
3. $\mathrm{e}^{j\theta_m} z_m$ for $m \in \mathscr{M}$ converges to the same nonzero value, and
4. $\boldsymbol{w}(t, x, y)$ asymptotically vanishes.

11.3.3 Feedback Control of Center Manifold Dynamics

In view of the assumption on the stability of A_m, let us assume that z_m is negligible for $m \notin \pm\mathscr{M}$ to avoid infinite-dimensionality [11]. We analyze the dynamics that z_m should obey when there exists no third order resonance: If $n_1,\ n_2,\ n_3 \in \mathscr{M}$ satisfy $n_1 + n_2 + n_3 = m \in \pm\mathscr{M}$, then at least one of n_i's is equal to m.

By putting $z_m \approx 0$ for $m \notin \pm\mathscr{M}$, the wave number m component appears only from the combination $(n_1, n_2, n_3) = (m, n, -n)$ and its permutation, where $n \in \pm\mathscr{M}$ is arbitrary. Therefore, we obtain the following approximation in the spatial frequency domain:

$$\frac{d}{dt}\begin{bmatrix} u_m \\ v_m \end{bmatrix} = A_m \begin{bmatrix} u_m \\ v_m \end{bmatrix} + \begin{bmatrix} -u_m \left(3|u_m|^2 + 6\sum_{n\neq m} |u_n|^2\right) \\ 0 \end{bmatrix} + \begin{bmatrix} w_m \\ 0 \end{bmatrix}, \tag{11.30}$$

where

$$w_m(t) := \int_\Omega w(t, x, y) p_m^*(x, y) dx dy.$$

Now, Problem 11.1 is naturally rephrased in this domain:

Problem 11.2 Consider the complex-valued system (11.30) and $\theta_m \in \mathbb{R},\ m \in \mathscr{M}$. Then, find a feedback control law

$$w_m(t) = W_m((u_n(t))_{n\in\mathscr{M}})$$

such that,

(1′) z_m is *ultimately bounded*, that is, there exists (initial state independent) $R > 0$ such that

$$\limsup_{t\to\infty} \|z_m\| < R \quad \text{for all} \quad m \in \mathscr{M}, \tag{11.31}$$

(3′-a)

$$\lim_{t\to\infty} |e^{j\theta_m} z_m(t) - e^{j\theta_n} z_n(t)| = 0 \quad \text{for all} \quad m, n \in \mathscr{M}, \tag{11.32}$$

(3′-b) the origin is locally unstable, and

(4′)

$$\lim_{t\to\infty} w_m(t) = 0 \quad \text{for all} \quad m \in \mathscr{M}. \tag{11.33}$$

For this problem, we can simply obtain a desired control law by considering the specific structure of Problem 11.2.

Theorem 11.8 *For* Problem 11.2, *consider the following diffusive coupling on* $\mathscr{M}$

$$w_m(t) = \sigma \sum_{n\in\mathscr{M}, n\neq m} \gamma_{mn}(e^{j\theta_n - j\theta_m} u_n(t) - u_m(t)), \tag{11.34}$$

where $\gamma_{mn} \geq 0$. *If* γ_{mn} *is associated to a strongly connected*[1] *graph on* $\mathscr{M}$, *then there exists* $\underline{\sigma} > 0$ *such that the control law* (11.34) *with any strength* $\sigma > \underline{\sigma}$ *satisfies the following:*

- *the condition* (1′) *in* Problem 11.2 *holds.*
- *If* $s_m = \bar{s}$ *(see* (11.28) *for the definition) for all* $m \in \mathscr{M}$, *then* (3′-a), (3′-b), (4′) *are also satisfied.*

Proof When we redefine $e^{j\theta_m} z_m$ as z_m, this theorem is equivalent to [12, Theorem1].

As a next step, we need to embed this proposed control law onto the original partial differential equation. The corresponding input pattern should be given as

$$w(t, x, y) := 2\sigma \sum_{m\in\mathscr{M}} Re\left(p_m\left(\sum_{n\neq m,\ n\in\mathscr{M}} \gamma_{mn}(e^{j(\theta_n-\theta_m)} u_n - u_m)\right)\right), \tag{11.35}$$

where u_m for $m \in \mathscr{M}$ is defined by (11.23). We have already verified numerically that this control law achieves the expected pattern formation.

[1] For any $m_i, m_j \in M$, there exists a sequence $\{i_1, i_2, \ldots, i_K\}$ on $\mathscr{M}$ such that $\gamma_{i_k, i_{k+1}} > 0$ for all $k = 1, \ldots, K-1$ and $i_1 = i, i_K = j$.

11.3.4 Numerical Example

The simulation in this section is executed under the following parameter settings: We take

$$A = \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}, \tag{11.36}$$

$$D_u = 0.2, \tag{11.37}$$

$$D_v = 1.5, \tag{11.38}$$

and

$$L_x = \frac{8\pi}{\sqrt{1.8}}, \quad L_y = \frac{L_x}{\sqrt{3}}. \tag{11.39}$$

For these parameters, A_m has one (real) unstable eigenvalue for $m = \pm m_i$

$$m_1 = (4, 0), \quad m_2 = (2, 2), \quad m_3 = (2, -2), \tag{11.40}$$

and A_m stable otherwise.

The initial patterns are randomly generated but sufficiently close to z_{eq}. We can expect this reaction-diffusion system can generate three roll patterns corresponding to m_i's in (11.40). Figure 11.9 is the snapshots of $\boldsymbol{u}(t, x, y)$ for the uncontrolled ($\boldsymbol{w} = 0$) reaction-diffusion dynamics. Actually, all of observed patterns (including transient ones) look like superpositions of these roll patterns.

Next, we attempt to stabilize another spacial pattern by feedback control. We implement the distributed actuation (11.35) where

$$\theta_{m_1} = 0, \quad \theta_{m_2} = 2\pi/3, \quad \theta_{m_3} = 4\pi/3$$

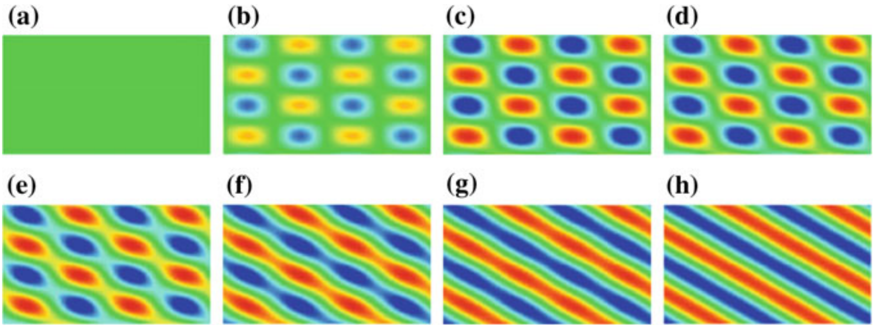

Fig. 11.9 Spatial pattern formation in uncontrolled reaction-diffusion systems. Depending on the initial profile, only one of the three base roll patterns appears. **a** $t = 0$. **b** $t = 3000$. **c** $t = 3500$. **d** $t = 3600$. **e** $t = 3700$. **f** $t = 4000$. **g** $t = 4300$. **h** $t = 5000$

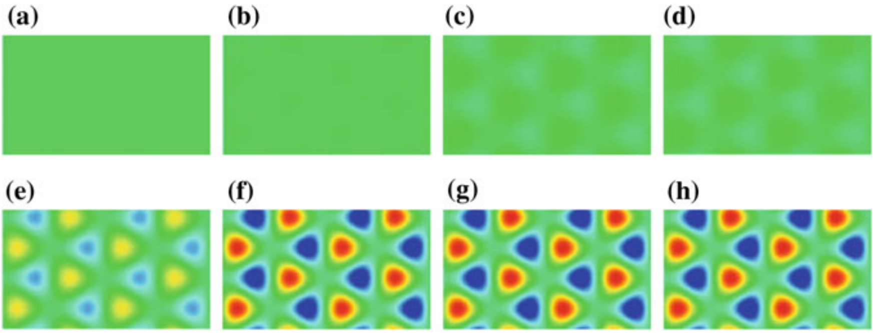

Fig. 11.10 Spatial pattern formation in controlled reaction-diffusion systems. Independent of the initial profile, the rotational symmetry is achieved. **a** $t = 0$. **b** $t = 2000$. **c** $t = 2500$. **d** $t = 2750$. **e** $t = 3000$. **f** $t = 3500$. **g** $t = 4000$. **h** $t = 5000$

and

$$\gamma_{mn} = 1, \quad m \neq n, \quad \sigma = 0.1. \tag{11.41}$$

Figure 11.10 shows the snapshots of $u(t, x, y)$ for the controlled dynamics. We can observe the convergence to another spatial pattern consisting of the three spatial spectra.

11.3.5 Summary

In this section, we formulated a feedback stabilization problem of unstable non-uniform spatial pattern in the reaction-diffusion systems. This problem was solved in the finite dimensionally approximated system. The proposed law, which is a diffusive coupling in the spatial spectrum, achieves desired spectrum consensus while preserving the instability of the trivial equilibrium pattern.

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